From Calculus to Cohomology

de Rham cohomology and characteristic classes

Ib Madsen and Jørgen Tornehave

University of Aarhus

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building,Trumpington Street,Cambridge,United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building,Cambridge CB2 2RU,UK http://www.cup.cam.ac.uk
40 West 20th Street,New York,NY 10011-4211,USA http://www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1997
Reprinted. 1998, 1999

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data
Madsen,I.H.(Ib Henning),1942-
From calculus to cohomology:de Rham cohomology and
characteristic classes / b Madsen and Jorgen Tornehave.
p. cm.
Includes bibliographical references and index.
ISBN 0-521-58059-5 (hc). – ISBN 0-521-58956-8 (pbk).
I. Homology theory. 2. Differential forms. 3. Characteristic classes.
I. Tornehave,Jergen. II. Title.
QA612.3.M33 1996
514.2-dc20 96-28589 CIP

ISBN 0 521 58059 5 hardback
ISBN 0 521 58956 8 paperback

Transferred to digital printing 2001

PREFACE

This text offers a self-contained exposition of the cohomology of differential forms, de Rham cohomology, and of its application to characteristic classes defined in terms of the curvature tensor. The only formal prerequisites are knowledge of standard calculus and linear algebra, but for the later part of the book some prior knowledge of the geometry of surfaces, Gaussian curvature, will not hurt the reader.

The first seven chapters present the cohomology of open sets in Euclidean spaces and give the standard applications usually covered in a first course in algebraic topology, such as Brouwer's fixed point theorem, the topological invariance of domains and the Jordan-Brouwer separation theorem. The next four chapters extend the definition of cohomology to smooth manifolds, present Stokes' theorem and give a treatment of degree and index of vector fields, from both the cohomological and geometric point of view.

The last ten chapters give the more advanced part of cohomology: the Poincaré-Hopf theorem, Poincare duality, Chern classes, the Euler class, and finally the general Gauss-Bonnet formula. As a novel point we prove the so called splitting principles for both complex and real oriented vector bundles. The text grew out of numerous versions of lecture notes for the beginning course in topology at Aarhus University. The inspiration to use de Rham cohomology as a first introduction to topology comes in part from a course given by G. Segal at Oxford many years ago, and the first few chapters owe a lot to his presentation of the subject. It is our hope that the text can also serve as an introduction to the modern theory of smooth four-manifolds and gauge theory.

The text has been used for third and fourth year students with no prior exposure to the concepts of homology or algebraic topology. We have striven to present all arguments and constructions in detail. Finally we sincerely thank the many students who have been subjected to earlier versions of this book. Their comments have substantially changed the presentation in many places.

Aarhus, January 1996

This page intentionally left blank.

CONTENTS

This page intentionally left blank.

1. INTRODUCTION

It is well-known that a continuous real function, that is defined on an open set of $\mathbb{R}$has a primitive function. How about multivariable functions? For the sake of simplicity we restrict ourselves to smooth (or C^∞-) functions, i.e. functions that have continuous partial derivatives of all orders. We begin with functions of two variables. Let $f : U \to \mathbb{R}^2$ be a smooth function .defined on an open set of $\mathbb{R}^2$.

Question 1.1 Is there a smooth function $F : U \to \mathbb{R}$, such that:

$$\frac{\partial F}{\partial x_1} = f_1 \text{ and } \frac{\partial F}{\partial x_2} = f_2, \text{ where } f = (f_1, f_2)? \tag{1}$$

Since

$$\frac{\partial^2 F}{\partial x_2 \partial x_1} = \frac{\partial^2 F}{\partial x_1 \partial x_2}$$

we must have

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1} \tag{2}$$

The correct question is therefore whether F exists, assuming $f = (f_1, f_2)$ satisfies (2). Is condition (2) also sufficient?

Example 1.2 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, \right)$$

It is easy to show that (2) is satisfied. However, there is no function $F : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ that satisfies (1). Assume there were; then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1, 0) - F(1, 0) = 0$$

On the other hand the chain rule gives

$$\begin{aligned}
\frac{d}{d\theta} F(\cos\theta, \sin\theta) &= \frac{dF}{dx} \cdot (-\sin\theta) + \frac{dF}{dy} \cdot \cos\theta \\
&= -f_1(\cos\theta, \sin\theta) \cdot \sin\theta + f_2(\cos\theta, \sin\theta) \cdot \cos\theta \\
&= 1
\end{aligned}$$

This contradiction can only be explained by the non-existence of F.

Definition 1.3 A subset $X \subseteq \mathbb{R}^n$ is said to be star-shaped with respect to the point $x_0 \in X$ if the line segemtn $\{tx_0 + (1-t)x | t \in [0,1]\}$ is contained in X for all $x \in X$.

Theorem 1.4 Let $U \subseteq \mathbb{R}^2$ be star-shaped. Then for any smooth function $f : U \to \mathbb{R}^2$ that satisfies (2), Question 1.1 has a solution.

PROOF. For the sake of simplicity we assume that $x_0 = 0 \in \mathbb{R}^2$. Consider the function $F : U \to \mathbb{R}$.

$$F(x_1, x_2) = \int_0^1 [x_1 f_1(tx_1, tx_2) + x_2 f_2(tx_1, tx_2)] \; dt.$$

Then one has

$$\frac{\partial F}{\partial x_1} = \int_0^1 \left[f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_2}{\partial x_1}(tx_1, tx_2) \right] \; dt$$

and

$$\frac{d}{dt} t f_1(tx_1, tx_2) = f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_2}{\partial x_1}(tx_1, tx_2)$$

Substituting this result into the formula, we get

$$\begin{aligned} \frac{\partial F}{\partial x_1}(x_1, x_2) &= \int_0^1 \left[\frac{d}{dt} t f_1(tx_1, tx_2) + tx_2 \left(\frac{\partial f_2}{\partial x_1}(tx_1, tx_2) - \frac{\partial f_1}{\partial x_2}(tx_1, tx_2) \right) \right] \; dt \\ &= f_1(tx_1, tx_2) \Big|_{t=0}^{1} \\ &= f_1(x_1, x_2) \end{aligned}$$

Analogously, $\frac{\partial F}{\partial x_2} = f_2(x_1, x_2)$. □

Example 1.2 and Theorem 1.4 suggest that the answer to Question 1.1 depends on the "shape" of "topology" of U. Instead of searching for a further examples or counterexamples of set U and fucntion f, we define an invariant of U, which tells us or not the question has an affirmative answer (for all f), assuming the necessary condition (2).
Give the open set $U \subseteq \mathbb{R}^2$, let $C^\infty(U, \mathbb{R}^k)$ denote the set of smooth functions $\phi : U \to \mathbb{R}^k$. This is a vecter space. If $k = 2$ one may consider $\phi : U \to \mathbb{R}^k$ as a vecter filed on U by plotting $\phi(u)$ from the point u. We define the *gradient* and *rotation*:

$$\text{grad} : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^2), \qquad \text{grad} : C^\infty(U, \mathbb{R}^2) \to C^\infty(U, \mathbb{R})$$

by

$$\text{grad}(\phi) = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right), \qquad \text{rot}(\phi) = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}$$

Note that $\mathrm{rot} \circ \mathrm{grad} = 0$. Hence the kernel of rot contains the image of grad,

$$\mathrm{Ker(rot)} = \text{Kernel of rot}$$
$$\mathrm{Im(grad)} = \text{Image of grad}$$

Since both rot and grad are linear operators, Im(grad) is a subspace of Ker(rot). Therefore we can consider the quotient vector space, i.e. the vector space of cosets 0: α + Im(grad) where 0: $\alpha \in$ Ker(rot):

$$H^1(U) = \mathrm{Ker(rot)}/\,\mathrm{Im(grad)}. \tag{3}$$

Both Ker(rot) and Im(grad) are infinite-dimensional vector spaces. It is remarkable that the quotient space $H^1(U)$ is usually finite-dimensional. We can now reformulate Theorem 1.4 as

$$H^1(U) = 0 \text{ where } U \subseteq \mathbb{R}^2 \text{ is star-shaped.} \tag{4}$$

On the other hand, Example 1.2 tells us that $H^1(\mathbb{R}^2 - \{0\}) \neq 0$. Later on we shall see that $H^1(\mathbb{R}^2 - \{0\})$ is 1-dimensional, and that $H^1(\mathbb{R}^2 - \cup_{i=1}^k \{x_i\}) \cong \mathbb{R}^k$. The dimension of $H^1(U)$ is the number of "holes" in U.
In analogy with (3) we introduce

$$H^0(U) = \mathrm{Ker(grad)} \tag{5}$$

This definition works for open sets U of $\mathbb{R}^k$ with $k \geqslant 1$, when we define

$$\mathrm{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Theorem 1.5 An open set $U \subseteq \mathbb{R}^k$ is connected if and only if $H^0(U) = \mathbb{R}$.

PROOF. Assume that $\mathrm{grad}(f) = 0$. Then f is locally constant: each $x_0 \in U$ has a neighborhood $V(x_0)$ with $f(x) = f(x_0)$ when $x \in V(x_0)$. If U is connected, then every locally constant function is constant. Indeed, for $x_0 \in U$ the set

$$\{x \in U | f(x) = f(x_0) = f^{-1}(f(x_0))\}$$

is closed because f is continuous, and open since f is locally constant. Hence it is equal to U, and $H^0(U) = \mathbb{R}$. Conversely, if U is not connected, then there exists a smooth, surjective function $f : U \to \{0, 1\}$. Such a function is locally constant, so $\mathrm{grad}(f) = 0$. It follows that $\dim H^0(U) > 1$. □

The reader may easily extend the proof of Theorem 1.5 to show that $\dim H^0(U)$ is precisely the number of connected components of U.
We next consider functions of three variables. Let $U \subseteq \mathbb{R}^3$ be an open set. A real function on U has three partial derivatives and (2) is replaced by three equations. We introduce the notation

$$\mathrm{grad} : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^3)$$
$$\mathrm{rot} : C^\infty(U, \mathbb{R}^3) \to C^\infty(U, \mathbb{R}^3)$$
$$\mathrm{div} : C^\infty(U, \mathbb{R}^3) \to C^\infty(U, \mathbb{R})$$

for the linear operators defined by

$$\operatorname{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$
$$\operatorname{rot}(f) = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$
$$\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

Note that $\operatorname{rot} \circ \operatorname{grad} = 0$ and $\operatorname{div} \circ \operatorname{rot} = 0$. We define $H^0(U)$ and set $H^1(U)$ as in Equations (3) and (5) and

$$H^2(U) = \operatorname{Ker}(\operatorname{div})/\operatorname{Im}(\operatorname{rot})$$

Theorem 1.6 For an open star-shaped set in $\mathbb{R}^3$ we have that $H^0(U) = R, H^1(U) = 0$ and $H^2(U) = 0$.

PROOF. The values of $H^0(U)$ and $H^1(U)$ are obtained as above, so we shall restrict ourselves to showing that $H^2(U) = 0$. It is convenient to assume that U is star-shaped with respect to 0. Consider a function $F : U \to \mathbb{R}^3$ with $\operatorname{div} F = 0$, and define $G : U \to \mathbb{R}^3$ by

$$G(x) = \int_0^1 (F(tx) \times tx)\, dt$$

where the $\times$ denotes the cross product.

$$(f_1, f_2, f_2) \times (x_1, x_2, x_3) = \begin{vmatrix} e_1 & f_1 & x_1 \\ e_2 & f_2 & x_2 \\ e_3 & f_3 & x_3 \end{vmatrix} = (f_2x_3 - f_3x_2, f_3x_1 - f_1x_3, f_1x_2 - f_2x_1)$$

Straightforward calculations give

$$\operatorname{rot}(F(tx) \times tx) = \frac{d}{dt}(t^2F(tx))$$

Hence

$$\operatorname{rot} G(x) = \int_0^1 \frac{d}{dt}(t^2F(tx))\, dt = F(x)$$

□

If $U \in \mathbb{R}^3$ is not star-shaped both $H^1(U)$ and $H^2(U)$ may be non-zero.

Example 1.7 Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 = 1, x_3 = 0\}$ be the unit circle in the (x_1, x_2)-plane. Consider the function

$$f(x_1, x_2, x_3) = \left(\frac{-2x_1x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{-2x_2x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{x_1^2 + x_2^2 - 1}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}\right)$$

on the open set $U = \mathbb{R}^3 - S$.

One finds that $\mathrm{rot}(f) = 0$. Hence f defines an element $[f] \in H^1(U)$. By integration along a curve γ in U, which is linked to S (as two links in a chain), we shall show that $[f] \neq 0$. The curve in question is

$$\gamma(t) = \left(\sqrt{1 + \cos t}, 0, \sin t\right), -\pi \leqslant t \leqslant \pi$$

Assume $\mathrm{grad}(F) = f$ as a function on U. We can determine the integral of $\frac{d}{dt}F(\gamma(t))$ in two ways. On the hand we have

$$\int_{\pi-\epsilon}^{-\pi+\epsilon} \frac{d}{dt}F(\gamma(t))\, dt = F(\gamma(-\pi+\epsilon)) - F(\gamma(\pi-\epsilon)) \to 0 \qquad \text{for } \epsilon \to 0$$

and on the other hand the chain rule gives

$$\begin{aligned} \frac{d}{dt}F(\gamma(t)) &= f_1(\gamma(t)) \cdot \gamma_1'(t) + f_2(\gamma(t)) \cdot \gamma_2'(t) + f_3(\gamma(t)) \cdot \gamma_3'(t) \\ &= \sin^2 t + 0 + \cos^2 t = 1. \end{aligned}$$

Therefore the integral also converges to 2π, which is a contradiction.

Example 1.8 Let U be an open set in $\mathbb{R}^k$ and $X : U \to \mathbb{R}^k$ a smooth function (a smooth vector field). Recall that the *energy* $A_\gamma(X)$, of X along a smooth curve $\gamma : [a, b] \to U$ is defined by the integral

$$A_\gamma(X) = \int_a^b \langle X \circ \gamma(t), \gamma'(t) \rangle\, dt$$

where $\langle \cdot \rangle$ denotes the standard product. If $X = \mathrm{grad}(\Phi)$ and $\Phi_\gamma(a) = \Phi_\gamma(b)$, then the energy is zero, since

$$\langle X \circ \gamma(t), \gamma'(t) \rangle = \frac{d}{dt}\Phi(\gamma(t))$$

by the rule; compare Example 1.2.

This page intentionally left blank.

