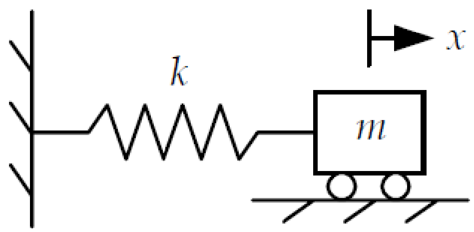
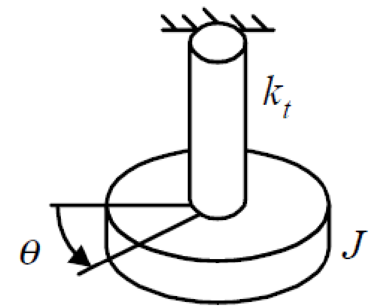
Mechanical stiffness

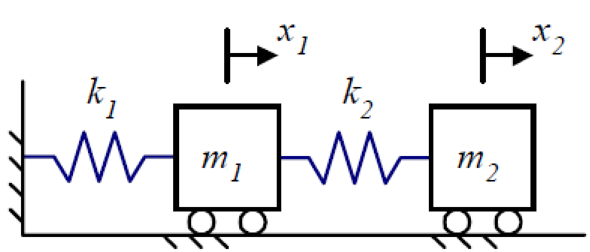
Degrees of freedom (DOF)

