CHAPTER 1

FIRST ORDER DIFFERENTIAL EQUATIONS

1.1 INTRODUCTION

Consider the initial value problem

$$y' = f(x,y),\ y(x_0) = y_0 \tag{1.1.1}$$

where the function $f(x,y)$ is at least continuous in a domain $D \subseteq R^2$, and $(x_0, y_0) \in D$. By a **solution** of (1.1.1) in an interval J containing x_0, we mean a function $y(x)$ satisfying :

(i) $y(x_0) = y_0$,

(ii) for all $x \in J$, the points $(x, y(x)) \in D$,

(iii) $y'(x)$ exists and continuous for all $x \in J$; and

(iv) $y'(x) = f(x, y(x))$.

If J is closed then at the end points of J only one - sided existence of $y'(x)$ is assumed.

It is well known that the continuity of $f(x, y)$ in a closed rectangle

$$\bar{S} : |x - x_0| \le a, |y - y_0| \le b$$

is sufficient for the existence of at least one solution of (1.1.1) in the interval $J_h : |x - x_0| \le h = \min(a, b/M)$, where $M = \sup_{\bar{S}} |f(x, y)|$. However, if $f(x, y)$ is discontinuous then the nature of the solutions is quite arbitrary. For example, the initial value problem $y' = 2(y - 1)/x,\ y(0) = 0$ has no solution; while the problem $y' = 2(y-1)/x,\ y(0) = 1$ has an infinite number of solutions $y(x) = 1 + cx^2$, where c is an arbitrary constant.

Although, continuity of $f(x, y)$ is sufficient for the existence of a solution of (1.1.1), it does not imply the uniqueness of the solutions. For example, the function $f(x, y) = y^{2/3}$ is continuous in R^2, but the problem $y' = y^{2/3},\ y(0) = 0$ has an infinite number of solutions $y(x) \equiv 0$, and

$$y(x) = \begin{cases} 0, & 0 \le x \le c \\ (x - c)^3/27, & x \ge c \end{cases}$$

where c is an arbitrary constant. In fact, the situation is rather bad, in the year 1925 Lavrentieff [96] gave an example of a continuous function $f(x, y)$ in the open rectangle

$$S : |x - x_0| < a,\ |y - y_0| < b$$

with the property that for every choice of the initial point (x_0, y_0) interior to S, the initial value problem (1.1.1) has more than one solution in every interval $[x_0, x_0 + \epsilon]$ and $[x_0 - \epsilon, x_0]$. Further, in the year 1963 Hartman [58] used methods somewhat similar to that of van Kampen [159] and constructed another continuous function $f(x, y)$ in R^2 so that the problem (1.1.1) has the same nonuniqueness property. Hence, to ensure the uniqueness also, besides continuity we need to impose some additional conditions on $f(x, y)$.

1.2 LIPSCHITZ UNIQUENESS THEOREM

To ensure the uniqueness of the solutions of (1.1.1) we begin with the assumption that the variation of the function $f(x,y)$ relative to y remains bounded. We state this condition in

Definition 1.2.1. The function $f(x,y)$ is said to satisfy the **uniform Lipschitz condition** in a domain D if

$$|f(x,y) - f(x,\bar{y})| \le L|y-\bar{y}| \tag{1.2.1}$$

for all $(x,y),(x,\bar{y})$ in D having the same x. The nonnegative constant L is called the **Lipschitz constant**.

The Lipschitz condition lies in between the continuity and the differentiability of f with respect to y. Indeed, from (1.2.1) the continuity of f with respect to y is obvious, and the continuous function $f(x,y) = y^{2/3}$ violates the Lipschitz condition in any domain containing $y = 0$. However, the nondifferentiable function $f(x,y) = |y|, (x,y) \in R^2$ satisfies (1.2.1) with $L = 1$. If the function f is differentiable with respect to y, then it is easy to compute the Lipschitz constant L. In fact, we shall prove

Lemma 1.2.1. Let the domain D be convex and the function $f(x,y)$ be differentiable with respect to y in D. Then, for the Lipschitz condition (1.2.1) to be satisfied, it is necessary and sufficient that

$$\sup_D \left|\frac{\partial f(x,y)}{\partial y}\right| \le L. \tag{1.2.2}$$

Proof. Since $f(x,y)$ is differentiable with respect to y and the domain D is convex, for all $(x,y),(x,\bar{y}) \in D$ the mean value theorem provides

$$f(x,y) - f(x,\bar{y}) = \frac{\partial f(x,y^*)}{\partial y}(y-\bar{y}), \tag{1.2.3}$$

where $\min\{y,\bar{y}\} < y^* < \max\{y,\bar{y}\}$. Thus, on using (1.2.2) in (1.2.3), we find (1.2.1). Conversely, from (1.2.1) we have

$$\left|\frac{\partial f(x,y)}{\partial y}\right| = \lim_{\bar{y}\to y}\left|\frac{f(x,y)-f(x,\bar{y})}{y-\bar{y}}\right| \le L. \qquad \blacksquare$$

To prove the first main result of this section we need the following two lemmas.

Lemma 1.2.2. Let $f(x,y)$ be continuous in the domain D, then any solution of (1.1.1) is also a solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \tag{1.2.4}$$

and conversely.

Proof. Any solution of the differential equation $y' = f(x,y)$ converts it into an identity in x, i.e., $y'(x) = f(x, y(x))$. An integration of this equality with $y(x_0) = y_0$ gives (1.2.4). Conversely, if $y(x)$ is any solution of (1.2.4), then the substitution $x = x_0$ in (1.2.4) gives $y(x_0) = y_0$. Further, since $f(x,y)$ is continuous, on differentiating (1.2.4), we find that $y'(x) = f(x, y(x))$. ∎

Lemma 1.2.3. Let $\phi(x)$ and $q(x)$ be nonnegative continuous functions in $|x - x_0| \leq a$, and satisfy

$$\phi(x) \leq \left| \int_{x_0}^{x} q(t)\phi(t)dt \right|. \tag{1.2.5}$$

Then, $\phi(x) = 0$ in $|x - x_0| \leq a$.

Proof. For $x \in [x_0, x_0 + a]$, we define $r(x) = \int_{x_0}^{x} q(t)\phi(t)dt$ so that $r(x_0) = 0$, and $r'(x) = q(x)\phi(x)$. Thus, from (1.2.5) we have $r'(x) \leq q(x)r(x)$. Multiplying both sides of this inequality by $\exp\left(-\int_{x_0}^{x} q(t)dt\right)$, we get

$$\frac{d}{dx}\left(\exp\left(-\int_{x_0}^{x} q(s)ds\right) r(x)\right) \leq 0.$$

Thus, $\exp\left(-\int_{x_0}^{x} q(s)ds\right) r(x)$ is nonincreasing. Since $r(x_0) = 0$, it follows that $r(x) \leq 0$, and hence $\phi(x) \leq r(x) \leq 0$. However, since the function $\phi(x)$ is nonnegative, we find that $\phi(x) = 0$ in $[x_0, x_0 + a]$. The proof is similar in $[x_0 - a, x_0]$. ∎

Theorem 1.2.4 (Lipschitz Uniqueness Theorem). Let $f(x,y)$ be continuous and satisfy the uniform Lipschitz condition (1.2.1) in $\bar{S}$. Then, the initial value problem (1.1.1) has at most one solution in $|x - x_0| \leq a$.

First Proof. Suppose that $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in

$|x - x_0| \leq a$, then from (1.2.4) and (1.2.1) it follows that

$$\begin{aligned} |y(x) - \bar{y}(x)| &\leq \left| \int_{x_0}^{x} |f(t, y(t)) - f(t, \bar{y}(t))| dt \right| \\ &\leq L \left| \int_{x_0}^{x} |y(t) - \bar{y}(t)| dt \right|. \end{aligned}$$

Now as an application of Lemma 1.2.3, we find that $|y(x) - \bar{y}(x)| \equiv 0$, and hence $y(x) = \bar{y}(x)$ in $|x - x_0| \leq a$. ∎

It is interesting to note that Theorem 1.2.4 can also be proved by using mean value theorem, and the alternative proof does not require the integral equation representation (1.2.4). Further, the novelty in the proof is in its simplicity.

Second Proof. We shall consider the interval $[x_0, x_0 + a]$, whereas the proof in the interval $[x_0 - a, x_0]$ is similar. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1). Then, $y'(x) = f(x, y(x))$ and $\bar{y}'(x) = f(x, \bar{y}(x))$ in $[x_0, x_0 + a]$ and $y(x_0) = \bar{y}(x_0) = y_0$. Suppose to the contrary that the function $y(x) - \bar{y}(x)$ (which is known to be continuously differentiable in $[x_0, x_0 + a]$ and vanishes at $x = x_0$) does not vanish for all x in $[x_0, x_0 + a]$. Then, there exists a number ε with $x_0 \leq \varepsilon < x_0 + a$ such that $|y(x) - \bar{y}(x)| = 0$ for $x_0 \leq x \leq \varepsilon$ and $|y(x) - \bar{y}(x)|$ is not identically zero in an interval $\varepsilon \leq x \leq \varepsilon + l \leq x_0 + a$, where $l > 0$. Choose $l > 0$ so small that $lL < 1$, where L is the Lipschitz constant of $f(x, y)$. Since the function $|y(x) - \bar{y}(x)|$ is continuous and not identically zero in the closed interval $\varepsilon \leq x \leq \varepsilon + l$, it has a positive maximum at some point $\hat{\varepsilon} > \varepsilon$ in this interval. By the mean value theorem for $\varepsilon \leq x \leq \hat{\varepsilon}$, there exists $\varepsilon < \varepsilon^* < \hat{\varepsilon}$ such that

$$y'(\varepsilon^*) - \bar{y}'(\varepsilon^*) = \frac{[y(\hat{\varepsilon}) - \bar{y}(\hat{\varepsilon})] - [y(\varepsilon) - \bar{y}(\varepsilon)]}{\hat{\varepsilon} - \varepsilon}.$$

Thus, it follows that

$$\begin{aligned} 0 &< |y(\hat{\varepsilon}) - \bar{y}(\hat{\varepsilon})| = |\hat{\varepsilon} - \varepsilon||y'(\varepsilon^*) - \bar{y}'(\varepsilon^*)| \\ &= (\hat{\varepsilon} - \varepsilon)|f(\varepsilon^*, y(\varepsilon^*)) - f(\varepsilon^*, \bar{y}(\varepsilon^*))| \\ &\leq lL|y(\varepsilon^*) - \bar{y}(\varepsilon^*)| \\ &< |y(\varepsilon^*) - \bar{y}(\varepsilon^*)|. \end{aligned}$$

Hence, the positive number $|y(\hat{\varepsilon}) - \bar{y}(\hat{\varepsilon})|$ is not the maximum value of the continuous function $|y(x) - \bar{y}(x)|$ in the interval $\varepsilon \le x \le \varepsilon + l$, which is a contradiction. ∎

We shall provide one more proof of Theorem 1.2.4 which does not require even the mean value theorem.

Third Proof. Once again, we shall consider only the interval $[x_0, x_0 + a]$. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1), then the function $v(x) = (y(x) - \bar{y}(x))^2$ satisfies

$$\begin{aligned} v'(x) &= 2(y(x) - \bar{y}(x))(y'(x) - \bar{y}'(x)) \\ &= 2(y(x) - \bar{y}(x))(f(x, y(x)) - f(x, \bar{y}(x))) \\ &\le 2Lv(x), \end{aligned}$$

which is the same as $(\exp(-2L(x - x_0))v(x))' \le 0$. Thus, the function $\exp(-2L(x - x_0))v(x)$ is nonincreasing, and hence the condition $v(x_0) = 0$ implies that $v(x) \le 0$. Therefore, $v(x) \equiv 0$, i.e., $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. ∎

We shall illustrate Theorem 1.2.4 with an example.

Example 1.2.1. Consider the initial value problem

$$y' = 1 + y^2, \; y(0) = 1. \tag{1.2.6}$$

The function $f(x, y) = 1 + y^2$ is continuously differentiable in R^2. Further, from Lemma 1.2.1, we find that it satisfies the Lipschitz condition (1.2.1) in $\bar{S}$ with $L = \sup_{\bar{S}} |2y| \le 2(1 + b)$. Thus, from Theorem 1.2.4, the problem (1.2.6) has at most one solution in $|x| \le a$. ∎

The solution $y(x) = \tan(x + \pi/4)$ of (1.2.6) exists only in the interval $(-3\pi/4, \pi/4)$. However, it does not contradict the conclusion of Theorem 1.2.4 because 'at most' does include nonexistence of the solutions.

If the function $f(x, y)$ satisfies one sided Lipschitz condition, then the uniqueness of the solutions of (1.1.1) is also one - sided. This is the content of our next result.

Theorem 1.2.5. Let $f(x,y)$ be continuous in

$$\bar{S}_+ : x_0 \leq x \leq x_0 + a, |y - y_0| \leq b$$

and for all $(x,y),(x,\bar{y})$ in $\bar{S}_+$ with $\bar{y} \geq y$ it satisfies **one - sided Lipschitz condition**

$$(1.2.7) \qquad f(x,\bar{y}) - f(x,y) \leq L(\bar{y} - y).$$

Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0 + a]$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) which differ somewhere in $[x_0, x_0 + a]$. We assume that $\bar{y}(x) > y(x)$ for $x_1 < x \leq x_1 + \varepsilon \leq x_0 + a$, and $y(x) = \bar{y}(x)$ for $x_0 \leq x \leq x_1$, i.e., x_1 is the greatest lower bound (glb) of the set A of values of x for which $\bar{y}(x) > y(x)$. This glb exists because the set A is bounded below by x_0 at least. Thus, for all $x \in (x_1, x_1 + \varepsilon]$, we have

$$\bar{y}(x) - y(x) = \int_{x_1}^{x} (f(t,\bar{y}(t)) - f(t,y(t)))dt \leq L \int_{x_1}^{x} (\bar{y}(t) - y(t))dt.$$

Applying Lemma 1.2.3 to the above inequality, we get $\bar{y}(x) - y(x) = 0$ in $[x_1, x_1 + \varepsilon]$. This contradiction shows that $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. ∎

The above proof of Theorem 1.2.5 is similar to that of the first proof of Theorem 1.2.4. Its two other proofs based on the second and the third proof of Theorem 1.2.4 can be rather easily extended.

The Lipschitz condition is only a sufficient but not a necessary condition to prove the uniqueness of the solutions of (1.1.1). For this, we present the following two interesting examples.

Example 1.2.2. Consider the initial value problem

$$(1.2.8) \qquad y' = 1 + y^{2/3}, \; y(0) = 0.$$

The function $f(x,y) = 1 + y^{2/3}$ is continuous in R^2. However, it does not satisfy the Lipschitz condition in any domain containing $y = 0$.

Separating the variables in the above differential equation and using the substitution $y = z^3$ immediately lead to the unique solution of (1.2.8) as

$$3(y^{1/3} - \tan^{-1} y^{1/3}) = x. \qquad ∎$$

Example 1.2.3. Consider the initial value problem

$$(1.2.9)\qquad \begin{aligned} y' &= f(x,y) = \begin{cases} y\ln(1/y), & 0<y<1 \\ 0, & y=0 \end{cases} \\ y(0) &= \alpha, \quad 0 \le \alpha < 1. \end{aligned}$$

Since $\lim_{y\to 0^+} y\ln(1/y) = \lim_{y\to 0^+} \frac{\ln(1/y)}{1/y} = \lim_{y\to 0^+} y\,\frac{-1/y^2}{-1/y^2} = \lim_{y\to 0^+} y = 0$, this function $f(x,y)$ is continuous in the strip $|x| < \infty, 0 \le y < 1$. However, it does not satisfy the Lipschitz condition in any domain containing $y = 0$. The problem (1.2.9) has a unique solution which can be written as

$$x = \ln\left[\ln(\frac{1}{\alpha})\Big/\ln(\frac{1}{y})\right]. \qquad \blacksquare$$

The conclusion of the Example 1.2.2 can be deduced from the following :

Theorem 1.2.6. Let the function $f(x,y) = f(x)g(y)$ be continuous in the domain D, and $g(y) \neq 0$. Then, the initial value problem (1.1.1) has a unique solution $y(x) = G^{-1}(F(x))$, where $F(x) = \int_{x_0}^{x} f(t)dt$ and $G(y) = \int_{y_0}^{y} \frac{dz}{g(z)}$, as long as $(x, F(x)) \in D$.

Proof. Since $g(y) \neq 0$, $\frac{y'}{g(y)} = f(x)$, and hence it follows that

$$G(y) = \int_{y_0}^{y} \frac{dz}{g(z)} = \int_{x_0}^{x} f(t)dt = F(x).$$

Obviously, the function $G(y)$ is strictly monotonic, and hence the conclusion follows by the inverse function theorem. $\blacksquare$

In Section 1.1 we have seen that the initial value problem $y' = g(y) = y^{2/3}$, $y(0) = 0$ has an infinite number of solutions. This function $g(y) = y^{2/3}$ vanishes at $y = 0$. In our next result we shall show that this property is in fact true for all such initial value problems.

Theorem 1.2.7. Let the function $g(y)$ be continuous in the interval $|y - y_0| < b$. Further, let the initial value problem

$$(1.2.10)\qquad y' = g(y), \; y(x_0) = y_0$$

has at least two solutions in every neighborhood of x_0. Then, $g(y_0) = 0$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.2.10) in $U : |x - x_0| < \varepsilon$. Let $g(y_0) \neq 0$, then there is a neighborhood V of y_0 on which $g(y)$ is bounded away from zero. Let $W : |x - x_0| < \varepsilon_1 \leq \varepsilon$ be such that $y(W) \subset V$ and $\bar{y}(W) \subset V$. For $x \in W$, we define $G(x) = \int_{y(x)}^{\bar{y}(x)} \frac{dz}{g(z)}$. Obviously, if $y(x) \neq \bar{y}(x)$ for some $x \in W$, then $G(x) \neq 0$. But, for all $x \in W$

$$G'(x) = \frac{\bar{y}'(x)}{g(\bar{y}(x))} - \frac{y'(x)}{g(y(x))} = 0.$$

Therefore, $G(x)$ is a constant on W. However, since $G(x_0) = 0$ it follows that $G(x) = 0, x \in W$. This contradiction completes the proof. ∎

From Example 1.2.3 it is clear that the converse of Theorem 1.2.7 is not true. However, the following holds.

Theorem 1.2.8. Let the function $g(y)$ be continuous in the interval $|y - y_0| \leq b$, and $g(y_0) = 0$. Further, let $g(y) \neq 0$ for $y_0 < y \leq y_0 + b$ $(y_0 - b \leq y < y_0)$ and the integral

$$\int_{y_0}^{y_0+b} \frac{dz}{g(z)} \qquad \left(\int_{y_0-b}^{y_0} \frac{dz}{g(z)}\right) \tag{1.2.11}$$

is divergent. Then, there is no solution of the initial value problem (1.2.10) which will from above (below) join $y = y_0$.

Proof. Assume that there exists a solution $y(x) \not\equiv y_0$ of the initial value problem (1.2.10). Then, let there be (to consider one of the four cases) right to x_0 a point x_1 with $y_0 < y(x_1) = y_1 < y_0 + b$ Thus, by the differential equation it follows that

$$\int_{y_1}^{y(x)} \frac{dz}{g(z)} = (x - x_1) \tag{1.2.12}$$

at least as long as $y(x)$ satisfies $y_0 < y(x) \leq y_0 + b$. Assume that $\bar{x}$ is the first point left to x_1 with $y(\bar{x}) = y_0$. Then, (1.2.12) leads to a contradiction, because as $x \to \bar{x}$ the integral in the left of (1.2.12) tends to ∞ whereas the right side remains bounded. ∎

From the conditions of Theorem 1.2.8 it is clear that if a solution $y(x) > y_0 (< y_0)$ at some point, then this inequality holds everywhere. Further, if both the integrals in (1.2.11) diverge then $y(x) \equiv y_0$ is the only solution of (1.2.10). As an example the initial value problem $y' = 2y, y(0) = 0$ has a unique solution $y(x) \equiv 0$, and the integral $\int_0^{\pm\alpha} \frac{dz}{z}, (\alpha > 0)$ diverges. Whereas, in Section 1.1 we have seen that the problem $y' = y^{2/3}, y(0) = 0$ has an infinite number of solutions and the integral $\int_0^{\pm\alpha} \frac{dz}{z^{2/3}}$, $(\alpha > 0)$ converges.

1.3 PEANO'S UNIQUENESS THEOREM

In the previous section, we have seen that the initial value problem (1.1.1) may have a unique solution even when $f(x, y)$ does not satisfy the Lipschitz condition. In this section we shall discuss Peano's uniqueness theorem which assumes apparently different conditions on the function $f(x, y)$, such as increasing or decreasing with respect to y.

Theorem 1.3.1 (Peano's Uniqueness Theorem). Let $f(x, y)$ be continuous in $\bar{S}_+$ and nonincreasing in y for each fixed x in $[x_0, x_0 + a]$. Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0 + a]$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) which differ somewhere in $[x_0, x_0+a]$. We assume that $\bar{y}(x) > y(x)$ for $x_1 < x \le x_1+\varepsilon \le x_0+a$, and $y(x) = \bar{y}(x)$ for $x_0 \le x \le x_1$. Thus, for all $x \in (x_1, x_1 + \varepsilon]$, we have $f(x, y(x)) \ge f(x, \bar{y}(x))$, and hence $y'(x) \ge \bar{y}'(x)$. This implies that the function $\phi(x) = \bar{y}(x) - y(x)$ is nonincreasing. Further, since $\phi(x_1) = 0$, we have $\phi(x) \le 0$ in $[x_1, x_1 + \varepsilon]$. This contradiction proves that $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. ∎

The above result is actually a particular case of Theorem 1.2.5, indeed if $L = 0$, then condition (1.2.7) reduces to the hypothesis of Theorem 1.3.1.

The 'nonincreasing' nature in Theorem 1.3.1 cannot be replaced by 'nondecreasing'. For this, we have

Example 1.3.1. Consider the initial value problem

$$y' = |y|^{1/2}\text{sgn } y, \; y(0) = 0 \qquad (1.3.1)$$

where $\operatorname{sgn} y = \begin{cases} 1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0 \end{cases}$. The function $f(x,y) = |y|^{1/2}\operatorname{sgn} y$ is continuous for $0 \leq x < \infty, |y| < \infty$ and it is nondecreasing in y for all $x \in [0,\infty)$. The problem (1.3.1) has two solutions $y(x) \equiv 0$ and $y(x) = x^2/4$ in $[0,\infty)$. ∎

We shall now illustrate an example where $f(x,y)$ does not satisfy the Lipschitz condition (1.2.1) but it satisfies the conditions of Theorem 1.3.1.

Example 1.3.2. Consider the initial value problem

$$y' = -|y|^{1/2}\operatorname{sgn} y, \ y(0) = 0. \tag{1.3.2}$$

The function $f(x,y) = -|y|^{1/2}\operatorname{sgn} y$ does not satisfy the Lipschitz condition (1.2.1) in any domain containing $y = 0$. Indeed, for $y > 0$ we have

$$\frac{|f(x,y) - f(x,0)|}{|y-0|} = \frac{|y|^{1/2}}{|y|} = \frac{1}{|y|^{1/2}},$$

which is unbounded as $y \to 0$. However, conditions of Theorem 1.3.1 are satisfied since $-|y|^{1/2}\operatorname{sgn} y$ is continuous for $0 \leq x < \infty, |y| < \infty$ and nonincreasing in y for all $x \in [0,\infty)$. Thus, from Theorem 1.3.1 it follows that $y(x) \equiv 0$ in $[0,\infty)$ is the only solution of the problem (1.3.2). ∎

Theorem 1.3.2. Suppose that $f(x,y)$ is continuous in

$$\bar{S}_- : x_0 - a \leq x \leq x_0, |y - y_0| \leq b$$

and nondecreasing in y for each fixed x in $[x_0 - a, x_0]$. Then, the initial value problem (1.1.1) has at most one solution in $[x_0 - a, x_0]$.

Proof. The proof is similar to that of Theorem 1.3.1. ∎

On combining the above results, we obtain

Theorem 1.3.3. If $f(x,y)$ is continuous in $\bar{S}$ and nondecreasing in y for each fixed x in $[x_0 - a, x_0]$, and nonincreasing in y for each fixed x in $[x_0, x_0 + a]$, then the initial value problem (1.1.1) has at most one solution in $|x - x_0| \leq a$. ∎

The conditions of Peano's uniqueness theorem are only sufficient. This is clear from the initial value problem (1.2.8). The function $f(x,y) = 1 + y^{2/3}$ is nondecreasing in y for each x in $0 \le x < \infty$. However, in Example 1.2.2 we have seen that (1.2.8) has a unique solution.

1.4 OSGOOD'S UNIQUENESS THEOREM

In this section we shall study a generalization of the Lipschitz uniqueness theorem which is due to Osgood . For this, we require the following :

Lemma 1.4.1. Let $g(z)$ be a continuous and nondecreasing function in the interval $[0, \infty)$ and $g(0) = 0$, $g(z) > 0$ for $z > 0$. Also,

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon \frac{dz}{g(z)} = \infty. \tag{1.4.1}$$

Let $\phi(x)$ be a nonnegative continuous function in $[0, a]$. Then,

$$\phi(x) \le \int_0^x g(\phi(t))dt,\ 0 < x \le a \tag{1.4.2}$$

implies that $\phi(x) = 0$ in $[0, a]$.

Proof. Define $\Phi(x) = \max_{0 \le t \le x} \phi(t)$ and assume that $\Phi(x) > 0$ for $0 < x \le a$. Then, $\phi(x) \le \Phi(x)$ and for each x there is an $x_1 \le x$ such that $\phi(x_1) = \Phi(x)$. From this, we have

$$\Phi(x) = \phi(x_1) \le \int_0^{x_1} g(\phi(t))dt \le \int_0^x g(\Phi(t))dt,$$

i.e., the increasing function $\Phi(x)$ satisfies the same inequality as $\phi(x)$ does. Let us set $\bar{\Phi}(x) = \int_0^x g(\Phi(t))dt$, then $\bar{\Phi}(0) = 0, \Phi(x) \le \bar{\Phi}(x)$ and $\bar{\Phi}'(x) = g(\Phi(x)) \le g(\bar{\Phi}(x))$. Hence, for $0 < \delta < a$, we have

$$\int_\delta^a \frac{\bar{\Phi}'(x)}{g(\bar{\Phi}(x))} dx \le a - \delta < a.$$

However, from (1.4.1), it follows that

$$\int_\delta^a \frac{\bar{\Phi}'(x)}{g(\bar{\Phi}(x))} dx = \int_\varepsilon^\alpha \frac{dz}{g(z)},\ \bar{\Phi}(\delta) = \varepsilon, \bar{\Phi}(a) = \alpha$$

becomes infinite when $\varepsilon \to 0^+ (\delta \to 0)$. This contradiction shows that $\Phi(x)$ cannot be positive and so $\Phi(x) \equiv 0$, and hence $\phi(x) = 0$ in $[0, a]$. ∎

Theorem 1.4.2 (Osgood's Uniqueness Theorem). Let $f(x,y)$ be continuous in $\bar{S}$ and for all $(x,y),(x,\bar{y}) \in \bar{S}$ it satisfies **Osgood's condition**

$$|f(x,y) - f(x,\bar{y})| \le g(|y - \bar{y}|), \tag{1.4.3}$$

where $g(z)$ is the same as in Lemma 1.4.1. Then, the initial value problem (1.1.1) has at most one solution in $|x - x_0| \le a$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in $|x - x_0| \le a$. We shall show that $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. From (1.4.3) it follows that

$$\begin{aligned} |y(x_0 + x) - \bar{y}(x_0 + x)| &\le \int_{x_0}^{x_0+x} |f(t, y(t)) - f(t, \bar{y}(t))| dt \\ &\le \int_{x_0}^{x_0+x} g(|y(t) - \bar{y}(t)|) dt \\ &= \int_0^x g(|y(z + x_0) - \bar{y}(z + x_0)|) dz. \end{aligned}$$

For x in $[0, a]$, we set $\phi(x) = |y(x + x_0) - \bar{y}(x + x_0)|$. Then, the nonnegative continuous function $\phi(x)$ satisfies the inequality (1.4.2), and therefore, Lemma 1.4.1 implies that $\phi(x) = 0$ in $[0, a]$, i.e., $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. If x is in $[x_0 - a, x_0]$, then the proof remains the same except that we need to define the function $\phi(x) = |y(x_0 - x) - \bar{y}(x_0 - x)|$ in $[0, a]$. ∎

Example 1.4.1. Consider the initial value problem

$$y' = Ly, \; y(0) = 0 \tag{1.4.4}$$

where $L > 0$. For this problem, we choose $g(z) = Lz$, which is clearly continuous and nondecreasing in the interval $[0, \infty)$. Further, since $g(0) = 0$, $g(z) > 0$ for $z > 0$, and $\lim_{\varepsilon \to 0^+} \int_\varepsilon [g(z)]^{-1} dz = \frac{1}{L} \lim_{\varepsilon \to 0^+} \; \ln \frac{1}{\varepsilon} = \infty$. This function $g(z)$ satisfies the conditions of Lemma 1.4.1. Next, for any y and $\bar{y}$ we have

$$|f(x,y) - f(x,\bar{y})| = |Ly - L\bar{y}| = g(|y - \bar{y}|)$$

and hence Osgood's condition (1.4.3) is also satisfied. Therefore, from Theorem 1.4.2 the problem (1.4.4) has a unique solution, namely, $y(x) \equiv 0$. ∎

The importance of Example 1.4.1 is in the fact that Osgood's condition (1.4.3) with $g(z) = Lz$ reduces to the Lipschitz condition (1.2.1).

