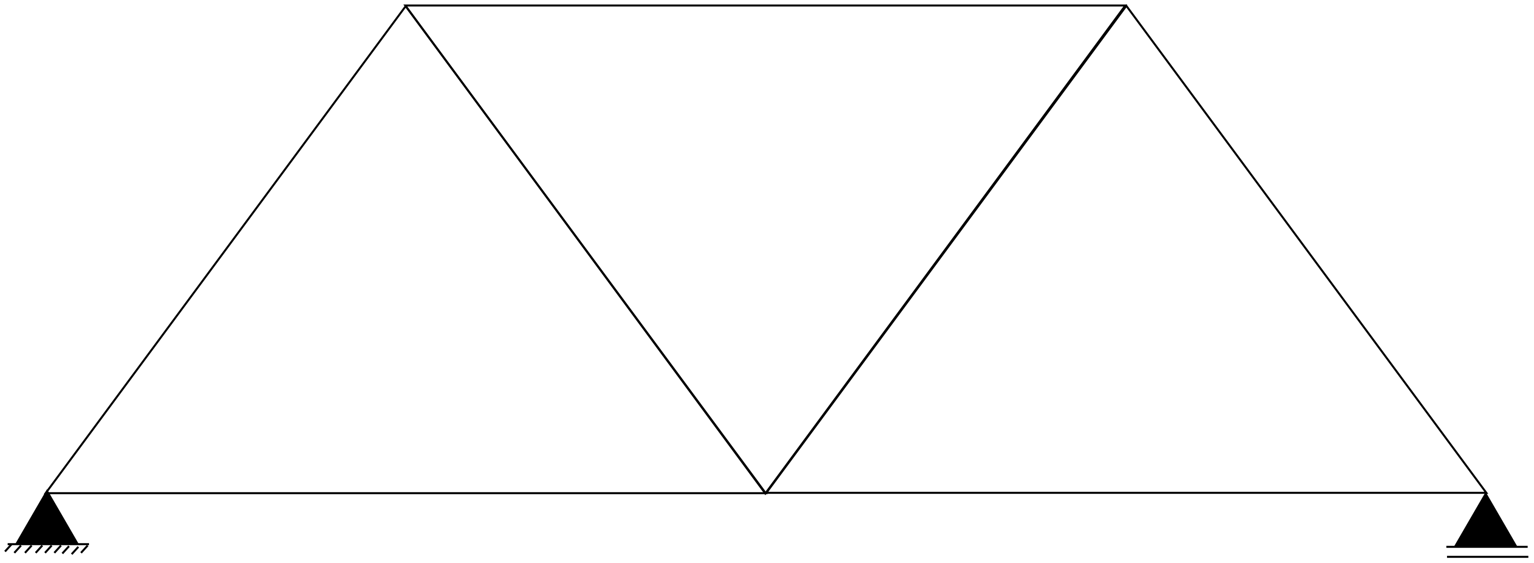
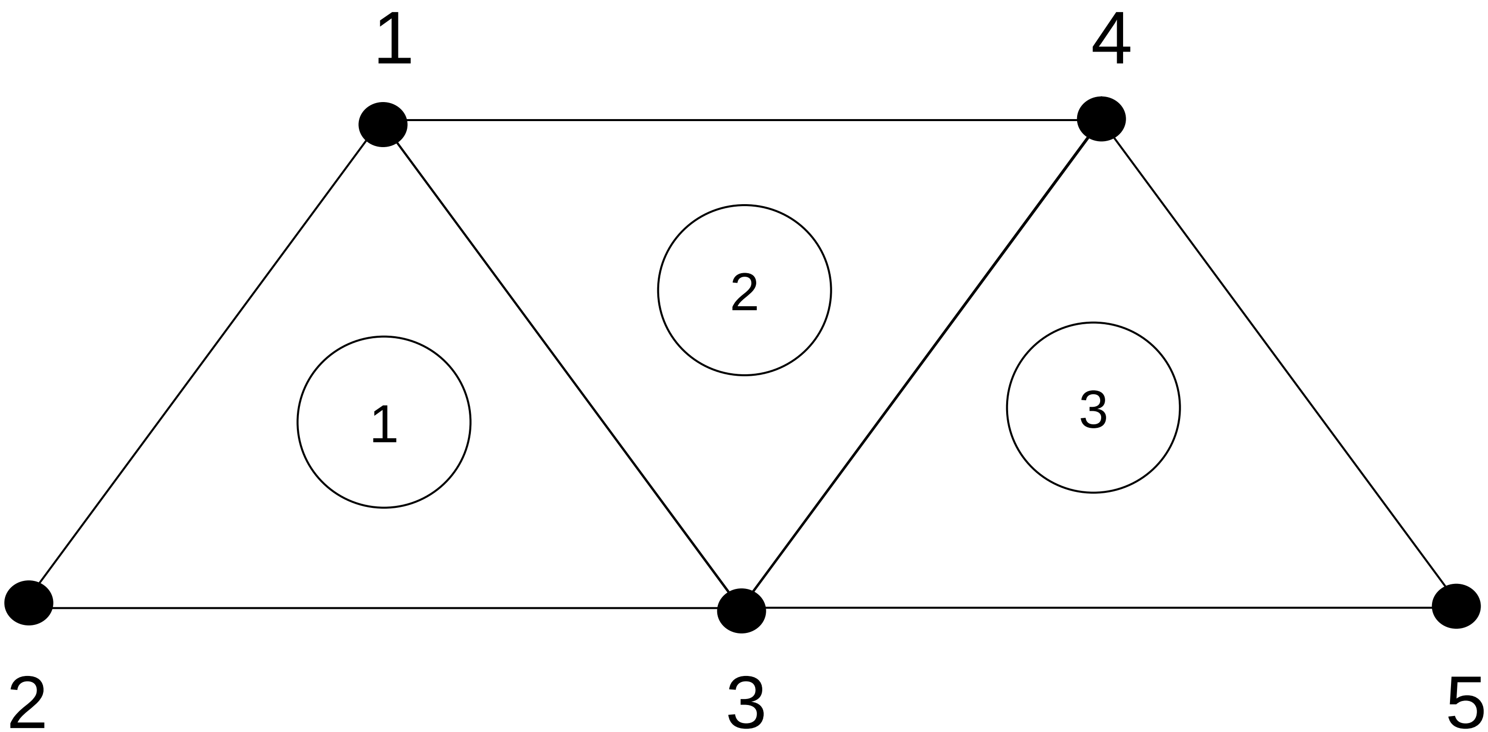
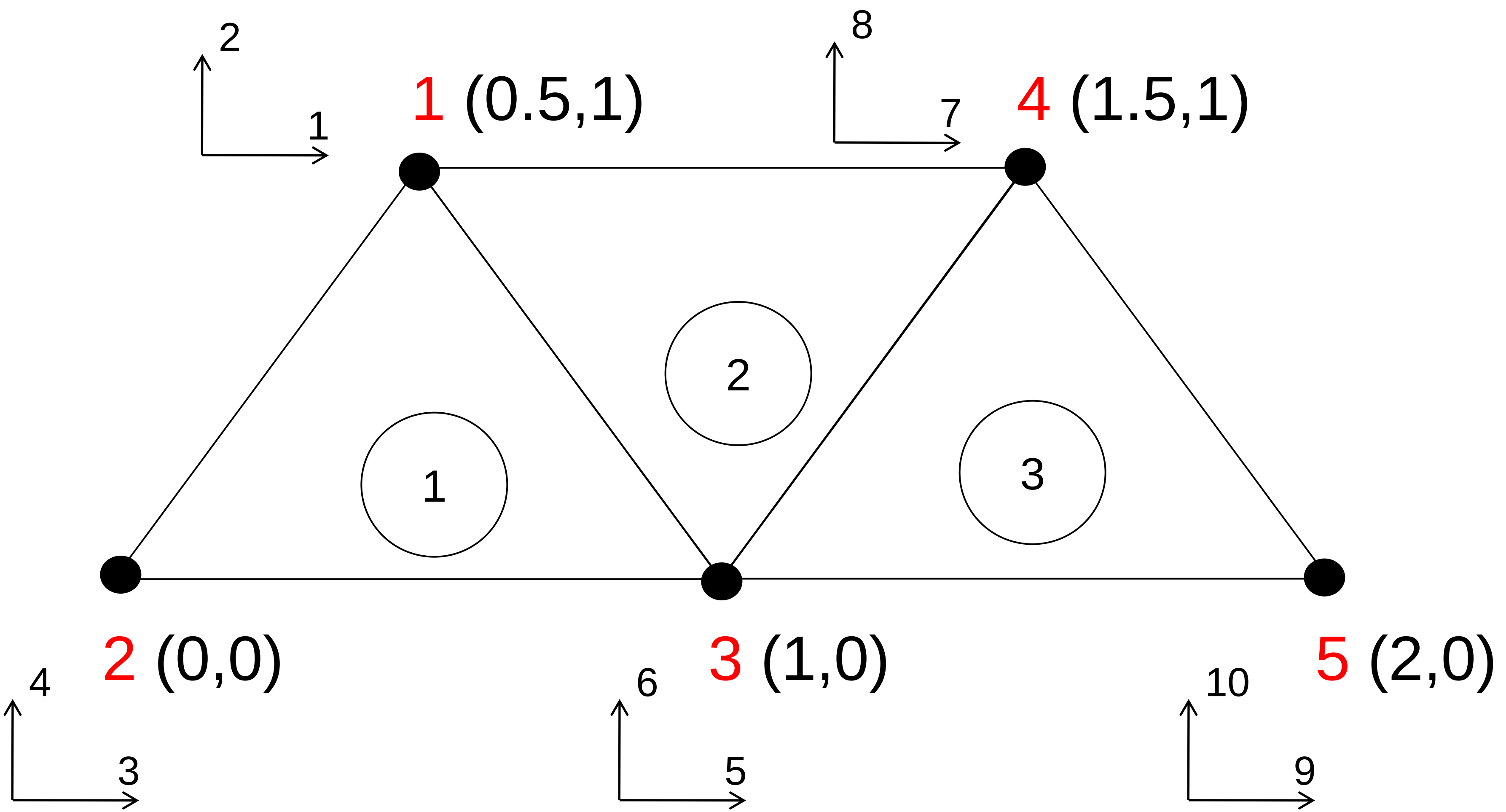
STIFFNESS MATRIX OF TRIANGULAR BIDIMENSIONAL ELEMENTS