2. THE ALTERNATING ALGEBRA

Let V be a vector space over $\mathbb{R}$. A map

$$f : \underbrace{V \times V \times \cdots \times V}_{\text{k times}} \to \mathbb{R}$$

is called k-*linear* (or *multilinear*), if f is linear in each factor.

Definition 2.1 A k-linear map $\omega : V^k \to \mathbb{R}$ is said to be alternating if $\omega(\xi_1, \cdots, \xi_k) = 0$ whenever $\xi_i = \xi_j$ for some pair $i \neq j$. The vector space of alternating, k-linear maps is denoted by $\mathrm{Alt}^k(V)$.

We immediately note that $\mathrm{Alt}^k(V) = 0$ if $k > \dim V$. Indeed, let $e_1, \cdots, e_n$ be a basis of V, and let $\omega \in \mathrm{Alt}^k(V)$. Using multilinearity,

$$\omega(\xi_1, \ldots, \xi_k) = \omega\Big(\sum \lambda_{i,1} e_i, \ldots, \sum \lambda_{i,k} e_i\Big) = \sum \lambda_J \omega(e_{j_1}, \ldots, e_{j_k})$$

with $\lambda_J = \lambda_{j_1,1}, \cdots, \lambda_{j_k,k}$. Since $k > n$, there must be at least one repetition among the elments $e_{j_1}, \cdots, e_{j_k}$. Hence $\omega(e_{j_1}, \cdots, e_{j_k}) = 0$.
The symmetric group of permutations of the set $\{1, \cdots, k\}$ is denoted by $S(k)$. We remind the reader that any permutation can be written as a composition of transpositions. The transposition that interchanges i and j will be denoted by (i, j). Furthemiore, and this fact will be used below, any permutation can be written as a composition of transpositions of the type $(i, i+1), (i, i+1) \circ (i+1, i+2) \circ (i, i+1) = (i, i+2)$ and so forth. The sign of a permutation:

$$\mathrm{sign} : S(k) \to \{\pm 1\} \tag{1}$$

is a homomorphism, $\mathrm{sign}(\sigma \circ \tau) = \mathrm{sign}(\sigma) \circ \mathrm{sign}(\tau)$, which maps every transposition to -1. Thus the sign of $\sigma \in S(k)$ is -1 precisely if σ decomposes into a product consisting of an odd number of transpositions.

Lemma 2.2 If $\omega \in \mathrm{Alt}^k(V)$ and $\sigma \in S(k)$, then

$$\omega\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}\big) = \mathrm{sign}(\sigma)\omega(\xi_1, \ldots, \xi_k).$$

PROOF. It is sufficient to prove the formula when $\sigma = (i, j)$. Let

$$\omega_{i,j}(\xi, \xi') = \omega(\xi_1, \ldots, \xi, \ldots, \xi', \ldots, \xi_k),$$

with ξ and ξ' eoccurring at positions i and j respectively. The remaining $\xi_p \in V$ are arbitrary but fixed vectors. From the definition it follows that $\omega_{i,j} \in \mathrm{Alt}^2(V)$. Hence $\omega_{i,j}(\xi_i + \xi_j, \xi_i + \xi_j) = 0$. Bilinearity yields that $\omega_{i,j}(\xi_i + \xi_j) + \omega_{i,j}(\xi_j + \xi_i) = 0$ $\square$

Example 2.3 Let $V = \mathbb{R}^k$ and $\xi_i = (\xi_{i1}, \cdots, \xi_{ik})$. The function $\omega(\xi_1, \cdots, \xi_k) = \det((\xi_{ij}))$ is alternating, by the calculational rules for determinants.

We want to define the *exterior product*

$$\wedge : \mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V) \to \mathrm{Alt}^{p+q}(V).$$

When $p = q = 1$, it is given by $(\omega_1 \wedge \omega_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_2(\xi_1)\omega_1(\xi_2)$.

Definition 2.4 A (p, q)-*shuffle* σ is a permutation of the set $\{1, \cdots, p + q\}$ satisfying

$$\sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

The set of all such permutations is denoted by $S(p, q)$. Since a (p, q)-shuffle is uniquely determined by the set $\{\sigma(1), \cdots, \sigma(p)\}$, the cardinality of $S(p, q)$ is $\binom{p+q}{p}$.

Definition 2.5 (Exterior product) For $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, we defined

$$(\omega_1 \wedge \omega_2)(\xi_1, \ldots, \xi_{p+q}) = \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}) \cdot \omega_2(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}).$$

It is obvious that $\omega_1 \wedge \omega_2$ is a $(p + q)$-linear map, but moreover.

Lemma 2.6 If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2\, \mathrm{Alt}^q(V)$ then $\omega_1 \wedge \omega_2 \in \mathrm{Alt}^{p+q}(V)$.

Proof. We first show that $\omega_1 \wedge \omega_2(\xi_1, , \xi_2, \cdots, \xi_{p+q}) = 0$ when $\xi_i = \xi_j$. We let

(i) $S_{12} = \{\sigma \in S(p, q) | \sigma(1) = 1, \sigma(p + 1) = 2\}$

(ii) $S_{21} = \{\sigma \in S(p, q) | \sigma(1) = 2, \sigma(p + 1) = 1\}$

(iii) $S_0 = S(p, q) - (S_{12} - S_{21})$

If $\sigma \in S_0$ then either $\omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}) = 0$ or $\omega_2(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}) = 0$, since $\xi_p\sigma(1) = \xi_{\sigma(2)}$ or $\xi_{\sigma(p+1)} = \xi_{\sigma(p+2)}$. Left composition with the transposition $\tau = (1, 2)$ is a bijecrtion $S_{12} \to S_{21}$. We therefore have

$$\begin{aligned}
&(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \ldots, \xi_{p+q}) \\
&= \sum_{\sigma \in S_{12}} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}) \\
&- \sum_{\sigma \in S_{12}} \mathrm{sign}(\sigma)\omega_1(\xi_{r\sigma(1)}, \ldots, \xi_{\tau\sigma(p)}) \cdot \omega_2(\xi_{\tau\sigma(p+1)}, \ldots \xi_{\tau\sigma(p+q)}).
\end{aligned}$$

Since $\sigma(1) = 1$ and $\sigma(p + 1) = 2$, while $\tau\sigma(1) = 2$ and $\tau\sigma(p + 1) = 1$, we see that $\tau\sigma(i) = \sigma(i)$ where $i \neq 1, p+1$. But $\xi_1 = \xi_2$ so the terms in the two sums cancel. The case $\xi_i = \xi_{i+1}$ is similar. Now $\omega_1 \wedge \omega_2$ is alternating according to Lemma 2.7 below. □

Lemma 2.7 A k-linear map ω is alternating if $\omega(\xi_1, \cdots, \xi_k) = 0$ for all k-tuples with $\xi_i = \xi_{i+1}$ for some $1 \leqslant i \leqslant k-1$.

PROOF. $S(k)$ is generated by the transpositions $(i, i+1)$, and by the argument of Lemma 2.6,

$$\omega(\xi_1, \cdots, \xi_i, \xi_{i+1}, \cdots, \xi_k) = -\omega(\xi_1, \cdots, \xi_{i+1}, \xi_i, \cdots, \xi_k).$$

Hence Lemma 2.6 holds for all $\sigma \in S(k)$, and ω is alternating. $\square$

It is clear from the definition that

$$\begin{aligned}(\omega_1 + \omega_1') \wedge \omega_2 &= \omega_1 \wedge \omega_2 + \omega_1' \wedge w_2 \\ (\lambda\omega_1) \wedge \omega_2 &= \lambda(\omega_1 \wedge \omega_2) = \omega_1 \wedge \lambda\omega_2 \\ \omega_1 \wedge (\omega_2 + \omega_2') &= \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_2'\end{aligned}$$

for $\omega_1, \omega_1' \in \mathrm{Alt}^p(V)$ and $\omega_2, \omega_2' \in \mathrm{Alt}^q(V)$.

Lemma 2.8 If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, then $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$.

PROOF. Let $\tau \in S(p+q)$ be the element with

$$\begin{array}{llll}\tau(1) = p+1, & \tau(2) = p+2, & \cdots, & \tau(q) = p+q. \\ \tau(q+1) = 1, & \tau(q+2) = 2, & \cdots, & \tau(p+q) = p.\end{array}$$

We have $\mathrm{sign}(\tau) = (-1)^{pq}$. Composition with τ defines bijiection

$$S(p,q) \xrightarrow{\cong} S(q,p), \quad \sigma \mapsto \tau \circ \sigma$$

Note that

$$\begin{aligned}\omega_2(\xi_{\sigma\gamma(1)}, \cdots, \xi_{\sigma\gamma(q)}) &= \omega_2(\xi_{\tau\sigma(p+1)}, \cdots, \xi_{\tau\sigma(p+q)}). \\ \omega_1(\xi_{\sigma\gamma(q+1)}, \cdots, \xi_{\sigma\gamma(p+q)}) &= \omega_1(\xi_{\tau\sigma(1)}, \cdots, \xi_{\tau\sigma(p)}).\end{aligned}$$

Hence

$$\begin{aligned}&\omega_2 \wedge \omega_1(\xi_1, \ldots, \xi_{p+q}) \\ &= \sum_{\sigma \in S(q,p)} \mathrm{sign}(\sigma)\omega_2\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(q)}\big)\omega_1\big(\xi_{\sigma(q+1)}, \ldots, \xi_{\sigma(p+q)}\big) \\ &= \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma\tau)\omega_2\big(\xi_{\sigma\tau(1)}, \ldots, \xi_{\sigma\tau(q)}\big)\omega_1\big(\xi_{\sigma\tau(q+1)}, \ldots, \xi_{\sigma\tau(p+q)}\big) \\ &= (-1)^{pq} \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma)\omega_1\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}\big)\omega_2\big(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}\big) \\ &= (-1)^{pq}\omega_1 \wedge \omega_2(\xi_1, \ldots, \xi_{p+q}).\end{aligned}$$

$\square$

Lemma 2.9 If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$ and $\omega_3 \in \mathrm{Alt}^r(V)$, then

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

Proof. Let $S(p,q,r) \subseteq S(p+q+r)$ consist of the permutations σ with

$$\begin{gathered}\sigma(1) <\cdot< \sigma(p)\\ \sigma(p+1) <\cdot< \sigma(p+q)\\ \sigma(p+q+1) <\cdot< \sigma(p+q+r).\end{gathered}$$

We will need the subset $S(\tilde{p},q,r)$ and $S(p,q,\sim r)$ of $S(p,q,r)$ given by

$$\begin{aligned}\sigma \in S(\tilde{p},q,r) &\iff \sigma \text{ is the identity on } \{1,\cdots,p\} \text{ and } \sigma \in S(p,q,r)\\ \sigma \in S(p,q,\tilde{r}) &\iff \sigma \text{ is the identity on } \{p+q+1,\cdots,p+q+r\}\\ &\qquad\qquad\qquad\qquad \text{and } \sigma \in S(p,q,r)\end{aligned}$$

There are bijections

$$\begin{aligned}S(p,q,r) \times S(p,q,r) &\xrightarrow{\cong} S(p,q,r); \quad (\sigma,\tau) \mapsto \sigma \circ \tau\\ S(p,q,r) \times S(p,q,\tilde{r}) &\xrightarrow{\cong} S(p,q,r); \quad (\sigma,\tau) \mapsto \tau \circ \sigma.\end{aligned} \tag{2}$$

With these notations we have

$$\begin{aligned}&[\omega_1 \wedge (\omega_2 \wedge \omega_3)](\xi_1,\ldots,\xi_{p+q+r})\\ &= \sum_{\sigma \in S(p,q+r)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})(\omega_2 \wedge \omega_3)(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q+r)})\\ &= \sum_{\sigma \in S(p,q+r)} \mathrm{sign}(\sigma) \sum_{\sigma \in S(p,q,r)} \mathrm{sign}(\tau)\Big[\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})\\ &\quad \omega_2(\xi_{\sigma\tau(p+1)},\ldots,\xi_{\sigma\tau(p+q)})\omega_3(\xi_{\sigma\tau(p+q+1)},\ldots,\xi_{\sigma\tau(p+q+r)})\Big]\\ &= \sum_{u \in S(p,q,r)} \Big[\mathrm{sign}(u)\omega_1(\xi_{u(1)},\ldots,\xi_{u(p)})\omega_2(\xi_{u(p+1)},\ldots,\xi_{u(p+q)})\\ &\quad \omega_3(\xi_{u(p+q+1)},\ldots,\xi_{u(p+q+r)})\Big]\end{aligned}$$

where the last equality follows from the first equation in (2). Quite analogously one can calculate $[(\omega_1 \wedge \omega_2) \wedge \omega_3](\xi_1,\cdots,\xi_{p+q+r})$, employing the second equation in (2). □

Remark 2.10 In other textbook on alternating functions one can often see the definition

$$\begin{aligned}&\omega_1 \bar{\wedge} \omega_2(\xi_1,\ldots,\xi_{p+q})\\ =&\frac{1}{p!q!} \sum_{\sigma \in S(p+q)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)}).\end{aligned}$$

Note that in this formula $\{\sigma(1), \cdots, \sigma(p)\}$ and $\{\sigma(p+1), \cdots, \sigma(p+q)\}$ are not ordered. There are exactly $S(p) \times S(q)$ ways to come from an ordered set to the arbitrary sequence above; this causes the factor $\frac{1}{p!q!}$, so $\omega_1 \bar{\wedge} \omega_2 = \omega_1 \wedge \omega_2$.

An $\mathbb{R}$-algebra A consists of a vector space over $\mathbb{R}$ and a bilinear map $\mu : A \times A \to A$ which is associative, $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ for every $a, b, c \in A$. The algebra is called *unitary* if there exists a unit element for $\mu, \mu(1, a) = \mu(a, 1) = a$ for all $a \in A$.

Definition 2.11

(i) A graded $\mathbb{R}$-algebra A_* is a sequence of vector spaces $A_k, k = 0, 1, \cdots,$ and bilinear maps $\mu : A_k \times A_l \to A_{k+l}$ which are associative.

(ii) The algebra A_* is called connected if there exists a unit element $1 \in A_0$ and if $\epsilon : \mathbb{R} \to A_0$, given by $\epsilon(r) = r \cdot 1$, is an isomorphism.

(iii) The algebra called (graded) commutative (or anti-commutative), if $\mu(a, b) = (-1)^{kl}\mu(b, a)$ for all $a \in A_k$ and $b \in A_l$.

