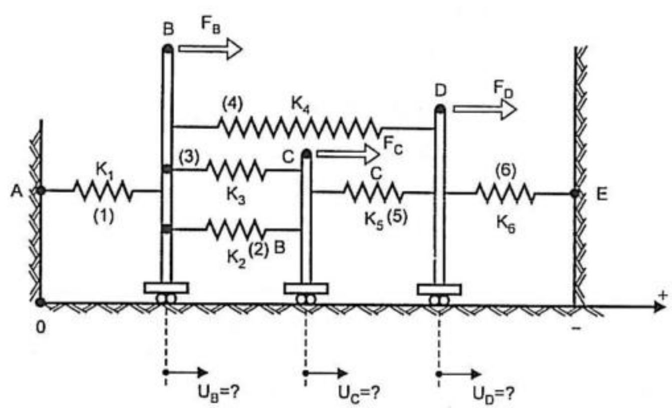
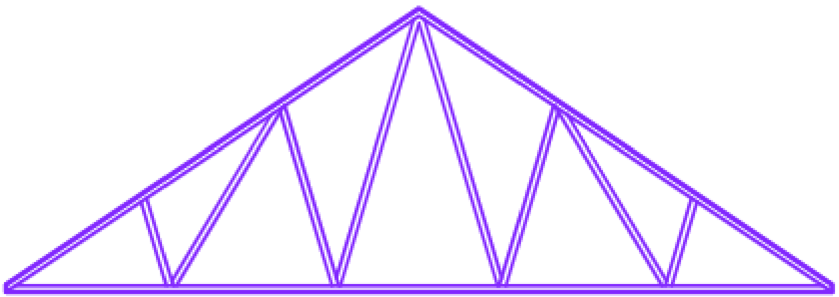
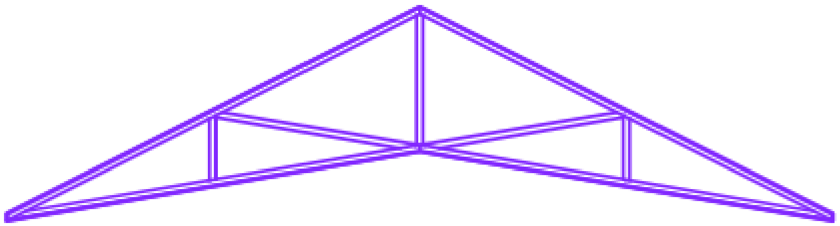
Proposed solution using sparse matrix

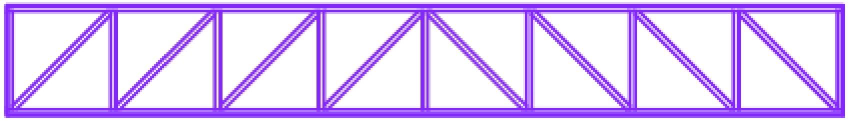
```
clear; clc;
k = [200  100  150  300  400  500];
cte = [ 1 -1 ; -1  1]
k1 = k(1)*cte';  k2 = k(2)*cte'; k3 = k(3)*cte';
k4 = k(4)*cte';  k5 = k(5)*cte';  k6 = k(6)*cte';
n = 5; // number of nodes
i=[1 1  2 2 ];
j=[1 2  1 2 ];
KL1=sparse([i(:),j(:)],k1,[n n]); //AB
KL2=sparse([i(:)+1 j(:)+1],k2,[n n]); //BC
KL3=sparse([i(:)+1 j(:)+1],k3,[n n]); //BC
KL4=sparse([i(:)*2 j(:)*2],k4,[n n]); //BD
KL5=sparse([i(:)+2 j(:)+2],k5,[n n]); //CD
KL6=sparse([i(:)+3 j(:)+3],k6,[n n]); //DE
KG = KL1+KL2+KL3+KL4+KL5+KL6
KG(5,:)=[]; KG(1,:)=[]; KG(:,5)=[]; KG(:,1)=[]
disp(full(KG));
```

