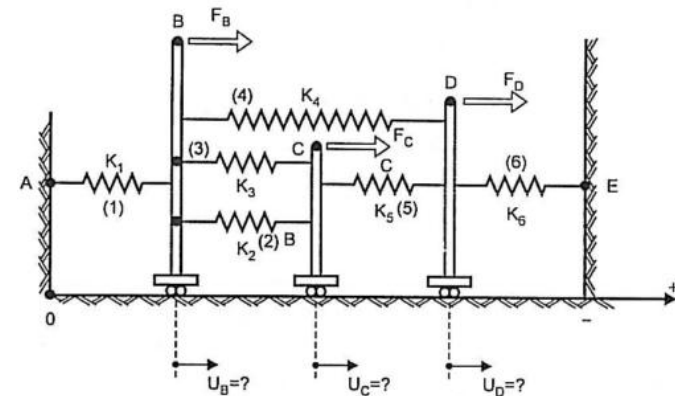


Proposed solution using sparse matrix

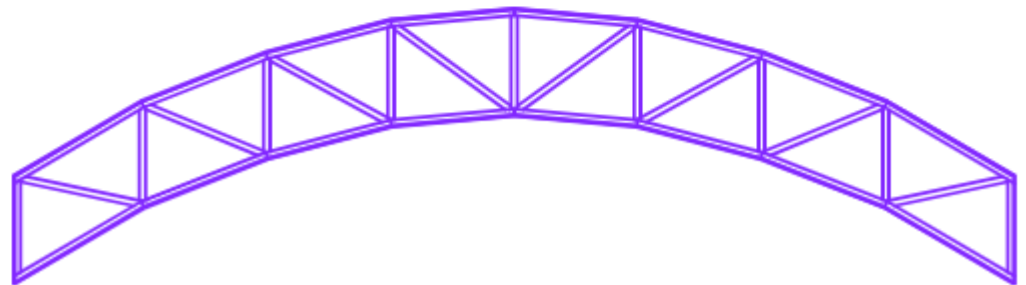
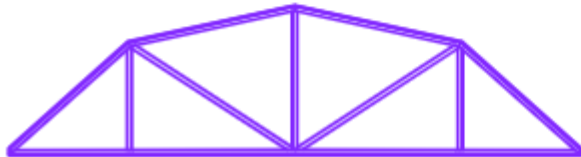
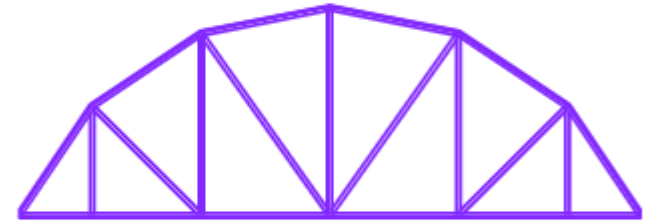
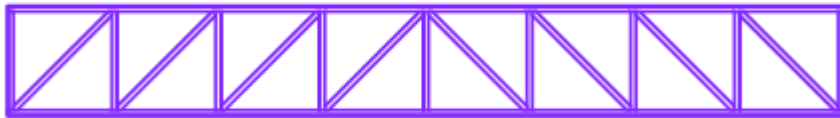
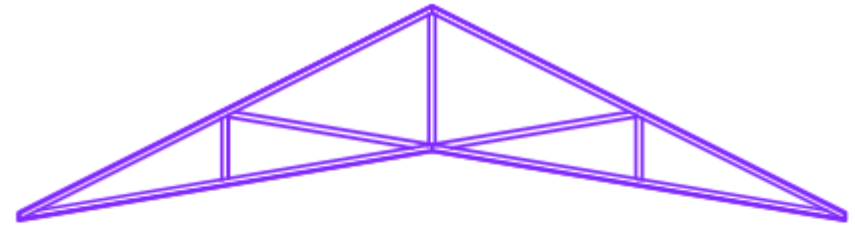
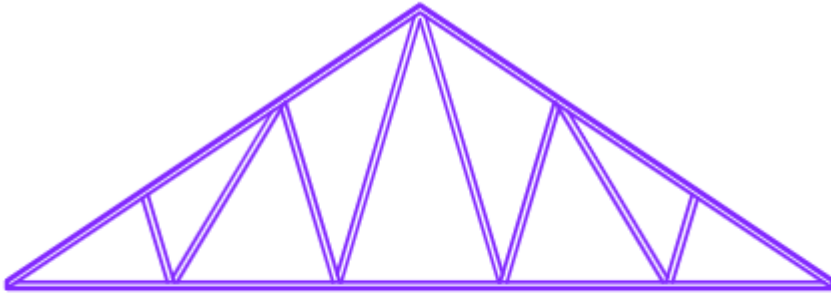
```

1 clear; -clc;
2 k = [200 -100 -150 -300 -400 -500]; -
3 cte = [-1 -1; -1 -1]
4 k1 = k(1)*cte'; - k2 = k(2)*cte'; - k3 = k(3)*cte'; -
5 k4 = k(4)*cte'; - k5 = k(5)*cte'; - k6 = k(6)*cte';
6 n = 5; -// -number -of -nodes
7 i = [1 1 -2 2 -]; -
8 j = [1 2 -1 2 -];
9 KL1 = sparse([i(:), j(:)], k1, [n - n]); -//AB
10 KL2 = sparse([i(:)+1 - j(:)+1], k2, [n - n]); -//BC
11 KL3 = sparse([i(:)+1 - j(:)+1], k3, [n - n]); -//BC
12 KL4 = sparse([i(:)*2 - j(:)*2], k4, [n - n]); -//BD
13 KL5 = sparse([i(:)+2 - j(:)+2], k5, [n - n]); -//CD
14 KL6 = sparse([i(:)+3 - j(:)+3], k6, [n - n]); -//DE
15 KG = -KL1+KL2+KL3+KL4+KL5+KL6
16 KG(5,:) = []; -KG(1,:) = []; -KG(:,5) = []; -KG(:,1) = []
17 disp(full(KG));