The elements in A_k are said to have degree k. The set $\mathrm{Alt}^k(V)$ is a vector space over $\mathbb{R}$ in the usual manner:

$$(\omega_1 + \omega_2)(\xi_1, \ldots, \xi_k) = \omega_1(\xi_1, \ldots, \xi_k) + \omega_2(\xi_1, \ldots, \xi_k)$$
$$(\lambda\omega)(\xi_1, \ldots, \xi_k) = \lambda\omega(\xi_1, \ldots, \xi_k), \quad \lambda \in \mathbb{R}.$$

The product from Definition 2.5 is a bilinear map from $\mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V)$ to $\mathrm{Alt}^{p+q}(V)$. We set $\mathrm{Alt}^0(V) = \mathbb{R}$ and expand the product to $\mathrm{Alt}^0(V) \times \mathrm{Alt}^p(V)$ by using the vector space structure. The basic formal properties of the alternating forms can now be summarized in.

Theorem 2.12 $\mathrm{Alt}^*(V)$ is an anti-commutative and connected graded algebra.

$\mathrm{Alt}^*(V)$ is called the exterior or alternating algebra associated to V.

Lemma 2.13 For 1-forms $\omega_1, \cdots, \omega_p \in \mathrm{Alt}^1(V)$, we have

$$(\omega_1 \wedge \cdots \wedge \omega_p)(\xi_1, \ldots, \xi_p) = \det \begin{pmatrix} \omega_1(\xi_1) & \omega_1(\xi_2) & \cdots & \omega_1(\xi_p) \\ \omega_2(\xi_1) & \omega_2(\xi_2) & \cdots & \omega_2(\xi_p) \\ \vdots & \vdots & & \vdots \\ \omega_p(\xi_1) & \omega_p(\xi_2) & \cdots & \omega_p(\xi_p) \end{pmatrix}$$

Proof. The case $p = 2$ is abvious. We procced by induction on p. According to Definition 2.5,

$$\begin{aligned} &\omega_1 \wedge (\omega_2 \wedge \ldots \wedge \omega_p)(\xi_1, \ldots, \xi_p) \\ =&\sum_{j=1}^{p} (-1)^{j+1}\omega_1(\xi_j)(\omega_2 \wedge \ldots \wedge \omega_p)\Big(\xi_1, \quad, \dot{\xi}_j, \ldots, \xi_p\Big) \end{aligned}$$

where $(\xi_1, \cdots, \hat{\xi_j}, \cdots, \xi_p)$ denotes the $p-1$-tuple where ξ_j has been omitted. The lemma follows by expanding the determinant by the first row. □

Note, from Lemma 2.13, that if the 1-forms $\omega_1, \cdot, \omega_p \in \mathrm{Alt}^1(V)$ are linearly independent then $\omega_1 \wedge \cdots \wedge \omega_p \neq 0$. Indeed, we can choose elements $\xi \in V$ with $\omega_i(\xi_i) = 0$ for $i \neq j$ and $\omega_j(\xi_j) = 0$, so that $\det(\omega_i(\xi_j)) = 1$. Conversely, if $\omega_1, \cdots, \omega_p$ are linearly dependent, we can express one of them, say ω_p, as a linear combination of the others. If $\omega_p = \sum_{i=1}^{p-1} r_i \omega_i$, then

$$\omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_p = \sum_{i=1}^{p-1} r_i \omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_i = 0,$$

as the determinant in Lemma 2.13 has two rows. We have proved.

Lemma 2.14 For 1-form $\omega_1, \cdots, \omega_p$ on V, $\omega_1 \wedge \cdots \wedge \omega_p \neq 0$ if and only if they are linearly independent.

Theorem 2.15 Let $e_1, \cdots, e_n$ be a basis of V and $\epsilon_1, \cdots, \epsilon_n$ the dual basis of V^*. Then

$$\left\{\epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(k)}\right\}_{\sigma \in S(p, n-p)}$$

is a basis of $\mathrm{Alt}^p(V^*)$. In particular

$$\dim \mathrm{Alt}^p(V^*) = \binom{\dim V}{p}.$$

Proof. Since $\epsilon_i e_j = 0$ when $i \neq j$, and $\epsilon_i e_j = 1$, Lemma 2.13 gives

$$\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_p}(e_{j_1}, \ldots, e_{j_p}) = \begin{cases} 0 & \text{if } \{i_1, \cdots, i_p\} \neq \{j_1, \cdots, j_p\}, \\ \mathrm{sign}(\sigma) & \text{if } \{i_1, \cdots, i_p\} = \{j_{\sigma(1)}, \cdots, j_{\sigma(p)}\}. \end{cases} \tag{3}$$

Here σ is the permutation $\epsilon(i_k) = j_k$. From Lemma 2.13 and (3) we get

$$\omega = \sum_{\sigma \in S(p, n-p)} \omega\big(e_{\sigma(1)}, \ldots, e_{\sigma(p)}\big) \epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)}$$

for any alternating p-form. Thus $\epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(p)}$ generates the vector space $\mathrm{Alt}^p(V)$. Linear independence follows from (3), since a relation

$$\sum_{\sigma \in S(p, n-p)} \lambda_\sigma \epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)} = 0, \quad \lambda_\sigma \in \mathbb{R}$$

evaluated on $(e_{\sigma(1)}, \cdots, e_{\sigma(p)})$ gives $\lambda_\sigma = 0$. □

Note from Theorem 2.15 that $\mathrm{Alt}^n(V) \xrightarrow{\cong} \mathbb{R}$ if $n = \dim V$ and, as mentioned earlier, that $\mathrm{Alt}^p(V) = 0$ if $p > n$. A basis of $\mathrm{Alt}^n(V)$ is given by $\epsilon_1 \wedge \cdots \wedge \epsilon_n$. In particular every alternating n-fonn on $\mathbb{R}^n$ is proportional to the form in Example 2.3.

A linear map $f : V \to W$ induces the linear map

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^p(W) \to \mathrm{Alt}^p(V) \tag{4}$$

by setting $\mathrm{Alt}^p(f)(W)(\xi_1, \cdots, \xi_p) = \omega(f(\xi_1), \cdots, f(\xi_p))$. For the composition of maps we have $\mathrm{Alt}^p(g \circ f) = \mathrm{Alt}^p(f) \circ \mathrm{Alt}^p(g)$, and $\mathrm{Alt}^P(\mathrm{id}) = \mathrm{id}$. These two properties are summarized by saying that $\mathrm{Alt}^p(-)$ is a *contravariant functor*. If $\dim V = n$ and $f : V \to V$ is a linear map then

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^n(V) \to \mathrm{Alt}^n(V)$$

is a linear endomorphism of 1-dimensional vector space and thus multiplication by a number d. From Theorem 2.16 below it follows that $d = \det(f)$. We shall also be using other maps

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^p(V) \to \mathrm{Alt}^p(V)$$

Let $\mathrm{tr}(g)$ denotes the trace of a linear endomorphism g.

Theorem 2.16 The characteristic polynomial ofa linear endomorphism $f : V \to V$ is given by

$$\det(f - t) = \sum_{i=0}^{n} (-1)^i \mathrm{tr}\Big(\mathrm{Alt}^{n-i}(f)\Big) t^i,$$

when $n = \dim V$.

Proof. Choose a basis $e_1, \cdots, e_n$ of V Assume first that $e_1, \cdots, e_n$ are eigenvectors of f,

$$f(e_i) = \lambda_i e_i, i = 1, \cdots, n.$$

Let $\epsilon_1, \cdots, \epsilon_n$ be the dual basis of $\mathrm{Alt}^1(V)$. Then

$$\mathrm{Alt}^p(f)\big(\epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(p)}\big) = \lambda_{\sigma(1)} \cdot \cdots \cdot \lambda_{\sigma(p)} \epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(p)}$$

and

$$\mathrm{tr}\, \mathrm{Alt}^p(f) = \sum_{\sigma \in S(p, n-p)} \lambda_{\sigma(1)} \cdot \cdots \cdot \lambda_{\sigma(p)}.$$

On the other hand

$$\det(f - t) = \prod_{i=1}^{n} (\lambda_i - t) = \sum (-1)^{n-p} \left(\sum \lambda_{\sigma(1)} \cdot \cdots \cdot \lambda_{\sigma(p)}\right) t^{n-p}.$$

This proves the formula when f is diagonal.
If f is replaced by gfg^{-l}, with g an isomorphism on V, then both sides of the equation of Theorem 2.16 remain unchanged. This is obvious for the left-hand side and follows for the right-hand side since

$$\mathrm{Alt}^p(gfg^{-1}) = \mathrm{Alt}^p(g)^{-1} \circ \mathrm{Alt}^p(f) \circ \mathrm{Alt}^p(g).$$

by the functor property. Hence $\mathrm{tr}\,\mathrm{Alt}^p(g \circ f \circ g^{-l}) = \mathrm{tr}\,\mathrm{Alt}^p(f)$. Consider the set

$$D = \{gf^{-1}g^{-1} | f \text{ diagonal }, g \in GL(V)\}.$$

If V is a vector space over $\mathbb{C}$ and all maps are complex linear, then D is dense in the set of linear endomorphisms on V. We shall not give a formal proof of this, but it follows since every matrix with complex entries can be approximated arbitrarily closely by a matrix for which all roots of the characteristic polynomial are distinct. Since eigenvectors belonging to different eigenvalues are linearly independent, V has a basis consisting of eigenvectors for such a matrix, which then belongs to D. For general $f \in \mathrm{End}(V)$ we can choose a sequence $d_n \in D$ with $d_n \to f$ (i.e. the (i, j)-th element in dn converges to the (i, j)-th element in f). Since both sides in the equation we want to prove are continuous, and since the equation holds for d_n , it follows for f. □

It is not true that the set of diagonalizable matrices over $\mathbb{R}$ is dense in the set of matrices over $\mathbb{R}-a$ matrix with imaginary eigenvalues cannot be approximated by a matrix of the form gfg^{-1}, with f a real diagonal matrix. Therefore in the proof of Theorem 2.16 we. must pass to complex linear maps, even if we are mainly interested in real ones.

3. DE RHAM COHOMOLOG

In this chapter U will denote an open set in $\mathbb{R}^n, e_1, \cdots, e_n$ the standard basis and $\varepsilon_1, \cdots, \varepsilon_n$ the dual basis of $\mathrm{Alt}^1(\mathbb{R}^n)$.

Definition 3.1 A differential p-form on U is a smooth map $w : U \to \mathrm{Alt}^1(\mathbb{R}^n)$. The vector space of all such maps is denoted by $\Omega^p(U)$.

If $p = 0$ then $\mathrm{Alt}^1(\mathbb{R}^n) = \mathbb{R}$ and $\Omega^0(U)$ is just the vector space of all smooth real-valued functions on U, $\Omega^0(U) = \Omega^0(U, \mathbb{R})$.
The usual derivative of a smooth map $\omega : U \to \mathrm{Alt}^p(\mathbb{R}^n)$ is denoted $D\omega$ and its value at x by $D_x\omega$. It is the linear map

$$D_x\omega : \mathbb{R}^n \to \mathrm{Alt}^p(\mathbb{R}^n)$$

with

$$(D_x\omega)(e_i) = \frac{d}{dt}\omega(x + te_i)\Big|_{t=0} = \frac{\partial\omega}{\partial x_i}(x)$$

In $\mathrm{Alt}^p(\mathbb{R}^n)$ we have the basis $\epsilon_1 \wedge \cdots \wedge \epsilon_p$ where I runs over all sequences with $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$. Hence every $\omega \in \Omega^p(U)$ can be written in the form $\omega(x) = \sum \omega_I(x)\epsilon_I$, with $\omega_I(x)$ smooth real-valued functions of $x \in U$. The differential $D_x\omega$ is the linear map

$$D_x\omega(e_j) = \sum_I \frac{\partial\omega_I}{\partial x_j}(x)\epsilon_I, j = 1, \cdots, n. \tag{1}$$

The function $x \mapsto D_x\omega$ is a smooth map from U to the vector space of linear maps from $\mathbb{R}^n$ to $\mathrm{Alt}^p(\mathbb{R}^n)$

Definition 3.2 The exterior differential $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is the linear operator

$$d_x\omega(\xi_1, \cdots, \xi_{p+1}) = \sum_{l=1}^{p+1} (-1)^{l-1} D_x\omega(\xi_l)(\xi_1, \cdots, \hat{\xi}_l, \cdots, \xi_{p+1})$$

with $(\xi_1, \cdots, \hat{\xi}_l, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \xi_{l-1}, \xi_{l+1}, \cdots, \xi_{p+1})$.

It follows from Lemma 2.7 that $d_x\omega \in \mathrm{Alt}^{p+1}\mathbb{R}^n$. Indeed, if $\xi_i = \xi_{i+1}$, then

$$\begin{aligned}&\sum_{l=1}^{p+1} (-1)^{l-1} D_x\omega(\xi_l)(\xi_1, \cdots, \hat{\xi}_l, \cdots, \xi_{p+1})\\ =&(-1)^{i-1}D_x\omega(\xi_i)(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1})\\ &+ (-1)^i D_x\omega(\xi_{i+1})(\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1})\end{aligned}$$

because $(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1})$.

Example 3.3 Let $x_i : U \to \mathbb{R}$ be the i-th projection. Then $dx_i\Omega^1(U)$ is the constant map $dx_i : x \to \epsilon_i$. This follows from (1). In general, for $f \in \Omega^0(U)$, (1) shows that

$$d_x f(\zeta) = \frac{\partial f}{\partial x_i}\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}\zeta^n \tag{2}$$

with $(\zeta^1, \cdots, \zeta^n) = \zeta$. In other words, $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

Lemma 3.4 If $\omega(x) = f(x)\epsilon_I$ then $d_x\omega = d_x f \wedge \epsilon_I$.

Proof. By (1) we have

$$D_x\omega(\zeta) = (D_x f)(\zeta)\epsilon_I = \left(\frac{\partial f}{\partial x_1}\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}\zeta^n\right)\epsilon_I = d_x f(\zeta)\epsilon_I$$

and Definition 3.2 gives

$$\begin{aligned} d_x\omega(\xi_1, \cdots, \xi_{p+1}) &= \sum_{k=1}^{p+1} (-1)^{k-1}\, d_x f(\xi_k)\epsilon_I\left(\xi_1, \cdots, \hat{\xi}_k, \cdots \xi_{p+1}\right) \\ &= [d_x f \wedge \epsilon_I](\xi_1, \cdots, \xi_{p+1}). \end{aligned}$$

□

Note for $\epsilon_I \in \mathrm{Alt}^p(\mathbb{R}^n)$ that

$$\epsilon_k \wedge \epsilon_I = \begin{cases} 0 & \text{if } k \in I \\ (-1)^r \epsilon_J & \text{if } k \notin I \end{cases}$$

with r the number determined by $i_r < k < i_{r+l}$ and $J = (i_1, \cdots, i_r, k, \cdots, i_p)$.