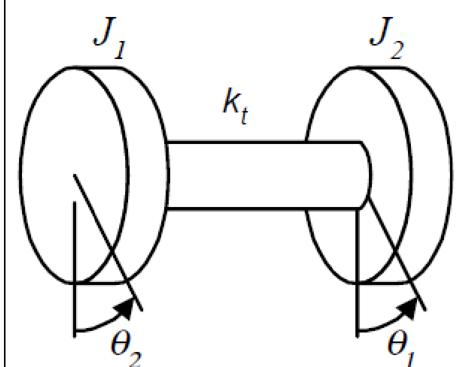
1 DOF



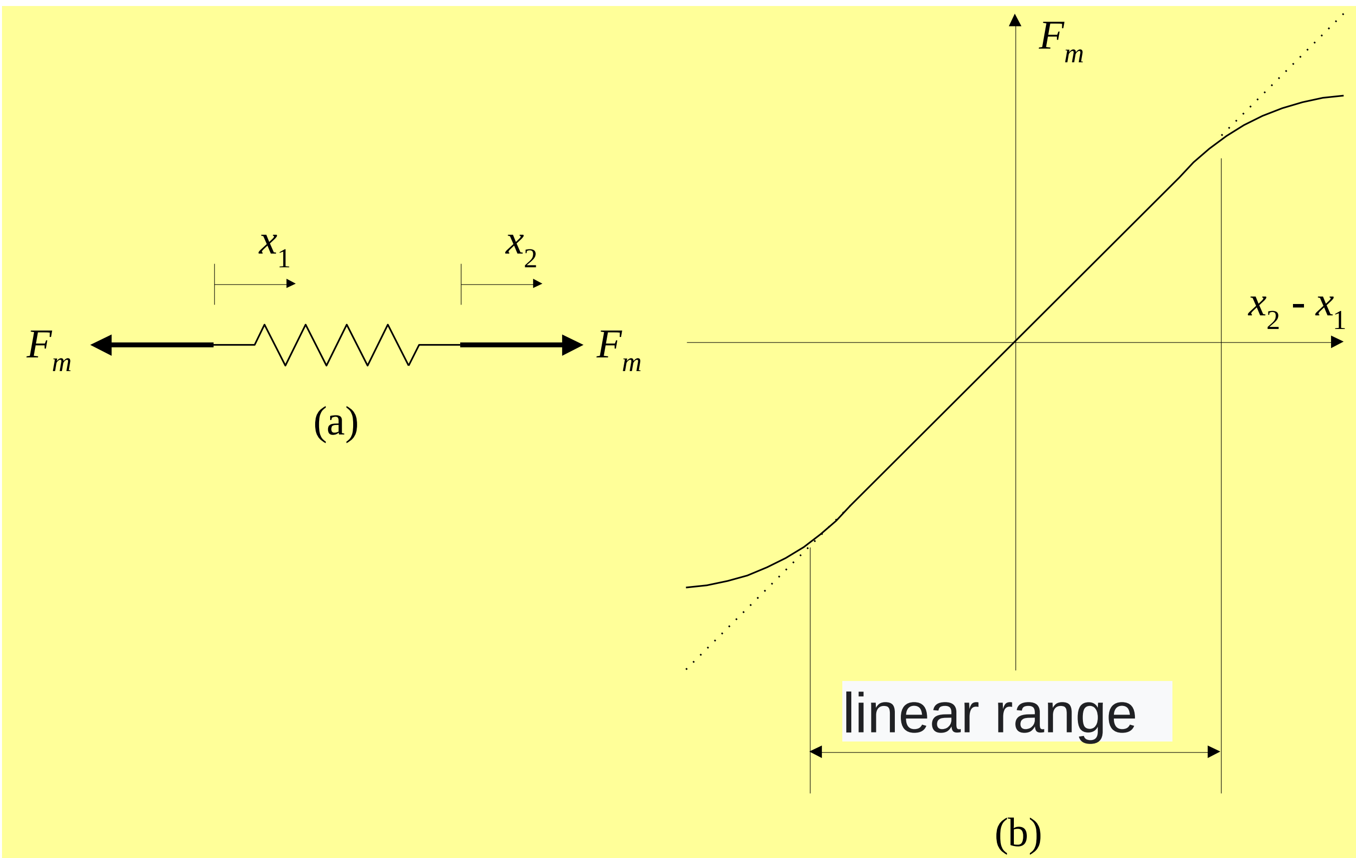


2 DOFs

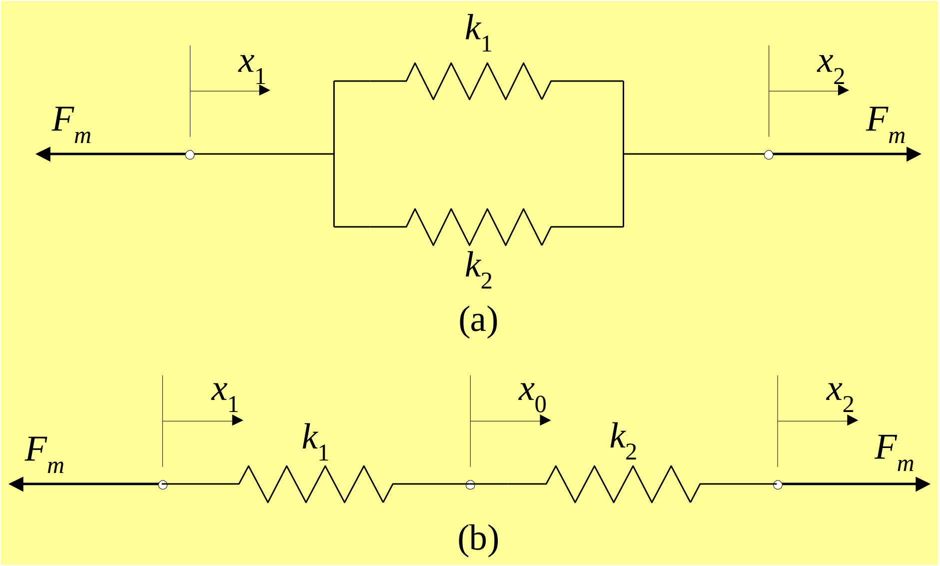




Spring element



Springs association



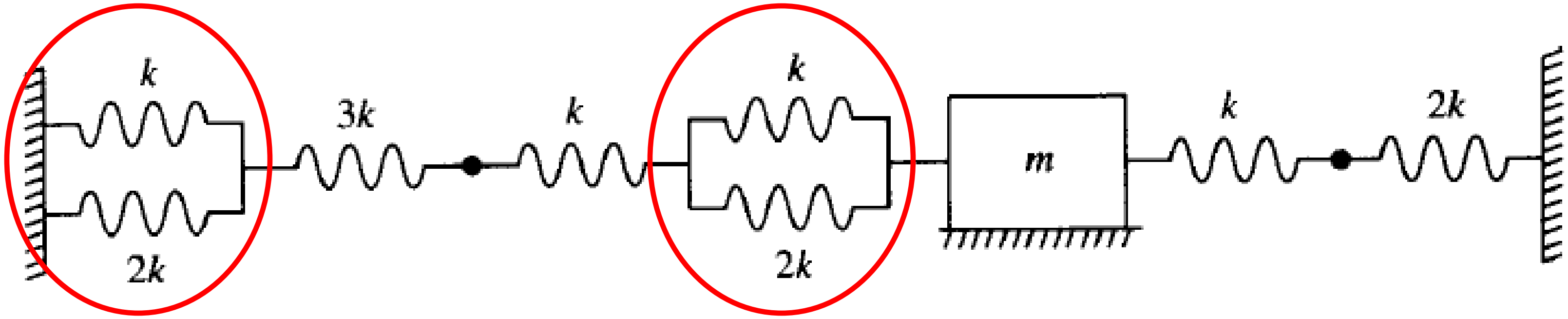
$$k_{eq} = k_1 + k_2$$

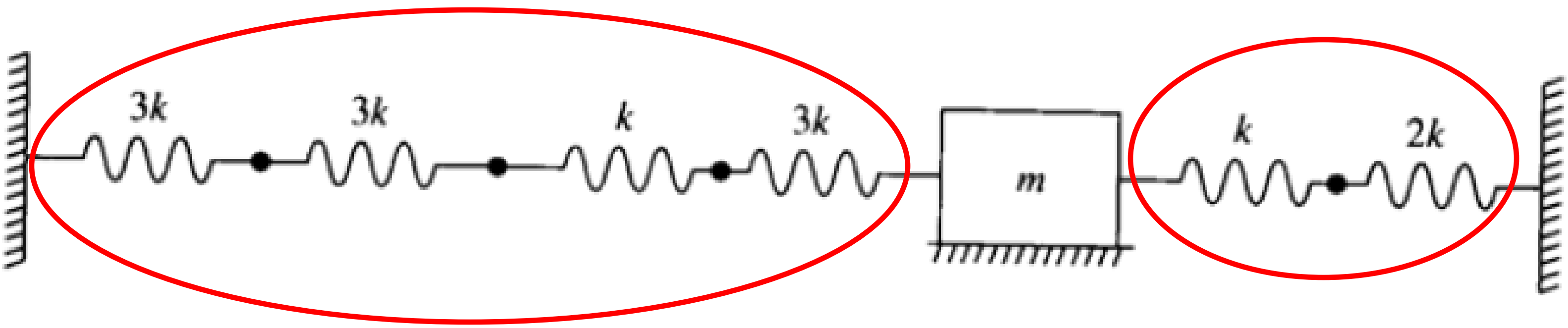
$$k_{eq} = \sum_{i=1}^{n} k_i$$

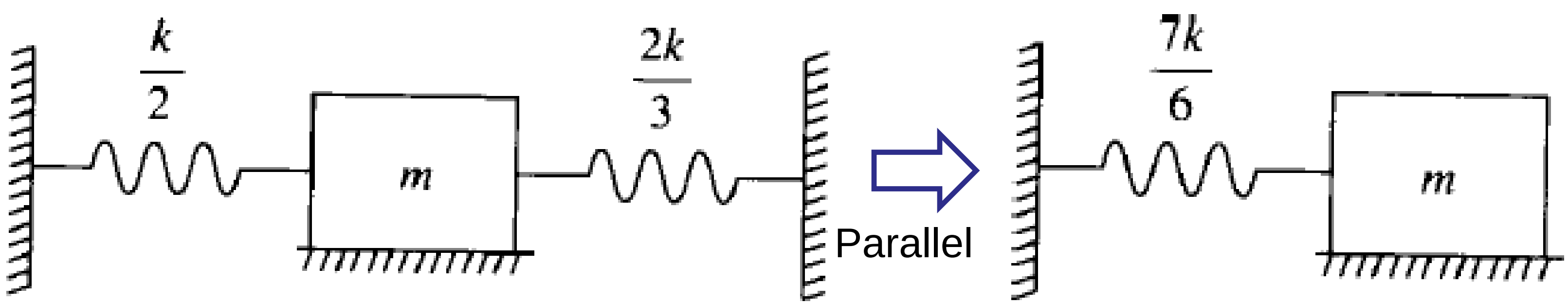
$$k_{eq} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

$$k_{eq} = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{k_i}}$$

Example







On Scilab

Determine the equivalent stiffness of the system:

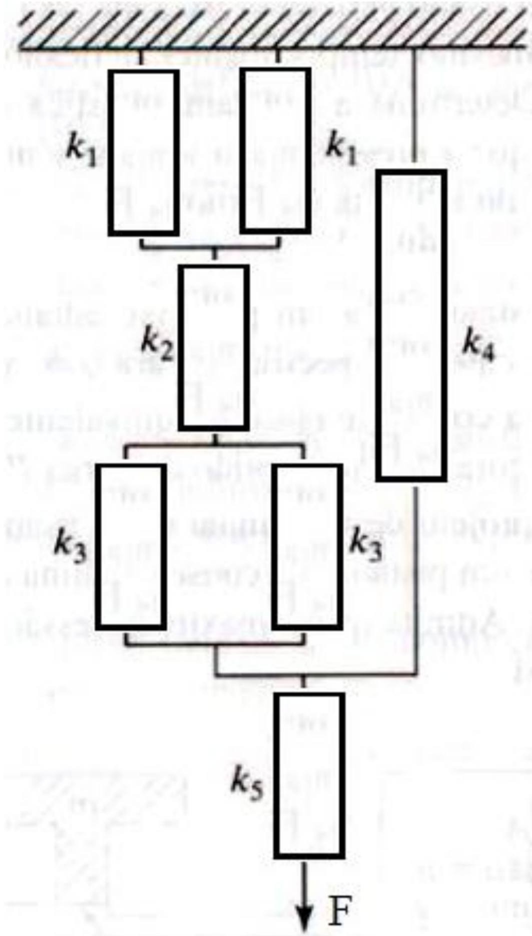
K1 = 100 N/m

K2 = 200 N/m

K3 = 300 N/m

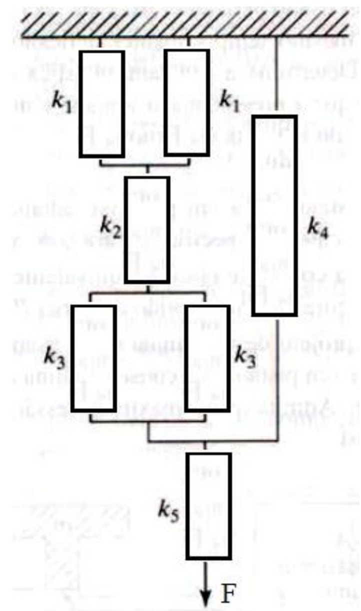
K4 = 400 N/m

K5 = 500 N/m



Solution

```
clear; clc;
K1 = 100; // N/m
K2 = 200; // N/m
K3 = 300; // N/m
K4 = 400; // N/m
K5 = 500; // N/m
keq1 = K1+K1; // N/m
keq3 = K3+K3; // N/m
keq123 = 1/((1/keq1)+(1/K2)+(1/keq3));
keq1234 = keq123 + K4; // N/m
keq = 1/((1/keq1234)+(1/K5))
```



K1	100
K2	200
K3	300
K4	400
K5	500
keq	246
keq1	200
keq123	85.7
keq1234	486
keq3	600

Torsional stiffness

$$\boxed{k_t = \frac{M_t}{\theta} = \frac{GI}{L}}$$

S.I. [Nm]

