Complex Geometry

Logarithmic vanishing theorems on compact Kähler manifolds

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Beautynote

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CHAP

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CH 1 Preliminaries

1.1 Proof of Theorem1.1

Lemma 1.1.1 – 3.3

$$\langle[\sqrt{-1}\Theta(V,h^V),\Lambda_{\tilde{\omega}_P}]\,u,u\rangle \geqslant C|u|^2.$$

Proof.

$$\Omega_Y^p\otimes K_Y^{-1}=\Omega_Y^p\otimes\Omega_Y^{-n}=\Omega_Y^{-(n-p)}=^a\bigoplus_{i_1,\cdots,i_{n-p}}(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1}).$$

$$\begin{aligned}\Omega_Y^p\otimes\Omega_Y^{-n} &= \Omega_Y^p\otimes(\Omega_Y^{-p}\otimes\Omega_Y^{-(n-p)})\\ &= (\Omega_Y^p\otimes 1)\otimes(\Omega_Y^{-p}\otimes\Omega_Y^{-(n-p)})\\ &= (\Omega_Y^p\otimes 1)\otimes(\Omega_Y^{-p}\otimes 1)\otimes(1\otimes\Omega_Y^{-(n-p)})\\ &= 1\otimes\Omega_Y^{-(n-p)}=\Omega_Y^{-(n-p)}.\end{aligned}$$

$$(\Omega_Y^p\otimes 1)\otimes(\Omega_Y^{-p}\otimes 1)=\Omega_Y^p(\Omega_Y^{-p})\otimes 1=1(\text{trivial holomorphic cotangent bundle})\otimes 1.$$

[a] For $(\mathrm{TY},\tilde{\omega}_P)=\bigoplus_{i=1}^n(F_i,\omega_i)$ by using (2.1), then in local case (Fixed a local coordinate $(W;z_1,\cdots,z_n)$), one has $F_i^*=F_i^{-1}$(Dual line bundle)[Huy05, §2.2, p71]. (Ω_Y is the dual of TY) The conclusion is clear through some easy computation.

We have

$$\sqrt{-1}\Theta(V,h^V)=\sum_{i_1,\cdots,i_{n-p}}\sqrt{-1}\Theta[L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1})]$$

and

$$\begin{aligned}\sqrt{-1}\Theta[L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1})] &= \sum_i\sqrt{-1}\partial\bar{\partial}\log(\omega_i)\\ (3.12) &\geqslant \alpha\tilde{\omega}_P-(n-p)C\tilde{\omega}_P\\ (\text{if }\alpha>(n-p+1)C) &> C\tilde{\omega}_P>0,\end{aligned}$$

where we denote

$$\sqrt{-1}\Theta[L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1})]=\sqrt{-1}\Theta[L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1}),h_Y^L\otimes\omega_{i_1}^*\otimes\cdots\otimes\omega_{i_{n-p}}^*].$$

Thus, the curvature of each summand of $L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1})$ is strictly positive, i.e.

$$\sqrt{-1}\Theta[L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1})]>0.$$

$$\left\langle [\sqrt{-1}\Theta(V,h^V),\Lambda_{\widetilde{\omega}_P}]\,u,u\right\rangle = \left\langle \left[\sum_{i_1,\cdots,i_{n-p}} \sqrt{-1}\Theta[L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1})],\Lambda_{\widetilde{\omega}_P}\right] u,u\right\rangle$$
$$\geqslant (q\cdot C-(n-n))|u|^2 \geqslant C|u|^2.(q\geqslant 1)$$

And the proof of the assertion that of the metric $\widetilde{h}_Y^L$ is Nakano positive is on following.