Lemma 3.5 For $p \geqslant 0$ the composition $\Omega^p(U) \to \Omega^{p+1}(U) \to \Omega^{p+2}(U)$ is indentity zero.

Proof. Let $\omega = f\epsilon_I$. Then

$$d\omega = \; df \wedge \epsilon_I = \frac{\partial f}{\partial x_1}\epsilon_1 \wedge \epsilon_I + \cdots + \frac{\partial f}{\partial x_n}\epsilon_n \wedge \epsilon_I$$

Now use $\epsilon_i \wedge \epsilon_i = 0$ and $\epsilon_i \wedge \epsilon_j = -\epsilon_j \wedge \epsilon_i$ to obtain that

$$\begin{aligned} d^2\omega &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\epsilon_i \wedge (\epsilon_j \wedge \epsilon_I) \\ &= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)\epsilon_i \wedge \epsilon_j \wedge \epsilon_I \\ &= 0. \end{aligned}$$

□

The exterior product in $\mathrm{Alt}^* \mathbb{R}^n$, induces an exterior product on $\Omega^*(U)$ upon defining

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x)$$

The exterior product of a differential p-form and a differential q-fonn is a differential $(p+q)$-form, so we get a bilinear map

$$\wedge : \Omega^p(U) \times \Omega^q(U) \to \Omega^{p+q}(U)$$

For a smooth function $f \in C^\infty(U, \mathbb{R})$, we have that

$$(f\omega_1) \wedge \omega_2 = f(\omega_1 \wedge \omega_2) = \omega_1 \wedge (f\omega_2)$$

This just expresses the bilinearity of the product in $\mathrm{Alt}^* \mathbb{R}^n$. Also note that $f \wedge \omega = f\omega$ when $f \in \Omega^0 U$ and $\omega \in \Omega^p(U)$.

Lemma 3.6 For $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^q(U)$,

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2$$

Proof. It is sufficient to show the formula when $\omega_1 = f\epsilon_I$ and $\omega_2 = g\epsilon_J$. But then $\omega_1 \wedge \omega_2 = fg\epsilon_I \wedge \epsilon_J$, and

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= d(fg) \wedge \epsilon_I \wedge \epsilon_J = ((df)g + fdg) \wedge \epsilon_I \wedge \epsilon_J \\ &= dfg \wedge \epsilon_I \wedge \epsilon_J + fdg \wedge \epsilon_I \wedge \epsilon_J \\ &= df \wedge \epsilon_I \wedge g\epsilon_J + (-1)^p f\epsilon_I \wedge dg \wedge \epsilon_J \\ &= d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2. \end{aligned}$$

□

Summing up, we have introduced an anti-commutative algebra $\Omega^*(U)$ with a *differential*,

$$\mathrm{d} : \Omega^*(U) \to \Omega^{*+1}(U), \qquad \mathrm{d} \circ \mathrm{d} = 0$$

and d is a *derivation* (satisfies Lemma 3.6): $(\Omega^*(U), \mathrm{d})$ is a commutative *DGA* (differential graded algebra). It is called the *de Rham complex* of U.

Theorem 3.7 There is precisely one linear operator $\mathrm{d} : \Omega^0(U) \to \Omega^{p+1}(U), p = 0, 1, \cdots$, such that

(i) $f \in \Omega^*(U), \mathrm{d}f = \frac{\partial f}{\partial x_i}\epsilon_1 + \cdots + \frac{\partial f}{\partial x_n}\epsilon_n$

(ii) $\mathrm{d} \circ \mathrm{d} = 0$

(iii) $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \mathrm{d}\omega_1 \wedge \mathrm{d}\omega_1$ if $\omega_1 \in \Omega^p(U)$.

PROOF. We have already defined d with the asserted properties. Conversely assume that d' is a linear operator satisfying (i), (ii) and (iii). We will show that d' is the exterior differential.
The first property tells us that $d = d'$ on $\Omega^0(V)$. In particular $d'x_i = dx_i$ for the i-th projection $x_i : U \to \mathbb{R}$. It follows from Example 3.3 that $d'x_i = \epsilon_i$, the constant function. Since $d' \circ d' = 0$ we have that $d'\epsilon_i$. Then (iii) gives $d'\epsilon_I = 0$. Now let $\omega = f\epsilon_I = f \wedge \epsilon_I$. Again by using (iii),

$$d'\omega = d'f \wedge \epsilon_I + f \wedge d'\epsilon_I = d'f \wedge \epsilon_I = df \wedge \epsilon_I = d\omega.$$

Since every p-form is the sum of such special p-forms, $d = d'$ on all of $\Omega^p(U)$. □

For an open set V in $\mathbb{R}^3$, $d : \Omega^1(U) \to \Omega^2(U)$ is given as

$$\begin{aligned} &d(f_1\epsilon_1 + f_2\epsilon_2 + f_3\epsilon_3) = df_1 \wedge \epsilon_1 + df_2 \wedge \epsilon_2 + df_3 \wedge \epsilon_3 \\ = &\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\epsilon_1 \wedge \epsilon_2 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\epsilon_2 \wedge \epsilon_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\epsilon_3 \wedge \epsilon_1. \end{aligned}$$

The first equality follows from Theorem 3.7.(iii), as $\epsilon_i : U \to \mathrm{Alt}^1(\mathbb{R}^3)$ is the constant map, and hence $d\epsilon_i = 0$, by (1). Alternatively, we have already noted that the 1-forms ϵ_i and dx_i agree, and hence $d\epsilon_i = d \circ d(x_i) = 0$ by Theorem 3.7.(ii). The second equality comes from the anti-commutativity, $\epsilon_i \wedge \epsilon_j = -\epsilon_j \wedge \epsilon_i$. and Theorem 3.7.(i).
Quite analogously we can calculate that

$$d(g_3\epsilon_1 \wedge \epsilon_2 + g_1\epsilon_2 \wedge \epsilon_3 + g_2\epsilon_3 \wedge \epsilon_1) = \left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3}\right)\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3.$$

Definition 3.8 The p-th (de Rham) cohomology group is the quotient vector space

$$H^p(U) = \frac{\mathrm{Ker}\left(d\colon \Omega^p(U) \to \Omega^{p+1}(U)\right)}{\mathrm{Im}(d\colon \Omega^{p-1}(U) \to \Omega^p(U))}.$$

In particular $H^p(U) = 0$ for $p < 0$, and $H^0(U)$ is the kernel of

$$d : C^\infty(U) \to \Omega^1(U).$$

and therefore is the vector space of maps $f \in C^\infty(U, \mathbb{R})$ with vanishing derivatives. This is precisely the space of locally constant maps.
Let $\sim$ be the equivalence relation on the open set V such that $q_1 \sim q_2$ if there exists a continuous curve $\alpha : [a, b] \to V$ with $\alpha(a) = q_1$ and $\alpha(b) = q_2$. The equivalence classes partition V into disjoint open subsets, namely the connected components of U. A connected component of U is a maximal non-empty subset W of U that cannot be written as the disjoint union of two non-empty open subsets of W (in the topology induced by $\mathbb{R}^n$). An open set $U \subseteq \mathbb{R}^n$ has at most countably many connected components (in each of them one can choose a point with rational coordinates.)

Lemma 3.9 $H^0(U)$ is the vector space of maps $U \to \mathbb{R}$ that are constant on each connected component of U.

PROOF. A locally constant function $f : U \to \mathbb{R}$ gives a partition of U into the mutually disjoint open sets $f^{-1}(c), c \in \mathbb{R}$. Consequently $f : U \to \mathbb{R}$ is locally constant precisely when f is constant on each connected component of U. □

It follows that $\dim_{\mathbb{R}} H^0(U)$ (considered as a non-negative integer or ∞) is precisely the number of connected components of U.
The elements in $\Omega^p(U)$ with $d\omega = 0$ are called the closed p-*forms*. The elements of the image $\Omega^{p-1}(U) \subset \Omega^p(U)$ are the *exact* p-*forms*. The p-th cohomology group thus measures whether every closed p-form is exact. This condition is satisfied precisely when $H^p(u) = 0$. A closed p-form $\omega \subset \Omega^p(U)$ gives a cohomology class, denoted by

$$[\omega] = \omega + d\Omega^{p-1}(U) \in H^p(U),$$

and $[\omega] = [\omega']$ if and only if $\omega - \omega'$ is exact. In general the vector space of *closed* p-*form* and the vector space of exact p-forms are infinite-dimensional. In contrast $H^p(U)$ usually has finite dimension.
We can define a bilinear, associative and anti-commutative product

$$H^p(U) \times H^q(U) \to H^{p+q}(U) \tag{3}$$

by setting $[\omega_1][\omega_2] = [\omega_1 \wedge \omega_2]$. This is well-defined because

$$\begin{aligned}(\omega_1 + d\eta_1) \wedge (\omega_2 + d\eta_2) &= \omega_1 \wedge \omega_2 + d\eta_1 \wedge \omega_2 + d\eta_1 \wedge \omega_2 + d\eta_1 \wedge d\eta_2 \\ &= \omega_1 \wedge \omega_2 + d(\eta_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \omega_2 + \eta_1 \wedge d\eta_2)\end{aligned}$$

We want to make $U \to H^p(U)$ into a *contravariant functor*. Thus to a smooth map $\phi : U_1 \to U_2$ between open sets $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$, we shall define a linear map

$$H^p(\phi) : H^p(U_2) \to H^p(U_1)$$

such that

$$\begin{aligned} H^p(\phi_2 \circ \phi_1) &= H^p(\phi_1) \circ H^p(\phi_2) \\ H^p(\mathrm{id}) &= \mathrm{id} \end{aligned} \tag{4}$$

We first make $\Omega^*(-)$ into a contravariant functor.

Definition 3.10 Let $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ be open sets and $\phi : U_1 \to U_2$ a smooth map. The induced morphism $\Omega^p(\phi) : \Omega^p(U_2) \to \Omega^p(U_1)$ is defines by

$$\Omega^p(\phi)(\omega)_x = \mathrm{Alt}^p(D_x\phi) \circ \omega(\phi(x)), \quad \Omega^0(\phi)(\omega)_x = \omega_{\phi(x)}.$$

Frequently one writes ϕ^* instead of $\Omega^p(\phi)$. We note that the analogue of (4) is satisfied. Indeed,

$$\phi^*(\omega)_x(\xi_1, \cdots, \xi_p) = \omega_{\phi(x)}(D_x\phi(\xi_1), \cdots, D_x\phi(\xi_p)),$$

and using the chain rule $D_x(\phi \circ \phi) = D_{\phi(x)}\phi \circ D_x\phi$, for $\phi : U_1 \to U_2$, $\psi : U_2 \to U_3$, it is easy to see that

$$\Omega^p(\psi \circ \phi) = \Omega^p(\phi) \circ \Omega^p(\psi), \qquad \Omega^p(\mathrm{id}_U) = \mathrm{id}_{\Omega^p(U)}.$$

It should be noteed that $\Omega^p(i)(\omega) = \omega \circ i$ when $i : U_1 \hookrightarrow U_2$ is an inclusion, since then $D_x i = \mathrm{id}$.

Example 3.11 For the constant 1-form $\epsilon_i \in \Omega^1(U_2)$, we have that

$$\phi^*(\epsilon_i) = \sum_{k=1}^{n} \frac{\partial\phi_i}{\partial x_k}\epsilon_k = d\phi_i$$

With ϕ_i the i-th coordinate function. To see this, let $\zeta \in \mathbb{R}^n$. Then

$$\begin{aligned}\phi^*(\epsilon_i)(\zeta) = \epsilon_i(D_x\phi(\zeta)) &= \epsilon_i\left(\sum_{k=1}^{m}\Big(\sum_{l=1}^{n}\frac{\partial\phi_k}{\partial x_l}\zeta^l\Big)e_k\right)\\ &= \sum_{l=1}^{n}\frac{\partial\phi_i}{\partial x_l}\zeta^l = \sum_{l=1}^{n}\frac{\partial\phi_i}{\partial x_l}\epsilon_l(\zeta) = d\phi_i(\zeta).\end{aligned}$$

Theorem 3.12 With Definition 3.10 we have the relations

(i) $\phi^*(\omega \wedge \tau) = \phi^*(\omega) \wedge \phi^*(\tau)$

(ii) $\phi^*(f) = f \circ \phi$ if $f \in \Omega^0(U_2)$

(iii) $d\phi^*(\omega) = \phi^*(d\omega)$

Conversely, if $\phi' : \Omega^*(U_2) \to \Omega^*(U_1)$ is a linear map satisfying three conditions, then $\phi' = \phi$.