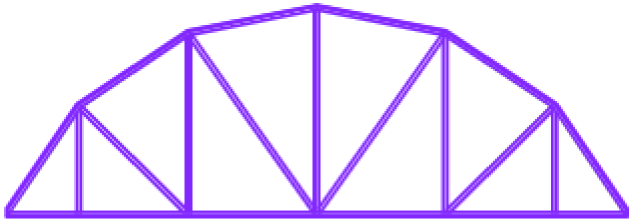


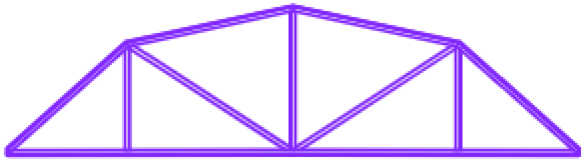
2D Trusses

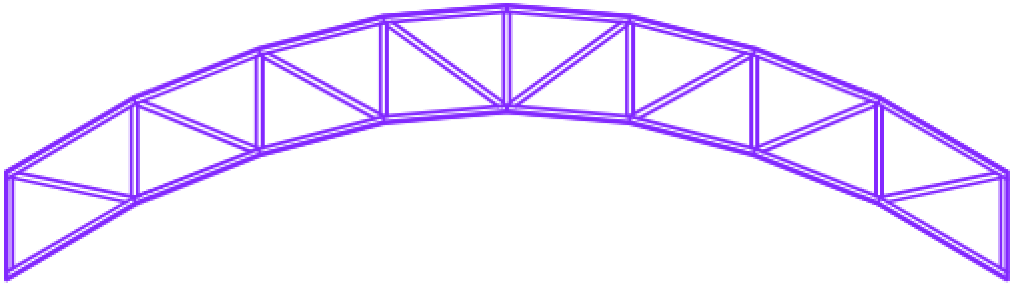








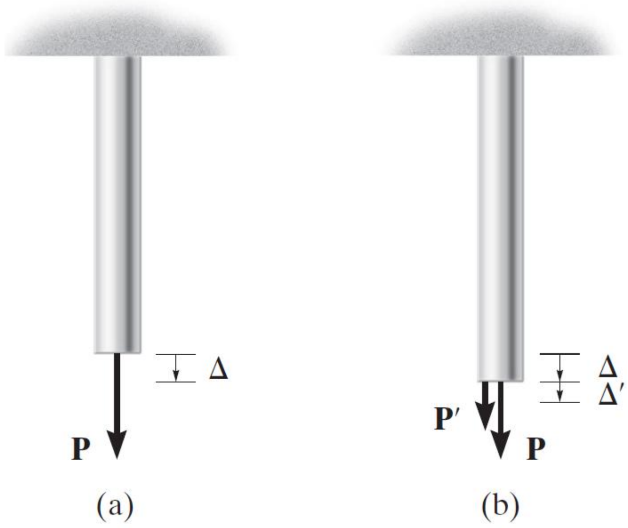




External work and deformation energy

- The work is caused by an internal load and moment.
- The load (P) promotes work when it undergoes a displacement (*dx*) in the same direction.

$$\boxed{U_e = \int_0^x F\,dx}$$

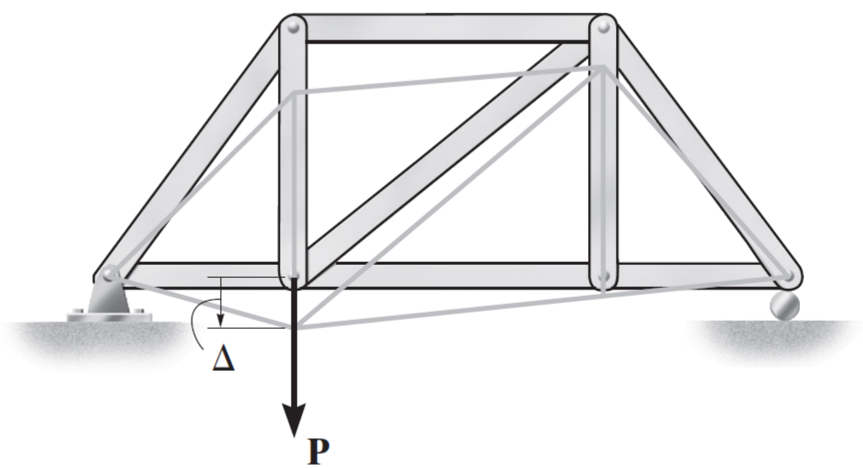


$$When\,F = P, the\ final\,displacement\,is:$$

$$\boxed{U_e = \frac{1}{2} P\Delta}$$

Energy conservation applied to trusses

- All energy methods used in mechanics are based on an energy balance, often called energy conservation.
- Energy conservation for the body is expressed as $U_e = U_i$
- Adding the energies of all the elements of the truss, we have :

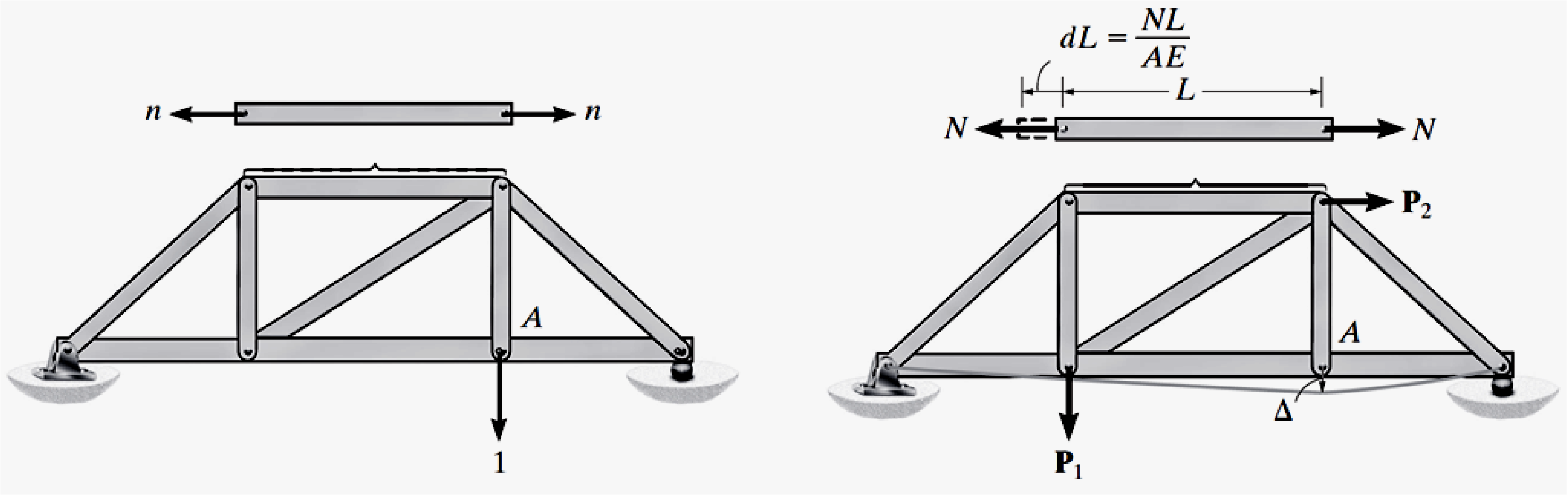


$$\boxed{\frac{1}{2}P\Delta = \sum \frac{N^2 L}{2AE}}$$

where: N is the internal load in each element

Principle of Virtual Work applied to trusses

The internal virtual work for a member is: $$\int_0^L \frac{nN}{AE}\,dx = \frac{nNL}{AE}$$