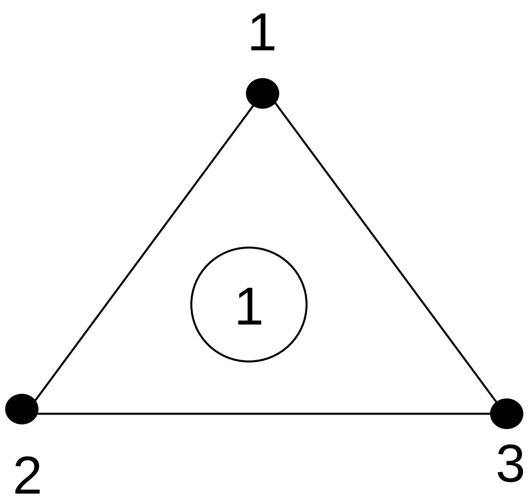
Elements numbering

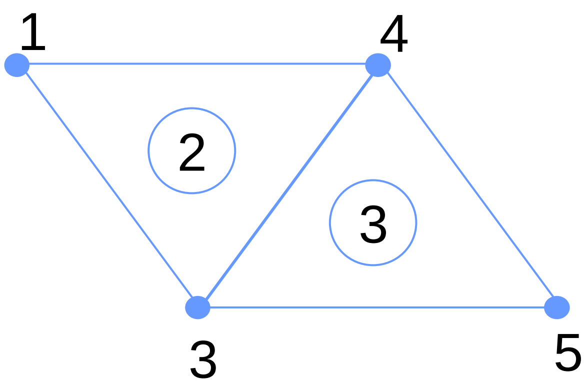


Nodes numbering



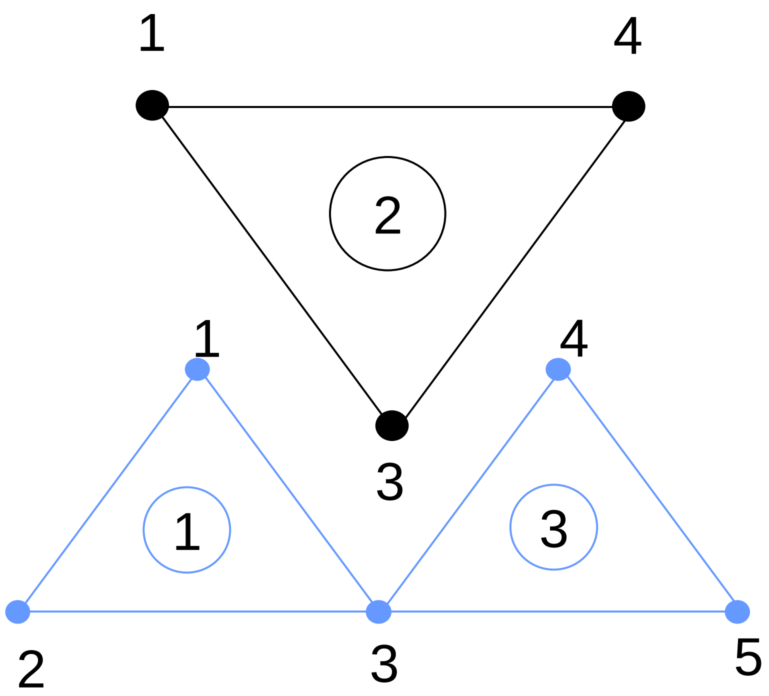
Definition of the element 1





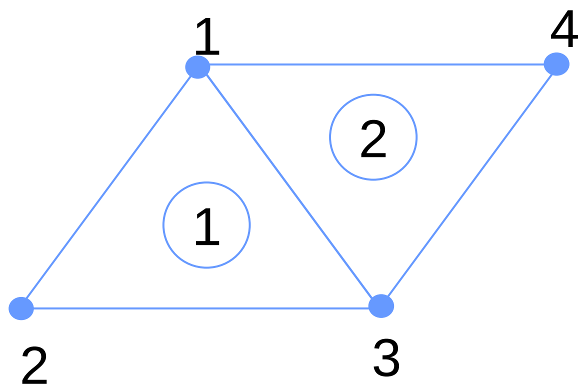
El.	Node 1	Node 2	Node 3
1	1	2	3

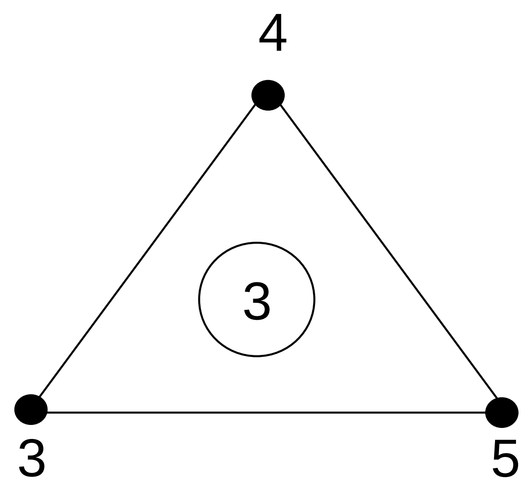
Definition of the element 2



El.	Node 1	Node 2	Node 3
1	1	2	3
2	1	3	4

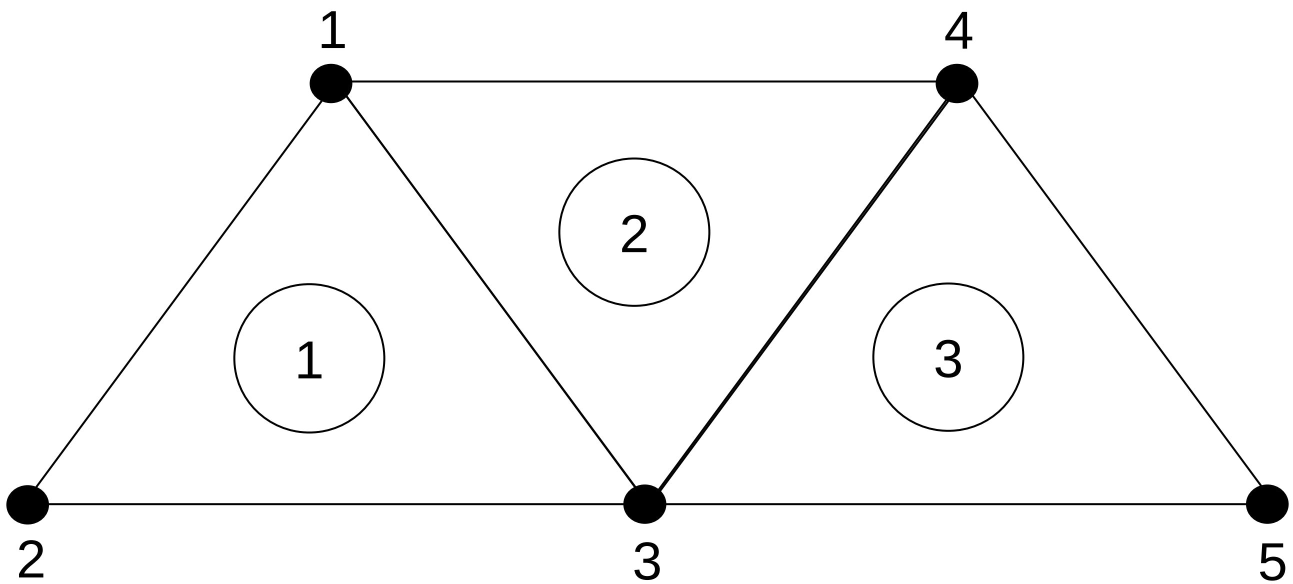
Definition of the element 3





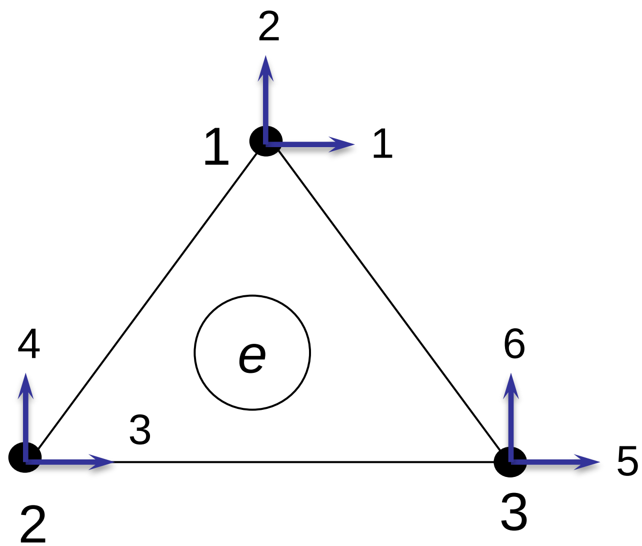
El.	Node 1	Node 2	Node 3
1	1	2	3
2	1	3	4
3	3	5	4

Elements definiton



El.	Node 1	Node 2	Node 3
1	1	2	3
2	1	3	4
3	3	5	4

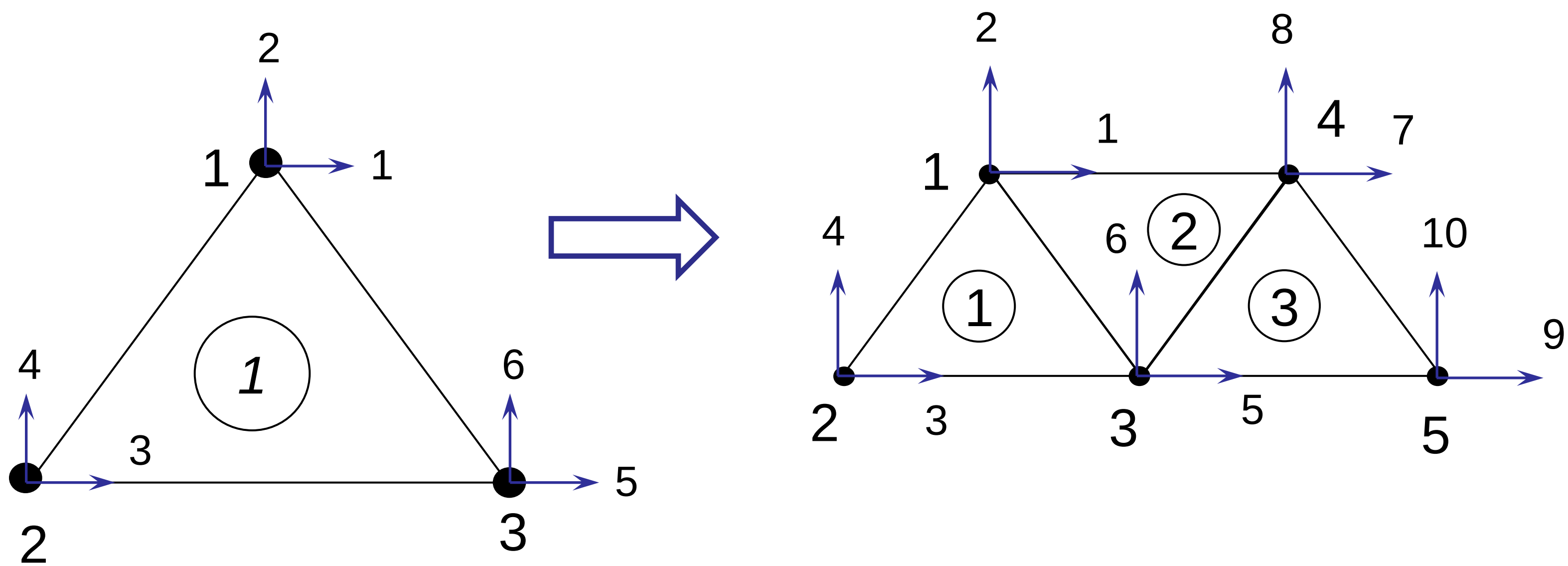
Element stiffness matrix



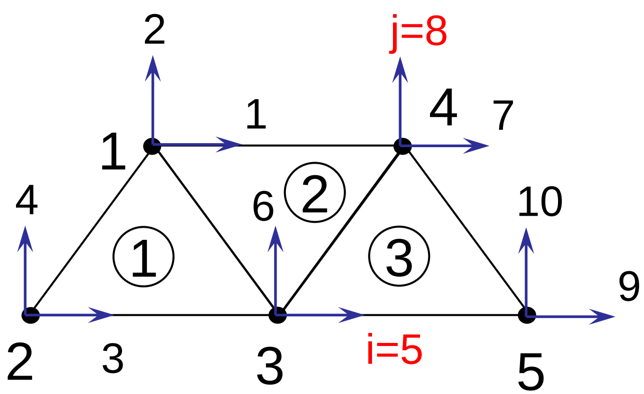
		Node 1		Node 2		Node 3	
	GL	1	2	3	4	5	6
Node 1	1	k_{11}	k_{12}	k_{13}	k_{14}	k_{15}	k_{16}
	2	k_{21}	k_{22}	k_{23}	k_{24}	k_{25}	k_{26}
Node 2	3	k_{31}	k_{32}	k_{33}	k_{34}	k_{35}	k_{36}
	4	k_{41}	k_{42}	k_{43}	k_{44}	k_{45}	k_{46}
Node 3	5	k_{51}	k_{52}	k_{53}	k_{54}	k_{55}	k_{56}
	6	k_{61}	k_{62}	k_{63}	k_{64}	k_{65}	k_{66}

The element k_{ij} of the stiffness matrix relates forces and displacements in degrees of freedom *i* and *j* of the element.

Structure stiffness matrix



Structure stiffness matrix - Example



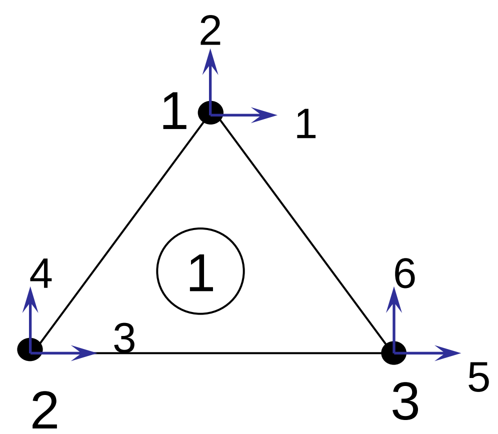
The element k_{58} of the stiffness matrix relates forces and displacements in degrees of freedom *5* and 8 of the structure.

		Node 1		Node 2		Node 3		Node 4		Node 5	
	GL	1	2	3	4	5	6	7	j= 8	9	10
Node 1	1										
	2										
Node 2	3										
	4										
Node 3	i=5								k_{ij}		
	6										
Node 4	7										
	8										
Node 5	9										
	10										

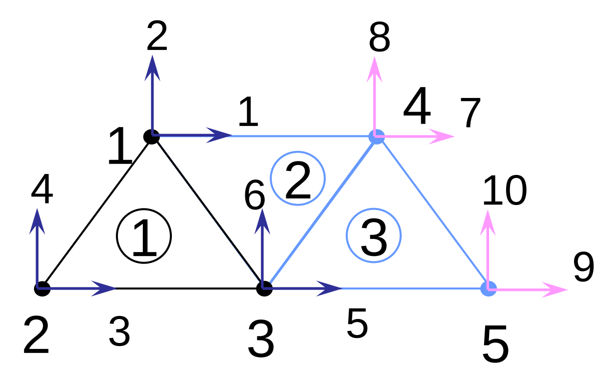
Element 1 contribution

Local numb.				Node 1		Node 2		Node 3					
				1	2	3	4	5	6				
		Global numb.		Node 1		Node 2		Node 3		Node 4		Node 5	
				1	2	3	4	5	6	7	8	9	10
Node 1	1	Node 1	1	k_{11}	k_{12}	k_{13}	k_{14}	k_{15}	k_{16}				
	2		2	k_{21}	k_{22}	k_{23}	k_{24}	k_{25}	k_{26}				
Node 2	3	Node 2	3	k_{31}	k_{32}	k_{33}	k_{34}	k_{35}	k_{36}				
	4		4	k_{41}	k_{42}	k_{43}	k_{44}	k_{45}	k_{46}				
Node 3	5	Node 3	5	k_{51}	k_{52}	k_{53}	k_{54}	k_{55}	k_{56}				
	6		6	k_{61}	k_{62}	k_{63}	k_{64}	k_{65}	k_{66}				
		Node 4	7										
			8										
		Node 5	9										
			10										

Local numbering



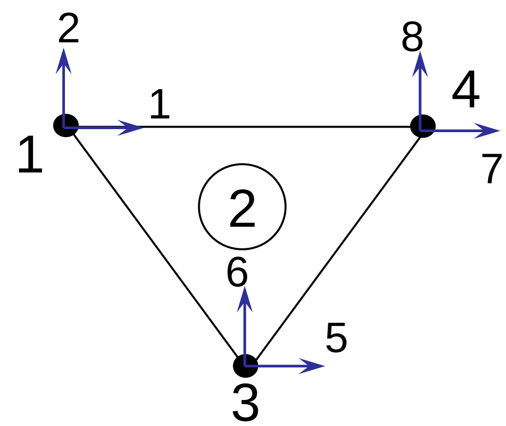
Global numbering



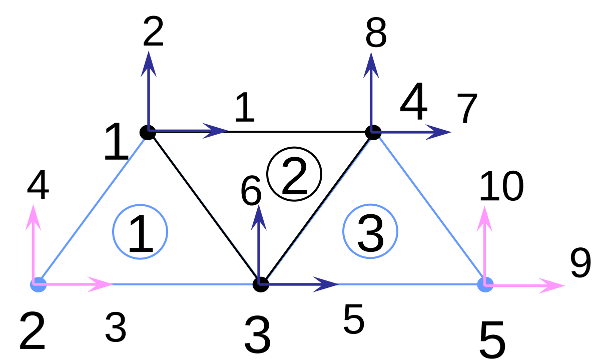
Element 2 contribution

Local numb.				Node 1				Node 2		Node 3			
				1	2			3	4	5	6		
		Global numb.		Node 1		Node 2		Node 3		Node 4		Node 5	
				1	2	3	4	5	6	7	8	9	10
Node 1	1	Node 1	1	k_{11}	k_{12}			k_{13}	k_{14}	k_{15}	k_{16}		
	2		2	k_{21}	k_{22}			k_{23}	k_{24}	k_{25}	k_{26}		
		Node 2	3										
			4										
Node 2	3	Node 3	5	k_{31}	k_{32}			k_{33}	k_{34}	k_{35}	k_{36}		
	4		6	k_{41}	k_{42}			k_{43}	k_{44}	k_{45}	k_{46}		
Node 3	5	Node 4	7	k_{51}	k_{52}			k_{53}	k_{54}	k_{55}	k_{56}		
	6		8	k_{61}	k_{62}			k_{63}	k_{64}	k_{65}	k_{66}		
		Node 5	9										
			10										