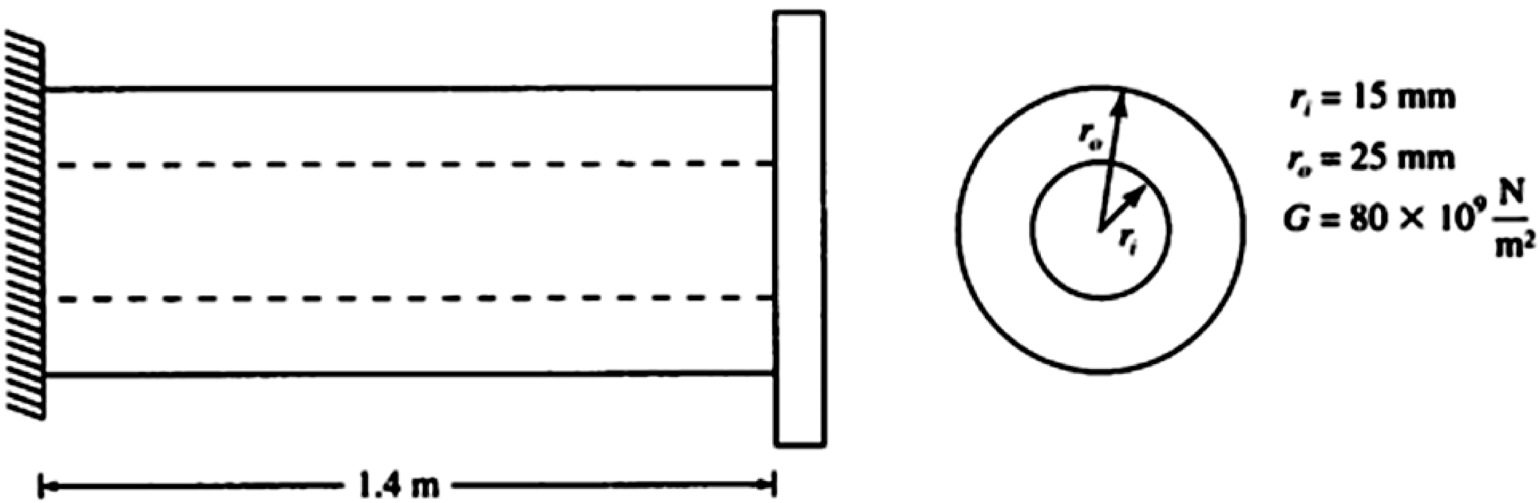
- Hollow shaft:

$$I = \frac{\pi}{32}\left(d_e{}^4 - d_i{}^4\right)$$

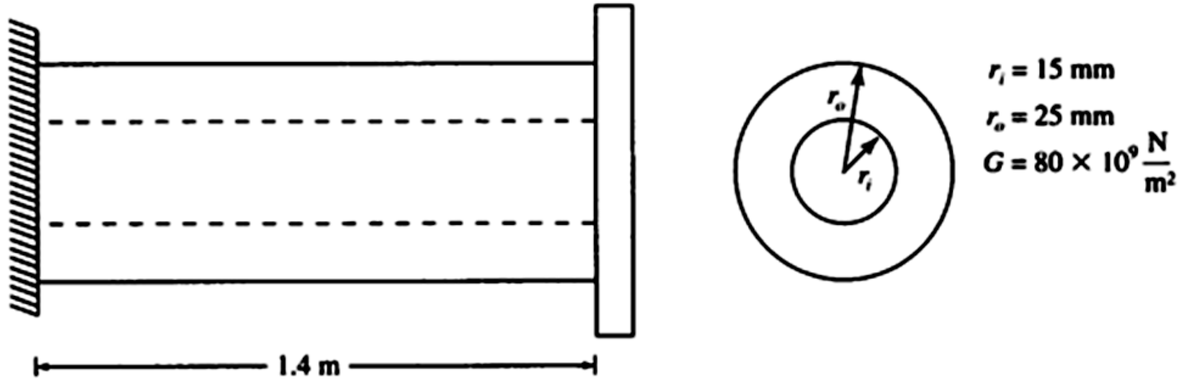
- Solid shaft :

$$I = \frac{\pi}{2}\left(r^4\right) = \frac{\pi}{32}\left(d^4\right)$$

Example: Determine the torsional stiffness of the hollow shaft, as shown in the Figure below.



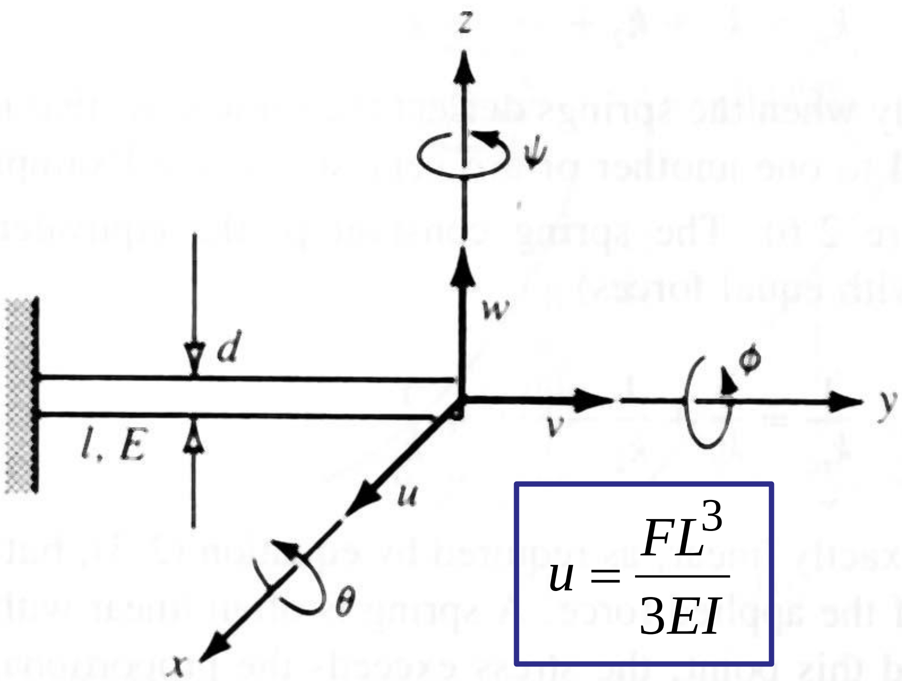
On Scilab:




```
clear; clc;
ri=15; // mm
re=25; // mm
L=1400;  // mm
G = 80000;  // MPa
J = (%pi/2)*(((re).^4)-((ri).^4));
k=(G*J/L)/1000; // 10^-3 (mm to m)
printf("k = %g Nm\n",k);
```

G	8e+04
J	5.34e+05
L	1.4e+03
k	3.05e+04
re	25
ri	15

Elastic systems

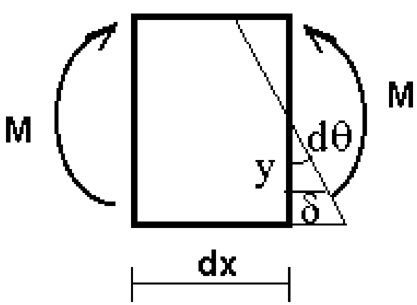


$$F_v = \frac{EAv}{L}, \qquad k_{vv} = \frac{EA}{L}$$

$$F_u = \frac{3EI_z u}{L^3}, \qquad k_{uu} = \frac{3EI_z}{L^3}$$

$$F_w = \frac{3EI_x w}{L^3}, \qquad k_{ww} = \frac{3EI_x}{L^3}$$

$$u = \frac{FL^3}{3EI}$$



$$\phi = \frac{ML}{GI}$$

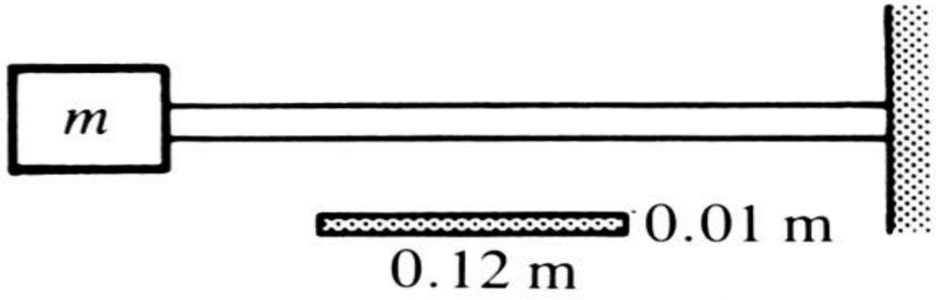
$$d\theta = \frac{M.dx}{EI}$$

$$M_\phi = \frac{GI_y \phi}{L}, \qquad k_{\phi\phi} = \frac{GI_y}{L}$$

$$M_\psi = \frac{EI_z \psi}{L}, \qquad k_{\psi\psi} = \frac{EI_z}{L}$$

$$M_\theta = \frac{EI_x \theta}{L}, \qquad k_{\theta\theta} = \frac{EI_x}{L}$$

Example: cantilever beam



$$m_{viga} = 100\,kg$$

$$m_{eff} = 3.12\,kg$$

$$L = 1m$$

$$E = 2.1x20^{11}\,N/m^2$$

$$I = \frac{bh^3}{12} = \frac{0.12 \times 0.01^3}{12} = 1 \times 10^{-8}\ m^4$$

$$k = \frac{3EI}{L^3} = \frac{3 \times 2.1 \times 10^{11} \times 10^{-8}}{1^3} = 6300\,N/m$$