$$\begin{aligned}
\sum_{i_1,\cdots,i_{n-p}} \sqrt{-1}\Theta[L|_Y\otimes(F_{i_1}^{-1}\otimes\cdots\otimes F_{i_{n-p}}^{-1})] &= \sum_{i,j,\alpha,\gamma}\sqrt{-1}R^{\gamma}_{i\bar{j}\alpha}\,\mathrm{d}z^i\wedge d\bar{z}^j\otimes e^{\alpha}\otimes e_{\gamma}(\text{cf P.5})\\
(R^{\gamma}_{i\bar{j}\alpha}=h^{\gamma\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}) &= \sum_{i,j,\alpha,\beta}\sqrt{-1}h^{\gamma\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}\,\mathrm{d}z^i\wedge d\bar{z}^j\otimes e^{\alpha}\otimes \bar{e}^{\beta}\\
\mathrm{d}z^i=\left(u^i\frac{\partial}{\partial z^i}\right) &= \sum_{i,j,\alpha,\beta}\sqrt{-1}h^{\gamma\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}\left(u^i\frac{\partial}{\partial z^i}\right)\wedge\left(\bar{u}^j\frac{\partial}{\partial\bar{z}^j}\right)\otimes e^{\alpha}\otimes\bar{e}^{\beta}\\
&= \sum_{i,j,\alpha,\beta}\sqrt{-1}h^{\gamma\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}u^{i\alpha}\bar{u}^{j\beta}\left(\frac{\partial}{\partial z^i}\otimes e^{\alpha}\right)\wedge\left(\frac{\partial}{\partial\bar{z}^j}\otimes\bar{e}^{\beta}\right)\\
&> \sum_{i_1,\cdots,i_{n-p}} C\widetilde{\omega}_P > 0
\end{aligned}$$

As $h^{\gamma\bar{\beta}}>0$, then we have $\sum_{i,j,\alpha,\beta}\sqrt{-1}h^{\gamma\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}u^{i\alpha}\bar{u}^{j\beta}>0$, which immediately shows that

$$\sum_{i,j,\alpha,\beta}R_{i\bar{j}\alpha\bar{\beta}}u^{i\alpha}\bar{u}^{j\beta}>0.$$

Thus $\widetilde{h}_Y^L$ is Nakano positive. ■

Theorem 1.1.1 – Main theorem *Let*

X	*A compact Kähler manifold*
D	*A small normal crossing (SNC) divisor*
N	*A line bundle*
$\Delta=\sum_{i=1}^s\alpha_iD_i$	*an $\mathbb{R}$-divisor with $\alpha_i\in[0,1]$ such that $N\otimes\mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle*
L	*A nef line bundle*

Then we have

$$H^q(X,\Omega_X^p(\log D)\otimes L\otimes N)=0,\ \textit{for any}\ p+q\geqslant n+k+1.$$

Proof: (A small sketch). The following computation is of (4.5).