Unitary virtual load

Real load

Virtual loads method applied to trusses

Virtual work equation

$$\boxed{1 \cdot \Delta = \sum \frac{nNL}{AE}}$$

1 = external virtual load

Δ = joint displacement

n = internal virtual load

N = internal load on a truss element

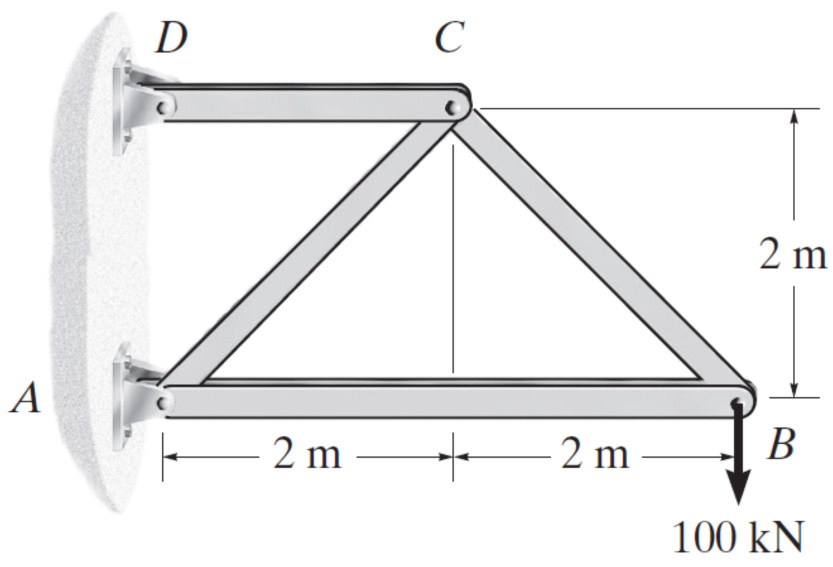
L = element length

A = cross-sectional area

E = modulus of elasticity of the element

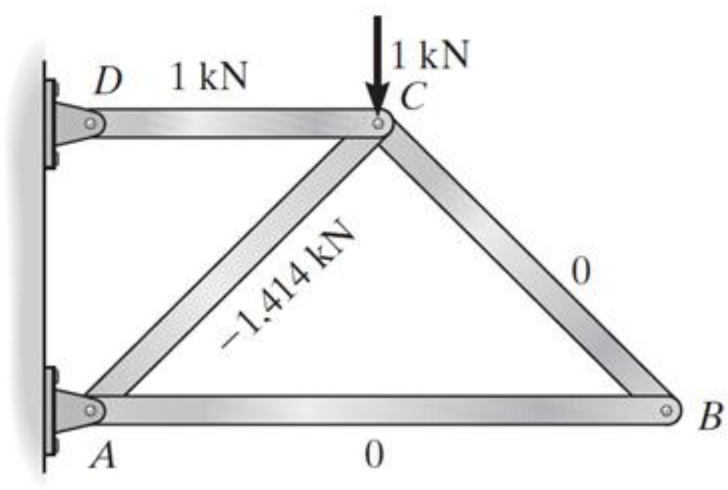
Example

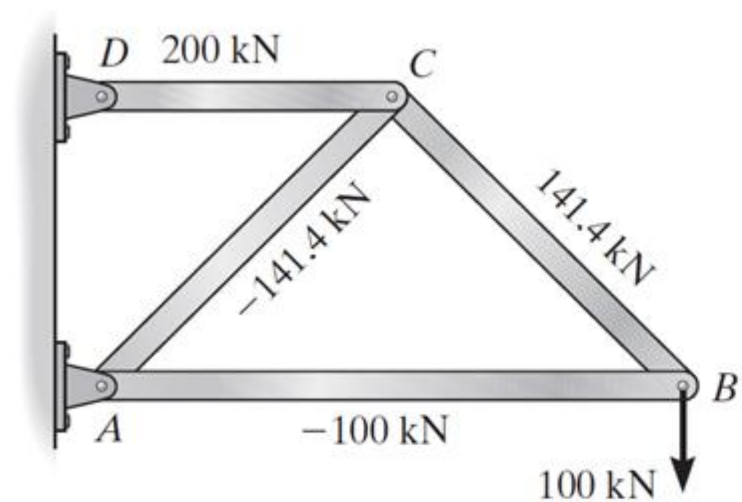
Determine the vertical displacement of the C joint of the steel truss shown below. The cross-sectional area of each element is A = 400 mm^2 and E = 200 GPa.



Solution

Vertical virtual load of 1 kN is added at C node and the load for each element is calculated using the method below.





Unitary virtual load at C Real loads

Virtual loads method applied to trusses

Arranging the data in tabular form, we have:

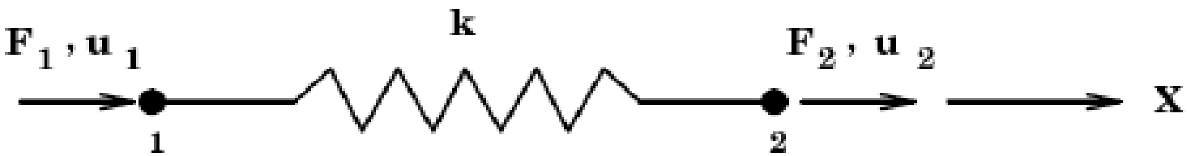
Member	n	N	L	nNL
AB	0	−100	4	0
BC	0	141.4	2.828	0
AC	−1.414	−141.4	2.828	565.7
CD	1	200	2	400
				Σ 965.7 kN$^2\cdot$m

Therefore, $1\text{ kN}\cdot\Delta_{C_v} = \sum\frac{nNL}{AE} = \frac{965.7\text{ kN}^2\cdot\text{m}}{AE}$