Example 1.4.2. Consider the initial value problem

$$(1.4.5) \qquad \begin{aligned} y' &= f(x,y) = \begin{cases} -y\ln y, & 0 < y \le 1/e \\ 0, & y = 0 \end{cases} \\ y(0) &= 0. \end{aligned}$$

From Example 1.2.3 it is clear that for this problem Lipschitz uniqueness theorem is not applicable. Further, since this function $f(x,y)$ is increasing with respect to y, Peano's uniqueness theorem also cannot be used. For this problem we choose $g(z) = \begin{cases} 0, & z = 0 \\ -z\ln z, & 0 < z \le 1/e \\ 1/e, & z > 1/e \end{cases}$. Clearly, $g(z)$ is continuous and nondecreasing in $[0,\infty)$ with $g(0) = 0,\ g(z) > 0$ for $z > 0$. Moreover, we have

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon \frac{dz}{g(z)} = \lim_{\varepsilon\to 0^+} \int_\varepsilon \frac{dz}{-z\ln z} = \lim_{\varepsilon\to 0^+} \ln(\ln(1/\varepsilon)) = \infty.$$

Thus, this function $g(z)$ satisfies the conditions of Lemma 1.4.1. Now, we shall verify the condition (1.4.3). For this, we note that $\frac{d^2 f}{dy^2} = -\frac{1}{y} < 0, 0 < y \le 1/e$ and therefore f is concave. Let $0 < y - \bar{y} < \bar{y} < y \le 1/e$, then it follows that

$$\frac{f(x, y-\bar{y}) - f(x,0)}{y-\bar{y}} \ge \frac{f(x,y) - f(x,\bar{y})}{y - \bar{y}},$$

which is the same as

$$-(y-\bar{y})\ln(y-\bar{y}) \ge -y\ln y + \bar{y}\ln\bar{y}$$

and hence

$$|f(x,y) - f(x,\bar{y})| = |(-y\ln y) - (-\bar{y}\ln\bar{y})| \le -|y-\bar{y}|\ln|y-\bar{y}| = g(|y-\bar{y}|).$$

Thus, for the problem (1.4.5) Osgood's uniqueness theorem is applicable. ∎

In Theorem 1.4.2 the condition that the function $g(z)$ is nondecreasing is in fact superfluous. To show this we need the following result which is in a sense complementary to Theorem 1.2.8.

Theorem 1.4.3. Let the function $g(y) > 0$ be continuous for $y_0 < y \le y_0 + \beta$ for some $\beta > 0$ and $g(y_0) = 0$. Further, let the integral $\int_{y_0+\epsilon} \frac{dz}{g(z)}$ diverges as $\varepsilon \to 0^+$. Then, the initial value problem $y' = g(y),\ y(x_1) = y_1$ where $y_0 < y_1 \leq y_0 + \beta$ has a unique solution. This solution $y(x)$ is such that $y_0 < y(x) \leq y_0 + \beta$ and tends to y_0 as $x \to -\infty$. Further, if $\int_{y_0+\varepsilon} \frac{dz}{g(z)}$ converges as $\varepsilon \to 0^+$, then there are infinite number of solutions of (1.2.10).

Proof. If $y(x)$ is a solution of $y' = g(y),\ y(x_1) = y_1$, then it follows that

$$\int_{y_1}^{y(x)} \frac{dz}{g(z)} = (x - x_1). \tag{1.4.6}$$

Since $g(y)$ is defined only for $y_0 \leq y \leq y_0 + \beta$, it is necessary that $y_0 \leq y(x) \leq y_0 + \beta$. From (1.4.6) and the fact that g is continuous and $g(y) > 0$ for $y_0 < y \leq y_0 + \beta$, we find that $y(x)$ exists for as long as $y_0 < y(x) \leq y_0 + \beta$. Since equation (1.4.6) defines only candidate for a solution (implicitly), and since by direct substitution it gives a solution, it follows that for $y' = g(y),\ y(x_1) = y_1$, (1.4.6) defines the unique solution.

To show that $\lim_{x \to -\infty} y(x) = y_0$, when

$$\lim_{\varepsilon \to 0^+} \int_{y_0+\varepsilon}^{y_1} \frac{dz}{g(z)} = \infty \tag{1.4.7}$$

we proceed as follows. The left side of (1.4.6) is finite provided $y_0 < y(x) \leq y_0 + \beta$. Since the integral in (1.4.7) diverges, this solution $y(x)$ cannot become y_0 for a finite value of x and at the same time satisfy (1.4.6). Thus, $y_0 < y(x) \leq y_0 + \beta$ for all $x < x_1$. As $x \to -\infty$ the right side of (1.4.6) tends to $-\infty$, and therefore $\lim_{x \to -\infty} y(x) = y_0$.

Since $g(y_0) = 0$, clearly $y(x) \equiv y_0$ is a solution of (1.2.10). If there is another solution $\phi(x)$ of (1.2.10) such that $y_0 < \phi(x) \leq y_0 + \beta$ for $x \neq x_0$, then it follows that

$$\lim_{\varepsilon \to 0^+} \int_{y_0+\varepsilon}^{\phi(x)} \frac{dz}{g(z)} = (x - x_0). \tag{1.4.8}$$

If the integral in (1.4.7) diverges, then (1.4.8) does not define a function. If the integral in (1.4.7) converges, then the equation (1.4.8) defines a function; in fact, then there is a family of solutions $\phi_c(x)$ of (1.2.10) defined by

$$\begin{cases} \phi_c(x) = y_0, & x \leq c \\ \displaystyle\int_{y_0}^{\phi_c(x)} \frac{dz}{g(z)} = (x - c), & x > c \end{cases}$$

for every $c \geq x_0$. Here, it is understood that

$$\int_{y_0}^{\phi_c(x)} \frac{dz}{g(z)} = \lim_{\epsilon \to 0+} \int_{y_0+\epsilon}^{\phi_c(x)} \frac{dz}{g(z)},$$

where this integral converges. ∎

As an example once again consider the initial value problem $y' = y^{2/3}, y(0) = 0$ discussed in Sections 1.1 and 1.2.

Theorem 1.4.4. Let $f(x,y)$ be continuous in $\bar{S}$ and for all $(x,y),(x,\bar{y}) \in \bar{S}$ it satisfies Osgood's condition (1.4.3), where $g(z) > 0$ is a continuous function for $0 < z \leq 2b$, $g(0) = 0$ and satisfies the condition (1.4.1). Then, the initial value problem (1.1.1) has at most one solution in $|x - x_0| \leq a$.

First Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in $|x-x_0| \leq a$. Once again we shall show that $y(x) = \bar{y}(x)$ in $[x_0, x_0+a]$. Let $\phi(x) = \bar{y}(x) - y(x)$ and assume that there is a point $x_1 \in (x_0, x_0+a]$ such that $\phi(x_1) > 0$. Then, we have

$$\begin{aligned} |\phi'(x)| &= |f(x,\bar{y}(x)) - f(x,y(x))| \leq g(|\bar{y}(x) - y(x)|) \\ &= g(|\phi(x)|) < 2g(|\phi(x)|). \end{aligned}$$

By Theorem 1.4.3 the initial value problem $z' = 2g(z)$, $z(x_1) = \phi(x_1)$ has a unique solution $z(x)$ existing for $-\infty < x \leq x_1$ and $z(x) > 0$. Since initially $z(x_1) = \phi(x_1)$, the solution $z(x)$ of $z' = 2g(z)$ may follow $\phi(x)$ for some $x \leq x_1$. Let $\bar{x} = \mathrm{glb}\{x : \phi(x) = z(x)\}$. Then, $x_0 < \bar{x} \leq x_1$, because $\phi(x_0) = 0$ and $z(x_0) > 0$. Since $z(x) > 0$ for $-\infty < x \leq x_1, \phi(\bar{x}) = z(\bar{x}) > 0$. But,

$$\phi'(\bar{x}) \leq |\phi'(\bar{x})| < 2g(|\phi(\bar{x})|) = 2g(\phi(\bar{x})) = 2g(z(\bar{x})) = z'(\bar{x});$$

thus $\phi'(\bar{x}) < z'(\bar{x})$.

By continuity this inequality must hold in some interval $\beta \leq x \leq \bar{x}$. Then,

$$\int_x^{\bar{x}} \phi'(t)dt < \int_x^{\bar{x}} z'(t)dt$$

implies that $\phi(\bar{x}) - \phi(x) < z(\bar{x}) - z(x)$ which yields the inequality $\phi(x) > z(x)$ in $[\beta, \bar{x}]$. But, $\bar{x}$ is the smallest value of x for which $\phi(x) = z(x)$ and so the graphs $y = \phi(x)$, $y = z(x)$ cannot cross again. Hence, we have $\phi(x) > z(x)$ in

$[x_0, \bar{x}]$. But, $\phi(x_0) = 0 < z(x_0)$, which is a contradiction. Thus, $\phi(x) \equiv 0$, and hence $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. ∎

Like Lipschitz uniqueness theorem, Theorem 1.4.4 can also be proved by using only the mean value theorem.

Second Proof. We shall consider the interval $[x_0, x_0 + a]$, whereas the proof in the interval $[x_0 - a, x_0]$ is similar. Let $G(z)$ be a function such that $G(z)$ has a continuous first order derivative in $(0, 2b], G(0^+) = -\infty$, and $G'(z) = \frac{1}{g(z)}$ for $0 < z \leq 2b$ (hence $G(z) > -\infty$ for $0 < z \leq 2b$). For example, we may simply choose

$$G(z) = \int_b^z \frac{dt}{g(t)} \quad \text{for } \ 0 < z \leq 2b.$$

Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) which differ somewhere in $[x_0, x_0 + a]$. As in Theorem 1.2.5 we assume that $\bar{y}(x) > y(x)$ for $x_1 < x \leq x_1 + \varepsilon \leq x_0 + a$, and $y(x) = \bar{y}(x)$ for $x_0 \leq x \leq x_1$. This means that we also have

$$e^{G(\bar{y}(x_1) - y(x_1))} = 0, \quad \text{while} \quad e^{G(\bar{y}(x) - y(x))} > 0 \ \text{ for } \ x_1 < x \leq x_1 + \varepsilon.$$

Then, by the mean value theorem, applied to the function $e^{-x}e^{G(\bar{y}(x)-y(x))}$ in the interval $[x_1, x_1 + \varepsilon]$, there exists a number $\bar{x}$, with $x_1 < \bar{x} < x_1 + \varepsilon$, such that

$$\begin{aligned}
0 \ &< \ e^{-(x_1+\varepsilon)}e^{G(\bar{y}(x_1+\varepsilon) - y(x_1+\varepsilon))} \\
&= \ e^{-(x_1+\varepsilon)}e^{G(\bar{y}(x_1+\varepsilon) - y(x_1+\varepsilon))} - e^{-x_1}e^{G(\bar{y}(x_1) - y(x_1))} \\
&= \ \varepsilon e^{-\bar{x}}e^{G(\bar{y}(\bar{x}) - y(\bar{x}))}\left[\frac{\bar{y}'(\bar{x}) - y'(\bar{x})}{g(\bar{y}(\bar{x}) - y(\bar{x}))} - 1\right] \\
&= \ \varepsilon e^{-\bar{x}}e^{G(\bar{y}(\bar{x}) - y(\bar{x}))}\left[\frac{f(\bar{x}, \bar{y}(\bar{x})) - f(\bar{x}, y(\bar{x}))}{g(\bar{y}(\bar{x}) - y(\bar{x}))} - 1\right] \\
&\leq \ 0,
\end{aligned}$$

where the last inequality follows from Osgood's condition (1.4.3). This contradiction proves the result. ∎

A one - sided analog of Theorem 1.4.4 is the following :

Theorem 1.4.5. Let $f(x,y)$ be continuous in $\bar{S}_+$ and for all $(x,y),(x,\bar{y})$ in $\bar{S}_+$ with $\bar{y} \ge y$ it satisfies **one - sided Osgood condition**

$$f(x,\bar{y}) - f(x,y) \le g(\bar{y} - y), \tag{1.4.9}$$

where $g(z) > 0$ is a continuous function for $0 < z \le 2b, g(0) = 0$ and satisfies the condition (1.4.1). Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0 + a]$. ∎

Often, the uniqueness of the solutions of (1.1.1) is achieved only in a small interval containing x_0, and not in the entire interval of interest. Such type of results are called **local uniqueness theorems.** In the following we state a local Osgood's uniqueness theorem.

Theorem 1.4.6. Let $f(x,y)$ be continuous in a domain D and for all $(x,y),(x,\bar{y}) \in D$ it satisfies Osgood's condition (1.4.3), where $g(z) > 0$ is a continuous function for $0 < z \le \alpha,\ g(0) = 0$ for some $\alpha > 0$ and satisfies the condition (1.4.1). Then, the initial value problem (1.1.1) has at most one solution in a small interval containing x_0. ∎

We note that the functions $Lz, Lz|\ln z|, Lz|\ln z|\ln|\ln z|, Lz|\ln z|\ln|\ln z|\cdot \ln\ln|\ln z|, \cdots$, where L is a positive constant, all have the properties required for the function $g(z)$ in Theorem 1.4.6. Further, in progression these functions provide weaker restrictions on the function $f(x,y)$, i.e., provide stronger theorems. However, there is no "strongest" theorem of this type, i.e., if $g(z)$ satisfies the conditions of Theorem 1.4.6, then there always exists another function $g_1(z)$ satisfying the same conditions, but

$$\lim_{z \to 0} \frac{g_1(z)}{g(z)} = \infty.$$

We also remark that if the graph of $g(z)$ is concave, then in Theorem 1.4.6 the condition (1.4.1) is not only sufficient but necessary also.

1.5 MONTEL - TONELLI'S UNIQUENESS THEOREM

A generalization of Osgood's uniqueness theorem is embodied in the following :

Theorem 1.5.1 (Montel - Tonelli's Uniqueness Theorem). Let $f(x,y)$ be continuous in $\bar{S}$ and for all $(x,y),(x,\bar{y}) \in \bar{S}$ it satisfies **Montel Tonelli's condition**

$$|f(x,y)-f(x,\bar{y})| \leq h(x)g(|y-\bar{y}|), \tag{1.5.1}$$

where the function $h(x) \geq 0$ is integrable in $|x-x_0| \leq a$, and $g(z) > 0$ is a continuous function for $0 < z \leq 2b$, $g(0) = 0$ and satisfies the condition (1.4.1). Then, the initial value problem (1.1.1) has at most one solution in $|x-x_0| \leq a$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in $|x-x_0| \leq a$. Once again we shall show that $y(x) = \bar{y}(x)$ in $[x_0, x_0+a]$. Let $\phi(x) = \bar{y}(x) - y(x)$ and assume that there is a point $x_1 \in (x_0, x_0+a]$ such that $\phi(x) > 0$ for $x_1 < x \leq x_1 + \varepsilon$ and $\phi(x) = 0$ for $x_0 \leq x \leq x_1$. Then, for all $x \in [x_1, x_1+\varepsilon]$ it follows that

$$\phi'(x) = f(x,\bar{y}(x)) - f(x,y(x)) \leq h(x)g(\phi(x))$$

and hence for $\bar{x} \in (x_1, x_1+\varepsilon)$, we have

$$\int_{\phi(\bar{x})}^{\phi(x_1+\varepsilon)} \frac{dz}{g(z)} \leq \int_{\bar{x}}^{x_1+\varepsilon} h(t)dt.$$

However, in the above inequality as $\bar{x}$ tends to x_1 the right side remains finite, whereas the left side in view of (1.4.1) becomes unbounded. This contradiction completes the proof. ∎

1.6 NAGUMO'S UNIQUENESS THEOREM

The main result in this section assumes that the variation of the function

$f(x, y)$ relative to y is bounded by a certain function of x. We state this condition in

Definition 1.6.1. The function $f(x, y)$ is said to satisfy **Nagumo's condition** in the domain D if

$$|f(x, y) - f(x, \bar{y})| \leq k|x - x_0|^{-1}|y - \bar{y}|; \;\; x \neq x_0, k \leq 1 \tag{1.6.1}$$

for all $(x, y), (x, \bar{y}) \in D$.

Lemma 1.6.1. Let $\phi(x)$ be a nonnegative continuous function in $|x - x_0| \leq a$, and $\phi(x_0) = 0$, and let $\phi(x)$ be differentiable at $x = x_0$ with $\phi'(x_0) = 0$. Then, the inequality

$$\phi(x) \leq \left| \int_{x_0}^{x} \frac{\phi(t)}{t - x_0} dt \right| \tag{1.6.2}$$

implies that $\phi(x) = 0$ in $|x - x_0| \leq a$.

Proof. For $x \in [x_0, x_0 + a]$, we define $\Phi(x) = \int_{x_0}^{x} \frac{\phi(t)}{t - x_0} dt$. This integral exists since $\lim_{x \to x_0} \frac{\phi(x)}{x - x_0} = \phi'(x_0) = 0$. Further, we have

$$\Phi'(x) = \frac{\phi(x)}{x - x_0} \leq \frac{\Phi(x)}{x - x_0}$$

and hence

$$\frac{d}{dx}\left(\frac{\Phi(x)}{x - x_0}\right) = \frac{(x - x_0)\Phi'(x) - \Phi(x)}{(x - x_0)^2} \leq 0,$$

which implies that $\frac{\Phi(x)}{x - x_0}$ is nonincreasing. Since $\Phi(x_0) = 0$, this gives $\Phi(x) \leq 0$, which is a contradiction to $\Phi(x) \geq 0$. So, $\Phi(x) \equiv 0$, and hence $\phi(x) = 0$ in $[x_0, x_0 + a]$. The proof is similar in $[x_0 - a, x_0]$. ∎

We shall use Lemma 1.6.1 to prove

Theorem 1.6.2 (Nagumo's Uniqueness Theorem). Let $f(x, y)$ be continuous and satisfy Nagumo's condition (1.6.1) in $\bar{S}$. Then, the initial value problem (1.1.1) has at most one solution in $|x - x_0| \leq a$.

First Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in $|x - x_0| \leq a$.

Then, from (1.6.1) it follows that

$$\begin{aligned} |y(x)-\bar{y}(x)| &\le \left|\int_{x_0}^{x} |f(t,y(t))-f(t,\bar{y}(t))|dt\right| \\ &\le \left|\int_{x_0}^{x} |t-x_0|^{-1}|y(t)-\bar{y}(t)|dt\right|. \end{aligned}$$

We set $\phi(x)=|y(x)-\bar{y}(x)|$, then the nonnegative continuous function $\phi(x)$ in $|x-x_0|\le a$ satisfies the inequality (1.6.2). Further, since $\phi(x)$ is continuous in $|x-x_0|\le a$, and $\phi(x_0)=0$, from the mean value theorem, we have

$$\begin{aligned} \phi'(x_0) &= \lim_{h\to 0}\frac{\phi(x_0+h)-\phi(x_0)}{h} = \lim_{h\to 0}\frac{|y(x_0+h)-\bar{y}(x_0+h)|}{h} \\ &= \lim_{h\to 0}\frac{|(y(x_0+h)-y(x_0))-(\bar{y}(x_0+h)-\bar{y}(x_0))|}{h} \\ &= \lim_{h\to 0}\frac{|hy'(x_0+\theta_1 h)-h\bar{y}'(x_0+\theta_2 h)|}{h},\ 0<\theta_1,\theta_2<1 \\ &= (\mathrm{sgn}\ h)\lim_{h\to 0}|y'(x_0+\theta_1 h)-\bar{y}'(x_0+\theta_2 h)| \\ &= 0. \end{aligned}$$

Thus, the conditions of Lemma 1.6.1 are satisfied, and hence $\phi(x)\equiv 0$, i.e., $y(x)=\bar{y}(x)$ in $|x-x_0|\le a$. ∎

Theorem 1.6.2 can also be proved without using integral equations and the integral inequality. The alternative proof we shall now give requires only the mean value theorem.

Second Proof. We shall consider the interval $[x_0,x_0+a]$, whereas the proof in the interval $[x_0-a,x_0]$ is similar. As before, let $y(x)$ and $\bar{y}(x)$ be solutions in $[x_0,x_0+a]$. Consider the function $v(x)$ defined by $|y(x)-\bar{y}(x)|/(x-x_0)$ in $(x_0,x_0+a]$, and the value zero at $x=x_0$. It will be first shown that this function $v(x)$ is continuous at $x=x_0$. For this, on using the mean value theorem, and the fact that $y(x)-\bar{y}(x)$ vanishes at $x=x_0$, we find that

$$\begin{aligned} \frac{|y(x)-\bar{y}(x)|}{x-x_0} &= \frac{|(y(x)-\bar{y}(x))-(y(x_0)-\bar{y}(x_0))|}{x-x_0} \\ &= |y'(x*)-\bar{y}'(x*)| \\ &= |f(x*,y(x*))-f(x*,\bar{y}(x*))|, \end{aligned}$$

where $x_0 < x* < x$. But, $f(x,y)$ is continuous at (x_0, y_0), therefore

$$\lim_{x \to x_0} \frac{|y(x) - \bar{y}(x)|}{x - x_0} = |f(x_0, y_0) - f(x_0, y_0)| = 0,$$

as desired.

Now, suppose contrary to what we wish to prove, that the function $y(x) - \bar{y}(x)$ is not identically zero in $[x_0, x_0 + a]$. Then, there exists a number $x_m > x_0$ such that $v(x)$ attains its positive maximum in $[x_0, x_0 + a]$ at $x = x_m$. Furthermore,

$$\frac{|y(x) - \bar{y}(x)|}{x - x_0} < \frac{|y(x_m) - \bar{y}(x_m)|}{x_m - x_0},$$

whenever $x_0 < x < x_m$. However, by the mean value theorem applied to the function $y(x) - \bar{y}(x)$ in the interval $x_0 \le x \le x_m$, and by Nagumo's condition (1.6.1) satisfied by f, we have

$$\begin{aligned} 0 < \frac{|y(x_m) - \bar{y}(x_m)|}{x_m - x_0} &= \frac{|(y(x_m) - \bar{y}(x_m)) - (y(x_0) - \bar{y}(x_0))|}{x_m - x_0} \\ &= |y'(\hat{x}) - \bar{y}'(\hat{x})| \\ &= |f(\hat{x}, y(\hat{x})) - f(\hat{x}, \bar{y}(\hat{x}))| \\ &\le \frac{|y(\hat{x}) - \bar{y}(\hat{x})|}{\hat{x} - x_0}, \end{aligned}$$

where $\hat{x}$ satisfies $x_0 < \hat{x} < x_m$. The last inequality is in direct contradiction to the way in which x_m was chosen, and so the theorem is proved. ∎

In condition (1.6.1), the constant $k = 1$ is the best possible, i.e., it cannot be replaced by $k > 1$. This is illustrated in the following :

Example 1.6.1. Consider the initial value problem

$$(1.6.3) \qquad y' = f(x,y) = \begin{cases} 0, & 0 \le x \le 1, y \le 0 \\ \dfrac{(1+\varepsilon)y}{x}, & 0 \le x \le 1,\ 0 < y < x^\delta, \\ & \qquad \delta = 1 + \varepsilon > 1 \\ (1+\varepsilon)x^{\delta - 1}, & 0 \le x \le 1, x^\delta \le y \end{cases}$$

$$y(0) = 0.$$

It is easily seen that this function $f(x,y)$ is continuous at $(0,y)$ for $0 < y < x^\delta$ since

$$\left|\frac{(1+\varepsilon)y}{x}\right| < \frac{(1+\varepsilon)x^\delta}{x} \to 0 \quad \text{as} \quad x \to 0^+$$

Hence, $f(x,y)$ is continuous in the strip $0 \le x \le 1, |y| < \infty$. Next, $f(x,y)$ satisfies the condition (1.6.1) in this strip except that $k = 1+\varepsilon > 1$. We shall verify this by considering the following cases :

Suppose $0 < y, \bar{y} < x^\delta$, then

$$|f(x,y) - f(x,\bar{y})| = \left|\frac{(1+\varepsilon)y}{x} - \frac{(1+\varepsilon)\bar{y}}{x}\right| = \frac{(1+\varepsilon)}{x}|y-\bar{y}|.$$

Suppose $0 < y < x^\delta \le \bar{y}$, then

$$\begin{aligned} |f(x,y) - f(x,\bar{y})| &= \left|\frac{(1+\varepsilon)y}{x} - (1+\varepsilon)x^{\delta-1}\right| = \frac{(1+\varepsilon)}{x}|y - x^\delta| \\ &= \frac{(1+\varepsilon)}{x}(x^\delta - y) \le \frac{(1+\varepsilon)}{x}|\bar{y} - y|. \end{aligned}$$

Suppose $y \le 0 < \bar{y} < x^\delta$, then

$$|f(x,y) - f(x,\bar{y})| = \left|0 - \frac{(1+\varepsilon)\bar{y}}{x}\right| \le \frac{(1+\varepsilon)}{x}|y-\bar{y}|.$$

Suppose $y \le 0, x^\delta \le \bar{y}$, then

$$\begin{aligned} |f(x,y) - f(x,\bar{y})| &= |0 - (1+\varepsilon)x^{\delta-1}| = \frac{(1+\varepsilon)}{x}x^\delta \\ &\le \frac{(1+\varepsilon)}{x}\bar{y} \le \frac{(1+\varepsilon)}{x}|y-\bar{y}|. \end{aligned}$$

The initial value problem (1.6.3) has an infinite number of solutions $y(x) = cx^\delta$ in the interval $0 \le x \le 1$, where c is an arbitrary constant such that $0 < c < 1$. ∎

The continuity requirement of the function $f(x,y)$ particularly at the point (x_0, y_0) is also essential for the validity of the uniqueness theorem. For this, we have

Example 1.6.2. Consider the initial value problem

$$(1.6.4) \qquad y' = f(x,y) = \begin{cases} 1, & 0 \le x \le 1, y > x \\ \dfrac{y}{x}, & 0 \le x \le 1, 0 < y \le x \\ 0, & 0 \le x \le 1, y \le 0 \end{cases}$$
$$y(0) = 0.$$

This function $f(x,y)$ satisfies Nagumo's condition (1.6.1) in the strip $0 \le x \le 1, |y| < \infty$. For this, we look at the following cases :

Suppose $0 < y, \bar{y} \le x$, then

$$|f(x,y) - f(x,\bar{y})| = \left|\frac{y}{x} - \frac{\bar{y}}{x}\right| = \frac{1}{x}|y - \bar{y}|.$$

Suppose $y > x \ge \bar{y} > 0$, then

$$|f(x,y) - f(x,\bar{y})| = \left|1 - \frac{\bar{y}}{x}\right| = \frac{1}{x}(x - \bar{y}) < \frac{1}{x}(y - \bar{y}) = \frac{1}{x}|y - \bar{y}|.$$

Suppose $y > x, \bar{y} \le 0$, then

$$|f(x,y) - f(x,\bar{y})| = |1 - 0| = \frac{x}{x} < \frac{1}{x}y \le \frac{1}{x}|y - \bar{y}|.$$

Suppose $0 < y \le x, \bar{y} \le 0$, then

$$|f(x,y) - f(x,\bar{y})| = \left|\frac{y}{x} - 0\right| \le \frac{1}{x}|y - \bar{y}|.$$

Since for $y > x$, $\lim_{(x,y)\to(0^+,0)} f(x,y) = 1$; and for $0 < y \le x$, $\lim_{\substack{y=x \\ x\to 0^+}} f(x,y) = \lim_{\substack{y=x \\ x\to 0^+}} \frac{y}{x} = 1$, but $f(0,0) = 0$ this function $f(x,y)$ is not continuous at $(0,0)$.

The initial value problem (1.6.4) has an infinite number of solutions $y(x) = cx$ in the interval $0 \le x \le 1$, where c is an arbitrary constant such that $0 \le c \le 1$. ∎

Once again, Nagumo's condition is only a sufficient condition to prove the uniqueness of the solutions of the initial value problems. To show this, we have

Example 1.6.3. Consider the initial value problem

$$
(1.6.5) \qquad y' = f(x,y) = \begin{cases} 0, & -\infty < x \le 0, |y| < \infty \\ 2x, & 0 < x \le 1, -\infty < y < 0 \\ 2x - 4\dfrac{y}{x}, & 0 < x \le 1, 0 \le y \le x^2 \\ -2x, & 0 < x \le 1, x^2 < y < \infty \end{cases}
$$

$$
y(0) = 0.
$$

This function $f(x,y)$ is continuous at $(0,y)$ for $0 \le y \le x^2$ since

$$
|2x - 4y/x| \le 2x + 4x^2/x = 6x \to 0 \quad \text{as} \quad x \to 0.
$$

Hence, $f(x,y)$ is continuous in the region $-\infty < x \le 1, |y| < \infty$.

Next, we shall verify that $f(x,y)$ satisfies condition (1.6.1) except that $k = 4$. Once again, it suffices to consider the following cases :

Suppose $0 \le y, \bar{y} \le x^2$, then

$$
|f(x,y) - f(x,\bar{y})| = \left| -\frac{4y}{x} + \frac{4\bar{y}}{x} \right| = \frac{4}{x}|y - \bar{y}|.
$$

Suppose $-\infty < y < 0 \le \bar{y} \le x^2$, then

$$
|f(x,y) - f(x,\bar{y})| = \left| 2x - 2x + \frac{4\bar{y}}{x} \right| < \frac{4}{x}|y - \bar{y}|.
$$

Suppose $-\infty < y < 0, x^2 < \bar{y} < \infty$, then

$$
|f(x,y) - f(x,\bar{y})| = 4x < \frac{4}{x}\bar{y} \le \frac{4}{x}|y - \bar{y}|.
$$

Suppose $0 \le y \le x^2 < \bar{y} < \infty$, then

$$
|f(x,y) - f(x,\bar{y})| = \left| 2x + 2x - \frac{4y}{x} \right| = \frac{4}{x}(x^2 - y) < \frac{4}{x}|y - \bar{y}|.
$$

The initial value problem (1.6.5) has a unique solution

$$
y(x) = \begin{cases} 0, & -\infty \le x \le 0 \\ x^2/3, & 0 < x \le 1. \end{cases} \qquad \blacksquare
$$

Another example where the function $f(x,y)$ satisfies the conditions of Theorem 1.6.2 except that $k = 1 + \varepsilon$, but has a unique solution is the following :

Example 1.6.4. The initial value problem

$$(1.6.6)\qquad y' = f(x,y) = \begin{cases} 0, & x = 0, |y| < \infty \\ x^{\varepsilon}, & 0 < x \le 1, -\infty < y < 0 \\ x^{\varepsilon} - (1+\varepsilon)\dfrac{y}{x}, & 0 < x \le 1, 0 \le y \le x^{1+\varepsilon} \\ -\varepsilon x^{\varepsilon}, & 0 < x \le 1, x^{1+\varepsilon} < y < \infty \end{cases}$$
$$y(0) = 0$$

has a unique solution $y(x) = \frac{1}{2(1+\varepsilon)}x^{1+\varepsilon}, 0 \le x \le 1.$ ∎

In our final example here the function $f(x,y)$ satisfies all the conditions of Theorem 1.6.2, and hence has at most one solution.