```



2D Trusses



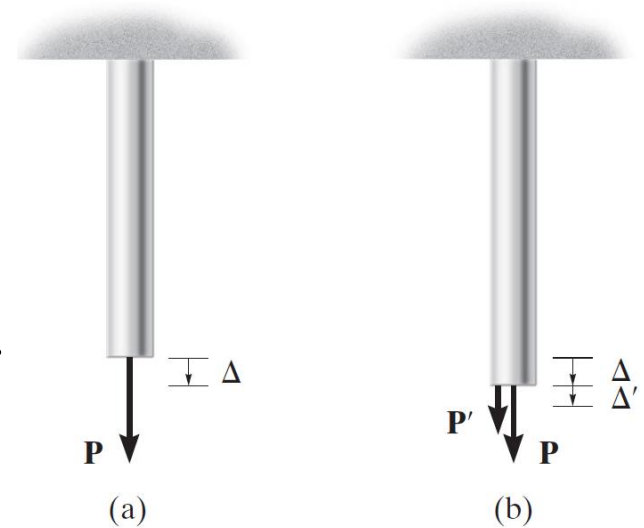
External work and deformation energy

- The work is caused by an internal load and moment.
- The load (P) promotes work when it undergoes a displacement (dx) in the same direction.

$$U_e = \int_0^x F dx$$

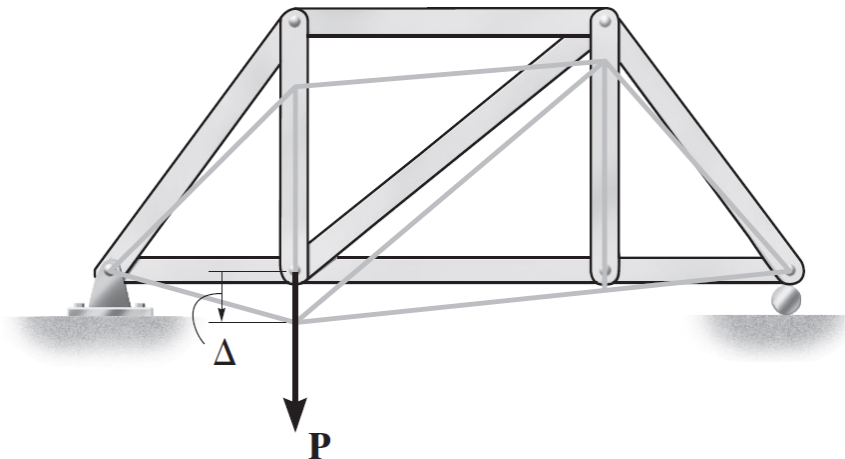
When $F = P$, the final displacement is :

$$U_e = \frac{1}{2} P \Delta$$



Energy conservation applied to trusses

- All energy methods used in mechanics are based on an energy balance, often called energy conservation.
- Energy conservation for the body is expressed as $U_e = U_i$
- Adding the energies of all the elements of the truss, we have :

