Complex Geometry

L^2 EXTENSION THEOREMS WITH AN OPTIMAL ESTIMATE AND APPLICATIONS

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BEAUTYNOTE

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CHAP

Contents

CH 1 Preliminaries

1.1 L^2-extension theorems with optimal constant

1.1.1 Some notations

Table 1.1: Terminologies Interpretation

Terminologies	Interpretations
M	complex n-dimensional manifold
S	a closed complex subvariety of M
S_{reg}	the regular part of S .
$\mathrm{d}V_M$	a contionuous volume form on M
$\#_A(S)$	the set of such a class of the upper-semi-continuous (polar) function $\Psi\colon M \to [-\infty, A), A \in (-\infty, +\infty]$ such that 1. $\Psi^{-1}(-\infty) \supset S$ and $\Psi^{-1}(-\infty)$ is a closed subset of M; 2. if S is l-dimensional around a point $x \in S_{reg}$, there exists a local coordinate $(z_1, \ldots, z_n)$ on a neighborhood U of x such that $z_{l+1} = \cdots = z_n = 0$ on $S \cap U$ and $$\sup_{U\setminus S} \left\|\Psi(z) - (n-l)\log \sum_{l+1}^{n} \|z_j\|^2\right\| < +\infty.$$
$\Delta_{A,h,\delta}(S)$	the subset of functions Ψ in $\#_A(S)$ which satisfies that both $he^{-\Psi}$ and $he^{-(1+\delta)\Psi}$ are semi-positive in the sense of Nakano on $M \setminus (X \cup S)$.
$\Delta_A(S)$	the subset of plurisubharmonic functions Ψ in $\#_A(S)$
Terminologies	*Interpretations*

For each $\Psi \in \#_A(S)$, one can associate a positive measure $dV_M[\Psi]$ on S_{reg} as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_l} f d\mu \geq \limsup_{t\to\infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_M$$

for any nonnegative continuous function f with $\mathrm{Supp}\, f \subset\subset M$, where $\mathbb{I}_{\{-1-t<\Psi<-t\}}$ is the characteristic function of the set $\{-1-t < \Psi < -t\}$. Here denote by S_l the l-dimensional component of S_{reg}, denote by σ_m the volume of the unit sphere in $\mathbb{R}^{m+1}$.

Let ω be a Kähler metric on $M \setminus (X \cup S)$, where X is a closed subset of M such that $S_{sing} \subset X$

(S_{sing} is the singular part of S).

We can define measure $dV_{\omega}[\Psi]$ on $S\setminus X$ as the minimum element of the partially ordered set of positive measures $d\mu'$ satisfying

$$\int_{S_l} f d\mu' \geq \limsup_{t\to\infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_{M\setminus(X\cup S)} f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_{\omega}$$

for any nonnegative continuous function f with $\mathrm{Supp}(f)\subset\subset M\setminus X$ (As

$$\mathrm{Supp}(\mathbb{I}_{\{-1-t<\Psi<-t\}})\cap \mathrm{Supp}(f)\subset\subset M\setminus(X\cup S),$$

right hand side of the above inequality is well-defined).

Let u be a continuous section of $K_M\otimes E$, where E is a holomorphic vector bundle equipped with a continuous metric h on M.

We define

$$|u|^2_h|_V := \frac{c_n h(e,e) v\wedge\bar{v}}{dV_M},$$

and

$$|u|^2_{h,\omega}|_V := \frac{c_n h(e,e) v\wedge\bar{v}}{dV_{\omega}},$$

where $u|_V = v\otimes e$ for an open set $V\subset M\setminus(X\cup S)$, v is a continuous section of $K_M|_V$ and e is a continuous section of $E|_V$ (especially, we define

$$|u|^2|_V := \frac{c_n u\wedge\bar{u}}{dV_M},$$

when u is a continuous section of K_M). It is clear that $|u|^2_h$ is independent of the choice of V.

The following argument shows a relationship between $dV_{\omega}[\Psi]$ and $dV_M[\Psi]$ (resp. dV_{ω} and dV_M), precisely

$$\int_{M\setminus(X\cup S)} f|u|^2_{h,\omega} dV_{\omega}[\Psi] = \int_{M\setminus(X\cup S)} f|u|^2_h dV_M[\Psi], \tag{1.1}$$

$$(resp. \int_{M\setminus(X\cup S)} f|u|^2_{h,\omega} dV_{\omega} = \int_{M\setminus(X\cup S)} f|u|^2_h dV_M) \tag{1.2}$$

where f is a continuous function with compact support on $M\setminus X$.

Given $\delta>0$, let

1. ***Positivity:*** $c_A(t)$ be a *positive* function on $(-A,+\infty)$ $(A\in(-\infty,+\infty))$
2. ***smoothness:*** $c_A(t)\in C^{\infty}((-A,+\infty))$
3. ***Integrablity:*** $\int_{-A}^{\infty} c_A(t)e^{-t}dt<\infty$
4. ***Inequality:***

$$\begin{aligned}&\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^t c_A(t_1)e^{-t_1}dt_1\right)^2 > \\ &c_A(t)e^{-t}\left[\int_{-A}^t \left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1\right)dt_2 + \frac{1}{\delta^2}c_A(-A)e^A\right],\end{aligned} \tag{1.3}$$

for any $t\in(-A,+\infty)$.