Example 1.6.5. For the initial value problem

$$(1.6.7)\qquad y' = f(x,y) = \begin{cases} 0, & x = 0, |y| < \infty \\ \frac{1}{2}x^{\varepsilon}, & 0 < x \le 1, -\infty < y < 0 \\ \frac{1}{2}x^{\varepsilon} - \dfrac{y}{x}, & 0 < x \le 1, 0 \le y \le x^{1+\varepsilon} \\ -\frac{1}{2}x^{\varepsilon}, & 0 < x \le 1, x^{1+\varepsilon} < y < \infty \end{cases}$$
$$y(0) = 0,$$

$y(x) = \frac{1}{2(2+\varepsilon)}x^{1+\varepsilon}, 0 \le x \le 1$ is the only solution. ∎

A one - sided analog of Nagumo's uniqueness theorem is the following :

Theorem 1.6.3. Let $f(x,y)$ be continuous in $\bar{S}_+$ and for all $(x,y), (x,\bar{y})$ in $\bar{S}_+$ with $\bar{y} \ge y$ it satisfies **one sided Nagumo condition**

$$(1.6.8)\qquad f(x,\bar{y}) - f(x,y) \le k(x-x_0)^{-1}(\bar{y}-y);\ x \ne x_0,\ k \le 1.$$

Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0+a]$. ∎

1.7 KRASNOSEL'SKII KREIN'S UNIQUENESS THEOREM

In the previous section we have seen that in condition (1.6.1) the Nagumo constant k must not be greater than one is a necessary condition. Thus, if k

is allowed to be any constant greater then zero, then an additional condition must be imposed on the function $f(x, y)$ to achieve the uniqueness of the solutions of the initial value problem (1.1.1). One such condition is added in the following :

Theorem 1.7.1 (Krasnosel'skii Krein's Uniqueness Theorem). Let $f(x, y)$ be continuous in $\bar{S}$ and for all $(x, y), (x, \bar{y}) \in \bar{S}$ it satisfies

$$|f(x,y) - f(x,\bar{y})| \leq k|x - x_0|^{-1}|y - \bar{y}|, \ x \neq x_0, k > 0 \tag{1.7.1}$$

and the Hölder condition

$$|f(x,y) - f(x,\bar{y})| \leq c|y - \bar{y}|^{\alpha}, \ c > 0, 0 < \alpha < 1, k(1-\alpha) < 1. \tag{1.7.2}$$

($k(1-\alpha) < 1$ is no restriction at all, when $k \leq 1$.)

Then, the initial value problem (1.1.1) has at most one solution in $|x - x_0| \leq a$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in $|x - x_0| \leq a$. We shall show that $y(x) = \bar{y}(x)$ in the interval $[x_0, x_0 + a]$. We let $\phi(x) = |y(x) - \bar{y}(x)|$. For this, from (1.7.2), we have

$$\begin{aligned} \phi(x) &\leq \int_{x_0}^{x} |f(t, y(t)) - f(t, \bar{y}(t))| dt \\ &\leq \int_{x_0}^{x} c|y(t) - \bar{y}(t)|^{\alpha} dt = \int_{x_0}^{x} c\phi^{\alpha}(t) dt. \end{aligned}$$

Now, we let $R(x) = \int_{x_0}^{x} c\phi^{\alpha}(t)dt$, then $R(x_0) = 0$, and $R'(x) = c\phi^{\alpha}(x) \leq cR^{\alpha}(x)$, which gives $R'(x) - cR^{\alpha}(x) \leq 0$. Since $R(x) > 0$ for $x > x_0$, on multiplying this inequality by $(1-\alpha)R^{-\alpha}(x)$, we obtain $(R^{1-\alpha}(x))' \leq c(1-\alpha) \leq c$, and hence $R^{1-\alpha}(x) \leq c(x - x_0)$. Thus, it follows that $\phi(x) \leq (c(x - x_0))^{(1-\alpha)^{-1}}$. Hence, the function $\psi(x) = \frac{\phi(x)}{(x-x_0)^k}$ satisfies the inequality

$$0 \leq \psi(x) \leq c^{(1-\alpha)^{-1}}(x - x_0)^{(1-\alpha)^{-1}-k}.$$

Since $k(1-\alpha) < 1$, it is immediate that $\lim_{x \to x_0^+} \psi(x) = 0$. Therefore, if we define $\psi(x_0) = 0$, then the function $\psi(x)$ is continuous in $[x_0, x_0 + a]$. We wish to show that $\psi(x) = 0$ in $[x_0, x_0 + a]$. If $\psi(x) > 0$ at any point in $[x_0, x_0 + a]$, then there exists a point $x_1 > x_0$ such that $0 < m = \psi(x_1) = \max_{x_0 \le x \le x_0+a} \psi(x)$. However, from (1.7.1) we have

$$\begin{aligned} m = \psi(x_1) = \frac{\phi(x_1)}{(x_1 - x_0)^k} &\le (x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{-1} \phi(t)dt \\ &\le (x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{k-1} \psi(t)dt \\ &< m(x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{k-1} dt \\ &= m, \end{aligned}$$

which is the desired contradiction. So, $\psi(x) \equiv 0$, and hence $\phi(x) = 0$ in $[x_0, x_0 + a]$. The proof is similar in the interval $[x_0 - a, x_0]$. ∎

We shall illustrate the above theorem with an example.

Example 1.7.1. Consider the initial value problem

$$(1.7.3) \quad \begin{aligned} y' = f(x,y) &= \begin{cases} 0, & 0 \le x \le 1, x^{(1-\alpha)^{-1}} < y < \infty \\ kx^{\alpha(1-\alpha)^{-1}} - k\dfrac{y}{x}, & 0 \le x \le 1, 0 \le y \le x^{(1-\alpha)^{-1}} \\ kx^{\alpha(1-\alpha)^{-1}}, & 0 \le x \le 1, -\infty < y < 0 \end{cases} \\ y(0) &= 0, \end{aligned}$$

where $0 < \alpha < 1, k > 0$ and $k(1 - \alpha) < 1$.

This function $f(x, y)$ is continuous at $(0, y)$ for $0 \le y \le x^{(1-\alpha)^{-1}}$ since

$$\left| kx^{\alpha(1-\alpha)^{-1}} - k\frac{y}{x} \right| \le kx^{\alpha(1-\alpha)^{-1}} + \frac{k}{x} x^{(1-\alpha)^{-1}} = 2kx^{\alpha(1-\alpha)^{-1}} \to 0 \text{ as } x \to 0^+.$$

Thus, it is clear that $f(x, y)$ is continuous in the strip $0 \le x \le 1, |y| < \infty$.

Next, we shall verify the conditions (1.7.1) and (1.7.2) by considering the following cases :

Suppose $0 \leq y, \bar{y} \leq x^{(1-\alpha)^{-1}}$, then

$$\begin{aligned}
|f(x,y) - f(x,\bar{y})| &= \left|-k\frac{y}{x} + k\frac{\bar{y}}{x}\right| \\
&= \frac{k}{x}|y - \bar{y}|; \text{ and} \\
|f(x,y) - f(x,\bar{y})| &= \frac{k}{x}|y - \bar{y}|^{1-\alpha}|y - \bar{y}|^{\alpha} \\
&\leq \frac{k}{x}(|y| + |\bar{y}|)^{1-\alpha}|y - \bar{y}|^{\alpha} \leq \frac{k}{x}\ 2^{1-\alpha}x|y - \bar{y}|^{\alpha} \\
&= 2^{1-\alpha}k|y - \bar{y}|^{\alpha}.
\end{aligned}$$

Suppose $x^{(1-\alpha)^{-1}} < y < \infty, -\infty < \bar{y} < 0$, then

$$\begin{aligned}
|f(x,y) - f(x,\bar{y})| &= |-kx^{\alpha(1-\alpha)^{-1}}| < \frac{k}{x}y \\
&< \frac{k}{x}|y - \bar{y}|; \text{ and} \\
|f(x,y) - f(x,\bar{y})| &= kx^{\alpha(1-\alpha)^{-1}} < ky^{\alpha} < k|y - \bar{y}|^{\alpha} \\
&< 2^{1-\alpha}k|y - \bar{y}|^{\alpha}
\end{aligned}$$

Suppose $x^{(1-\alpha)^{-1}} < y < \infty, 0 \leq \bar{y} \leq x^{(1-\alpha)^{-1}}$, then

$$\begin{aligned}
|f(x,y) - f(x,\bar{y})| &= \left|-kx^{\alpha(1-\alpha)^{-1}} + k\frac{\bar{y}}{x}\right| = \frac{k}{x}\left|x^{(1-\alpha)^{-1}} - \bar{y}\right| \\
&< \frac{k}{x}|y - \bar{y}|; \text{ and} \\
|f(x,y) - f(x,\bar{y})| &= k\left[\frac{x^{(1-\alpha)^{-1}} - \bar{y}}{(x^{(1-\alpha)^{-1}})^{(1-\alpha)}}\right] \\
&\leq k\left[x^{(1-\alpha)^{-1}} - \bar{y}\right]^{\alpha} < k(y - \bar{y})^{\alpha} \\
&< 2^{1-\alpha}k|y - \bar{y}|^{\alpha}.
\end{aligned}$$

Suppose $0 \le y \le x^{(1-\alpha)^{-1}}, -\infty < \bar{y} < 0$, then

$$\begin{aligned}|f(x,y) - f(x,\bar{y})| &= \left|kx^{(1-\alpha)^{-1}} - k\frac{y}{x} - kx^{(1-\alpha)^{-1}}\right| = k\frac{y}{x} \\ &< \frac{k}{x}|y-\bar{y}|; \text{ and} \\ |f(x,y) - f(x,\bar{y})| &= k\frac{y}{x} \le ky^{\alpha} \\ &< 2^{1-\alpha}k|y-\bar{y}|^{\alpha}.\end{aligned}$$

Since all the conditions of Theorem 1.7.1 are satisfied, the initial value problem (1.7.3) has a unique solution in $[0,1]$, namely

$$y(x) = \frac{k(1-\alpha)}{1+k(1-\alpha)}x^{(1-\alpha)^{-1}}. \qquad \blacksquare$$

For the uniqueness of the solutions of (1.1.1) in Theorem 1.7.1 the condition $k(1-\alpha) < 1$ is not necessary. For this, it suffices to note that the function $f(x,y)$ in (1.7.3) is nonincreasing in y for each x. Thus, by Theorem 1.3.1 the problem (1.7.3) has at most one solution.

1.8 KOOI'S UNIQUENESS THEOREM

A generalization of Krasnosel'skii - Krein's uniqueness theorem is the following :

Theorem 1.8.1 (Kooi's Uniqueness Theorem). Let $f(x,y)$ be continuous in $\bar{S}$ and for all $(x,y), (x,\bar{y}) \in \bar{S}$ it satisfies

$$(1.8.1) \qquad |x-x_0||f(x,y)-f(x,\bar{y})| \le k|y-\bar{y}|, \; k>0; \text{ and}$$

$$(1.8.2) \qquad |x-x_0|^{\beta}|f(x,y)-f(x,\bar{y})| \le c|y-\bar{y}|^{\alpha}, \; c>0$$

where the constants k, α and β satisfy the inequalities

$$(1.8.3) \qquad 0<\alpha<1, \; \beta<\alpha \quad \text{and} \quad k(1-\alpha) < 1-\beta.$$

Then, the initial value problem (1.1.1) has at most one solution in $|x-x_0| \le a$.

Proof. The proof is similar to that of Theorem 1.7.1. ∎

Example 1.8.1. Consider the initial value problem

$$(1.8.4) \qquad y' = f(x,y) = \begin{cases} 0, & 0 \le x \le 1, \\ & x^{(1-\beta)(1-\alpha)^{-1}} < y < \infty \\ kx^{(\alpha-\beta)(1-\alpha)^{-1}} - k\dfrac{y}{x}, & 0 \le x \le 1, \\ & 0 \le y \le x^{(1-\beta)(1-\alpha)^{-1}} \\ kx^{(\alpha-\beta)(1-\alpha)^{-1}}, & 0 \le x \le 1, -\infty < y < 0 \end{cases}$$
$$y(0) = 0,$$

where the constants k, α and β satisfy the inequalities (1.8.3). As in Example 1.7.1 it is easily seen that this function $f(x,y)$ is continuous in the strip $0 \le x \le 1, |y| < \infty$ and satisfies the inequalities (1.8.1),(1.8.2). Thus, the initial value problem (1.8.4) has a unique solution $y(x) = \frac{\gamma}{1+\gamma}x^{(1-\beta)(1-\alpha)^{-1}}$, where $\gamma = k(1-\alpha)(1-\beta)^{-1}$. ∎

From Example 1.8.1 it is clear that for the conclusion of Theorem 1.8.1 also the condition $k(1-\alpha) < 1-\beta$ is not necessary.

The next theorem we shall prove is based on Kooi's work.

Theorem 1.8.2. Let $f(x,y)$ be continuous in $\bar{S}$ and

$$(1.8.5) \qquad |f(x,y)| \le A|x-x_0|^p; \; p > -1, A > 0.$$

Further, let for all $(x,y), (x,\bar{y}) \in \bar{S}$ it satisfy

$$(1.8.6) \qquad |f(x,y) - f(x,\bar{y})| \le \frac{c}{|x-x_0|^r}|y-\bar{y}|^q; \; q \ge 1, c > 0$$

where

$$(1.8.7) \qquad q(1+p) - r = p, \quad \text{and} \quad \rho = \frac{c(2A)^{q-1}}{(p+1)^q} < 1.$$

Then, the initial value problem (1.1.1) has at most one solution in $|x-x_0| \le a$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in $|x-x_0| \le a$. We shall show that $y(x) = \bar{y}(x)$ in $[x_0 - a, x_0]$. For this, from (1.8.5) we have

$$\begin{aligned} \phi(x) = |y(x) - \bar{y}(x)| \ &\le \int_x^{x_0} |f(t,y(t)) - f(t,\bar{y}(t))| dt \\ &\le 2A \int_x^{x_0} (x_0 - t)^p dt \\ &= \frac{2A}{p+1}(x_0 - x)^{p+1}. \end{aligned}$$

Using this estimate and (1.8.6), we obtain

$$\begin{aligned} \phi(x) \ &\le c\int_x^{x_0} \frac{1}{(x_0-t)^r}\phi^q(t)dt \\ &\le c\left(\frac{2A}{p+1}\right)^q \int_x^{x_0} (x_0-t)^{q(p+1)-r}dt \\ &= \rho\left(\frac{2A}{p+1}\right)(x_0-x)^{p+1}. \end{aligned}$$

Now on using this new estimate and (1.8.6), we get

$$\begin{aligned} \phi(x) \ &\le c\rho^q\left(\frac{2A}{p+1}\right)^q \int_x^{x_0} (x_0-t)^{q(p+1)-r}dt \\ &= \rho^{1+q}\left(\frac{2A}{p+1}\right)(x_0-x)^{p+1}. \end{aligned}$$

Continuing in this way, we find that

$$\phi(x) \le \rho^{1+q+q^2+\cdots+q^m}\left(\frac{2A}{p+1}\right)(x_0-x)^{p+1};\ m = 1,2,\cdots .$$

Since $q \ge 1$ and $\rho < 1$, it follows that $\phi(x) = 0$ in $[x_0 - a, x_0]$. The same is true for $x \in [x_0, x_0 + a]$. ∎

Example 1.8.2. Consider the initial value problem (1.7.3). The function $f(x,y)$ in (1.7.3) satisfies (1.8.5) and (1.8.6) with $A = 2k, p = \alpha(1-\alpha)^{-1}, c = k, r = q = 1$, and therefore $q(1+p) - r = p$ and $\rho = \frac{c(2A)^{q-1}}{(p+1)^q} = k(1-\alpha) < 1$. Thus, from Theorem 1.8.2 also we can conclude that the problem (1.7.3) has at most one solution. We also note that for this problem the conditions of

Theorem 1.8.2 are also satisfied with $A = 2k, p = \alpha(1-\alpha)^{-1}, c = 2^{1-\alpha}k, q = \alpha$ and $r = 0$. ∎

1.9 ROGER'S UNIQUENESS THEOREM

Nagumo's condition (1.6.1) for $x \geq x_0 = 0$ is the same as

$$(1.9.1) \qquad |f(x,y) - f(x,\bar{y})| \leq \frac{1}{x}|y - \bar{y}|, \ x \neq 0.$$

In this section, we shall establish a uniqueness criterion when $\frac{1}{x}$ is replaced by $\frac{1}{x^2}$ in the above condition. For this, we require the following :

Lemma 1.9.1. If

(i) $\phi(x)$ is continuous and nonnegative in $[0, a]$,

(ii) $\phi(x) \leq \int_0^x \frac{1}{t^2}\phi(t)dt$; and

(iii) $\phi(x) = o(e^{-1/x})$, as $x \to 0$,

then $\phi(x) \equiv 0$.

Proof. Let $\psi(x) = \int_0^x \frac{1}{t^2}\phi(t)dt$. Differentiating $\psi(x)$ with respect to x and using (ii), we obtain for $x > 0, \psi'(x) = \frac{1}{x^2}\phi(x) \leq \frac{1}{x^2}\psi(x)$, which is the same as $(e^{1/x}\psi(x))' \leq 0$, so that $e^{1/x}\psi(x)$ is nonincreasing. If $\varepsilon > 0$, then from (iii), we have for small x, that

$$e^{1/x}\psi(x) = e^{1/x}\int_0^x \frac{1}{t^2}\phi(t)dt \leq e^{1/x}\int_0^x \frac{1}{t^2}\varepsilon e^{-1/t}dt = \varepsilon.$$

Hence, $\lim_{x\to 0^+} e^{1/x}\psi(x) = 0$, and this implies that $e^{1/x}\psi(x) \leq 0$ for $x > 0$. This gives $\psi(x) \leq 0$. However, $\psi(x)$ is nonnegative, and thus $\psi(x) \equiv 0$. The result now follows from (i). ∎

Theorem 1.9.2 (Roger's Uniqueness Theorem). Let $f(x,y)$ be continuous in

$$\bar{\Delta} : 0 \leq x \leq a, \ |y| < \infty$$

and satisfy the condition

$$f(x,y) = o(e^{-1/x}x^{-2}) \tag{1.9.2}$$

uniformly for $0 \le y \le \delta$, $\delta > 0$ arbitrary. Further, let

$$|f(x,y) - f(x,\bar{y})| \le \frac{1}{x^2}|y - \bar{y}|, \ x \neq 0 \tag{1.9.3}$$

for all $(x,y), (x,\bar{y}) \in \bar{\Delta}$. Then, the initial value problem

$$y' = f(x,y), \ y(0) = 0 \tag{1.9.4}$$

has at most one solution in $[0,a]$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.9.4) in $[0,a]$. Then, from (1.2.4) and (1.9.3), we get

$$|y(x) - \bar{y}(x)| \le \int_0^x \frac{1}{t^2}|y(t) - \bar{y}(t)|dt.$$

Also, if $\varepsilon > 0$, then from (1.2.4) and (1.9.2) for small x, we have

$$|y(x) - \bar{y}(x)| \le 2\varepsilon \int_0^x e^{-1/t}t^{-2}dt = 2\varepsilon e^{-1/x}.$$

Now as an application of Lemma 1.9.1, we find that $|y(x) - \bar{y}(x)| \equiv 0$, and hence $y(x) = \bar{y}(x)$ in $[0,a]$. ∎

We shall illustrate Theorem 1.9.2 with an example.

Example 1.9.1. Consider the initial value problem

$$y' = f(x,y) = \begin{cases} \left(1 + \dfrac{1}{x}\right) e^{-1/x}, & 0 \le x \le 1, xe^{-1/x} \le y < \infty \\ \dfrac{y}{x^2} + e^{-1/x}, & 0 \le x \le 1, 0 \le y \le xe^{-1/x} \\ e^{-1/x}, & 0 \le x \le 1, -\infty < y \le 0 \end{cases} \tag{1.9.5}$$

$$y(0) = 0.$$

Since $\lim_{x \to 0^+} e^{-1/x} = 0$, and $\lim_{x \to 0^+} \frac{1}{x}e^{-1/x} = 0$, and for $0 \le y \le xe^{-1/x}$,

$$\left|\frac{y}{x^2} + e^{-1/x}\right| \le \frac{xe^{-1/x}}{x^2} + e^{-1/x} \to 0 \quad \text{as} \quad x \to 0^+$$

it is clear that this function $f(x,y)$ is continuous at $(0,y)$ for $|y| < \infty$, and hence it is continuous in the strip $0 \le x \le 1, |y| < \infty$.

Next, we shall verify that $f(x,y)$ satisfies condition (1.9.2). For this, since

$$\frac{f(x,y)}{e^{-1/x}x^{-2}} = \begin{cases} x + x^2, & 0 < x \le 1, xe^{-1/x} \le y < \infty \\ ye^{1/x} + x^2, & 0 < x \le 1, 0 \le y \le xe^{-1/x} \\ x^2, & 0 < x \le 1, -\infty < y \le 0 \end{cases} \tag{1.9.6}$$

and for $0 \le y \le xe^{-1/x}$,

$$|ye^{1/x} + x^2| \le (xe^{-1/x})e^{1/x} + x^2 = x + x^2 \to 0 \quad \text{as} \quad x \to 0^+,$$

we find that each expression on the right of the equation (1.9.6) tends to zero as x tends to zero. Thus, (1.9.2) holds.

The condition (1.9.3) can be verified by considering the following cases :
Suppose $0 \le y, \bar{y} \le xe^{-1/x}$, then

$$|f(x,y) - f(x,\bar{y})| = \left|\frac{y}{x^2} - \frac{\bar{y}}{x^2}\right| = \frac{1}{x^2}|y - \bar{y}|.$$

Suppose $xe^{-1/x} \le y < \infty, 0 \le \bar{y} \le xe^{-1/x}$, then

$$\begin{aligned} |f(x,y) - f(x,\bar{y})| &= \left|\frac{e^{-1/x}}{x} - \frac{\bar{y}}{x^2}\right| = \frac{1}{x^2}(xe^{-1/x} - \bar{y}) \\ &\le \frac{1}{x^2}(y - \bar{y}) = \frac{1}{x^2}|y - \bar{y}|. \end{aligned}$$

Suppose $xe^{-1/x} \le y < \infty, -\infty < \bar{y} \le 0$, then

$$|f(x,y) - f(x,\bar{y})| = \frac{1}{x^2}xe^{-1/x} \le \frac{1}{x^2}y \le \frac{1}{x^2}|y - \bar{y}|.$$

Suppose $0 \le y \le xe^{-1/x}, -\infty < \bar{y} \le 0$, then

$$|f(x,y) - f(x,\bar{y})| = \frac{1}{x^2}y \le \frac{1}{x^2}|y - \bar{y}|.$$

Thus, all conditions of Theorem 1.9.2 are satisfied, and hence in conclusion this initial value problem has a unique solution in $[0,1]$, namely, $y(x) = xe^{-1/x}$ ∎

In Theorem 1.9.2 the assumption $f(x,y) = o(e^{-1/x}x^2)$ is necessary. For this, we have the following :

Example 1.9.2. Consider the initial value problem

$$(1.9.7) \qquad y' = f(x,y) = \begin{cases} 0, & 0 \le x \le 1, -\infty < y \le 0 \\ \dfrac{y}{x^2}, & 0 \le x \le 1, 0 \le y \le e^{-1/x} \\ \dfrac{1}{x^2}e^{-1/x}, & 0 \le x \le 1, e^{-1/x} \le y < \infty \end{cases}$$

$$y(0) = 0.$$

Since $\lim_{x\to 0^+} \frac{1}{x^2}e^{-1/x} = 0$, it follows that for $0 \le y \le e^{-1/x}$, $\left|\frac{y}{x^2}\right| \le \frac{1}{x^2}e^{-1/x} \to 0$ as $x \to 0^+$. Thus, it is clear that this function $f(x,y)$ is continuous in the strip $0 \le x \le 1, |y| < \infty$.

As usual, we shall verify condition (1.9.3) by considering the following cases :

Suppose $0 \le y, \bar{y} \le e^{-1/x}$, then

$$|f(x,y) - f(x,\bar{y})| = \left|\frac{y}{x^2} - \frac{\bar{y}}{x^2}\right| = \frac{1}{x^2}|y - \bar{y}|.$$

Suppose $-\infty < y \le 0, 0 \le \bar{y} \le e^{-1/x}$, then

$$|f(x,y) - f(x,\bar{y})| = \left|-\frac{\bar{y}}{x^2}\right| = \frac{1}{x^2}|y - \bar{y}|.$$

Suppose $-\infty < y \le 0, e^{-1/x} \le \bar{y} < \infty$, then

$$|f(x,y) - f(x,\bar{y})| = \left|-\frac{1}{x^2}e^{-1/x}\right| \le \frac{1}{x^2}\bar{y} \le \frac{1}{x^2}|y - \bar{y}|.$$

Suppose $0 \le y \le e^{-1/x}, e^{-1/x} \le \bar{y} < \infty$, then

$$\begin{aligned} |f(x,y) - f(x,\bar{y})| &= \left|\frac{y}{x^2} - \frac{e^{-1/x}}{x^2}\right| = \frac{1}{x^2}(e^{-1/x} - y) \\ &\le \frac{1}{x^2}(\bar{y} - y) \le \frac{1}{x^2}|y - \bar{y}|. \end{aligned}$$

This function $f(x,y)$ does not satisfy the condition (1.9.2). Indeed, for $e^{-1/x} \le y < \infty, \lim_{x\to 0^+} \frac{f(x,y)}{e^{-1/x}x^{-2}} \ne 0$.

The initial value problem (1.9.7) has an infinite number of solutions $y(x) = ce^{-1/x}$ in $[0,1]$, where c is an arbitrary constant such that $0 \le c \le 1$. ∎

1.10 WITTE'S UNIQUENESS THEOREM

In the year 1974 the following result was established.

Theorem 1.10.1 (Witte's Uniqueness Theorem). Let $f(x,y)$ be continuous in

$$\Delta : 0 < x \le a,\ |y| < \infty$$

and satisfy the conditions

$$(1.10.1) \qquad |f(x,y) - f(x,\bar{y})| \le h(x)|y-\bar{y}|,\ (x,y),(x,\bar{y}) \in \Delta;\ \text{and}$$

$$(1.10.2) \qquad |f(x,y)| \le \rho(x)h(x)\exp\left(\int_a^x h(t)dt\right) \quad \text{in} \quad \Delta,$$

where $h(x) > 0$ is continuous in $(0,a]$ and $\rho(x)$ is continuous in $[0,a]$ with $\rho(0) = 0$. Then, the initial value problem (1.1.1) with $x_0 = 0$ has at most one solution in $[0,a]$.

We shall generalize this result slightly. For this, we need the following :

Lemma 1.10.2. Let $\phi(x)$ be a nonnegative continuous function in $[0,a]$ and let

(i) $h(x) > 0$ be a continuous function in $(0,a]$,

(ii) there exist a function $H(x)$ in $(0,a]$ such that $H'(x) = h(x)$ for almost all $x \in (0,a]$, and $\lim_{x\to 0^+} H(x)$ exists (finite or infinite),

(iii) $\phi(x) \le \int_0^x h(t)\phi(t)dt,\ x \in [0,a];$ and

(iv) $\phi(x) = o(\exp(H(x)))$ as $x \to 0^+$.

Then, $\phi(x) = 0$ in $[0, a]$.

Proof. Let $\psi(x) = \int_0^x h(t)\phi(t)dt$, $x \in [0, a]$. The existence and the continuity of $\psi(x)$ is clear from the given hypotheses. Thus, it follows that

$$\psi'(x) = h(x)\phi(x) \le h(x)\psi(x), \ x \in (0, a]$$

and hence

$$\frac{d}{dx}(e^{-H(x)}\psi(x)) \le 0, \text{ for almost all } x \in [0, a].$$

This allows us to deduce that the function $\exp(-H(x))\psi(x)$ is nonincreasing. Hence, choosing $\varepsilon > 0$ and taking x sufficiently small, in view of (iv) we get

$$\begin{aligned} e^{-H(x)}\psi(x) = &\ e^{-H(x)} \int_0^x h(t)\phi(t)dt \\ &\le \varepsilon e^{-H(x)} \int_0^x h(t)e^{H(t)}dt \\ &\le \varepsilon e^{-H(x)} e^{H(x)} \\ &= \varepsilon, \end{aligned}$$

so that $\lim_{x \to 0^+} \exp(-H(x))\psi(x) = 0$. Consequently, $\exp(-H(x))\psi(x) \le 0$ for $x > 0$, which implies that $\int_0^x h(t)\phi(t)dt \le 0$. Therefore, $\phi(x) \equiv 0$. ∎

If $h(x)$ is continuous in $[0, a]$ and $H(x) = \int_0^x h(t)dt$, then Lemma 1.10.2 is the same as Lemma 1.2.3. Taking $h(x) = x^{-1}$ and $H(x) = \ln x$ so that condition (iv) becomes $\phi(x) = o(x)$ as $x \to 0^+$. Thus, in this case Lemma 1.10.2 reduces to Lemma 1.6.1. Further, we note that in the case $h(x) = x^{-1-\alpha}$, $\alpha > 0$ and $H(x) = -\alpha^{-1}x^{-\alpha}$ the condition (iv) has the form $\phi(x) = o(\exp(-\alpha^{-1}x^{-\alpha}))$ as $x \to 0^+$. Thus, for $\alpha = 1$ we obtain Lemma 1.9.1. Finally, consider $h(x) = kx^{-1}$, where $k > 1$ and $H(x) = x^k$ so that condition (iv) has the form $\phi(x) = o(x^k)$. Thus, Lemma 1.10.2 also covers situations where Lemma 1.6.1 is not applicable.

In view of the above remarks the following result not only unifies Theorems 1.2.4, 1.6.2 and 1.9.2 but also provides new uniqueness criterion.