Proof. Let $x \in U_1$ and let $\xi_1, \cdots, \xi_{p+q}$ be vectors in $\mathbb{R}^n$. Then

$$\begin{aligned}&\phi^*(\omega \wedge \tau)_x(\xi_1, \cdots, \xi_{p+q}) = (\omega \wedge \tau)_{\phi(x)}(D_x\phi(\xi_1), \cdots, D_x\phi(\xi_{p+q}))\\ &= \sum \mathrm{sign}(\sigma)\Big[\omega_{\phi(x)}\big(D_x\phi\big(\xi_{\sigma(1)}\big), \cdots, D_x\phi\big(\xi_{\sigma(p)}\big)\big)\\ &\qquad\qquad \tau_{\phi(x)}(D_x\phi(\xi_{\sigma(p+1)}), \cdots, D_x\phi(\xi_{\sigma(p+q)}))\Big]\\ &= \sum \mathrm{sign}(\sigma)\phi^*(\omega)_x\big(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}\big)\phi^*(\tau)_x\big(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}\big)\\ &= (\phi^*(\omega)_x \wedge \phi^*(\tau)_x)(\xi_1, \cdots, \xi_{p+q}).\end{aligned}$$

This shows (i) when $p > 0$ and $q > O$. If $p = 0$ or $q = 0$ the proof is quite analogous, but easier. Property (ii) is contained in the definition of ϕ^* for degree

0. So we are left with (iii). We shall first show that $d\phi^*(f) = \phi^*(df)$ when $f \in \Omega^0(U_2)$. We have that

$$df = \sum_{i=k}^{n} \frac{\partial f}{\partial x_k} \epsilon_k = \sum_{i=k}^{n} \frac{\partial f}{\partial x_k} \wedge \epsilon_k,$$

when ϵ_k is considered as the element in $\Omega^1(U_2)$ with constant value ϵ_k. From (i) and (ii) we obtain

$$\begin{aligned}
\phi^*(df) &= \sum_{k=1}^{m} \phi^*\left(\frac{\partial f}{\partial x_k}\right) \wedge \phi^*(\epsilon_k) = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \circ \phi\right) \wedge \left(\sum_{l=1}^{n} \frac{\partial \phi_k}{\partial x_l} \epsilon_l\right) \\
&= \sum_{k=1}^{m} \sum_{l=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ \phi\right) \left(\frac{\partial \phi_k}{\partial x_l}\right) \epsilon_l = \sum_{l=1}^{n} \left(\sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \circ \phi\right) \frac{\partial \phi_k}{\partial x_l}\right) \epsilon_l \\
&= \sum_{l=1}^{n} \frac{\partial (f \circ \phi)}{\partial x_l} \epsilon_l = d(f \circ \phi) = d(\phi^*(f)).
\end{aligned}$$

In more general case $\omega = f\epsilon_I = f \wedge \epsilon_I$, Lemma 3.6 gives $d\omega = df \wedge \epsilon_I$, because $d\epsilon_I = 0$. Hence

$$\begin{aligned}
\phi^*(\,d\omega) = \phi^*(\,df) \wedge \phi^*(\epsilon_I) &= d(\phi^*(f)) \wedge \phi^*(\epsilon_I) \\
&= d(\phi^*(f) \wedge \phi^*(\epsilon_I)) = d(\phi^*\omega)
\end{aligned}$$

The second last equality uses Lemma 3.6 and the fact that $d\epsilon_I = 0$:

$$\begin{aligned}
d\phi^*(\epsilon_I) &= d\big(\phi^*(\epsilon_{i_1}) \wedge \ldots \wedge \phi^*(\epsilon_{i_p})\big) \\
&= \sum (-1)^{k-1}\, \phi^*(\epsilon_{i_1}) \wedge \ldots \wedge d\phi^*(\epsilon_{i_k}) \wedge \ldots \wedge \phi^*(\epsilon_{i_p}) \\
&= 0
\end{aligned}$$

since $d\phi^*(\epsilon_{i_k}) = 0$ by Example 3.11 and Lemma 3.5. □

In the following it will be convenient to use the notation of Example 3.3 and write

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

instead of the (constant) p-form $\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_p}$. An arbitrary p-forrn can then be written as

$$\omega(x) = \sum \omega_I(x)\, dx_I$$

and Example 3.11 becomes $\phi^*(dy_i) = d\phi_i$ when $y_i : U_2 \to \mathbb{R}$ is the i-th coordinate function and $\phi_i = y_i \circ \phi$ the i-th coordinate of ϕ; cf. Theorem 3.12.(ii),(iii)

Example 3.13

(i) Let $\gamma : (a, b) \to U$ be an smooth curve in U, $\gamma = (\gamma_1, \cdots, \gamma_n)$, and that

$$\omega = f_1\, dx_1 + \cdots + f_n\, dx_n$$

be a 1-form on U. Then we have that

$$\begin{aligned} \gamma^\star(\omega) &= \gamma^*(f_1) \wedge \gamma^*(\, dx_1) + \cdots + \gamma^*(f_n) \wedge \gamma^*(\, dx_n) \\ &= \gamma^*(f_1)\, d(\gamma^*(x_1)) + \cdots + \gamma^*(f_n)\, d(\gamma^*(x_n)) \\ &= (f_1 \circ \gamma)\, d\gamma_1 + \cdots + (f_n \circ \gamma)\, d\gamma_n \\ &= [(f_1 \circ \gamma)\gamma_1' + \cdots + (f_n \circ \gamma)\gamma_n']\, dt \\ &= \langle f(\gamma(t)), \gamma'(t) \rangle\, dt. \end{aligned}$$

Here $\langle\cdot\rangle$ is the usual inner product. Compare Example 1.8

(ii) Let $\phi : U_1 \to U_2$ be a smooth map between open sets in $\mathbb{R}^n$. Then

$$\phi^*(\, dx_1 \wedge \cdots \wedge\, dx_n) = \det(D_x\phi)\, dx_1 \wedge \cdots \wedge\, dx_n.$$

indeed, from Theorem 3.12,

$$\begin{aligned} \phi^*(dx_1 \wedge \ldots \wedge dx_n) = \phi^*(\, dx_1) \wedge \ldots \wedge \phi^*(\, dx_n) &= \, d\phi^*(x_1) \wedge \ldots \wedge\, d\phi^*(x_n) \\ &= \, d\phi_1 \wedge \ldots \wedge\, d\phi_n = \det(D_x\phi)\, dx_1 \wedge \ldots \wedge\, dx_n. \end{aligned}$$

The last equality is a consequence of Lemma 2.13.

Example 3.14 If $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is given by $\phi(x, t) = \psi(t)x$, where $\phi(t)$ is a smooth real valued function, Then

$$\phi^*(\, dx_i) = x_i\psi'(t)\, dt + \psi(t)\, dx_i.$$

To a smooth map $\phi : U_1 \to U_2$ we can now associate a linear map

$$H^p(\phi) : H^p(U_2) \to H^p(U_1)$$

by setting $H^p(\phi)[\omega] = [\Omega^p(\phi)(\omega)](= \phi^*(\omega))$. The definition is independent of the choice of representative, since $\phi^*(\omega + d\nu) = \phi^*(\omega) + \phi^*(\omega) + d\phi^*(\nu)$. Furthermore,

$$H^{p+q}(\phi)([\omega_1][\omega_2]) = (H^p(\phi)[\omega_1])(H^q(\phi)[\omega_2])$$

such that $H^*(\phi) : H^*(U_2) \to H^*(U_1)$ is a homomorphism of graded algebras.

Theorem 3.15 (Poincaré's Lemma) If U is a star-shaped open set then $H^p(U) = 0$ for $p > 0$, and $H^0(U) = \mathbb{R}$.

PROOF. We may assume U to be star-shaped with respect to the origin $0 \in \mathbb{R}^n$, and wish to construct a linear operator

$$S_p : \Omega^p(U) \to \Omega^{p-1}(U)$$

such that $dS_p + S_{p+1}d = \mathrm{id}$ when $p > 0$ and $S_1 d = \mathrm{id} - e$, where $e(\omega) = \omega(0)$ for $\omega \in \Omega^0(U)$. Such an operator immediately implies our theorem, since $dS_p(\omega) = \omega$ for a closed p-form, $p > 0$, and hence $[\omega] = 0$. If $p = 0$ we have $\omega - \omega(0) = S_1 d\omega = 0$, and ω must be constant.
First we construct

$$\hat{S}_p : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U).$$

Every $\omega \in \Omega^p(U \times \mathbb{R})$ can be written in the form

$$\omega = \sum f_I(x,t)\, dx_I + \sum g_J(x,t)\, dx_J \wedge\, dt$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_{p-1})$. We define

$$\hat{S}_p(\omega) = \sum \left(\int_0^1 g_J(0,t)\, dt \right) dx_J$$

Then we have that

$$\begin{aligned} d\hat{S}_p(\omega) + \hat{S}_{p+1}\, d(\omega) &= \sum_{J,i} \left(\int_0^1 \frac{\partial g_J(x,t)}{\partial x_i}\, dt \right) dx_i \wedge\, dx_J \\ &+ \sum_I \left(\int_0^1 \frac{\partial f_I(x,t)}{\partial t}\, dt \right) dx_I - \sum_{J,i} \left(\int_0^1 \frac{\partial g_J}{\partial x_i}\, dt \right) dx_i \wedge\, dx_J \\ &= \sum \left(\int_0^1 \frac{\partial f_I(x,t)}{\partial t}\, dt \right) dx_I \\ &= \sum f_I(x,1)\, dx_I - \sum f_I(x,0)\, dx_I. \end{aligned}$$

We apply this result to $\phi^*(\omega)$, where

$$\phi : U \times \mathbb{R} \to U, \quad \phi(x,t) = \psi(t)x.$$

and $\psi(t)$ is a smooth function for which

$$\begin{cases} \psi(t) = 0, & \text{if } t \leqslant 0 \\ \psi(t) = 1, & \text{if } t \geqslant 1 \\ 0 \leqslant \psi(t) \leqslant 1, & \text{otherwise} \end{cases} \tag{5}$$

Define $S_p(\omega) = \hat{S}_p(\phi^*(\omega))$ with $\hat{S}_p : \Omega(U \times \mathbb{R}) \to \Omega^{p-1}(U)$ as above. Assume that $\omega = \sum h_I(x) dx_I$. Form Example 3.14 we have

$$\phi^*(\omega) = \sum h_I(\psi(t)x)(d\psi(t)x_{i_1} + \psi(t)dx_{i_1}) \wedge \cdots \wedge \left(d\psi(t)x_{i_p} + \psi(t)dx_{i_p}\right)$$

In the notation used above we then get that

$$\sum f_I(x,t)\, dx_I = \sum h_I(\psi(t)x)\psi(t)^p\, dx_I$$

This implies that

$$dS_p(\omega) + S_{p+1}\, d\omega = \begin{cases} \sum h_I(x)\, dx_I = \omega & p > 0 \\ \omega(x) - \omega(0) & p = 0 \end{cases}$$

$\square$

4. CHAIN COMPLEXES AND THEIR COHOMOLOGY

In this chapter we present some general algebraic definitions and viewpoints, which should illuminate some of the constructions of Chapter 3. The algebraic results will be applied later to de Rham cohomology in Chapters 5 and 6.
A sequence of vector spaces and linear maps

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1}$$

is said to be *exact* when $\operatorname{Im} f = \operatorname{Ker} g$, where as above

$$\begin{aligned} \operatorname{Ker} g &= \{b \in B | g(b) = 0\} \qquad \text{the kernel of } g \\ \operatorname{Im} f &= \{f(a) | a \in A\} \qquad \text{the image of } f \end{aligned}$$

Note that $A \xrightarrow{f} B \to 0$ is exact precisely when f is surjective and that $0 \to B \xrightarrow{g} C$ is exact precisely when g is injective. A sequence $A^* = \{A^i, d^i\}$,

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots \tag{2}$$

of vector spaces and linear maps is called a *chain complex* provided $d^{i+1} \circ d^i = 0$ for all i. It is exact if

$$\operatorname{Ker}\ d^i = \operatorname{Im}\ d^{i-1}$$

for all i. An exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{3}$$

is called *short exact*. This is equivalent to requiring that

$$f \text{ is injective, } g \text{ is surjective and } \operatorname{Im} f = \operatorname{Ker} g$$

The *cokernel* of a linear map $f : A \to B$ is

$$\operatorname{Cok}(f) = B/\operatorname{Im}(f).$$

For a short exact sequence, g induces an isomorphism

$$g : \operatorname{Cok}(f) \xrightarrow{\cong} C.$$

Every (long) exact sequence, as in (2), induces short *exact sequences* (which can be used to calculate A^i)

$$0 \to \operatorname{Im}\ d^{i-1} \to \operatorname{Im}\ d^i \to 0$$

Furthennore the isomorphisms

$$A^{i-1}/\operatorname{Im} d^{i-1} \cong A^{i-1}/\operatorname{Ker} d^{i-1} \xrightarrow{\cong} d^{i-1} \operatorname{Im} d^{i-1}$$

are frequently applied in concrete calculations.
The *direct sum* of vector spaces A and B is the vector space

$$\begin{aligned} &A \oplus B = \{(a,b) | a \in A, b \in B\} \\ &\lambda(a,b) = (\lambda a, \lambda b), \qquad \lambda \in \mathbb{R} \\ &(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \end{aligned}$$

If $\{a_i\}$ and $\{b_j\}$ are bases of A and B, respectively, then $\{(a_i, 0), (0, b_j)\}$ is a basis of $A \oplus B$. In particular

$$\dim(A \oplus B) = \dim A + \dim B$$

Lemma 4.1 Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence ofvector spaces. Then B is finite-dimensional if both A and C are, and $B \cong A \oplus C$.

Proof. Choose a basis $\{a_i\}$ of A and $\{c_j\}$ of C. Since g is surjective there exist $b_j \in B$ with $g(b_j) = c_j$. Then $\{f(a_i), b_j\}$ is a basis of B: For $b \in B$ we have $g(b) = \sum \lambda_i c_j$. Hence $b - \sum \lambda_i b_i \in \operatorname{Ker} g$. Since $\operatorname{Ker} g = \operatorname{Im} f$, $b - \sum \lambda_i b_i = f(a)$, so

$$b - \sum \lambda_j b_j = f\left(\sum \mu_i a_i\right) = \sum \mu_i f(a_i).$$

This shows that b can be written as a linear combination of $\{b_j\}$ and $\{f(a_i)\}$. It is left to the reader to show that $\{b_j, f(a_i)\}$ are linearly independent. $\square$

Definition 4.2 For a chain complex $A^* = \{\cdots \to A^{p-1} \xrightarrow{d^{p-1}} A^p \xrightarrow{d^p} A^{p+1} \to \cdots\}$, we define the p-th cohomology vector space to be

$$H^p(A^*) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

The elements of Ker d^p are called p-cycles (or are said to be closed) and the elements of Im d^{p-1} are called p-boundaries (or said to be exact). The elements of $H^p(A^*)$ are called *cohomology classes*.
A chain map $f : A^* \to B^*$ between chain complexes consists of a family $f^p : A^p \to B^p$ of linear maps, satisfying $d_B^p \circ f^p = f^{p+1} \circ d_A^p$. A chain map is illustrated as the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{p-1} & \xrightarrow{d^{p-1}} & A^p & \xrightarrow{d^p} & A^{p+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{p-1}} & & \downarrow{\scriptstyle f^p} & & \downarrow{\scriptstyle f^{p+1}} & & \\ \cdots & \longrightarrow & B^{p-1} & \xrightarrow{d^{p-1}} & B^p & \xrightarrow{d^p} & B^{p+1} & \longrightarrow & \cdots \end{array}$$

Lemma 4.3 A chain map $f : A^* \to B^*$ induces a linear map

$$f^* = H^*(f) : H^p(A^*) \to H^p(B^*), \text{ for all } p$$

PROOF. Let $\alpha \in A^p$ be a cycle ($d^p a = 0$) and $[a] = a + \mathrm{Im}\, d^{p-l}$ its corresponding cohomology class in $H^p(A^*)$. We define $f^*([a]) = [f^p(a)]$. Two remarks are needed. First, we have $d_B^p f^p(a) = f^{p+1} d_A^p(a) = f^{p+1}(0) = 0$. Hence $f^p(a)$ is a cycle. Second, $[f^p(a)]$ is independent of which cycle a we choose in the class $[a]$. If $[a_1] = [a_2]$ then $a_1 - a_2 \in \mathrm{Im}\, d_A^{p-1}$, and $f^p(a_1 - a_2) = f^p d_A^{p-1}(x) = d_B^{p-1} f^{p-1}(x)$. Hence $f^p(a_1) - f^p(a_2) \in \mathrm{Im}\, d_B^{p-1}$, and $f^p(a_1), f^p(a_2)$ define the same cohomology class. $\square$

A *category* C consists of "objects" and "morphisms" between them, such that "composition" is defined. If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morphisms, then there exists a morphism $g \circ f : C_1 \to C3$. Furthermore it is to be assumed that $\mathrm{id}_C : C \to C$ is a morphism for every object C of C. The concept is best illustrated by examples:

- The category of open sets in Euclidean spaces, where the morphisms are the smooth maps.
- The category of vector spaces, where the morphisms are the linear maps.
- The category of abelian groups, where the morphisms are homomor phisms.
- The category of chain complexes, where the morphisms are the chain maps.
- A category with just one object is the same as a semigroup, namely the semigroup of morphisms of the object.
- Every partially ordered set is a category with one morphism from c to d, when $c \leqslant d$.

