

Tensor Product Kernels: Characteristic Property and Universality

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Motivation: 'Classical' Information Theory

- Kullback-Leibler divergence:

$$KL(\mathbb{P}, \mathbb{Q}) = \int_{\mathbb{R}^d} p(x) \log \left[\frac{p(x)}{q(x)} \right] dx.$$

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Alternatives: Rényi, Tsallis, L^2 divergence... Typically: $\mathcal{X} = \mathbb{R}^d$.

Euclidean Space → Inner Product → Kernel

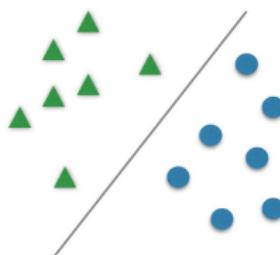
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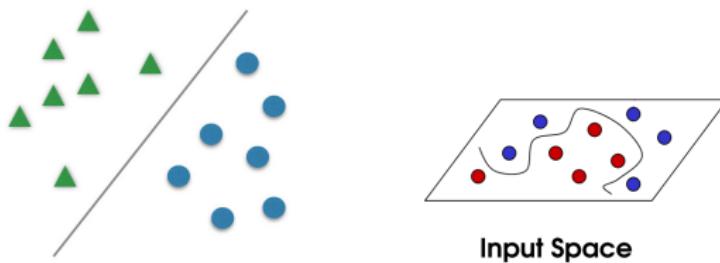
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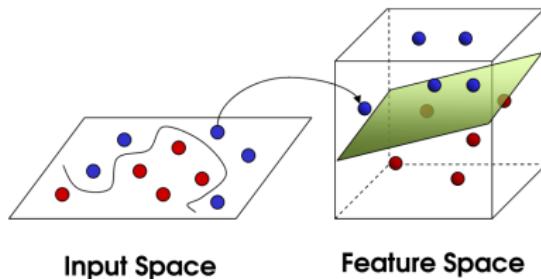
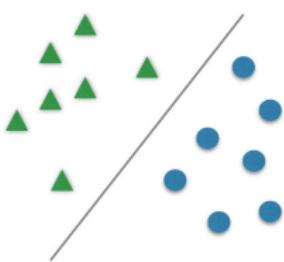
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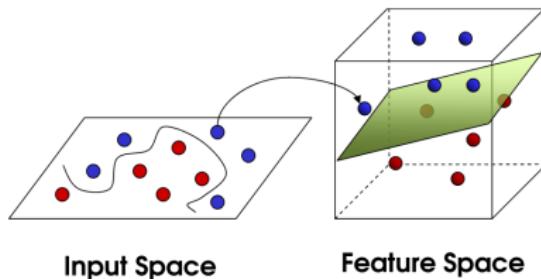
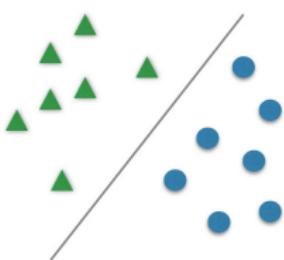
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② Representation of distributions (this talk):

$$\mathbb{P} \mapsto \mathbb{E}_{\mathbf{x} \sim \mathbb{P}} \varphi(\mathbf{x}), \quad \mathbb{Q} \mapsto \mathbb{E}_{\mathbf{x} \sim \mathbb{Q}} \varphi(\mathbf{x}).$$

$\varphi(\mathbf{x}) = \mathbf{x}$: mean, $\varphi(\mathbf{x}) = e^{i\langle \cdot, \mathbf{x} \rangle}$: characteristic function.

Diverse Set of Domains, Kernel Examples



- $\mathcal{X} = \mathbb{R}^d, \gamma > 0$:

$$k_p(\mathbf{x}, \mathbf{y}) = (\langle \mathbf{x}, \mathbf{y} \rangle + \gamma)^p, \quad k_G(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x} - \mathbf{y}\|_2^2},$$

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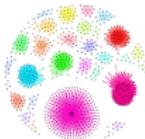
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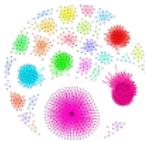
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- $\mathcal{X} = \text{trees, graphs, dynamical systems, sets, permutations, ...}$

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'KL divergence & mutual information' on kernel-endowed domains.

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When is HSIC an independence measure? Conditions on k_m -s?

Ingredients

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Assumption

\mathcal{X}_m : kernel-enriched domains.

Ingredients: Kernel, RKHS ($\mathcal{X} := \mathcal{X}_m$, $k := k_m$)

Given: \mathcal{X} set. \mathcal{H} (ilbert space).

- Kernel:

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Equivalent definitions. We represent distributions in an RKHS...