Theorem 1.10.3. Let $f(x, y)$ be continuous in Δ and in addition to (1.10.1) it satisfies the condition

$$(1.10.3) \quad |f(x,y) - f(x,\bar{y})| = o(h(x)\exp(H(x))) \text{ as } x \to 0^+ \text{ uniformly with respect to } y, \bar{y} \in [-\delta, \delta],\ \delta > 0 \text{ arbitrary,}$$

where $h(x)$ and $H(x)$ are such as in Lemma 1.10.2. Then, the initial value problem (1.9.4) has at most one solution in $[0, a]$.

Proof. As usual let $y(x)$ and $\bar{y}(x)$ be two solutions of (1.9.4). Then, by (1.10.1), we have

$$|y(x) - \bar{y}(x)| \le \int_0^x h(t)|y(t) - \bar{y}(t)|dt.$$

Further, for an arbitrary $\varepsilon > 0$ and x sufficiently small, from (1.10.3) we get

$$|y(x) - \bar{y}(x)| \le \varepsilon \int_0^x h(t)e^{H(t)}dt \le \varepsilon e^{H(x)}.$$

The result is now an immediate consequence of Lemma 1.10.2. ∎

It is clear that (1.10.2) implies (1.10.3), however the converse is not true. For this, we illustrate the following :

Example 1.10.1. Consider the initial value problem

$$(1.10.4) \qquad y' = f(x,y) = \begin{cases} \left(1 + \dfrac{1}{x}\right) e^{-1/x} + 1, & 0 \le x \le 1, \\ & xe^{-1/x} \le y < \infty \\ \dfrac{y}{x^2} + e^{-1/x} + 1, & 0 \le x \le 1, \\ & 0 \le y \le xe^{-1/x} \\ e^{-1/x} + 1, & 0 \le x \le 1, -\infty < y \le 0 \end{cases}$$
$$y(0) = 0.$$

From Example 1.9.1 it is clear that this function $f(x,y)$ is continuous in the strip $0 \le x \le 1, |y| < \infty$. Further, it satisfies the conditions (1.10.1) and (1.10.3) with $h(x) = x^{-2}$ and $H(x) = -x^{-1}$. The problem (1.10.4) has a unique solution $y(x) = x(1 + e^{-1/x})$. It is of interest to note that for this function $f(x,y)$ the condition (1.9.2) of Roger's uniqueness theorem as well as condition (1.10.2) of Witte's uniqueness theorem is not satisfied.

Corollary 1.10.4. In Theorem 1.10.3 conditions (1.10.1) and (1.10.3) can be replaced by the following one - sided conditions

(1.10.5) $(f(x,y) - f(x,\bar{y}))(y - \bar{y}) \le h(x)(y - \bar{y})^2$, $(x,y),(x,\bar{y}) \in \Delta$; and

(1.10.6) $f(x,y) - f(x,\bar{y}) = o(h(x)\exp(H(x)))$ as $x \to 0^+$ uniformly with respect to $y, \bar{y} \in [-\delta, \delta]$, $\delta > 0$ arbitrary.

Proof. Let $y(x)$ and $\bar{y}(x)$ be two solutions of (1.9.4). We define $v(x) = (y(x) - \bar{y}(x))^2$ so that in view of (1.10.5) it follows that

$$\begin{aligned} v'(x) &= 2(y(x) - \bar{y}(x))(y'(x) - \bar{y}'(x)) \\ &= 2(y(x) - \bar{y}(x))(f(x, y(x)) - f(x, \bar{y}(x))) \\ &\le 2h(x)(y(x) - \bar{y}(x))^2 \\ &= 2h(x)v(x). \end{aligned}$$

Thus, we have $v'(x) - 2h(x)v(x) \le 0$, and consequently

$$\frac{d}{dx}(\exp(-2H(x))v(x)) \le 0$$

for almost all $x \in [0, a]$. The above inequality implies that the function $e^{-2H(x)}v(x)$ is nonincreasing. On the other hand, taking $\varepsilon > 0$ arbitrary and x sufficiently small, from (1.10.6) we find

$$\begin{aligned} e^{-2H(x)}v(x) &= e^{-2H(x)}(y(x) - \bar{y}(x))^2 \\ &= e^{-2H(x)} \left(\int_0^x (f(t, y(t)) - f(t, \bar{y}(t)))dt \right)^2 \\ &\le e^{-2H(x)} \left(\int_0^x \varepsilon h(t) e^{H(t)} dt \right)^2 \\ &\le \varepsilon^2 e^{-2H(x)} e^{2H(x)} \\ &= \varepsilon^2 \end{aligned}$$

and hence $\lim_{x \to 0^+} \exp(-2H(x))v(x) = 0$. Therefore, $v(x) \equiv 0$, i.e., $y(x) = \bar{y}(x)$ in $[0, a]$. ∎

Finally, we remark that in Theorem 1.10.1 the condition (1.10.2) can be replaced by any one of the following :

(i) **Lemmert's Condition.**

The function

$$F(x,y) = f(x,y)\exp\left(\int_x^a h(t)dt\right) \Big/ h(x) \tag{1.10.7}$$

is uniformly continuous for $0 < x \le a,\ |y| \le b,\ b > 0$ arbitrary.

(ii) **Došlà - Došlý's Condition.**

$|f(x,y)| = o(H(x))$ as $x \to 0^+$ uniformly with respect to $y \in [-\delta, \delta],\ \delta > 0$ arbitrary, where $H(x)$ is a nonnegative function defined in $(0,a]$ and satisfies

$$\int_0^a H(t)dt < \infty; \quad \text{and} \tag{1.10.8}$$

$$\liminf_{x\to 0^+}\left[\int_0^x H(t)dt\ \exp\left(\int_x^a h(t)dt\right)\right] < \infty.$$

1.11 PERRON'S UNIQUENESS THEOREM

To prove the main result of this section we need the following :

Definition 1.11.1. A solution $r(x)$ $(\rho(x))$ of the initial value problem (1.1.1) which exists in an interval J is said to be maximal (minimal) solution if for an arbitrary solution $y(x)$ of (1.1.1) existing in J, the inequality $y(x) \le r(x)$ $(\rho(x) \le y(x))$ holds for all $x \in J$.

Clearly, the maximal solution $r(x)$ and the minimal solution $\rho(x)$ if they exist, are unique.

Theorem 1.11.1 [92]. Let $f(x,y)$ be continuous in $\bar{S}_+$, then there exists a maximal solution $r(x)$ and a minimal solution $\rho(x)$ of the initial value problem (1.1.1) in the interval $[x_0, x_0+\alpha]$ for some positive α. ∎

Let $y(x)$ be a continuous function in an interval J. We shall adopt the following notation for Dini derivatives

$$D^+y(x) = \limsup_{h\to 0^+}\left(\frac{y(x+h)-y(x)}{h}\right),$$

$$D_+y(x) = \liminf_{h\to 0^+}\left(\frac{y(x+h)-y(x)}{h}\right),$$

$$D^-y(x) = \limsup_{h\to 0^-}\left(\frac{y(x+h)-y(x)}{h}\right),$$

$$D_-y(x) = \liminf_{h\to 0^-}\left(\frac{y(x+h)-y(x)}{h}\right).$$

When $D^+y(x) = D_+y(x)$, the right derivative of $y(x)$ exists and it will be denoted by $y'_+(x)$. Similarly, $y'_-(x)$ denotes the left derivative.

As an application of the maximal solution $r(x)$, we have

Theorem 1.11.2 [92]. Let $f(x,y)$ be continuous in a domain D, and let $r(x)$ be the maximal solution of (1.1.1) in the interval $J = [x_0, x_0 + a)$. Also, let $y(x)$ be a solution of the differential inequality

$$D^+y(x) \le f(x, y(x))$$

in J. Then, $y(x_0) \le y_0$ implies that $y(x) \le r(x)$, for all $x \in J$. ∎

Theorem 1.11.3 [92]. Let $f(x,y)$ be continuous in a domain D, and let $\rho(x)$ be the minimal solution of (1.1.1) in the interval $J = (x_0 - a, x_0]$. Also, let $y(x)$ be a solution of the differential inequality

$$D_-y(x) \le f(x, y(x))$$

in J. Then, $y_0 \le y(x_0)$ implies that $\rho(x) \le y(x)$, for all $x \in J$. ∎

Theorem 1.11.4 (Perron's uniqueness Theorem). Assume that

(i) the function $g(x,z)$ is continuous and nonnegative in $x_0 \le x \le x_0+a$, $0 \le z \le 2b$ and for every x_1, $x_0 < x_1 < x_0 + a$, $z(x) \equiv 0$ is the only differentiable function in $[x_0, x_1)$, which satisfies

$$\begin{aligned} z'(x) &= g(x, z(x)),\ x_0 \le x < x_1 \\ z(x_0) &= 0, \end{aligned} \tag{1.11.1}$$

(ii) the function $f(x,y)$ is continuous in $\bar{S}_+$ and for all $(x,y),(x,\bar{y}) \in \bar{S}_+$ satisfies **Perron's condition**

$$(1.11.2) \qquad |f(x,y) - f(x,\bar{y})| \le g(x, |y - \bar{y}|).$$

Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0 + a]$.

Proof. Suppose there are two solutions $y(x)$ and $\bar{y}(x)$ of (1.1.1) in the interval $[x_0, x_0 + a]$. Define $\phi(x) = |y(x) - \bar{y}(x)|$. Clearly, $\phi(x_0) = 0$ and from (1.11.2), we have

$$\begin{aligned} D^+\phi(x) &\le |y'(x) - \bar{y}'(x)| = |f(x, y(x)) - f(x, \bar{y}(x))| \\ &\le g(x, |y(x) - \bar{y}(x)|) \\ &= g(x, \phi(x)). \end{aligned}$$

Now, for any x_1 such that $x_0 < x_1 < x_0 + a$, we obtain from Theorem 1.11.2 the inequality $\phi(x) \le r(x)$, $x_0 \le x < x_1$, where $r(x)$ is the maximal solution of (1.1.1). However, from the hypothesis of the theorem, $r(x) \equiv 0$, and hence $\phi(x) = 0$ in $[x_0, x_1)$, which proves the result. ∎

Corollary 1.11.5. The uniform Lipschitz condition (1.2.1) is a particular case of Perron's condition (1.11.2). ∎

1.12 GYÖRI'S UNIQUENESS THEOREM

We begin with the following generalization of the well - known Viswanatham's lemma [162].

Lemma 1.12.1. Assume that

(i) $\phi(x)$ is a nonnegative and bounded function in $[x_0, x_0 + a]$,

(ii) $\psi(x)$ is increasing and continuous in $(x_0, x_0 + a)$; and

(iii) $g(x,z) \geq 0$ is continuous for $x_0 \leq x \leq x_0 + a,\ 0 \leq z \leq k + \sup_{x_0 \leq x \leq x_0 + a} \phi(x) = \phi_0$, where k is an arbitrary positive constant. Further, for fixed $x,\ g(x,z)$ is a nondecreasing function of z, and $g(x,\ \phi(x))$ is integrable.

Then, the inequality

$$\phi(x) \leq k + \int_{x_0}^{x} g(t, \phi(t)) d\psi(t),\ x_0 \leq x \leq x_0 + a \tag{1.12.1}$$

implies that for some $a < x_1 \leq x_0 + a$

$$\phi(x) \leq r(x),\ x_0 \leq x \leq x_1 \tag{1.12.2}$$

where $r(x)$ is the maximal solution of the integral equation

$$z(x) = k + \int_{x_0}^{x} g(t, z(t)) d\psi(t) \tag{1.12.3}$$

in the interval $x_0 \leq x \leq x_1$, where this solution exists.

Proof. Since the notion of the maximal solution of an integral equation is analogous to that of in the case of differential equations, it suffices to show that (1.12.3) has a solution satisfying (1.12.2). For this, we define the following successive approximations

$$z_{n+1}(x) = k + \int_{x_0}^{x} g(t, z_n(t)) d\psi(t);\ n = 0, 1, 2, \cdots \tag{1.12.4}$$

where $z_0(x) = \phi(x)$.

These functions $z_n(x)$ are obviously continuous. Let $M > 0$ be the maximum of $g(x,z)$ for $x_0 \leq x \leq x_0 + a,\ 0 \leq z \leq \phi_0$. We shall show that for a suitable $x_0 < x_1 \leq x_0 + a$, we have

$$0 \leq z_n(x) \leq \phi_0 \tag{1.12.5}$$

for all positive integers n, whenever $x_0 \leq x \leq x_1$. For this, we note that the case $n = 0$ is obvious. Thus, if (1.12.5) is true for some $n \geq 1$, then by (1.12.4) it follows that

$$0 \leq z_{n+1}(x) \leq k + M(\psi(x) - \psi(x_0)) \leq \phi_0,$$

whenever $x \geq x_0$ is so small that

$$\psi(x) \leq \frac{\phi_0 - k}{M} + \psi(x_0) = \frac{\sup_{x_0 \leq x \leq x_0+a} \phi(x)}{M} + \psi(x_0).$$

The continuity of $\psi(x)$ implies the existence of such a x. This shows that the sequence $\{z_n(x)\}$ is bounded in the interval $x_0 \leq x \leq x_1$, for some $x_1 > x_0$. Moreover, for every x and n the points $(x, z_n(x))$ belong to the domain of $g(x, z)$.

Now we shall show that $z_n(x) \leq z_{n+1}(x)$ for every $x_0 \leq x \leq x_1$, $n \geq 0$. For this, the case $n = 0$ is obvious. Thus, if $z_{n-1}(x) \leq z_n(x)$, $n \geq 1$ then by the monotonic nature of $g(x, z)$ and $\psi(x)$, we have

$$\begin{aligned} z_n(x) &= k + \int_{x_0}^{x} g(t, z_{n-1}(t)) d\psi(t) \\ &\leq k + \int_{x_0}^{x} g(t, z_n(t)) d\psi(t) \\ &= z_{n+1}(x), \quad x_0 \leq x \leq x_1. \end{aligned}$$

The definition of the functions $\{z_n(x)\}$ implies that for $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x_0 \leq \bar{x}, \hat{x} \leq x_1$ and $n \geq 1$ the inequality

$$|z_n(\bar{x}) - z_n(\hat{x})| \leq M|\psi(\bar{x}) - \psi(\hat{x})| < \varepsilon$$

holds, provided $|\bar{x} - \hat{x}| \leq \delta$. Therefore, Arzela's theorem implies that this sequence of functions converges uniformly. Let $\bar{z}(x) = \lim_{n\to\infty} z_n(x)$, $x_0 \leq x \leq x_1$. It is clear that this function is a solution of the equation (1.12.3). Further, in view of

$$\phi(x) = z_0(x) \leq z_1(x) \leq \ldots \leq \bar{z}(x) \leq r(x)$$

the inequality (1.12.2) is satisfied. ∎

Corollary 1.12.2. Assume that

(i) $\phi(x)$ is a nonnegative and bounded function in $[x_0, x_0 + a]$,

(ii) $\psi(x)$ is a continuous nondecreasing function in $[x_0, x_0 + a]$; and

(iii) $g(z) \geq 0$ is a continuous nondecreasing function for $0 \leq z < \infty$ and there exists a $\phi_0 > 0$ such that $g(z) > 0$ for $z \geq \phi_0$, moreover $g(\phi(x))$ is integrable.

Then, for an arbitrary positive constant k the inequality

$$\phi(x) \leq k + \int_{x_0}^{x} g(\phi(t))d\psi(t),\ x_0 \leq x \leq x_0 + a \tag{1.12.6}$$

implies that for some $a < x_1 \leq x_0 + a$

$$\phi(x) \leq G^{-1}(G(k) + \psi(x) - \psi(x_0)),\ x_0 \leq x \leq x_1 \tag{1.12.7}$$

where

$$G(\phi) = \int_{\phi_0}^{\phi} \frac{dz}{g(z)},\quad \phi \geq 0 \tag{1.12.8}$$

and G^{-1} is the inverse of G; moreover x_1 is chosen so that $G(k)+\psi(x)-\psi(x_0)$ belongs to the domain of G^{-1} whenever $x_0 \leq x \leq x_1$.

Proof. If $r(x)$ is the maximal solution of the integral equation

$$z(x) = k + \int_{x_0}^{x} g(z(t))d\psi(t), \tag{1.12.9}$$

then in view of Lemma 1.12.1 we have $\phi(x) \leq r(x)$. Therefore, it suffices to show that any solution $\bar{z}(x)$ of (1.12.9) satisfies the inequality

$$\bar{z}(x) < G^{-1}(G(k+\varepsilon) + \psi(x) - \psi(x_0)) = z_\varepsilon(x), \tag{1.12.10}$$

where $z_\varepsilon(x)$ is a solution of the integral equation

$$z_\varepsilon(x) = k + \varepsilon + \int_{x_0}^{x} g(z_\varepsilon(t))d\psi(t). \tag{1.12.11}$$

However, since

$$\int_{x_0}^{x} g(G^{-1}(G(k+\varepsilon)+\psi(t)-\psi(x_0)))d\psi(t) = \int_{\psi(x_0)}^{\psi(x)} g(G^{-1}(G(k+\varepsilon)+t-\psi(x_0)))dt$$

and

$$\frac{dG^{-1}(\phi)}{d\phi} = g(G^{-1}(\phi))$$

it follows that

$$\begin{aligned}
&\int_{x_0}^{x} g(z_\varepsilon(t))d\psi(t)\\
&= \int_{x_0}^{x} g(G^{-1}(G(k+\varepsilon)+\psi(t)-\psi(x_0)))d\psi(t)\\
&= G^{-1}(G(k+\varepsilon)+\psi(x)-\psi(x_0)) - G^{-1}(G(k+\varepsilon)+\psi(x_0)-\psi(x_0))\\
&= -k-\varepsilon+G^{-1}(G(k+\varepsilon)+\psi(x)-\psi(x_0))\\
&= -k-\varepsilon+z_\varepsilon(x). \qquad \blacksquare
\end{aligned}$$

Remark 1.12.1. If $q(x) \geq 0$ is continuous and $\psi(x) = \int_{x_0}^{x} q(t)dt$, then Corollary 1.12.2 is the same as LaSalle's lemma (better known as Bihari's lemma). Further, in addition if $g(z) = z$, then the inequality (1.12.6) reduces to

$$\phi(x) \leq k + \int_{x_0}^{x} q(t)\phi(t)dt \tag{1.12.12}$$

for which (1.12.7) becomes

$$\phi(x) \leq k \exp\left(\int_{x_0}^{x} q(t)dt\right), \tag{1.12.13}$$

which is the famous **Gronwall's inequality**. Obviously, if $k = 0$, then (1.12.12) is the same as (1.2.5) and then (1.12.13) also gives $\phi(x) \equiv 0$.

Theorem 1.12.3 (Győri's Uniqueness Theorem). Assume that the function $f(x,y)$ is continuous in $\bar{S}_+$ and

(i) for every sufficiently small δ, $0 < \delta < a$, the function $\psi(x;\delta)$ is continuous and nondecreasing in x, $x_0 + \delta \leq x \leq x_0 + a$,

(ii) $g(z) \geq 0$ is continuous and nondecreasing for $0 \leq z < \infty$,

(iii) there is no positive constant k such that the inequality

$$\int_{\varepsilon+\delta\mu(\delta)}^{\varepsilon+k} \frac{dz}{g(z)} \leq \psi(x_0+a;\delta) - \psi(x_0+\delta;\delta) \tag{1.12.14}$$

holds for any ε, $\delta > 0$ and for any function $\mu(\delta)$ tending to 0 as $\delta \to 0$; and

(iv) for all $x_0+\delta \le x \le x_0+a$ and $y_0-b \le y, \bar{y} \le y_0+b$ the function $f(x,y)$ satisfies the inequality

$$|f(x,\bar{y})-f(x,y)| \le \psi'(x;\delta)g(|\bar{y}-y|). \tag{1.12.15}$$

Then, there exists a $0<\eta\le a$ such that the initial value problem (1.1.1) has at most one solution in the interval $[x_0, x_0+\eta]$.

Proof. Let $y(x)$ and $\bar{y}(x)$ be two solutions of the initial value problem (1.1.1). Let $x>x_0$ and fix $\delta>0$ so that $x_0+\delta<x$. Then, in view of (1.12.15) and on omitting the singular part of $\psi(x;\delta)$ from its canonical decomposition as a continuous and monotone function, we find that

$$\begin{aligned}\int_{x_0+\delta}^{x}|f(t,\bar{y}(t))-f(t,y(t))|dt &\le \int_{x_0+\delta}^{x} g(|\bar{y}(t)-y(t)|)\psi'(t;\delta)dt\\ &\le \int_{x_0+\delta}^{x} g(|\bar{y}(t)-y(t)|)d\psi(t;\delta).\end{aligned}$$

Thus, it follows that

$$\begin{aligned}&|\bar{y}(x)-y(x)| \le \int_{x_0}^{x}|f(t,\bar{y}(t))-f(t,y(t))|dt\\ &\le \delta \sup_{x_0\le x\le x_0+\delta}|f(x,\bar{y}(x))-f(x,y(x))| + \int_{x_0+\delta}^{x} g(|\bar{y}(t)-y(t)|)d\psi(t;\delta).\end{aligned}$$

Let the function $\mu(\delta)$ in (1.12.14) be $\sup_{x_0\le x\le x_0+\delta}|f(x,\bar{y}(x))-f(x,y(x))|$ then, from the above inequality we have

$$|\bar{y}(x)-y(x)| \le \delta\mu(\delta) + \int_{x_0+\delta}^{x} g(|\bar{y}(t)-y(t)|)d\psi(t;\delta).$$

Thus, if $\varepsilon>0$ is an arbitrary but fixed number, then

$$|\bar{y}(x)-y(x)|+\varepsilon \le \delta\mu(\delta)+\varepsilon + \int_{x_0+\delta}^{x} g(|\bar{y}(t)-y(t)|+\varepsilon)d\psi(t;\delta)$$

holds. Now an application of Corollary 1.12.2 gives

$$G(|\bar{y}(x)-y(x)|+\varepsilon) \le G(\delta\mu(\delta)+\varepsilon)+\psi(x;\delta)-\psi(x_0+\delta;\delta),$$

where the function $G(\phi)$ is defined in (1.12.8). However, since for every ε, $\delta>0$

$$\begin{aligned}\int_{\varepsilon+\delta\mu(\delta)}^{|\bar{y}(x)-y(x)|+\varepsilon}\frac{dz}{g(z)} &\le \psi(x;\delta)-\psi(x_0+\delta;\delta)\\ &\le \psi(x_0+a;\delta)-\psi(x_0+\delta;\delta)\end{aligned}$$

hypothesis (iii) implies that $|\bar{y}(x) - y(x)| \equiv 0$, i.e., $y(x) = \bar{y}(x)$, $x \geq x_0$. Since $x > x_0$ is arbitrary the result follows. ∎

Corollary 1.12.4. Assume that $g(z)$ is as in Lemma 1.4.1 and Osgood's condition (1.4.3) is satisfied. Then, the conclusion of Theorem 1.12.3 holds.

Proof. It suffices to note that the conditions of Theorem 1.12.3 with

$$\psi(x;\delta) - \psi(x_0+\delta;\delta) = x - x_0 - \delta$$

are satisfied. ∎

Remark 1.12.2. For $g(z) = \sqrt{z}$, and

$$\psi(x;\delta) - \psi(x_0+\delta;\delta) = (x-x_0)^{\delta} - \delta^{\delta}, \quad x > x_0 + \delta \ (\delta > 0)$$

the inequality (1.12.14) for ε, $\delta > 0$ becomes

$$a^{\delta} - \delta^{\delta} \geq \int_{\varepsilon+\delta\mu(\delta)}^{\varepsilon+k} \frac{dz}{\sqrt{z}} = 2\sqrt{\varepsilon+k} - 2\sqrt{\varepsilon+\delta\mu(\delta)},$$

which implies that $k = 0$. Therefore, the condition

$$|f(x,\bar{y}) - f(x,y)| \leq \delta(x-x_0)^{\delta-1}\sqrt{|\bar{y}-y|}$$

implies the conclusion of Theorem 1.12.3. Hence, in Corollary 1.12.4 it is not necessary that the function $g(z)$ must satisfy the condition (1.4.1).

Corollary 1.12.5. Assume that $g(z) = z$ and Nagumo's condition (1.6.1) is satisfied. Then, the conclusion of Theorem 1.12.3 holds.

Proof. It suffices to note that the conditions of Theorem 1.12.3 with

$$\psi(x;\delta) - \psi(x_0+\delta;\delta) = \int_{x_0+\delta}^{x} \frac{dt}{t-x_0}, \ x_0 + \delta < x$$

are satisfied. ∎

Corollary 1.12.6 (Yang's Uniqueness Theorem). Assume that $g(z) = z^{\beta}$ and **Yang's condition**

$$|f(x,\bar{y}) - f(x,y)| \leq \frac{L}{(x-x_0)^{\alpha}}|\bar{y}-y|^{\beta} \tag{1.12.16}$$

is satisfied, where $\beta > 1$, $0 \le \alpha < \beta$, $\alpha \neq 1$ and $L > 0$ are arbitrary constants. Then, the conclusion of Theorem 1.12.3 holds.

Proof. It suffices to note that the conditions of Theorem 1.12.3 with

$$\psi(x;\delta) - \psi(x_0+\delta;\delta) = L\int_{x_0+\delta}^{x} \frac{dt}{(t-x_0)^{\alpha}},\ x_0+\delta \le x \le x_0+a$$

are satisfied. For this, (i) and (ii) are obvious, whereas for (iii) we assume the contrary, i.e., that for some positive k the inequality (1.12.14) holds for any $\varepsilon,\ \delta > 0$ and for any function $\mu(\delta)$ tending to 0 as $\delta \to 0$. But, then for $x > x_0 + \delta$, we have

$$\frac{1}{(1-\beta)(\varepsilon+k)^{\beta-1}} - \frac{1}{(1-\beta)(\varepsilon+\delta\mu(\delta))^{\beta-1}} \le \frac{L}{1-\alpha}\left[\frac{1}{a^{\alpha-1}} - \frac{1}{\delta^{\alpha-1}}\right],$$

which is the same as

$$\frac{1}{(1-\beta)(\varepsilon+k)^{\beta-1}} - \frac{L}{(1-\alpha)a^{\alpha-1}} \le \frac{1}{(1-\beta)(\varepsilon+\delta\mu(\delta))^{\beta-1}} - \frac{L}{(1-\alpha)\delta^{\alpha-1}}$$

for all $\varepsilon,\ \delta > 0$. Let $\varepsilon = \delta - \delta\mu(\delta)$, then the above inequality becomes

$$\text{(1.12.17)}\quad \frac{1}{(1-\beta)(\delta-\delta\mu(\delta)+k)^{\beta-1}} - \frac{L}{(1-\alpha)a^{\alpha-1}} \le \frac{1}{(1-\beta)\delta^{\beta-1}} - \frac{L}{(1-\alpha)\delta^{\alpha-1}}.$$

However, from the assumptions on the constants we find that

$$\lim_{\delta\to 0}\left[\frac{1}{(1-\beta)\delta^{\beta-1}} - \frac{L}{(1-\alpha)\delta^{\alpha-1}}\right] = \lim_{\delta\to 0}\frac{(1-\alpha) - L(1-\beta)\delta^{\beta-\alpha}}{(1-\alpha)(1-\beta)\delta^{\beta-1}} = -\infty.$$

Thus, if $\delta \to 0$, then (1.12.17) implies that

$$\frac{1}{(1-\beta)k^{\beta-1}} - \frac{L}{(1-\alpha)a^{\alpha-1}} \le -\infty,$$

which is an obvious contradiction. The case $x < x_0$ can be treated similarly. ∎

1.13 IYANAGA'S UNIQUENESS THEOREM

This is the first general uniqueness criterion from which several earlier results, with strict inequalities only, can be deduced.

Theorem 1.13.1 (Iyanaga's Uniqueness Theorem). Assume that

(i) the function $g(x,z)$ is continuous and nonnegative in $x_0 < x \le x_0+a$, $0 \le z \le 2b$ and for every x_1, $x_0 < x_1 \le x_0 + a$ and z_1, $0 < z_1 \le 2b$ solutions $z(x) = z(x, x_1, z_1)$ of the differential equation

$$z' = g(x,z) \tag{1.13.1}$$

exist for $x_0 < x \le x_1$, $0 \le z(x) \le 2b$, and $\lim_{x\to x_0^+} z(x) > 0$, or $\lim_{x\to x_0^+} z(x) = 0$ and $\lim_{x\to x_0^+} z'(x) > 0$; and

(ii) the function $f(x,y)$ is continuous in $\bar{S}_+$ and for all (x,y), $(x,\bar{y})$ in $\bar{S}_+$ with $\bar{y} > y$ the following inequality is satisfied

$$g(x, \bar{y} - y) > f(x,\bar{y}) - f(x,y). \tag{1.13.2}$$

Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0 + a]$.

Proof. Let $y(x)$ and $\bar{y}(x)$ be two solutions of (1.1.1) in the interval $[x_0, x_0+a]$. Then, the function $\phi(x) = \bar{y}(x) - y(x)$ satisfies $\phi(x_0) = \phi'(x_0) = 0$. We assume that $\phi(x) > 0$ for $x_1 < x \le x_2 \le x_0 + a$, and $\phi(x) = 0$ for $x_0 \le x \le x_1$. Let $\rho(x)$ be the minimal solution of (1.13.1) satisfying the initial condition $\rho(x_2) = z_2 = \phi(x_2)$. We note that from the given hypotheses $\rho(x_0 + \varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$, and hence we can choose $\varepsilon > 0$ so small that $\rho(x_0+\varepsilon) > \phi(x_0+\varepsilon)$. Since the function $g(x,z)$ is nonnegative it is clear that $\rho(x) > 0$ in the interval $[x_0+\varepsilon, x_2]$. For $x \in (x_1, x_2]$ the inequality (1.13.2) implies that

$$\phi'(x) = f(x,\bar{y}(x)) - f(x,y(x)) < g(x,\bar{y}(x) - y(x)) = g(x,\phi(x)).$$

Thus, in view of Theorem 1.11.3 it follows that $\rho(x) \leq \phi(x)$ for all $x \in (x_1, x_2]$. However, since $\phi(x_1) = 0$, from the continuity we find that $\rho(x_1) = 0$. Therefore, if at all x_1 must be the same as x_0. But, then it contradicts $\rho(x_0 + \varepsilon) > \phi(x_0 + \varepsilon)$ for small ε. ∎

Corollary 1.13.2. For the uniqueness of the solutions of the initial value problem (1.9.4) each of the following conditions is sufficient.