$$
\begin{aligned}
&\sqrt{-1}\Theta(\mathcal{F}, h^{\mathcal{F}}_{\alpha,\varepsilon,\tau}) \\
&= \sqrt{-1}\Theta(L \otimes F \otimes \mathcal{O}_X(-[\Delta]), h^{\mathcal{F}}_{\alpha,\varepsilon,\tau}) \\
&= \sqrt{-1}\partial\overline{\partial}\log\left(h^{\mathcal{F}}_{\alpha,\varepsilon,\tau}\right) \\
&= \sqrt{-1}\partial\overline{\partial}\log\left(h^{L}\right) + \sqrt{-1}\partial\overline{\partial}\log\left(h^{F}\right) + \sqrt{-1}\partial\overline{\partial}\log\left(h^{\Delta}\right)^{-1} + \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}\|\sigma\|^{2\tau_i}_{D_i}(\log^2(\varepsilon\|\sigma_i\|^2_{D_i}))^{\frac{\alpha}{2}}\right) \\
&= \sqrt{-1}\partial\overline{\partial}\log\left(h^{L}\right) + \sqrt{-1}\partial\overline{\partial}\log\left(h^{F}\right) + \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}h^{a_i}_{D_i}\right)^{-1} + \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}\|\sigma\|^{2\tau_i}_{D_i}\right) \\
&+ \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}(\log^2(\varepsilon\|\sigma_i\|^2_{D_i}))^{\frac{\alpha}{2}}\right) \\
&= \sqrt{-1}\partial\overline{\partial}\log\left(h^{L}\right) + \sqrt{-1}\partial\overline{\partial}\log\left(h^{F}\right) - \sum_{i=1}^{s}a_i\sqrt{-1}\partial\overline{\partial}\log(h_{D_i}) + \sum_{i=1}^{s}\tau_i\sqrt{-1}\partial\overline{\partial}\log\left(\|\sigma\|^2_{D_i}\right) \\
&+ \sum_{i=1}^{s}\alpha\sqrt{-1}\partial\overline{\partial}\log\left(\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)\right) \\
&= \sqrt{-1}\partial\overline{\partial}\log\left(h^{L}\right) + \sqrt{-1}\partial\overline{\partial}\log\left(h^{F}\right) - \sum_{i=1}^{s}a_ic_1(D_i) + \sum_{i=1}^{s}\tau_i\sqrt{-1}\partial\overline{\partial}\log(h_{D_i}) \\
&+ \sum_{i=1}^{s}\alpha\sqrt{-1}\partial\left(\frac{\overline{\partial}\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}{\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}\right) \\
&= \sqrt{-1}\partial\overline{\partial}\log\left(h^{L}\right) + \sqrt{-1}\partial\overline{\partial}\log\left(h^{F}\right) - \sum_{i=1}^{s}a_ic_1(D_i) + \sum_{i=1}^{s}\tau_ic_1(D_i) \\
&+ \sum_{i=1}^{s}\alpha\sqrt{-1}\left(\frac{\partial\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)\cdot\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right) - \overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)\wedge\partial\log\left(\|\sigma_i\|^2_{D_i}\right)}{\left(\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)\right)^2}\right) \\
&= \sqrt{-1}\Theta(L, h^{L}) + \sqrt{-1}\Theta(F, h^{F}) + \sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) + \sum_{i=1}^{s}\left(\frac{\alpha\sqrt{-1}\partial\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)}{\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}\right) \\
&+ \sqrt{-1}\sum_{i=1}^{s}\left(\frac{\alpha\partial\log\left(\|\sigma_i\|^2_{D_i}\right)\wedge\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)}{\left(\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)\right)^2}\right) \\
&= \sqrt{-1}\Theta(L, h^{L}) + \sqrt{-1}\Theta(F, h^{F}) + \sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) + \sum_{i=1}^{s}\left(\frac{\alpha c_1(D_i)}{\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}\right) \\
&+ \sqrt{-1}\sum_{i=1}^{s}\left(\frac{\alpha\partial\log\left(\|\sigma_i\|^2_{D_i}\right)\wedge\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)}{\left(\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)\right)^2}\right).
\end{aligned}
$$

where $h_{D_i} = \|\sigma_i\|^2_{D_i}$.

■

Remark

$$\mathcal{O}_X([D]) \otimes \mathcal{O}_X(-[\Delta]) = \mathcal{O}_X(\sum_{i=1}^{s}[D_i]) \otimes \mathcal{O}_X(-\sum_{i=1}^{s} a_i[D_i])$$

$$(\text{Dual line bundle}) = \sum_{i=1}^{s} a_i \mathcal{O}_X([D_i]) \otimes \mathcal{O}_X(-[D_i])$$

$$= \sum_{i=1}^{s} a_i \mathcal{O}_X(1). \text{ (trivial line bundle)}$$

Definition 1.1.1 – Poincaré Type Metric *A metric ω_Y is of Poincaré Type along D if for each local coordinate chart $(W; z_1, \cdots, z_n)$ along D, the restriction $\omega|_{W^*_{1/2}}$ is equivalent to the usual Poincaré type metric ω_P defined by*

$$\omega_P = \sqrt{-1}\sum_{i=1}^{k} \frac{\mathrm{d}z_j \wedge d\overline{z_j}}{|z_j|^2 \cdot \log^2 |z_j|^2} + \sqrt{-1} \sum_{j=k+1}^{n} \mathrm{d}z_j \wedge d\overline{z_j}.$$

Where $W^*_r = Y \cap W = (\Delta^*_r)^k \times (\Delta_r)^{n-k}, r \in (0, \frac{1}{2}]$.