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- Applications:
 - two-sample testing [Gretton et al., 2012], domain adaptation [Zhang et al., 2013],
 - kernel belief propagation [Song et al., 2011], kernel Bayes' rule [Fukumizu et al., 2013], model criticism [Lloyd et al., 2014],
 - approximate Bayesian computation [Park et al., 2016], probabilistic programming [Schölkopf et al., 2015],
 - distribution classification [Muandet et al., 2011], distribution regression [Szabó et al., 2016], topological data analysis [Kusano et al., 2016].
- Review [Muandet et al., 2017].

Let us switch to HSIC.

MMD with $\mathbf{k} = \otimes_{m=1}^M k_m$:

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Applications:

- blind source separation [Gretton et al., 2005],
- feature selection [Song et al., 2012], post selection inference [Yamada et al., 2016],
- independence testing, causal inference [Gretton et al., 2008, Mooij et al., 2016, Pfister et al., 2017].

Central in Applications: Characteristic Property

- MMD: k is called **characteristic** [Fukumizu et al., 2008] if

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Wanted

- $\otimes_{m=1}^M k_m$ is **\mathcal{I} -characteristic**: conditions in terms of k_m -s?
- $\otimes_{m=1}^M k_m$ is **characteristic**: relation?

Characteristic Property: Description on \mathbb{R}^d

For continuous bounded shift-invariant kernels on \mathbb{R}^d :

$$k(\mathbf{x}, \mathbf{x}') = k_0(\mathbf{x} - \mathbf{x}') \stackrel{(*)}{=} \int_{\mathbb{R}^d} e^{-i\langle \mathbf{x} - \mathbf{x}', \boldsymbol{\omega} \rangle} d\Lambda(\boldsymbol{\omega})$$

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Theorem ([Sriperumbudur et al., 2010])

k is characteristic iff. $\text{supp}(\Lambda) = \mathbb{R}^d$.

Examples on \mathbb{R} ; Similarly \mathbb{R}^d

kernel name k_0	$\hat{k}_0(\omega)$	$supp(\hat{k}_0)$
Gaussian	$e^{-\frac{x^2}{2\sigma^2}}$	$\sigma e^{-\frac{\sigma^2 \omega^2}{2}}$
Laplacian	$e^{-\sigma x }$	$\sqrt{\frac{2}{\pi}} \frac{\sigma}{\sigma^2 + \omega^2}$
B_{2n+1} -spline	$*^{2n+2} \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)$	$\frac{4^{n+1}}{\sqrt{2\pi}} \frac{\sin^{2n+2}\left(\frac{\omega}{2}\right)}{\omega^{2n+2}}$
Sinc	$\frac{\sin(\sigma x)}{x}$	$\sqrt{\frac{\pi}{2}} \chi_{[-\sigma, \sigma]}(\omega)$
Fejér	$\frac{1}{n+1} \frac{\sin^2 \frac{(n+1)x}{2}}{\sin^2\left(\frac{x}{2}\right)}$	$\sqrt{2\pi} \sum_{j=-n}^n \left(1 - \frac{ j }{n+1}\right) \delta(\omega - j)$ $\{0, \pm 1, \pm 2, \dots, \pm n\}$

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Challenge

Characteristic/ \mathcal{I} -characteristic/universality of $\bigotimes_{m=1}^M k_m$ in terms of k_m -s!

Existing Results, $M = 2$

- [Waegeman et al., 2012, Gretton, 2015]:
 $k_1 \& k_2$: universal $\Rightarrow k_1 \otimes k_2$: universal ($\Rightarrow \mathcal{I}$ -characteristic).

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Main Challenge

' $\otimes k_m$: \mathcal{I} -characteristic $\Leftrightarrow k_m$: characteristic ($\forall m$)' does NOT hold.

Idea: Characteristic Property as lspd

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- Observation [Sriperumbudur et al., 2010]: k is characteristic iff.

$$\|\mu_{\mathbb{F}}\|_k^2 > 0, \forall \underbrace{\mathbb{F} \in \mathcal{M}_b(\mathcal{X}) \setminus \{0\} \quad \mathbb{F}(\mathcal{X}) = 0}_{\mathcal{F}_1}.$$

Idea: Characteristic Property as $\text{Is}pd$

- Characteristic property:

$$\mathbb{F} = \mathbb{P}_1 - \mathbb{P}_2 \neq 0 \Rightarrow \mu_{\mathbb{F}} \neq 0.$$

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$$\underbrace{\|\mu_{\mathbb{F}}\|_k^2}_{\int_{\mathcal{X}} \int_{\mathcal{X}} k(x, x') d\mathbb{F}(x) d\mathbb{F}(x')} > 0, \quad \underbrace{\forall \mathbb{F} \in \mathcal{M}_b(\mathcal{X}) \setminus \{0\} \quad \mathbb{F}(\mathcal{X}) = 0}_{\mathcal{F}_1}.$$