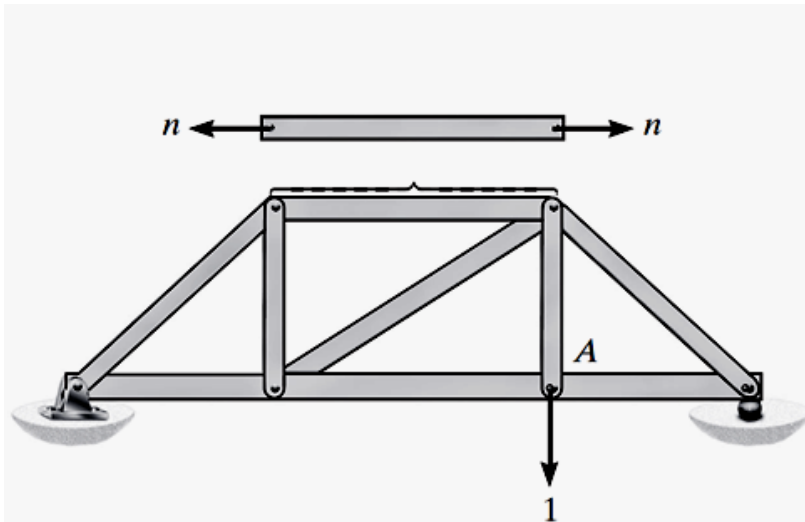


$$\frac{1}{2} P \Delta = \sum \frac{N^2 L}{2AE}$$

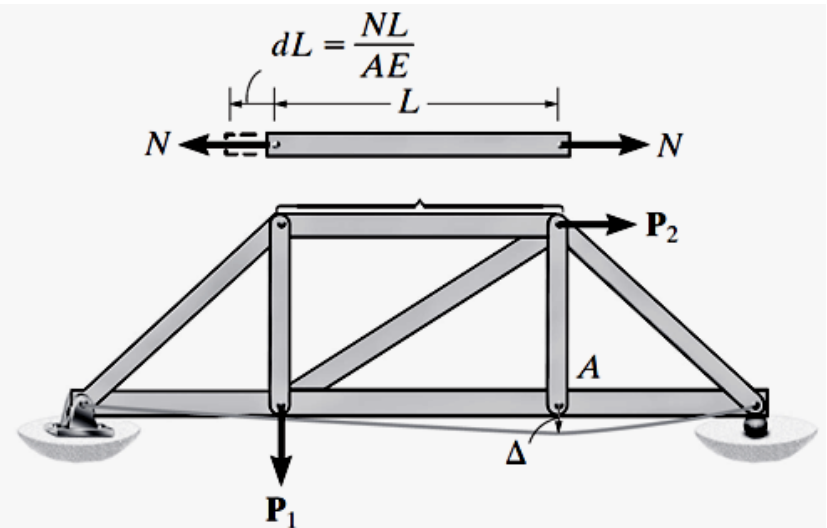
where: N is the internal load in each element

Principle of Virtual Work applied to trusses

The internal virtual work for a member is:
$$\int_0^L \frac{nN}{AE} dx = \frac{nNL}{AE}$$



Unitary virtual load



Real load

Virtual loads method applied to trusses

Virtual work equation

$$1 \cdot \Delta = \sum \frac{nNL}{AE}$$

1 = external virtual load

Δ = joint displacement

n = internal virtual load

N = internal load on a truss element

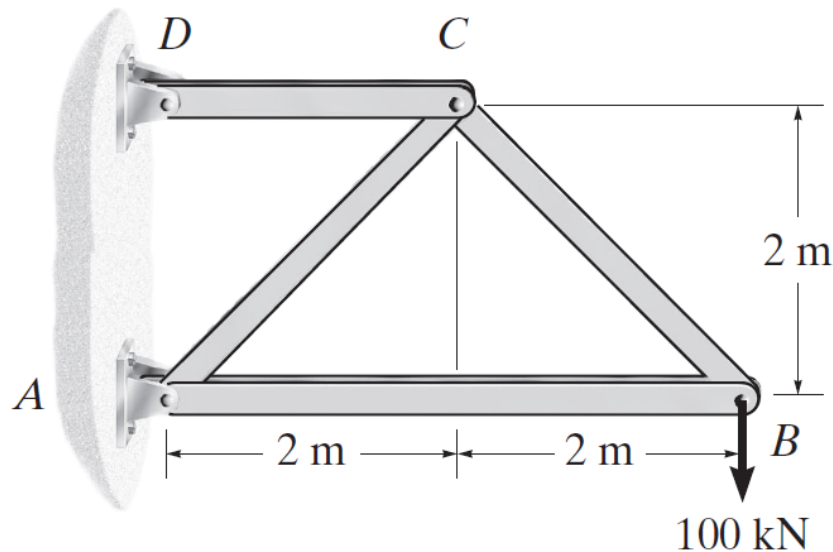
L = element length

A = cross-sectional area

E = modulus of elasticity of the element

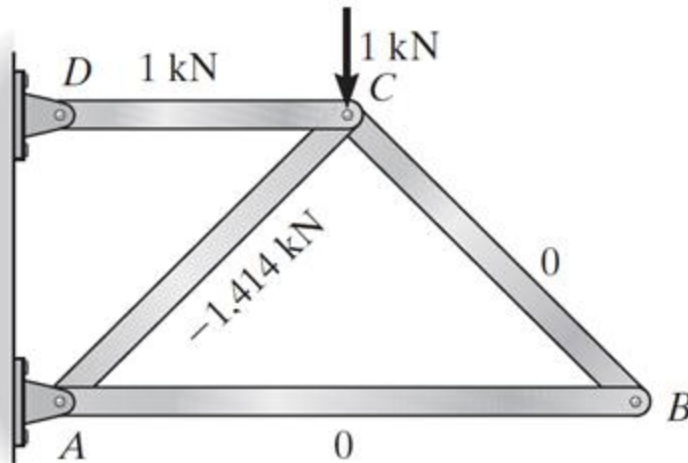
Example

Determine the vertical displacement of the C joint of the steel truss shown below. The cross-sectional area of each element is $A = 400 \text{ mm}^2$ and $E = 200 \text{ GPa}$.