Local numbering



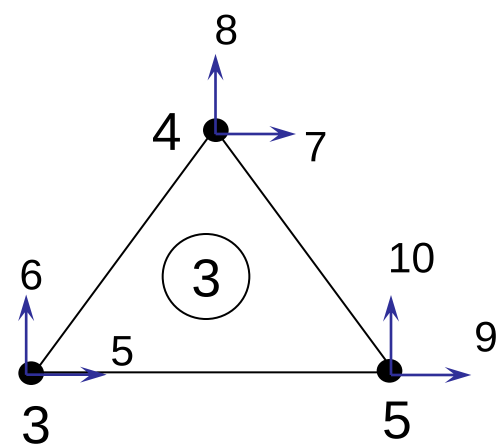
Global numbering



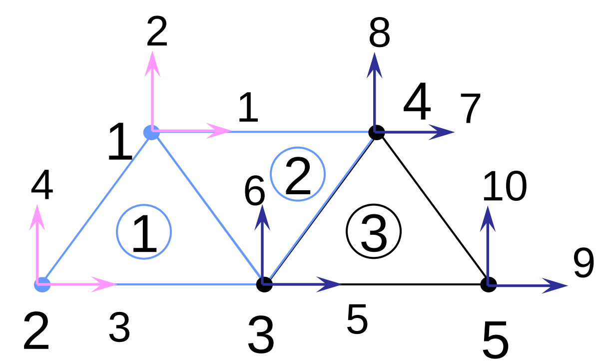
Element 3 contribution

Local numb.								Node 1		Node 3		Node 2	
								1	2	5	6	3	4
		Global Numb.		Node 1		Node 2		Node 3		Node 4		Node 5	
				1	2	3	4	5	6	7	8	9	10
		Node 1	1										
			2										
		Node 2	3										
			4										
Node 1	1	Node 3	5					k_{11}	k_{12}	k_{15}	k_{16}	k_{13}	k_{14}
	2		6					k_{21}	k_{22}	k_{25}	k_{26}	k_{23}	k_{24}
Node 3	5	Node 4	7					k_{51}	k_{52}	k_{55}	k_{56}	k_{53}	k_{54}
	6		8					k_{61}	k_{62}	k_{65}	k_{66}	k_{63}	k_{64}
Node 2	3	Node 5	9					k_{31}	k_{32}	k_{35}	k_{36}	k_{33}	k_{34}
	4		10					k_{41}	k_{42}	k_{45}	k_{46}	k_{43}	k_{44}

Local numbering



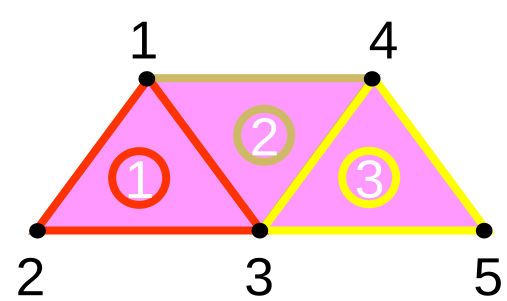
Global numbering



Contribution of all elements

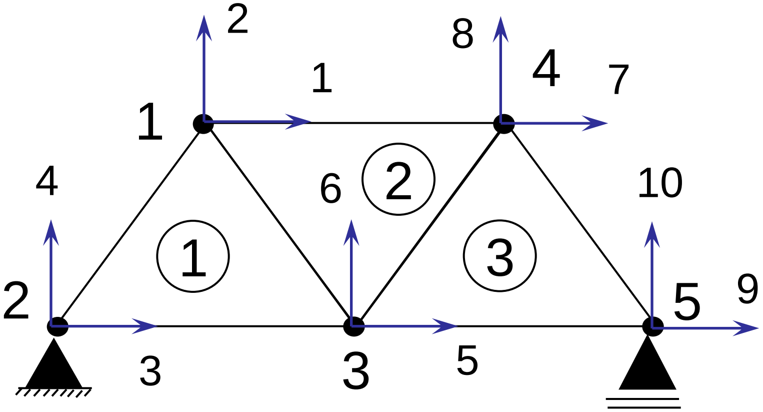
Local numb.								Node 1		Node 3		Node 2	
								1	2	5	6	3	4
		Global numb.		Node 1		Node 2		Node 3		Node 4		Node 5	
				1	2	3	4	5	6	7	8	9	10
		Node 1	1										
			2										
		Node 2	3										
			4										
	1	Node 3	5										
	2		6										
	5	Node 4	7										
	6		8										
	3	Node 5	9										
	4		10										

$$K_{11} = K_{11}^{e_1} + K_{11}^{e_2}$$



Boundary conditions

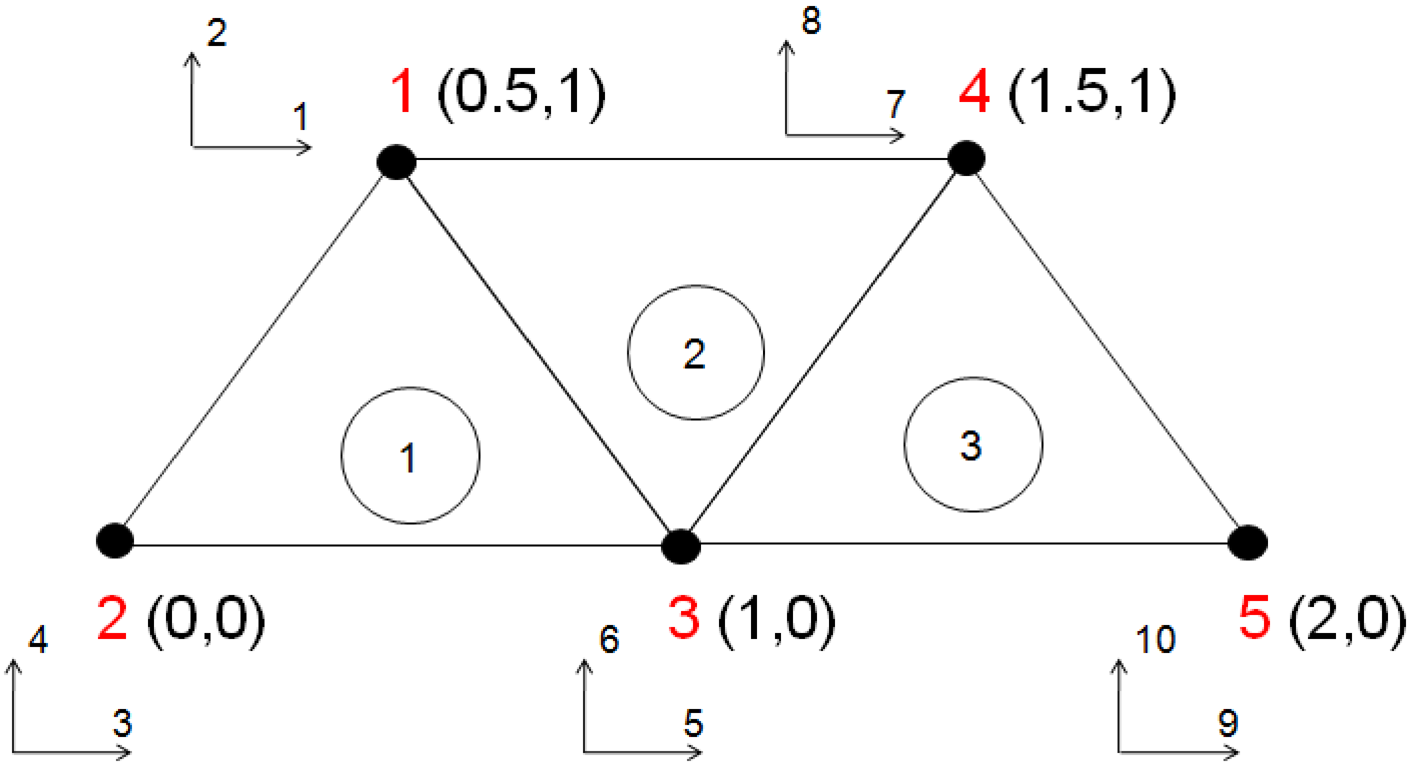
The rows and columns of the global stiffness matrix corresponding to the restricted degrees of freedom are deleted.



Global numb.		Node 1		Node 2		Node 3		Node 4		Node 5	
		1	2	3	4	5	6	7	8	9	10
Node 1	1										
	2										
Node 2	3										
	4										
Node 3	5										
	6										
Node 4	7										
	8										
Node 5	9										
	10										

	1	2	5	6	7	8	9
1							
2							
5							
6							
7							
8							
9							

Example: Develop a Scilab code to return the stiffness matrix of the structure below, considering the restricted degrees of freedom and $k_{ij} = 1$.



Solution

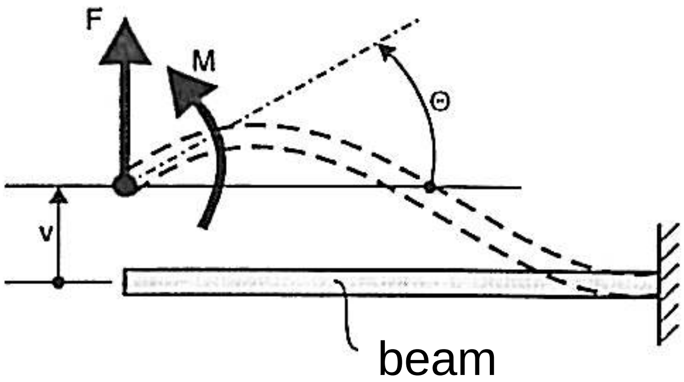
```
clear; clc;
k = ones(6,6)
i = [1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 3 4 4 4 4 4 4 5 5 5 5 5 5 6 6 6 6 6 6];
j = [1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6];

k1=sparse([i(:),j(:)],k,[10,10]);

v1 = [i(1:12),i(13:36)+2]; 
v2 = [j(1:2),j(3:6)+2,j(1:2),j(3:6)+2,j(1:2),j(3:6)+2,j(1:2),j(3:6)+2,j(1:2),...
      j(3:6)+2,j(1:2),j(3:6)+2,] ;
k2=sparse([v1(:),v2(:)],k,[10,10]);

v1 = i+4; v2 = j+4;
k3=sparse([v1(:),v2(:)],k,[10,10]);

k_g = k1+k2+k3;

//Restricted DoFs (3,4 and 10)
k_g(10,:) = []; k_g(:,10) = [];
k_g(3:4,:) = []; k_g(:,3:4) = [];
```

K=

$$
\begin{matrix}
2 & 2 & 2 & 2 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 & 1 & 1 & 0 \\
2 & 2 & 3 & 3 & 2 & 2 & 1 \\
2 & 2 & 3 & 3 & 2 & 2 & 1 \\
1 & 1 & 2 & 2 & 2 & 2 & 1 \\
1 & 1 & 2 & 2 & 2 & 2 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{matrix}
$$

Principle of Virtual Works

$$\boxed{\tau_{EXT} = \tau_{INT}}$$

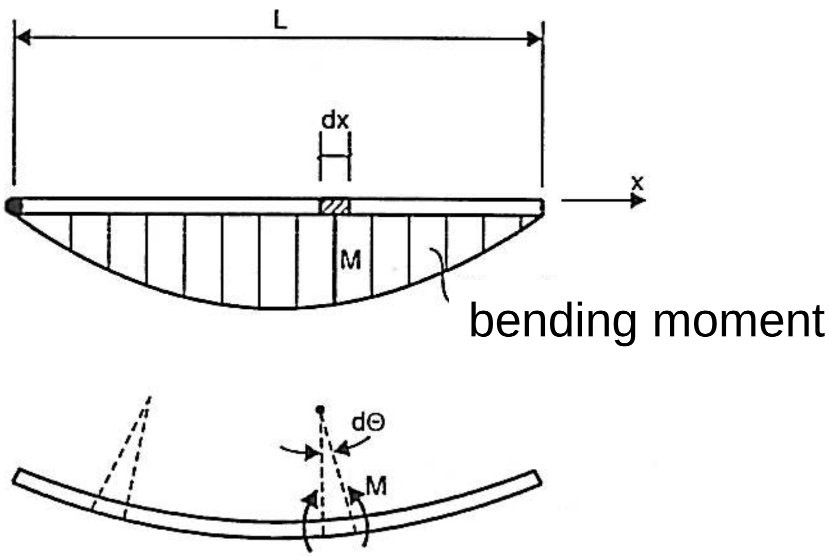


Work of external loads

$$\boxed{\tau_{EXT} = F \cdot v + M \cdot \theta}$$

Load

Moment



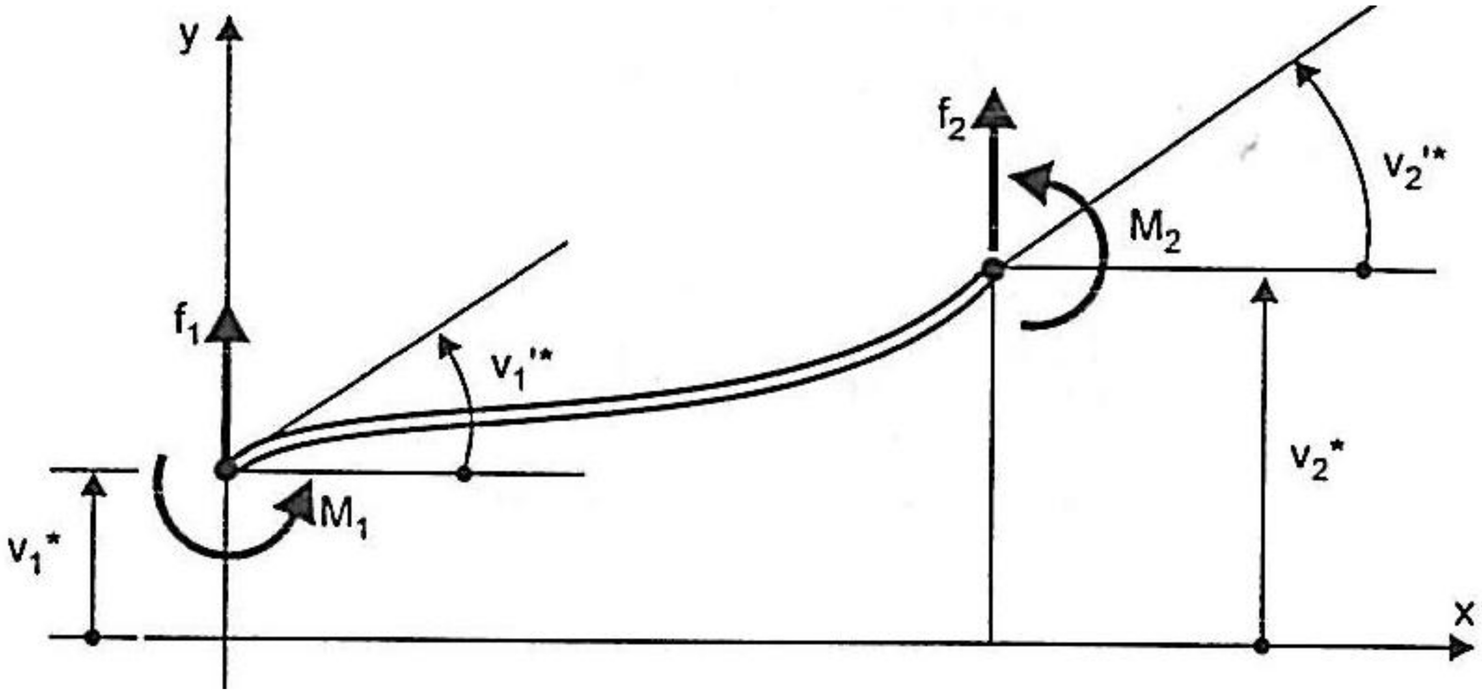
Work of internal loads

$$\boxed{\tau_{INT} = \int_0^L M \cdot d\theta}$$

External virtual works

$$\tau_{\text{external}} = f_1.v_1^* + M_1.v_1'^* + f_2.v_2^* + M_2.v_2'^*$$

v^* = virtual displacements



External virtual works

The product of the transposed virtual displacements vector by the vector of forces can be expressed by:

$$\tau_{external} = \left\{ \begin{matrix} v_1^* & v'^*_1 & v_2^* & v'^*_2 \end{matrix} \right\} . \left\{ \begin{matrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{matrix} \right\}$$

$$\boxed{\tau_e = \left\{\delta^*\right\}^T . \left\{f\right\}}$$

Internal virtual works

$$\boxed{\tau_i = \int_0^L \left\{ v''(x)^* \right\}^T .M(x)dx}$$

* The transposition of the product from two matrices is equal to the product of the transpositions in reverse order.

$$\left\{ v''(x)^* \right\}^T = \left([B] \cdot \left\{ \delta^* \right\} \right)^T = \left\{ \delta^* \right\}^T \cdot [B]^T$$

$$\mathcal{M}(x) = E \cdot I \cdot v''(x),$$ being: $''(x) = [B(x)] \cdot \{\delta\},$

$$\mathcal{M}(x) = E \cdot I \cdot [B(x)] \cdot \{\delta\},$$

** $[S(x)] = E \cdot I \cdot [B(x)],$

$$\boxed{\tau_{int} = \int_0^L \left\{ \delta^* \right\}^T \cdot [B]^T \cdot E \cdot I \cdot [B] \{\delta\} \cdot dx}$$

$$\boxed{\mathcal{M}(x) = [S(x)] \cdot \{\delta\}}$$

Equating the equations (internal and external work), we have:

$$\cancel{\left\{\delta^*\right\}^T}\cdot\{f\}=\int_0^L \cancel{\left\{\delta^*\right\}^T}\cdot[B]^T\cdot E\cdot I\cdot[B]\{\delta\}\cdot dx$$

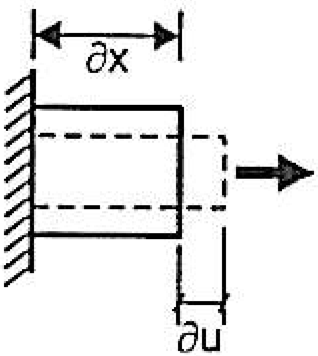
$$\boxed{\{f\}=\left[\int_0^L [B]^T\cdot E\cdot I\cdot[B]\cdot dx\right]\cdot\{\delta\}}$$

nodal loads in the element

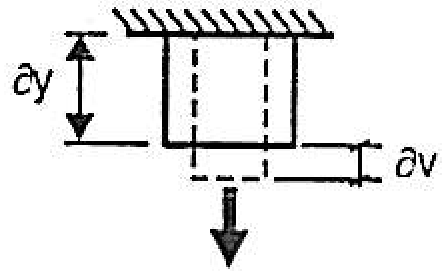
nodal displacements

Stiffness matrix of the element

Knowing that:

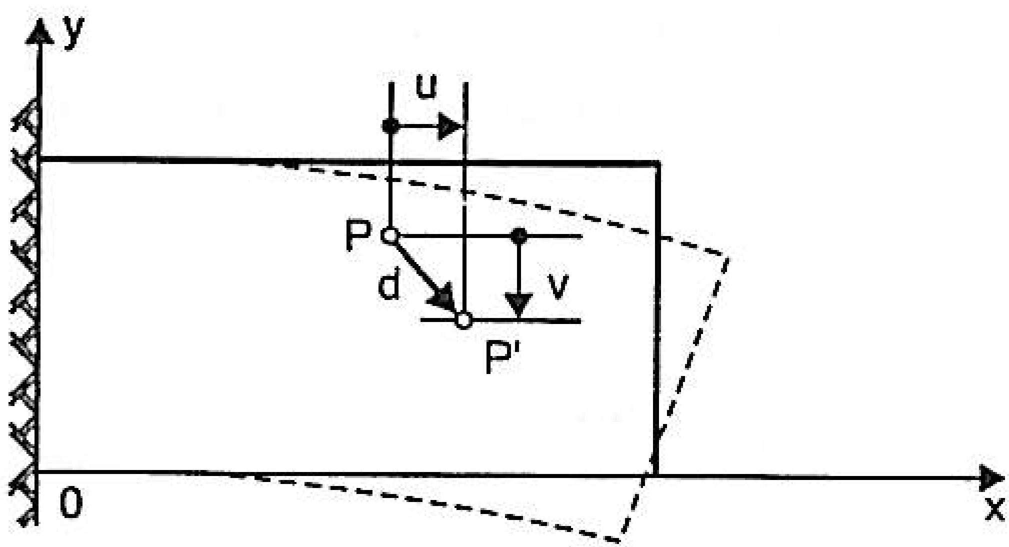


$$\varepsilon_x = \frac{\partial u}{\partial x}$$

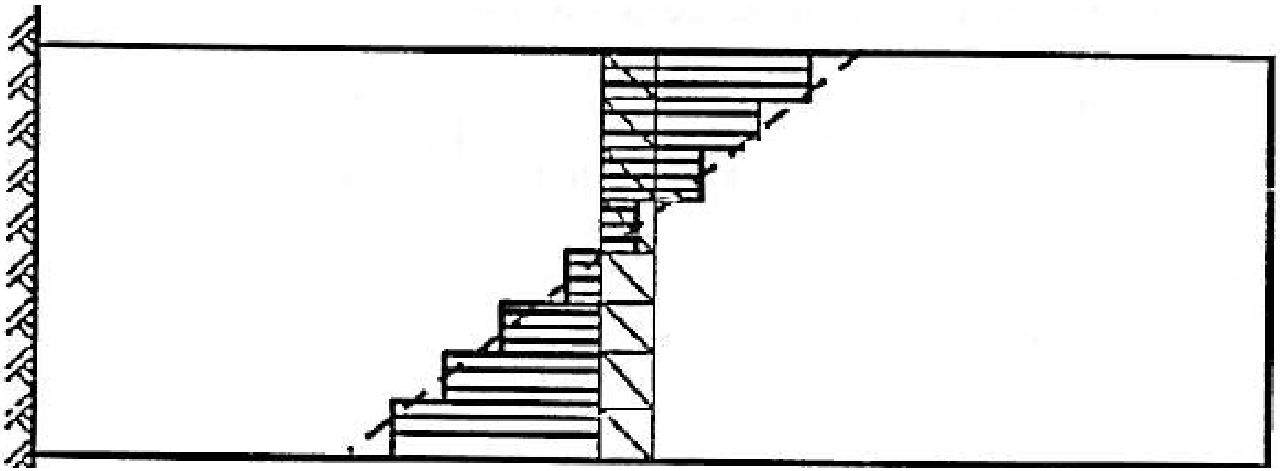


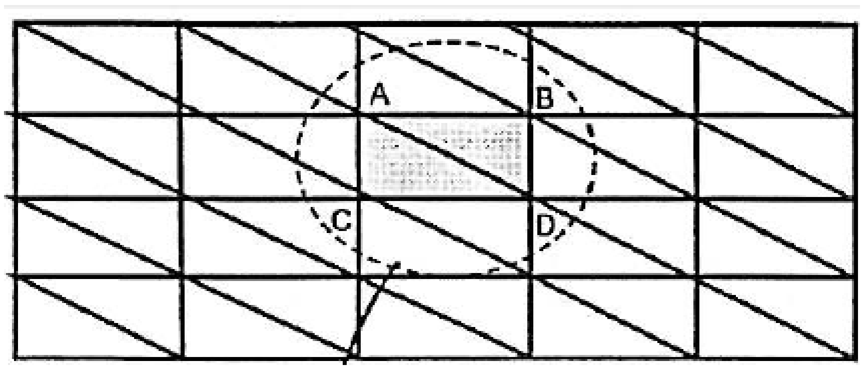
$$\varepsilon_y = \frac{\partial v}{\partial y}$$

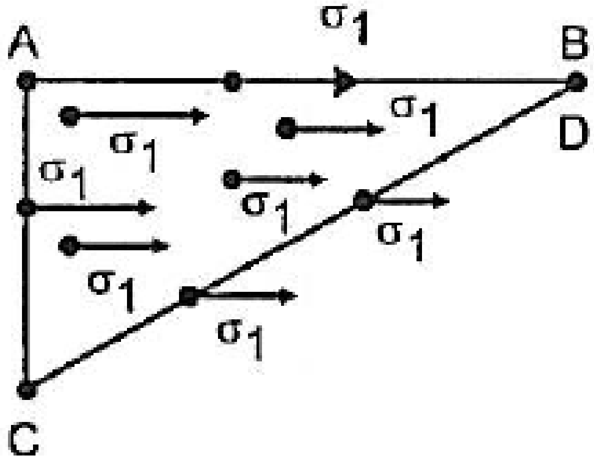
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



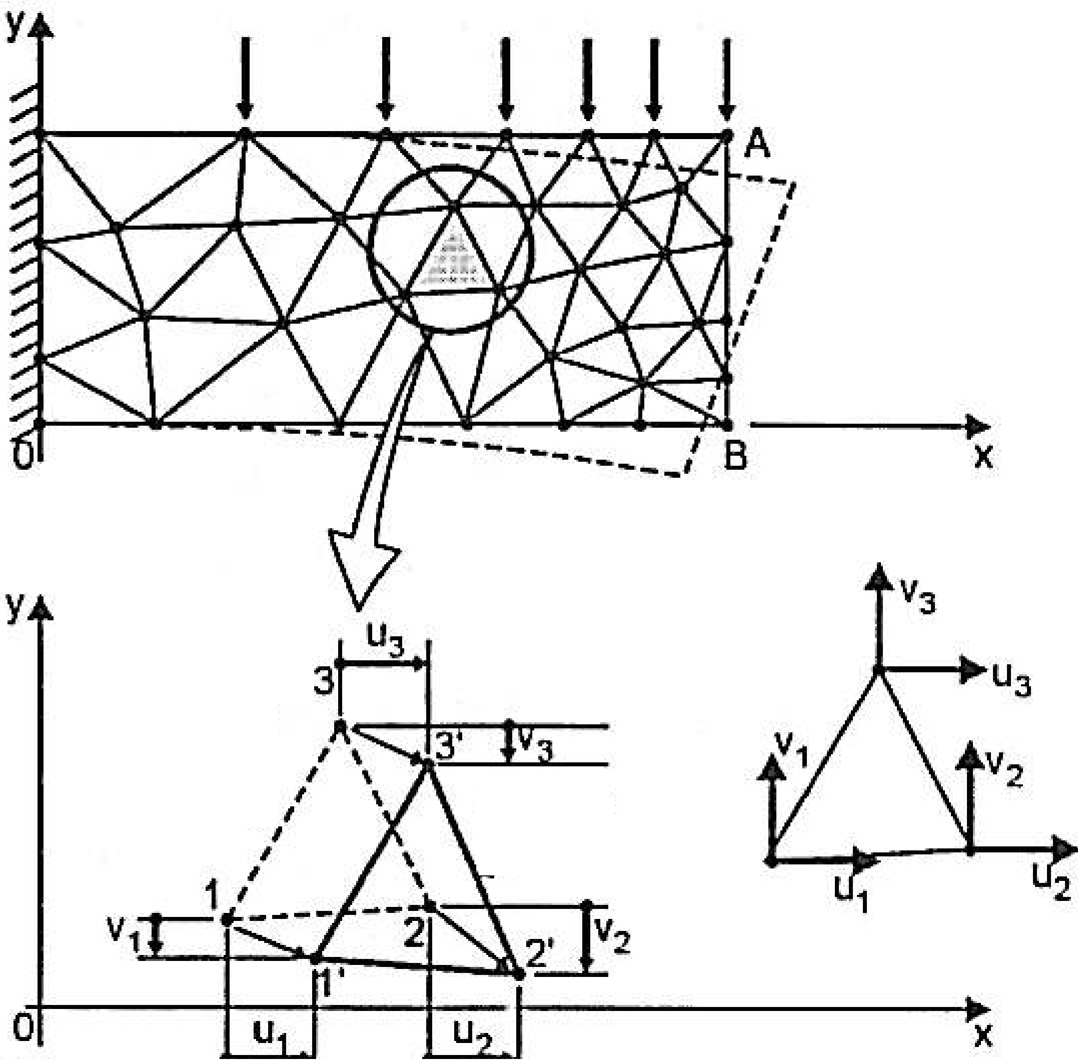
BEAM'S DISCRETIZED TRIANGULAR MODEL







For a linear two-dimensional triangular element with 6 DoFs



$$\{f\}_{6\times 1} = [k]_{6\times 6} \cdot \{\delta\}_{6\times 1}$$

$$\boxed{\begin{aligned} u(x,y) &= C_1 + C_2 \cdot x + C_3 \cdot y \\ v(x,y) &= C_4 + C_5 \cdot x + C_6 \cdot y \end{aligned}}$$