Idea: Characteristic Property as lspd

- Characteristic property:

$$\mathbb{F} = \mathbb{P}_1 - \mathbb{P}_2 \neq 0 \Rightarrow \mu_{\mathbb{F}} \neq 0.$$

Here: $\mathbb{F} \in \mathcal{M}_b(\mathcal{X})$, $\mathbb{F}(\mathcal{X}) = \underbrace{\mathbb{P}_1(\mathcal{X})}_1 - \underbrace{\mathbb{P}_2(\mathcal{X})}_1 = 0$.

- Observation [Sriperumbudur et al., 2010]: k is characteristic iff.

$$\|\mu_{\mathbb{F}}\|_k^2 > 0, \forall \underbrace{\mathbb{F} \in \mathcal{M}_b(\mathcal{X}) \setminus \{0\} \quad \mathbb{F}(\mathcal{X}) = 0}_{\mathcal{F}_1}.$$

- We saw: k is universal iff.

$$\|\mu_{\mathbb{F}}\|_k^2 > 0, \forall \underbrace{\mathbb{F} \in \mathcal{M}_b(\mathcal{X}) \setminus \{0\}}_{\mathcal{F}_2}.$$

\mathcal{F} -ispd Tensor Product Kernels

From now on: $\mathcal{X} = \times_{m=1}^M \mathcal{X}_m$. Let $\mathcal{F} \subseteq \mathcal{M}_b(\mathcal{X})$, $0 \in \mathcal{F}$.

Definition

$k = \otimes_{m=1}^M k_m$ is called \mathcal{F} -ispd if

$$\|\mu_k(\mathbb{F})\|_{\mathcal{H}_k}^2 > 0, \quad \forall \mathbb{F} \in \mathcal{F} \setminus \{0\}$$

\mathcal{F} -ispd Tensor Product Kernels

From now on: $\mathcal{X} = \times_{m=1}^M \mathcal{X}_m$. Let $\mathcal{F} \subseteq \mathcal{M}_b(\mathcal{X})$, $0 \in \mathcal{F}$.

Definition

$k = \otimes_{m=1}^M k_m$ is called \mathcal{F} -ispd if

$$\begin{aligned}\|\mu_k(\mathbb{F})\|_{\mathcal{H}_k}^2 &> 0, \quad \forall \mathbb{F} \in \mathcal{F} \setminus \{0\}, \text{ equivalently} \\ \mu_k(\mathbb{F}) &= 0 \Rightarrow \mathbb{F} = 0 \quad (\mathbb{F} \in \mathcal{F}).\end{aligned}$$

Examples

\mathcal{F}	\mathcal{F} -ispd k
$\mathcal{M}_b(\mathcal{X})$	universal
$[\mathcal{M}_b(\mathcal{X})]^0$	characteristic

$$\subseteq \quad \subseteq [\mathcal{M}_b(\mathcal{X})]^0 \subseteq \mathcal{M}_b(\mathcal{X}).$$

\cup

$$\Leftarrow \quad \Leftarrow \text{ characteristic} \Leftarrow \text{ universal}.$$

\Downarrow

Examples

\mathcal{F}	$\mathcal{F}\text{-ispd } k$
$\mathcal{M}_b(\mathcal{X})$	universal
$[\mathcal{M}_b(\mathcal{X})]^0$	characteristic
$\mathcal{I} := \{\mathbb{P} - \bigotimes_{m=1}^M \mathbb{P}_m\}$	\mathcal{I} -characteristic

$$\subseteq \quad \subseteq \quad [\mathcal{M}_b(\mathcal{X})]^0 \subseteq \mathcal{M}_b(\mathcal{X}) .$$

UI

\mathcal{I}

$$\Leftarrow \quad \Leftarrow \quad \text{characteristic} \Leftarrow \text{universal}.$$

↓

\mathcal{I} -characteristic

Examples

\mathcal{F}	$\mathcal{F}\text{-ispd } k$
$\mathcal{M}_b(\mathcal{X})$	universal
$[\mathcal{M}_b(\mathcal{X})]^0$	characteristic
$\mathcal{I} := \{\mathbb{P} - \bigotimes_{m=1}^M \mathbb{P}_m\}$	\mathcal{I} -characteristic
$[\bigotimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)]^0$	\otimes -characteristic

$$\subseteq [\bigotimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)]^0 \subseteq [\mathcal{M}_b(\mathcal{X})]^0 \subseteq \mathcal{M}_b(\mathcal{X}) .$$

