Lecture 10, MATH 5010

Option Sensitivities, Greeks, and Their Hedging

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1 Greeks or Options Sensitivities

1.1 Introduction

The graphs below show call price 6 month, 2 month, and 2 hours before expiration.

The call has strike price $K = 100$, volatility $s = 30\%$, interest rate $r = 6\%$, and no dividends.

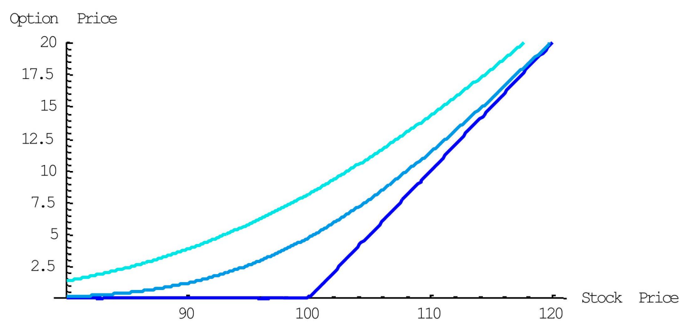


Figure 1: Yo

One can see that before expiration there is a **convexity** in the graph. The convexity is described by the second derivative of the call price with respect to stock price X that is also called Gamma and denoted by a greek letter Γ. This second derivative is positive for call owner, (and put owner too).

The holder of the call makes more money when stock price goes \$1 up then loses less money when stock price goes \$1 down. However for that positive convexity the holder of the option pays with time decay.

If the stock price stays the same as time passes, the call will lose value. The price curve for a call 6 months before expiration is above the curve 2 moths before expiration etc.

1.2 Definition

Option sensitivities. Greeks: Delta, Gamma, Vega, Theta, Rho

Formulas expressing theoretical European option (call or put) price f depend on three "main" variables:

- Time remaining to expiry T
- Current price of the underlying asset X
- Option strike price K

They also depend the following "secondary" variables:

- Volatility of the underlying asset σ
- Current risk-free rate r
- Continuous dividend yield q

Thus we have the aggregate function:

$$f = f(X, K, r, q, \sigma, T)$$

We define "Greeks" or sensitivities of option price with respect to input variables:

- $\theta = \frac{\partial f}{\partial t}$ is "Theta", measuring time-decay of an option.
- $\Delta = \frac{\partial f}{\partial X}$ is "Delta", responsible for sensitivity to changes in underlying
- $\Lambda = \frac{\partial f}{\partial \sigma}$ is "Vega", responsible for sensitivity towards volatility changes
- $\rho = \frac{\partial f}{\partial r}$ is "Rho", measures sensitivity to changes in interest rate
- $\rho_1 = \frac{\partial f}{\partial q}$ is "Rho 1", measures sensitivity to changes in dividend yield
- $\Gamma = \frac{\partial^2 f}{\partial X^2}$ is "Gamma", measures the convexity of the option price.

1.3 Delta

Delta (Δ) is the rate of change of the option price with respect to the underlying

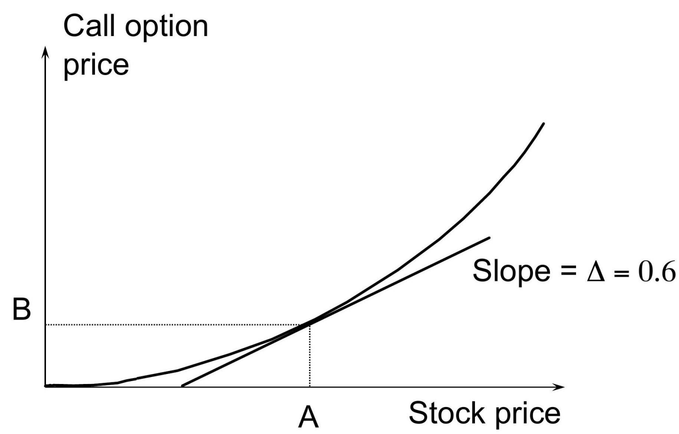


Figure 2: Diagram of Delta

Delta of an option is often measured in percents. For example, if an option to buy 100 shares of XYZ stock has Delta of 45% then in the first approximation the value of the option for the small price move in the stock will change the same way as the value of 45 shares of XYZ. Delta can also be measured in units of the underlying, for stocks in shares.