Definition 1.1.1 – Condition (ab) *Let M be a complex manifold with a continuous volume form dV_M, and S be a closed complex subvariety of M. We call (M, S) satisfies condition (ab) if M and S satisfy the following conditions:*

There exists a closed subset $X \subset M$ such that:

(a) X is locally negligible with respect to L^2 holomorphic functions, i.e., for any local coordinate neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U \setminus X$, there exists an L^2 holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U\setminus X} = f$ with the same L^2 norm.

(b) $M \setminus X$ is a Stein manifold which intersects with every component of S, such that $S_{sing} \subset X$.

If $c_A(t)e^{-t}$ is decreasing with respect to t, then inequality 1.3 holds.

We establish the following L^2 extension theorem with an optimal estimate as follows:

Theorem 1.1.1 – main theorem 1 *Let (M, S) satisfy condition (ab), h be a smooth metric on a holomorphic vector bundle E on M with rank r. Let $\Psi \in \#_A(S) \cap C^\infty(M \setminus S)$, which satisfies*

1) *$he^{-\Psi}$ is semi-positive in the sense of Nakano on $M \setminus (S \cup X)$ (X is as in the definition of condition (ab)),*
2) *there exists a continuous function $a(t)$ on $(-A, +\infty]$, such that $0 < a(t) \leq s(t)$ and $a(-\Psi)\sqrt{-1}\Theta_{he^{-\Psi}} + \sqrt{-1}\partial\bar{\partial}\Psi$ is semi-positive in the sense of Nakano on $M \setminus (S \cup X)$, where*

$$s(t) = \frac{\int_{-A}^{t}(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1)dt_2 + \frac{1}{\delta^2}c_A(-A)e^A}{\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1}.$$

Then there exists a uniform constant $\boldsymbol{C} = 1$, which is optimal, such that, for any holomorphic section f of $K_M \otimes E|_S$ on S satisfying

$$\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty, \tag{1.4}$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M c_A(-\Psi)|F|_h^2 dV_M \leq \boldsymbol{C}\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi], \tag{1.5}$$

where $c_A(t)$ satisfies $c_A(-A)e^A := \lim_{t\to -A^+} c_A(t)e^{-t} < \infty$ and $c_A(-A)e^A \neq 0$.

Remark We need to clasify the following questions:

i) What are the theorem say and its signification?

Now we consider a useful and simpler class of functions as follows:

Let $c_A(t)$ be a positive function in $C^\infty((-A, +\infty))$ ($A \in (-\infty, +\infty]$), satisfying

$$\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$$

and

$$\left(\int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1\right)^2 > c_A(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1dt_2, \tag{1.6}$$

for any $t \in (-A, +\infty)$.

When $c_A(t)e^{-t}$ is decreasing with respect to t and A is finite, inequality 1.6 holds.

For such a simpler and sufficiently useful class of functions, we establish the following L^2 extension theorem with an optimal estimate

Theorem 1.1.2 – The simple version of the main theorem *Let (M, S) satisfy condition (ab), and Ψ be a plurisubharmonic function in $\Delta_A(S) \cap C^\infty(M \setminus (S \cup X))$ (X is as in the definition of condition (a, b)), Let h be a smooth metric on a holomorphic vector bundle E on M with rank r, such that $he^{-\Psi}$ is semi-positive in the sense of Nakano on $M \setminus (S \cup X)$, (when E is a line bundle, h can be chosen as a semipositive singular metric). Then there exists a uniform constant $\mathbf{C} = 1$, which is optimal, such that, for any holomorphic section f of $K_M \otimes E|_S$ on S satisfying condition 1.4 there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and*

$$\int_M c_A(-\Psi)|F|_h^2 dV_M \leq \mathbf{C} \int_{-A}^{\infty} c_A(t)e^{-t}dt \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

1.1.2 Proof of Lemma 4.4

Lemma 1.1.1 – Lemma 4.4 [GZ15] *Let Δ be the unit disc and Δ_r be the disc with radius r. Then for any holomorphic function f on Δ, which satisfies $\int_\Delta |f|^2 \mathrm{d}\lambda < \infty$, we have a uniformly constant $C_r = \frac{1}{1-r^2}$, which is only dependent on r, such that*

$$\int_\Delta |f|^2 \mathrm{d}\lambda \leq C_r \int_{\Delta \setminus \Delta_r} |f|^2 \mathrm{d}\lambda,$$

where λ is the Lebsgue measure on $\mathbb{C}$.

1.1.3 Proof of Lemma 4.5

Let

$$L_h^2(M) := \left\{ \alpha \mid \alpha \in \Omega_M^{n,0}(E), \int_E \{\alpha, \alpha\}_h < \infty \right\}.$$

where $\{\alpha, \alpha\}_h =: |\alpha|_h^2 \mathrm{d}V_M$.

Proof. We can choose a covering $\{U_i\}_{i=1,2,\ldots}$ of M, which satisfies

(1) $U_i \subset\subset M$, and $\exists K_i \subset\subset U_i$, such that $\bigcup_{i=1}^\infty K_i = M$;
(2) $E|_{U_i}$ is trivial with holomorphic basis $e_1^i, \ldots, e_r^i$;
(3) $K_M|_{U_i}$ is trivial with holomorphic basis v^i.