⋃
 \mathcal{I}

$$\Leftarrow \otimes\text{-characteristic} \Leftarrow \text{characteristic} \Leftarrow \text{universal}.$$

\Downarrow
 \mathcal{I} -characteristic

Examples

\mathcal{F}	\mathcal{F} -ispd k
$\mathcal{M}_b(\mathcal{X})$	universal
$[\mathcal{M}_b(\mathcal{X})]^0$	characteristic
$\mathcal{I} := \{\mathbb{P} - \bigotimes_{m=1}^M \mathbb{P}_m\}$	\mathcal{I} -characteristic
$[\bigotimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)]^0$	\otimes -characteristic
$\bigotimes_{m=1}^M \mathcal{M}_b^0(\mathcal{X}_m)$	\otimes_0 -characteristic

$$\bigotimes_{m=1}^M \mathcal{M}_b^0(\mathcal{X}_m) \subseteq [\bigotimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)]^0 \subseteq [\mathcal{M}_b(\mathcal{X})]^0 \subseteq \mathcal{M}_b(\mathcal{X}).$$

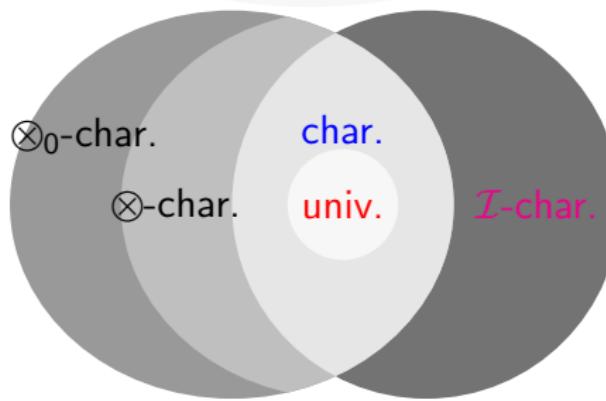
UI

\mathcal{I}

$$\otimes_0\text{-characteristic} \Leftarrow \otimes\text{-characteristic} \Leftarrow \text{characteristic} \Leftarrow \text{universal}.$$

↓

\mathcal{I} -characteristic

$\mathcal{M}_b(\mathcal{X})$ $[\mathcal{M}_b(\mathcal{X})]^0$ $[\otimes_m \mathcal{M}_b(\mathcal{X}_m)]^0$ $\otimes_m \mathcal{M}_b^0(\mathcal{X}_m)$ \mathcal{I} 

Results

Proposition

- (i) $\bigotimes_{m=1}^M k_m$: characteristic \Rightarrow \bigotimes -characteristic.
- (ii) $\bigotimes_{m=1}^M k_m$: \bigotimes -characteristic \Rightarrow \bigotimes_0 -characteristic.
- (iii) $\bigotimes_{m=1}^M k_m$: \bigotimes_0 -characteristic $\Leftrightarrow (k_m)_{m=1}^M$ are characteristic.

Various Characteristic Properties of $\bigotimes_{m=1}^M k_m$

Proposition

- (i) $\bigotimes_{m=1}^M k_m$: characteristic \Rightarrow \otimes -characteristic.
- (ii) $\bigotimes_{m=1}^M k_m$: \otimes -characteristic \Rightarrow \otimes_0 -characteristic.
- (iii) $\bigotimes_{m=1}^M k_m$: \otimes_0 -characteristic $\Leftrightarrow (k_m)_{m=1}^M$ are characteristic.

(iii) remains. Idea: with $k = \bigotimes_{m=1}^M k_m$, $\mathbb{F} = \bigotimes_{m=1}^M \mathbb{F}_m$,

$$\underbrace{\|\mu_k(\mathbb{F})\|_{\mathcal{H}_k}^2}_{>0} = \underbrace{\prod_{m=1}^M \|\mu_{k_m}(\mathbb{F}_m)\|_{\mathcal{H}_{k_m}}^2}_{\forall >0},$$

Reverse of (ii) does not hold.

Example

- $\mathcal{X}_m = \{1, 2\}$, $\tau_{\mathcal{X}_m} = \mathcal{P}(\{1, 2\})$, $k_m(x, x') = 2\delta_{x,x'} - 1$, $M = 2$.
- $k_1 = k_2$: characteristic, but $k_1 \otimes k_2$ is not \otimes -characteristic.
- $k_1 \otimes k_2$ is \mathcal{I} -characteristic.