(i) $|f(x,y) - f(x,\bar{y})| < (1 + \varepsilon(x))\dfrac{|y-\bar{y}|}{x}$, where $\varepsilon(x) > 0$ and $\lim_{x\to 0^+} \int_\delta^x \dfrac{\varepsilon(t)}{t} dt > -M$, $0 < M < \infty$, $\delta > 0$;

(ii) $|f(x,y) - f(x,\bar{y})| < \alpha\dfrac{|y-\bar{y}|}{x} + \beta|y-\bar{y}|\ln\dfrac{1}{|y-\bar{y}|}$, $0 \leq \alpha < 1$, $0 \leq \beta$;

(iii) $|f(x,y) - f(x,\bar{y})| < \dfrac{|y-\bar{y}|}{x} \times \dfrac{\left|\ln\dfrac{1}{|y-\bar{y}|}\right|^l}{\left|\ln\dfrac{1}{x}\right|^k}$, $0 < l < k$, $l \leq 1$.

Proof. For the proof of (i) we can apply Theorem 1.13.1 and consider the differential equation

$$z' = g(x,z) = (1 + \varepsilon(x))\frac{z}{x} \tag{1.13.3}$$

and put an indefinite integral $\int \dfrac{\varepsilon(x)}{x} dx = I(x)$.

The general solution of (1.13.3) is $z(x) = xe^{c+I(x)}$, where c is an arbitrary constant. Since $I(x) > -M$ for any $c \neq -\infty$, we have

$$\lim_{x\to 0^+} \frac{dz}{dx} = e^{c+I(x)} + \varepsilon(x)e^{c+I(x)} > 0.$$

For the proof of (ii) we consider the differential equation

$$z' = g(x,z) = \alpha\frac{z}{x} - \beta z \ln z. \tag{1.13.4}$$

The general solution of (1.13.4) is $z(x) = \exp\left(\alpha e^{-\beta x} G(x) + ce^{-\beta x}\right)$, where $G(x)$ is the indefinite integral $\int \dfrac{e^{\beta x}}{x} dx$, and c is an arbitrary constant. Since

$$\begin{aligned} \ln\frac{dz}{dx} &= \alpha e^{-\beta x}G(x) + ce^{-\beta x} + \ln\left(-\alpha\beta e^{-\beta x}G(x) + \frac{\alpha}{x} - c\beta e^{-\beta x}\right) \\ &= \alpha e^{-\beta x}G(x) - \ln x + O(1) \end{aligned}$$

on putting $\alpha = 1/(1+\gamma)$, $\gamma > 0$ and choosing $\varepsilon < \gamma$ and then $\delta(\varepsilon)$ so that

$$\ln \frac{\delta(\varepsilon)}{x} < G(\delta(\varepsilon)) - G(x) < (1+\varepsilon) \ln \frac{\delta(\varepsilon)}{x},$$

we find

$$\ln \frac{dz}{dx} = \left(1 - \frac{1+\varepsilon}{1+\gamma} e^{-\beta x}\right) \ln \frac{1}{x} + O(1) \quad \longrightarrow \infty.$$

Hence, $\lim_{x \to 0^+} \dfrac{dz}{dx} = \infty$.

For the proof of (iii), we consider

$$\frac{dz}{dx} = \frac{z}{x} \frac{(-\ln z)^l}{(-\ln x)^k}. \tag{1.13.5}$$

The general solution of (1.13.5) for $0 < l < k < 1$, is

$$z(x) = \exp\left(-\left\{\frac{1-l}{1-k}(-\ln x)^{1-k} + c\right\}^{1/(1-l)}\right),$$

where c is an arbitrary constant. Since,

$$\ln \frac{dz}{dx} = -\left\{\frac{1-l}{1-k}(-\ln x)^{1-k} + c\right\}^{1/(1-l)} + \ln(-\ln x)^{-k}$$
$$+ \ln\left\{\frac{1-l}{1-k}(-\ln x)^{1-k} + c\right\}^{l/(1-l)} + \ln \frac{1}{x} \quad \longrightarrow \infty$$

we find that $\lim_{x \to 0^+} \dfrac{dz}{dx} = \infty$. A similar situation holds for $l < k = 1$ as well as $1 = l < k$. ∎

Finally, we note that the condition $|f(x,y) - f(x,\bar{y})| \le L(x)|y - \bar{y}|$, where $\lim_{x \to 0^+} x \exp(-\int_\delta^x L(t)dt) < M$, is identical with the condition (i). For this, putting $L(x) = (1+\varepsilon(x))/x$ we have $\int \frac{\varepsilon(x)}{x} dx > -M$, and conversely, putting $(1+\varepsilon(x))/x = L(x)$, we have $\lim_{x \to 0^+} x \exp(-\int_\delta^x L(t)dt) < M$.

1.14 INABA'S UNIQUENESS THEOREM

The motivation of the following result comes from the Corollary 1.13.2.

Theorem 1.14.1 (Inaba's Uniqueness Theorem). Assume that

(i) the function $\varepsilon(x)$ is nonnegative in $[0,a]$ and for every $\delta > 0$ there exists a positive number M such that

$$\lim_{x\to 0^+}\int_x^\delta \frac{\varepsilon(t)}{t}dt < M \ , \tag{1.14.1}$$

(ii) the function $h(x)$ is nonnegative in $[0,a]$ and for every $\delta > 0$

$$\lim_{x\to 0^+}\int_x^\delta h(t)dt \ , \tag{1.14.2}$$

is a finite positive number N, and there exists a positive number $\bar{N}$ such that

$$\lim_{x\to 0^+}\int_x^\delta \frac{\int_0^t h(s)ds}{t}dt < \bar{N}, \tag{1.14.3}$$

(iii) the function $f(x,y)$ is continuous in $\bar{S}_+$ where $(x_0,y_0)=(0,0)$ and for all (x,y), $(x,\bar{y})$ in $\bar{S}_+$ with $\bar{y} > y$ the following inequality is satisfied

$$f(x,\bar{y}) - f(x,y) \le \frac{1+\varepsilon(x)}{x}(\bar{y}-y) + h(x)(\bar{y}-y)\ln\frac{1}{\bar{y}-y}. \tag{1.14.4}$$

Then, the initial value problem (1.1.1) with $(x_0,y_0)=(0,0)$ has at most one solution in $[0,a]$.

Proof. Let $y(x)$ and $\bar{y}(x)$ be two solutions of (1.1.1) in the interval $[0,a]$. Then, the function $\phi(x) = \bar{y}(x) - y(x)$ satisfies $\phi(0) = \phi'(0) = 0$. We assume that $\phi(x) > 0$ for $x_1 < x \le x_2 \le a$, and $\phi(x) = 0$ for $0 \le x \le x_1$. Since for all $x \in (x_1,x_2]$, $\phi(x) > 0$ the inequality (1.14.4) implies that

$$\phi'(x) = f(x,\bar{y}(x)) - f(x,y(x)) \le \frac{1+\varepsilon(x)}{x}\phi(x) + h(x)\phi(x)\ln\frac{1}{\phi(x)},$$

which is the same as

$$-\frac{d}{dx}\ln\frac{1}{\phi(x)} - h(x)\ln\frac{1}{\phi(x)} \le \frac{1+\varepsilon(x)}{x}.$$

Further, since by assumption, $\lim_{x\to 0^+}\int_x^\delta h(t)dt$ is finite, $\int_0^x h(t)dt$ is continuous and $\frac{d}{dx}\int_0^x h(t)dt = h(x)$. Therefore, it follows that

$$-\frac{d}{dx}\left[\exp\left(\int_0^x h(t)dt\right)\ln\frac{1}{\phi(x)}\right] \le \frac{1+\varepsilon(x)}{x}\exp\left(\int_0^x h(t)dt\right)$$

and hence an integration from x to δ gives

$$\exp\left(\int_0^x h(t)dt\right)\ln\frac{1}{\phi(x)} - \exp\left(\int_0^\delta h(t)dt\right)\ln\frac{1}{\phi(\delta)} \tag{1.14.5}$$

$$\leq \int_x^\delta \frac{1}{t}\exp\left(\int_0^t h(s)ds\right)dt + \int_x^\delta \frac{\varepsilon(t)}{t}\exp\left(\int_0^t h(s)ds\right)dt.$$

If $\phi(x_1) = 0$ for $x_1 > 0$, then as $x \to x_1$, the left side of (1.14.5) tends to ∞, whereas the right side remains finite. This contradiction forces that x_1 must be 0. Thus, if $\phi(x) > 0$ for $x_1 < x \leq x_2$, then it is positive for $0 < x \leq x_2$ also. Now, since $\exp(\int_0^x h(t)dt) > 1$, for small x, conditions (1.14.2) and (1.14.3) give

$$\int_x^\delta \frac{1}{t}\exp\left(\int_0^t h(s)ds\right)dt$$

$$\leq \int_x^\delta \frac{1}{t}\left[1 + \int_0^t h(s)\exp\left(\int_0^s h(\tau)d\tau\right)ds\right]dt$$

$$\leq \ln\delta + \ln\frac{1}{x} + \exp\left(\int_0^\delta h(t)dt\right)\int_x^\delta \frac{\exp\left(\int_0^t h(s)ds\right)}{t}dt$$

$$< \exp\left(\int_0^x h(t)dt\right)\ln\frac{1}{x} + \ln\delta + e^N \times \bar{N}.$$

Also, in view of (1.14.1) and (1.14.2), we have

$$\int_x^\delta \frac{\varepsilon(t)}{t}\exp\left(\int_0^t h(s)ds\right)dt \leq \exp\left(\int_0^\delta h(t)dt\right)\int_x^\delta \frac{\varepsilon(t)}{t}dt \quad < e^N M.$$

Using the above estimates in (1.14.5), we obtain

$$\exp\left(\int_0^x h(t)dt\right)\ln\frac{x}{\phi(x)} < \ln\delta + e^N\left(M + \bar{N} + \ln\frac{1}{\phi(\delta)}\right) \quad < G,$$

where G is positive and depends only on δ. Therefore, it follows that

$$\ln\frac{x}{\phi(x)} < G\exp\left(-\int_0^x h(t)dt\right) \quad < G.$$

In this inequality as $x \to 0^+$ the right side remains finite, whereas for the left side we find that

$$\lim_{x\to 0^+}\ln\frac{x}{\phi(x)} = -\ln\lim_{x\to 0^+}\frac{\phi(x)}{x} = -\ln\lim_{x\to 0^+}\frac{\phi'(x)}{1} = \infty.$$

This contradiction completes the proof. ∎

Corollary 1.14.2. Let the function $f(x,y)$ be continuous in $\bar{S}$, where $(x_0, y_0) = (0,0)$ and for all (x,y), $(x,\bar{y})$ in $\bar{S}$ the following inequality is satisfied

$$(1.14.6) \qquad |f(x,y) - f(x,\bar{y})| \le \frac{1+\varepsilon(|x|)}{|x|}|y-\bar{y}| + h(|x|)|y-\bar{y}|\ln\frac{1}{|y-\bar{y}|},$$

where the functions $\varepsilon(x)$ and $h(x)$ satisfy the conditions of Theorem 1.14.1. Then, the initial value problem (1.1.1) with $(x_0, y_0) = (0,0)$ has at most one solution in $|x| \le a$.

Proof. Let $\phi(x) = \bar{y}(x) - y(x)$ in a small interval $(x_1, x_2]$ so that

$$\left|\frac{d\phi(x)}{dx}\right| \le \frac{1+\varepsilon(|x|)}{|x|}\phi(x) + h(|x|)\phi(x)\ln\frac{1}{\phi(x)}.$$

If we set $\xi = |x|$, then since $\frac{d\xi}{dx} = \pm 1$ it follows that

$$\left|\frac{d\phi(x)}{dx}\right| = \left|\frac{d\bar{\phi}(\xi)}{d\xi} \times \frac{d\xi}{dx}\right| = \left|\frac{d\bar{\phi}(\xi)}{d\xi}\right| \ge \frac{d\bar{\phi}(\xi)}{d\xi}.$$

Hence, in the interval (ξ_1, ξ_2) where $\xi_1 = \min(|x_1|, |x_2|)$ and $\xi_2 = \max(|x_1|, |x_2|)$ it follows that

$$\frac{d\bar{\phi}(\xi)}{d\xi} \le \frac{1+\varepsilon(\xi)}{\xi}\bar{\phi}(\xi) + h(\xi)\bar{\phi}(\xi)\ln\frac{1}{\bar{\phi}(\xi)}.$$

But, this implies as in the proof of Theorem 1.14.1 that $\phi(x) \equiv 0$. ∎

1.15 KAMKE'S UNIQUENESS THEOREM

The main result of this section is more general than those of Perron and Iyanaga, also it includes as special cases many known criteria.

Theorem 1.15.1 (Kamke's Uniqueness Theorem). Assume that

(i) the function $g(x,z)$ is continuous and nonnegative in $x_0 < x \le x_0 + a$, $0 \le z \le 2b$ and for every x_1, $x_0 < x_1 < x_0 + a$, $z(x) \equiv 0$ is the only differentiable function in $x_0 < x < x_1$ and continuous in $x_0 \le x \le x_1$ for which $z'_+(x_0) = \lim_{x \to x_0^+} \frac{z(x) - z(x_0)}{x - x_0}$ exists; and

$$z'(x) = g(x, z(x)),\ x_0 < x < x_1 \tag{1.15.1}$$

$$z(x_0) = z'_+(x_0) = 0, \tag{1.15.2}$$

(ii) the function $f(x,y)$ is continuous in $\bar{S}_+$ and for all $(x,y),(x,\bar{y}) \in \bar{S}_+$ where $x \neq x_0$ the inequality (1.11.2) is satisfied.

Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0 + a]$.

To prove this result we need the following :

Lemma 1.15.2. Let the function $g(x,z)$ satisfy the hypothesis (i) of Theorem 1.15.1. Further, let the function $g_1(x,z)$ be continuous and nonnegative in $x_0 \le x \le x_0 + a,\ 0 \le z \le 2b,\ g_1(x,0) \equiv 0$, and

$$g_1(x,z) \le g(x,z),\ x \neq x_0. \tag{1.15.3}$$

Then, for every $x_1,\ x_0 < x_1 < x_0 + a,\ z(x) \equiv 0$ is the only differentiable function in $x_0 \le x \le x_1$, which satisfies

$$z' = g_1(x,z),\ z(x_0) = 0. \tag{1.15.4}$$

Proof. It suffices to show that the maximal solution $r(x)$ of (1.15.4) is identically zero. On the contrary, suppose that there exists a $\sigma,\ x_0 < \sigma < x_0 + a$, such that $r(\sigma) > 0$. Because of the inequality (1.15.3), we have

$$r'(x) \le g(x, r(x)),\ x_0 < x \le \sigma.$$

If $\rho(x)$ is the minimal solution of $z' = g(x,z),\ z(\sigma) = r(\sigma)$ then an application of Theorem 1.11.3 gives that

$$\rho(x) \le r(x) \tag{1.15.5}$$

as far as $\rho(x)$ exists to the left of σ. The solution $\rho(x)$ by using standard arguments can be continued up to $x = x_0$. If $\rho(\tau) = 0$, for some $\tau, x_0 < \tau < \sigma$, we can effect the continuation by defining $\rho(x) = 0$ for $x_0 < x < \tau$. Otherwise, (1.15.5) ensures the possibility of continuation. Since $r(x_0) =$

0, $\lim_{x \to x_0^+} \rho(x) = 0$, and we define $\rho(x_0) = 0$. Furthermore, since $g_1(x, z)$ is continuous at $(x_0, 0)$ and $g_1(x_0, 0) = 0$, $r'_+(x_0)$ exists and is equal to zero. Thus, because of (1.15.5), it follows that $\rho'_+(x_0)$ exists and $\rho'_+(x_0) = 0$. But, we have assumed that $g(x, z)$ satisfies the hypothesis of Theorem 1.15.1. Hence, $\rho(x) \equiv 0$. This contradicts the fact that $\rho(\sigma) = r(\sigma) > 0$. Therefore, $r(x) \equiv 0$. ∎

Proof of Theorem 1.15.1. Define the function

$$g_f(x, z) = \sup_{|y - \bar{y}| = z} |f(x, y) - f(x, \bar{y})| \tag{1.15.6}$$

in $x_0 \le x \le x_0 + a$, $0 \le z \le 2b$. Since $f(x, y)$ is continuous in $\bar{S}_+$, $g_f(x, z)$ is continuous in $x_0 \le x \le x_0 + a$, $0 \le z \le 2b$. In view of (1.15.6) it is clear that (1.11.2) holds with the function $g_f(x, z)$. Moreover, $g_f(x, z) \le g(x, z)$ for $x_0 < x \le x_0 + a$, $0 \le z \le 2b$. Thus, Lemma 1.15.2 is applicable with $g_1(x, z) \le g_f(x, z)$, and therefore $g_f(x, z)$ satisfies the assumptions of Theorem 1.11.4. Hence, the result follows. ∎

Corollary 1.15.3. In Theorem 1.15.1 condition (1.15.2) cannot be replaced by

$$\lim_{x \to x_0^+} z(x) = \lim_{x \to x_0^+} z'(x) = 0. \tag{1.15.7}$$

Proof. We shall consider $(x_0, y_0) = (0, 0)$ and $a = 1$. Let $\phi(x)$ be a function defined in $(0, 1]$ and satisfy the following conditions :

(i) $x^2 < \phi(x) \le \frac{4}{3}x^2$,

(ii) $\phi'(x)$ is continuous,

(iii) $\phi'(x) \ge 2x$; and

(iv) $\lim_{x \to x_0^+} \phi'(x)$ does not exist.

The construction of such a function is not difficult. For $x > 0$ and $z \ge 0$, let

$$g(x, z) = \frac{\phi'(x)}{\phi(x)} z. \tag{1.15.8}$$

From (i), (ii) and (iii) it is clear that this function $g(x,z)$ is continuous and nonnegative for $0 < x \leq 1,\ z \geq 0$. Further, from (i) and (iii), for $0 < x \leq 1,\ z \geq 0$ we have

$$g(x,z) \geq \frac{3z}{2x}. \tag{1.15.9}$$

Since all solutions of the differential equation (1.15.1) with $g(x,z)$ defined in (1.15.8) are of the form $z(x) = c\phi(x)$, from (i) it follows that all solutions of (1.15.1) satisfy (1.15.2). Therefore, the hypotheses of Theorem 1.15.1 are not satisfied. However, from (iv) we see that (1.15.7) is satisfied only for the solution $z(x) \equiv 0$.

Now in the strip $0 \leq x \leq 1, |y| < \infty$ we define a continuous function

$$f(x,y) = \begin{cases} \dfrac{4y}{3x} & 0 < x \leq 1, 0 \leq y \leq x^{4/3} \\[2ex] \dfrac{4}{3}x^{1/3} & 0 < x \leq 1, x^{4/3} < y < \infty \\[2ex] 0 & 0 \leq x \leq 1, -\infty < y \leq 0. \end{cases} \tag{1.15.10}$$

For this function, in view of (1.15.9), it follows that for $x \neq 0$

$$|f(x,y) - f(x,\bar{y})| \leq \frac{4}{3x}|y - \bar{y}| \leq \frac{3}{2x}|y - \bar{y}| \leq g(x, |y - \bar{y}|).$$

The initial value problem (1.9.4) with this function $f(x,y)$ has two distinct solutions $y(x) \equiv 0$ and $\bar{y}(x) = x^{4/3}$ in $[0,1]$. ∎

Corollary 1.15.4. The function $g(x,z) = \lambda(x)z$, where $\lambda(x) \geq 0$ and continuous for $x_0 < x \leq x_0 + a$ satisfies the conditions of Theorem 1.15.1 provided

$$\limsup_{x \to x_0^+} [1 + \lambda(x)] e^{-\mu(x)} > 0, \tag{1.15.11}$$

where

$$\mu(x) = \int_x^{x_0+a} \lambda(t)dt,\ x \neq x_0. \tag{1.15.12}$$

Proof. Consider the differential equation

$$z' = \lambda(x)z. \tag{1.15.13}$$

The nontrivial solutions of (1.15.13) are nonvanishing constant multiples of the function $e^{-\mu(x)}$, where $\mu(x)$ is given in (1.15.12). The derivative of this function is $\lambda(x)e^{-\mu(x)}$. Since $\lambda(x) \geq 0$, it follows from the assumption (1.15.11) that every solution $z(x) \not\equiv 0$ of (1.15.13) violates at least one of the two limiting conditions (1.15.2). ∎

If $\lambda(x) = L \geq 0$, then $\mu(x) = L(x_0 + a - x)$ so that the condition (1.15.11) becomes $\limsup_{x\to x_0^+}(1 + L)e^{-L(x_0+a-x)} > 0$, which certainly holds. Thus, in view of Corollary 1.15.4 the Lipschitz Uniqueness Theorem 1.2.4 is a particular case of Theorem 1.15.1. Similarly, if $\lambda(x) = k(x - x_0)^{-1}, (k \geq 0)$ then $\mu(x) = k\ln(a(x - x_0)^{-1})$ and the condition (1.15.11) reduces to $\limsup_{x\to x_0^+}(1 + k(x - x_0)^{-1})\frac{(x-x_0)^k}{a^k} > 0$, which is true if $k \leq 1$. Therefore, once again in view of Corollary 1.15.4 Nagumo's Uniqueness Theorem 1.6.2 is included in Theorem 1.15.1. We further note that the condition (1.15.11) is definitely satisfied provided

$$\limsup_{x\to x_0^+} e^{-\mu(x)} > 0, \quad \text{i.e.,} \ \liminf_{x\to x_0^+} \mu(x) < \infty \tag{1.15.14}$$

which is the same as

$$\lim_{x\to x_0^+} \int_x^{x_0+a} \lambda(t)dt < \infty. \tag{1.15.15}$$

Of course, the function $\lambda(x) = k(x - x_0)^{-1}$, $0 < k \leq 1$ fails to satisfy the condition (1.15.15).

Corollary 1.15.5. For the condition (1.15.11) to be satisfied it is sufficient that

$$\ln(x - x_0) + \int_x^{x_0+a} \lambda(t)dt \not\to \infty, \quad \text{as} \quad x \to x_0^+. \tag{1.15.16}$$

Proof. If (1.15.15) holds then (1.15.16) is obvious. Thus, it suffices to exclude the case (1.15.15). This in view of (1.15.14) means that $e^{-\mu(x)} \to 0$ as $x \to x_0^+$. Hence the condition for the applicability of the one sided version of l'Hospital's rule

$$\limsup_{x\to x_0^+} \frac{F'(x)}{G'(x)} \geq \limsup_{x\to x_0^+} \frac{F(x)}{G(x)}$$

is satisfied by $F(x) = e^{-\mu(x)}$ and $G(x) = (x - x_0)$. Therefore, it follows that

$$\limsup_{x\to x_0^+} \lambda(x)e^{-\mu(x)} \geq \limsup_{x\to x_0^+} e^{-\mu(x)}(x - x_0)^{-1}.$$

Thus, the condition (1.15.11) is certainly satisfied if

$$\limsup_{x\to x_0^+} e^{-\mu(x)}(x - x_0)^{-1} > 0. \tag{1.15.17}$$

But, (1.15.17) is equivalent to $\liminf_{x\to x_0^+} e^{\mu(x)}(x - x_0) < \infty$, which in turn implies that

$$-\infty \leq \liminf_{x\to x_0^+} \left[\ln(x - x_0) + \int_x^{x_0+a} \lambda(t)dt\right] < \infty. \quad \blacksquare \tag{1.15.18}$$

Corollary 1.15.6. If $x_0 = 0$, then the function $g(x, z) = \lambda(x)\phi(z)$ is admissible in Theorem 1.15.1 provided that $\lambda(x) \geq 0$ is continuous for $0 < x \leq a$; $\phi(z)$ is continuous for $0 \leq z \leq 2b, \phi(0) = 0, \phi(z) > 0$ for $0 < z \leq 2b$; and either of the following holds

$$\limsup_{x\to 0^+} \int_x \left[\frac{1}{\phi(t)} - \lambda(t)\right] dt = \infty; \quad \text{or} \tag{1.15.19}$$

$$\limsup_{x\to 0^+} \int_x \left[\frac{1}{\phi(t)} - \lambda(t)\right] dt > -\infty \ \text{ and } \phi(z) \leq z. \tag{1.15.20}$$

Proof. From Theorem 1.15.1 it suffices to show that the only solution of the differential equation $z' = \lambda(x)\phi(z)$ satisfying (1.15.2) is $z(x) \equiv 0$. Assume the contrary, i.e., there is a solution $z(x)$ such that $z(\xi) \neq 0$, where $0 < \xi < a$. Let c be the first zero of $z(x)$ to the left of ξ. Then, $z(x) > 0$ for $0 \leq c < x \leq \xi$, and $z'(x) = \lambda(x)\phi(z(x))$. If $c = 0$, then by the hypothesis $z(c) = z'_+(c) = 0$. If $c > 0$, then $z'(c) = \lambda(c)\phi(0) = 0$. Hence, for all sufficiently small positive ε, $z(c + \varepsilon) < \varepsilon$ and

$$\int_\varepsilon^{z(\xi)} \frac{dt}{\phi(t)} \leq \int_{z(c+\varepsilon)}^{z(\xi)} \frac{dt}{\phi(t)} = \int_{c+\varepsilon}^{\xi} \lambda(t)dt.$$

Therefore, we find that

$$\int_\xi^{z(\xi)} \frac{dt}{\phi(t)} \leq \int_{c+\varepsilon}^{\xi} \lambda(t)dt - \int_\varepsilon^\xi \frac{dt}{\phi(t)} \leq \int_\varepsilon^\xi \left[\lambda(t) - \frac{1}{\phi(t)}\right] dt. \tag{1.15.21}$$

But this contradicts $z(\xi) \neq 0$, since (1.15.19) implies that the right side of (1.15.21) tends to $-\infty$ as $\varepsilon \to 0^+$. If $c > 0$, then from (1.15.20) also the same contradiction holds. If $c = 0$, then since

$$\int_{z(\varepsilon)}^{\varepsilon} \frac{dt}{\phi(t)} = \int_{\varepsilon}^{\xi} \left[\lambda(t) - \frac{1}{\phi(t)}\right] dt - \int_{\xi}^{z(\xi)} \frac{dt}{\phi(t)}$$

in view of $z'_+(0) = 0$ and (1.15.20) it follows that the left side tends to ∞ and the right side remains bounded as $\varepsilon \to 0^+$. ∎

The following particular cases of Corollary 1.15.6 give several known uniqueness criteria.

(i) **Generalized Lipschitz Uniqueness**

$$\lambda(x) \quad \text{satisfies (1.15.15);} \quad \phi(z) = z.$$

(ii) **Osgood's Uniqueness**

$$\lambda(x) = \lambda, \quad \text{a constant;} \quad \limsup_{x \to 0^+} \int_x \frac{dt}{\phi(t)} = \infty.$$

(Thus, as in Theorem 1.4.4 we note that in Theorem 1.4.2 the function g need not be nondecreasing.)

(iii) **Montel - Tonelli's Uniqueness**

$$\limsup_{x \to 0^+} \int_x \lambda(t)dt < \infty; \quad \limsup_{x \to 0^+} \int_x \frac{dt}{\phi(t)} = \infty.$$

(iv) **Nagumo's Uniqueness**

$$\lambda(x) = \frac{1}{x}; \quad \phi(z) = z.$$

(v) **van Kampen's Uniqueness**

$$\lambda(x) = \frac{1}{x} + g(x); \quad \phi(z) = z,$$

where $g(x)$ is continuous and integrable for $0 < x \leq a$.

(vi) **Iyanaga's Uniqueness**

$$\lambda(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n; \quad \phi(z) = b_1 z + b_2 z^2 + \cdots + b_m z^m, \ 0 \le \frac{b_1}{a_1} \le 1.$$

Corollary 1.15.7. The Krasnosel'skii - Krein Uniqueness Theorem 1.7.1 is a particular case of Theorem 1.15.1.

Proof. It suffices to note that in Theorem 1.15.1 the function $g(x,z) = \min\{cz^\alpha, kz/(x-x_0)\}$ for $x > x_0$ with $c > 0, k > 0, 0 < \alpha < 1$ and $k(1-\alpha) < 1$ satisfies all the required hypotheses. ∎

Corollary 1.15.8. Kooi's Uniqueness Theorem 1.8.1 is a particular case of Theorem 1.15.1.

Proof. It suffices to note that in Theorem 1.15.1 the function $g(x,z) = \min\{c(x-x_0)^{-\beta} z^\alpha, kz/(x-x_0)\}$ for $x > x_0$ with $c > 0, k > 0, 0 < \alpha < 1, \beta < \alpha$ and $k(1-\alpha) < 1 - \beta$ satisfies all the required hypotheses. ∎

1.16 SHEN'S UNIQUENESS THEOREM

A nontrivial generalization of Kamke's Uniqueness Theorem 1.15.1 is embodied in the following :

Theorem 1.16.1 (Shen's Uniqueness Theorem). Assume that

(i) the function $g(x,z)$ is continuous and nonnegative in $0 < x \le a$, $0 < z \le 2b$ and for every ξ, $0 < \xi < a$ the differential equation (1.13.1) has a sequence of positive solutions $z_{n,\xi}(x)$ satisfying :

  (a) either $\overline{\lim}_{x\to 0}\ z_{n,\xi}(x) > 0$, or $\lim_{x\to 0}\ z_{n,\xi}(x) = 0$ and $\overline{\lim}_{x\to 0}\ \frac{z_{n,\xi}(x)}{x} > 0$; and

  (b) $\lim_{n\to 0}\ z_{n,\xi}(\xi) = 0$,

(ii) the function $f(x,y)$ is continuous in $\bar{S}_+$ with $(x_0, y_0) = (0,0)$ and for all $0 < x \le a$, $-b \le y < \bar{y} \le b$ the inequality (1.13.2) is satisfied.

Then, the initial value problem (1.9.4) has at most one solution in $[0, a]$.

