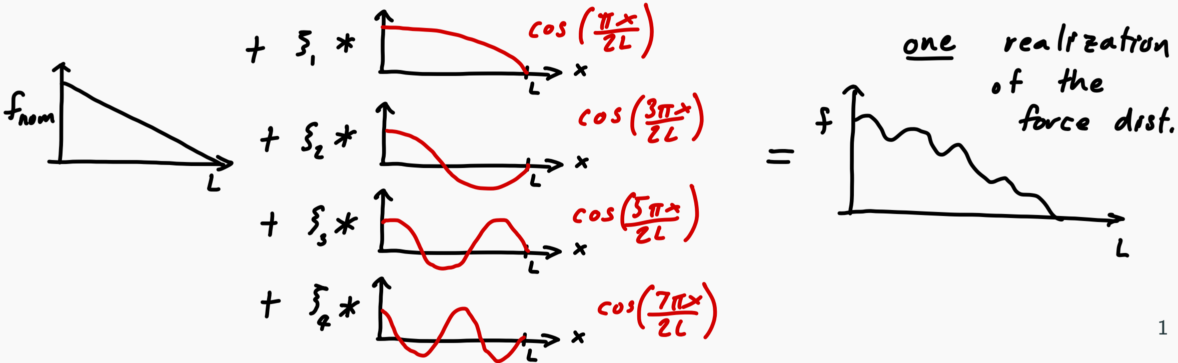
Project #4 Model

* Use Gauss-Hermite quadrature,
  not Gauss-Legendre (as in Assign. 21)

First, Let's Discuss the Model Adopted For the Uncertain Load

The loading on the wing is modelled as $f(x,\xi) = f_{\text{nom}}(x) + \delta_f(x,\xi)$ where the probabilistic perturbation has the form

$$\delta_f(x,\xi) = \sum_{n=1}^{4} \xi_n \cos\left(\frac{(2n-1)\pi x}{2L}\right), \qquad \text{with} \quad \xi_n \sim \mathcal{N}\left(0, \frac{f_{\text{nom}}(0)}{10n}\right).$$



Example: Statistics For A Function Of Multivariate Random Variables

Suppose we have two random variables, ξ_1 and ξ_2, that are normally distributed as follows:

$$\xi_1 \sim \mathcal{N}(2,1), \qquad \text{and} \qquad \xi_2 \sim \mathcal{N}(-1,4).$$

Consider the function

$$f(\xi_1,\xi_2) = \frac{(\xi_1 - 2)^2}{1+\xi_2^2}$$

← like stress at one node

Let's estimate the mean, or expected value, of f.

We Will Need the Gauss-Hermite Quadrature Rules

$$\int_{-\infty}^{\infty} f(u)e^{-u^2}\,du \approx \sum_{i=1}^{m} w_i f(u_i).$$

Recall that we need to adjust the weights and quadrature locations, because e^{-u^2} is not quite the same as a Gaussian probability density.

$$u = \frac{\xi - \mu}{\sqrt{2}\,\sigma} \implies \xi = \sqrt{2}\,\sigma u + \mu, \quad \frac{d\xi}{du} = \sqrt{2}\,\sigma$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(\xi-\mu)^2}{2\sigma^2}}\, d\xi = \frac{1}{\cancel{\sigma}\cancel{\sqrt{2}}\sqrt{\pi}} \int_{-\infty}^{\infty} f(\sqrt{2}\,\sigma u + \mu)\, e^{-u^2}\, \cancel{\sqrt{2}}\,\cancel{\sigma}\, du$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\sqrt{2}\,\sigma u + \mu)\, e^{-u^2}\, du$$

Make the Substition Into the Integral For the Mean of f

Adjusting the weights and quadrature locations we get

$$
\begin{aligned}
\mu_f = \quad E(f) &= \int_{-\infty}^{\infty} f(\xi) P(\xi; \mu, \sigma)\, d\xi \\
&= \int_{-\infty}^{\infty} f(\xi) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}}\, d\xi \quad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\sqrt{2}\sigma u + \mu) e^{-u^2} du \\
&\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{m} w_i f\left(\sqrt{2}\sigma\xi_i + \mu\right).
\end{aligned}
$$