Proof Idea: $k_1 \otimes k_2$: not \otimes -characteristic

Goal: construct a witness $0 \neq \mathbb{F} = \mathbb{F}_1 \otimes \mathbb{F}_2 \in \otimes_{m=1}^2 \mathcal{M}_b(\mathcal{X}_m)$ s.t.

$$0 = \mathbb{F}(\mathcal{X}_1 \times \mathcal{X}_2) = \mathbb{F}_1(\mathcal{X}_1)\mathbb{F}_2(\mathcal{X}_2),$$

$$0 = \int_{\mathcal{X}_1 \times \mathcal{X}_2} \int_{\mathcal{X}_1 \times \mathcal{X}_2} k_1(x_1, x'_1) k_2(x_2, x'_2) d\mathbb{F}(x_1, x_2) d\mathbb{F}(x'_1, x'_2).$$

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Finite signed measures on $\{1, 2\}$:

$$\mathbb{F}_1(\mathbf{a}) = a_1\delta_1 + a_2\delta_2, \quad \mathbb{F}_2(\mathbf{b}) = b_1\delta_1 + b_2\delta_2.$$

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$$\mathbb{F}_1(\mathbf{a}) = a_1\delta_1 + a_2\delta_2, \quad \mathbb{F}_2(\mathbf{b}) = b_1\delta_1 + b_2\delta_2.$$

This gives

$$0 = (a_1 + a_2)(b_1 + b_2), \quad 0 = (a_1 - a_2)^2(b_1 - b_2)^2.$$

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Goal: construct a witness $0 \neq \mathbb{F} = \mathbb{F}_1 \otimes \mathbb{F}_2 \in \otimes_{m=1}^2 \mathcal{M}_b(\mathcal{X}_m)$ s.t.

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This gives

$$0 = (a_1 + a_2)(b_1 + b_2), \quad 0 = (a_1 - a_2)^2(b_1 - b_2)^2.$$

\Rightarrow Two symmetric solutions ($\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$):

$$a_1 + a_2 = 0,$$

$$b_1 = b_2.$$

$$a_1 = a_2,$$

$$b_1 + b_2 = 0.$$

Towards \mathcal{I} -characteristicity

In the previous example:

$$k_1, k_2: \text{characteristic} \Rightarrow k_1 \otimes k_2: \mathcal{I}\text{-characteristic}.$$

In fact:

- this holds for any bounded kernel,
- +converse for any $M \geqslant 2!$ Formally, ...

Proposition

- (i) k_1, k_2 : characteristic $\Rightarrow k_1 \otimes k_2$: \mathcal{I} -characteristic.
- (ii) $\otimes_{m=1}^M k_m$: \mathcal{I} -characteristic $\Rightarrow (k_m)_{m=1}^M$ are characteristic.

Proposition

- (i) k_1, k_2 : characteristic $\Rightarrow k_1 \otimes k_2$: \mathcal{I} -characteristic.
- (ii) $\otimes_{m=1}^M k_m$: \mathcal{I} -characteristic $\Rightarrow (k_m)_{m=1}^M$ are characteristic.

Proof idea:

- (i) Induction: see later universality ($\mathbb{F} = \mathbb{P} - \mathbb{P}_1 \otimes \mathbb{P}_2$).

\mathcal{I} -characteristic Property

Proposition

- (i) k_1, k_2 : characteristic $\Rightarrow k_1 \otimes k_2$: \mathcal{I} -characteristic.
- (ii) $\otimes_{m=1}^M k_m$: \mathcal{I} -characteristic $\Rightarrow (k_m)_{m=1}^M$ are characteristic.

Proof idea:

- (i) Induction: see later universality ($\mathbb{F} = \mathbb{P} - \mathbb{P}_1 \otimes \mathbb{P}_2$).
- (ii) If a k_m is not characteristic, witness can be constructed.

k_1, k_2, k_3 : characteristic $\Rightarrow \otimes_{m=1}^3 k_m$: \mathcal{I} -characteristic

Example

- $\mathcal{X}_m = \{1, 2\}$, $\tau_{\mathcal{X}_m} = \mathcal{P}(\{1, 2\})$, $k_m(x, x') = 2\delta_{x,x'} - 1$, $M = 3$.
- Then
 - $(k_m)_{m=1}^3$: characteristic.
 - $\otimes_{m=1}^3 k_m$: is **not** \mathcal{I} -characteristic. Witness:

$$p_{1,1,1} = \frac{1}{5}, \quad p_{1,1,2} = \frac{1}{10}, \quad p_{1,2,1} = \frac{1}{10}, \quad p_{1,2,2} = \frac{1}{10},$$
$$p_{2,1,1} = \frac{1}{5}, \quad p_{2,1,2} = \frac{1}{10}, \quad p_{2,2,1} = \frac{1}{10}, \quad p_{2,2,2} = \frac{1}{10}.$$

Non- \mathcal{I} -characteristicity: Analytical Solution

Parameter: $\mathbf{z} = (z_0, z_1, \dots, z_5) \in [0, 1]^6$.