Then we may write $F_j|_{U_i} = f_{j,i}^k e_k^i \otimes v^i$, where $f_{j,i}^k$ are holomorphic functions on U_i. As h is a Hermitian metric and $U_i \subset\subset M$, there exists a constant $B_K > 0$, such that

$$\sum_{1 \leq k,l \leq r} h\left(e_k^i, e_l^i\right) f_{j,i}^k \bar{f}_{j,i}^l \geq B_K \sum_{k=1}^{r} \left|f_{j,i}^k\right|^2.$$

It may be computed by this way:

$$\begin{aligned}\left(F_j|_{U_i}, \bar{F}_j|_{U_i}\right)_h &= \left(f^k_{j,i}e^i_k \otimes v^i, \bar{f}^l_{j,i}e^i_l \otimes \bar{v}^i\right)_h \\ &= f^k_{j,i}\bar{f}^l_{j,i}h(e^i_k \otimes v^i, e^i_l \otimes \bar{v}^i) \\ &= f^k_{j,i}\bar{f}^l_{j,i}h(e^i_k, e^i_l) \otimes h(v^i, \bar{v}^i) \\ &= f^k_{j,i}\bar{f}^l_{j,i}h(e^i_k, e^i_l).\end{aligned}$$

It is worth to note that $\left(F_j|_{U_i}, \bar{F}_j|_{U_i}\right)_h = |F_j|_{U_i}|^2_h$ and

$$\int_{U_j} |F_j|_{U_j}|^2_h \mathrm{d}V_M = \int_{U_j} \left|\sum_{k=1}^{r} f^k_{j,i}e^i_k \otimes v^i\right|^2_h \mathrm{d}V_M \leqslant C_K.$$

By inequality (4.2), it follows that

$$\begin{aligned}\int_{U_j} \left|\sum_{k=1}^{r} f^k_{j,i}e^i_k \otimes v^i\right|^2_h \mathrm{d}V_M &\leqslant \int_{U_j} \sum_{k=1}^{r} |f^k_{j,i}e^i_k \otimes v^i|^2_h \mathrm{d}V_M \\ &= \sum_{k=1}^{r} \int_{U_j} \left\{f^k_{j,i}e^i_k \otimes v^i, f^k_{j,i}e^i_k \otimes v^i\right\}_h \\ &= \sum_{k=1}^{r} \int_{U_j} \left\{v^i \otimes f^k_{j,i}e^i_k, v^i \otimes f^k_{j,i}e^i_k\right\}_h \\ &= \sum_{k=1}^{r} \int_{U_j} \left\langle f^k_{j,i}e^i_k, f^k_{j,i}e^i_k\right\rangle_h \sqrt{-1}^{(\dim U_j)^2} v^i \wedge \bar{v}^i \\ &= \sum_{k=1}^{r} \int_{U_j} |f^k_{j,i}|^2_h c_n v^i \wedge \bar{v}^i\end{aligned}$$

Then we have

$$\int_{U_j} \sum_{k=1}^{r} \left|f^k_{j,i}\right|^2 c_n v^i \wedge \bar{v}^i \leq \frac{C_K}{B_K} \tag{1.7}$$

for any $j = 1, 2, \ldots$. We can obtain a subsequence of $\{F_j\}_{j=1,2,\ldots}$ which is uniformly convergent on any compact subset of M by the following steps:

(1) On U_1, by inequality (1.7), we can obtain subsequence $\left\{F'_{1_j}\right\}_{j=1,2,\ldots}$ of $\{F_j\}_{j=1,2,\ldots}$ which is uniformly convergent on K_1;

(2) On U_2, by inequality (1.7), we can obtain subsequence $\left\{F'_{2_j}\right\}_{j=1,2,\ldots}$ of $\left\{F'_{1,j}\right\}_{j=1,2,\ldots}$ which is uniformly convergent on K_2;

(3) On U_3, by inequality (1.7), we can obtain subsequence $\left\{F'_{3_j}\right\}_{j=1,2,\ldots}$ of $\left\{F'_{2,j}\right\}_{j=1,2,\ldots}$ which is uniformly convergent on K_3 ....

As the transition matrix of E is invertible, we see that $\left\{F'_{j_j}\right\}_{j=1,2,\ldots}$ is uniformly convergent on any compact subset of M. Thus we have proved the lemma. ■

1.1.4 Proof of Lemma 4.6

Lemma 1.1.2 – Lemma 4.6. of [GZ15] *Let M be a complex manifold. Let S be a closed complex submanifold of M. Let $\{U_j\}_{j=1,2,\ldots}$ be a sequence of open subsets on M, which satisfies*

$$U_1 \subset U_2 \subset \cdots \subset U_j \subset U_{j+1} \subset \cdots,$$

and $\bigcup_{j=1}^{\infty} U_j = M\backslash S$. Let $\{V_j\}_{j=1,2,\ldots}$ be a sequence of open subsets on M, which satisfies

1. *$V_1 \subset V_2 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots$,*
2. *$V_j \supset U_j$, and $\bigcup_{j=1}^{\infty} V_j = M$.*

Let $\{g_j\}_{j=1,2,\ldots}$ be a sequence of positive Lebesgue measurable functions on U_k, which satisfies that g_j are almost everywhere convergent to g on any compact subset of $U_k (j \geq k)$, and g_j have uniformly positive lower and upper bounds on any compact subset of $U_k (j \geq k)$, where g is a positive Lebesgue measurable function on $M\backslash S$.