Replacing A and E for their numerical values, we have:

$$1\text{ kN}\cdot\Delta_{C_v} = \frac{965.7\text{ kN}^2\cdot\text{m}}{[400(10^{-6})\text{ m}^2]\,200(10^6)\text{ kN/m}^2}$$

$$\Delta_{C_v} = 0.01207\text{ m} = \boxed{12.1\text{ mm}}$$



$$\underbrace{\left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\}}_{\text{Loads}} = \underbrace{\left[\begin{array}{cc} k_{11} & k_{12} \\ k_{21} & k_{22} \end{array} \right]}_{\text{Stiffness matrix}} \underbrace{\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}}_{\text{Nodal displacements}}$$

1) $u2 = 0$

$$k = \frac{F_1}{u_1} \quad \rightarrow \quad F_1 = k u_1$$

$$\sum F_X = 0 : \quad F_1 + F_2 = 0 \quad \rightarrow \quad F_1 = -F_2$$

$$F_1 = -F_2 = k u_1$$

2) $u1 = 0$

$$F_2 = -F_1 = k u_2$$

Truss linear system

$$\begin{cases} F_1 = ku_1 - ku_2 \\ F_2 = -ku_1 + ku_2 \end{cases}$$

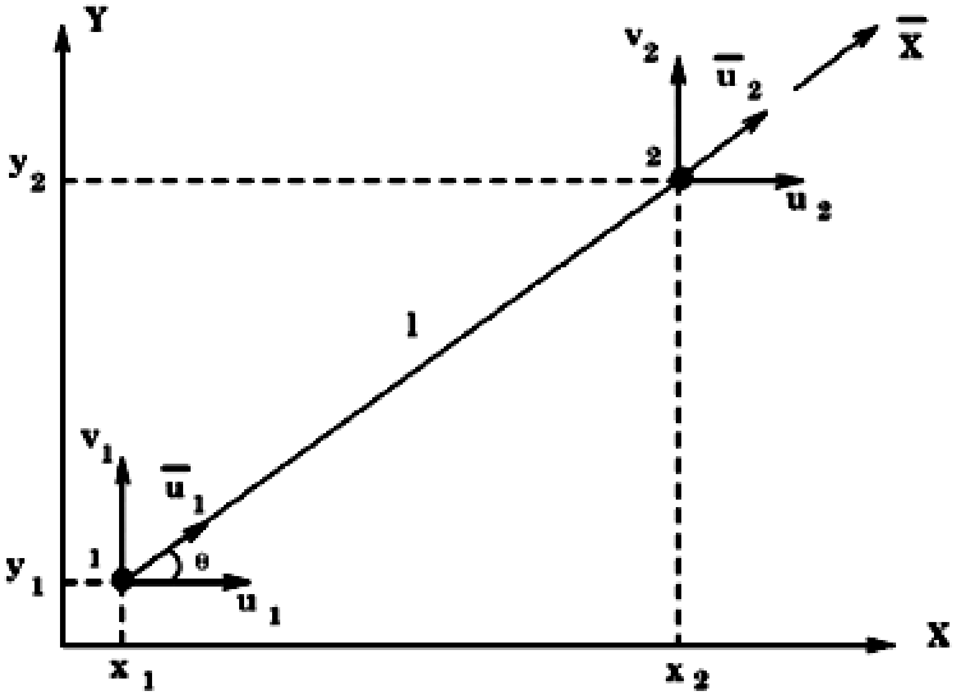
$$\left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\} = \left[\begin{array}{rr} k & -k \\ -k & k \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} \rightarrow \left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\} = k \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}$$

$$\bar{\epsilon} = \frac{\delta}{l} = \frac{1}{l}(\bar{u}_2 - \bar{u}_1)$$

$$\bar{\epsilon} = \frac{1}{l} \left[\begin{array}{cc} -1 & 1 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_1 \\ \bar{u}_2 \end{array} \right\}$$

$$\bar{\sigma} = \frac{E}{l} \left[\begin{array}{cc} -1 & 1 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_1 \\ \bar{u}_2 \end{array} \right\}$$

Local displacement and global displacement



$$\bar{u}_1 = u_1 \cos\theta + v_1 \sin\theta$$
$$\bar{u}_2 = u_2 \cos\theta + v_2 \sin\theta$$

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\{\bar{u}\} = [T]\{u\}$$

u = local displacement
$\bar{u}$ = global displacement

Stiffness matrix in the global system

$$\{\bar{P}_e\} = [\bar{K}_e]\{\bar{u}\} \quad (1)$$

$$\{\bar{u}\} = [T]\{u\} \quad (2)$$

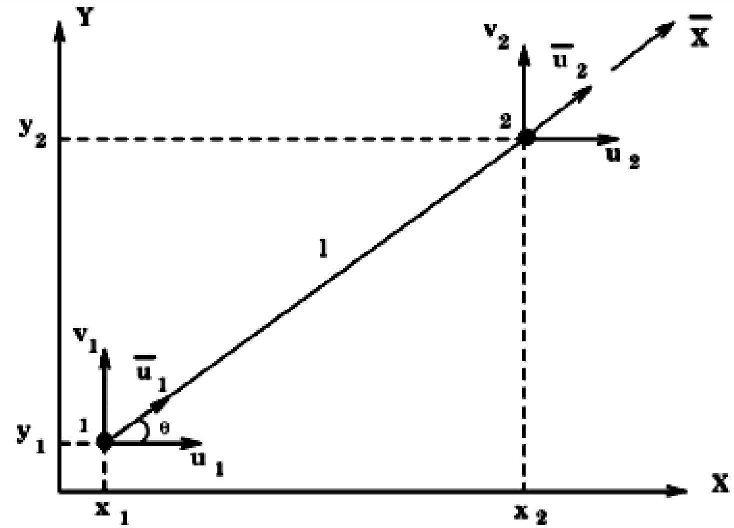
Replacing (2) in (1), and multiplying the two members of Eq. (1) by $[T]^T$, we have: $[T]^T[\bar{K}_e][T]\{u\} = [T]^T\{\bar{P}_e\}$

$$[K_e] = [T]^T[\bar{K}_e][T] = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}$$