Non- \mathcal{I} -characteristicity: Analytical Solution

Parameter: $\mathbf{z} = (z_0, z_1, \dots, z_5) \in [0, 1]^6$. Example: $p_{1,1,1} =$

$$\frac{z_2 + z_1 + z_4 + z_5 - 3z_2z_1 - 4z_2z_4 - 4z_1z_4 - z_2z_3 - 2z_2z_0 - 2z_1z_3 - 3z_2z_5 - 2z_4z_3 - z_1z_0 - 3z_1z_5 - 2z_4z_0 - 4z_4z_5 - z_3z_0 - z_3z_5 - z_0z_5 + 2z_2z_1^2 + 2z_2^2z_1 + 4z_2z_4^2 + 2z_2^2z_4 + 4z_1z_4^2 + 2z_1^2z_4 + 2z_2^2z_0 + 2z_1^2z_3 + 2z_2z_5^2 + 2z_2^2z_5 + 2z_4^2z_3 + 2z_1z_5^2 + 2z_1^2z_5 + 2z_4z_0^2 + 2z_4z_5^2 + 4z_4^2z_5 - z_2^2 - z_1^2 - 3z_4^2 + 2z_4^3 - z_5^2 + 6z_2z_1z_4 + 2z_2z_1z_3 + 2z_2z_4z_3 + 2z_2z_1z_0 + 4z_2z_1z_5 + 4z_2z_4z_0 + 4z_1z_4z_3 + 6z_2z_4z_5 + 2z_1z_4z_0 + 6z_1z_4z_5 + 2z_2z_3z_0 + 2z_2z_3z_5 + 2z_1z_3z_0 + 2z_2z_0z_5 + 2z_1z_3z_5 + 2z_4z_3z_0 + 2z_4z_3z_5 + 2z_1z_0z_5 + 2z_4z_0z_5}{2z_2z_1 - z_1 - 2z_4 - z_3 - z_0 - 2z_5 - z_2 + 2z_2z_4 + 2z_1z_4 + 2z_2z_0 + 2z_1z_3 + 2z_2z_5 + 2z_4z_3 + 2z_1z_5 + 2z_4z_0 + 4z_4z_5 + 2z_3z_0 + 2z_3z_5 + 2z_0z_5 + 2z_4^2 + 2z_5^2}.$$

Non- \mathcal{I} -characteristicity: Analytical Solution

Parameter: $\mathbf{z} = (z_0, z_1, \dots, z_5) \in [0, 1]^6$. Example: $p_{1,1,1} =$

$$\frac{z_2 + z_1 + z_4 + z_5 - 3z_2z_1 - 4z_2z_4 - 4z_1z_4 - z_2z_3 - 2z_2z_0 - 2z_1z_3 - 3z_2z_5 - 2z_4z_3 - z_1z_0 - 3z_1z_5 - 2z_4z_0 - 4z_4z_5 - z_3z_0 - z_3z_5 - z_0z_5 + 2z_2z_1^2 + 2z_2^2z_1 + 4z_2z_4^2 + 2z_2^2z_4 + 4z_1z_4^2 + 2z_1^2z_4 + 2z_2^2z_0 + 2z_1^2z_3 + 2z_2z_5^2 + 2z_2^2z_5 + 2z_4^2z_3 + 2z_1z_5^2 + 2z_1^2z_5 + 2z_4^2z_0 + 2z_4z_5^2 + 4z_4^2z_5 - z_2^2 - z_1^2 - 3z_4^2 + 2z_4^3 - z_5^2 + 6z_2z_1z_4 + 2z_2z_1z_3 + 2z_2z_4z_3 + 2z_2z_1z_0 + 4z_2z_1z_5 + 4z_2z_4z_0 + 4z_1z_4z_3 + 6z_2z_4z_5 + 2z_1z_4z_0 + 6z_1z_4z_5 + 2z_2z_3z_0 + 2z_2z_3z_5 + 2z_1z_3z_0 + 2z_2z_0z_5 + 2z_1z_3z_5 + 2z_4z_3z_0 + 2z_4z_3z_5 + 2z_1z_0z_5 + 2z_4z_0z_5}{2z_2z_1 - z_1 - 2z_4 - z_3 - z_0 - 2z_5 - z_2 + 2z_2z_4 + 2z_1z_4 + 2z_2z_0 + 2z_1z_3 + 2z_2z_5 + 2z_4z_3 + 2z_1z_5 + 2z_4z_0 + 4z_4z_5 + 2z_3z_0 + 2z_3z_5 + 2z_0z_5 + 2z_4^2 + 2z_5^2}.$$

We chose: $\mathbf{z} = \left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$.