Let E be a holomorphic vector bundle on M, with Hermitian metric h. Let $\{F_j\}_{j=1,2,\ldots}$ be a sequence of holomorphic $(n,0)$-form on V_j with values in E. Assume that $\lim_{j\to\infty} \int_{U_j} \{F_j, F_j\}_h\, g_j = C$, where C is a positive constant.

Then there exists a subsequence $\{F_{j_l}\}_{l=1,2,\ldots}$ of $\{F_j\}_{j=1,2,\ldots}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to an $(n,0)$-form F on M with value in E on any compact subset of M when $l \to +\infty$, such that

$$\int_M \{F,F\}_h g \leq C.$$

Proof. As $\liminf_{j\to\infty} \int_{U_j} \{F_j, F_j\}_h g_j = C < \infty$, it follows that there exists a subsequence of $\{F_{j_k}\}$ such that

$$\lim_{k\to\infty} \int_{U_{j_k}} \{F_{j_k}, F_{j_k}\}_h g_{j_k} = C.$$

Then ***by Lemma 4.4***, for any compact subset $K_k \subset U_k \subset M$, it follows that there exists $K_{j_k} \subset M\backslash S$ satisfying $K_{j_k} \subset U_{j_k}$, and

$$\int_{K_k} \{F_j, F_j\}_h \leqslant \int_{K_{j_k}} \{F_{j_k}, F_{j_k}\}_h g_{j_k} \to \int_{K_{j_k}} \{F_j, F_j\}_h g_j$$

when j efficiently large enough, that is $j \geqslant j_k$. So by using ***Lemma 4.5***, we have a subsequence of $\{F_j\}$ which is convergent on any compact subset $K_k^{\bullet} \subset M$. Without loose generality, we shall assume that

1. $\bigcup_{k=1}^{\infty} K_k^{\bullet} = M$;
2. $K_k \subset K_{k+1}$.

Thus we have a subsequence $\{F_j\}$ which is convergent to a holomorphic $(n,0)$-form with values in E on any compact subset of M.

Given $K \subset M\backslash S$, as $\{F_j\}$ (resp. g_j) is uniformly convergent to F (resp. g) for $j \geqslant k_K$, we have

$$\int_K \{F,F\}_h g \leqslant \lim_{j\to\infty} \int_{U_j} \{F_j, F_j\}_h g_j,$$

where k_K satisfies $U_{k_K} \subset K$. It is clear that

$$\int_M \{F,F\}_h g \leqslant \lim_{j\to\infty} \int_{U_j} \{F_j, F_j\}_h g_j.$$

■

1.1.5 Proof of Lemma 4.8

Lemma 1.1.3 – LEMMA 4.8. of [GZ15] *Let $c_A(t)$ be a positive function in $C^\infty((-A,+\infty))$, which satisfies $\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$ and inequality (1.6), for any $t \in (-A,+\infty)$. Then there exists a sequence of positive C^∞ smooth functions $\{c_{A,m}(t)\}_{m=1,2,\dots}$ on $(-A,+\infty)$, which satisfies*

(i) *$c_{A,m}(t)$ are continuous near $+\infty$ and $\lim_{t\to+\infty} c_{A,m}(t) > 0$;*
(ii) *$c_{A,m}(t)$ are uniformly convergent to $c_A(t)$ on any compact subset of $(-A,+\infty)$, when m goes to ∞;*
(iii) *$\int_{-A}^{\infty} c_{A,m}(t)e^{-t}dt$ is convergent to $\int_{-A}^{\infty} c_A(t)e^{-t}dt$ when m approaches to ∞;*
(iv) *for any $t \in (-A,+\infty)$,*

$$\left(\int_{-A}^{t} c_{A,m}\left(t_1\right)e^{-t_1}dt_1\right)^2 > c_{A,m}(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} c_{A,m}\left(t_1\right)e^{-t_1}dt_1dt_2$$

holds.

Proof. **Construction of the function $c_{A,m}(t)$**

Let

$$g_B(t) = \begin{cases} c_A(t), & t \in (-A,-A+B], \\ g_B(t), & t \in (-A+B,+\infty) \end{cases}$$

satisfies the following conditions

1. $g_B(t)$ is ***positive continuous decreasing function*** on $t \in [-A+B,+\infty)$;
2. $g_B(t)$ is ***smooth*** on $t \in (-A+B,+\infty)$;
3. $\lim_{t\to\infty} g_B(t) > 0$ and

$$\int_{-A+B}^{\infty} g_B(t)e^{-t}\mathrm{d}t < B^{-1}, \tag{1.8}$$

where $B > 0$.

When $t \in (-A,-A+B)$, by Inequality (1.6) we have

$$g_B(t) = c_A(t)$$

and

$$\left(\int_{-A}^{t} g_B(t_1)e^{-t_1}dt_1\right)^2 > g_B(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} g_B(t_1)e^{-t_1}dt_1dt_2 \tag{1.9}$$

holds for any $t \in t \in (-A,-A+B)$.