Delta hedging involves maintaining a Delta neutral portfolio i.e. maintaining of total Delta of option+(stock hedge) zero when stock price change. Delta hedging of a written (sold) option involves a "**buy high, sell low**" trading rule. The delta of a European call on a non-dividend paying stock is

$$N(d_1)$$

The delta of a European put on the non-dividend paying stock is

$$N(d_1) - 1$$

1.4 Theta

Theta Θ or Time Decay of a derivative security is the rate of change of the value with respect to the passage of time

The theta of a long call or long put is usually negative. If time passes and the price of the underlying asset and its volatility remain the same, the value of a long call or long put option declines.

In practice Theta of an option is often calculated for 1 day, thus we have $\Theta =$ Opt price Tomorrow - Opt price Today $=$ Opt price $(T - \frac{1}{365})$to expiry - Opt price **T** to expiry

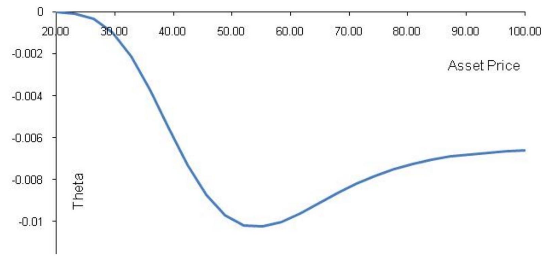


Figure 3: Theta for Call Option: $\mathrm{K} = 50, \sigma = 25\%, \mathrm{r} = 5\%,\ \mathrm{T} = 1$

1.5 Gamma

Gamma Γ is the rate of change of delta Δ with respect to the price of the underlying asset. Gamma is greatest for options that are close to the money.

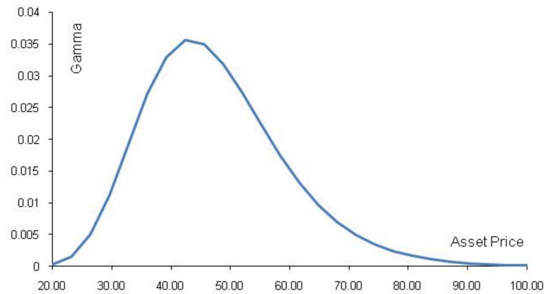
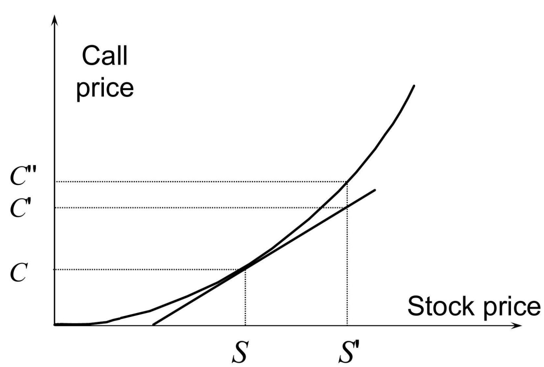


Figure 4: Gamma for Call or Put Option: $\mathrm{K} = 50, \sigma = 25\%, \mathrm{r} = 5\%,\ \mathrm{T} = 1$

Gamma Corrects Delta Hedging Errors Caused By Curvature:



1.5.1 Interpretation of Gamma

For a delta neutral portfolio, $\Delta\Pi \approx \Theta\Delta t + 1/2\Gamma\Delta S^2$

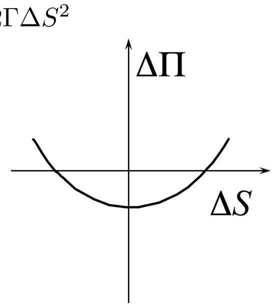


Figure 5: Positive Gamma

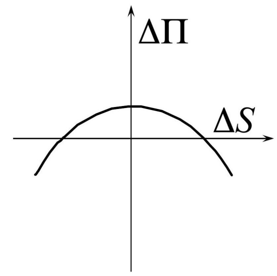


Figure 6: Negative Gamma

1.6 Relationship Between Delta, Gamma, and Theta

For a portfolio Π of options on a stock paying a continuous dividend yield at rate q it follows from the Black-Scholes partial differential equation that:

$$\Theta + (r-q)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Where S is a stock price also denoted by X sometimes.