A contravariant functor $F : \mathcal{C} \to \mathcal{V}$ between two categories maps every object $C \in \mathrm{ob}\,\mathcal{C}$ to an object $F(C) \in \mathrm{ob}\,\mathcal{V}$, and every morphism $f : C_1 \to C_2$ in $\mathcal{C}$ to a morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathcal{V}$, such that

$$F(g \circ f) = F(f) \circ F(g), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}.$$

A *covariant functor* $F : \mathcal{C} \to \mathcal{V}$ is an assignment in which $F(f) : F(C_1) \to F(C_2)$, and

$$F(g \circ f) = F(g) \circ F(f), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}.$$

Functors thus are the "structure-preserving" assignments between categories. The contravariant ones change the direction of the arrows, the covariant ones preserve directions. We give a few examples:

- Let A be a vector space and $F(C) = \mathrm{Hom}(C, A)$, the linear maps from C to A. For $\phi : C_1 \to C_2$, $\mathrm{Hom}(\phi, A) : \mathrm{Hom}(C_2, A) \to \mathrm{Hom}(C_2, A)$ is given by $\mathrm{Hom}(\phi, A)(\psi) = \psi \circ \phi$. This is a contravariant functor from the category of vector spaces to itself.

- $F(C) = \mathrm{Hom}(C, A), F(\phi) : \psi \to \phi \circ \psi$. This is a covariant functor from the category of vector spaces to itself.

- Let $\mathcal{U}$ be the category of open sets in Euclidean spaces and smooth maps, and *Vect* the category of vector spaces. The vector space of differential p-forms on $U \in \mathcal{U}$ defines a contravariant functor

$$\Omega^p(U) : \mathcal{U} \to \mathrm{Vect}.$$

- Let Vect^* be the category of chain complexes. The de Rham complex defines a contravariant functor $\Omega^* : \mathcal{U} \to \mathrm{Vect}^*$.

- For every p the homology $H^p : \mathrm{Vect}^* \to \mathrm{Vect}$ is a covariant functor.

- The composition of the two functors above is exactly the de Rham cohomology functor $H^p : \mathcal{U} \to \mathrm{Vect}$. It is contravariant.

A short exact sequence of chain complexes

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$$

consists of chain maps f and g such that $0 \to A^p \xrightarrow{f} B^p \xrightarrow{g} C^p \to 0$ is exact for every p.

Lemma 4.4 For a short exact sequence of chain complexes the sequence

$$H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*)$$

is exact.

Proof. Since $g^p \circ f^p = 0$, we have

$$g^* \circ f^*([a]) = g^*([f^p(a)]) = [g^p(f^p(a))] = 0$$

for every cohomology class $[a] \in H^p(A^*)$. Conversely, assume for $[b] \in H^p(B)$ that $g^*[b] = 0$. Then $g^p(b) = d_C^{p-1}(c)$. Since g^{p-1} is surjective, there exists $b_1 \in B^{p-1}$ with $g^{p-1}(b_1) = c$. It follows that $g^p(b - d_B^{p-1}(b_1)) = 0$. Hence there exists $a \in A^p$ with $f^p(a) = b - d_B^{p-1}(b_1)$. We will show that a ia a p-cycle. Since f^{p+1} is injective, it is sufficient to note that $f^{p+1}(d_A^p(a)) = 0$. But

$$f^{p+1}\big(d_A^p(a)\big) = d_B^p\big(f^p(a)\big) = d_B^p\big(b - d_B^{p-1}(b_1)\big) = 0$$

since b is a p-cycle and $d^p \circ d^{p-1} = 0$. We have thus found a cohomology class $[a] \in H^p(A)$, and $f^*([a]) = [b - d_B^{p-1}(b_1)]$. $\square$

One might expect that the sequence of Lemma 4.4 could be extended to a short exact sequence, but this is not so. The problem is that, even though $g^p : B^p \to C^p$ is surjective, the pre-image $(g^P)^{-1}(c)$ of a p-cycle with $c \in C^p$ need not contain a cycle. We shall measure when this is the case by introducing.

Definition 4.5 For a short exact sequence of chain complexes $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$, we define

$$\partial^* : H^p(C^*) \to H^{p+1}(A^*)$$

to be the linear map given by

$$\partial^*([c]) = \left[\left(f^{p+1}\right)^{-1}\left(d_B^p\left((g^p)^{-1}(c)\right)\right)\right]$$

There are several things to be noted. The definition expresses that for every $b \in (g^p)^{-1}(c)$ we have $d_B^p(b) \in \mathrm{Im}(f^{p+1})$, and that the uniquely determined $a \in A^{p+1}$ with $f^{p+1}(a) = d_B^p(b)$ is a $p+1$-cycle. Finally it is postulated that $[a] \in H^{p+1}(A^*)$ is independent of the choice of $b \in (g^b)^{-1}(c)$.
In order to prove these assertions it is convenient to write the given short exact sequence in a diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & A^{p-1} \arrow[r, "f^{p-1}"] \arrow[d, "d_A^{p-1}"] & B^{p-1} \arrow[r, "g^{p-1}"] \arrow[d, "d_B^{p-1}"] & C^{p-1} \arrow[r] \arrow[d, "d_C^{p-1}"] & 0 \\
0 \arrow[r] & A^p \arrow[r, "f^p"] \arrow[d, "d_A^p"] & B^p \arrow[r, "g^p"] \arrow[d, "d_B^p"] & C^p \arrow[r] \arrow[d, "d_C^p"] \arrow[dll] & 0 \\
0 \arrow[r] & A^{p+1} \arrow[r, "f^{p+1}"] \arrow[d] & B^{p+1} \arrow[r, "g^{p+1}"] \arrow[d] & C^{p+1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

The slanted arrow indicates the definition of ∂^*. We shall now prove the necessary assertions which, when combined, make ∂^* well-defined. Namely:

(i) If $g^p(b) = c$ and $d_C^p(c) = 0$, then $d_B^p(b) \in f^{p+1}$.

(ii) If $f^{p+1}(a) = d_B^p(b)$, then $d_A^{p+1}(a) = 0$.

(iii) If $g^p(b_1)g^p(b) = c$ and $f^{p+1}(a_i) = d_B^p(b_i)$, then $[a_1] = [a_2] \in H^{p+1}(A^*)$.

The first assertion follows, because $g^{p+1}d_B^p(b) = d_C^p(c) = 0$, and $\mathrm{Ker}\, g^{p+1} = \mathrm{Im}\, f^{p+1}$; (ii) uses the injectivity of f^{p+2} and that $f^{p+2}d_A^{p+1}(a) = d_B^{p+1}f^{p+1}(a) = d_B^{p+1}d_B^p(b) = 0$; (iii) follows since $b_1 - b_2 = f^p(a)$ so that $d_B^p(b_1) - d_B^p(b_2) = d_B^p f^p(a) = f^{p+1}d_A^p(a)$, and therefore $(f^{p+1})^{-1}(d_B^p(b_1)) = (f^{p+1})^{-1}(d_B^p(b_2)) + d_A^p(a)$.

Example 4.6 Here is a short exact sequence of chain complexes (the dots indicate that the chain groups are zero) with $\partial^* \neq 0$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathbb{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

One can easily verify that $\partial^* : \mathbb{R} \to \mathbb{R}$ is an isomorphism.

Lemma 4.7 The sequence $H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*)$ is exact.

Proof. We have $\partial^* g^*([b]) = \partial^* g^p([b]) = [(f^{p+1})^{-1}(d_B(b))] = 0$. Conversely assume that $\partial^*([c]) = 0$. Choose $b \in B^p$ with $g^p(b) = c$ and $a \in A^p$, such that

$$d_B^p(b) = f^{p+1}(\, d_A^p a).$$

Now we have $d_B^p(b - f^p(a)) = 0$ and $g^p(b - f^p(a)) = c$. Hence $g^*[b - f^p(a)] = [c]$. $\square$

Lemma 4.8 The sequence $H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*)$ is exact.

Proof. We have $f^*\partial^*([c]) = [d_B^p(b)] = 0$, where $g^p(b) = c$. Conversely assume that $f^*([a]) = 0$. i.e., $f^{p+1}(a) = d_B^p(b)$. Then $d_C^p(g^p(b)) = g^{p+1}f^{p+1}(a) = 0$, and $\partial^*[g^p(b)] = [a]$. $\square$

We can sum up Lemmas 4.4, 4.7 and 4.8 in the important.

Theorem 4.9 (Long exact homology sequence) Let $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$ be a short exact sequence of chain complexes. Then the sequence

$$\cdots \to H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*) \to \cdots$$

is exact.

Definition 4.10 Two chain maps $f, g : A^* \to B^*$ are said to be *chain homotopic* if there exists a linear map $s : A^p \to B^{p-1}$ satisfying

$$d_B s + s\, d_A = f - g : A^p \to B^p$$

for every p.

In the form of a diagram, a chain homotopy is given by the slanted arrows.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p-1} \arrow[r] \arrow[d, "f-g"] & A^{p} \arrow[r] \arrow[d, "f-g"] \arrow[dl] & A^{p+1} \arrow[r] \arrow[d, "f-g"] \arrow[dl] & A^{p+2} \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & B^{p-1} \arrow[r] & B^{p} \arrow[r] & B^{p+1} \arrow[r] & B^{p+2} \arrow[r] & \cdots
\end{tikzcd}$$

The name *chain homotopy* will be explained in Chapter 6.

Lemma 4.11 For two chain-homotopic chain maps $f, g : A^* \to B^*$ we have that

$$f^* = g^* : H^p(A^*) \to H^p(B^*).$$

Proof. If $[a] \in H^p(A^*)$, then

$$(f^* - g^*)[a] = [f^p(a) - g^p(a)] = [(\, d_B^{p-1} s(a) + s\, d_A^p(a)] = [\, d_B^{p-1} s(a)] = 0.$$

□

Remark 4.12 In the proof of the Poincare lemma in Chapter 3 we constructed linear maps

$$S^p : \Omega^p(U) \to \Omega^{p-1}(U)$$

with $d^{p-1}S^p + S^{p+1}d^p = \mathrm{id}$ for $p > 0$. Hence id $= 0$ on $H^p(U)$, and $H^p(U) = 0$ when $p > 0$.

Lemma 4.13 If A^* and B^* are chain complexes then

$$H^p(A^* \otimes B^*) = H^p(A^*) \otimes H^p(B^*).$$

Proof. It is obvious that

$$\begin{aligned} \mathrm{Ker}(\, d^p_{A\otimes B}) &= \mathrm{Ker}\ d^p_A \otimes \mathrm{Ker}\ d^p_B \\ \mathrm{Im}(\, d^{p-1}_{A\otimes B}) &= \mathrm{Im}\ d^{p-1}_A \otimes \mathrm{Im}\ d^{p-1}_B. \end{aligned}$$

and the lemma follows. □

This page intentionally left blank.

5. THE MAYER-VIETORIS SEQUENCE

This chapter introduces a fundamental calculational technique for de Rham cohomology, namely the so-called Mayer-Vietoris sequence, which calculates $H^*(U_1 \cup U_2)$ as a "function" of $H^*(U_1), H^*(U_2)$ and $H^*(U_1 \cap U_2)$. Here UI and U_2 are open sets in $\mathbb{R}^n$. By iteration we get a calculation of $H^*(U_1 \cap \cdots \cap U_n)$ as a "function" of $H^*(u_\alpha)$, where 0: runs over the subsets of $1, \cdots, n$ and $U_{i_1} \cap \cdots \cap U_{i_r}$ when $\alpha = i_1, \cdots, i_r$. Combined with the Poincaré lemma, this yields a *principal* calculation of $H^*(U)$ for quite general open sets in $\mathbb{R}^n$. If, for instance, U can be covered by a finite number of convex open sets U_i, then every U_α will also be convex and $H^*(U_\alpha)$ thus known from the Poincaré lemma

Theorem 5.1 Let U_1 and U_2 be open sets in $\mathbb{R}^n$ with union $U = U_1 \cup U_2$. For $\nu = 1, 2$, let $i_\nu : U_\nu \to U$ and $j_\nu : U_1 \cap U_2 \to U_\nu$ be the corresponding inclusions. Then the sequence

$$0 \to \Omega^p(U) \xrightarrow{I^p} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{J^p} \Omega^p(U_1 \cap U_2) \to\to 0$$

is exact, where $I^p(\omega) = (i_1^*(\omega), i_2^*(\omega)), J^p(\omega_1, \omega_2) = j_1^*(\omega_1) - j_2^*(\omega_2)$.