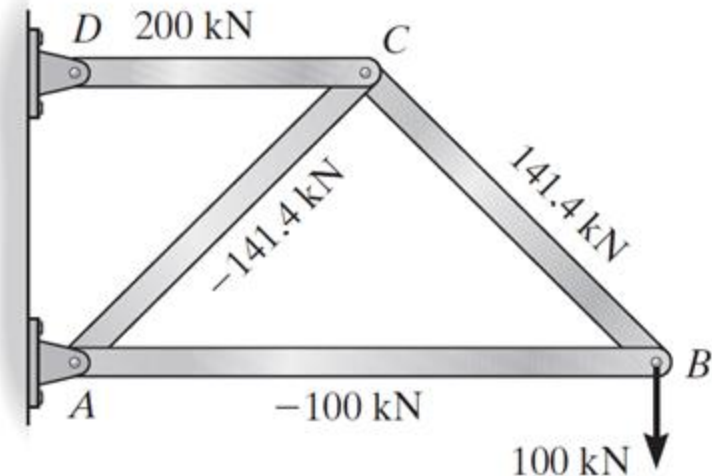


Solution

Vertical virtual load of 1 kN is added at C node and the load for each element is calculated using the method below.



Unitary virtual load at C



Real loads

Virtual loads method applied to trusses

Arranging the data in tabular form, we have:

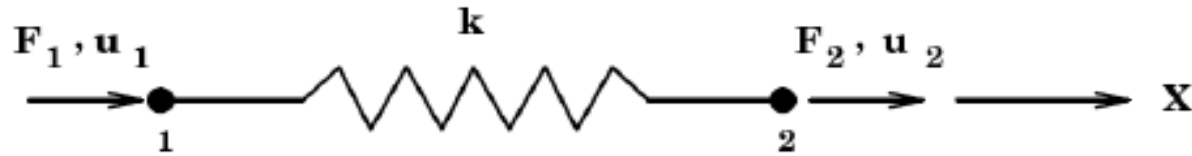
Member	n	N	L	nNL
AB	0	-100	4	0
BC	0	141.4	2.828	0
AC	-1.414	-141.4	2.828	565.7
CD	1	200	2	400
				$\Sigma 965.7 \text{ kN}^2 \cdot \text{m}$

$$\text{Therefore, } 1 \text{ kN} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{965.7 \text{ kN}^2 \cdot \text{m}}{AE}$$

Replacing A and E for their numerical values, we have:

$$1 \text{ kN} \cdot \Delta_{C_v} = \frac{965.7 \text{ kN}^2 \cdot \text{m}}{[400(10^{-6}) \text{ m}^2] 200(10^6) \text{ kN/m}^2}$$

$$\Delta_{C_v} = 0.01207 \text{ m} = \boxed{12.1 \text{ mm}}$$



$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}}_{\text{Stiffness matrix}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

↓
Loads
↓
Nodal displacements

1) $u_2 = 0$

$$k = \frac{F_1}{u_1} \rightarrow F_1 = ku_1$$

$$\sum F_X = 0 : F_1 + F_2 = 0 \rightarrow F_1 = -F_2$$

$$F_1 = -F_2 = ku_1$$

2) $u_1 = 0$

$$F_2 = -F_1 = ku_2$$

Truss linear system

$$\begin{cases} F_1 = ku_1 - ku_2 \\ F_2 = -ku_1 + ku_2 \end{cases}$$

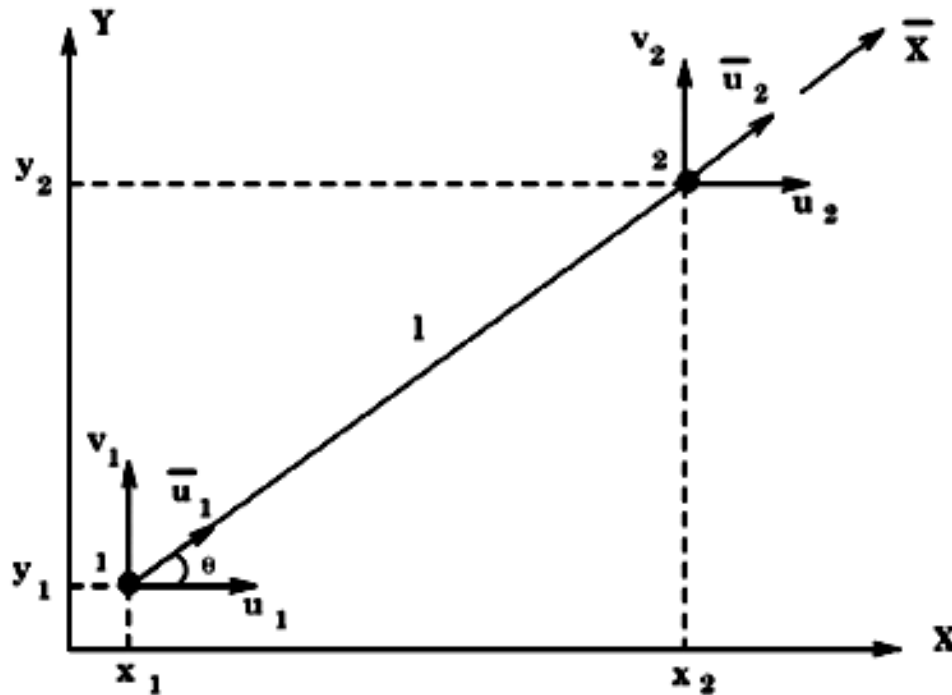
$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \rightarrow \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\bar{\epsilon} = \frac{\delta}{l} = \frac{1}{l}(\bar{u}_2 - \bar{u}_1)$$

$$\bar{\epsilon} = \frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}$$

$$\bar{\sigma} = \frac{E}{l} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}$$