$$[K_e] = \frac{EA}{l} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

being $c = \cos\theta$, $s = \sin\theta$

Stiffness matrix in the global system

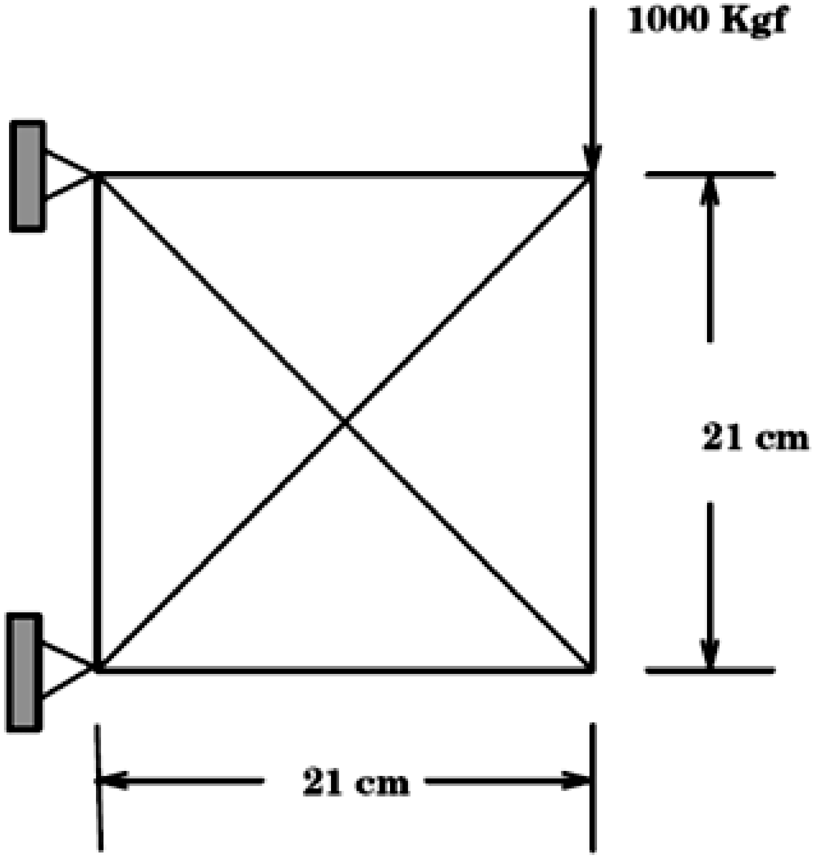


$$\{P_e\} = \begin{Bmatrix} cP_1 \\ sP_1 \\ cP_2 \\ sP_2 \end{Bmatrix}$$

$$\epsilon = \frac{1}{l}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

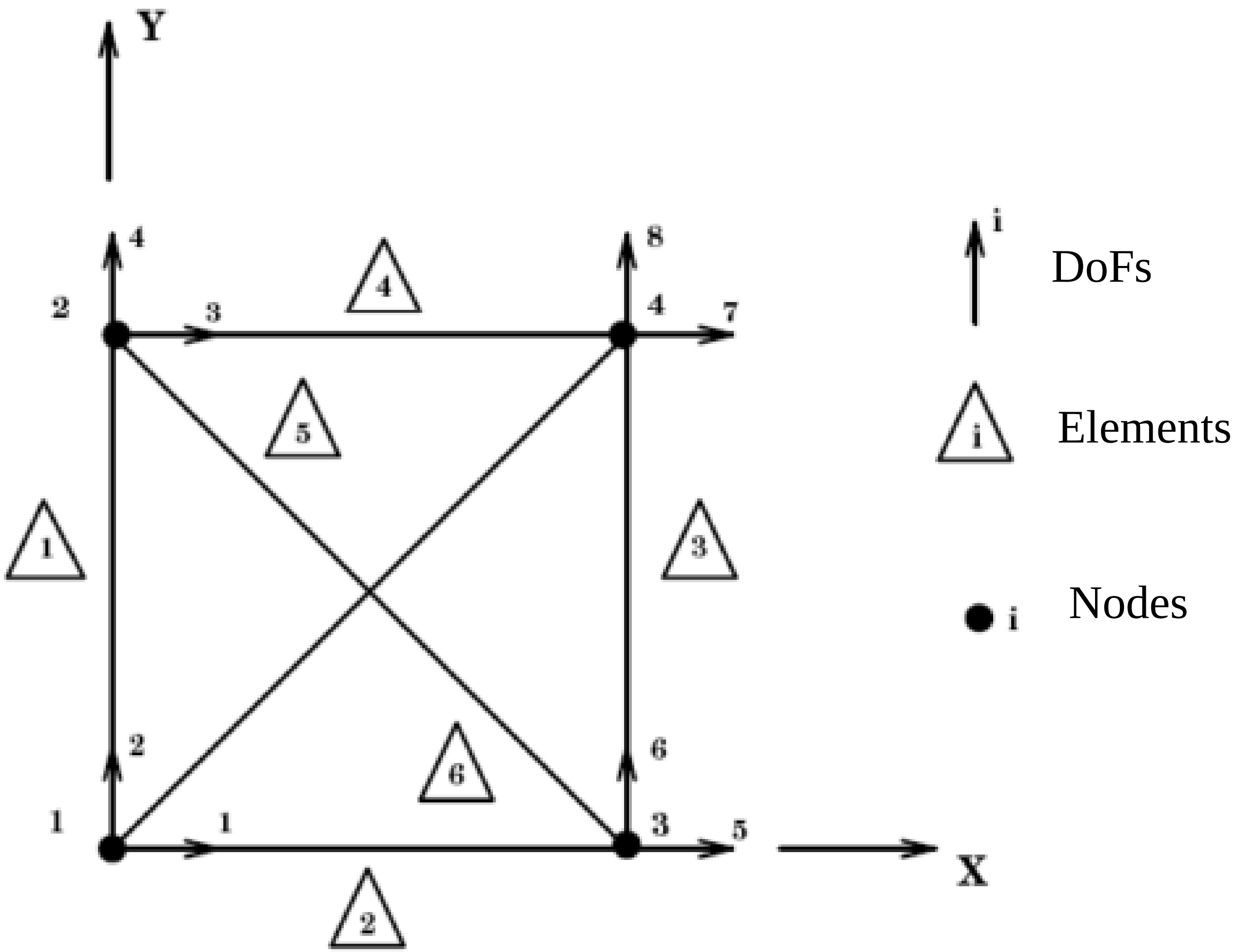
$$\sigma = \frac{E}{l}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Example