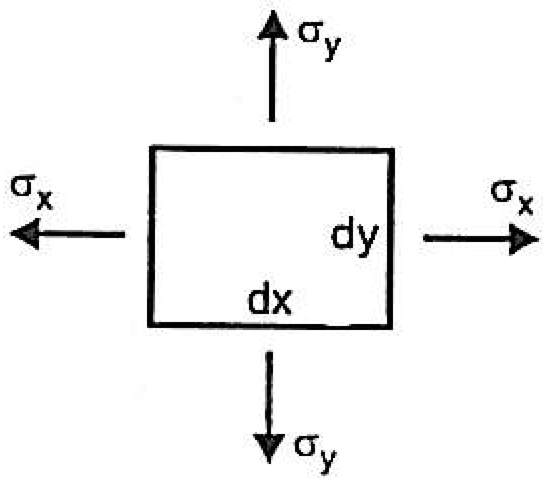
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = C_3 + C_5$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = C_2$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = C_6$$

Two-dimensional Finite Element Formulation

Analyzing the plane state of stresses, it seems that the side contraction, given by **$\nu\sigma_y$/E**, must be considered, as well as the shear stresses.



$$\boxed{\varepsilon_x = \frac{\sigma_x}{E} - \nu \cdot \frac{\sigma_y}{E}} \quad (1)$$

$$\boxed{\varepsilon_y = \frac{\sigma_y}{E} - \nu \cdot \frac{\sigma_x}{E}} \quad (2)$$

$$\boxed{\tau_{xy} = G \cdot \gamma_{xy}}$$

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

(3)

$$\boxed{\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E} \cdot \tau_{xy}}$$

The Equations (1), (2) and (3) can be described in the matrix form as follows.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2(1+\upsilon) \end{bmatrix} . \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Inverting the coefficient matrix, we have:

$$\begin{Bmatrix} \sigma x \\ \sigma y \\ \tau xy \end{Bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \dfrac{1-\upsilon}{2} \end{bmatrix} . \begin{Bmatrix} \varepsilon x \\ \varepsilon y \\ \gamma xy \end{Bmatrix}$$

$\sigma(x, y) = [D].\varepsilon(x, y)$

$[D] =$ elasticity matrix of the material

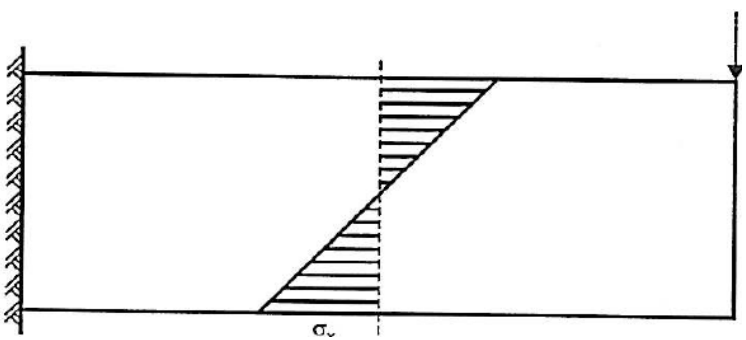
Knowing that:

$$\varepsilon_x = \frac{\partial u}{\partial x} = C_2 \qquad \varepsilon_y = \frac{\partial v}{\partial y} = C_6 \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = C_3 + C_5$$

We are able to calculate the stresses in a point within the element

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{cases} \sigma_x = \dfrac{E}{1-\nu^2}(\varepsilon_x + \nu.\varepsilon_y) = \dfrac{E}{1-\nu^2}(C_2 + \nu.C_6) = \text{CONSTANT} \\ \sigma_y = \dfrac{E}{1-\nu^2}(\nu.\varepsilon_x + \varepsilon_y) = \dfrac{E}{1-\nu^2}(\nu.C_2 + C_6) = \text{CONSTANT} \\ \tau_{xy} = \dfrac{E}{1-\nu^2}.(\dfrac{1-\nu}{2}).(C_3 + C_5) = \text{CONSTANT} \end{cases}$$



INTERPOLATION FUNCTIONS

$$u = C_1 + C_2 \cdot x + C_3 \cdot y$$

$$v = C_4 + C_5 \cdot x + C_6 \cdot y$$

$$\{\delta(x,y)\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}}_{[H(x,y)]} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = [H(x,y)] \cdot \{C\}$$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}}_{(A)} \cdot \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix}$$

$$\boxed{\{\delta_{(x,y)}\} = [H(x,y)] \cdot [A]^{-1} \cdot \{\delta\}}$$

$$\{\delta\} = [A] \cdot \{C\} \rightarrow \{C\} = [A]^{-1} \cdot \{\delta\} \quad (4)$$

Inverting matrix A (Equation 4), we have:

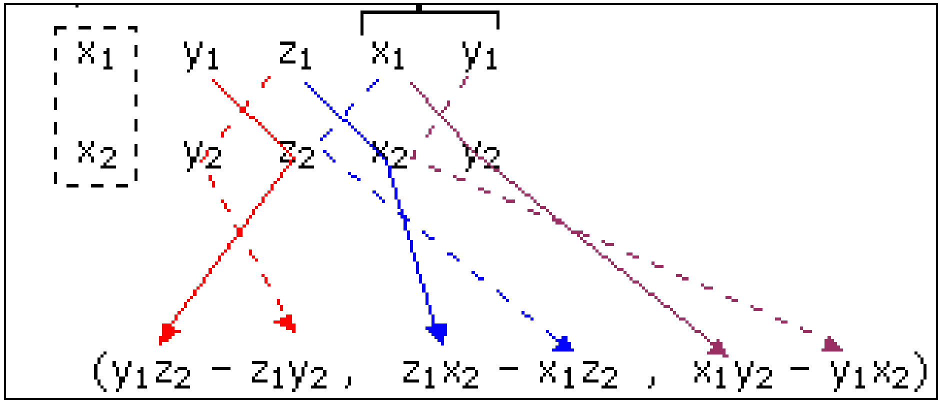
$$[A]^{-1} = \frac{1}{2\cdot\Delta}\begin{bmatrix} x_2 \cdot y_3 - x_3 y_2 & 0 & -x_1 \cdot y_3 + x_3 \cdot y_1 & 0 & x_1 \cdot y_2 - x_2 \cdot y_1 & 0 \\ y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 & 0 \\ 0 & x_2 \cdot y_3 - x_3 y_2 & 0 & -x_1 \cdot y_3 + x_3 \cdot y_1 & 0 & x_1 \cdot y_2 - x_2 \cdot y_1 \\ 0 & y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \end{bmatrix}$$

Being:

$$2\cdot\Delta = 2\cdot(\text{ Triangle area } 123) = (x_2 \cdot y_3 - x_3 \cdot y_2) - (x_1 \cdot y_3 - x_3 \cdot y_1) + (x_1 \cdot y_2 - x_2 \cdot y_1)$$

From linear algebra, we have:

Triangle area = Vector product /2



STRESS CALCULATION

$$\varepsilon_x = C_2$$
$$\varepsilon_y = C_6$$
$$\delta_{xy} = C_3 + C_5$$

$$\Rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \delta_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}}_{[G]} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} \Rightarrow \{\varepsilon(x,y)\} = [G] \cdot [C] \quad (5)$$

Replacing the Eq. (4) in Eq. (5), we have:

$$\boxed{\{\varepsilon(x,y)\} = \underbrace{[G] \cdot [A]^{-1}}_{[B]} \cdot \{\delta\}} \Rightarrow \boxed{\{\varepsilon(x,y)\} = [B] \cdot \{\delta\}}$$

$$[B] = \frac{1}{2 \cdot \Delta} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

$$\boxed{\{\sigma(x,y)\} = [D] \cdot \{\varepsilon(x,y)\}}$$

Virtual work: Two-dimensional triangular element

$$\tau_{EXT} = f_{x1} \cdot u_1^* + f_{y1} \cdot v_1^* + f_{x2} \cdot u_2^* + f_{y2} \cdot v_2^* + f_{x3} \cdot u_3^* + f_{y3} \cdot v_3^*$$

$$\boxed{\tau_{EXT} = \{\delta^*\}^T \cdot \{f\}} \quad (6)$$

$$\tau_{EXT} = \{u_1^* \;\; V_1^* \;\; u_2^* \;\; V_2^* \;\; u_3^* \;\; V_3^*\} \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix}$$

$$\tau_i = \int \{\varepsilon(x,y)^*\}^T . \{\sigma(x,y)\} . dV$$

Being: $\varepsilon(x,y)^* = [B].\{\delta\}^*$ and $\sigma(x,y) = [D].[B].\{\delta\}$

$$\boxed{\tau_i = \int \{[B].\{\delta\}^*\}^T [D].[B].\{\delta\}.dV} \quad (7)$$

Equating Equations (6) and (7), we have :

$$\{f\} = \left[\int_V [B]^T . [D] . [B] . dV \right] . \{\delta\}$$

Being: $[K]^e = \int_V [B]^T . [D] . [B] . dV$

The matrices [B] and [D] contain constant terms. For an element of constant thickness, the volume integral is equal to the product of the triangle area (A) by the thickness (t). Therefore: $\int_V dV = A.t$

Being: $\boxed{[k]^e = [k]^e_\varepsilon + [k]_\gamma}$ **and** $\left| \begin{array}{l} x_{ij} = x_i - x_j \\ y_{ij} = y_i - y_j \end{array} \right.$ **, we have:**

$$[K]^e_\varepsilon = \frac{E.t}{4.A_{123}(1-\nu^2)} \cdot \begin{bmatrix} y^2_{32} & & & & & \\ -\nu.y_{32}.x_{32} & x^2_{32} & & & \text{"Sim"} & \\ -y_{32}.y_{31} & \nu.x_{32}.y_{31} & y^2_{31} & & & \\ \nu.y_{32}.x_{31} & -x_{32}.x_{31} & -\nu.y_{31}.x_{31} & x^2_{31} & & \\ y_{32}.y_{21} & -\nu.x_{32}.y_{21} & -y_{31}.y_{21} & \nu.x_{31}.y_{21} & y^2_{21} & \\ -\nu.y_{32}.x_{21} & x_{32}.x_{21} & \nu.y_{31}.x_{21} & -x_{31}.x_{21} & -\nu.y_{21}.x_{21} & x^2_{21} \end{bmatrix}$$

$$[K]^e_\gamma = \frac{E.t}{8.A_{123}(1+\nu)} \cdot \begin{bmatrix} x^2_{32} & & & & & \\ -x_{32}.y_{32} & y^2_{32} & & & \text{"Sim"} & \\ -x_{32}.x_{31} & y_{32}.x_{31} & x^2_{31} & & & \\ x_{32}.y_{31} & -y_{32}.y_{31} & -x_{31}.y_{31} & y^2_{31} & & \\ x_{32}.x_{21} & -y_{32}.x_{21} & -x_{31}.x_{21} & y_{31}.x_{21} & x^2_{21} & \\ -x_{32}.y_{21} & y_{32}.y_{21} & x_{31}.y_{21} & -y_{31}.y_{21} & -x_{21}.y_{21} & y^2_{21} \end{bmatrix}$$

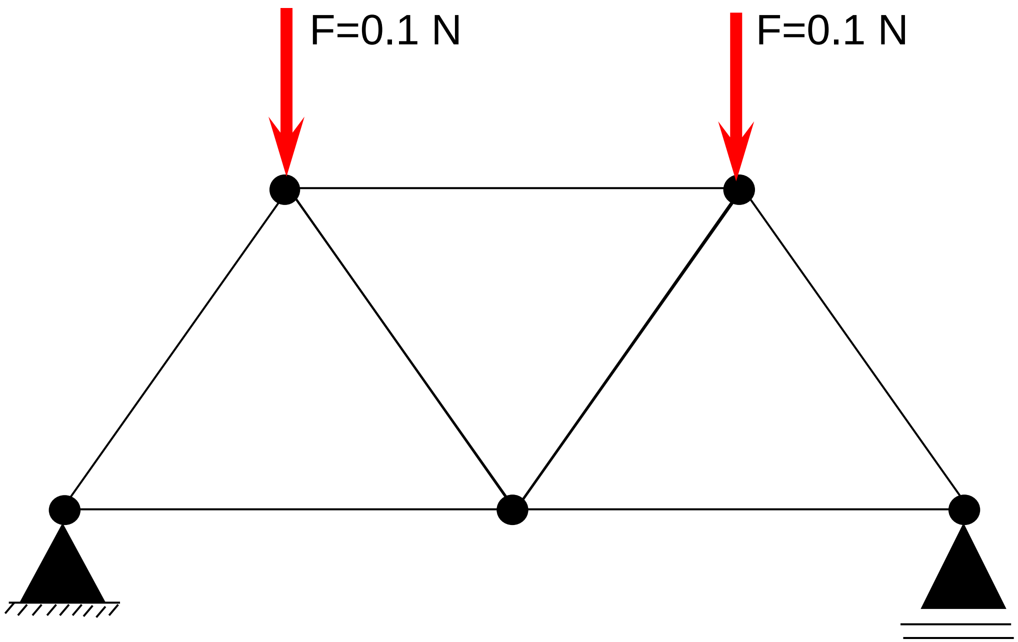
Remembering that:

$$\sigma(x,y) = [D].\varepsilon(x,y)$$

$$\begin{Bmatrix} \sigma x \\ \sigma y \\ \tau xy \end{Bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} . \begin{Bmatrix} \varepsilon x \\ \varepsilon y \\ \gamma xy \end{Bmatrix}$$

Example

Based on the previous code, print the global stiffness matrix for the following input data: E = 200; **n**= 0.3; t = 1.