Local displacement and global displacement



$$\bar{u}_1 = u_1 \cos \theta + v_1 \sin \theta$$

$$\bar{u}_2 = u_2 \cos \theta + v_2 \sin \theta$$

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\{\bar{u}\} = [T]\{u\} \quad \begin{cases} u = \text{local displacement} \\ \bar{u} = \text{global displacement} \end{cases}$$

Stiffness matrix in the global system

$$\{\bar{P}_e\} = [\bar{K}_e]\{\bar{u}\} \quad (1)$$

Replacing (2) in (1), and multiplying the two members

$$\{\bar{u}\} = [T]\{u\} \quad (2)$$

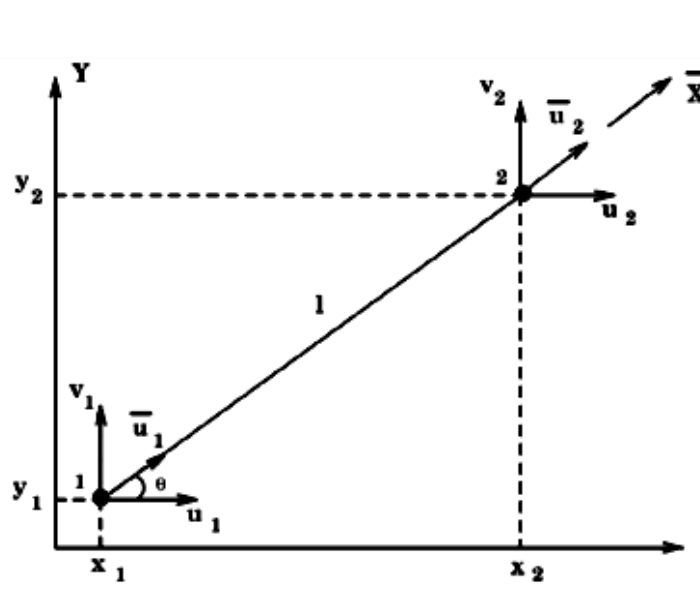
of Eq. (1) by $[T]^T$, we have: $[T]^T[\bar{K}_e][T]\{u\} = [T]^T\{\bar{P}_e\}$

$$[K_e] = [T]^T[\bar{K}_e][T] = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & \cos \theta \\ 0 & \sin \theta \end{bmatrix} \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \end{bmatrix}$$

$$[K_e] = \frac{EA}{l} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

being $c = \cos \theta$, $s = \sin \theta$

Stiffness matrix in the global system

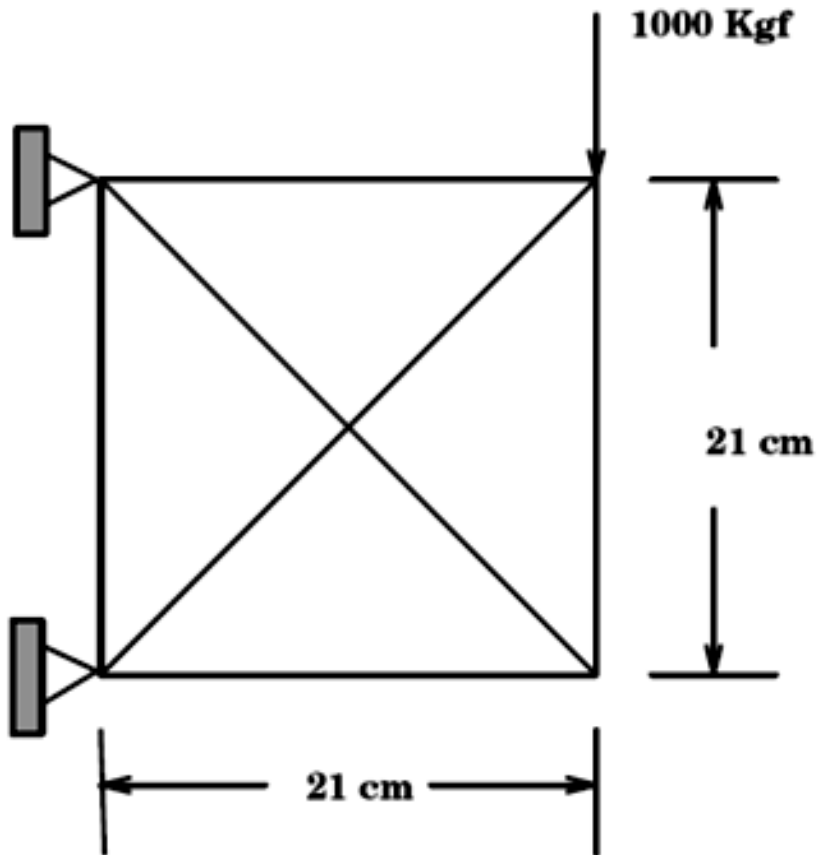


$$\{P_e\} = \begin{Bmatrix} cP_1 \\ sP_1 \\ cP_2 \\ sP_2 \end{Bmatrix}$$

$$\epsilon = \frac{1}{l} \begin{bmatrix} -c & -s & c & s \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