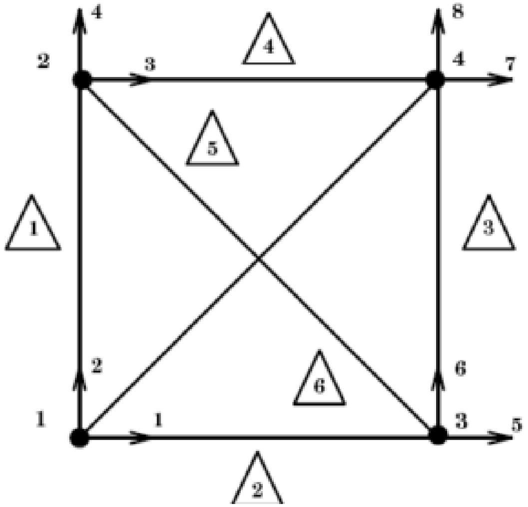


- Beams 1 to 4: 1 cm^2
- Beams 5 and 6: sqrt(2) cm^2
- E = 21e5 kgf/cm^2

Solution: Definition of nodes, elements and DoFs



Nodes	x [cm]	y [cm]
1	0	0
2	0	21
3	21	0
4	21	21



Element	Nodes conection	l [cm]	Area (cm^2)	c	s
1	1-2	21	1	0	1
2	1-3	21	1	1	0
3	3-4	21	1	0	1
4	2-4	21	1	1	0
5	2-3	$21\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}/2$	$\sqrt{2}/2$
6	1-4	$21\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}/2$	$\sqrt{2}/2$

Local stiffness matrices

$$[K_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \times 10^5 \qquad [K_2] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^5$$

$$[K_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \times 10^5 \qquad [K_4] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^5$$

$$[K_5] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{2} \times 10^5 \qquad [K_6] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \frac{1}{2} \times 10^5$$

Global stiffness matrices

$$[K_g] = \frac{1}{2} \times 10^5 \begin{bmatrix} 3 & 1 & 0 & 0 & -2 & 0 & -1 & -1 \\ 1 & 3 & 0 & -2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & 1 & -2 & 0 \\ 0 & -2 & -1 & 3 & 1 & -1 & 0 & 0 \\ -2 & 0 & -1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 3 & 0 & -2 \\ -1 & -1 & -2 & 0 & 0 & 0 & 3 & 1 \\ -1 & -1 & 0 & 0 & 0 & -2 & 1 & 3 \end{bmatrix}$$

Applying boundary conditions

$$[K_g] = \frac{1}{2} \times 10^5 \begin{bmatrix} 3 & 1 & 0 & 0 & -2 & 0 & -1 & -1 \\ 1 & 3 & 0 & -2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & 1 & -2 & 0 \\ 0 & -2 & -1 & 3 & 1 & -1 & 0 & 0 \\ -2 & 0 & -1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 3 & 0 & -2 \\ -1 & -1 & -2 & 0 & 0 & 0 & 3 & 1 \\ -1 & -1 & 0 & 0 & 0 & -2 & 1 & 3 \end{bmatrix}$$

$$\{P_g\} = \{ \cancel{0} \quad \cancel{0} \quad \cancel{0} \quad \cancel{0} \quad 0 \quad 0 \quad 0 \quad -1000 \ \}^T$$

Strains and stresses calculation

$$\frac{1}{2} \times 10^5 \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 0 & -2 \\ 0 & 0 & 3 & 1 \\ 0 & -2 & 1 & 3 \end{bmatrix} \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{Bmatrix}$$

$$\{U\} = 1 \times 10^{-3} \{\ 0 \quad 0 \quad 0 \quad 0 \quad -4.286 \quad -12.857 \quad 5.714 \quad -17.143\ \}^T$$

Element	$(\epsilon \times 10^{-4})$	$(\sigma\ [Kgf/cm^2])$
1	0	0
2	−2.04	−428.57
3	−2.04	−428.57
4	2.72	571.43
5	2,72	428.57
6	−2.72	−571.43