As $g_B(t)$ is decreasing on $[-A+B,+\infty)$, it implices that inequality (1.9) holds for any $t \in (-A,+\infty)$ (cf. The property of Inequality (1.6) of the situation that $c_A(t)e^{-t}$ is decreasing with respect to t.) and by Inequality (1.8) we have

$$\begin{aligned}\int_{-A}^{\infty} g_B(t)e^{-t}\mathrm{d}t &= \int_{-A}^{-A+B} g_B(t)e^{-t}\mathrm{d}t + \int_{-A+B}^{\infty} g_B(t)e^{-t}\mathrm{d}t \\ &< \int_{-A}^{\infty} c_A(t)e^{-t}\mathrm{d}t + B^{-1} \to \int_{-A}^{\infty} c_A(t)e^{-t}\mathrm{d}t\ (B\to\infty).\end{aligned}$$

Thus
$$\lim_{B\to\infty}\int_{-A}^{\infty} g_B(t)e^{-t}\mathrm{d}t=\int_{-A}^{\infty} c_A(t)e^{-t}\mathrm{d}t.$$

Now we focus on the left problem : What is the situation in the neighborhood of the point $-A+B$? Fixing a small enough $\varepsilon_B>0$ such that $[(-A+B)-\varepsilon_B,(-A+B)+\varepsilon_B]\subset(-A,+\infty)$, one can find a sequence of functions $\{g_{B,j}(t)\}_{j=1,2,\cdots}$ in $C^\infty(-A,+\infty)$, satisfying $g_{B,j}(t)=g_B(t)$ when $t\notin[-A+B-\varepsilon_B,-A+B+\varepsilon_B]$, which is uniformly convergent to $g_B(t)$. In other words, $\{g_{B,j}(t)\}_{j=1,2,\cdots}$ are almost everywhere equal to $g_B(t)$. Then it is clear that for j big enough
$$(\int_{-A}^{t} g_{B,j}(t_1)e^{-t_1}dt_1)^2>g_{B,j}(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} g_{B,j}(t_1)e^{-t_1}dt_1dt_2,$$
holds for any $t\in(-A,+\infty)$.

For any given B, we can choose j_B large enough such that

(1.10)
$$1).\left|\int_{-A}^{\infty} g_{B,j_B}(t)e^{-t}\mathrm{d}t-\int_{-A}^{\infty} g_B(t)\mathrm{d}t\right|<B^{-1};$$
$$2).\max_{t\in(-A,+\infty)}|g_{B,j_B}(t)-g_B(t)|<B^{-1};$$
$$3).\left(\int_{-A}^{t} g_{B,j_B}(t_1)e^{-t_1}dt_1\right)^2>g_{B,j_B}(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} g_{B,j_B}(t_1)e^{-t_1}dt_1dt_2,\ \forall t\in(-A,+\infty).$$

Let $c_{A,m}:=g_{m,j_m}$, thus we have proved the case that $A<+\infty$.

Secondly, we consider the case that $A=+\infty$. Let $g_B(t):=c_\infty(t)$ when $t\in(-\infty,B)$, $g_B(t):=c_\infty(B)$ when $t\in[B,\infty)$, where $B>0$.

Using the same construction as the case $A<+\infty$, we obtain the the case that $A=+\infty$. ■

1.1.6 Proof of Lemma 4.11

Now we introduce a relationship between inequality 1.6 and 1.3.

Lemma 1.1.4 *Let $c_A(t)$ satisfy $\int_{-A}^{+\infty} c_A(t)e^{-t}dt<\infty$ and inequality 1.6 ($A\in(-\infty,+\infty]$). For each $A'<A$, there exists A'' and $\delta''>0$, such that $A>A''>A'$ and there exists $c_{A''}(t)\in C^0([-A'',+\infty))$ satisfying*

i) $c_{A''}(t)=c_A(t)|_{[-A',+\infty)}$;

ii)
$$\int_{-A''}^{+\infty} c_{A''}(t)e^{-t}dt+\frac{1}{\delta''}c_{A''}(-A'')e^{A''}=\int_{-A}^{+\infty} c_A(t)e^{-t}\mathrm{d}t;$$

iii)
$$\int_{-A''}^{t}\left(\frac{1}{\delta''}c_{A''}(-A'')e^{A''}+\int_{-A''}^{t_2} c_{A''}(t_1)e^{-t_1}dt_1\right)dt_2+\frac{1}{\delta''^2}c_{A''}(-A'')e^{A''}$$
$$<\int_{-A}^{t}\left(\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1\right)dt_2.$$

Proof. Given $A'<A$. Let $g(t)|_{[-A',+\infty)}:=c_A(t)|_{[-A',+\infty)}$. As $c_A(t)$ satisfies $\int_{-A}^{+\infty} c_A(t)e^{-t}dt<\infty$ and inequality 1.6 holds ($A\in(-\infty,+\infty]$), we can choose a continuous function $g(t)$ such that it's decreasing rapidly enough on $[A'',A']$ (A'' can be chosen near A' enough), and the following holds:

(1).

$$\int_{-A''}^{+\infty} c_{A''}(t)e^{-t}dt + \frac{1}{\delta''}c_{A''}(-A'')e^{A''} = \int_{-A}^{+\infty} c_A(t)e^{-t}\mathrm{d}t;$$

(2).

$$\int_{-A''}^{t}\left(\frac{1}{\delta''}c_{A''}(-A'')e^{A''} + \int_{-A''}^{t_2} c_{A''}(t_1)e^{-t_1}dt_1\right)dt_2 + \frac{1}{\delta''^2}c_{A''}(-A'')e^{A''} < \int_{-A}^{t}\left(\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1\right)dt_2.$$

Thus we have proved the lemma. ■

Remark – TODO I How can we do that? Why?