Solution for each element

```
Area = [x1  y1    1;
          x2  y2    1
          x3  y3    1 ];
A=abs(0.5*det(Area));

cte1=(E*t)/(4*A*(1-nu^2));
cte2=(E*t)/(8*A*(1+nu));

x21=x2-x1; x31=x3-x1; x32=x3-x2; y21=y2-y1; y31=y3-y1;  y32=y3-y2;

ke_1=cte1*[y32^2 -nu*y32*x32 -y32*y31 nu*y32*x31 y32*y21 -nu*y32*x21 -
           nu*y32*x32 x32^2 nu*x32*y31 -x32*x31 -nu*x32*y21 x32*x21
           -y32*y31 nu*x32*y31 y31^2 -nu*y31*x31 -y31*y21 nu*y31*x21
           nu*y32*x31 -x32*x31 -nu*y31*x31 x31^2 nu*x31*y21 -x31*x21
           y32*y21 -nu*x32*y21 -y31*y21 nu*x31*y21 y21^2 -nu*y21*x21
           -nu*y32*x21 x32*x21 nu*y31*x21 -x31*x21 -nu*y21*x21 x21^2];

ke_2 = cte2*[x32^2 -x32*y32 -x32*x31 x32*y31 x32*x21 -x32*y21
            -x32*y32 y32^2 y32*x31 -y32*y31 -y32*x21 y32*y21
           -x32*x31 y32*x31 x31^2 -x31*y31 -x31*x21 x31*y21
           x32*y31 -y32*y31 -x31*y31 y31^2 y31*x21 -y31*y21
           x32*x21 -y32*x21 -x31*x21 y31*x21 x21^2 -x21*y21
           -x32*y21 y32*y21 x31*y21 -y31*y21 -x21*y21 y21^2];

k = ke_1+ke_2
```

Scilab code for the element 1

```
clear; clc;
E  = 10;   nu= 0.3;  t = 1;
i = [1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 3 4 4 4 4 4 4 5 5 5 5 5 5 6 6 6 6 6 6];
j = [1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6];

// Element 1
x1=0.5; y1=1 ;x2= 0 ;y2 =0; x3=1 ; y3=0;
Area = [x1  y1    1;
        x2  y2    1
        x3  y3    1 ];
A=abs(0.5*det(Area));
cte1=(E*t)/(4*A*(1-nu^2));
cte2=(E*t)/(8*A*(1+nu));

x21=x2-x1; x31=x3-x1; x32=x3-x2; y21=y2-y1; y31=y3-y1;  y32=y3-y2;

ke_1 = cte1 * [y32^2 -nu*y32*x32 -y32*y31 nu*y32*x31 y32*y21 -nu*y32*x21
       -nu*y32*x32 x32^2 nu*x32*y31 -x32*x31 -nu*x32*y21 x32*x21
       -y32*y31 nu*x32*y31 y31^2 -nu*y31*x31 -y31*y21 nu*y31*x21
       nu*y32*x31 -x32*x31 -nu*y31*x31 x31^2 nu*x31*y21 -x31*x21
       y32*y21 -nu*x32*y21 -y31*y21 nu*x31*y21 y21^2 -nu*y21*x21
       -nu*y32*x21 x32*x21 nu*y31*x21 -x31*x21 -nu*y21*x21 x21^2];

ke_2 = cte2 * [x32^2 -x32*y32 -x32*x31 x32*y31 x32*x21 -x32*y21
        -x32*y32 y32^2 y32*x31 -y32*y31 -y32*x21 y32*y21
        -x32*x31 y32*x31 x31^2 -x31*y31 -x31*x21 x31*y21
        x32*y31 -y32*y31 -x31*y31 y31^2 y31*x21 -y31*y21
        x32*x21 -y32*x21 -x31*x21 y31*x21 x21^2 -x21*y21
        -x32*y21 y32*y21 x31*y21 -y31*y21 -x21*y21 y21^2];
k = ke_1 + ke_2;
k1 = sparse([i(:), j(:)],k,[10, 10]);
```

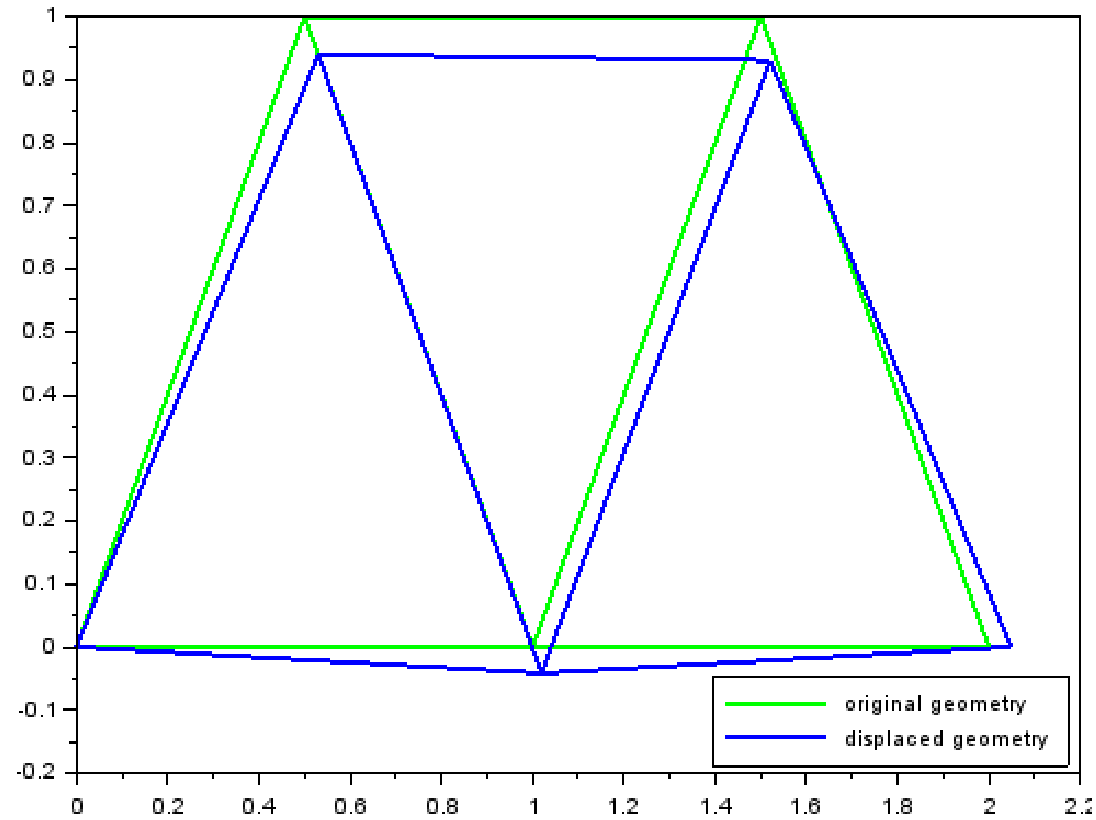
Scilab code for the element 2

```
// Element 2
x1=0.5; y1=1 ;x2= 1 ;y2 =0; x3=1.5 ; y3=1;
Area = [x1  y1    1;
        x2  y2    1
        x3  y3    1 ];
A=abs(0.5*det(Area));
cte1=(E*t)/(4*A*(1-nu^2));
cte2=(E*t)/(8*A*(1+nu));

x21=x2-x1; x31=x3-x1; x32=x3-x2; y21=y2-y1; y31=y3-y1;  y32=y3-y2;
ke_1 = cte1 * [y32^2 -nu*y32*x32 -y32*y31 nu*y32*x31 y32*y21 -nu*y32*x21
       -nu*y32*x32 x32^2 nu*x32*y31 -x32*x31 -nu*x32*y21 x32*x21
       -y32*y31 nu*x32*y31 y31^2 -nu*y31*x31 -y31*y21 nu*y31*x21
       nu*y32*x31 -x32*x31 -nu*y31*x31 x31^2 nu*x31*y21 -x31*x21
       y32*y21 -nu*x32*y21 -y31*y21 nu*x31*y21 y21^2 -nu*y21*x21
       -nu*y32*x21 x32*x21 nu*y31*x21 -x31*x21 -nu*y21*x21 x21^2];

ke_2 = cte2 * [x32^2 -x32*y32 -x32*x31 x32*y31 x32*x21 -x32*y21
        -x32*y32 y32^2 y32*x31 -y32*y31 -y32*x21 y32*y21
        -x32*x31 y32*x31 x31^2 -x31*y31 -x31*x21 x31*y21
        x32*y31 -y32*y31 -x31*y31 y31^2 y31*x21 -y31*y21
        x32*x21 -y32*x21 -x31*x21 y31*x21 x21^2 -x21*y21
        -x32*y21 y32*y21 x31*y21 -y31*y21 -x21*y21 y21^2];
k = ke_1+ke_2
v1 = [i(1:12),i(13:36)+2];
v2 = [j(1:2),j(3:6)+2,j(1:2),j(3:6)+2,j(1:2),j(3:6)+2,j(1:2),j(3:6)+2,j(1:2),...
     j(3:6)+2,j(1:2),j(3:6)+2,] ;
k2=sparse([v1(:),v2(:)],k,[10,10]);
```

Scilab code for the element 3

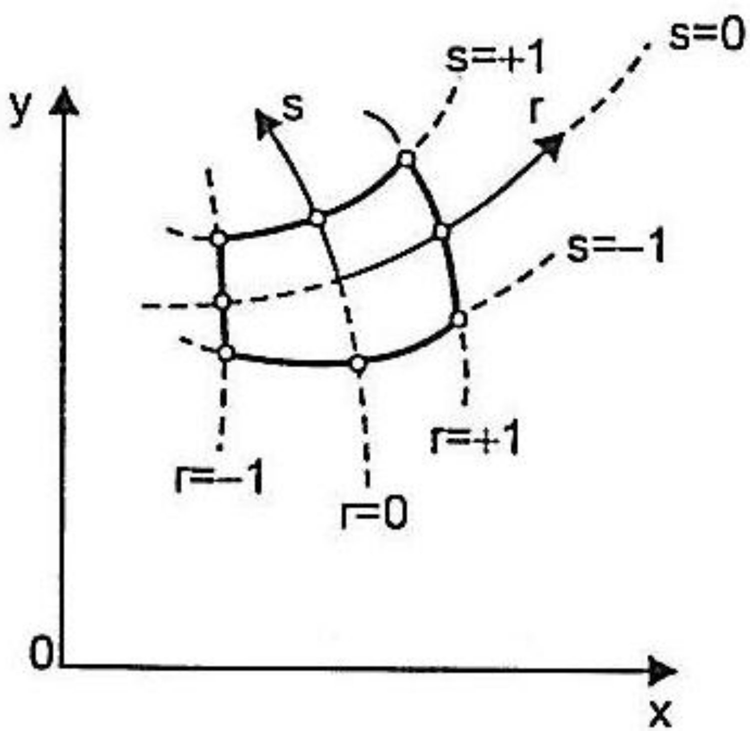
```
// Element 3
x1=1; y1=0 ;x2= 2 ;y2 =0; x3=1.5 ; y3=1;
Area = [x1  y1    1;
        x2  y2    1
        x3  y3    1 ];
A=abs(0.5*det(Area));
cte1=(E*t)/(4*A*(1-nu^2));
cte2=(E*t)/(8*A*(1+nu));

x21=x2-x1; x31=x3-x1; x32=x3-x2; y21=y2-y1; y31=y3-y1;  y32=y3-y2;
ke_1 = cte1 * [y32^2 -nu*y32*x32 -y32*y31 nu*y32*x31 y32*y21 -nu*y32*x21
       -nu*y32*x32 x32^2 nu*x32*y31 -x32*x31 -nu*x32*y21 x32*x21
       -y32*y31 nu*x32*y31 y31^2 -nu*y31*x31 -y31*y21 nu*y31*x21
       nu*y32*x31 -x32*x31 -nu*y31*x31 x31^2 nu*x31*y21 -x31*x21
       y32*y21 -nu*x32*y21 -y31*y21 nu*x31*y21 y21^2 -nu*y21*x21
       -nu*y32*x21 x32*x21 nu*y31*x21 -x31*x21 -nu*y21*x21 x21^2];


ke_2 = cte2 * [x32^2 -x32*y32 -x32*x31 x32*y31 x32*x21 -x32*y21
        -x32*y32 y32^2 y32*x31 -y32*y31 -y32*x21 y32*y21
        -x32*x31 y32*x31 x31^2 -x31*y31 -x31*x21 x31*y21
        x32*y31 -y32*y31 -x31*y31 y31^2 y31*x21 -y31*y21
        x32*x21 -y32*x21 -x31*x21 y31*x21 x21^2 -x21*y21
        -x32*y21 y32*y21 x31*y21 -y31*y21 -x21*y21 y21^2];
k = ke_1+ke_2
v1 = i+4; v2 = j+4;
k3=sparse([v1(:),v2(:)],k,[10,10]);
```

Results




```
k_g = k1+k2+k3; // Global stiffness matrix

//Restricted DoFs (3,4 and 10)
k_g(10,:) = []; k_g(:,10) = [];
k_g(3:4,:) = []; k_g(:,3:4) = [];
f = -0.1; // N
F = [0 f 0 0 0 f 0]' // Load vector considering restricted DoFs (3,4 and 10)
u = inv(k_g)*F;

// Displacements plotting
Y = [0.5,      1     ; 0 ,0; 1     , 0     ; 1.5     , 1     ; 2     , 0; 1     , 0     ; 0.5,      1     ; 1.5,      1     ];
// displaced geometry
U = [0.5+u(1), 1+u(2); 0 ,0; 1+u(3), 0+u(4); 1.5+u(5), 1+u(6); 2+u(7), 0; 1+u(3), 0+u(4); 0.5+u(1), 1+u(2); 1.5+u(5), 1+u(6)];

plot(Y(:,1),Y(:,2),"g-",U(:,1),U(:,2),"b-","LineWidth",2);
legend("original geometry","displaced geometry");
xgrid
```