$$A_4 = \frac{(571.43)(1)}{500.00} = 1.14\ cm^2,$$

$$A_6 = \frac{(571,43)(\sqrt{2})}{500.00} = 1.61\ cm^2$$

Scilab implementation

```
clear; clc
E = 2100000;// kgf/cm2
A = [1,1,1,1,sqrt(2),sqrt(2)];// cm2
X = [0,0;
     0,21;
     21,0;
     21,21];//mm
i = [1,1,1,1,2,2,2,2,3,3,3,3,4,4,4,4];
j = [1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4];

// stiffness matrix of the element 1
deltax = X(2,1)-X(1,1);
deltay = X(2,2)-X(1,2);
L1 = sqrt(deltax^2+deltay^2);// mm
C = deltax/L1;// slope of the element (cos)
S = deltay/L1;// slope of the element (sin)
k = [C*C,C*S,-C*C,-C*S;
     C*S,S*S,-C*S,-S*S;
     -C*C,-C*S,C*C,C*S;
     -C*S,-S*S,C*S,S*S];
cte = (E*A(1))/L1;
k1 = cte*k;
K1 = sparse([i(:),j(:)],k1,[8,8]);
K1 = full(K1);
KSte1 = [-C,-S,C,S];
```

```
// stiffness matrix of the element 2
deltax = X(3,1)-X(1,1);
deltay = X(3,2)-X(1,2);
L2 = sqrt(deltax^2+deltay^2);// mm
C = deltax/L2;// slope of the element (cos)
S = deltay/L2;// slope of the element (sin)
k = [C*C,C*S,-C*C,-C*S;
     C*S,S*S,-C*S,-S*S;
     -C*C,-C*S,C*C,C*S;
     -C*S,-S*S,C*S,S*S];
cte = (E*A(2))/L2;
k2 = cte*k;
v1 = [i(1:8),i(9:16)+2];
v2 = [j(1:2),j(3:4)+2,j(1:2),j(3:4)+2,j(1:2),j(3:4)+2,j(1:2),j(3:4)+2];
K2 = sparse([v1(:),v2(:)],k2,[8,8]);
K2 = full(K2);
KSte2 = [-C,-S,C,S];

// stiffness matrix of the element 3
deltax = X(4,1)-X(3,1);
deltay = X(4,2)-X(3,2);
L3 = sqrt(deltax^2+deltay^2);// mm
C = deltax/L3;// slope of the element (cos)
S = deltay/L3;// slope of the element (sin)
k = [C*C,C*S,-C*C,-C*S;
     C*S,S*S,-C*S,-S*S;
     -C*C,-C*S,C*C,C*S;
     -C*S,-S*S,C*S,S*S];
cte = (E*A(3))/L3;
k3 = cte*k;
v1 = i+4;
v2 = j+4;
K3 = sparse([v1(:),v2(:)],k3,[8,8]);
K3 = full(K3);
KSte3 = [-C,-S,C,S];
```

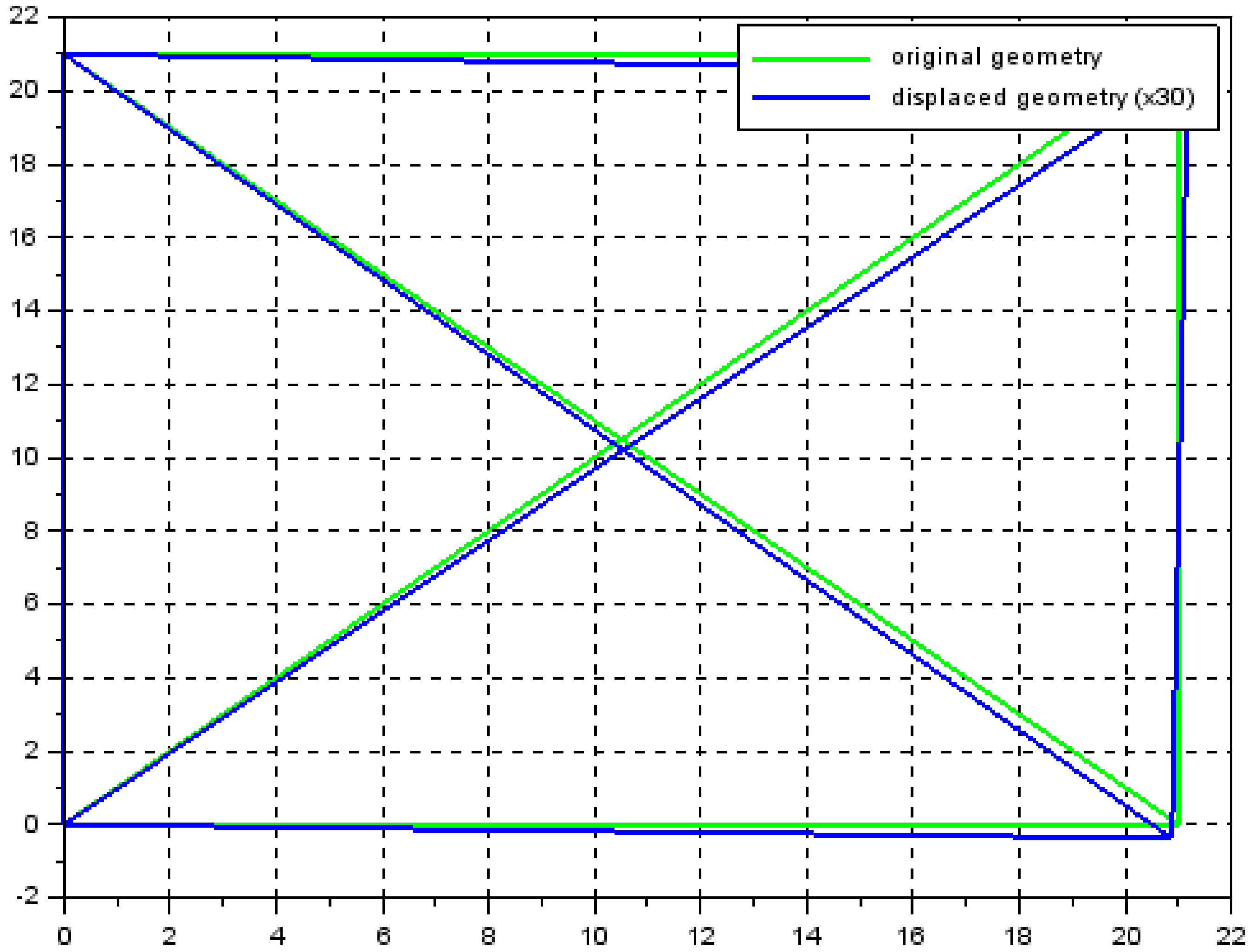
```
// stiffness matrix of the element 4
deltax = X(4,1)-X(2,1);
deltay = X(4,2)-X(2,2);
L4 = sqrt(deltax^2+deltay^2);// mm
C = deltax/L4;// slope of the element (cos)
S = deltay/L4;// slope of the element (sin)
k = [C*C,C*S,-C*C,-C*S;
     C*S,S*S,-C*S,-S*S;
     -C*C,-C*S,C*C,C*S;
     -C*S,-S*S,C*S,S*S];
cte = (E*A(4))/L4;
k4 = cte*k;
v1 = [i(1:8)+2,i(9:16)+4];
v2 = [j(1:2)+2,j(3:4)+4,j(1:2)+2,j(3:4)+4,j(1:2)+2,j(3:4)+4,j(1:2)+2,j(3:4)+4];
K4 = sparse([v1(:),v2(:)],k4,[8,8]);
K4 = full(K4);
KSte4 = [-C,-S,C,S];

// stiffness matrix of the element 5
deltax = X(3,1)-X(2,1);
deltay = X(3,2)-X(2,2);
L5 = sqrt(deltax^2+deltay^2);// mm
C = deltax/L5;// slope of the element (cos)
S = deltay/L5;// slope of the element (sin)
k = [C*C,C*S,-C*C,-C*S;
     C*S,S*S,-C*S,-S*S;
     -C*C,-C*S,C*C,C*S;
     -C*S,-S*S,C*S,S*S];
cte = (E*A(5))/L5;
k5 = cte*k;
v1 = i+2;
v2 = j+2;
K5 = sparse([v1(:),v2(:)],k5,[8,8]);
K5 = full(K5);
KSte5 = [-C,-S,C,S];
```

```
// stiffness matrix of the element 6
deltax = X(4,1)-X(1,1);
deltay = X(4,2)-X(1,2);
L6 = sqrt(deltax^2+deltay^2);// mm
C = deltax/L6;// slope of the element (cos)
S = deltay/L6;// slope of the element (sin)
k = [C*C,C*S,-C*C,-C*S;
     C*S,S*S,-C*S,-S*S;
     -C*C,-C*S,C*C,C*S;
     -C*S,-S*S,C*S,S*S];
cte = (E*A(6))/L6;
k6 = cte*k;
v1 = [i(1:8),i(9:16)+4];
v2 = [j(1:2),j(3:4)+4,j(1:2),j(3:4)+4,j(1:2),j(3:4)+4,j(1:2),j(3:4)+4];
K6 = sparse([v1(:),v2(:)],k6,[8,8]);
K6 = full(K6);
KSte6 = [-C,-S,C,S];

K = K1+K2+K3+K4+K5+K6;
K = full(K);
K(1:4,:) = [];
K(:,1:4) = [];
F = [0,0,0,-1000]';
u = inv(K)*F;
```

```
// Stresses calculation
u = [0;0;0;0;v];// clamping
sigma12 = (E/L1)*(KSte1*u(1:4));
sigma13 = (E/L2)*(KSte2*[u(1:2);u(5:6)]);
sigma34 = (E/L3)*(KSte3*u(5:8));
sigma24 = (E/L4)*(KSte4*[u(3:4);u(7:8)]);
sigma23 = (E/L5)*(KSte5*u(3:6));
sigma14 = (E/L6)*(KSte6*[u(1:2);u(7:8)]);
sigma = [sigma12;sigma13;sigma34;sigma24;sigma23;sigma14];
disp (sigma)

// Displacement plot
u = v*30;//scale
U = [0,0;0,21;21+u(3),21+u(4);21+u(1),0+u(2);0,0;21+u(3),21+u(4);0,21;21+u(1),0+u(2)];// displaced geometry
Y = [0,0;0,21;21,21;21,0;0,0;21,21;0,21;21,0];// original geometry
plot(Y(:,1),Y(:,2),"g-",U(:,1),U(:,2),"b-","LineWidth",2);
legend("original geometry","displaced geometry (x30)");
xgrid
```

```
   0.
 - 428.57143
 - 428.57143
   571.42857
   428.57143
 - 571.42857
```