Theorem 1.1.2 – The key theorem for the proof of the main theorem *Let*

(X, ω)	*A compact Kähler manifold of dimension n*
D	*A SNC divisor in X*
ω_P	*A smooth Kähler metric on $Y = X - D$ which is Poincaré Type along D*

Then there exists a smooth Hermitian metric h^L_Y on $L|_Y$ such that the sheaf $\Omega^p(\log D) \otimes \mathcal{O}(L)$ over X has a fine resolution given by the L^2 Dolbeault complex $(\Omega^{p,*}_{(2)}(X, L, \omega_P, h^L_Y), \overline{\partial})$.

*In other words, we have an **exact sequence of sheaf over** X*

$$0 \to \Omega^p(\log D) \otimes \mathcal{O}(L) \to \Omega^{p,*}_{(2)}(X, L, \omega_P, h^L_Y)$$

*such that $\Omega^{p,q}_{(2)}(X, L, \omega_P, h^L_Y)$ is a **fine sheaf** for any $0 \le p, q \le n$. In particular,*

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}(L)) \cong H^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y) \cong \mathbb{H}^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y). \tag{1.1}$$

***Note:** The isomorphism holds up to equivalence of metrics, i.e. if $\widetilde{\omega}_P \sim \omega_P$ and $\widetilde{h}^L_Y \sim h^L_Y$, then*

$$\mathbb{H}^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y) \cong \mathbb{H}^{p,q}_{(2)}(Y, L, \widetilde{\omega}_P, \widetilde{h}^L_Y).$$

! Replacing the line bundle with vector bundle is still valid.

Problem 1.1.1 – Why $\Omega^{p,q}_{(2)}(X,E)$ is a fine sheaf over X? In the paper [Hua+16, §2.3, P7], the author asserts that if $u \in \Gamma(U, \Omega^{p,q}_{(2)}(X,E))$ and $f \in C^\infty(X)$, then $fu \in \Gamma(U, \Omega^{p,q}_{(2)}(X,E))$. This demonstrates the existence of a partition of unity in $\Omega^{p,q}_{(2)}(X,E)$. Subsequently, the author claims that $\Omega^{p,q}_{(2)}(X,E)$ is a fine sheaf over X.

Proof. The basis for this assertion lies in the properties of fine sheaves and the specific construction of $\Omega^{p,q}_{(2)}(X,E)$:

1. **Definition of a Fine Sheaf**: *A sheaf $\mathcal{F}$ is considered "fine" if it satisfies certain partition of unity properties.* In this context, it means that for any open cover $\{U_i\}$ of the underlying topological space X, there exist smooth functions $\rho_i \in C^\infty(X)$ with specific properties:
   - $0 \leq \rho_i \leq 1$ for all i.
   - $\operatorname{supp}(\rho_i) \subseteq U_i$ (the support of ρ_i is contained in U_i).
   - $\sum \rho_i(x) = 1$ for all x in X (the sum of ρ_i at each point x is 1).

   These partition of unity functions ρ_i are crucial for gluing together local sections of the sheaf to obtain global sections.
2. **Construction of $\Omega^{p,q}_{(2)}(X,E)$**: This sheaf represents smooth differential forms of type (p,q) with values in a vector bundle E over the manifold X. *Its construction involves defining local sections on coordinate patches and specifying how these sections transition between overlapping patches.*
3. **Demonstrating Fine Sheaf Property**: In Section 2.3 of the paper, the author asserts that if u is a section in $\Omega^{p,q}_{(2)}(X,E)$ and f is a smooth function on X, then the product fu is also a section in $\Omega^{p,q}_{(2)}(X,E)$. *This demonstrates compatibility with the fine sheaf property because it shows that you can use smooth functions (such as the ρ_i functions from the partition of unity) to combine sections locally without leaving the sheaf $\Omega^{p,q}_{(2)}(X,E)$.*

   Essentially, this step ensures that $\Omega^{p,q}_{(2)}(X,E)$ is closed under multiplication by smooth functions, which is a crucial property for a sheaf to be fine.