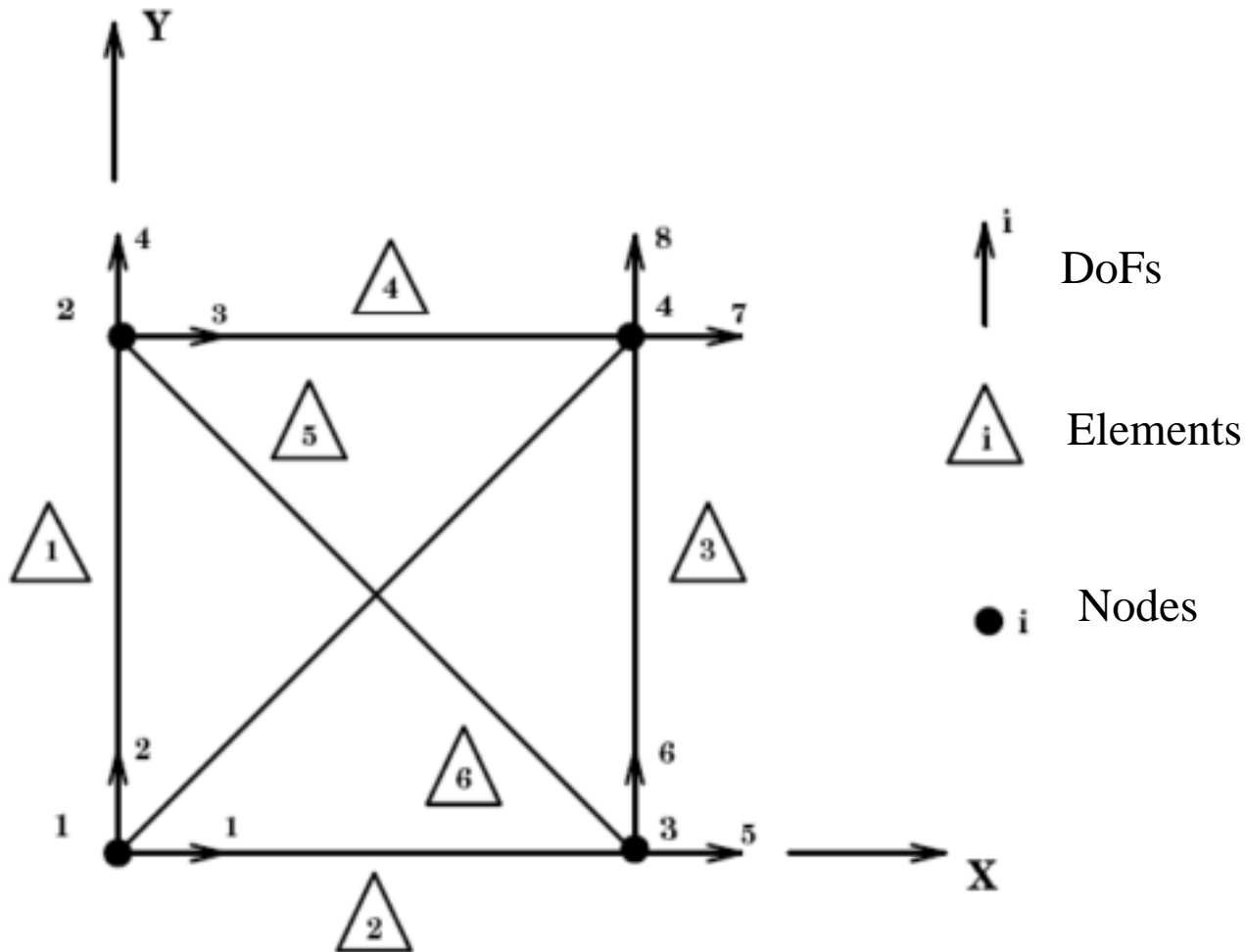
$$\sigma = \frac{E}{l} \begin{bmatrix} -c & -s & c & s \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Example

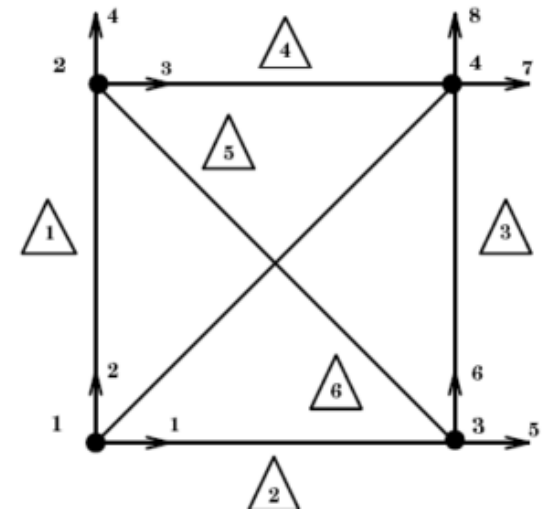


- Beams 1 to 4: 1 cm²
- Beams 5 and 6: $\sqrt{2}$ cm²
- $E = 21e5$ kgf/cm²

Solution: Definition of nodes, elements and DoFs



Nodes	x [cm]	y [cm]
1	0	0
2	0	21
3	21	0
4	21	21



Element	Nodes conection	l [cm]	Area (cm^2)	c	s
1	1-2	21	1	0	1
2	1-3	21	1	1	0
3	3-4	21	1	0	1
4	2-4	21	1	1	0
5	2-3	$21\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}/2$	$\sqrt{2}/2$
6	1-4	$21\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}/2$	$\sqrt{2}/2$