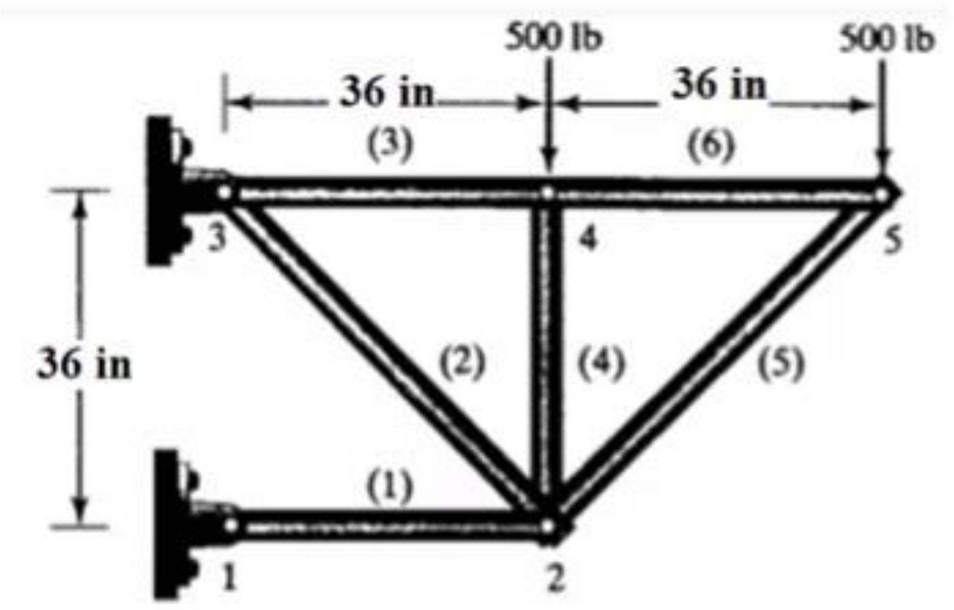
Displacements plot



Exercise

Write a Scilab code to calculate stresses and plot displacements, considering:

- Beams 1 to 6 cross-sectional area: 8 in^2
- E = 1.9 e5 lb/in^2



Solution code details can be found at:
<https://www.youtube.com/watch?v=rKeFEOKY88I>