Proposition

Assume $k_m : \mathbb{R}^{d_m} \times \mathbb{R}^{d_m} \rightarrow \mathbb{R}$ are continuous, translation-invariant kernels. Then the followings are equivalent:

- (i) $(k_m)_{m=1}^M$ -s are characteristic.
- (ii) $\otimes_{m=1}^M k_m$: \otimes_0 -characteristic.
- (iii) $\otimes_{m=1}^M k_m$: \otimes -characteristic.
- (iv) $\otimes_{m=1}^M k_m$: \mathcal{I} -characteristic.
- (v) $\otimes_{m=1}^M k_m$: characteristic.

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- (v) $\otimes_{m=1}^M k_m$: characteristic.

Proof idea: We already know

$$(v) \Rightarrow (iii) \Rightarrow (ii) \Leftrightarrow (i), \quad (v) \Rightarrow (iv) \Rightarrow (i).$$

Remains: (i) \Rightarrow (v).

$(k_m)_{m=1}^M$: characteristic $\Rightarrow \otimes_{m=1}^M k_m$: characteristic

- Since k_m is characteristic

$$k_m \xrightarrow{\text{Bochner thm}} \Lambda_m, \text{ supp}(\Lambda_m) = \mathbb{R}^{d_m}.$$

$(k_m)_{m=1}^M$: characteristic $\Rightarrow \otimes_{m=1}^M k_m$: characteristic

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- Tensor kernel:

$$\otimes_{m=1}^M k_m \xrightarrow{\text{Bochner thm}} \Lambda = \otimes_{m=1}^M \Lambda_m.$$

$(k_m)_{m=1}^M$: characteristic $\Rightarrow \otimes_{m=1}^M k_m$: characteristic

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- Tensor kernel:

$$\otimes_{m=1}^M k_m \xrightarrow{\text{Bochner thm}} \Lambda = \otimes_{m=1}^M \Lambda_m.$$

- $\text{supp}(\Lambda) = \times_{m=1}^M \underbrace{\text{supp}(\Lambda_m)}_{\mathbb{R}^{d_m}} = \mathbb{R}^d.$

Universality of $\otimes_{m=1}^M k_m$

We saw: for $M \geq 3$

$(k_m)_{m=1}^M$ are characteristic $\Rightarrow \otimes_{m=1}^M k_m$: \mathcal{I} -characteristic.

Proposition

$\otimes_{m=1}^M k_m$: universal $\Leftrightarrow (k_m)_{m=1}^M$ are universal.

The tricky direction: if $(k_m)_{m=1}^M$ are universal . . .

Goal: injectivity of $\mu = \mu_{\otimes_{m=1}^M k_m}$ on $\mathcal{M}_b(\mathcal{X})$, i.e.

$$\mu(\mathbb{F}) = 0 \stackrel{?}{\Rightarrow} \mathbb{F} = 0.$$

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$$\mu(\mathbb{F}) = 0 \stackrel{?}{\Rightarrow} \mathbb{F} = 0.$$

Enough:

$$\mathbb{F}\left(\times_{m=1}^M B_m\right) = 0, \quad \forall B_m.$$

Proof idea

$$\otimes_{m=1}^M \mathcal{H}_{k_m} \ni 0 = \mu(\mathbb{F}) = \int_{\mathcal{X}} \otimes_{m=1}^M \textcolor{blue}{k_m}(\cdot, x_m) d\mathbb{F}(x),$$

$$\mathbb{R} \ni 0 = \mathbb{F} \left(\times_{m=1}^M B_m \right) = \int_{\mathcal{X}} \times_{m=1}^M \textcolor{blue}{\chi_{B_m}}(x_m) d\mathbb{F}(x), \forall B_m.$$

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$$\otimes_{m=1}^M \mathcal{H}_{k_m} \ni 0 = \mu(\mathbb{F}) = \int_{\mathcal{X}} \otimes_{m=1}^M k_m(\cdot, x_m) d\mathbb{F}(x),$$

$$\otimes_{m=\textcolor{red}{J}+1}^M \mathcal{H}_{k_m} \ni 0 = \int_{\mathcal{X}} \prod_{m=1}^J \chi_{B_m}(x_m) \otimes_{m=J+1}^M k_m(\cdot, x_m) d\mathbb{F}(x), \forall B_m,$$

$$\mathbb{R} \ni 0 = \mathbb{F}\left(\times_{m=1}^M B_m\right) = \int_{\mathcal{X}} \times_{m=1}^M \chi_{B_m}(x_m) d\mathbb{F}(x), \forall B_m.$$