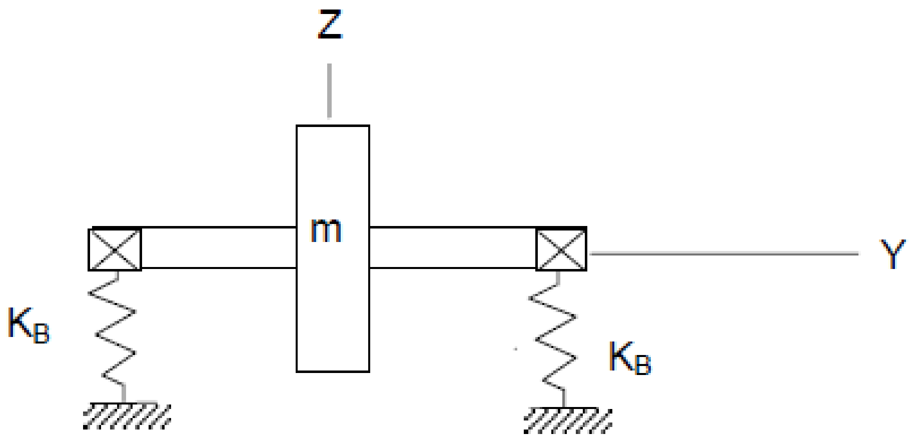
Without mass (m)

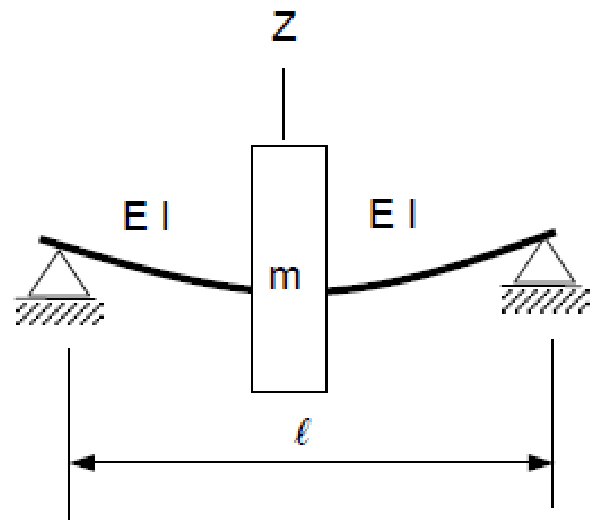
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6300}{100}} = 7.94\ rad/s$$

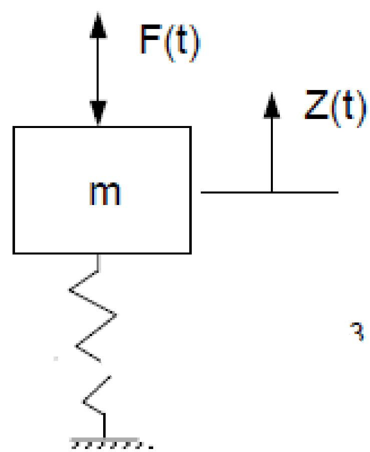
With mass (m)

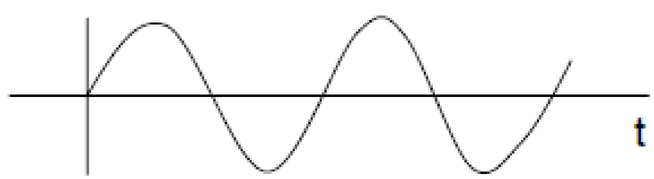
$$\omega_n = \sqrt{\frac{k}{m_{eq}}} = \sqrt{\frac{6300}{100 + 3.12}} = 7.82\ rad/s$$

Point load stiffness

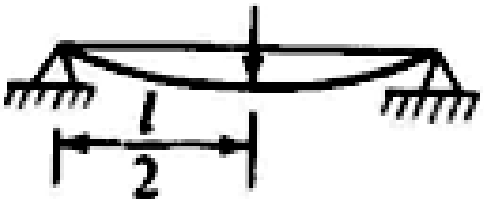








Equations development



$$M_{max} = R_A x$$

$$R_A = R_B = P/2$$

Applying the double integration method, we have :

$$EI\frac{d^2v}{dx^2} = (P/2)x$$

$$EI\frac{dv}{dx} = \frac{Px^2}{4} + C_1 \qquad (1)$$

$$EIv = \frac{Px^3}{12} + C_1 x + C_2 \qquad (2)$$

Considering the boundary conditions, we have:

- ✓ $\frac{dv}{dx} = 0$ in $x = L/2$

$$0 = \frac{PL^2}{16} + C_1$$ From Equation (1)

- ✓ $v = 0$ in $x = 0$

$$0 = \frac{Px^3}{12} + C_1 \cdot x + C_2$$ From Equation (2)

Isolating the value of C_1 e C_2, we have: $C_1 = -\frac{PL^2}{16}$; $C_2 = 0$

Replacing values of C_1 e C_2 in the Equation (2), we have:

$$EIv = \frac{Px^3}{12} - \frac{PL^2}{16}x + 0$$

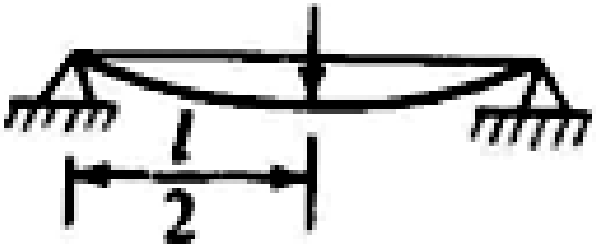
Knowing that the maximum displacements occur in L/2:

$$EIv = \frac{PL^3}{96} - \frac{PL^3}{32}$$

$$v = \frac{(-P)L^3}{48EI} \Rightarrow -P = \frac{48EIv}{L^3}$$

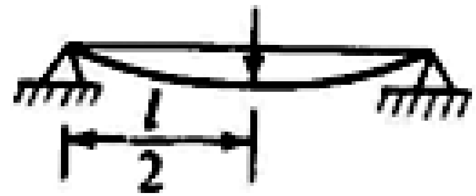
$$\boxed{k = \frac{48EI}{L^3}}$$

Example: Determine the stiffness of a bar of length L = 300 mm and E = 210 GPa, with a rectangular cross section (10x5 mm^2). Check what happens when the original position of the bar is rotated 90 degrees (cross section = 5x10 mm^2) .