■

Remark In summary, *the assertion that $\Omega^{p,q}_{(2)}(X,E)$ is a fine sheaf is based on the construction of $\Omega^{p,q}_{(2)}(X,E)$ and the demonstration that it satisfies the necessary partition of unity property when sections are multiplied by smooth functions.* This property is essential for many purposes in differential geometry and *allows for the gluing of local sections to obtain global sections over a manifold X.*

Problem 1.1.2
1. How to get (3.5)?
2. How to obtain the Laurentz series representation of $\sigma_I(z)$ on $W^*_{1/2}$?
3. Why σ is L^2 integrable on W^*_r iff $\beta_j > -\tau_j$ along D_j by using polar coordinates and Fubini Theorem (Example 2.4)?
4. Why σ and $\nabla\sigma$ have only logarithimic pole and σ is a section of $\Omega^p(\log D) \otimes \mathcal{O}(L)$ on W?

Solution.
1.
2. The Laurentz series equation for several variables is

$$f(z_1,\cdots,z_n) = \sum_{J=-\infty}^{\infty} a_J(z_1 - z_{1_0})^{j_1}\cdots(z_n - z_{n_0})^{j_n}, \quad R_1 \leqslant |z_i - z_{i_0}| \leqslant R_2,$$

where $J=(j_1,\cdots,j_n)$. $f(z_1,\cdots,z_n)$ is single-valued analysis in the annulus centered at every point z_{i_0}. The coefficients are

$$a_J=\frac{1}{(2\pi i)^n}\int_{\Omega_1}\cdots\int_{\Omega_n}\frac{f(z_1,\cdots,z_n)}{(z_1-z_{1_0})^{j_1+1}\cdots(z_n-z_{n_0})^{j_n+1}}\mathrm{d}z_J,$$

where $\mathrm{d}z_J=\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n$ and $\Omega_1,\cdots,\Omega_n$ are counter-clockwise closed curves surrounding the expansion point $(z_{1_0},\cdots,z_{n_0})$ in each variable, and the order of inegration can be interchanged.
By using the above equation, for a fixed point $(0,\cdots,0)\in W^*_{1/2}=\Delta^{*t}_{1/2}\times\Delta^{n-t}_{1/2}$, we have

$$\sigma_I(z)=\sum_{J=-\infty}^{\infty}a_J(z_1)^{j_1}\cdots(z_t)^{j_t},\quad J=(j_1,\cdots,j_t),$$

where $a_J=\sigma_{IJ}(z_{t+1},\cdots,z_n)$ is a holomorphic function on $\Delta^{n-t}_{1/2}$. Thus $\sigma_I(z)$ is bounded on $W^*_r\subset W^*_{1/2}$, i.e. there exists a positive constant M such that $|\sigma_I(z)|\leqslant M$.