Proof. As usual suppose that $y(x)$ and $\bar{y}(x)$ are two solutions of the initial value problem (1.9.4) in the interval $[0, a]$. Let $\xi \in (0, a)$ be such that $y(\xi) \neq \bar{y}(\xi)$. Without loss of generality we may assume that $\bar{y}(x) \geq y(x)$ in $(0, a)$. In fact, if these $y(x)$ and $\bar{y}(x)$ do not satisfy this inequality then we can choose the maximal and the minimal solutions of (1.9.4). Let $\phi(x) = \bar{y}(x) - y(x)$, then $\phi(x) \geq 0,\ x \in (0, a),\ \phi(0) = 0,\ \phi(\xi) > 0$, and

$$\begin{aligned} \phi'(x) &= \bar{y}'(x) - y'(x) = f(x, \bar{y}(x)) - f(x, y(x)) \\ &= f(x, y(x) + \phi(x)) - f(x, y(x)). \end{aligned}$$

Let $h(x, \phi) = f(x, y(x) + \phi) - f(x, y(x))$, then $\phi(x)$ satisfies the differential equation $\phi'(x) = h(x, \phi(x))$.

Since $\xi \in (0, a)$ and $\phi(\xi) > 0$ from the given hypothesis on the function $g(x, z)$ the differential equation (1.13.1) must have a positive solution $z_{n_0,\xi}(x)$ such that

(a) either $\overline{\lim}_{x \to 0} z_{n_0,\xi}(x) > 0$, or $\lim_{x \to 0} z_{n_0,\xi}(x) = 0$ and $\overline{\lim}_{x \to 0} \frac{z_{n_0,\xi}(x)}{x} > 0$; and

(b) $z_{n_0,\xi}(\xi) < \phi(\xi)$.

Now we shall show that (a) ensures the existence of an $\eta \in (0, \xi)$ such that $z_{n_0,\xi}(\eta) > \phi(\eta)$. For this, if $\overline{\lim}_{x \to 0} z_{n_0,\xi}(x) > 0$, then for sufficiently small positive η it is impossible to have $z_{n_0,\xi}(\eta) \leq \phi(\eta)$. Since otherwise, in the limit we will have $z_{n_0,\xi}(0) \leq \phi(0) = 0$, which is certainly not true. If $\lim_{x \to 0} z_{n_0,\xi}(x) = 0$ and $\overline{\lim}_{x \to 0} \frac{z_{n_0,\xi}(x)}{x} > 0$ then we need to consider $\frac{z_{n_0,\xi}(\eta)}{\eta} \leq \frac{\phi(\eta)}{\eta}$. However, since $\phi'(0) = 0$ it follows that $\overline{\lim}_{\eta \to 0} \frac{z_{n_0,\xi}(\eta)}{\eta} \leq \phi'(0) = 0$. But, this contradicts $\overline{\lim}_{x \to 0} \frac{z_{n_0,\xi}(x)}{x} > 0$.

On the other hand due to (b) we have $z_{n_0,\xi}(\xi) < \phi(\xi)$ and therefore there must be a $\mu \in (\eta, \xi)$ such that $z_{n_0,\xi}(\mu) = \phi(\mu)$.

Now let $\nu = \text{glb}\{x : z_{n_0,\xi}(x) = \phi(x),\ x \in (\eta, \xi)\}$, then for $x \in (\eta, \nu)$ it follows that $z_{n_0,\xi}(x) > \phi(x)$, and $z_{n_0,\xi}(\nu) = \phi(\nu)$. Thus, we have

$$\frac{z_{n_0,\xi}(x) - z_{n_0,\xi}(\nu)}{x - \nu} < \frac{\phi(x) - \phi(\nu)}{x - \nu},\ x \in (\eta, \nu).$$

In the above inequality as $x \to \nu$, we find that $z'_{n_0,\xi}(\nu) \le \phi'(\nu)$. But, since

$$z'_{n_0,\xi}(\nu) = g(\nu, z_{n_0,\xi}(\nu)) = g(\nu, \phi(\nu))$$

and on the other hand in view of (1.13.2)

$$\phi'(\nu) = h(\nu, \phi(\nu)) < g(\nu, \phi(\nu))$$

we get the contradiction $g(\nu, \phi(\nu)) < g(\nu, \phi(\nu))$. Therefore, $\phi(x) \equiv 0$, or $y(x) = \bar{y}(x),\ x \in [0, a]$. ∎

If the function $g(x, z)$ satisfies the condition (i) of Theorem 1.15.1 then the condition (i) of Theorem 1.16.1 is also satisfied. In order to show this, we note that the differential equation (1.13.1) has the solutions $y_{n,\xi}(x)$ satisfying $y_{n,\xi}(\xi) = \frac{1}{n}$. Obviously this sequence of functions satisfies the condition (b). To show that it satisfies the condition (a) also, we note that if $\overline{\lim}_{x\to 0}\, z_{n,\xi}(x) > 0$ is not true then $z_{n,\xi}(0) = 0$. Hence, we must have $\overline{\lim}_{x\to 0}\, \frac{z_{n,\xi}(x)}{x} > 0$. Because, if this is not true then necessarily $z'_{n,\xi}(0) = 0$. But, then from the hypothesis (i) of Theorem 1.15.1, $z_{n,\xi}(x) \equiv 0$, and this contradicts the initial condition $y_{n,\xi}(\xi) = \frac{1}{n}$.

Example 1.16.1. The differential equation

$$z' = g(x, z) = \begin{cases} 0, & z \ge 2x^3 \\ 3(x^2 - (z - x^3)^{2/3}), & z \le 2x^3 \end{cases} \tag{1.16.1}$$

has the following sequence of positive solutions

$$z_{n,\xi}(x) = \begin{cases} x^3 + \left(\dfrac{1}{n} - x\right)^3, & x \ge \dfrac{1}{2n} \\ \dfrac{1}{4n^3}, & x \le \dfrac{1}{2n} \end{cases}$$

and this sequence satisfies the condition (i) of Theorem 1.16.1 (in condition (a) $\overline{\lim}_{x\to 0}\, z_{n,\xi}(x) > 0$ holds). However, for the differential equation (1.16.1), $z(x) = x^3$ is a nontrivial solution which satisfies the conditions $z(0) = z'(0) = 0$, and hence for (1.16.1) the condition (i) of Theorem 1.15.1 is not satisfied. ∎

Example 1.16.2. The differential equation

(1.16.2) $z' = g(x,z)$

$$= \begin{cases} \left(\frac{x^2+(x^4+4xz)^{1/2}}{x}\right)\left(\frac{x^2+(x^4+4xz)^{1/2}}{4x} - x\right), & z \geq 2x^3 \\ 3(x^2-(z-x^3)^{2/3}), & z \leq 2x^3 \end{cases}$$

has the following sequence of positive solutions

$$z_{n,\xi}(x) = \begin{cases} x^3 + \left(\frac{1}{n} - x\right)^3, & x \geq \frac{1}{2n} \\ -\frac{1}{n}\left(x - \frac{1}{2n}\right)^2 + \frac{1}{4n^3}, & x \leq \frac{1}{2n} \end{cases}$$

and this sequence satisfies the condition (i) of Theorem 1.16.1 (in condition (a) $\overline{\lim}_{x\to 0} \frac{z_{n,\xi}(x)}{x} > 0$ holds). However, for the differential equation (1.16.2), $z(x) = x^3$ is a nontrivial solution which satisfies the conditions $z(0) = z'(0) = 0$, and hence for (1.16.2) the condition (i) of Theorem 1.15.1 is not satisfied. ∎

1.17 MIKOLAJSKA'S UNIQUENESS THEOREM

Here we shall present a far reaching extension of Witte's Uniqueness Theorem 1.10.1. For this, we begin with the following result which provides sufficient conditions on the function $g(x,z)$ so that the condition (i) of Theorem 1.15.1 is guaranteed.

Theorem 1.17.1. Assume that

(i) the function $h(x,z)$ is continuous for $0 < x \leq a$, $0 \leq z < \infty$, increasing with respect to z and $h(x,0) \equiv 0$,

(ii) $\rho(x, x_1, c)$, $x_1 \in (0, a]$ is the left minimal solution of the problem

(1.17.1) $$z' = h(x,z), \ z(x_1) = c > 0,$$

(iii) the function $g(x,z)$ is continuous for $0 < x \leq a,\ 0 \leq z \leq 2b$ and that

$$g(x,z) \leq h(x,z), \tag{1.17.2}$$

(iv) for each solution $z(x)$ of the equation (1.13.1) defined in $(0, x_1]$, $0 < x_1 \leq a$ such that $\lim_{x\to 0} z(x) = 0$ there exists a function $\mu(x, z(\cdot)) \geq 0$ which is continuous in $[0,a]$, $\mu(0, z(\cdot)) = 0$, and a number δ, $0 < \delta \leq a$ such that

$$g(x, z(x)) < h(x, \rho(x, x_1, \mu(x, z(\cdot)))),\quad 0 < x \leq \delta. \tag{1.17.3}$$

Then, each solution of (1.13.1) defined in the interval $(0,\alpha), 0 < \alpha < a$ such that $\lim_{x\to 0} z(x) = 0$ is equal to 0 for $x \in (0,\alpha)$.

Proof. Suppose there exists a solution $z_1(x)$ of (1.13.1) which is defined in $(0,\alpha]$, $0 < \alpha < a$ such that $z_1(x) \not\equiv 0$, $\lim_{x\to 0} z_1(x) = 0$. Then, there exists $\xi_0 \in (0,\alpha]$ such that $c_0 = z_1(\xi_0) > 0$. Because of (1.17.2) we have $z_1'(x) \leq h(x, z_1(x))$, $x \in (0, \xi_0]$ such that $z_1(x) \geq 0$ and consequently

$$z_1(x) \geq \rho(x, \xi_0, c_0) \geq 0,\quad 0 < x \leq \xi_0. \tag{1.17.4}$$

From the hypothesis (iv) it follows that there exists $\delta_0 > 0$ such that

$$z_1'(x) < h(x, \rho(x, \xi_0, \mu(x, z_1(\cdot)))),\quad 0 < x \leq \delta_0. \tag{1.17.5}$$

Consider $\varepsilon \in (0, c_0)$. Then, there exists $x_2 \in (0, \delta_0)$ such that $0 \leq \mu(x, z_1(\cdot)) \leq \varepsilon$, $0 < x \leq x_2$.

Since $\rho(x, x_1, \zeta)$ is increasing with respect to ζ it follows that

$$\rho(x, \xi_0, \mu(x, z_1(\cdot))) \leq \rho(x, \xi_0, \varepsilon) \leq \rho(x, \xi_0, c_0),\quad 0 < x \leq x_2. \tag{1.17.6}$$

Now we will show that there exists $x_1 \in (0, x_2]$ such that

$$\rho_0 = z_1(x_1) - \rho(x_1, \xi_0, \varepsilon) > 0. \tag{1.17.7}$$

From (1.17.4) and (1.17.6) it follows that for each $x \in (0, x_2]$, $z_1(x) - \rho(x, \xi_0, \varepsilon) \geq 0$. Suppose that $z_1(x) - \rho(x, \xi_0, \varepsilon) = 0$, $0 < x \leq x_2$. Then, $z_1'(x) = \rho'(x, \xi_0, \varepsilon)$

for $0 < x \le x_2$, and hence $h(x, \rho(x, \xi_0, \varepsilon)) \equiv g(x, z_1(x))$. Thus, in view of (1.17.5) we obtain

$$h(x, \rho(x, \xi_0, \varepsilon)) < h(x, \rho(x, \xi_0, \mu(x, z_1(\cdot)))), \quad 0 < x \le x_2.$$

Hence, from (1.17.6) and the increasing nature of h we find that

$$h(x, \rho(x, \xi_0, \varepsilon)) < h(x, \rho(x, \xi_0, \varepsilon)), \quad 0 < x \le x_2$$

which is impossible, and thus there exists $x_1 \in (0, x_2]$ such that ρ_0 defined in (1.17.7) is positive.

Next, in $(0, x_2]$ we have

$$z_1'(x) < h(x, \rho(x, \xi_0, \mu(x, z_1(\cdot)))) \le h(x, \rho(x, \xi_0, \varepsilon))$$

and thus

$$\begin{aligned} z_1(x) &\ge \rho(x, \xi_0, \varepsilon) + z_1(x_1) - \rho(x_1, \xi_0, \varepsilon) \\ &= \rho(x, \xi_0, \varepsilon) + \rho_0, \ 0 < x \le x_1 \end{aligned}$$

i.e., $z_1(x) \ge \rho_0 > 0, \ 0 < x \le x_1$. But, then $0 = \lim_{x \to 0} z_1(x) = \rho_0 > 0$, and hence $z_1(x) = 0, \ 0 < x \le a$. ∎

If $g(x, z)$ is continuous for $0 < x \le a, \ 0 \le z \le 2b$ then the conditions (i) (iv) in Theorem 1.17.1 are necessary for the uniqueness of the solution $z(x)$ of (1.13.1) satisfying $\lim_{x \to 0} z(x) = 0$. This means that if $z(x) = 0$ is the unique solution of the equation (1.13.1) satisfying $\lim_{x \to 0} z(x) = 0$ then there exists a function $h(x, z)$ continuous for $0 < x \le a, \ 0 \le z < \infty$, continuously differentiable with respect to z, increasing with respect to z, and there exists $\mu(x)$ continuous in $[0, a]$, $\mu(x) > 0$ for $(0, a]$, $\mu(0) = 0$ such that

$$0 = g(x, 0) < h(x, \rho(x, x_1, \mu(x, 0)))$$

for each $x_1 \in (0, a]$. For this, it is sufficient to take for $h(x, z)$ any function which is continuous for $0 < x \le a, \ 0 \le z < \infty$, continuously differentiable with respect to z, increasing with respect to z and such that

$$h(x, z) \ge z + \max_{0 \le t \le z} |g(x, t)|.$$

It can easily be seen that such a function exists. Consider any $\mu(x)$ continuous in $[0,a]$ such that $\mu(0)=0<\mu(x)$, $0<x\le a$. From the uniqueness of the solutions of the equation $z'=h(x,z)$ it follows that $\rho'(x,x_1,\mu(x))>0$, $0<x\le a$, and hence

$$h(x,\rho(x,x_1,\mu(x)))>0=g(x,z(x))=g(x,0),\ \ 0<x\le a.$$

We further note that in Theorem 1.17.1 the inequality (1.17.3) cannot be replaced by the weak inequality. To show this we have the following :

Example 1.17.1. Let in (1.13.1) the function $g(x,z)$ be defined as follows

$$g(x,z)=\begin{cases}2x, & z\ge x^2,\ 0<x\le 1\\ 2[x-\sqrt{x^2-z}], & 0\le z\le x^2.\end{cases}$$

We set $h(x,z)=g(x,z)$, $z\ge 0$. The function $h(x,z)$ thus defined is increasing in z and continuous for $0<x\le 1$, $z\ge 0$. For (x_1,z_1) such that $x_1^2\le z_1$ we have $\rho(x,x_1,z_1)=x^2-x_1^2+z_1$, and hence $\lim_{x\to 0}\rho(x,x_1,z_1)=z_1-x_1^2$ for $z_1\ge x_1^2$. For $0\le z_1<x_1^2$ we have $z'=2x-2\sqrt{x^2-z}$ for x such that $z\le x^2$ from which it is easy to obtain

$$z(x)=2\left(x_1-\sqrt{x_1^2-z_1}\right)(x-x_1)+z_1,\ x_1\ge x\ge x_1-\sqrt{x_1^2-z_1}$$

and hence

$$\rho(x,x_1,c)=\begin{cases}2\left(x_1-\sqrt{x_1^2-c}\right)(x-x_1)+c, & x_1-\sqrt{x_1^2-z_1}\le x\le 1\\ x^2, & 0<x\le x_1-\sqrt{x_1^2-z_1}\end{cases}$$

i.e., $\rho(x,x_1,c)\to 0$ as $x\to 0$ provided $c\le x_1^2$. Each solution of (1.13.1) such that $\lim_{x\to 0}z(x)=0$, $z(x)\ne 0$ is of the form $z(x)=\rho(x,x_1,c_0)$, $0<c_0\le x_1^2$ and $z'=2x$, $0<x\le x_1-\sqrt{x_1^2-c_0}=\delta_0$.

We set

$$\mu(x,z(\cdot))=\begin{cases}x_1^2-(x-x_1)^2, & 0\le x\le\delta_0\\ c_0, & \delta_0\le x\le x_1\end{cases}$$

so that

$$\rho(x,x_1,\mu(x,z(\cdot)))=\rho(x,x_1,c_0),\ \delta_0\le x\le x_1$$

and for $0 < x \le \delta_0$,

$$x_1 - \sqrt{x_1^2 - \mu(x, z(\cdot))} = x_1 - \sqrt{x_1^2 - x_1^2 + (x - x_1)^2} = x.$$

Consequently, we find that $\rho(x, x_1, \mu(x, z(\cdot))) = x^2$ from which we obtain

$$h(x, \rho(x, x_1, \mu(x, z(\cdot)))) = 2x = z'(x),\ 0 < x \le \delta_0$$

and $\lim_{x\to 0} \mu(x, z(\cdot)) = \lim_{x\to 0}(x_1^2 - (x - x_1)^2) = 0$. ∎

If the left solutions of the equation $z' = h(x, z)$ are unique for $0 < x \le a,\ z \ge 0$ then it is possible to replace the inequality (1.17.3) be the weaker inequality

$$g(x, z(x)) \le h(x, \rho(x, x_1, \mu(x, z(\cdot)))),\ \ 0 < x < \delta_0. \tag{1.17.8}$$

Indeed in this case

$$0 < \rho(x_1, \xi_0, \varepsilon) < \rho(x_1, \xi_0, c_0) \le z_1(x_1)$$

hence $\rho_0 > 0$ for each $x_1 \in (0, \xi_0]$ and consequently the proof in this case is simpler than that of Theorem 1.17.1.

The following result is an application of Theorem 1.17.1.

Theorem 1.17.2. Let $g(x, z)$ be continuous for $0 < x \le a,\ 0 \le z \le 2b$, increasing in z and that

$$0 \le g(x, z) \le g(x)z^2,\ 0 < x \le a,\ z^2 \le (2b)^2. \tag{1.17.9}$$

Further, let for each $x_1 \in (0, a]$ there exists $x_2 \in (0, x_1)$ such that

$$g(x, 2b) < \frac{g(x)}{[2\int_x^{x_1} g(t)dt]^2},\ 0 < x \le x_2. \tag{1.17.10}$$

Then, for (1.13.1), $z(x) \equiv 0$ is the only solution satisfying the condition $\lim_{x\to 0} z(x) = 0$.

Proof. In this case we have $h(x, z) = g(x)z^2,\ 0 < x \le a$ and hence

$$\rho(x, x_1, c) = \frac{c}{1 - c\int_{x_1}^{x} g(t)dt},\ 0 < x \le x_1. \tag{1.17.11}$$

Suppose that $z(x)$ is a solution of (1.13.1) such that $0 < z(x_1) = z_1 \leq 2b$ and that $\lim_{x\to 0} z(x) = 0$. Set $\mu(x) = \max(x, 2z(x))$. From (1.17.9) it follows that

$$\mu(x) \geq 2z(x) \geq 2\sqrt{\frac{g(x,z(x))}{g(x)}}, \quad 0 < x \leq x_1.$$

But, because of (1.17.10) and the fact that $g(x,z)$ is increasing, we have

$$\sqrt{\frac{g(x,z(x))}{g(x)}} \leq \sqrt{\frac{g(x,2b)}{g(x)}} < \frac{1}{2\int_x^{x_1} g(t)dt}, \quad 0 < x \leq x_2$$

and hence

$$1 - \int_x^{x_1} g(t)dt \sqrt{\frac{g(x,z(x))}{g(x)}} > \frac{1}{2}$$

from which we obtain, finally

$$2\sqrt{\frac{g(x,z(x))}{g(x)}} \geq \frac{\sqrt{\frac{g(x,z(x))}{g(x)}}}{1 - \int_x^{x_1} g(t)dt \sqrt{\frac{g(x,z(x))}{g(x)}}}$$

so that

$$\mu(x) > \frac{\sqrt{\frac{g(x,z(x))}{g(x)}}}{1 - \int_x^{x_1} g(t)dt \sqrt{\frac{g(x,z(x))}{g(x)}}}, \quad 0 < x \leq x_2.$$

Consequently,

$$g(x,z(x)) < \left[\frac{\mu(x)}{1 + \mu(x)\int_x^{x_1} g(t)dt}\right]^2 g(x), \quad 0 < x \leq x_2.$$

Because of (1.17.9) and (1.17.11) we therefore have

$$g(x,z(x)) < h(x, \rho(x, x_1, \mu(x))).$$

From the definition of $\mu(x)$ and the fact that $\lim_{x\to 0} z(x) = 0$ it follows that $\lim_{x\to 0}\mu(x) = 0$. Thus, the conditions of Theorem 1.17.1 are satisfied, and hence it follows that $z(x) = 0, \ 0 < x \leq x_1$. ∎

It is clear that in the above theorem the increasing nature of g is not necessary. In fact, it suffices to replace (1.17.10) by

$$|g(x,z)| < \frac{g(x)}{[\int_x^{x_1} g(t)dt]^2}, \ 0 < x \leq x_2 < x_1, \ |z| \leq 2b$$

and (1.17.9) by $|g(x,z)| \le g(x)z^2$.

Theorem 1.17.3 (Mikolajska's Uniqueness Theorem). Assume that

(i) $f(x,y)$ is continuous for $0 < x \le a,\ |y - y_0| \le b_1$,

(ii) the function $h_1(x,z)$ is continuous for $0 < x \le a,\ 0 \le z < \infty$, increasing with respect to z and $h_1(x,0) \equiv 0$,

(iii) $\rho_1(x, x_1, c),\ x_1 \in (0,a],\ c \ge 0$ is the minimal solution of the initial value problem

$$z' = h_1(x,z),\ z(x_1) = c \tag{1.17.12}$$

(iv) for all $(x,y),\ (x,\bar{y})$ such that $0 < x \le a,\ y_0 - b_1 \le y,\ \bar{y} \le y_0 + b_1$ the following inequality holds

$$|f(x,y) - f(x,\bar{y})| \le h_1(x, |y-\bar{y}|) \tag{1.17.13}$$

(v) for each $\alpha \in (0,a]$ and each pair of solutions $y(x),\ \bar{y}(x)$ of $y' = f(x,y)$ defined in the interval $(0,\alpha]$ such that $\lim_{x\to 0} y(x) = \lim_{x\to 0} \bar{y}(x) = y_0$ there exists a $\delta_1 \in (0,\alpha]$ and a function $\mu_1(x,y,\bar{y}) \ge 0$, continuous in $(0,\delta_1],\ \mu_1(0,y,\bar{y}) = 0$, such that

$$\begin{aligned} &|f(x,y(x)) - f(x,\bar{y}(x))| \\ &\qquad < h_1(x, \rho_1(x, \alpha, \mu_1(x,y,\bar{y}))),\quad 0 < x \le \delta_1. \end{aligned} \tag{1.17.14}$$

Then, the initial value problem

$$y' = f(x,y),\ \lim_{x\to 0} y(x) = y_0 \tag{1.17.15}$$

has at most one solution $y(x)$ in $(0,\alpha_0],\ 0 < \alpha_0 \le a$.

Proof. Suppose $y(x)$ is a solution of (1.17.15) in the interval $(0,\lambda]$. Then, there exists $0 < \lambda_0 \le \lambda$ such that $|y(x) - y_0| \le b_1/2,\ 0 < x \le \lambda_0$. Suppose $w(x)$ is another solution of (1.17.15) in the interval $(0,\bar{\lambda}]$, where $\bar{\lambda} \le \lambda_0$. In the interval $(0,\bar{\lambda}]$ we set $y_1(x) = \min(y(x), w(x))$ and $y_2(x) = \max(y(x), w(x))$. Then, $y_1(x)$ and $y_2(x)$ are solutions of (1.17.15) and $y_1(x) \le y_2(x),\ 0 < x \le \bar{\lambda}$.

Let τ be the largest s such that $|y_i(x) - y_0| \leq b_1/2,\ 0 < x \leq s;\ i = 1,2$. We shall show that $y_1(x) = y_2(x)$ in $(0,\tau]$ from which it will follow that $|y_i(\tau) - y_0| = |y(\tau) - y_0| \leq b_1/2,\ i = 1,2$ and hence $\tau = \bar{\lambda}$.

Consider the function $z(x) = y_2(x) - y_1(x)$. It satisfies the equation

$$z' = f(x, z + y_1(x)) - f(x, y_1(x)),\ 0 < x \leq \tau.$$

We define

$$g(x,z) = \begin{cases} f(x, z + y_1(x)) - f(x, y_1(x)),\ 0 < x \leq \tau,\ 0 < z \leq b_1/2 \\ f(\tau, z + y_1(\tau)) - f(\tau, y_1(\tau)),\ \tau < x \leq a,\ 0 < z \leq b_1/2. \end{cases}$$

Clearly, $z' = g(x,z)$ satisfies the hypotheses of Theorem 1.17.1 with $b = b_1/4$, $\delta = \min(\tau, \delta_1)$, and

$$h(x,z) = \begin{cases} h_1(x,z), & 0 < x \leq \tau \\ h_1(\tau,z), & \tau < x \leq a. \end{cases}$$

Thus, each solution $z(x) \geq 0$ of $z' = g(x,z)$ such that $\lim_{x\to 0} z(x) = 0$ is equal to 0 in $(0,a]$ from which it follows that $y_1(x) = y_2(x),\ 0 < x \leq \tau$. This proves the result with $\alpha_0 = \tau$. ∎

Corollary 1.17.4. If the conditions of Theorem 1.17.3 are satisfied except that $\rho(x, x_1, c),\ x_1 \in (0,a],\ c \geq 0$ is the only solution of (1.17.12) then the inequality (1.17.14) can be replaced by the weaker inequality, i.e.,

$$(1.17.16)\quad |f(x,y(x)) - f(x,\bar{y}(x))| \leq h_1(x, \rho_1(x, \alpha, \mu_1(x, y, \bar{y}))),\ 0 < x \leq \delta_1. \quad ∎$$

If $y_0 = 0$ then the condition (1.17.16) can be simplified. Indeed, in this case it is sufficient to assume

$$|f(x,y(x))| \leq h_1(x, \rho_2(x, \mu_2(x, y(\cdot)))),$$

where $\mu_2(x, y(\cdot)) \to 0$ as $x \to 0$ and $h_1(x, \rho_2(x, c_1)) + h_1(x, \rho_2(x, c_2)) \leq h_1(x, \rho_2(x, c_1 + c_2))$, $\rho_2(x,c)$ is the solution of the initial value problem $z' = h_1(x,z),\ z(a) = c$.

In particular, if $h_1(x,z) = h(x)z$, $h(x) \geq 0$ then we have $\rho_2(x,c) = c\exp\left(\int_a^x h(t)dt\right)$, $h_1(x,\rho_2(x,c_1)) + h_1(x,\rho_2(x,c_2)) = h(x)(c_1+c_2)\exp\left(\int_a^x h(t)dt\right) = h_1(x,\rho_2(x,c_1+c_2))$.

From these observations it is clear that Witte's Uniqueness Theorem 1.10.1 is a particular case of the Corollary 1.17.4.

1.18 BRAUER'S UNIQUENESS THEOREM

The motivation of the following result is due to the Krasnosel'skii - Krein uniqueness theorem.

Theorem 1.18.1 (Brauer's Uniqueness Theorem). Suppose that

(i) the functions $A(x)$ and $B(x)$ are continuous and nonnegative in $[x_0, x_0+a]$ such that $A(x_0) = B(x_0) = 0, B(x) > 0,\ x > x_0$ and

$$\lim_{x \to x_0^+} \frac{A(x)}{B(x)} = 0, \tag{1.18.1}$$

(ii) the functions $g_1(x,z)$, $g_2(x,z)$ are continuous and nonnegative for $x_0 < x \leq x_0 + a$, $0 \leq z \leq 2b$,

(iii) all the solutions $z(x)$ of

$$z' = g_1(x,z) \tag{1.18.2}$$

with $z(x_0) = 0$ obey $z(x) \leq A(x)$ in $[x_0, x_0+a]$,

(iv) the only solution $w(x)$ of

$$w' = g_2(x,w) \tag{1.18.3}$$

in $[x_0, x_0+a]$ such that

$$\lim_{x \to x_0^+} \frac{w(x)}{B(x)} = 0 \tag{1.18.4}$$

is the trivial solution; and

(v) the function $f(x,y)$ is continuous in $\bar{S}_+$ and for $(x,y),(x,\bar{y}) \in \bar{S}_+$ where $x \neq x_0$

$$|f(x,y) - f(x,\bar{y})| \leq \begin{cases} g_1(x,|y-\bar{y}|) \\ g_2(x,|y-\bar{y}|). \end{cases} \tag{1.18.5}$$

Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0+a]$.