1.1.7 Proof of Remark 4.12

Since A may be chosen as positive infinity, we have a sufficient condition for inequality 1.6 holding:

Remark – Remark 4.12 Assume that

$$\frac{d}{dt}c_{+\infty}(t)e^{-t}\begin{cases} > 0, & t \in (-\infty, a), \\ \leq 0, & t \in [a, +\infty). \end{cases},$$

where $a \neq -\infty$. Assume that $\frac{d^2}{dt^2}\log(c_A(t)e^{-t}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c'_A(t)}{c_A(t)}\right) < 0$ for $t \in (-A, a)$. Then inequality 1.6 holds.

Proof. Let

$$H(t,f) := \left(\int_{-A}^{t} f(t_1)dt_1\right)^2 - f(t)\int_{-A}^{t}\left(\int_{-A}^{t_2} f(t_1)dt_1\right)dt_2, \tag{1.11}$$

where $f(t)$ is a positive smooth function on $(-A, +\infty)$.

Inequality 1.6 becomes $H(t, c_A(t)e^{-t}) > 0$ for any $t \in (-A, +\infty)$. That is $\frac{H(t,c_A(t)e^{-t})}{c_A(t)e^{-t}} > 0$ for any $t \in (-A, +\infty)$.

It suffices to prove $\frac{d}{dt}\frac{H(t,c_A(t)e^{-t})}{c_A(t)e^{-t}} > 0$ for any $t \in (-\infty, a)$, therefore

$$H(t, \frac{d}{dt}(c_A(t)e^{-t})) > 0$$

for any $t \in (-\infty, a)$.

As $\frac{d}{dt}(c_A(t)e^{-t}) > 0$ for any $t \in (-\infty, a)$, it suffices to prove that

$$\frac{d}{dt}\frac{H(t, \frac{d}{dt}(c_A(t)e^{-t}))}{\frac{d}{dt}(c_A(t)e^{-t})} > 0$$

for any $t \in (-\infty, a)$, which is $H(t, \frac{d}{dt}\frac{d}{dt}(c_A(t)e^{-t})) > 0$ for any $t \in (-\infty, a)$.

Note that $H(t, \frac{d}{dt}\frac{d}{dt}(c_A(t)e^{-t})) = -(c_A(t)e^{-t})^2\frac{d}{dt}\frac{d}{dt}\log(c_A(t)e^{-t})$. Thus we have proved the present Remark. ■

1.1.8 Proof of Lemma 4.14

Lemma 1.1.5 *Let M be a complex manifold of dimension n and S be an $(n-l)-$dimensional closed complex submanifold. Let $\Psi \in \Delta(S)$. Assume that there exists a local coordinate $(z_1, \cdots, z_n)$ on a neighborhood U of $x \in M$ such that*

$$\{z_{n-l+1} =, \cdots, z_n = 0\} = S \cap U$$

and $\psi := \Psi - l \log(|z_{n-l+1}|^2 + \cdots + |z_n|^2)$ is continuous on U.

Then we have

$$d\lambda_z[\Psi] = e^{-\psi} d\lambda_{z'},$$

where $d\lambda_z$ and $d\lambda_{z'}$ denote the Lebesgue measures on U and $S \cap U$. Especially,

$$|f \wedge dz_{n-l+1} \wedge \cdots \wedge dz_n|_h^2 d\lambda_z[\Psi] = 2^l \{f, f\}_h e^{-\psi},$$

where f is a continuous $(n-l, 0)$ form with value in the Hermitian vector bundle (E, h) on $S \cap U$.

Proof. Note that $d\lambda_z[l \log(|z_{n-l+1}|^2 + \cdots + |z_n|^2)] = d\lambda_{z'}$ for $z = (z', z_{n-l+1}, \cdots, z_n)$. According to the definition of ***generalized residue volume form*** $d\lambda_z[\Psi]$ and the continuity of ψ, the lemma follows. ∎

Remark

$$\begin{aligned} |f \wedge dz_{n-l+1} \wedge \cdots \wedge dz_n|_h^2 d\lambda_z[\Psi] &\overset{\text{By def}}{=} |f \wedge dz_{n-l+1} \wedge \cdots \wedge dz_n|_h^2 \left(e^{-\psi} \mathrm{d}\lambda_{z'}\right) \\ &= e^{-\psi} \cdot \{f, f\}_h^2 \cdot |dz_{n-l+1} \wedge \cdots \wedge dz_n|^2 \\ &= 2^l e^{-\psi} \cdot \{f, f\}_h^2. \end{aligned}$$

1.1.9 Proof of Lemma 4.16

Lemma 1.1.6 *Let $d_1(t)$ and $d_2(t)$ be two positive continuous functions on $(0, +\infty)$, which satisfy*

$$\int_0^{+\infty} d_1(t) e^{-t} dt = \int_0^{+\infty} d_2(t) e^{-t} dt < \infty,$$

and

$$\begin{cases} d_1(t) = d_2(t), & \{t < r_3\} \cup \{t > r_1\}; \\ d_1(t) < d_2(t), & \{r_3 < t < r_2\}; \\ d_1(t) > d_2(t), & \{r_2 < t < r_1\}. \end{cases}$$

where $0 < r_3 < r_2 < r_1 < +\infty$.