3. By using polar coordinates, we obtain that

$$\begin{aligned}
&\|\sigma\|^2_{L^2(W^*_r)}\\
&=\sum_{|I|=p}\int_{W^*_r}|e|^2_{h^L}\left(|\sigma_I(z)|^2\prod_{\nu=1}^{b}\log^2|z_{i_{p\nu}}|^2\prod_{i=1}^{t}|z_i|^{2\tau_i}(\log^2|z_i|^2)^{\alpha/2}\right)\omega_P^n\\
&\leqslant\sum_{|I|=p}\int_{W^*_r}\left(|\sigma_I(z)|^2\prod_{\nu=1}^{b}\log^2|z_{i_{p\nu}}|^2\prod_{i=1}^{t}|z_i|^{2\tau_i}(\log^2|z_i|^2)^{\alpha/2}\right)\omega_P^n\\
&\leqslant\sum_{|I|=p}\left(\underbrace{\int_0^{2\pi}\cdots\int_0^{2\pi}}_{b+t}\right)\left(\underbrace{\int_0^{\frac12}\cdots\int_0^{\frac12}}_{b+t}\right)\left(|\sigma_I(z)|^2\prod_{\nu=1}^{b}\log^2 r^2_{i_{p\nu}}\prod_{i=1}^{t}r_i^{2\tau_i}(\log^2 r_i^2)^{\alpha/2}\right)\omega_P^n\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{r}\\
&=\sum_{|I|=p}|\sigma_I(z)|^2\prod_{\nu=1}^{b}\left(\int_0^{2\pi}\int_0^{\frac12}\log^2 r^2_{i_{p\nu}}\mathrm{d}\theta_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\cdot\prod_{i=1}^{t}\left(\int_0^{2\pi}\int_0^{\frac12}r_i^{2\tau_i}(\log^2 r_i^2)^{\alpha/2}\mathrm{d}\theta_i\mathrm{d}r_i\right)\omega_P^n\\
&=\sum_{|I|=p}(2\pi)^{b+t}|\sigma_I(z)|^2\prod_{\nu=1}^{b}\left(\int_0^{\frac12}\log^2 r^2_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\cdot\prod_{i=1}^{t}\left(\int_0^{\frac12}r_i^{2\tau_i}(\log^2 r_i^2)^{\alpha/2}\mathrm{d}r_i\right)\omega_P^n\\
&=\sum_{|I|=p}(2\pi)^{b+t}2^{2b+\alpha t}|\sigma_I(z)|^2\prod_{\nu=1}^{b}\left(\int_0^{\frac12}\log^2 r_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\cdot\prod_{i=1}^{t}\left(\int_0^{\frac12}r_i^{2\tau_i}(\log r_i)^{\alpha}\mathrm{d}r_i\right)\omega_P^n\\
&\leqslant\sum_{|I|=p}(2\pi)^{b+t}2^{2b+\alpha t}M^2\left[\prod_{\nu=1}^{b}\left(\int_0^{\frac12}\log^2 r_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\cdot\prod_{i=1}^{t}\left(\int_0^{\frac12}r_i^{2\tau_i}(\log r_i)^{\alpha}\mathrm{d}r_i\right)\right]\left(\frac12\right)^n\\
&<+\infty\quad\text{(By using Example 2.4)}
\end{aligned}$$

where $|e|^2_{h^L}\in[\frac12,1]$ over W by hypothesis and $r_i=|z_i|$, $\mathrm{d}\boldsymbol{r}=\mathrm{d}r_{i_{p1}}\wedge\cdots\wedge\mathrm{d}r_{i_{pb}}\wedge\mathrm{d}r_1\wedge\cdots\wedge\mathrm{d}r_t$, $\mathrm{d}\boldsymbol{\theta}=\mathrm{d}\theta_{i_{p1}}\wedge\cdots\wedge\mathrm{d}\theta_{i_{pb}}\wedge\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_t$. $(r\le\frac12)$ Thus σ is L^2 integrable on W^*_r iff $\beta_j>-\tau_j$ along D_j.

4.

$$
\begin{aligned}
\nabla\sigma(z) &= \sum_{|I|=p} \nabla\left(\sigma_I(z)\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes e\right)\\
&= \sum_{|I|=p} \mathrm{d}\sigma_I(z)\wedge\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes e + \sum_{|I|=p}\sigma_I(z)\wedge\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes \mathrm{d}e\\
&+ \sum_{\nu=1}^{p}\left(\sum_{|I|=p}\sigma_I(z)\zeta_{i_1}\wedge\cdots\wedge(\mathrm{d}\zeta_{i_\nu})\wedge\cdots\wedge\zeta_{i_p}\otimes e\right)\\
&= \sum_{|I|=p} \mathrm{d}\sigma_I(z)\wedge\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes e
\end{aligned}
$$

,where

$$
\begin{aligned}
\mathrm{d}\sigma_I(z) &= \sum_{J=-\infty}^{\infty} \mathrm{d}\left(a_J(z_1)^{j_1}\cdots(z_t)^{j_t}\right)\\
&= \sum_{J=-\infty}^{\infty} \mathrm{d}\left(a_J\right)(z_1)^{j_1}\cdots(z_t)^{j_t} + \sum_{J=-\infty}^{\infty} a_J\mathrm{d}\left((z_1)^{j_1}\cdots(z_t)^{j_t}\right)\\
&= \sum_{J=-\infty}^{\infty} \mathrm{d}\left(\sigma_{IJ}(z_{t+1},\cdots,z_n)\right)(z_1)^{j_1}\cdots(z_t)^{j_t} + \sum_{J=-\infty}^{\infty}\sigma_{IJ}(z_{t+1},\cdots,z_n)\mathrm{d}\left((z_1)^{j_1}\cdots(z_t)^{j_t}\right)
\end{aligned}
$$

and

$$
\sigma_{IJ}(z_{t+1},\cdots,z_n) = \frac{1}{(2\pi i)^{n-t}}\int_{W^*_{1/2}} \frac{\sigma_I(z_{t+1},\cdots,z_n)}{(z_{t+1}-z_{t+1_0})^{j_{t+1}+1}\cdots(z_n-z_{n_0})^{j_n+1}}\mathrm{d}z_{t+1}\wedge\cdots\wedge\mathrm{d}z_n.
$$