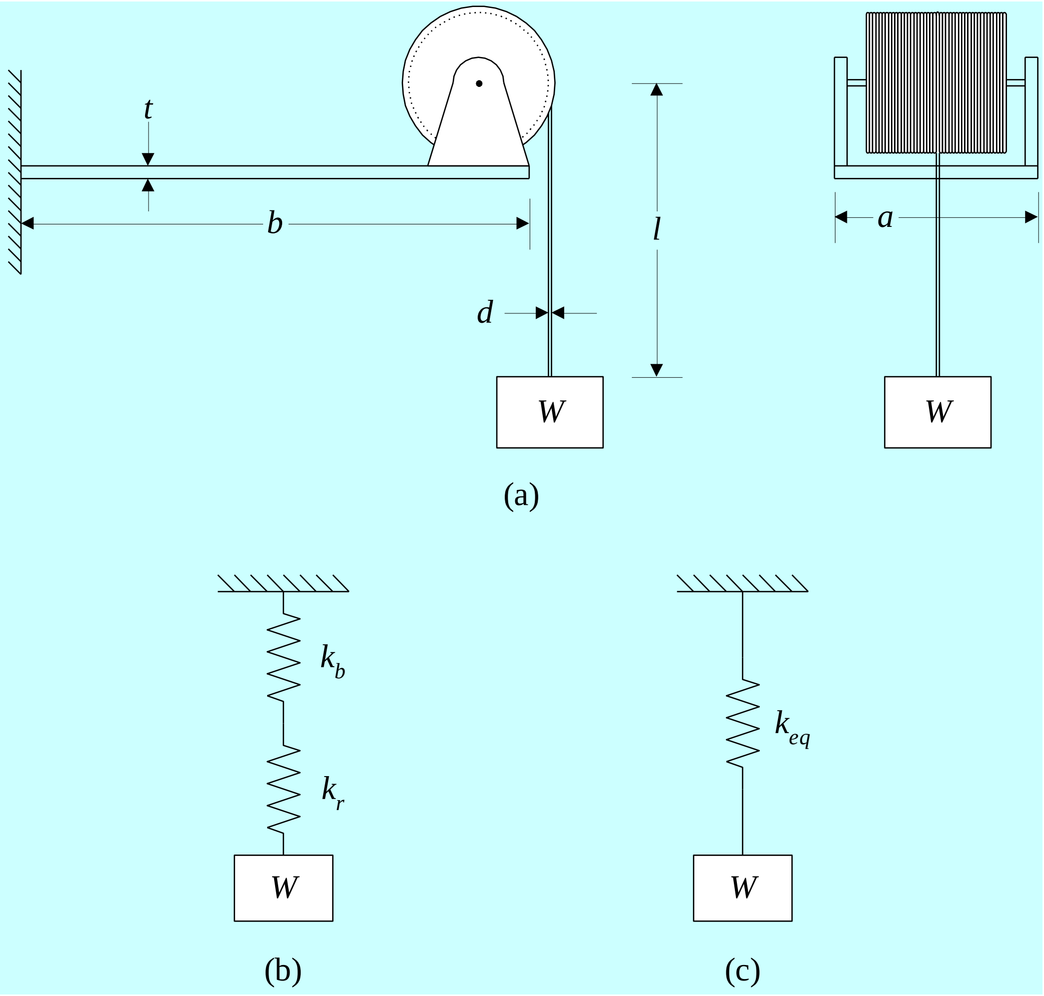
Solution on Scilab

```
clear; clc;
E = 210000; // MPa
L = 300;  // mm
b1 = 10;  // mm
h1 = 5;  // mm
I1 = (b1*(h1^3))/12;// mm^4
k1 = (48*E*I1)/(L^3); // N/mm
printf("k1 = %g N/mm\n",k1);
b2 = 5;  // mm
h2 = 10;  // mm
I2 = (b2*(h2^3))/12;// mm^4
k2 = (48*E*I2)/(L^3); // N/mm
printf("k2 = %g N/mm\n",k2);
```



Nome	Value
E	2.1e+05
I1	104
I2	417
L	300
b1	10
b2	5
h1	5
h2	10
k1	38.9
k2	156

Load lifting system

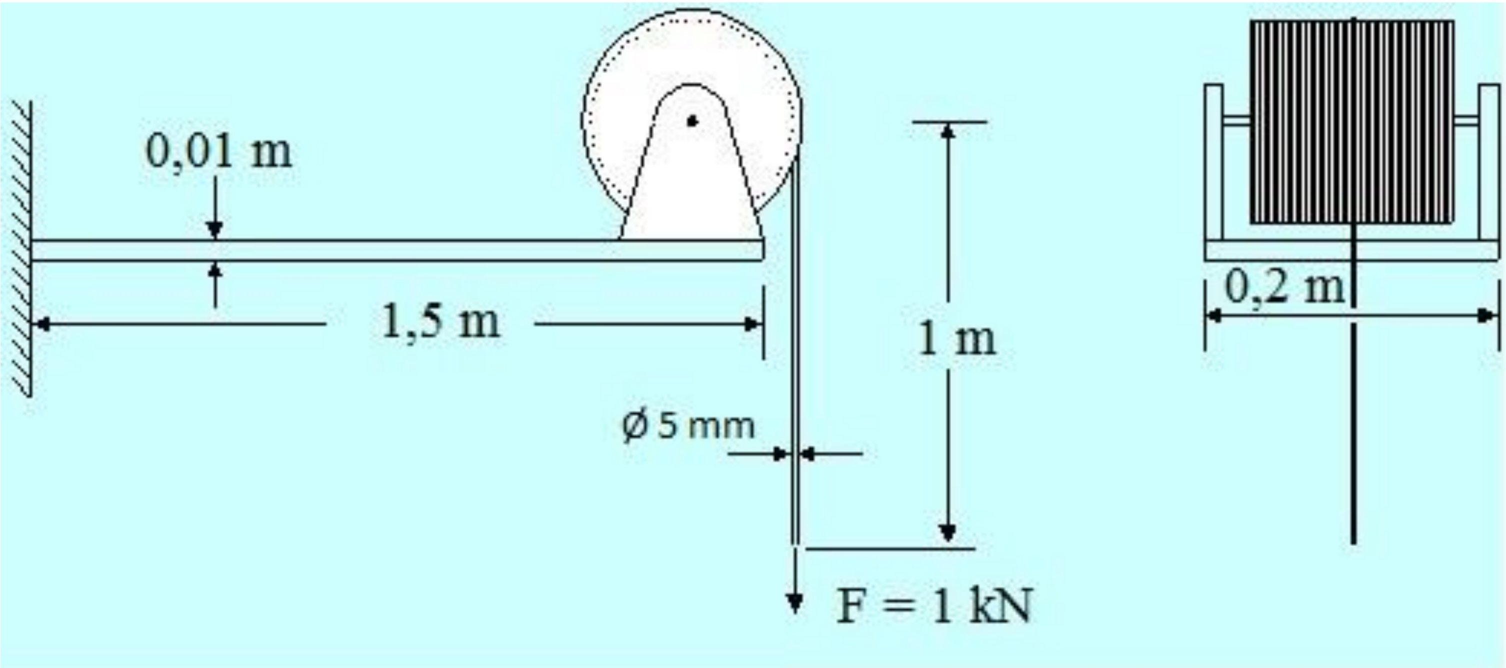


$$k_b = \frac{3EI}{L^3}$$

$$k_r = \frac{EA}{L}$$

$$k_{eq} = \frac{k_b k_r}{k_b + k_r}$$

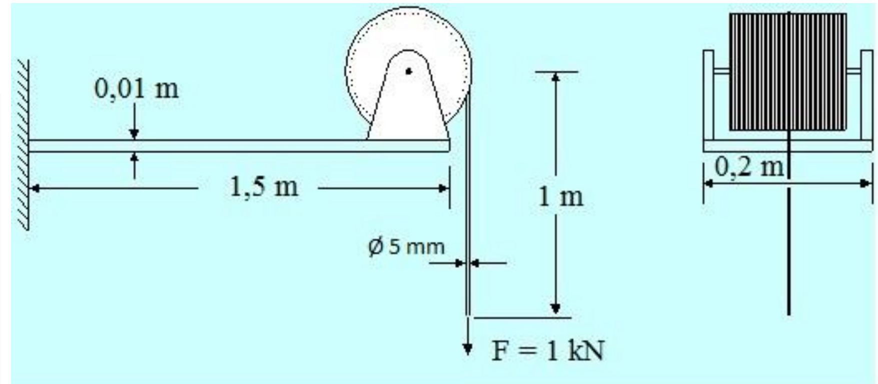
Example:



Considering E = 210 GPa for the entire lifting system, calculate the equivalent stiffness.

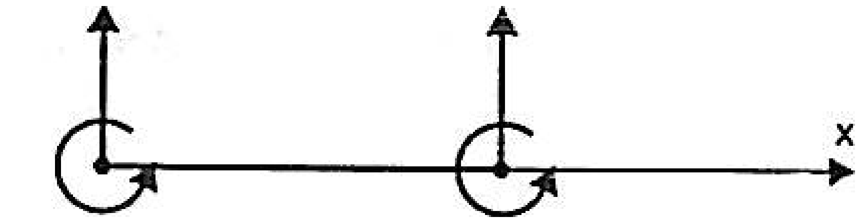
Solution on Scilab

```
clear; clc;
E = 210000; // MPa
// beam (bending)
Lb = 1500;  // mm
b = 200;  // mm
h = 10;  // mm
I = (b*(h^3))/12;// mm^4
kb = (3*E*I)/(Lb^3); // N/mm
// cable (traction)
r = 2.5; // mm
Lr = 1000;  // mm
A = %pi*(r^2); // mm^2
kr = (E*A)/Lr; // N/mm
keq = (kb*kr)/(kb+kr)
printf("k = %g N/mm\n",keq);
```



A	19.6
E	2.1e+05
I	1.67e+04
Lb	1.5e+03
Lr	1e+03
b	200
h	10
kb	3.11
keq	3.11
kr	4.12e+03
r	2.5