To prove this result it is convenient to prove the following :

Lemma 1.18.2. Let the functions $A(x), B(x), g_1(x,z)$ and $g_2(x,z)$ satisfy the hypotheses of Theorem 1.18.1. Further, let the function $g(x,z)$ be continuous and nonnegative in $x_0 \leq x \leq x_0+a,\ 0 \leq z \leq 2b,\ g(x,0) \equiv 0$, and for $x \neq x_0$

$$g(x,z) \leq \begin{cases} g_1(x,z) \\ g_2(x,z). \end{cases} \tag{1.18.6}$$

Then, for every $x_1,\ x_0 < x_1 \leq x_0+a,\ z(x) \equiv 0$ is the only differentiable function in $x_0 \leq x < x_1$, which satisfies

$$z' = g(x,z),\ z(x_0) = 0. \tag{1.18.7}$$

Proof. We shall show that the maximal solution $r(x)$ of (1.18.7) is identically zero. On the contrary, suppose that there exists a $\sigma,\ x_0 < \sigma < x_0+a$ such that $r(\sigma) > 0$. Now proceeding as in Lemma 1.15.2 and using (1.18.6) and (1.18.7), we obtain

$$\rho_2(x) \leq r(x) \tag{1.18.8}$$

as far as $\rho_2(x)$ exists to the left of σ, where $\rho_2(x)$ is the minimal solution of (1.18.3) such that $\rho_2(\sigma) = r(\sigma)$. As before, we can continue $\rho_2(x)$ up to x_0 by defining $\rho_2(x_0) = 0$. Since $\rho_2(x) \not\equiv 0$, we have $\lim_{x\to x_0^+} \rho_2(x)/B(x) \neq 0$, which in view of (1.18.8) implies that $\lim_{x\to x_0^+} r(x)/B(x) \neq 0$. This together with (1.18.1) shows that there exists a x_2 such that

$$r(x_2) > A(x_2). \tag{1.18.9}$$

Let $\rho_1(x)$ be the minimal solution of (1.18.2) such that $\rho_1(x_2) = r(x_2)$. Then it can be shown, arguing similarly, that $\rho_1(x)$ can be continued up to x_0, $\rho_1(x_0) = 0$, and

$$0 \leq \rho_1(x) \leq r(x), \ x_0 \leq x \leq x_2. \tag{1.18.10}$$

Since, by hypothesis (iii) of Theorem 1.18.1 all solutions $z(x)$ of (1.18.2) must obey $z(x) \leq A(x), x_0 \leq x \leq x_0 + a$ we must have

$$\rho_1(x) \leq A(x), \ x_0 \leq x \leq x_0 + a.$$

But this is not possible because of (1.18.9) and the fact that $\rho_1(x_2) = r(x_2)$. Hence, $\rho_1(x_0) > 0$, which implies in view of (1.18.10) that $0 < \rho_1(x_0) \leq r(x_0)$, contradicting the assumption $r(x_0) = 0$. Therefore, $r(x) \equiv 0$. ∎

Proof of Theorem 1.18.1. Consider the function $g(x,z) = g_f(x,z)$, where $g_f(x,z)$ is defined in (1.15.6). Now combining the arguments in the proofs of Theorem 1.15.1 and Lemma 1.18.2, it is clear that $g(x,z)$ verifies the conditions of Theorem 1.11.4, which is sufficient to establish the uniqueness of the solutions. ∎

Corollary 1.18.3. The Krasnosel'skii Krein Uniqueness Theorem 1.7.1 is a particular case of Theorem 1.18.1.

Proof. It suffices to note that the functions $A(x) = [c(1-\alpha)(x-x_0)]^{(1-\alpha)^{-1}}$, $B(x) = (x-x_0)^k$, $g_1(x,z) = cz^\alpha$ and $g_2(x,z) = kz/(x-x_0)$ where $c > 0$, $k > 0$, $0 < \alpha < 1$ and $k(1-\alpha) < 1$ are admissible in Theorem 1.18.1. ∎

Corollary 1.18.4. Kooi's Uniqueness Theorem 1.8.1 is a particular case of Theorem 1.18.1.

Proof. It suffices to note that the functions $A(x) = [c(1-\alpha)(1-\beta)^{-1}(x-x_0)^{1-\beta}]^{(1-\alpha)^{-1}}$, $B(x) = (x-x_0)^k, g_1(x,z) = cz^\alpha(x-x_0)^{-\beta}$ and $g_2(x,z) = kz/(x-x_0)$ where $c > 0$, $k > 0$, $0 < \alpha < 1$, $\beta < \alpha$ and $k(1-\alpha) < 1-\beta$ are admissible in Theorem 1.18.1. ∎

Corollary 1.18.5. Kamke's Uniqueness Theorem 1.15.1 is a particular case of Theorem 1.18.1.

Proof. Define the function

$$g(x,z) = \min\{g_1(x,z), g_2(x,z)\}. \tag{1.18.11}$$

This function $g(x,z)$ satisfies (1.18.6). Thus, it suffices to show that no nontrivial solution of (1.15.1) fulfills the limiting conditions (1.15.2). The assumption that there exists a differentiable function $z(x)$ satisfying the differential equation (1.15.1) and the conditions (1.15.2) for which $z(\sigma) > 0$, $x_0 < \sigma < x_0 + a$ leads to the contradiction that $z(x_0) > 0$ on following the proof of Lemma 1.18.2. ∎

1.19 WEND'S UNIQUENESS THEOREM

From Example 1.3.1 it is clear that in Peano's Uniqueness Theorem 1.3.1 the nonincreasing nature of the function $f(x,y)$ with respect to y cannot be replaced by nondecreasing. Here, we shall show that under suitable restrictions one - sided uniqueness of (1.1.1) still holds when $f(x,y)$ is nondecreasing in both x and y.

Theorem 1.19.1 (Wend's Uniqueness Theorem). Let $f(x,y)$ be continuous and nondecreasing in both x and y in $\bar{S}_+$. Further, let $f(x,y)$ be nonnegative in $\bar{S}_+$ and $f(x,y_0) \neq 0$ for all $x \in (x_0, x_0+a]$. Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0+a]$.

Proof. We may assume without loss of generality that $x_0 = 0$, since otherwise a translation would give us this case. From the given hypotheses if $y(x)$ is a solution of (1.1.1) then $y'(x) = f(x, y(x)) > 0$ in $(0,a]$. This implies that $y(x)$ is strictly increasing in x for $0 < x \leq a$. Since $y(x)$ is continuous and increasing, for $\delta > 0$ sufficiently small, there exists a η such that $0 < \eta < a$ and $y(\eta) = y_0 + \delta$. We define $y_\delta(x) = y(x+\eta)$, $0 \leq x \leq a - \eta$. Then, $y_\delta(0) = y(\eta) = y_0 + \delta$. Using the hypothesis that $f(x,y)$ is nondecreasing in x

and the fact that $y_\delta'(x) = y'(x+\eta) = f(x+\eta, y_\delta(x))$, it follows that

$$\begin{aligned} y_\delta(x) &= y_\delta(0) + \int_0^x f(t+\eta, y_\delta(t))dt \\ &= y_0 + \delta + \int_0^x f(t+\eta, y_\delta(t))dt, \end{aligned}$$

which is the same as

$$\begin{aligned} (1.19.1) \quad y_\delta(x) - y(x) &= \delta + \int_0^x (f(t+\eta, y_\delta(t)) - f(t, y(t)))dt \\ &\geq \delta + \int_0^x (f(t, y_\delta(t)) - f(t, y(t)))dt. \end{aligned}$$

Suppose $y_\delta(t) > y(t)$ for $0 \leq t < x$ and $y_\delta(x) = y(x)$, then (1.19.1) gives $f(t, y_\delta(t)) - f(t, y(t)) < 0$ for some t, $0 < t < x$, which is a contradiction since f is nondecreasing in y. Thus, it follows that $y_\delta(x) > y(x)$ for $0 \leq x \leq a - \eta$. Now let $r(x)$ be the maximal solution of $y' = f(x, y), y(0) = y_0$. Then, by the same argument for $0 < \delta' < \delta$, we have

$$y(x) \leq r(x) < y_{\delta'}(x) < y_\delta(x), \ 0 \leq x \leq a - \eta.$$

But $y_\delta(x) \to y(x)$ as $\eta \to 0$ and $\delta \to 0$. So, $y(x) \equiv r(x)$, and hence (1.1.1) has at most one solution in $[x_0, x_0 + a]$. ∎

From Theorem 1.2.6 it is clear that if $f(x, y) = f(x)g(y)$ then in Theorem 1.19.1 the nondecreasing condition with respect to x as well as y can be dropped.

1.20 van KAMPEN'S UNIQUENESS THEOREM

So far all the results we have proved impose conditions on the function $f(x, y)$. In the following result the conditions are imposed on the family of the solutions of (1.1.1).

Theorem 1.20.1 (van Kampen's Uniqueness Theorem). Let $f(x, y)$ be continuous in $\bar{S}_+$. Further, let there exist a function $\eta(x, x_1, y_1)$ in $x_0 \leq x, x_1 \leq x_0 + a$, $|y_1 - y_0| \leq \beta(< b)$ with the following properties :

(i) for a fixed (x_1, y_1), $y(x) = \eta(x, x_1, y_1)$ is a solution of

$$y' = f(x, y), \ y(x_1) = y_1 \tag{1.20.1}$$

(ii) $\eta(x, x_1, y_1)$ is uniformly Lipschitz continuous with respect to y_1; and

(iii) no two solution arcs $y(x) = \eta(x, x_1, y_1)$, $y(x) = \eta(x, x_2, y_2)$ pass through the same point (x, y) unless $\eta(x, x_1, y_1) = \eta(x, x_2, y_2)$ for $x_0 \le x \le x_0 + a$.

Then, $y(x) = \eta(x, x_0, y_0)$ is the only solution of (1.1.1) in $[x_0, x_0 + a]$.

Proof. Let $y(x)$ be any solution of (1.1.1). We will show that $y(x) = \eta(x, x_0, y_0)$ for small $x - x_0 \ge 0$. Condition (ii) implies that there exists a constant K such that

$$|\eta(x, x_1, y_1) - \eta(x, x_1, y_2)| \le K|y_1 - y_2| \tag{1.20.2}$$

for $x_0 \le x$, $x_1 \le x_0 + a$ and $|y_1 - y_0| \le \beta, |y_2 - y_0| \le \beta$. Let $|f(x, y)| \le M$ in $\bar{S}_+$. Then, any solution $y(x)$ of (1.1.1) satisfies $|y(x) - y_0| \le M(x - x_0) \le \frac{1}{2}\beta$ if $x_0 \le x \le x_0 + \frac{\beta}{2M}$. Thus, $\eta(x, s, y(s))$ is defined and $|\eta(x, s, y(s)) - y(s)| \le M|x - s| \le \frac{1}{2}\beta$ if $x_0 \le x, s \le x_0 + \frac{\beta}{2M}$. Hence, $|\eta(x, s, y(s)) - y_0| \le \beta$ if $x_0 \le x, s \le x_0 + \gamma$, where $\gamma = \min(a, \frac{\beta}{2M})$. Condition (iii) means that any point on any of the arcs $y(x) = \eta(x, x_1, y_1)$ can be used to determine this arc. Thus, (1.20.2) with $y_1 = y(x_1)$ and $y_2 = \eta(x_1, s, y(s))$ gives

$$|\eta(x, x_1, y(x_1)) - \eta(x, s, y(s))| \le K|y(x_1) - \eta(x_1, s, y(s))| \tag{1.20.3}$$

if $x_0 \le x, x_1, s \le x_0 + \gamma$. Now, let x be fixed in $[x_0, x_0 + \gamma]$. We will show that

$$\tau(x) = \eta(x, x_0, y_0) - y(x) = 0. \tag{1.20.4}$$

For this, we put

$$\sigma(s) = \eta(x, x_0, y_0) - \eta(x, s, y(s)) \tag{1.20.5}$$

for $x_0 \le s \le x$ $(\le x_0 + \gamma)$, so that $\sigma(x_0) = 0$ and $\sigma(x) = \tau(x)$. Then, (1.20.3) and (1.20.5) imply that

$$|\sigma(x_1) - \sigma(s)| \le K|y(x_1) - \eta(x_1, s, y(s))|. \tag{1.20.6}$$

Since $y(x) = \eta(x, s, y(s))$ is a solution of $y' = f(x,y)$ through the point $(s, y(s))$, it is seen that $\eta(x_1, s, y(s)) = y(s) + (x_1 - s)[f(s, y(s)) + o(1)]$ as $x_1 \to s$. Also, $y(x_1) = y(s) + (x_1 - s)[f(s, y(s)) + o(1)]$ as $x_1 \to s$. Hence, (1.20.6) gives that $\sigma(x_1) - \sigma(s) = Ko(1)|x_1 - s|$ as $x_1 \to s$; i.e., $\frac{d\sigma}{ds}$ exists and is 0. Thus, $\sigma(s)$ is the constant $\sigma(x_0) = 0$ for $x_0 \le s \le x$. In particular, $\tau(x) = \sigma(x)$ satisfies (1.20.4), as was to be proved. ∎

The existence of a continuous function $\eta(x, x_1, y_1)$ satisfying the conditions (i) and (iii) of Theorem 1.20.1 is not sufficient for the uniqueness of the solutions of (1.1.1). For this, we note that the continuous function $y(x) = \eta(x, x_1, y_1) = [y_1^{1/3} + \frac{1}{3}(x - x_1)]^3$ is a solution of $y' = y^{2/3}$, $y(x_1) = y_1$. Obviously, this function $\eta(x, x_1, y_1)$ is not uniformly Lipschitz continuous with respect to y_1. Further, $\eta(x^*, x_1, y_1) = \eta(x^*, x_2, y_2)$ holds provided that $y_1^{1/3} - y_2^{1/3} = \frac{1}{3}(x_1 - x_2)$. But, this condition implies that $\eta(x, x_1, y_1) = \eta(x, x_2, y_2)$ for all $x \ge 0$. Of course, the initial value problem $y' = y^{2/3}$, $y(0) = 0$ has an infinite number of solutions.

If $f(x, y)$ satisfies the uniform Lipschitz condition (1.2.1), then for small $\beta > 0$ a function $\eta(x, x_1, y_1)$ satisfying the conditions of Theorem 1.20.1 exists. However, the converse is not true, i.e., the existence of $\eta(x, x_1, y_1)$ satisfying the conditions (i) - (iii) of Theorem 1.20.1 does not imply that $f(x, y)$ is uniformly Lipschitz continuous with respect to y. For this it is sufficient to consider the differential equation $y' = g(y)$, where $g(y) > 0$ is continuous for all y.

A one - sided analog of van Kampen's uniqueness theorem is the following :

Theorem 1.20.2. Let $f(x, y)$ be continuous in $\bar{S}_+$. Further, let there exist a function $\eta(x, x_1, y_1)$ in $x_0 \le x_1 \le x \le x_0 + a$, $|y_1 - y_0| \le \beta (< b)$ with the following properties :

(i) for a fixed (x_1, y_1), $y(x) = \eta(x, x_1, y_1)$ is a solution of (1.20.1); and

(ii) there exists a constant K such that for $\max(x_1, x_2) \le x^* \le x \le x_0 + a$ and $|y_1 - y_0| \le \beta$, $|y_2 - y_0| \le \beta$

$$|\eta(x,x_1,y_1)-\eta(x,x_2,y_2)| \leq K|\eta(x^*,x_1,y_1)-\eta(x^*,x_2,y_2)|.$$

Then, $y(x)=\eta(x,x_1,y_1)$ is the only solution of (1.20.1) for sufficiently small intervals $[x_1, x_1+\varepsilon]$, $\varepsilon > 0$ to the right of x_1 (but not necessarily to the left of x_1). ∎

1.21 YOSIE'S UNIQUENESS THEOREM

So far, we have discussed several sufficient conditions for the uniqueness of the solutions of the initial value problem (1.1.1). Here we shall give the first known result which provides the necessary as well as sufficient conditions. To begin with, we note that if $f(x,y)$ is continuous in $\bar{S}_+$ then for the initial value problem (1.1.1) Theorem 1.11.1 ensures the existence of the maximal solution $r(x)$ and the minimal solution $\rho(x)$ in the interval $[x_0, x_0+\alpha]$. For simplicity, hereafter we shall assume that $\alpha = a$. Thus, if the initial value problem has any other solution $y(x)$ in the interval $[x_0, x_0+a]$, then it is necessary that

$$\rho(x) \leq y(x) \leq r(x), \ x \in [x_0, x_0+a]. \tag{1.21.1}$$

Therefore, a necessary and sufficient condition for the uniqueness of the solutions of the initial value problem (1.1.1) is

$$\rho(x) = r(x), \ x \in [x_0, x_0+a]. \tag{1.21.2}$$

This condition can be written in a different form, for this we need the following :

Definition 1.21.1. Suppose that the function $y(x)$ is continuous in the interval $[x_0, x_0+a]$, and $y'(x)$ exists. If $y(x)$ is such that $y(x_0)=y_0$, and satisfies the inequality

$$y'_+(x) < f(x,y(x)), \ x \in [x_0, x_0+a]$$

then it is said to be a lower - function with respect to the initial value problem (1.1.1). On the other hand, if

$$y'_+(x) > f(x,y(x)), \ x \in [x_0, x_0+a]$$

then $y(x)$ is said to be an upper - function.

Theorem 1.21.1 [92]. Let $\phi(x)$ and $\psi(x)$ be lower - and upper functions with respect to the initial value problem (1.1.1) in the interval $[x_0, x_0 + a]$. If $y(x)$ is any solution of (1.1.1) in the interval $[x_0, x_0 + a]$, then

$$\phi(x) < y(x) < \psi(x), \; x \in (x_0, x_0 + a]. \quad \blacksquare \tag{1.21.3}$$

Thus, in view of (1.21.1) and (1.21.3) and the fact that $\rho(x)$ and $r(x)$ are solutions of (1.1.1) it follows that

$$\phi(x) < \rho(x) \le y(x) \le r(x) < \psi(x), \; x \in (x_0, x_0 + a]. \tag{1.21.4}$$

Therefore, if for every $\varepsilon > 0$ there exists a pair of lower - and upper - functions $\phi(x)$ and $\psi(x)$ with respect to the initial value problem (1.1.1) such that $0 < \psi(x) - \phi(x) < \varepsilon$ in the interval $(x_0, x_0 + a]$ then, the condition (1.21.2) must be satisfied. Hence, the following result holds.

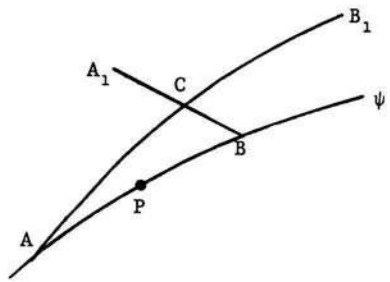
Theorem 1.21.2 (Yosie's Uniqueness Theorem). The initial value problem (1.1.1) has at most one solution in the interval $[x_0, x_0 + a]$ if and only if for every $\varepsilon > 0$ there exists a pair of lower - and upper - functions $\phi(x)$ and $\psi(x)$ with respect to the initial value problem (1.1.1) such that $0 < \psi(x) - \phi(x) < \varepsilon$ in the interval $(x_0, x_0 + a]$. $\blacksquare$

Note that by definition, although, for an arbitrary but fixed function $\psi(x)$

$$\psi'_+(x) - f(x, \psi(x)) > 0, \; x \in [x_0, x_0 + a] \tag{1.21.5}$$

the lower limit of $\psi'_+(x) - f(x, \psi(x))$ may be zero. Such an upper - function will now be replaced by $\bar{\psi}(x)$ which is close to $\psi(x)$, but $\bar{\psi}'_+(x) - f(x, \bar{\psi}(x))$ is greater than a fixed positive number for all $x \in [x_0, x_0 + a]$.

If the lower limit of $\psi'_+(x) - f(x, \psi(x))$ is zero, then on the curve $y = \psi(x)$, $x_0 \le x \le x_0 + a$ we can find a sequence of points $\{P_1, P_2, \cdots\}$ for which the lower limit of this difference is zero. Let P be the limit of this sequence. Obviously, at P this difference is non - zero since (1.21.5) holds everywhere on the curve $y = \psi(x)$ in the interval $[x_0, x_0 + a]$.



Assume that P is different from the end points of the curve $y = \psi(x)$, $x_0 \leq x \leq x_0 + a$. Then, we take a sufficiently small piece AB of this curve which contains P. In the following we shall denote by x_A, x_B, f_A, f_B etc. the values of x, y, f etc. at the points A, B and so on. We also assume that $x_A < x_P < x_B$. Since the difference $(\psi'_+ - f)_B$ is non zero we can set $(\psi'_+ - f)_B \geq \eta_2$, where η_2 is a sufficiently small positive number.

The line BA_1

$$y = y_B + (f_B + \eta_2)(x - x_B) \tag{1.21.6}$$

through the point B is above the curve $y = \psi(x)$ if $x_B - x$ is sufficiently small and positive. This is because ψ'_+ is always greater than $f(x, \psi)$, and hence also greater than $f(x, \psi) + \eta_2$ for sufficiently small η_2. From the continuity of the function f it follows that

$$\psi'_+(x) > f_B + \eta_2$$

for sufficiently small $x_B - x$. We also assume that $x_B - x_A$ is small enough so that the line (1.21.6) is above the curve $y = \psi(x)$.

If η_1 is small and positive then the curve AB_1

$$y = \psi(x) + \eta_1(x - x_A) \tag{1.21.7}$$

passes through the point A and is above the curve $y = \psi(x)$, as long as $x - x_A > 0$. One can bring (1.21.7) close enough to the curve $y = \psi(x)$ by choosing η_1 small enough. The difference of the ordinates AB_1 and A_1B

$$[\psi(x) + \eta_1(x - x_A)] - [y_B + (f_B + \eta_2)(x - x_B)]$$

is obviously a continuous function of x, and is negative at A and is positive at B because AB_1 and A_1B in the interval (x_A, x_B) are above the curve $y = \psi(x)$. Hence, it vanishes at least once in the interval (x_A, x_B), which means that AB_1 and A_1B have at least one common point C where $x_A < x_C < x_B$. If η_1 is small enough than C is arbitrarily close to the point B, along the line A_1B.

Since $f(x,y)$ is continuous one can choose small convex area G around B in which for every point Q the condition

$$|f_Q - f_B| < \eta_3 < \eta_2 \tag{1.21.8}$$

is satisfied. Here η_3 is arbitrarily small.

Now we take η_1 so small that the point C is in G. Then, the equation of the dotted line ACB can be written as

$$y = \begin{cases} \psi(x) + \eta_1(x - x_A), & x_A \le x \le x_C \\ y_B + (f_B + \eta_2)(x - x_B), & x_C \le x \le x_B. \end{cases}$$

Along AC one has

$$\begin{aligned} y'_+ - f(x,y) &= \psi'_+(x) + \eta_1 - f(x, \psi(x) + \eta_1(x - x_A)) \\ &= [\psi'_+(x) - f(x,\psi)] + \eta_1 + [f(x,\psi) - f(x, \psi + \eta_1(x - x_A))] \\ &> \eta_1 + [f(x,\psi) - f(x, \psi + \eta_1(x - x_A))]. \end{aligned}$$

From the continuity of $f(x,y)$, for an arbitrarily small ρ $(0 < \rho < \eta_1)$ one can set

$$|f(x,\psi) - f(x, \psi + \eta_1(x - x_A))| < \eta_1 - \rho, \ x_A \le x \le x_C$$

provided $(x_B - x_A)$ is small enough.

Then, it follows that

$$y'_+ - f(x,y) > \rho, \ x_A \le x \le x_C$$

and therefore along CB in view of (1.21.8) and the fact that C is in G, one has

$$\begin{aligned} y'_+ - f(x,y) &= f_B + \eta_2 - f(x, y_B + (f_B + \eta_2)(x - x_B)) \\ &> \eta_2 - \eta_3. \end{aligned}$$

Thus, along the dotted line ACB the difference $y'_+ - f(x,y)$ is always positive. Moreover, it is greater than a fixed positive value, because ρ and η_3 can always be chosen so that

$$0 < \rho < \eta_1, \quad 0 < \eta_3 < \eta_2.$$

Finally, we remark that the dotted line ACB can be pushed arbitrarily close to the curve $y = \psi(x)$. Thus, if the piece AB of the curve $y = \psi(x)$ is replaced by the dotted line ACB then, one can avoid the point at which the limit of $y'_+ - f(x,y)$ equals zero.

The above method with slight modifications is equally valid if P equals to A or B.

If such points P are everywhere dense on a part of the curve, then we divide it into several parts such that every part is small enough. Since the value of $y'_+ - f(x,y)$ is positive at the end points of these parts, one can replace each part (as above) by the dotted line on which the lower limit of $y'_+ - f(x,y)$ is not equal to zero, and which is close enough to the curve $y = \psi(x)$.

From these considerations in the uniqueness condition we may assume that the lower - and upper - functions are such that in the interval $x_0 \le x \le x_0 + a$ the differences $\psi'_+(x) - f(x,\psi(x))$ and $f(x,\phi(x)) - \phi'_+(x)$ have positive lower limits, i.e., there exist μ_1 and μ_2 positive numbers such that

$$\begin{aligned} &\psi'_+(x) - f(x,\psi(x)) \ge \mu_1 \\ &\phi'_+(x) - f(x,\phi(x)) \le -\mu_2, \quad x_0 \le x \le x_0 + a. \end{aligned} \tag{1.21.9}$$

Now let A and B be two points on the curve $y = \psi(x)$ and let ρ be a positive number less than μ_1.

Let G be the convex area containing the line AB. If G is small enough then since $f(x,y)$ is continuous, for each pair of points M and N in G we have

$$|f_M - f_N| < \rho < \mu_1. \tag{1.21.10}$$

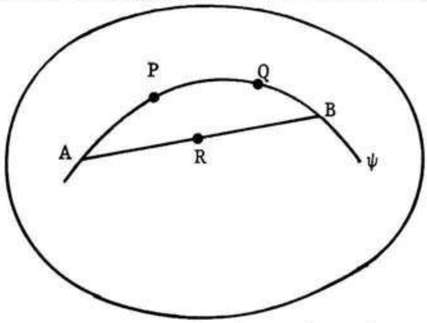
Let

$$t = \frac{y_B - y_A}{x_B - x_A},$$

then on the piece AB of the curve $y = \psi(x)$ we can find two points P and Q such that

$$(\psi'_+(x))_P \geq t \geq (\psi'_+(x))_Q \tag{1.21.11}$$

otherwise the two lines starting from A would not meet again for $x > x_A$.



Let R be any point of AB then by (1.21.10), we have

$$f_Q > f_R - \rho.$$

Moreover, from (1.21.9) and (1.21.11) it follows that

$$(\psi'_+)_Q - f_Q \geq \mu_1 \quad \text{and} \quad t - (\psi'_+)_Q \geq 0.$$

From these three inequalities, we find that

$$t - f_Q \geq \mu_1 \quad \text{and} \quad t - f_R > \mu_1 - \rho.$$

Along AB then one has the inequality

$$y'_+ - f(x, y) > \mu_1 - \rho > 0. \tag{1.21.12}$$

If we replace the curve piece AB of $y = \psi(x)$ by the line AB then for the curve $y = \bar{\psi}(x)$ obtained in this way we have the inequality $\bar{\psi}'_+ - f(x, \bar{\psi}) > 0$. This inequality characterizes the upper - function. Of course, the line AB can be chosen arbitrarily close to $\psi(x)$ if $|x_A - x_B|$ is small enough.

Because of this reason one can replace the function $\psi(x)$ from the uniqueness condition by such a function $\bar{\psi}(x)$. A similar remark holds for the function $\phi(x)$.

Now we divide the curves $y = \psi(x)$ and $y = \phi(x)$ in to a finite number of small pieces, and replace each piece by its chord so that one gets two polygons passing through the point (x_0, y_0). Thus, we can reformulate Theorem 1.21.2 as follows :

Theorem 1.21.3. The initial value problem (1.1.1) has at most one solution in the interval $[x_0, x_0 + a]$ if and only if for every $\varepsilon > 0$ there exists a pair of polygons $y = \bar{\phi}(x)$ and $y = \bar{\psi}(x)$ such that

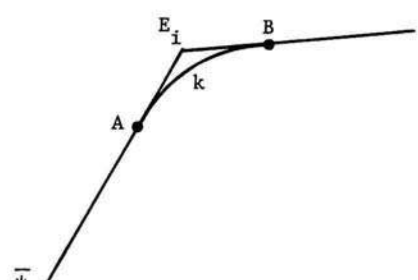
$$
\begin{aligned}
&0 < \bar{\psi}(x) - \bar{\phi}(x) < \varepsilon, && x \in (x_0, x_0 + a] \\
&\bar{\psi}'_+(x) - f(x, \bar{\psi}(x)) > 0, && x \in [x_0, x_0 + a] \\
&\bar{\phi}'_+(x) - f(x, \bar{\phi}(x)) < 0, && x \in [x_0, x_0 + a] \\
&\bar{\psi}(x_0) = \bar{\phi}(x_0) = y_0
\end{aligned}
\tag{1.21.13}
$$

and these polygons consist of a finite number of chords. The number of these chords depends on ε. ∎

The functions $\bar{\phi}(x)$ and $\bar{\psi}(x)$ are continuous in the interval $x_0 \leq x \leq x_0 + a$ whereas their derivatives make finite jumps at the vertex of these polygons $y = \bar{\phi}(x)$ and $y = \bar{\psi}(x)$. If the vertex of $y = \bar{\psi}(x)$ are $E_1,\ E_2, \cdots$ then from (1.21.12) it follows that

$$
\bar{\psi}'_+ - f(x, \bar{\psi}) > \mu, \quad x_0 \leq x \leq x_0 + a \tag{1.21.14}
$$

where μ is a small positive number.



For the vertex E_i, we choose two points A and B on the polygon $y = \bar{\psi}(x)$ having the same distance from E_i. We connect these points by a circular arc

$y = k(x)$ which meets the line AE_i at A and E_iB at B. In the interval $x_A \leq x \leq x_B$ the difference $|f(x,k) - f(x,\bar{\psi})|$ will be small provided A and B are close enough. Now we set

$$|f(x,k) - f(x,\bar{\psi})| < \varepsilon_i, \; x_A \leq x \leq x_B$$

where $\varepsilon_i > 0$ and less than μ. Then, from (1.21.14) we get

$$\bar{\psi}'_+ - f(x,k) > \mu + \theta\varepsilon_i > 0, \; -1 < \theta < 1. \tag{1.21.15}$$

Let t_1 and t_2 be the direction coefficients of the lines AE_i and E_iB and let t be the minimum of these two, then obviously $k'_+ = \frac{dk}{dx}$ has the value between t_1 and t_2 at each point of the interval $x_A < x < x_B$. Hence, $k'_+ - t > 0$ for $x_A < x < x_B$ and 0 at x_A and x_B. Thus, from (1.21.15), we have

$$t - f(x,k) > \mu + \theta\varepsilon_i$$

and hence

$$k'_+ - f(x,k) > \mu + \theta\varepsilon_i > 0.$$

Hence, we can replace the dotted line AB by the circular arc $y = k(x)$ so that the obtained curve $y = \xi_i(x)$ has all the properties of an upper - function. Following the same at every E_i leads to an upper - function $\xi(x)$ which in the whole interval $x_0 \leq x \leq x_0 + a$ has continuous derivative and is close to the curve $y = \bar{\psi}(x)$. Similarly, one can construct a lower - function which is close to $y = \bar{\phi}(x)$ and is continuously differentiable.