PROOF. For a smooth map $\phi : V \to W$ and a p-form $\omega = \sum f_I \, dx_I \in \Omega^p(W)$,

$$\Omega^p(\phi)(\omega) = \phi^*(\omega) = \sum (f_I \circ \phi) \, d\phi_{i_1} \wedge \ldots \wedge d\phi_{i_p}.$$

In particular, if ϕ is an inclusion of open sets in $\mathbb{R}^n$, i.e., $\phi_i(x) = x_i$, then

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_p} = dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Hence

$$\phi^*(\omega) = \sum f_I \circ \phi \, dx_I. \tag{1}$$

This will be used for $\phi = i_\nu, j_\nu, \nu = 1, 2$. It follows from (1) that I^p is injective. If namely $I^p(\omega) = 0$ then $i_1^*(\omega) = 0 = i_2^*(\omega)$, and

$$i^*(\nu) = \sum (f_I \circ i_\nu) \, dx_I = 0$$

If and only if $f_I \circ i_\nu = 0$ for all I. However $f_I \circ i_1 = 0$ and $f_I \circ i_2 = 0$ imply that $f_I = 0$ on all of U, since U_1 and U_2 cover U. Similarly, we show that $\operatorname{Ker} J^p = \operatorname{Im} I^p$. First

$$J^p \circ I^p(\omega) = j_2^* i_2^*(\omega) - j_1^* i_1^*(\omega) = j^*(\omega) - j_1^*(\omega) = 0$$

where $j : U_1 \cap U_2 \to U$ is the inclusion. Hence $\operatorname{Im} I^p \subseteq \operatorname{Ker} J^p$. To show the converse inclusion we start with two p-form $\omega_\nu \in \Omega^p(U_\nu)$.

$$\omega_1 = \sum f_I \, dx_I, \quad \omega_2 = \sum g_I \, dx_I.$$

Since $J^p(\sum h_I dx_I) = (\omega_1, \omega_2)$, we have that $j_1^*(\omega_1) = j_2^*(\omega_2)$, which by (1) translates into $f_I \circ j_1 = g_I \circ j_2$ or $f_I(x) = g_I(x)$ for $x \in U_1 \cap U_2$. We define a smooth function $h_I : U \to \mathbb{R}^n$ by

$$h_I(x) = \begin{cases} f_I(x), & x \in U_1, \\ g_I(x), & x \in U_2. \end{cases}$$

Then $I^p(\sum h_I dx_I) = (\omega_1, \omega_2)$. Finally we show that J^p is surjective. To this end we use a partition of unity $\{p-1, p_2\}$ with support in $\{U_1, U_2\}$. i.e., smooth functions $h_I : U \to \mathbb{R}^n$ by

$$p_\nu : U \to \{0, 1\}, \quad \nu = 1, 2$$

for which $\text{supp}_U(p_\nu) \subset U_\nu$, and such that $p_1(x) + p_2(x) = 1$ for $x \in U$ (cf. Appendix A).
Let $f : U_1 \cap U_2 \to \mathbb{R}$ be a smooth function. We use $\{p_1, p_2\}$ to extend f to U_1 and U_2. Since $\text{supp}_U(p_1) \cap U_2 \subset U_1 \cap U_2$, we can define a smooth function by

$$f_2(x) = \begin{cases} -f(x)p_1(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_2 - \text{supp}_U(p_1) \end{cases}$$

Analogously we define

$$f_1(x) = \begin{cases} f(x)p_2(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_1 - \text{supp}_U(p_2) \end{cases}$$

Note that $f_1(x) - f_2(x) = f(x)$ when $x \in U_1 \cap U_2$, because $p_1(x) + p_2(x) = 1$. For a differential form $\omega \in \Omega^p(U_1 \cap U_2)$, $\omega = \sum f_I dx_I$, we can apply the above to each of the functions $f_I : U_1 \cap U_2 \to \mathbb{R}$. This yields the functions $f_{I\nu} : U_\nu \to \mathbb{R}$, and thus the differential form $\omega_\nu = \sum f_{I,\nu} dx_I \in \Omega^p(U_\nu)$. With this choice $J^p(\omega_1, \omega_2) = \omega$. □

It is clear that

$$\begin{aligned} I&:\Omega^*(U) \to \Omega^*(U_1) \oplus \Omega^*(U_2) \\ J&:\Omega^*(U_1) \oplus \Omega^*(U_2) \to \Omega^*(U_1 \cap U_2) \end{aligned}$$

are chain maps, so that Theorem 5.1 yields a short exact sequence of chain complexes. From Theorem 4.9 one thus obtains a long exact sequence of cohomology vector spaces. Finally Lemma 4.13 tells us that

$$H^p(U)(\Omega^*(U_1) \oplus \Omega^*(U_2)) = H^p(U_1) \oplus H^p(U_2)$$

We have proved:

Theorem 5.2 (Mayer-Vietoris) Let U_1 and U_2 be open sets in $\mathbb{R}^n$ and $U = U_1 \cup U_2$. There exists an exact sequence of cohomology vector spaces

$$\cdots \to H^p(U) \xrightarrow{I^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}(U) \to \cdots$$

Here $I^*(\omega) = (i_1^*([\omega]), i_2^*([\omega]))$ and $J^*([\omega_1], [\omega_2]) = [j_1^*(\omega_1) - j_2^*(\omega_2)]$ in the notation of Theorem 5.1.

Corollary 5.3 If U_1 and U_2 are disjoint open sets in $\mathbb{R}^n$ then

$$I^* : H^p(U_1 \cup U_2) \to H^p(U_1) \oplus H^p(U_2)$$

is an isomorphism.

Proof. It follows from the Theorem 5.1 that

$$I^p : \Omega^p(U_1 \cup U_2) \to \Omega^p(U_1) \oplus \Omega^p(U_2)$$

is an isomorphism, and Lemma 4.13 gives that corresponding map on cohomology is also an isomorphism. □

Example 5.4 We use Theorem 5.2 to calculate the de Rham cohomology vector spaces of the punctured plane $\mathbb{R}^n - \{0\}$. Let

$$U_1 = \mathbb{R}^2 - \{(x_1, x_2) \mid x_1 \geqslant 0, x_2 = 0\}$$
$$U_2 = \mathbb{R}^2 - \{(x_1, x_2) \mid x_1 \leqslant 0, x_2 = 0\}.$$

These are star-shaped open sets, such that $H^p(U_1) = H^p(U_2) = 0$ for $p > 0$ and $H^0(U_1) = H^0(U_2) = \mathbb{R}$. Their intersection

$$U_1 \cap U_2 = \mathbb{R}^2 - \mathbb{R} = \mathbb{R}^2_+ \cup \mathbb{R}^2_-$$

is disjoint union of the open half-planes $x_2 > 0$ and $x_2 < 0$. Hence

$$H^p(U_1 \cap U_2) = \big\{0 \quad \text{if } p > 0. \quad \mathbb{R} \oplus \mathbb{R} \text{ if } \quad p = 0 \tag{2}$$

by the Poincaré lemma and Corollary 5.3. From the *Mayer-Vietoris sequence* we have

$$\cdots \to H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cap U_2) \xrightarrow{\partial^*}$$
$$H^{p+1}(\mathbb{R}^2 - \{0\}) \xrightarrow{I^*} H^{p+1}(U_1) \oplus H^{p+1}(U_2) \to \cdots$$

For $p > 0$,

$$0 \to H^p(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}(\mathbb{R}^2 - \{0\}) \to 0$$

is exact, i.e., ∂^* is an isomorphism and $H * q(\mathbb{R}^2 - \{0\}) = 0$ for $q > 0$ according to (2).

If $p = 0$, one gets the exact sequence

$$\begin{gathered} H^{-1}(U_1 \cap U_2) \to H^0(\mathbb{R}^2 - \{0\}) \xrightarrow{I^0} H^0(U_1) \oplus H^0(U_2) \xrightarrow{J^0} \\ H^0(U_1 \cap U_2) \xrightarrow{\partial^*} H^1(\mathbb{R}^2 - \{0\}) \xrightarrow{I^1} H^1(U_1) \oplus H^1(U_2) \end{gathered} \tag{3}$$

Since $H^{-1}(U) = 0$ for all open sets, and in particular $H^{-1}(U_\nu)$, I^0 is injective. Since $H^{-1}(U_\nu) = 0$, ∂^* is surjective, and the sequence (3) reduces to the exact sequence

$$0 \to H^0(\mathbb{R}^2 - \{0\}) \xrightarrow{I^0} \underbrace{H^0(U_1) \oplus H^0(U_2)}_{\mathbb{R} \otimes \mathbb{R}} \xrightarrow{J^0} \underbrace{H^0(U_1 \cap U_2)}_{\mathbb{R} \otimes \mathbb{R}} \xrightarrow{\partial^*} H^1(\mathbb{R}^2 - \{0\}) \to 0$$

However, $\mathbb{R}^2 - \{0\}$ is connected. Hence $H^0(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}$, and since I^0 is injective we must have that $\operatorname{Im} J^0 \cong \mathbb{R}$. Exactness gives $\operatorname{Ker} J^0 \cong \mathbb{R}$, so that J^0 has rank 1. Therefore $\operatorname{Im} J^0 \cong \mathbb{R}$ and, once again, by exactness

$$\partial^* : H^0(U_1 \cap U_2)/\operatorname{Im} J^0 \to H^1(\mathbb{R}^2 - \{0\})$$

Since $H^0(U_1 \cap U_2)/\operatorname{Im} J^0 \cong \mathbb{R}$, we have shown

$$H^p(\mathbb{R} - \{0\}) = \begin{cases} 0 & \text{if } p > 2, \\ \mathbb{R} & \text{if } p = 1 \\ \mathbb{R} & \text{if } p = 0 \end{cases}$$

In the proof above we could alternatively have calculated

$$J^0 : H^0(U_1) \oplus H^0(U_2) \to H^0(U_1 \cap U_2)$$

by using Lemma 3.9: $H^0(U)$ consists of locally constant functions. If f_i is a constant function on U_i, then

$$J^0(f_1) = f_{1|U_1 \cap U_2} \text{ and } J^0(f_2) = f_{2|U_1 \cap U_2}$$

so that $J^0(a, b) = a - b$.

Theorem 5.5 Assume that the open set U is covered by convex open sets $U_1, \cdots, U_r$. Then $H^p(U)$ is finitely generated.

PROOF. We use induction on the number of open sets. If $r = 1$ the assertion follows from the Poincaré lemma. Assume the assertion is proved for $r - 1$ and let $V = U_1 \cup \cdots \cup U_{r-1}$, such that $U = V \cup U_r$. From Theorem 5.2 we have the exact sequence

$$H^{p-1}(V \cup U_r) \xrightarrow{\partial^*} H^p(U) \xrightarrow{I^*} H^p(V) \oplus H^p(U_r)$$

which by Lemma 4.1 yields

$$H^p(U) \backsimeq \operatorname{Im} \partial^* \oplus \operatorname{Ker} I^*.$$

Now both V and $V \cap U_r = (U_1 \cap U_r) \cup \cdots \cup (U_{r-1} \cap U_r)$ are unions by $r-1$ convex open sets. Therefore Theorem 5.5 holds for $H^*(V \cap U_r), H^*(V)$ and $H^*(U_r)$, and hence also for $H^*(U)$. $\square$

This page intentionally left blank.

6. HOMOTOPY

In this chapter we show that de Rham cohomology is functorial on the category of continuous maps between open sets in Euclidean spaces and calculate $H^*(\mathbb{R}^n - \{0\})$.

Definition 6.1 Two continuous maps $f_\nu : X \to Y, \nu = 0, 1$ between topological spaces are said to be homotopic, if there exists a continuous map

$$F : X \times [0, 1] \to Y$$

such that $F(x, \nu) = f_\nu(x)$ for $\nu = 0, 1$ and all $x \in X$.

This is denoted by $f_0 \simeq f_1$, and F is called a *Homotopy* from f_0 to f_1. It is convenient to think of F as a family of continuous maps $f_t : X \to Y (0 \leqslant t \leqslant 1)$, given by $f_t(x) = F(x, t)$, which deform f_0 to f_1.

Lemma 6.2 Homotopy is an equivalence relation.

PROOF. If F is a homotopy from f_0 to f_1, a homotopy from f_1 to f_0 is defined by $G(x, t) = F(x, 1 - t)$. If $f_0 \simeq f_1$ via F and $f_1 \simeq f_2$ via G, then $f_0 \simeq f_2$ via

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leqslant t \leqslant \dfrac{1}{2} \\ G(x, 2t - 1) & \dfrac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Finally we have that $f \simeq f$ via $F(x, t) = f(x)$. □

Lemma 6.3 Let X, Y and Z be topological spaces and let $f_\nu : X \to Y$ and $g_\nu : Y \to Z$ be continuous maps for $\nu = 0, 1$. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

PROOF. Given homotopies F from f_0 to f_1 and G from g_0 to g_1, the homotopy H from $g_0 \circ f_0$ to $g_1 \circ f_1$ can be defined by $H(x, t) = G(F(x, t), t)$. □

Definition 6.4 A continuous map $f : X \to Y$ is called a *homotopy equivalence*, if there exists a continuous map $g : Y \to X$, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Such a map g is said to be a *homotopy inverse* to f.

Two topological spaces X and Y are called *homotopy equivalent* if there exists a homotopy equivalence between them. We say that X is contractible, when X is homotopy equivalent to a single-point space. This is the same as saying that id_X is homotopic to a constant map. The equivalence classes of topological spaces defined by the relation homotopy equivalence are called *homotopy types*.

Example 6.5 Let $Y \subseteq \mathbb{R}^m$ have the topology induced by $\mathbb{R}^m$. If, for the continuous maps $f_\nu : X \to Y, \nu = 0, 1$, the line segement in $\mathbb{R}^m$ from $f_0(x)$ to $f_1(x)$ is contained in Y for all $x \in X$, we can define a homotopy $F : X \times [0,1] \to Y$ from f_0 to f_1 by

$$F(x,t) = (1-t)f_0(x) + tf_1(x).$$

In particular this shows that a star-shaped set in $\mathbb{R}^m$ is contractible.

Lemma 6.6 If U, V are open sets in Euclidean spaces, then

(i) Every continuous map $f : U \to V$ is homotopic to a smooth map.

(ii) If two smooth maps $f_\nu : U \to V, \nu = 0,1$ are homotopic, then there exists a smooth map $F : U \times \mathbb{R} \to U$ with $F(x,\nu) = f_\nu(x)$ for $\nu = 0,1$ and all $x \in U$. (F is called a *smooth homotopy* from f_0 to f_1).