Local stiffness matrices

$$[K_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \times 10^5$$

$$[K_2] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^5$$

$$[K_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \times 10^5$$

$$[K_4] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^5$$

$$[K_5] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{2} \times 10^5$$

$$[K_6] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \frac{1}{2} \times 10^5$$

Global stiffness matrices

$$[K_g] = \frac{1}{2} \times 10^5 \begin{bmatrix} 3 & 1 & 0 & 0 & -2 & 0 & -1 & -1 \\ 1 & 3 & 0 & -2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & 1 & -2 & 0 \\ 0 & -2 & -1 & 3 & 1 & -1 & 0 & 0 \\ -2 & 0 & -1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 3 & 0 & -2 \\ -1 & -1 & -2 & 0 & 0 & 0 & 3 & 1 \\ -1 & -1 & 0 & 0 & 0 & -2 & 1 & 3 \end{bmatrix}$$

Applying boundary conditions

$$[K_g] = \frac{1}{2} \times 10^5 \begin{bmatrix} 3 & 1 & 0 & 0 & -2 & 0 & -1 & -1 \\ 1 & 3 & 0 & -2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & 1 & -2 & 0 \\ 0 & -2 & -1 & 3 & 1 & -1 & 0 & 0 \\ -2 & 0 & -1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 3 & 0 & -2 \\ -1 & -1 & -2 & 0 & 0 & 0 & 3 & 1 \\ -1 & -1 & 0 & 0 & 0 & -2 & 1 & 3 \end{bmatrix}$$

$$\{P_g\} = \{ \cancel{0} \quad \cancel{0} \quad \cancel{0} \quad \cancel{0} \quad 0 \quad 0 \quad 0 \quad -1000 \}^T$$

Strains and stresses calculation

$$\frac{1}{2} \times 10^5 \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 0 & -2 \\ 0 & 0 & 3 & 1 \\ 0 & -2 & 1 & 3 \end{bmatrix} \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{Bmatrix}$$

$$\{U\} = 1 \times 10^{-3} \{ 0 \quad 0 \quad 0 \quad 0 \quad -4.286 \quad -12.857 \quad 5.714 \quad -17.143 \}^T$$

Element	$(\epsilon \times 10^{-4})$	$(\sigma \text{ [Kgf/cm}^2\text{)])}$
1	0	0
2	-2.04	-428.57
3	-2.04	-428.57
4	2.72	571.43
5	2.72	428.57
6	-2.72	-571.43

$$A_4 = \frac{(571.43)(1)}{500.00} = 1.14 \text{ cm}^2,$$

$$A_6 = \frac{(571.43)(\sqrt{2})}{500.00} = 1.61 \text{ cm}^2$$

Scilab implementation

```

1  clear;-clc
2  E=-2100000; //-kgf/cm2
3  A=-[1,1,1,1,sqrt(2),sqrt(2)]; //-cm2
4  X=-[0,0;
5     -0,21;
6     -21,0;
7     -21,21]; //mm
8  i=-[1,1,1,1,2,2,2,2,3,3,3,3,4,4,4,4];
9  j=-[1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4];
10
11 //stiffness-matrix-of-the-element-1
12 deltax=X(2,1)-X(1,1);
13 deltay=X(2,2)-X(1,2);
14 L1=-sqrt(deltax^2+deltay^2); //-mm
15 C=-deltax/L1; //-slope-of-the-element-(cos)
16 S=-deltay/L1; //-slope-of-the-element-(sin)
17 k=-[C*C,C*S,-C*C,-C*S;
18     -C*S,S*S,-C*S,-S*S;
19     -C*C,-C*S,C*C,C*S;
20     -C*S,-S*S,C*S,S*S];
21 cte=- (E*A(1))/L1;
22 k1=-cte*k;
23 K1=-sparse([i(:),j(:)],k1,[8,8]);
24 K1=-full(K1);
25 KSte1=-[-C,-S,C,S];

```

```

27 //stiffness-matrix-of-the-element-2
28 deltax = X(3,1)-X(1,1);
29 deltax = X(3,2)-X(1,2);
30 L2 = sqrt(deltax^2+deltay^2); //mm
31 C = deltax/L2; //slope-of-the-element- (cos)
32 S = deltax/L2; //slope-of-the-element- (sin)
33 k = [C*C,C*S,-C*C,-C*S;
34      ....C*S,S*S,-C*S,-S*S;
35      ....-C*C,-C*S,C*C,C*S;
36      ....-C*S,-S*S,C*S,S*S];
37 cte = (E*A(2))/L2;
38 k2 = cte*k;
39 v1 = [i(1:8), i(9:16)+2];
40 v2 = [j(1:2), j(3:4)+2, j(1:2), j(3:4)+2, j(1:2), j(3:4)+2, j(1:2), j(3:4)+2];
41 K2 = sparse([v1(:), v2(:)], k2, [8, 8]);
42 K2 = full(K2);
43 KSte2 = [-C, -S, C, S];
44
45 //stiffness-matrix-of-the-element-3
46 deltax = X(4,1)-X(3,1);
47 deltax = X(4,2)-X(3,2);
48 L3 = sqrt(deltax^2+deltay^2); //mm
49 C = deltax/L3; //slope-of-the-element- (cos)
50 S = deltax/L3; //slope-of-the-element- (sin)
51 k = [C*C,C*S,-C*C,-C*S;
52      ....C*S,S*S,-C*S,-S*S;
53      ....-C*C,-C*S,C*C,C*S;
54      ....-C*S,-S*S,C*S,S*S];
55 cte = (E*A(3))/L3;
56 k3 = cte*k;
57 v1 = i+4;
58 v2 = j+4;
59 K3 = sparse([v1(:), v2(:)], k3, [8, 8]);
60 K3 = full(K3);
61 KSte3 = [-C, -S, C, S];