Since the derivative $\xi'(x)$ is continuous in $[x_0, x_0 + a]$, for a given small $\bar{\varepsilon} > 0$ we can construct a polynomial $Q(x)$ such that

$$|\xi'(x) - Q(x)| < \frac{\bar{\varepsilon}}{a}.$$

Thus, by an integration it follows that

$$|\xi(x) - P(x)| < \bar{\varepsilon}, \; x_0 \leq x \leq x_0 + a$$

where

$$P(x) = \int_{x_0}^{x} Q(t)dt + \xi(x_0).$$

Since $P(x)$ is also a polynomial and $P'(x) = Q(x)$ it is clear that $P(x_0) = \xi(x_0) = y_0$. Therefore, the uniqueness result can be restated as follows :

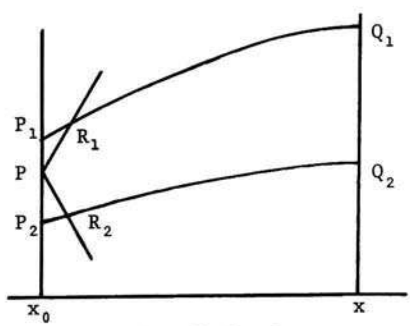
Theorem 1.21.4. The initial value problem (1.1.1) has at most one solution in the interval $[x_0, x_0 + a]$ if and only if for every $\varepsilon > 0$ there exists a pair of polynomials $P_1(x)$ and $P_2(x)$ such that

$$\begin{aligned} & 0 < P_1(x) - P_2(x) < \varepsilon, && x \in (x_0, x_0 + a] \\ & P'_{1+}(x) - f(x, P_1(x)) > 0, && x \in [x_0, x_0 + a] \\ & P'_{2+}(x) - f(x, P_2(x)) < 0, && x \in [x_0, x_0 + a] \\ & P_1(x_0) = P_2(x_0) = y_0. && \blacksquare \end{aligned} \tag{1.21.16}$$

Remark 1.21.1. We note that the fourth condition in (1.21.13) as well as (1.21.16) is not necessary. In fact these conditions can be replaced by

$$\begin{aligned} & \bar{\psi}(x_0) \geq y_0 \geq \bar{\phi}(x_0) \quad \text{and} \\ & P_1(x_0) \geq y_0 \geq P_2(x_0), \end{aligned} \tag{1.21.17}$$

respectively.



To show this, let P_1Q_1 and P_2Q_2 be the curves $y = P_1(x)$ and $y = P_2(x)$, and let P be the point (x_0, y_0). Let M be a positive number such that $|f(x, y)| < M$, $(x, y) \in \bar{S}_+$. Take two lines PR_1 and PR_2 through P which correspond to upper - and lower - functions and whose direction coefficients are M and $-M$. Then,

$$y' - f(x, y) = M - f(x, y) > 0 \quad \text{on} \quad PR_1,$$

and

$$y' - f(x,y) = -M - f(x,y) < 0 \quad \text{on} \quad PR_2$$

holds. Let R_1 and R_2 be the points where PR_1 and PR_2 intersect P_1Q_1 and P_2Q_2, respectively. Then, the curves PR_1Q_1 and PR_2Q_2 give the required upper - and lower - functions. ∎

Example 1.21.1. If the function $f(x,y)$ is monotonically decreasing in y, then the initial value problem (1.1.1) has at most one solution in $[x_0, x_0+a]$. For this, let $y(x)$ be a solution of (1.1.1), then consider the curves $y = y(x) \pm \nu$ $(\nu > 0)$, $x_0 \le x \le x_0 + a$. Since by assumption we have

$$f(x, y+\nu) < f(x,y)$$

$$f(x, y-\nu) > f(x,y), \quad x_0 \le x \le x_0 + a$$

it follows that

$$(y+\nu)'_+ = y'_+ = f(x,y) > f(x, y+\nu)$$

$$(y-\nu)'_+ = y'_+ = f(x,y) < f(x, y-\nu), \ x_0 \le x \le x_0 + a.$$

If $\nu > 0$ is sufficiently small then the curves $y = y(x) \pm \nu$ are very close, moreover

$$y(x_0) + \nu > y(x_0) = y_0 > y(x_0) - \nu.$$

Therefore, in view of Theorem 1.21.2 and the Remark 1.21.1, the conclusion follows.

Example 1.21.2. If the function $f(x,y)$ satisfies the Lipschitz condition (1.2.1) in $\bar{S}_+$ then the initial value problem (1.1.1) has at most one solution in $[x_0, x_0+a]$. For this, once again let $y(x)$ be a solution of (1.1.1). Since $y(x)$ is continuously differentiable in $[x_0, x_0+a]$, for every $\nu_1 > 0$ there exists a polynomial $P(x)$ such that

$$(1.21.18) \qquad |y(x) - P(x)| < \nu_1, \quad \text{and} \quad |y'(x) - P'(x)| < \nu_1.$$

By the continuity of $f(x,y)$ we can set

$$(1.21.19) \qquad |f(x,y) - f(x,P)| < \nu_2,$$

where ν_2 tends to zero if ν_1 does. From (1.21.18) and (1.21.19) and $y' = f(x, y)$ it follows that

$$|P'(x) - f(x, P(x))| < \nu_1 + \nu_2, \quad x_0 \leq x \leq x_0 + a. \tag{1.21.20}$$

Now consider the polynomial $P_1(x) = P(x) + \alpha(x - x_0)$, $\alpha > 0$. For sufficiently small α the curve $y = P_1(x)$ is in $\bar{S}_+$, and hence by (1.2.1), we find that

$$|f(x, P) - f(x, P_1)| < L|P - P_1| < L\alpha|x - x_0|, \; x_0 \leq x \leq x_0 + a. \tag{1.21.21}$$

From (1.21.20) and (1.21.21) it follows that

$$\begin{aligned} P'(x) &= f(x, P) \pm \theta_1(\nu_1 + \nu_2) \\ f(x, P_1) &= f(x, P) \pm \theta_2 L\alpha|x - x_0|, \; |\theta_1|, \; \theta_2| < 1. \end{aligned} \tag{1.21.22}$$

Now we choose an interval (x_0, x_1) so small that $L|x_1 - x_0| < 1$ and ν_1 and ν_2 small enough so that

$$\nu_1 + \nu_2 + L\alpha|x_1 - x_0| < \alpha.$$

Then, from (1.21.22) we get

$$|P'(x) - f(x, P_1(x))| < \alpha, \quad x_0 \leq x \leq x_0 + a. \tag{1.21.23}$$

However, since $P_1'(x) = P'(x) + \alpha$, we have $P_1'(x) > f(x, P_1(x))$. Similarly, one can find a polynomial $P_2(x) = P(x) - \beta(x - x_0)$, $\beta > 0$ such that $P_2'(x) < f(x, P_2(x))$. Finally, we note that for a given $\varepsilon > 0$ the difference $P_1(x) - P_2(x) = (\alpha + \beta)(x - x_0)$ in the interval (x_0, x_1) can be made less than ε, only if α and β are small. The conclusion now follows from Theorem 1.21.4 and the Remark 1.21.1.

Finally, we remark that on following as in Example 1.21.2 the conclusion of Theorem 1.4.4 can also be deduced from Theorem 1.21.4 and the Remark 1.21.1.

1.22 KITAGAWA'S UNIQUENESS THEOREM

Let the function $f(x,y)$ be continuous for $x_0 \le x \le x_0 + a$, $\mu(x) \le y \le \nu(x)$, where $\mu(x)$ and $\nu(x)$ are two continuous functions in the interval $[x_0, x_0 + a]$ and satisfy the following conditions :

1^o. $\mu(x_0) = y_0 = \nu(x_0)$,

2^o $\mu'_+(x) \le f(x, \mu(x))$; and

3^o $\nu'_+(x) \ge f(x, \nu(x))$.

Then, it is well known that for the initial value problem (1.1.1) there exist maximal and minimal solutions $\rho(x)$ and $r(x)$ and sequences $\{\phi_n(x)\}$ and $\{\psi_n(x)\}$ of lower - and upper - functions which converge uniformly to $\rho(x)$ and $r(x)$, respectively. Here we shall combine this result with Yosie's Uniqueness Theorem 1.22.2 to deduce a property of the maximal and minimal solutions when the solutions of (1.1.1) are not unique in the interval $[x_0, x_0+a]$. Further, as an immediate consequence of this property, we shall obtain two sets of sufficient conditions for the uniqueness of the solutions of (1.1.1). For this, throughout it is necessary to assume that

$$D_+\mu(x_0) < f(x_0, y_0) < D_+\nu(x_0). \tag{1.22.1}$$

Let $n(u, v, x_1)$ be an auxiliary neighborhood of the minimal solution $\rho(x)$, defined by

$$\begin{cases} x_0 \le x \le x_1 \le x_0 + a \\ u(x) < y < v(x), \end{cases}$$

where $u(x)$, $v(x)$ are two continuous functions in $[x_0, x_1]$ such that

1. $D_+\mu(x_0) \le D_+u(x_0) < f(x_0, y_0) < D_+v(x_0) \le D_+\nu(x_0)$,
2. $\mu(x) \le u(x) < \rho(x) < v(x) \le \nu(x)$, $x \in (x_0, x_1]$; and
3. $u(x_0) = v(x_0) = y_0$.

Similarly, let $N(U, V, X_1)$, be an auxiliary neighborhood of the maximal solution $r(x)$, defined by

$$\begin{cases} x_0 \le x \le X_1 \le x_0 + a \\ U(x) < y < V(x), \end{cases}$$

where $U(x),\ V(x)$ are two continuous functions in $[x_0, X_1]$ such that

1. $D_+\mu(x_0) \le D_+U(x_0) < f(x_0, y_0) < D_+V(x_0) \le D_+\nu(x_0)$,
2. $\mu(x) \le U(x) < r(x) < V(x) \le \nu(x),\ x \in (x_0, X_1]$; and
3. $U(x_0) = V(x_0) = y_0$.

Theorem 1.22.1. If the initial value problem (1.1.1) has more than one solutions in the interval $[x_0, x_0 + a]$, then the following hold

(i) for any $n(u, v, x_1)$ there exists at least one point ξ such that

$$\begin{vmatrix} f(\xi, y) & y & 1 \\ f(\xi, y_\rho) & y_\rho & 1 \\ f(\xi, \bar{y}) & \bar{y} & 1 \end{vmatrix} < 0$$

where $u(\xi) < y < y_\rho = \rho(\xi) < \bar{y} < v(\xi)$; and

(ii) for any $N(U, V, X_1)$ there exists at least one point Ξ such that

$$\begin{vmatrix} f(\Xi, Y) & Y & 1 \\ f(\Xi, Y_r) & Y_r & 1 \\ f(\Xi, \bar{Y}) & \bar{Y} & 1 \end{vmatrix} > 0$$

where $U(\Xi) < Y < Y_r = r(\Xi) < \bar{Y} < V(\Xi)$.

To prove this result we need the following :

Lemma 1.22.2. For any small $\varepsilon > 0$ and any given neighborhood $n(u, v, x_1)$ there exists a lower - function $\bar{\alpha}(x)$ such that, for

$$\begin{cases} \rho(x) - \bar{\alpha}(x) = d(x) \\ \min[\varepsilon,\ \rho(x) - u(x),\ v(x) - \rho(x)] = b(x) \end{cases}$$

it follows that

(i) $0 < d(x) < b(x),\ x_0 < x \le x_1$,

(ii) $d(x_0) = b(x_0) = 0$,

(iii) $D_+d(x_0) < D_+b(x_0)$; and

(iv) $u(x) < \bar{\alpha}(x) = \rho(x) - d(x) < \rho(x) < \rho(x) + d(x) < v(x),\ x_0 < x \le x_1$.

Proof. First we shall show that (iv) is a consequence of (i) (iii). For this, from (i) we have $0 < d(x) = \rho(x) - \bar{\alpha}(x) < b(x) \le \rho(x) - u(x)$, and hence $u(x) < \bar{\alpha}(x) = \rho(x) - d(x) < \rho(x) < \rho(x) + d(x) < \rho(x) + b(x) \leq \rho(x) + v(x) - \rho(x) = v(x)$.

Next, since the function $f(x,y)$ is continuous at the point (x_0, y_0), we can find positive numbers δ_1 and η such that

$$f(x,y) > f(x_0,y_0) - \eta > f(x_0,y_0) - D_+b(x_0)$$

in the domain $x_0 \leq x < x_0 + \delta_1,\ \rho(x) \leq y \leq r(x)$.

Now consider the function $L(x) = y_0 + (f(x_0,y_0) - \eta)(x - x_0)$, which has the following properties :

1. $L(x_0) = y_0$,
2. $L'(x) < f(x, L(x))$; and
3. $\rho(x) - b(x) < L(x)$ in $x_0 < x \leq x_0 + \delta$ $(\leq x_0 + \delta_1)$.

On the other hand from the existence of the sequence of lower - functions uniformly converging to the minimal solution $\rho(x)$ there must exist a lower - function $\bar{\phi}(x)$ such that

$$0 < \rho(x) - \bar{\phi}(x) < \min_{x_0+\delta \leq x \leq x_1} [b(x), \{f(x_0,y_0) - \eta\}\delta].$$

Now we define $\bar{\alpha}(x)$ as follows : If $\bar{\phi}(x) > L(x)$ in $(x_0, x_0 + \delta)$, then $\bar{\alpha}(x) = \bar{\phi}(x)$; and if there exists x_2 such that

$$\begin{cases} \bar{\phi}(x) > L(x),\ x_0 < x_2 \leq x_0 + \delta \\ \bar{\phi}(x_2) = L(x_2) \end{cases}$$

then

$$\bar{\alpha}(x) = \begin{cases} L(x),\ x_0 \leq x \leq x_2 \\ \bar{\phi}(x),\ x_2 \leq x \leq x_1. \end{cases}$$

This $\bar{\alpha}(x)$ is a required lower - function. ∎

Proof of Theorem 1.22.1. The proof of only (i) will be given, whereas the proof of (ii) is similar. For this, we shall show that if (i) is not true, i.e., if for every ξ in $(x_0, x_1]$ and every y and $\bar{y}$ in $(u(\xi), v(\xi))$

$$\begin{vmatrix} f(\xi,y) & y & 1 \\ f(\xi,y_\rho) & y_\rho & 1 \\ f(\xi,\bar{y}) & \bar{y} & 1 \end{vmatrix} \geq 0$$

then we can find, for any given positive number ε, a pair of lower - and upper - functions $\phi(x)$, $\psi(x)$ such that $0 < \psi(x) - \phi(x) < 2\varepsilon$ in $(x_0, x_1]$. But, then in view of Theorem 1.22.2 this will lead to the uniqueness of the solutions of (1.1.1).

To prove the existence of such a pair, we define the functions $p(x)$, $q(x)$, $\psi(x)$ and y and $\bar{y}$ in terms of the lower function $\bar{\alpha}(x)(= \rho(x) - d(x))$ as follows :

1. $\psi(x) = \rho(x) + d(x)$ in $[x_0, x_1]$,
2. $f(x, \bar{\alpha}(x)) = \bar{\alpha}'_+(x) + p(x)$, where $p(x) > 0$ in $[x_0, x_1]$ since $\bar{\alpha}(x)$ is a lower - function,
3. $f(x, \psi(x)) = \psi'_+(x) - q(x)$; and
4. $y = \bar{\alpha}(\xi) = \rho(\xi) - d(\xi)$, $\bar{y} = \psi(\xi) = \rho(\xi) + d(\xi)$.

Now we will show that $\psi(x)$ is a required upper - function, i.e., it satisfies the following conditions :

(a) $0 < \psi(x) - \bar{\alpha}(x) < 2\varepsilon$ for $x_0 < x \leq x_1$, where ε is a small positive number; and

(b) $\psi'_+(x) - f(x, \psi(x)) = q(x) > 0$, for $x_0 \leq x \leq x_1$.

Since, in $(x_0, x_1]$

$$0 < \psi(x) - \bar{\alpha}(x) = (\rho(x) + d(x)) - (\rho(x) - d(x)) = 2d(x) < 2\varepsilon$$

(a) follows. To show (b), we note that

$$\begin{vmatrix} f(\xi, y) & y & 1 \\ f(\xi, y_\rho) & y_\rho & 1 \\ f(\xi, \bar{y}) & \bar{y} & 1 \end{vmatrix} = \begin{vmatrix} f(\xi, \bar{\alpha}(\xi)) & \bar{\alpha}(\xi) & 1 \\ f(\xi, \rho(\xi)) & \rho(\xi) & 1 \\ f(\xi, \psi(\xi)) & \psi(\xi) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{\alpha}'_+(\xi) + p(\xi) & \rho(\xi) - d(\xi) & 1 \\ \rho'(\xi) & \rho(\xi) & 1 \\ \psi'_+(\xi) - q(\xi) & \rho(\xi) + d(\xi) & 1 \end{vmatrix} = - \begin{vmatrix} -d'_+(\xi) + p(\xi) & -d(\xi) \\ d'_+(\xi) - q(\xi) & d(\xi) \end{vmatrix}$$

$$= d(\xi)(q(\xi) - p(\xi)) \geq 0$$

by the hypothesis. As $d(\xi) > 0$ in $(x_0, x_1]$, it follows that $q(\xi) \geq p(\xi) > 0$ for $x_0 \leq \xi \leq x_1$. ∎

Theorem 1.22.3 (Kitagawa's Uniqueness Theorem). Let in addition to the above hypotheses one of the following conditions hold

(1) $f(x, y)$ is concave function of y; or

(2) $f(x, y)$ is convex function of y.

Then, the initial value problem (1.1.1) has at most one solution in $[x_0, x_0 + a]$.

Proof. The conclusion (i) of Theorem 1.22.1 for $y < y_\rho < \bar{y}$ can be written as

$$f(\xi, y)(y_\rho - \bar{y}) + f(\xi, y_\rho)(\bar{y} - y) + f(\xi, \bar{y})(y - y_\rho) < 0.$$

If $f(x, y)$ is concave function of y, then we have

$$f(\xi, y_\rho) = f\left(\xi, y\frac{\bar{y} - y_\rho}{\bar{y} - y} + \bar{y}\frac{y_\rho - y}{\bar{y} - y}\right)$$

$$\geq f(\xi, y)\frac{\bar{y} - y_\rho}{\bar{y} - y} + f(\xi, \bar{y})\frac{y_\rho - y}{\bar{y} - y},$$

which is the same as

$$f(\xi, y)(y_\rho - \bar{y}) + f(\xi, y_\rho)(\bar{y} - y) + f(\xi, \bar{y})(y - y_\rho) \geq 0.$$

If $f(x, y)$ is convex function of y, then a similar contradiction follows from the conclusion (ii) of Theorem 1.22.1. ∎

Example 1.22.1. For the initial value problem

$$\begin{aligned} y' &= f(x, y) = \frac{4x^3 y}{x^4 + y^2} \\ y(0) &= 0 \end{aligned} \tag{1.22.2}$$

it is easily seen that $\rho(x) = -x^2$ and $r(x) = x^2$ are the minimal and the maximal solutions for $x > 0$. As a function of y it is clear that $\dfrac{4x^3 y}{x^4 + y^2}$ is concave for $y < -\sqrt{3}x^2$ or $0 < y < \sqrt{3}x^2$, and convex for $-\sqrt{3}x^2 < y < 0$ or $y > \sqrt{3}x^2$.

If we take the functions $u(x), v(x)$ such that

$$\begin{cases} u(x) < \rho(x) = -x^2 < v(x), \ x \in (0, x_1] \\ u(0) = \rho(0) = v(0) = 0 \\ D_+u(0) < \rho'(0) = 0 < D_+v(0) \end{cases}$$

then there exists ξ such that $u(\xi) < 0$, and $v(\xi) > 0$. Since $f(\xi, y)$ is not always concave with respect to y in $(u(\xi), v(\xi))$, the conclusion (i) of Theorem 1.22.1 is justified. The conclusion (ii) of Theorem 1.22.1 can be similarly verified. ∎

Example 1.22.2. Consider the initial value problem

$$\begin{aligned} &y' = f(x,y) = b(1 + |y - ax|^{1/3}), \ b, a > 0 \\ &y(0) = 0. \end{aligned} \tag{1.22.3}$$

If $b > a$, then we can take $\mu(x) = ax$ and $\nu(x) = (a + b)x$. For sufficiently small x_1, we have

$$\begin{aligned} &\mu'(x) = a < b = f(x, \mu(x)) \\ &\nu'(x) = a + b > b(1 + |bx|^{1/3}) = f(x, \nu(x)), \quad 0 \le x \le x_1 \end{aligned}$$

and this $f(x, y)$ is a concave function of y, when b and $y - ax$ are positive. In this case the initial value problem (1.22.3) has a unique solution follows from Theorem 1.22.3.

If $b < a$, then we choose η such that $a - \eta < b$ and $\mu(x) = (a - \eta)x$ and $\nu(x) = ax$. For such a choice, we have

$$\begin{aligned} &\mu'(x) = a - \eta < b(1 + |-\eta x|^{1/3}) = f(x, \mu(x)) \\ &\nu'(x) = a > b = f(x, \nu(x)) \end{aligned}$$

and this $f(x, y)$ is a convex function of y, when b and $ax - y$ are positive. In this case also the initial value problem (1.22.3) has a unique solution follows from Theorem 1.22.3.

If $b = a$, then Theorem 1.22.3 is not applicable. However, in this case the initial value problem (1.22.3) has at least two solutions $y(x) = ax$ and $\bar{y}(x) = ax + (\frac{2}{3}ax)^{3/2}$. ∎

Example 1.22.3. The condition (1.22.1) is essential for the Theorem 1.22.3. To show this we consider the initial value problem

$$\begin{aligned} & y' = f(x,y) = (1-(y-1)^2)^{1/2} \\ (1.22.4) \qquad & \\ & y(0) = 0 \end{aligned}$$

Clearly, this $f(x,y)$ is continuous and is a convex function of y in the domain D defined by $x \geq 0,\ |y-1| \leq 1$. For this problem, let $\mu(x) = 0$ and $\nu(x) = \begin{cases} x, 0 \leq x \leq 2 \\ 2, x \geq 2 \end{cases}$ so that

$$D_+\mu(x) = f(x,\mu(x)) = 0.$$

For the initial value problem (1.22.4), $\rho(x) = 0$ and $r(x) = \begin{cases} 1-\cos x, 0 \leq x \leq \pi \\ 2, x \geq \pi \end{cases}$ are the minimal and the maximal solutions. ∎

1.23 TAMARKINE'S NONUNIQUENESS THEOREM

In Section 1.2 we have seen that if the function $f(x,y)$ does not satisfy the Lipschitz condition (1.2.1), then the initial value problem (1.1.1) may have several solutions. In the year 1922, Tamarkine [156] pointed out that the same situation prevails if in Osgood's condition (1.4.3) the integral $\int_\varepsilon \frac{dz}{g(z)}$ converges as $\varepsilon \to 0$. Specifically, his result is

Theorem 1.23.1 (Tamarkine's Nonuniqueness Theorem). Let $f(x,y)$ be continuous in $\bar{S}$ with $(x_0,y_0) = (0,0)$, and for all $(x,y) \in \bar{S}$

$$|f(x,y) - f(x,y_0(x))| \geq g(|y - y_0(x)|),$$

where $y_0(x)$ is a solution of (1.9.4); $g(z)$ is an increasing continuous function for $z \geq 0$, $g(0) = 0$ and the integral $\int_\varepsilon \frac{dz}{g(z)}$ converges as $\varepsilon \to 0$. Then, the initial value problem (1.9.4) has at least two solutions in $|x| \leq a$.

Proof. The proof is parallel to a more general result which we shall prove in the next section. ∎

Example 1.23.1. For the initial value problem

$$y' = y^{1/2},\ y(0) = 0 \tag{1.23.1}$$

$y_0(x) \equiv 0$ is a solution. Since the function $f(x,y) = y^{1/2}$ satisfies all the conditions of Theorem 1.23.1 with $g(z) = z^{1/2}$, and the integral $\int_\epsilon z^{-1/2}dz$ converges, (1.23.1) has at least two solutions. ∎

1.24 LAKSHMIKANTHAM'S NONUNIQUENESS THEOREM

If certain conditions of Kamke's uniqueness theorem are violated, then the initial value problem (1.1.1) has more than one solutions. Indeed, this is the content of the following result in which we shall assume that $(x_0, y_0) = (0,0)$.

Theorem 1.24.1 (Lakshmikantham's Nonuniqueness Theorem). Assume that

(i) the function $g(x,z)$ is continuous in $0 < x \le a,\ 0 \le z \le 2b,\ g(x,0) \equiv 0$, and $g(x,z) > 0$ for $z > 0$,

(ii) for each $x_1,\ 0 < x_1 < a,\ z(x) \not\equiv 0$ is a differentiable function in $0 < x < x_1$, and continuous in $0 \le x \le x_1$ for which $z'_+(0)$ exists; and

$$\begin{aligned} z'(x) &= g(x, z(x)),\ 0 < x < x_1 \\ z(0) &= z'_+(0) = 0, \end{aligned}$$

(iii) the function $f(x,y)$ is continuous in $\bar{S}_+$ with $(x_0, y_0) = (0,0)$ and for all $(x,y), (x,\bar{y}) \in \bar{S}_+,\ x \ne 0$

$$|f(x,y) - f(x,\bar{y})| \ge g(x, |y - \bar{y}|). \tag{1.24.1}$$

Then, the initial value problem (1.9.4) has at least two solutions in $[0,a]$.

Proof. Let us first assume that $f(x,0) \equiv 0$, so that, putting $\bar{y} = 0$ in (1.24.1) leads to the inequality $|f(x,y)| \ge g(x,|y|)$. Since $f(x,y)$ is continuous and

$g(x,z) > 0$ for $z > 0$, it follows that either $f(x,y) < 0$ or $f(x,y) > 0$ for $y \neq 0$. This implies that

$$f(x,y) \geq g(x,|y|), \quad \text{or} \tag{1.24.2}$$

$$f(x,y) \leq -g(x,|y|). \tag{1.24.3}$$

By hypothesis, there exists a σ, $0 < \sigma < a$ such that $z(\sigma) > 0$. Let $\rho(x)$ be the minimal solution of $y' = f(x,y), y(\sigma) = z(\sigma)$. Then, using an argument similar to that in the proof of Lemma 1.15.2, and the inequality (1.24.2), it can be shown that $\rho(x) \leq z(x)$ to left of σ as far as $\rho(x)$ exists. Moreover, $\rho(x)$ can be continued up to $x = 0$ and $0 \leq \rho(x) \leq z(x)$, $0 \leq x \leq \sigma$. Since $z(0) = z'_+(0) = 0$, we obtain from this relation that $\rho(0) = \rho'_+(0) = 0$. This proves that the initial value problem (1.9.4) has a solution $\rho(x)$, not identically zero. On the other hand, since $f(x,0) \equiv 0$, (1.9.4) admits the identically zero solution. Hence, we have two different solutions for the problem (1.9.4). Corresponding to the case (1.24.3), we can employ a similar reasoning to arrive at the same conclusion.

We shall now remove the restriction that $f(x,0) \equiv 0$. Let $y_0(x)$ be a solution of (1.9.4), existing in $0 \leq x \leq a$. Using the transformation $w = y - y_0(x)$, we get

$$\begin{aligned} w' = y' - y'_0(x) &= f(x,y) - f(x,y_0(x)) = f(x, w + y_0(x)) - f(x, y_0(x)) \\ &= F(x,w), \quad \text{say} . \end{aligned} \tag{1.24.4}$$

Evidently, $F(x,0) \equiv 0$. It follows, therefore that $w(x) \equiv 0$ is one solution of (1.24.4) passing through $(0,0)$. But the previous considerations show that (1.24.4) has a solution $w(x)$ which is different from the identically zero solution. This implies that $y(x) = w(x) + y_0(x)$ is not identically equal to $y_0(x)$. ∎

Corollary 1.24.2. Tamarkine's Nonuniqueness Theorem 1.23.1 is a particular case of Theorem 1.24.1.

Proof. It suffices to note that the initial value problem $z' = g(z)$, $z(0) = 0$ has a unique nontrivial solution $z(x) = G^{-1}(x)$, where $G(z) = \int_0^z \frac{dt}{g(t)}$. ∎

1.25 SAMIMI'S NONUNIQUENESS THEOREM

The following result is a generalization of Lakshmikantham's nonuniqueness theorem.

Theorem 1.25.1 (Samimi's Nonuniqueness Theorem). Let in Theorem 1.24.1 the condition (ii) be replaced by

(ii)′ for each x_1, $0 < x_1 < a$, $z(x) \not\equiv 0$ is a differentiable function in $0 < x < x_1$ and continuous in $0 \le x \le x_1$ for which $\lim_{x\to 0^+} \frac{z(x)}{B(x)} = 0$, where the function $B(x)$ is continuous in $[0,a]$ and $B(x) > 0$ for $x > 0$, and

$$\begin{aligned} z'(x) &= g(x, z(x)),\ 0 < x < x_1 \\ z(0) &= 0. \end{aligned}$$

Then, the initial value problem (1.9.4) has at least two solutions $y(x)$ and $\bar{y}(x)$ in $[0,a]$ such that $\lim_{x\to 0^+} \frac{|y(x)-\bar{y}(x)|}{B(x)} = 0$.

Proof. Imitating the proof of Theorem 1.24.1 we arrive at $0 \le \rho(x) \le z(x)$, $0 \le x \le \sigma$ which in view of $\rho(0) = 0$ implies that $z(0) = 0$. Thus, from the condition (ii)′ it follows that

$$0 \le \lim_{x\to 0^+} \frac{\rho(x)}{B(x)} \le \lim_{x\to 0^+} \frac{z(x)}{B(x)} = 0$$

and hence, we have $\lim_{x\to 0^+} \frac{\rho(x)}{B(x)} = 0$. This proves that the initial value problem (1.9.4) has a solution $\rho(x)$, not identically zero such that $\lim_{x\to 0^+} \frac{\rho(x)}{B(x)} = 0$. Since $f(x,0) \equiv 0$, (1.9.4) admits the identically zero solution $y_0(x)$ which obviously satisfies $\lim_{x\to 0^+} \frac{y_0(x)}{B(x)} = 0$. Hence, $\lim_{x\to 0^+} \frac{|y_0(x)-\rho(x)|}{B(x)} = 0$. The restriction $f(x,0) \equiv 0$ can easily be removed as in Theorem 1.24.1. ∎