Stiffness matrix

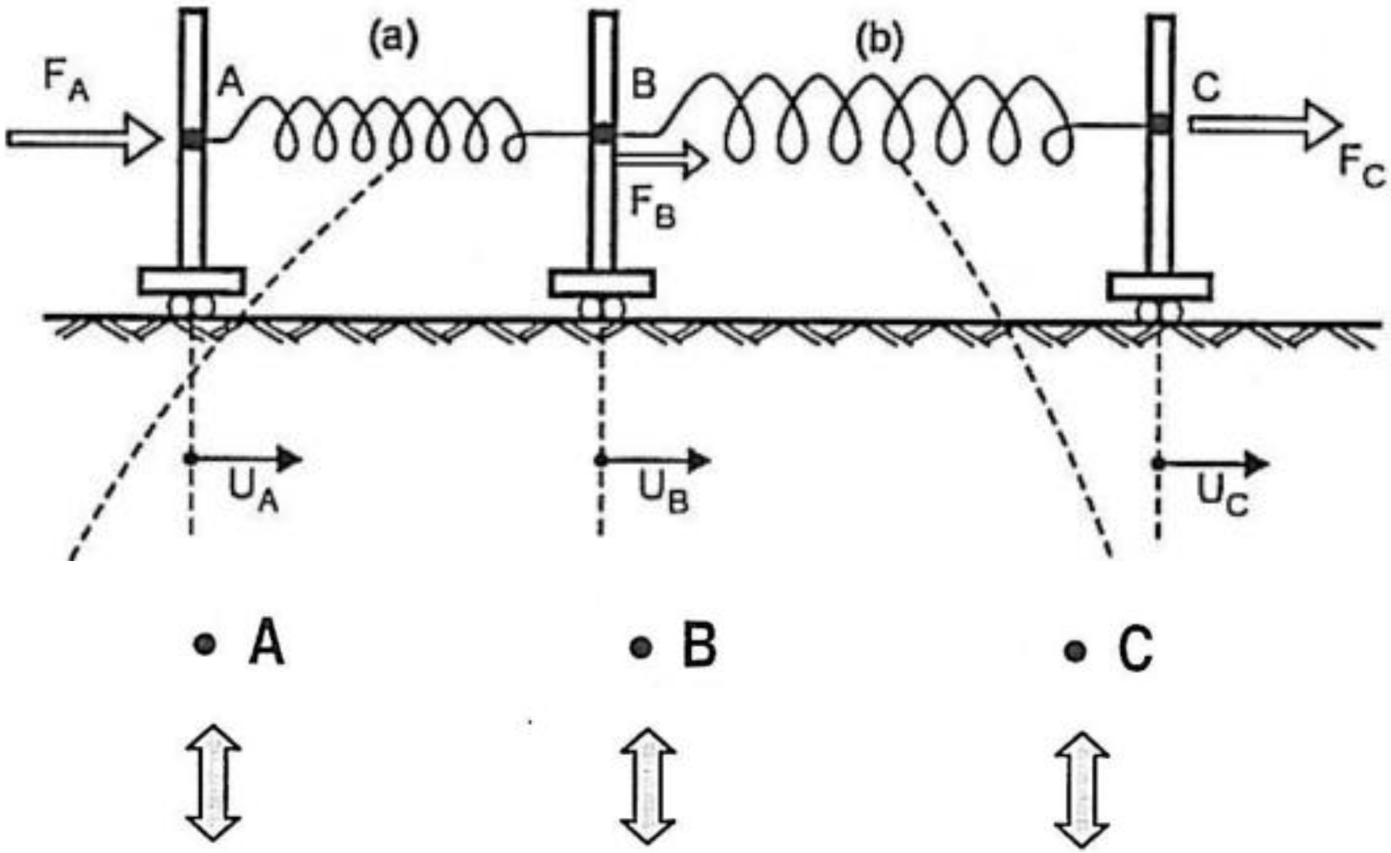


Node 1 Node 2

$$\begin{cases} F_1 = \dfrac{EA}{L} u_1 \\ F_1 = -\dfrac{EA}{L} u_2 \end{cases} \quad \begin{cases} F_2 = \dfrac{EA}{L} u_2 \\ F_2 = -\dfrac{EA}{L} u_1 \end{cases} \Rightarrow \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_{11} & -K_{12} \\ -K_{21} & K_{22} \end{bmatrix} . \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

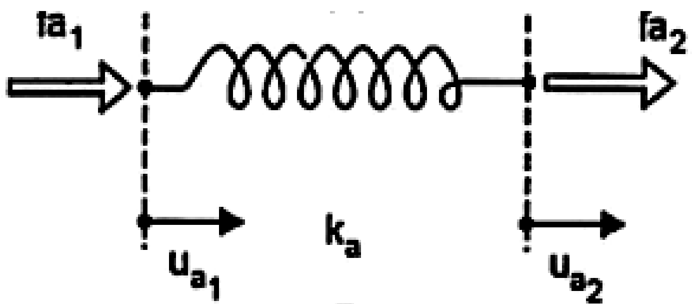
K_{21} is the load from node 2 due to the unit displacement of node 1 (u_1 = 1), keeping u_2 = 0.

Two elements stiffness matrix



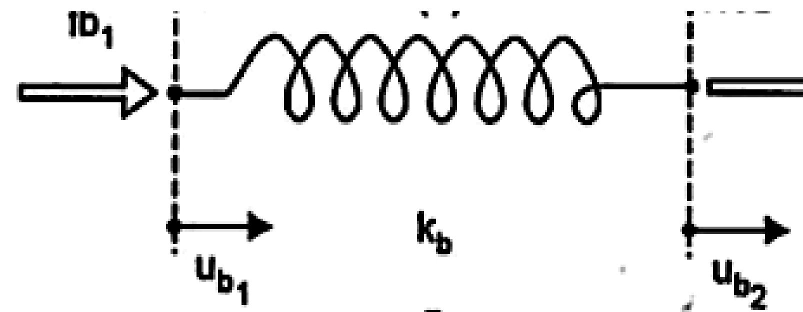
$F_A = f_{a1}$ $F_B = f_{a2} + f_{b1}$ $F_C = f_{b2}$

Two elements stiffness matrix



$$\begin{Bmatrix} f_{a_1} \\ f_{a_2} \end{Bmatrix} = \begin{bmatrix} k_a & -k_a \\ -k_a & k_a \end{bmatrix} \cdot \begin{Bmatrix} u_{a_1} \\ u_{a_2} \end{Bmatrix}$$

$$\begin{cases} f_{a_1} = k_a \cdot u_{a_1} - k_a \cdot u_{a_2} \\ f_{a_2} = -k_a \cdot u_{a_1} + k_a \cdot u_{a_2} \end{cases}$$



$$\begin{Bmatrix} f_{b_1} \\ f_{b_2} \end{Bmatrix} = \begin{bmatrix} k_b & -k_b \\ -k_b & k_b \end{bmatrix} \cdot \begin{Bmatrix} u_{b_1} \\ u_{b_2} \end{Bmatrix}$$

$$\begin{cases} f_{b_1} = k_b \cdot u_{b_1} - k_b \cdot u_{b_2} \\ f_{b_2} = -k_b \cdot u_{b_1} + k_b \cdot u_{b_2} \end{cases}$$

Global stiffness matrix

Stiffness matrix (Element "a")

Stiffness matrix (Element "b")

$$\begin{Bmatrix} F_A \\ F_B \\ F_C \end{Bmatrix} = \left[\begin{array}{ccc} \begin{array}{|cc|}\hline k_a & -k_a \\ -k_a & k_a \\ \hline \end{array} & & 0 \\ & + \begin{array}{|cc|}\hline k_b & -k_b \\ -k_b & k_b \\ \hline \end{array} & \\ 0 & & \end{array}\right] \cdot \begin{Bmatrix} U_A \\ U_B \\ U_C \end{Bmatrix}$$

$$\begin{Bmatrix} F_A \\ F_B \\ F_C \end{Bmatrix} = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \cdot \begin{Bmatrix} U_A \\ U_B \\ U_C \end{Bmatrix}$$

Solution

$$\begin{Bmatrix} F_A \\ F_B \\ F_C \end{Bmatrix} = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \cdot \begin{Bmatrix} U_A \\ U_B \\ U_C \end{Bmatrix}$$