Vega usually denoted by greek letters Lambda Λ or Nu υ, as there is no greek letter vega, is the rate of change of the value of a derivatives portfolio with respect to volatility

In practice Vega of an option is often calculated for increase in volatility from current value σ, to $\sigma + 1\% = \sigma + 0.01$. $Vega =$ Option price with $(\sigma + 0.01)-$ Option price with σ

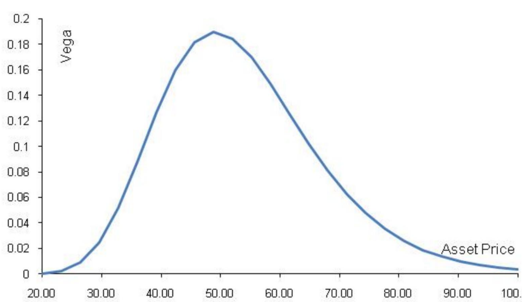


Figure 7: Vega for Call or Put Option: $K = 50, \sigma = 25\%, r = 5\%, T = 1$

2 Hedging the Greeks

2.1 Taylor Series Expansion of Derivatives Portfolio Change

The change in value of a portfolio of derivatives dependent on an asset as a function of of the asset price S, its volatility σ, and time t

$$\begin{aligned}\Delta\Pi &= \frac{\partial \Pi}{\partial S}\Delta\mathbf{S} + \frac{\partial \Pi}{\partial \sigma}\Delta\sigma + \frac{\partial \Pi}{\partial \mathbf{t}}\Delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial \mathbf{S}^2}(\Delta\mathbf{S})^2 + \dots \\ &= \text{Delta} \times \Delta\mathbf{S} + \text{Vega} \times \Delta\sigma + \text{Theta} \times \Delta t + \frac{1}{2}\text{Gamma} \times (\Delta\mathbf{S})^2 + \dots\end{aligned}$$

- Delta $\times$ (Change in Underlying Price). This part depends on the delta of the overall position, and, is a directional component. A positive delta means that the position is bullish and benefits from an increase in the underlying price. A negative delta means that the position is bearish and benefits from a decrease in the underlying price.

- Vega $\times$ Change in Volatility. If the volatility suddenly increases, then, a long derivative position will benefit from it. If the volatility suddenly decreases, then, a long derivative position will lose money as a result of it.

- Theta × (Number of Days Gone by) is the time decay. For a long position in option, theta is negative, and this term is loosing money as time passes. For a short position in option, this term is, respectively, making money.
- Gamma ×1/2 (Change in Underlying Price)ˆ2. The so-called, "gamma term" which reflects the convexity of the option prices. This is term is very important, since it is arises in hedging.

2.1.1 Table of Greeks for European Options

Greeks for European Options on an Asset that Provides a Continuous Yield at a Rate q, T time to expiration:

Greek Letter	Call Option	Put Option
Delta	$e^{-qT}N(d_1)$	$e^{-qT}\left[N(d_1)-1\right]$
Gamma	$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	
Theta	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})\ +qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})\ +qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Table 1: Greeks for Call and Put Options

2.2 Managing Delta, Gamma, and Vega in a Portfolio of Derivatives

Delta can be changed by taking a position in the underlying asset. To adjust gamma and vega it is necessary to take a position in an option or other derivative

2.2.1 Example 1

	Delta	Gamma	Vega
Portfolio	0	-5000	-8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral? Answer: Long 10,000 options, short 6000 of the asset.

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral? Answer: Long 4000 options, short 2400 of the asset.

2.2.2 Example 2

	Delta	Gamma	Vega	WEIGHT
Portfolio	0	-5000	-8000	1
Option 1	0.6	0.5	2.0	w_1
Option 2	0.5	0.8	1.2	w_2
Stock	1	0	0	w_3

What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral?
We solve:

$$
\begin{aligned}
&-5000+0.5w_1+0.8w_2=0\\
&-8000+2.0w_1+1.2w_2=0
\end{aligned}
$$

to get $w_1 = 400$ and $w_2 = 6000$. We require long positions of 400 and 6000 in option 1 and option 2. A short position of 3240 in the asset is then required to make the portfolio delta neutral.