```

```

63 ///stiffness-matrix-of-the-element-4
64 deltax = X(4,1)-X(2,1);
65 deltax = X(4,2)-X(2,2);
66 L4 = sqrt(deltax^2+deltax^2); ///mm
67 C = deltax/L4; ///slope-of-the-element (cos)
68 S = deltax/L4; ///slope-of-the-element (sin)
69 k = [C*C,C*S,-C*C,-C*S;
70      ....C*S,S*S,-C*S,-S*S;
71      ....-C*C,-C*S,C*C,C*S;
72      ....-C*S,-S*S,C*S,S*S];
73 cte = (E*A(4))/L4;
74 k4 = cte*k;
75 v1 = [i(1:8)+2,i(9:16)+4];
76 v2 = [j(1:2)+2,j(3:4)+4,j(1:2)+2,j(3:4)+4,j(1:2)+2,j(3:4)+4,j(1:2)+2,j(3:4)+4];
77 K4 = sparse([v1(:),v2(:)],k4,[8,8]);
78 K4 = full(K4);
79 KSte4 = [-C,-S,C,S];
80
81 ///stiffness-matrix-of-the-element-5
82 deltax = X(3,1)-X(2,1);
83 deltax = X(3,2)-X(2,2);
84 L5 = sqrt(deltax^2+deltax^2); ///mm
85 C = deltax/L5; ///slope-of-the-element (cos)
86 S = deltax/L5; ///slope-of-the-element (sin)
87 k = [C*C,C*S,-C*C,-C*S;
88      ....C*S,S*S,-C*S,-S*S;
89      ....-C*C,-C*S,C*C,C*S;
90      ....-C*S,-S*S,C*S,S*S];
91 cte = (E*A(5))/L5;
92 k5 = cte*k;
93 v1 = i+2;
94 v2 = j+2;
95 K5 = sparse([v1(:),v2(:)],k5,[8,8]);
96 K5 = full(K5);
97 KSte5 = [-C,-S,C,S];

```



```

99 // stiffness matrix of the element 6
100 deltax = X(4,1)-X(1,1);
101 deltax = X(4,2)-X(1,2);
102 L6 = sqrt(deltax^2+deltay^2); // mm
103 C = deltax/L6; // slope of the element (cos)
104 S = deltax/L6; // slope of the element (sin)
105 k = [C*C, C*S, -C*C, -C*S;
106      -C*S, S*S, -C*S, -S*S;
107      -C*C, -C*S, C*C, C*S;
108      -C*S, -S*S, C*S, S*S];
109 cte = (E*A(6))/L6;
110 k6 = cte*k;
111 v1 = [i(1:8), i(9:16)+4];
112 v2 = [j(1:2), j(3:4)+4, j(1:2), j(3:4)+4, j(1:2), j(3:4)+4, j(1:2), j(3:4)+4];
113 K6 = sparse([v1(:), v2(:)], k6, [8, 8]);
114 K6 = full(K6);
115 KSte6 = [-C, -S, C, S];
116
117 K = K1+K2+K3+K4+K5+K6;
118 K = full(K);
119 K(1:4, :) = [];
120 K(:, 1:4) = [];
121 F = [0, 0, 0, -1000]';
122 v = inv(K)*F;

```

```

124 //Stresses calculation
125 u = [0;0;0;0;v]; //clamping
126 sigma12 = (E/L1) * (KSte1*u(1:4));
127 sigma13 = (E/L2) * (KSte2*[u(1:2);u(5:6)]);
128 sigma34 = (E/L3) * (KSte3*u(5:8));
129 sigma24 = (E/L4) * (KSte4*[u(3:4);u(7:8)]);
130 sigma23 = (E/L5) * (KSte5*u(3:6));
131 sigma14 = (E/L6) * (KSte6*[u(1:2);u(7:8)]);
132 sigma = [sigma12;sigma13;sigma34;sigma24;sigma23;sigma14];
133 disp(sigma)
134
135 //Displacement plot
136 u = v*30; //scale
137 U = [0,0;0,21;21+u(3),21+u(4);21+u(1),0+u(2);0,0;21+u(3),21+u(4);0,21;21+u(1),0+u(2)]; //displaced geometry
138 Y = [0,0;0,21;21,21;21,0;0,0;21,21;0,21;21,0]; //original geometry
139 plot(Y(:,1),Y(:,2),"g-",U(:,1),U(:,2),"b-", "LineWidth",2);
140 legend("original geometry","displaced geometry (x30)");
141 xgrid

```

```

0.
- 428.57143
- 428.57143
 571.42857
 428.57143
- 571.42857

```

Displacements plot