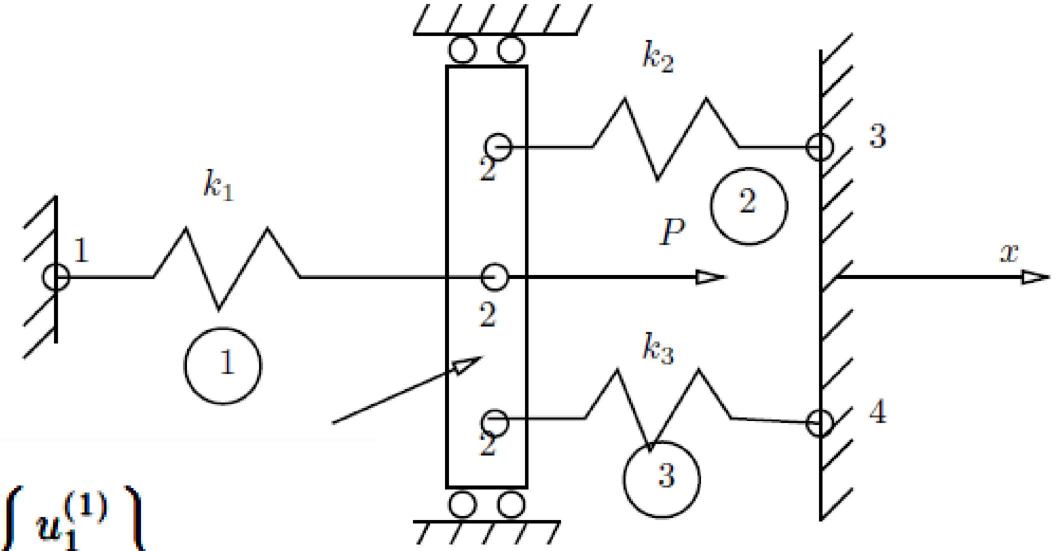
$$\{F_A\} = [k_a] \, . \, \{U_A\} + [-k_a \quad 0] \, . \begin{Bmatrix} U_B \\ U_C \end{Bmatrix}$$

e

$$\begin{Bmatrix} F_B \\ F_C \end{Bmatrix} = \begin{bmatrix} -k_a \\ 0 \end{bmatrix} . \, \{U_A\} + \begin{bmatrix} k_a + k_b & -k_b \\ k_b & -k_b \end{bmatrix} . \begin{Bmatrix} U_B \\ U_C \end{Bmatrix}$$

$$\boxed{U_A = 0}$$

Example

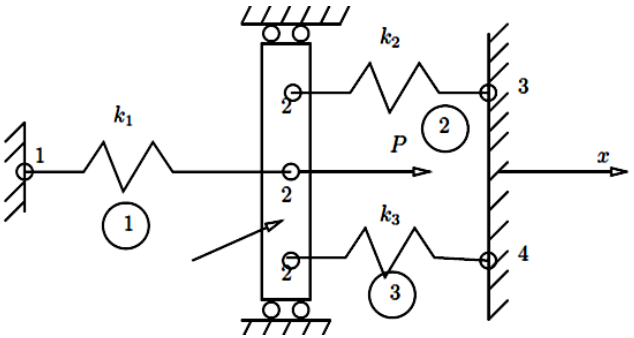


$$\begin{Bmatrix} R_1^{(1)} \\ R_2^{(1)} \end{Bmatrix} = k^{(1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} R_1^{(2)} \\ R_2^{(2)} \end{Bmatrix} = k^{(2)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix}$$

$$\begin{Bmatrix} R_1^{(3)} \\ R_2^{(3)} \end{Bmatrix} = k^{(3)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^{(3)} \\ u_2^{(3)} \end{Bmatrix}$$

Solution



K_{12}, K_{23} , K_{24}

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \\ F_4 \end{Bmatrix}$$

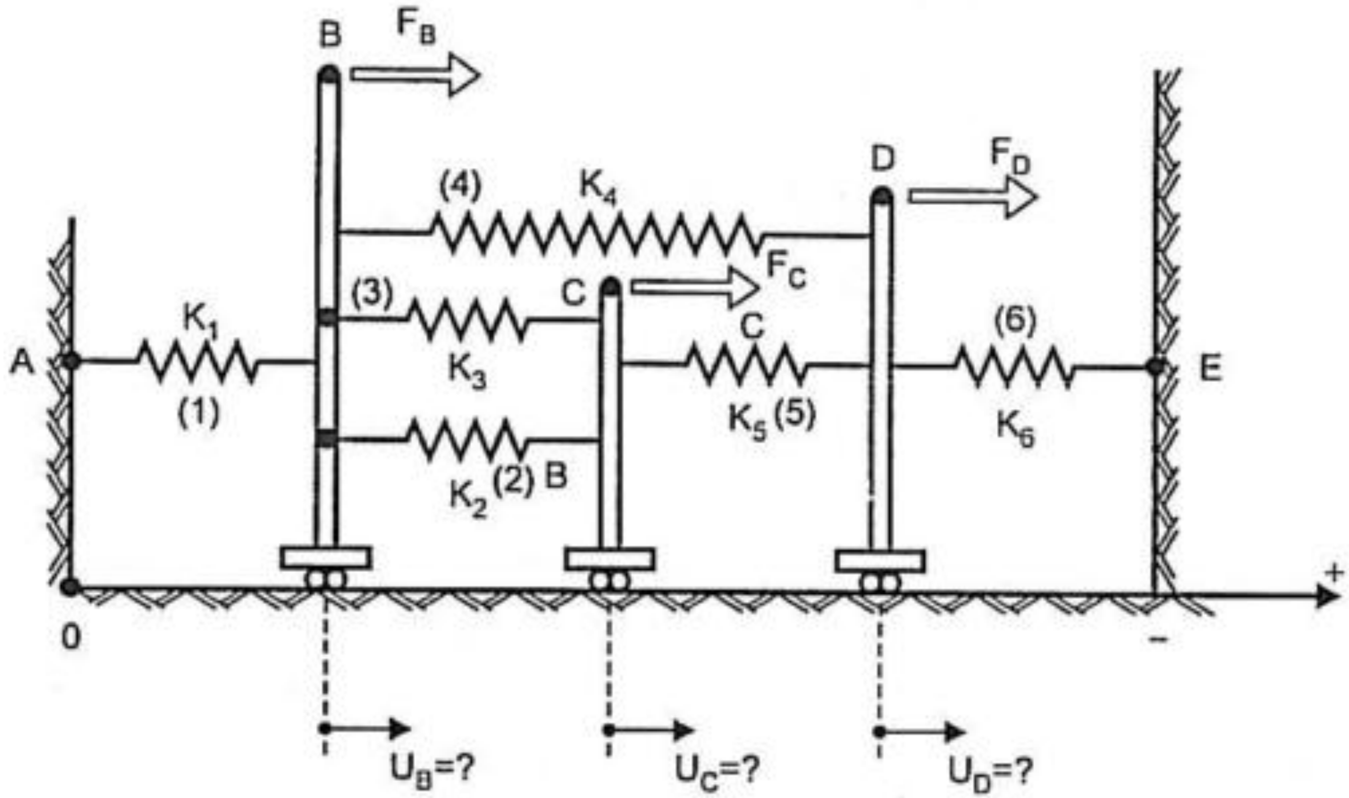
$$u_1 = u_3 = u_4 = 0,$$

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\boxed{(k_1+k_2+k_3)u_2 = P}$$

$$\boxed{-k_1 u_2 = F_1; \quad -k_2 u_2 = F_3; \quad -k_3 u_2 = F_4}$$

Example 2



K_1 = 200 Kgf/mm
K_2 = 100 Kgf/mm
K_3 = 150 Kgf/mm
K_4 = 300 Kgf/mm
K_5 = 400 Kgf/mm
K_6 = 500 Kgf/mm

F_B = 400 Kgf

F_C = 300 Kgf

F_D = 500 Kgf

Stiffness matrix of each (n) element

$$(1): [K]^1 = \begin{array}{c|c} \begin{array}{cc} A & B \end{array} & \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} & \begin{array}{c} A \\ B \end{array} \end{array}$$

$$(2): [K]^2 = \begin{array}{c|c} \begin{array}{cc} B & C \end{array} & \\ \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} & \begin{array}{c} B \\ C \end{array} \end{array}$$

$$(3): [K]^3 = \begin{array}{c|c} \begin{array}{cc} B & C \end{array} & \\ \begin{bmatrix} 150 & -150 \\ -150 & 150 \end{bmatrix} & \begin{array}{c} B \\ C \end{array} \end{array}$$

$$(4): [K]^4 = \begin{array}{c|c} \begin{array}{cc} B & D \end{array} & \\ \begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} & \begin{array}{c} B \\ D \end{array} \end{array}$$

$$(5): [K]^5 = \begin{array}{c|c} \begin{array}{cc} C & D \end{array} & \\ \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} & \begin{array}{c} C \\ D \end{array} \end{array}$$

$$(6): [K]^6 = \begin{array}{c|c} \begin{array}{cc} D & E \end{array} & \\ \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} & \begin{array}{c} D \\ E \end{array} \end{array}$$