Two-dimensional isoparametric element



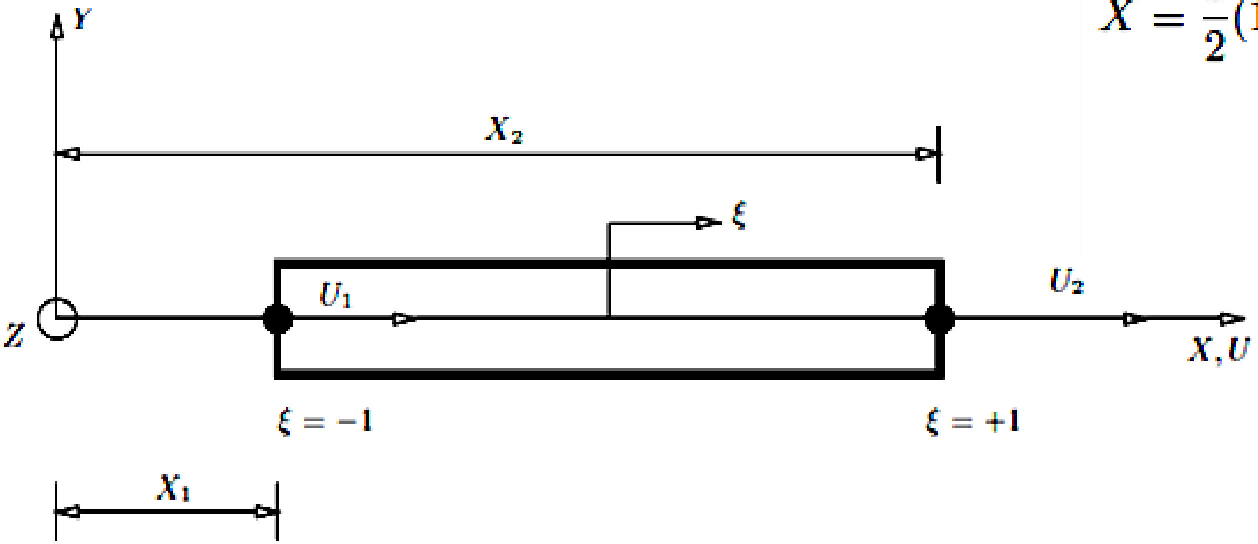
$$x = \sum_{i=1}^{p} h_i \cdot x_i \qquad u = \sum_{i=1}^{p} h_i \cdot u_i$$

$$y = \sum_{i=1}^{p} h_i \cdot y_i \qquad v = \sum_{i=1}^{p} h_i \cdot v_i$$

p = number of nodes from the element

h_i = element shape functions

Shape functions



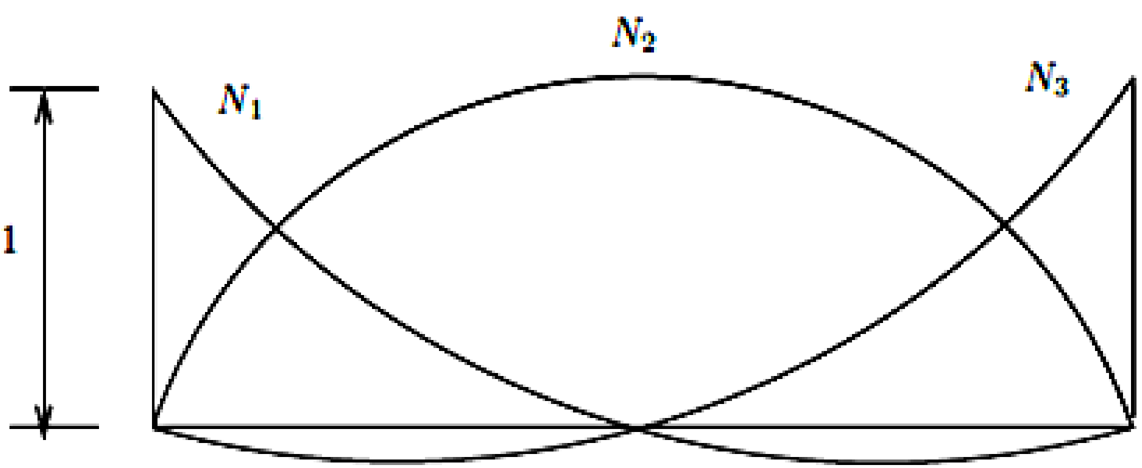
$$X = \frac{1}{2}(1-\xi)X_1 + \frac{1}{2}(1+\xi)X_2$$

$$X = \sum_{i=1}^{2} N_i X_i$$

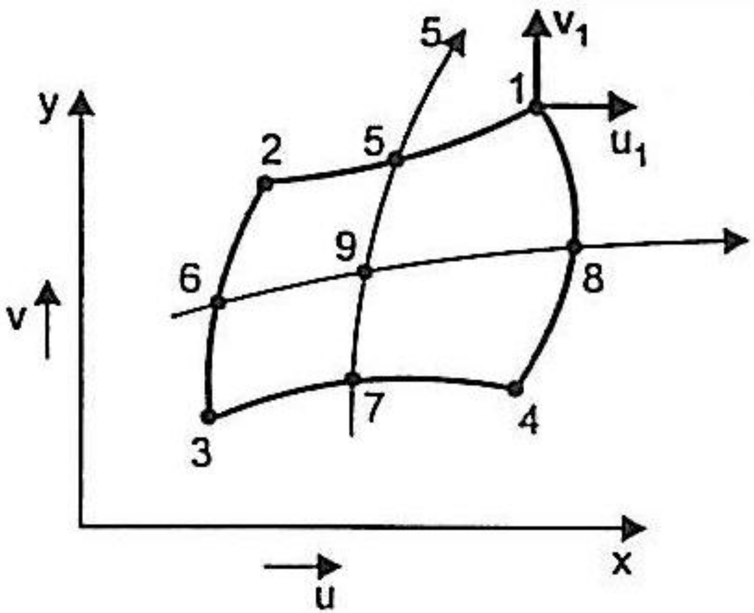
$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = (1-\xi).(1+\xi)$$

$$N_3 = \frac{1}{2}(1+\xi)$$



Isoparametric formulation



$$
\begin{cases}
h_1 = \frac{1}{4}(1+r)(1+s) \\
h_2 = \frac{1}{4}(1-r)(1+s) \\
h_3 = \frac{1}{4}(1-r)(1-s) \\
h_4 = \frac{1}{4}(1+r)(1-s)
\end{cases}
$$

$$h_5 = \frac{1}{2}(1-r^2)(1+s)$$

$$h_6 = \frac{1}{2}(1-s^2)(1-r)$$

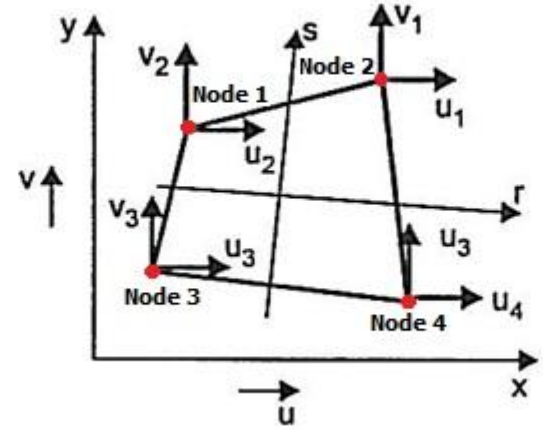
$$h_7 = \frac{1}{2}(1-r^2)(1-s)$$

$$h_8 = \frac{1}{2}(1-s^2)(1+r)$$

$$h_9 = \frac{1}{2}(1-r^2)(1-s^2)$$

$$\boxed{\begin{aligned} u &= h_1 \cdot u_1 + h_2 \cdot u_2 + h_3 \cdot u_3 + h_4 \cdot u_4 + h_5 \cdot u_5 + h_6 \cdot u_6 + h_7 \cdot u_7 + \\ &\quad h_8 \cdot u_8 + h_9 \cdot u_9 \\ v &= h_1 \cdot v_1 + h_2 \cdot v_2 + h_3 \cdot v_3 + h_4 \cdot v_4 + h_5 \cdot v_5 + h_6 \cdot v_6 + h_7 \cdot v_7 + \\ &\quad h_8 \cdot v_8 + h_9 \cdot v_9 \end{aligned}}$$

Isoparametric formulation of a two-dimensional element of four nodes



- Interpolation of element geometry:

$$\begin{cases} x = \frac{1}{4}(1+r)(1+s)\cdot x_1 + \frac{1}{4}(1-r)(1+s)\cdot x_2 + \frac{1}{4}(1-r)(1-s)\cdot x_3 + \frac{1}{4}(1+r)[1-s]\cdot x_4 \\ y = \frac{1}{4}(1+r)(1+s)\cdot y_1 + \frac{1}{4}(1-r)(1+s)\cdot y_2 + \frac{1}{4}(1-r)(1-s)\cdot y_3 + \frac{1}{4}(1+r)[1-s]\cdot y_4 \end{cases}$$

- Interpolation of the displacement field:

$$\begin{cases} u = \frac{1}{4}(1+r)(1+s)\cdot u_1 + \frac{1}{4}(1-r)(1+s)\cdot u_2 + \frac{1}{4}(1-r)(1-s)\cdot u_3 + \frac{1}{4}(1+r)[1-s]\cdot u_4 \\ v = \frac{1}{4}(1+r)(1+s)\cdot v_1 + \frac{1}{4}(1-r)(1+s)\cdot v_2 + \frac{1}{4}(1-r)(1-s)\cdot v_3 + \frac{1}{4}(1+r)[1-s]\cdot v_4 \end{cases}$$

Generalizing, we have:

$$
\begin{cases}
u(r,s) = \sum_i h_i\,(r,s)\cdot u_i = h_1\cdot u_1 + h_2\cdot u_2 + h_3\cdot u_3 + h_4\cdot u_4 \\
v(r,s) = \sum_i h_i\,(r,s)\cdot v_i = h_1\cdot v_1 + h_2\cdot v_2 + h_3\cdot v_3 + h_4\cdot v_4
\end{cases}
$$

By deriving the previous expressions in relation to x and y, deformations are obtained as a function of nodal displacements.

$$
\begin{cases}
\varepsilon_x = \dfrac{\partial u}{\partial x} = \dfrac{\partial h_1}{\partial x}\cdot u_1 + \dfrac{\partial h_2}{\partial x}\cdot u_2 + \dfrac{\partial h_3}{\partial x}\cdot u_3 + \dfrac{\partial h_4}{\partial x}\cdot u_4 \\
\varepsilon_y = \dfrac{\partial v}{\partial y} = \dfrac{\partial h_1}{\partial y}\cdot v_1 + \dfrac{\partial h_2}{\partial y}\cdot v_2 + \dfrac{\partial h_3}{\partial y}\cdot v_3 + \dfrac{\partial h_4}{\partial y}\cdot v_4
\end{cases}
$$

Isoparametric formulation in finite elements

By the chain rule, we have**:**

$$\begin{cases} \dfrac{\partial h_i}{\partial r} = \dfrac{\partial h_i}{\partial x} \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial h_i}{\partial y} \cdot \dfrac{\partial y}{\partial r} \\ \dfrac{\partial h_i}{\partial s} = \dfrac{\partial h_i}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial h_i}{\partial y} \cdot \dfrac{\partial y}{\partial s} \end{cases}$$

Matrix representiaton**:**

$$\begin{Bmatrix} \dfrac{\partial h_i}{\partial r} \\ \dfrac{\partial h_i}{\partial s} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix}}_{\text{Jacobian}} \cdot \begin{Bmatrix} \dfrac{\partial h_i}{\partial x} \\ \dfrac{\partial h_i}{\partial y} \end{Bmatrix}$$

The square matrix that contains derivatives of *x* and *y* as a function of *r* and *s* relates those derived in the local system and those derived in the natural system. It constitutes the Jacobian operator [*J*].

Isoparametric formulation in finite elements

For the calculus of deformations (dh_i/d_x and dh_i/d_y) matrix expression must be inverted, such as:

$$\boxed{\begin{Bmatrix} \dfrac{\partial h_i}{\partial x} \\ \dfrac{\partial h_i}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial h_i}{\partial r} \\ \dfrac{\partial h_i}{\partial s} \end{Bmatrix} \cdot [J]^{-1}}$$

Being:

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix}$$

The matrix expression will only have a solution if $[J]^{-1}$ exists. In this case, the matrix [J] must have an inverse and can't be singular.

Stiffness matrix - isoparametric element

$$[k]^e = \int_{volume} [B]^T \cdot [D] \cdot [B] \cdot dvol$$

$$\begin{cases} dvol = dx \cdot dy \cdot dz \\ dV = \det[J] \cdot dr \cdot ds \cdot dt \end{cases}$$

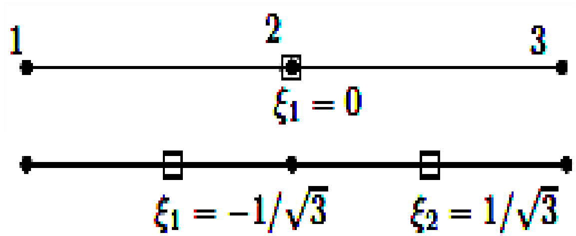
$$[k]^e = \int_{volume} [B]^T \cdot [D] \cdot [B] \cdot |J| \cdot dr \cdot ds \cdot dt$$

$$[k]^e = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \cdot [B]^T [D] \cdot [B] \cdot |J| \cdot dr \cdot ds \cdot dt$$

$$[F] = [B]^T \cdot [D] \cdot [B] \cdot \det[J] \quad \Rightarrow \quad [k]^e = \int_{vol} [F] \cdot dr \cdot ds \cdot dt$$

Stiffness matrix - isoparametric element

$$[k]^e = \sum_{i,j,k} \alpha_{ijk} \cdot [F]_{ijk}$$



$$\int_a^b F(r) \cdot dr = \alpha_1 \cdot F(r_1) + \alpha_2 \cdot F(r_2) + \cdots + \alpha_n \cdot F(r_n) + R_n$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} F(r,s,t) \cdot dr \cdot ds \cdot dt = \sum_{i,j,k} \alpha_i \cdot \alpha_j \cdot \alpha_k \cdot F(r_i, s_j, t_k)$$

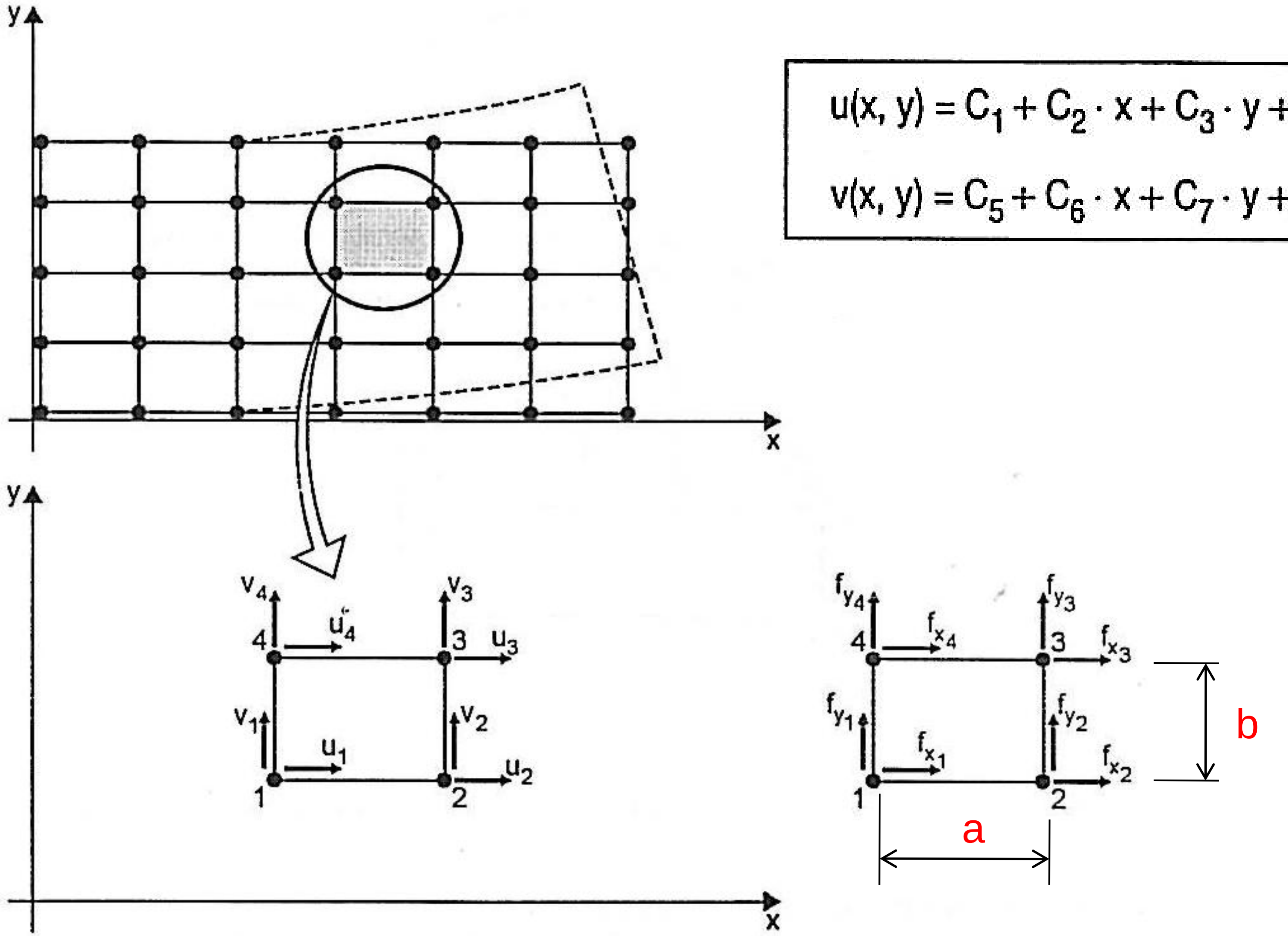
$F(r_i)$ are the values of the function of chosen points and α (or w) are the weights of each values in the sum.