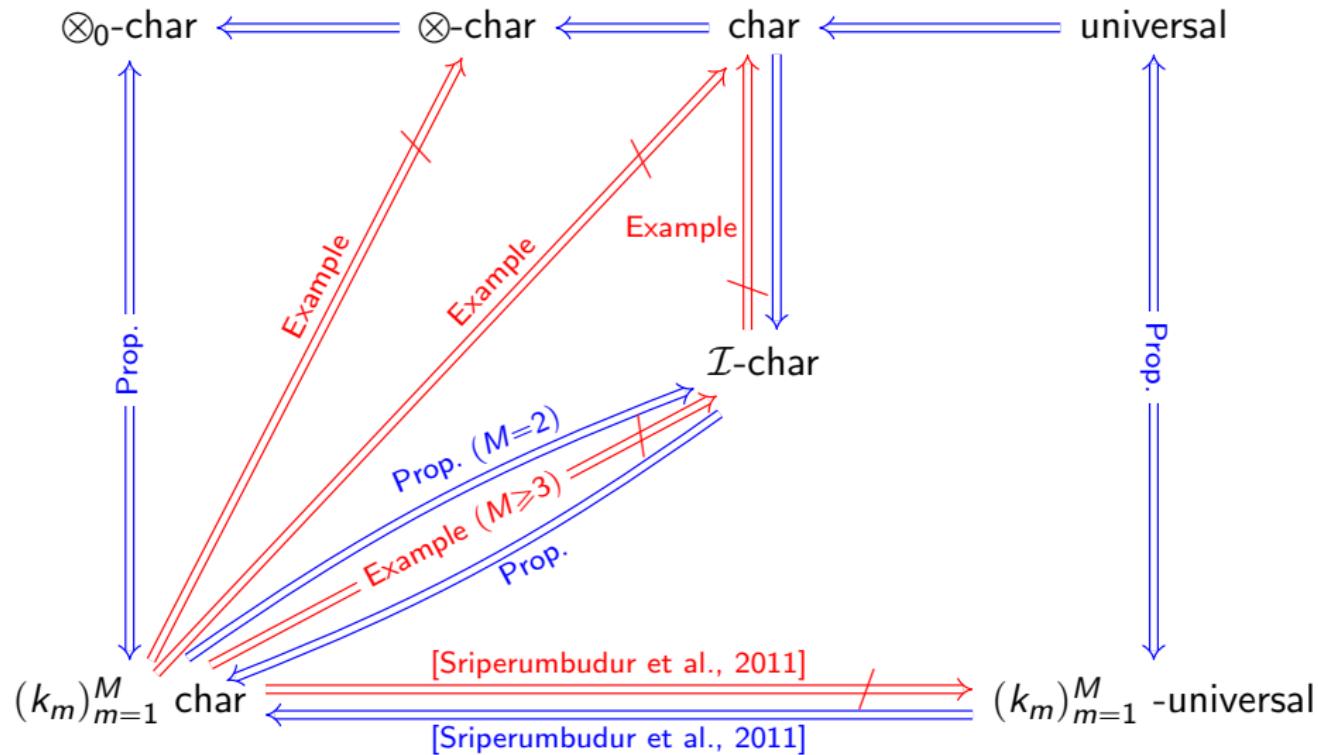
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$$\otimes_{m=\textcolor{red}{J}+1}^M \mathcal{H}_{k_m} \ni 0 = \int_{\mathcal{X}} \prod_{m=1}^J \chi_{B_m}(x_m) \otimes_{m=\textcolor{red}{J}+1}^M k_m(\cdot, x_m) d\mathbb{F}(x), \forall B_m,$$

$$\mathbb{R} \ni 0 = \mathbb{F}\left(\times_{m=1}^M B_m\right) = \int_{\mathcal{X}} \times_{m=1}^M \chi_{B_m}(x_m) d\mathbb{F}(x), \forall B_m.$$

We proceed by induction ($\textcolor{red}{J} = 0, \dots, M$).



Summary

We studied the validness of HSIC.

- HSIC \Rightarrow product structure:
 - Space: $\mathcal{X} = \times_{m=1}^M \mathcal{X}_m$.
 - Kernel: $k = \otimes_{m=1}^M k_m$.
- \mathcal{F} -ispd property \Rightarrow complete answer in terms of k_m -s.

Summary

We studied the validness of HSIC.

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 - Space: $\mathcal{X} = \times_{m=1}^M \mathcal{X}_m$.
 - Kernel: $k = \otimes_{m=1}^M k_m$.
- \mathcal{F} -ispd property \Rightarrow complete answer in terms of k_m -s.
- ITE toolkit, preprint (maths):

<https://bitbucket.org/szzoli/ite/>

<http://arxiv.org/abs/1708.08157>

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