As $\sigma_{IJ}(z_{t+1},\cdots,z_n)$ is a holomorphic function on $\Delta^{n-t}_{1/2}$, thus it has only removable singularity, and so as to $\sigma_I(z)$, which shows that σ and $\nabla\sigma$ have only logarihmic pole.

■

CH 2 Additional Material

2.1 Definition of Nef line bundle

Definition 2.1.1 – Nef line bundle (Algebraic version) *More generally, a line bundle L on a proper scheme X over a field k is said to be nef if it has nonnegative degree on every (closed irreducible) curve in X (The degree of a line bundle L on a proper curve C over k is the degree of the divisor (s) of any nonzero rational section s of L.) A line bundle may also be called an invertible sheaf.*

The term "nef" was introduced by Miles Reid as a replacement for the older terms "arithmetically effective" (Zariski 1962) and "numerically effective", as well as for the phrase "numerically eventually free". The older terms were misleading, in view of the examples below.

Every line bundle L on a proper curve C over k which has a global section that is not identically zero has nonnegative degree. As a result, a basepoint-free line bundle on a proper scheme X over k has nonnegative degree on every curve in X; that is, it is nef. More generally, a line bundle L is called semi-ample if some positive tensor power $L^{\otimes a}$ is basepoint-free. It follows that a semi-ample line bundle is nef. Semi-ample line bundles can be considered the main geometric source of nef line bundles, although the two concepts are not equivalent; see the examples below.

A Cartier divisor D on a proper scheme X over a field is said to be nef if the associated line bundle $O(D)$ is nef on X. Equivalently, D is nef if the intersection number $D \cdot C$ is nonnegative for every curve C in X.

To go back from line bundles to divisors, the first Chern class is the isomorphism from the Picard group of line bundles on a variety X to the group of Cartier divisors modulo linear equivalence. Explicitly, the first Chern class $\mathrm{C}_1(\mathrm{~L})$ is the divisor (s) of any nonzero rational section s of L.

2.1.1 The nef cone

To work with inequalities, it is convenient to consider $\mathbf{R}$-divisors, meaning finite linear combinations of Cartier divisors with real coefficients. The $\mathbf{R}$-divisors modulo numerical equivalence form a real vector space $N^1(X)$ of finite dimension, the Néron-Severi group tensored with the real numbers. (Explicitly: two $\mathbf{R}$-divisors are said to be numerically equivalent if they have the same intersection number with all curves in X.) An $\mathbf{R}$-divisor is called nef if it has nonnegative degree on every curve. The nef $\mathbf{R}$-divisors form a closed convex cone in $N^1(X)$, the nef cone $\mathrm{Nef}(X)$.

The cone of curves is defined to be the convex cone of linear combinations of curves with nonnegative real coefficients in the real vector space $N_1(X)$ of 1-cycles modulo numerical equivalence. The vector spaces $N^1(X)$ and $N_1(X)$ are dual to each other by the intersection pairing, and the nef cone is (by definition) the dual cone of the cone of curves.

A significant problem in algebraic geometry is to analyze which line bundles are ample, since that amounts to describing the different ways a variety can be embedded into projective space. One answer is Kleiman's criterion (1966): for a projective scheme X over a field, a line bundle (or $\mathbf{R}$-divisor) is ample if and only if its class in $N^1(X)$ lies in the interior of the nef cone. (An $\mathbf{R}$-divisor is called ample if it can be written as a positive linear combination of ample Cartier divisors.) It follows from Kleiman's criterion that, for X projective, every nef $\mathbf{R}$-divisor on X is a limit of ample $\mathbf{R}$-divisors in $\mathrm{N}^1(\mathrm{X})$. Indeed, for D nef and A ample, $D + cA$ is ample for all real numbers $c > 0$.