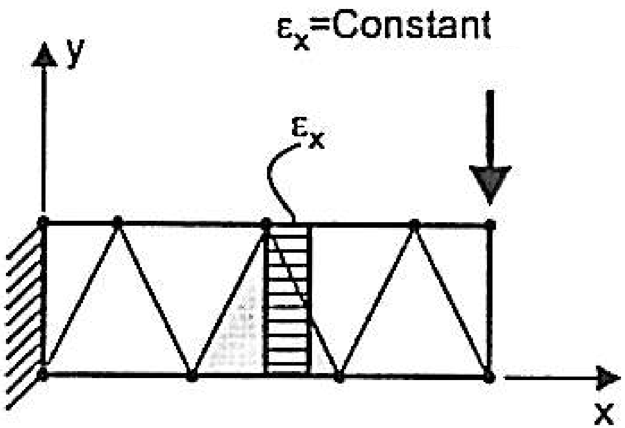
By using the Gauss integration, the stiffness matrix is obtained by the sum of the contributions of each element and each point of Gauss, which are spread in the appropriate positions, taking into account the DoF of each element.

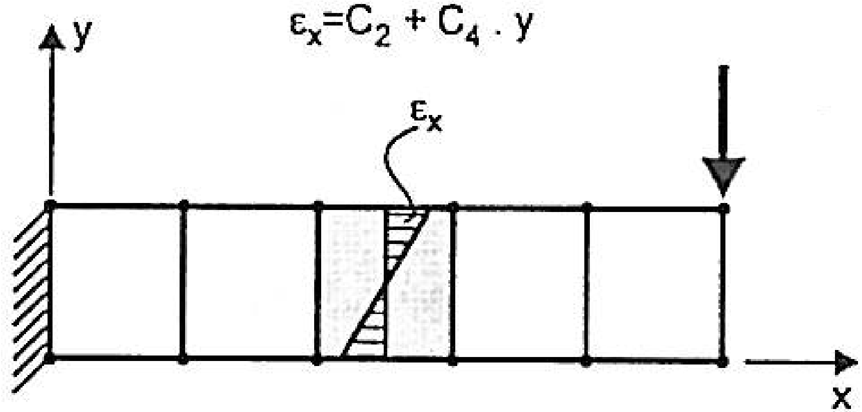
Two-dimensional linear rectangular element (8 DoFs)

$$u(x, y) = C_1 + C_2 \cdot x + C_3 \cdot y + C_4 \cdot x \cdot y$$

$$v(x, y) = C_5 + C_6 \cdot x + C_7 \cdot y + C_8 \cdot x \cdot y$$



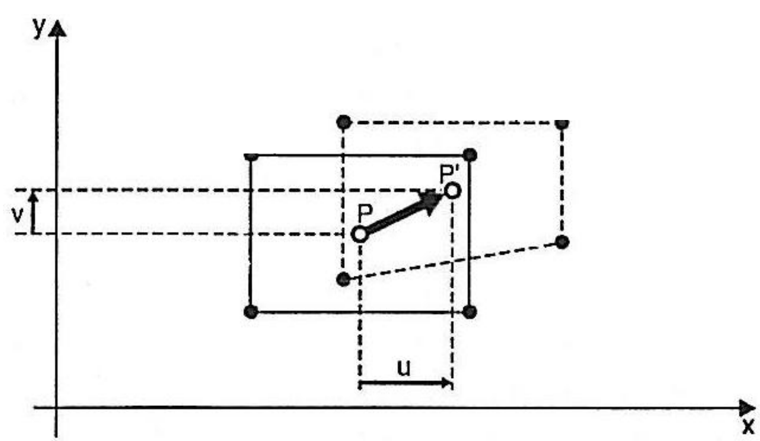




$$\varepsilon_x = \frac{\partial u}{\partial x} = C_2 + C_4 \cdot y$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = C_7 + C_8 \cdot x$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = C_3 + C_4 \cdot x + C_6 + C_8 \cdot y$$



Equations of the two-dimensional rectangular element

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Performing the matrix product, we have:

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu \cdot \varepsilon_y) = \frac{E}{1-\nu^2}\{C_2 + C_4 \cdot y + \nu(C_7 + C_8 \cdot x)\}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\nu \cdot \varepsilon_x + \varepsilon_y) = \frac{E}{1-\nu^2}\{\nu(C_2 + C_4 \cdot y) + C_7 + C_8 \cdot x\}$$

$$\tau_{xy} = \frac{E}{1-\nu^2} \cdot \left(\frac{1-\nu}{2}\right) \cdot \gamma_{xy} = \frac{E}{1-\nu^2} \cdot \left(\frac{1-\nu}{2}\right) \cdot (C_3 + C_4 \cdot x + C_6 + C_8 \cdot y)$$

Equations of the two-dimensional rectangular element

$$u(x,y) = C_1 + C_2 \cdot x + C_3 \cdot y + C_4 \cdot x \cdot y$$

$$v(x,y) = C_5 + C_6 \cdot x + C_7 \cdot y + C_8 \cdot x \cdot y$$

$$\{\delta\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}; \quad \{f\} = \begin{Bmatrix} f_{x_1} \\ f_{y_1} \\ f_{x_2} \\ f_{y_2} \\ f_{x_3} \\ f_{y_3} \\ f_{x_4} \\ f_{y_4} \end{Bmatrix} \quad \therefore \quad \{f\} = [k]^e \cdot \{\delta\}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \cdot \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{Bmatrix}$$

$$\{\delta(x,y)\} = [H(x,y)] \cdot \{C\}$$

Determination of coefficients

$$p / x = 0, y = 0 \Rightarrow u(x, y) = u_1 ; v(x, y) = v_1$$

$$p / x = 0, y = b \Rightarrow u(x, y) = u_2 ; v(x, y) = v_2$$

$$p / x = a, y = 0 \Rightarrow u(x, y) = u_3 ; v(x, y) = v_3$$

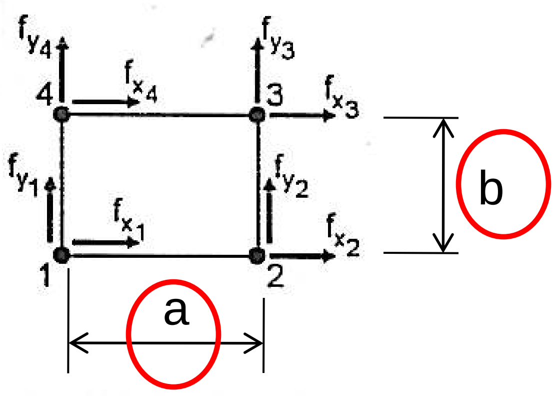
$$p / x = a, y = b \Rightarrow u(x, y) = u_4 ; v(x, y) = v_4$$

$$u(x,y) = C_1 + C_2 \cdot x + C_3 \cdot y + C_4 \cdot x \cdot y$$

$$v(x,y) = C_5 + C_6 \cdot x + C_7 \cdot y + C_8 \cdot x \cdot y$$

$$u_1 = C_1$$
$$v_1 = C_5$$
$$u_2 = C_1 + bC_3$$
$$v_2 = C_5 + bC_7$$
$$u_3 = C_1 + aC_4$$
$$v_3 = C_5 + aC_3$$
$$u_4 = C_1 + aC_2 + bC_3 + abC_4$$
$$v_4 = C_5 + aC_6 + bC_7 + abC_8$$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & b & 0 \\ 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a & 0 & 0 \\ 1 & a & b & ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a & b & ab \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{Bmatrix}$$



$$\{\delta\} = [A] \cdot \{C\}$$

$$\{C\} = [A]^{-1} \{\delta\}$$

Rewriting the matrix A^{-1}, we have:

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/a & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ -1/b & 0 & 1/b & 0 & 0 & 0 & 0 & 0 \\ 1/ab & 0 & 1/a & 0 & -1/ab & 0 & 1/ab & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/a & 0 & 0 & 0 & 1/a & 0 & 0 \\ 0 & -1/b & 0 & 1/b & 0 & 0 & 0 & 0 \\ 0 & 1/ab & 0 & -1/ab & 0 & -1/ab & 0 & 1/ab \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{Bmatrix} = \begin{Bmatrix} C_2 + C_4 \cdot y \\ C_7 + C_8 \cdot y \\ C_3 + C_4 \cdot x + C_6 + C_8 \cdot y \end{Bmatrix}$$

$$\downarrow$$

$$\{\varepsilon(x,y)\} = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \cdot \{C\} \Rightarrow \{\varepsilon(x,y)\} = [G] \cdot \{C\}$$

$$\{C\} = [A]^{-1} \cdot \{\delta\} \rightarrow \{\varepsilon(x,y)\} = \underbrace{[G] \cdot [A]^{-1}}_{} \cdot \{\delta\}$$

$$\boxed{[B] = [G] \cdot [A]^{-1}}$$

$$\downarrow$$

$$\boxed{\{\varepsilon(x,y)\} = [B] \cdot \{\delta\}}$$

Performing the matrix product [G].[A]$^{-1}$, we have:

$$[B] = \frac{1}{a \cdot b}\begin{bmatrix} -b+y & 0 & b-y & 0 & y & 0 & -y & 0 \\ 0 & -a+x & 0 & -x & 0 & x & 0 & a-x \\ -a+x & -b+y & -x & b-y & x & y & a-x & -y \end{bmatrix}$$

$$\{\sigma(x,y)\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \underbrace{\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}}_{[D]} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \sigma \end{Bmatrix}$$

$$\{\sigma\} = [D] \cdot \{\varepsilon\}$$

$$\{\varepsilon\} = [B] \cdot \{\delta\}$$

$$\boxed{\{\sigma(x,y)\} = \underbrace{[D] \cdot [B]} \cdot \{\delta\}}$$

$$\{S(x,y)\} = [D] \cdot [B]$$

Stiffness matrix relating loads and nodal displacements

$$[k]^e = \int_{vol} [B]^T \cdot [D] \cdot [B] \cdot dvol$$

Considering thickness(t) constant:

$$\boxed{[k]^e = t \cdot \left(\int_0^b \int_0^a [B]^T \cdot [D] \cdot [B] dxdy \right)}$$

The matrix [B] contains the terms x and y which are not constant. In this case, we need first to calculate the product $[B]^T.[D].[B]$ and after the integral considering the matrix terms (a and b).

Strain stiffness matrix:

$$\boxed{[k]^e = \boxed{[k]_\varepsilon} + [k]_\gamma}$$

$$[k]_\varepsilon = \frac{Et}{12(1-\nu^2)} \cdot \begin{bmatrix} \frac{4}{\alpha} & & & & & & & \\ 3\nu & 4\alpha & & & & & & \\ -\frac{4}{\alpha} & -3\nu & \frac{4}{\alpha} & & & & & \\ 3\nu & 2\alpha & -3\nu & 4\alpha & & & & \\ -\frac{2}{\alpha} & -3\nu & \frac{2}{\alpha} & -3\nu & \frac{4}{\alpha} & & & \\ -3\nu & -2\alpha & 3\nu & -4\alpha & 3\nu & \frac{4}{\alpha} & & \\ \frac{2}{\alpha} & 3\nu & -\frac{2}{\alpha} & 3\nu & -4\alpha & -3\nu & \frac{4}{\alpha} & \\ -3\nu & -4\alpha & 3\nu & -2\alpha & 3\nu & 2\alpha & -3\nu & 4\alpha \end{bmatrix}$$

$$\alpha = \frac{a}{b}$$

Gamma stiffness matrix

$$[k]^e = [k]_\varepsilon + \boxed{[k]_\gamma}$$

$$[k]_\gamma = \frac{E \cdot t}{24(1+\nu)} \begin{bmatrix} 4\alpha & & & & & & & \\ 3 & \frac{4}{\alpha} & & & & & & \\ 2\alpha & 3 & 4\alpha & & & & & \\ -3 & -\frac{4}{\alpha} & -3 & \frac{4}{\alpha} & & & & \\ -2\alpha & -3 & -4\alpha & 3 & 4\alpha & & & \\ -3 & -\frac{2}{\alpha} & -3 & \frac{2}{\alpha} & 3 & \frac{4}{\alpha} & & \\ -4\alpha & -3 & -2\alpha & 3 & 2\alpha & 3 & 4\alpha & \\ 3 & \frac{2}{\alpha} & 3 & -\frac{2}{\alpha} & -3 & -\frac{4}{\alpha} & -3 & \frac{4}{\alpha} \end{bmatrix}$$

$$\alpha = \frac{a}{b}$$

Exercise on Scilab:

Write a code to plot displacements considering E = 200;

n= 0.3; t = 1, F = 1.