Let f be a holomorphic function on Δ, then we have

$$\int_\Delta d_1(-\ln(|z|^2)) |f|^2 d\lambda \le \int_\Delta d_2(-\ln(|z|^2)) |f|^2 d\lambda < +\infty,$$

where λ is the Lebesgue measure on Δ. Moreover, the equality holds if and only if $f \equiv f(0)$.

Proof. Set

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

a Taylor expansion of f at 0, which is uniformly convergent on any given compact subset of Δ.

As

$$\int_{\Delta} d_1(-\ln(|z|^2))z^{k_1}\bar{z}^{k_2}d\lambda = 0$$

when $k_1 \neq k_2$, it follows that

$$\begin{aligned}
\int_{\Delta} d_1(-\ln(|z|^2))|f|^2 d\lambda &= \int_{\Delta} \sum_{k=0}^{\infty} d_1(-\ln(|z|^2))|a_k|^2|z|^{2k}d\lambda \\
&= \int_0^{2\pi}\int_0^{+\infty}\sum_{k=0}^{\infty} d_1(-\ln(|z|^2))|a_k|^2|z|^{2k}|z|\mathrm{d}|z|\mathrm{d}\theta \\
&= \int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\sum_{k=0}^{\infty} d_1(t)|a_k|^2 e^{-kt}\sqrt{e^{-t}}\mathrm{d}\left|\sqrt{e^{-t}}\right| \\
&= \int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\sum_{k=0}^{\infty} d_1(t)|a_k|^2 e^{-kt}\sqrt{e^{-t}}\left(\left|-\frac{1}{2}\right|\sqrt{e^{-t}}\right)\mathrm{d}t \\
&= \pi\sum_{k=0}^{\infty}|a_k|^2\int_0^{+\infty} d_1(t)e^{-kt}e^{-t}dt,
\end{aligned} \tag{1.12}$$

where $r = |z|$, and

$$\begin{aligned}
\int_{\Delta} d_2(-\ln(|z|^2))|f|^2 d\lambda &= \int_{\Delta} \sum_{k=0}^{\infty} d_2(-\ln(|z|^2))|a_k|^2|z|^{2k}d\lambda \\
&= \pi\sum_{k=0}^{\infty}|a_k|^2\int_0^{+\infty} d_2(t)e^{-kt}e^{-t}dt.
\end{aligned} \tag{1.13}$$

As

$$\int_0^{+\infty} d_1(t)e^{-t}dt = \int_0^{+\infty} d_2(t)e^{-t}dt < \infty,$$

$$d_1(t)|_{\{r_2<t<r_1\}} > d_2(t)|_{\{r_2<t<r_1\}},$$

and

$$d_1(t)|_{\{r_3<t<r_2\}} < d_2(t)|_{\{r_3<t<r_2\}},$$

it follows that

$$\int_{r_3}^{r_2}(d_2(t)-d_1(t))e^{-kt}e^{-t}dt > \int_{r_3}^{r_2}(d_2(t)-d_1(t))e^{-kr_2}e^{-t}dt, \tag{1.14}$$

$$\int_{r_2}^{r_1}(d_1(t)-d_2(t))e^{-kr_2}e^{-t}dt > \int_{r_2}^{r_1}(d_1(t)-d_2(t))e^{-kt}e^{-t}dt, \tag{1.15}$$

thus by adding (1.14) and (1.15) we have

$$\int_{r_3}^{r_2}(d_2(t)-d_1(t))e^{-kt}e^{-t}dt > \int_{r_2}^{r_1}(d_1(t)-d_2(t))e^{-kt}e^{-t}dt.$$

Then we have

$$\int_{r_3}^{r_2}(d_2(t)-d_1(t))e^{-kt}e^{-t}dt > \int_{r_2}^{r_1}(d_1(t)-d_2(t))e^{-kt}e^{-t}dt$$

$$\Longleftrightarrow \text{ split and reorganization}$$

$$\int_{r_3}^{r_2} d_2(t)e^{-kt}e^{-t}dt - \int_{r_3}^{r_2} d_1(t)e^{-kt}e^{-t}dt > \int_{r_2}^{r_1} d_1(t)e^{-kt}e^{-t}dt - \int_{r_2}^{r_1} d_2(t)e^{-kt}e^{-t}dt$$

$$\int_{r_3}^{r_2} d_2(t)e^{-kt}e^{-t}dt + \int_{r_2}^{r_1} d_2(t)e^{-kt}e^{-t}dt > \int_{r_2}^{r_1} d_1(t)e^{-kt}e^{-t}dt + \int_{r_3}^{r_2} d_1(t)e^{-kt}e^{-t}dt$$

Therefore

$$\int_{r_3}^{r_1} d_1(t)e^{-kt}e^{-t}dt < \int_{r_3}^{r_1} d_2(t)e^{-kt}e^{-t}dt,$$

for every $k \geq 1$.