Definition 2.1.2 – Metric definition of nef line bundles (Geometry version) *Let X be a compact complex manifold with a fixed Hermitian metric, viewed as a positive $(1,1)$-form ω. Following Jean-Pierre Demailly, Thomas Peternell and Michael Schneider,* ***a holomorphic line bundle L on X is said to be nef if for every $\varepsilon > 0$ there is a smooth Hermitian metric h_ε on L whose curvature satisfies $\Theta_h(L) \geqslant -\varepsilon\omega$.*** *When X is projective over C, this is equivalent to the previous definition (that L has nonnegative degree on all curves in X) which explains the more complicated definition just given.*

Definition 2.1.3 – Logarithmic pole *For a complex function $f(z)$, if there exists a pole at z_0 with the following form:*

$$f(z) \sim \frac{C}{(z - z_0)\log(z - z_0)},$$

where $\sim$ denotes that the ratio tends to 1 as $z \to z_0$, C is a nonzero complex number, and $\log(z - z_0)$ represents the logarithmic function, then z_0 is called a logarithmic pole of the function $f(z)$.

Note that the characteristic of a logarithmic pole is that the function becomes very large in magnitude as we approach points near z_0.

2.2 Some computation

$$\begin{aligned}
& e(\eta)[\mathrm{ie}(\partial\bar\partial\varphi), \Lambda] + e(\bar\partial\eta)\bar\partial^*_\varphi - e(\bar\partial\eta)^*\bar\partial - e(\partial\eta)\partial^* + e(\partial\eta)^*\partial_\varphi \\
&= e(\eta)[\mathrm{ie}(\partial\bar\partial\varphi), \Lambda] + \bar\partial e(\eta)\bar\partial^*_\varphi - \bar\partial^* e(\eta)\bar\partial - \partial e(\eta)\partial^* + \partial^* e(\eta)\partial_\varphi \\
&= e(\eta)[\mathrm{ie}(\partial\bar\partial\varphi), \Lambda] + e(\eta)\left[\bar\partial\bar\partial^*_\varphi - \bar\partial^*\bar\partial - \partial\partial^* + \partial^*\partial_\varphi\right] \\
&= e(\eta)[\bar\partial\bar\partial^*_\varphi + \bar\partial^*_\varphi\bar\partial - \partial_\varphi\partial^* - \partial^*\partial_\varphi] + e(\eta)\left[\bar\partial\bar\partial^*_\varphi - \bar\partial^*\bar\partial - \partial\partial^* + \partial^*\partial_\varphi\right] \\
&= \ldots + e(\eta)\left[\bar\partial(-\bar{*}(\bar\partial - e(\bar\partial\varphi))\bar{*}) - \bar\partial^*\bar\partial - \partial\partial^* + \partial^*(\partial - e(\partial\varphi))\right] \\
&= \cdots + e(\eta) \cdot -\bar{*}\left[\bar\partial(\bar\partial - e(\bar\partial\varphi)) - \bar\partial\bar\partial - \partial\partial + \partial(\partial - e(\partial\varphi))\right]\bar{*} \\
&= \cdots - e(\eta) \cdot -\bar{*}[\bar\partial e(\bar\partial\varphi) + \partial e(\partial\varphi)]\bar{*} \\
&= \cdots - e(\eta) \cdot e(\varphi) - \bar{*}[\bar\partial\bar\partial + \partial\partial]\bar{*} \\
&= 0?
\end{aligned}$$

$$\begin{aligned}
& \bar\partial e(\eta)\bar\partial^*_\varphi + \bar\partial^*_\varphi e(\eta)\bar\partial - \partial_\varphi e(\eta)\partial^* - \partial^* e(\eta)\partial_\varphi \\
&= e(\eta)[\bar\partial\bar\partial^*_\varphi + \bar\partial^*_\varphi\bar\partial - \partial_\varphi\partial^* - \partial^*\partial_\varphi] \\
&= \ldots
\end{aligned}$$

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Bibliography

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