Global stiffness matrix

$$
[K] = \begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\left[\begin{array}{c|c|c|c|c}
200 & -200 & 0 & 0 & 0 \\ \hline
-200 & \begin{array}{c}200+100\\+150+300\end{array} & \begin{array}{c}-100\\-150\end{array} & -300 & 0 \\ \hline
0 & \begin{array}{c}-100\\-150\end{array} & \begin{array}{c}100+150\\+400\end{array} & -400 & 0 \\ \hline
0 & -300 & -400 & \begin{array}{c}300+\\400+\\500\end{array} & -500 \\ \hline
0 & 0 & 0 & -500 & 500
\end{array}\right]
\end{array}
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}
$$

$$\Downarrow$$

$$
\begin{Bmatrix} F_A \\ \boxed{\begin{array}{c} F_B \\ F_C \\ F_D \end{array}} \\ F_E \end{Bmatrix}
=
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix}
200 & -200 & 0 & 0 & 0 \\
-200 & 750 & -250 & -300 & 0 \\
0 & -250 & 650 & -400 & 0 \\
0 & -300 & -400 & 1200 & -500 \\
0 & 0 & 0 & -500 & 500
\end{bmatrix}
\end{array}
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}
\begin{Bmatrix} U_A \\ \boxed{\begin{array}{c} U_B \\ U_C \\ U_D \end{array}} \\ U_E \end{Bmatrix}
$$

$$F_A = 200 \cdot U_A - 200 \cdot U_B + 0 \cdot U_C + 0 \cdot U_D + 0 \cdot U_E$$

$$F_B = -200 \cdot U_A + 750 \cdot U_B - 250 \cdot U_C - 300 \cdot U_D + 0 \cdot U_E$$

$$F_C = 0 \cdot U_A - 250 \cdot U_B + 650 \cdot U_C - 400 \cdot U_D + 0 \cdot U_E$$

$$F_D = 0 \cdot U_A - 300 \cdot U_B - 400 \cdot U_C + 1200 \cdot U_D - 500 \cdot U_E$$

$$F_E = 0 \cdot U_A + 0 \cdot U_B + 0 \cdot U_C - 500 \cdot U_D + 500 \cdot U_E$$

$$\begin{Bmatrix} F_A \\ F_B \\ F_C \\ F_D \\ F_E \end{Bmatrix} = \begin{array}{c} \begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 750 & -250 & -300 & 0 \\ 0 & -250 & 650 & -400 & 0 \\ 0 & -300 & -400 & 1200 & -500 \\ 0 & 0 & 0 & -500 & 500 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \cdot \begin{Bmatrix} U_A \\ U_B \\ U_C \\ U_D \\ U_E \end{Bmatrix}$$

$$\begin{array}{ccc} B & C & D \end{array}$$

$$\begin{Bmatrix} F_B \\ F_C \\ F_D \end{Bmatrix} = \underbrace{\begin{bmatrix} 750 & -250 & -300 \\ -250 & 650 & -400 \\ -300 & -400 & 1200 \end{bmatrix}}_{[K]_\Delta} \cdot \begin{Bmatrix} U_B \\ U_C \\ U_D \end{Bmatrix} \begin{array}{c} B \\ C \\ D \end{array}$$

$$\det [K]_\Delta = [750 \times 650 \times 1200 + (-250) \cdot (-400) \cdot (-300) + (-250)\ (-400)\ (-300)] -$$
$$- [(-300) \cdot (650)\ (-300) + (750) \cdot (-400) \cdot (-400) + (-250) \cdot (-250) \cdot (1200)]$$

$$\boxed{\det\ [K]_A = 271.500.000}$$

$$(\mathrm{cof}\ K_{BB}) = (-1)^{1+1} \cdot \begin{vmatrix} 650 & -400 \\ -400 & 1200 \end{vmatrix} = 650 \times 1200 - 400 \times 400 = 620000$$

$$(\mathrm{cof}\ K_{BC}) = (-1)^{1+2} \cdot \begin{vmatrix} -250 & -400 \\ -300 & 1200 \end{vmatrix} = (-1)\,[-250 \times 1200 - 300 \times 400] = 420.000$$

$$(\mathrm{cof}\ K_{BD}) = (-1)^{1+3} \cdot \begin{vmatrix} -250 & 650 \\ -300 & -400 \end{vmatrix} = [400 \times 250 - (-300) \times (650)] = 295.000$$

$$(\mathrm{cof}\ K_{CB}) = (-1)^{2+1} \cdot \begin{vmatrix} -250 & -300 \\ -400 & 1200 \end{vmatrix} = (-1)\,[(-250) \times 1200 - 400 \times 300] = 420.000$$

$$(\mathrm{cof}\ K_{cc}) = (-1)^{2+2} \cdot \begin{vmatrix} 750 & -300 \\ -300 & 1200 \end{vmatrix} = [750 \times 1200 - 300 \times 300] = 810.000$$

$$(\mathrm{cof}\ K_{CD}) = (-1)^{2+3} \cdot \begin{vmatrix} 750 & -250 \\ -300 & -400 \end{vmatrix} = (-1) \cdot [750 \times (-400) - 300 \times 250] = 375.000$$

$$(\mathrm{cof}\ K_{DB}) = (-1)^{3+1} \cdot \begin{vmatrix} -250 & -300 \\ 650 & -400 \end{vmatrix} = [400 \times 250 - (650) \times (-300)] = 295000$$

$$(\mathrm{cof}\ K_{DC}) = (-1)^{3+2} \cdot \begin{vmatrix} 750 & -300 \\ -250 & -400 \end{vmatrix} = (-1)\,[750 \times (-400) - 250 \times 300] = 375.000$$

$$(\mathrm{cof}\ K_{DD}) = (-1)^{3+3} \cdot \begin{vmatrix} 750 & -250 \\ -250 & 650 \end{vmatrix} = [750 \times 650 - 250 \times 250] = 425000$$

$$[K]_{\Delta}^{-1} = \frac{1}{271.500.000}\begin{bmatrix} 620000 & 420000 & 295000 \\ 420000 & 810000 & 375000 \\ 295000 & 375000 & 425000 \end{bmatrix} = \begin{bmatrix} 0{,}0022836 & 0{,}0015470 & 0{,}0010866 \\ 0{,}0015470 & 0{,}0029834 & 0{,}0013812 \\ 0{,}0010866 & 0{,}0013812 & 0{,}0015653 \end{bmatrix}$$

$$F_B = +400 \text{ kgf} \quad ; \quad F_C = +300 \text{ kgf} \quad ; \quad F_D = +500 \text{ kgf}$$

$$\{U\} = \begin{Bmatrix} U_B \\ U_C \\ U_D \end{Bmatrix} = \begin{bmatrix} 0{,}0022836 & 0{,}0015470 & 0{,}0010866 \\ 0{,}0015470 & 0{,}0029834 & 0{,}0013812 \\ 0{,}0010866 & 0{,}0013812 & 0{,}0015653 \end{bmatrix} \cdot \begin{Bmatrix} 400 \\ 300 \\ 500 \end{Bmatrix}$$

$$U_B = 0{,}0022836 \times 400 + 0{,}0015470 \times 300 + 0{,}0010866 \times 500 \quad \Rightarrow \quad \boxed{U_B = 1{,}92084 \text{ mm}}$$

$$U_C = 0{,}0015470 \times 400 + 0{,}0029834 \times 300 + 0{,}0013812 \times 500 \quad \Rightarrow \quad \boxed{U_C = 2{,}20442 \text{ mm}}$$

$$U_D = 0{,}0010866 \times 400 + 0{,}0013812 \times 300 + 0{,}0015653 \times 500 \quad \Rightarrow \quad \boxed{U_D = 1{,}63165 \text{ mm}}$$