■

Remark

$$\int_{\Delta} d_1(-\ln(|z|^2))z^{k_1}\bar{z}^{k_2}d\lambda = \begin{cases} 0, & k_1 \neq k_2, \\ |z|^{2k}, & k_1 = k_2 = k. \end{cases}$$

The equation

$$\int_{\Delta} z^{k_1}\cdot \bar{z}^{k_2}\, d\lambda = 0$$

is generally true when $k_1 \neq k_2$, especially under certain specific conditions and for appropriate integration regions Δ. This equation relies on principles of ***orthogonality*** in complex analysis, involving integrals of analytic functions and their conjugates over specific domains.

In the complex plane, z^{k_1} and $\bar{z}^{k_2}$ represent the k_1-th power of z and the complex conjugate of z raised to the k_2-th power, respectively. *When $k_1 \neq k_2$, these functions are orthogonal over the entire region, meaning that the integral of their product over that region is zero. This orthogonality arises due to the oscillatory nature of these functions in the complex plane, resulting in mutual cancellation over the entire region.*

1.1.10 Proof of Lemma 4.17

Let Ω be an open Riemann surface. Let $z_0 \in \Omega$, and V_{z_0} be a neighborhood of z_0 with local coordinate w, such that $w(z_0) = 0$.

Using the lemma 1.1.6, we have the following lemma on open Riemann surfaces:

Lemma 1.1.7 *Assume that there is a negative subharmonic function Ψ on Ω, such that $\Psi|_{V_{z_0}} = \ln|w|^2$, and $\Psi|_{\Omega\setminus V_{z_0}} \geq \sup_{z\in V_{z_0}}\Psi(z)$. Let $d_1(t)$ and $d_2(t)$ be two positive continuous functions on $(0,+\infty)$ as in lemma 1.1.6. Assume that $\{\Psi < -r_3+1\} \subset\subset V_{z_0}$ is a disc with the coordinate z. Let F be a holomorphic $(1,0)$ form, which satisfies $F|_{z_0} = dw$, then we have*

$$\int_{\Omega} d_1(-\Psi)\sqrt{-1}F\wedge\bar{F} \leq \int_{\Omega} d_2(-\Psi)\sqrt{-1}F\wedge\bar{F} < +\infty,$$

Moreover, the equality holds if and only if $F|_{V_{z_0}} = dw$.

Proof. It is clear that

$$\begin{aligned}&\int_\Omega d_1(-\Psi)\sqrt{-1}F\wedge\bar{F} = \\ &\int_{\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\sqrt{-1}|\frac{F}{dw}|^2 dw\wedge d\bar{w} + \int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\sqrt{-1}F\wedge\bar{F},\end{aligned} \tag{1.16}$$

$$\begin{aligned}&\int_\Omega d_2(-\Psi)\sqrt{-1}F\wedge\bar{F} = \\ &\int_{\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\sqrt{-1}|\frac{F}{dw}|^2 dw\wedge d\bar{w} + \int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\sqrt{-1}F\wedge\bar{F}.\end{aligned} \tag{1.17}$$

Note that $-\Psi|_{\Omega\setminus\{\log|w|^2<-r_3+1\}} < r_3-1 < r_3$, then by the definition we have $d_1(t) = d_2(t), t\in \{t<r_3\}\cup\{t>r_1\}$, then

$$\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\sqrt{-1}F\wedge\bar{F} = \int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\sqrt{-1}F\wedge\bar{F}.$$

Applying Lemma 1.1.6 to the rest parts of equalities 1.16 and 1.17, we get the present lemma. ■

1.1.11 Proof of Lemma 4.18

Using Lemma 1.1.6, we obtain the following lemma.

Lemma 1.1.8 *Let Ω be an open Riemann surface with Green function G_Ω. Let $z_0\in\Omega$, and V_{z_0} be a neighborhood of z_0 with local coordinate w, such that $w(z_0)=0$. Assume that there is a negative subharmonic function Ψ on Ω, such that $\Psi|_{V_{z_0}} = \ln|w|^2$ and $\Psi|_{\Omega\setminus V_{z_0}} \geq \sup_{z\in V_{z_0}}\Psi(z)$. Let $d_1(t)$ and $d_2(t)$ be two positive continuous functions on $(0,+\infty)$ as in Lemma 1.1.6. Assume that $\{\Psi<-r_3+1\}\subset\subset V_{z_0}$, which is a disc with the coordinate w. Let F be a holomorphic $(1,0)$ form, which satisfies $((p_j)_*(f_{-h,j}))F|_{z_0} = \mathrm{d}w$, then we have*

$$\int_\Omega d_1(-\Psi)\sqrt{-1}\rho F\wedge\bar{F} \leq \int_\Omega d_2(-\Psi)\sqrt{-1}\rho F\wedge\bar{F},$$

Moreover, the equality holds if and only if $((p_j)_*(f_{-h,j}))F|_{V_{z_0}} = \mathrm{d}w$.

The proof is the same as above lemma, they are all the corollaries of 1.1.6.

1.2 The proof of the main theorem I

CHAP

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