The 2D Integral for $E(f)$ Can Be Written As An Iterated Integral

$$
\begin{aligned}
E(f) &= \int_{\xi_2=-\infty}^{\infty} \int_{\xi_1=-\infty}^{\infty} f(\xi_1, \xi_2) P(\xi_1; \mu_1, \sigma_1) P(\xi_2; \mu_2, \sigma_2)\, d\xi_1 d\xi_2 \\
&= \int_{\xi_2=-\infty}^{=\infty} g(\xi_2) P(\xi_2; \mu_2, \sigma_2)\, d\xi_2
\end{aligned}
$$

where

$$
g(\xi_2) = \int_{\xi_1=-\infty}^{\infty} f(\xi_1, \xi_2) P(\xi_1; \mu_1, \sigma_1)\, d\xi_1
$$

In the Implementation, Each Dimension Requires A Loop

```matlab
% define function and Gaussian variables
f = @(xi1,xi2) (xi1-2.0).^2./(1.0 + xi2.^2);
mu1 = 2.0; sigma1 = 1; mu2 = -1.0; sigma2 = 4.0;
% using a 3 point Gauss-Hermite quadrature
xi = [-1.22474487139; 0.0; 1.22474487139];
wts = [0.295408975151; 1.1816359006; ...
  0.295408975151]./sqrt(pi); % adjusted weights!
mean_f = 0.0;
for i1 = 1:size(xi,1)
  pt1 = sqrt(2)*sigma1*xi(i1) + mu1;
  for i2 = 1:size(xi,1)
    pt2 = sqrt(2)*sigma2*xi(i2) + mu2;
    mean_f = mean_f + wts(i1)*wts(i2)*f(pt1,pt2);
  end
end
```

for ξ_1

$= \xi_1^{(i1)}$

for ξ_2

$= \xi_2^{(i2)}$

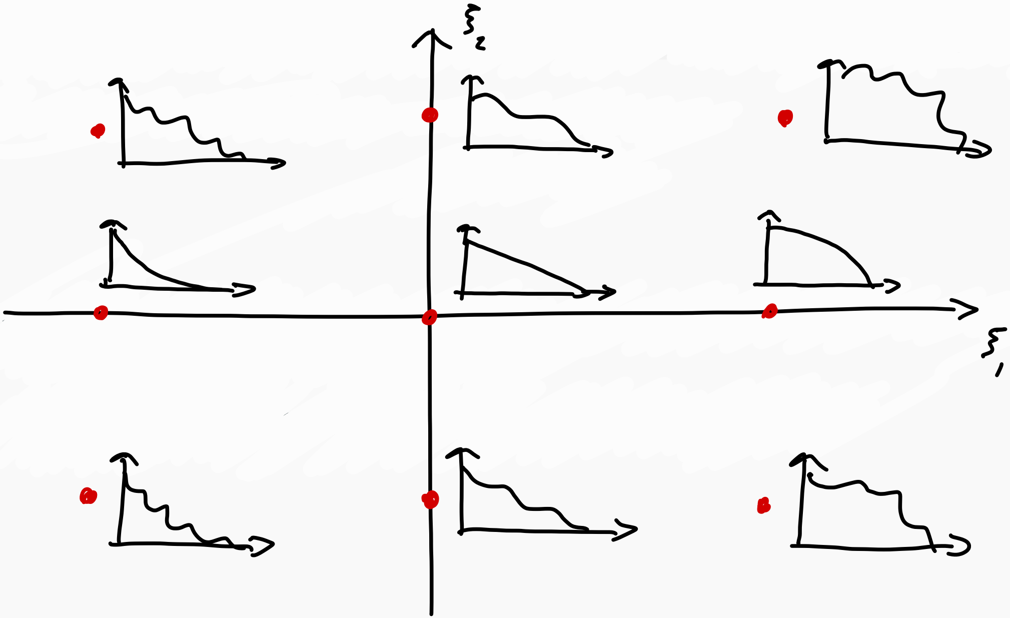
mean_f2 = mean_f2 + wts(i1)*wts(i2)*f(pt1,pt2)^2

What About Computing the Standard Deviation?

Here is a useful formula for computing the standard deviation

$$\sigma = \sqrt{E(f^2) - E(f)^2}$$

- During the quadrature loop, compute the expected value of the function squared at the same time you are computing the mean
- After the loop, use the above formula to get the standard deviation.


ξ_2
ξ_1