PROOF. We use Lemma A.9 to approximate h by a smooth map $f : U \to V$. We can choose f such that V contains the line segment from $h(x)$ to $f(x)$ for every $x \in U$. Then $h \simeq f$ by Example 6.5.
Let G be a homotopy from f_0 to f_1. Use continuous function $\psi : \mathbb{R} \to [0,1]$ with $\psi(t) = 0$ for $t \leqslant \frac{1}{3}$ and $\psi(t) = 1$ for $t \geqslant \frac{2}{3}$ to construct

$$H : U \times \mathbb{R} \to V, \quad H(x,t) = G(x, \psi(t)).$$

Since $H(x,t) = f_0(x)$ for $t \leqslant \frac{1}{3}$ and $H(x,t) = f_1(x)$ for $t \geqslant \frac{2}{3}$, H is smooth on $U \times (-\infty) \cup U \times (\frac{2}{3}, \infty)$. Lemma A.9 allows us to approximate H by a smooth map $F : U \times \mathbb{R} \to V$ such that F and H have the same restiction on $U \times \{0,1\}$. For $\nu = 0,1$ and $x \in U$ we have that $F(x,\nu) = H(x,\nu) = f_\nu(x)$. □

Theorem 6.7 If $f, g : U \to V$ are smooth maps and $f \simeq g$ then the induced chain maps

$$f^*, g^* : \Omega^*(V) \to \Omega^*(U)$$

are chain-homotopic (see Definition 4.10).

PROOF. Recall, from the proof of Theorem 3.15, that every p-form ω on $U \times \mathbb{R}$ can be written as

$$\omega = \sum f_I(x,t)\, dx_I + \sum g_J(x,t)\, dt \wedge dx_J$$

If $\phi : U \to U \times \mathbb{R}$ is the inlclusion map $\phi(x) = \phi_0(x) = (x,0)$, then

$$\phi^*(\omega) = \sum f_I(x,0)\, d\phi_I = \sum F_I(x,0)\, dx_I.$$

Indeed, $\phi^*(dt \wedge dx_J) = 0$ since the last component (the t-component) of ϕ is constant; see Example 3.11. Analogously, for $\phi_1(x) = (x,1)$, we have that

$$\phi_1^*(\omega) = \sum F_I(x,1)\, dx_I.$$

In the proof of Theorem 3.15 we constructed

$$\hat{S}_p^* : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$$

such that

$$(\,d\hat{S}_p + \hat{S}_{p+1}\,d)(\omega) = \phi_1^*(\omega) - \phi_0^*(\omega). \tag{1}$$

Consider the composition $U \xrightarrow{\phi_\nu} U \times \mathbb{R} \xrightarrow{F} V$, where F is a smooth homotopy between f and g. The we have that $F \circ \phi_0 = f$ and $F \circ \phi_1 = g$. We define

$$S_p : \Omega^p(V) \to \Omega^{p-1}(U)$$

to be $S_p = \hat{S}_p \circ F$, and assert that

$$\begin{aligned} d\hat{S}_p(F^*(\omega)) + \hat{S}_{p+1}\,dF^*(\omega) &= \phi_1^*(\omega) - \phi_0^*(\omega) \\ &= (F \circ \phi_0)^*(\omega) - (F \circ \phi_0)^*(\omega) \\ &= g^*(\omega) - f^*(\omega). \end{aligned}$$

Furthermore $\hat{S}_{p+1}F^*(\omega) = \hat{S}_{p+1}F^*\,d(\omega) = S_{p+1}\,d(\omega)$, since F^* is a chain map. □

In the situation of Theorem 6.7, Lemma 4.11 states that $f^* = g^* : H^p(V) \to H^(U)$. For a continuous map $\phi : U \to V$ we can find a smooth map $f : U \to V$ with $\phi \backsimeq f$ by (i) of Lemma 6.6, and by Lemma 6.2 and the result above we see that $f^* : H^p(V) \to H^(U)$ is independent of the choice of f. Hence we can define

$$\phi^* = H^p(\phi) : H^p(V) \to H^p(U).$$

be setting $\phi^* = f^*$, where $f : U \to V$ is a smooth map homotopy to ϕ.

Theorem 6.8 For $p \in \mathbb{Z}$ and open sets U, V, W in Euclidean spaces we have

(i) If $\phi_0, \phi_1 : U \to V$ are homotopic continuous maps, then

$$\phi_0^* = \phi_1^* : H^p(V) \to H^p(U).$$

(ii) If $\phi : U \to V$ and $\psi : V \to W$ are continuous, then $(\phi \circ \psi)^* = \psi^* \circ \phi^* : H^p(W) \to H^p(U)$.

(iii) If the continuous map $\phi : U \to V$ is a homotopy equivalence, then

$$\phi^* : H^p(V) \to H^p(U)$$

is an isomorphism.

Proof. Choose a smooth map $f : U \to V$ with $\phi \simeq f$. Lemma 6.2 gives that $\phi_1 \simeq f$ and (i) immediately follows. Part (ii), with smooth ϕ and ψ, follows from the formula

$$\Omega^p(\phi \circ \psi) = \Omega^p(\phi) \circ \Omega^p(\psi).$$

In the general case, choose smooth maps $f : U \to V$ and $g : V \to W$ with $\phi \simeq f$ and $\psi \simeq g$. Lemma 6.3 shows that $\phi \circ \psi \simeq g \circ f$, and we get

$$(\phi \circ \psi)^* = (g \circ f)^* = f^* \circ g^* = \psi^* \circ \phi^*.$$

If $\psi : U \to V$ is a homotopy inverse to ϕ, i.e.,

$$\psi \circ \phi \simeq \mathrm{id}_U, \text{ and } \phi \circ \psi \simeq \mathrm{id}_V,$$

then it follows from (ii) that $\psi^* : H^p(U) \to H^p(V)$ is inverse to ϕ^*. □

This result shows that $H^p(U)$ depends only on the homotopy type of U. In particular we have:

Corollary 6.9 (Topological invariance) A homeomorphism $h : U \to V$ between open sets in Euclidean spaces induces isomorphisms $h^* : H^p(U) \to H^p(V)$ for all p.

Proof. The corollary follows from Theorem 6.8.(iii), as $h^{-1} : V \to U$ is a homotopy inverse to h. □

Corollary 6.10 If $U \subseteq \mathbb{R}$ is an open contractible set, then $H^p(U) = 0$ when $p > 0$ and $H^0(U) = \mathbb{R}$.

Proof. Let $F : U \times [0,1] \to U$ be a homotopy from $f_0 = \mathrm{id}_U$ to a constant map f_1 with value $x_0 \in U$. For $x \in U, F(x,t)$ defines a continuous curve in U, which connects x to x_0. Hence U is connected and $H^0(U) = \mathbb{R}$ by Lemma 3.9. If $p > 0$ then $\Omega^p(f_1) : \Omega^p(U) \to \Omega^p(U)$ is the zero map. Hence by Theorem 6.8.(i) we get that

$$\mathrm{id}_{H^p(U)} = f_0^* = f_1^* = 0.$$

and thus $H^p(U) = 0$. □

In the proposition below, $\mathbb{R}^n$ is identified with the subspace $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^{n+1}$ and $\mathbb{R} \cdot 1$ denotes the 1-dimensional subspace consisting of constant functions.

Proposition 6.11 For an arbitrary closed subset A of $\mathbb{R}$ with $A \neq \mathbb{R}^n$ we have isomorphisms

$$\begin{aligned} H^p(\mathbb{R}^n - A) &\cong H^p(\mathbb{R}^n - A) \qquad \text{for} p \geqslant 1 \\ H^1(\mathbb{R}^{n+1} - A) &\cong H^0(\mathbb{R}^n - A)/\mathbb{R} \cdot 1 \\ H^0(\mathbb{R}^{n+1} - A) &\cong \mathbb{R}. \end{aligned}$$

PROOF. Define open subsets of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$,

$$\begin{aligned} U_1 &= \mathbb{R}^n \times (0,\infty) \cup (\mathbb{R}^n - A) \times (-1,\infty) \\ U_2 &= \mathbb{R}^n \times (-\infty,0) \cup (\mathbb{R}^n - A) \times (-\infty,1). \end{aligned}$$

Then $U_1 \cup U_2 = \mathbb{R}_{n+l} - A$ and $U_1 \cap U_2 = (\mathbb{R}_n - A)x(-1,1)$. Let $\phi : U_1 \to U_1$ be given by adding 1 to the $(n+l)$-st coordinate. For $x \in U_1$, U_1 contains the line segments from x to $\phi(x)$ and from $\phi(x)$ to a fixed point in $\mathbb{R}_n \times (0,\infty)$. As in Example 6.5 we get homotopies from id_{U_1} to ϕ and from ϕ to a constant map. It follows that U_1 is contractible. Analogously U_2 is contractible, and $H^p(U_\nu)$ is described in Corollary 6.10.
Let pr be the projection of $U_1 \cap U_2 = (\mathbb{R}^n - A) \times (-1,1)$ on $\mathbb{R}^n - A$. Define $i : \mathbb{R}^n - A \to U_1 \cap U_2$ by $i(y) = (y,0)$. We have $\mathrm{pr} \circ i = \mathrm{id}_{\mathbb{R}^n - A}$ and $i \circ \mathrm{pr} \backsimeq \mathrm{id}_{U_1 \cap U_2}$. From Theorem 6.8 (iii) we conclude that

$$\mathrm{pr}^* : H^p(\mathbb{R}^n - A) \to H^p(U_1 \cap U_2)$$

is an isomorphism for every p. Theorem 5.2 gives isomorphism

$$\partial^* : H^p(U_1 \cap U_2) \to H^{p+1}(\mathbb{R}^{n+1} - A)$$

for $p \geqslant 1$. By composition with pr^* one obtains the first part of Proposition 6.11. Consider the exact suquence

$$0 \to H^0(\mathbb{R}^{n+1} - A) \xrightarrow{I^*} H^0(U_1) \oplus H^0(U_2) \xrightarrow{J^*} H^0(U_1 \cap U_2) \xrightarrow{\partial^*} H^1(\mathbb{R}^{n+1} - A) \to 0.$$

An element of $H^0(U_1) \oplus H^0(U_2)$ is given by a pair of constant functions on U_1 and U_2 with values a_1 and a_2. Their images under J^* is by Theorem 5.2 the constant function on $U_1 \cap U_2$ with value $a_1 - a_2$. This shows that

$$\mathrm{Ker}\, \partial^* = \mathrm{Im}\, J^* = \mathbb{R} \cdot 1,$$

and we obtain the isomorphism

$$H^1(\mathbb{R}^{n+1} - A) \backsimeq H^0(U_1 \cap U_2)/\mathbb{R} \cdot 1 \cong H^0(\mathbb{R}^n - A)/\mathbb{R} \cdot 1.$$

We also have that $\dim(\mathrm{Im}(I^*)) = \dim(\mathrm{Ker}(J^*)) = 1$, so $H^0(\mathbb{R}^{n+1} - A) = \mathbb{R}$. □

Addendum 6.12 In the situation of Proposition 6.11 we have a *diffeomorphism*

$$R : \mathbb{R}^{n+1} - A \to \mathbb{R}^{n+1} - A$$

defined by $R(x_1, \cdots, x_n, x_{n+1}) = (x_1, \cdots, x_n, -x_{n+1})$. The induced linear map

$$R^* : H^{p+1}(\mathbb{R}^{n+1} - A) \to H^{p+1}(\mathbb{R}^{n+1} - A)$$

is multiplication by (-1) for $p \geqslant 0$.

PROOF. In the notation of the proof above we have commutative diagrams, in which the horizontal diffeomorphisms are restrictions of R:

$$\begin{tikzcd} \mathbb{R}^{n+1}-A \arrow[r, "R"] & \mathbb{R}^{n+1}-A \\ U_1 \arrow[u, "i_1"] \arrow[r, "R_1"] & U_2 \arrow[u, "i_2"] \\ U_1\cap U_2 \arrow[u, "j_1"] \arrow[r, "R_0"] & U_1\cap U_2 \arrow[u, "j_2"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^{n+1}-A \arrow[r, "R"] & \mathbb{R}^{n+1}-A \\ U_1 \arrow[u, "i_1"] \arrow[r, "R_2"] & U_2 \arrow[u, "i_2"] \\ U_1\cap U_2 \arrow[u, "j_1"] \arrow[r, "R_0"] & U_1\cap U_2 \arrow[u, "j_2"] \end{tikzcd}$$

In the proof of Proposition 6.11 we saw that

$$\partial^* : H^p(U_1\cap U_2) \to H^{p+1}(\mathbb{R}^{n+1}-A)$$

is surjective. Therefore it is sufficient to show that $R^*\circ\partial^*([\omega]) = -\partial^*([\omega])$ for arbitary closed p-form ω on $U_1\cap U_2$.
Using Theorem 5.1 we can find $\omega_\nu \in \Omega^p(U_\nu), \nu = 0,1,$ with $\omega = j_1^*(\omega_1) - j_2^*(\omega_2)$. The definition of ∂^* (see Definition 4.5) show that $\partial^*([\omega]) = [\tau]$ where $\tau\in\Omega^{p+1}(\mathbb{R}^{n+1}-A)$ is determined by $i_\nu^*(\tau) = d\omega_\nu$ for $\nu = 1,2$. Furthermore we get

$$\begin{aligned} -R_0^*\omega &= R_0^*\circ j_2^*(\omega_2) - R_0^*\circ j_1^*(\omega_1) = j_1^*\circ R_1^*(\omega_2) - j_2^*\circ R_2^*(\omega_1) \\ i_1^*(R^*\tau) &= R_1^*(i_2^*\tau) = R_1^*(\,d\omega_2) = \,d(R_1^*\omega_2) \\ i_2^*(R^*\tau) &= R_2^*(i_1^*\tau) = R_2^*(\,d\omega_1) = \,d(R_2^*\omega_1) \end{aligned}$$

These equations and the definition of ∂^* give $\partial^*(-[R_0^*\omega]) = [\partial^*\tau]$. Hence

$$\partial^*\circ R_0^*([\omega]) = -R_0^*\circ\partial^*([\omega]). \tag{2}$$

For the projection $\mathrm{pr} : U_1\times U_2 \to \mathbb{R}^n - A$ we have that $\mathrm{pr}\circ R_0 = \mathrm{pr}$ and therefore

$$H^p(\mathbb{R}^n - A) \xrightarrow{\mathrm{pr}^*} H^p(U_1\cap U_2) \xrightarrow{R_0^*} H^p(U_1\cap U_2)$$

is identical with pr^*. Since pr^* is an isomorphism, R_0^* is forced to be the identity map on $H^p(U_1\cap U_2)$, and the left-hand side in (2) is $\partial^*[\omega]$. This completes the proof □

Theorem 6.13 For $n \geqslant 2$ we have the isomorphisms

$$H^p(\mathbb{R}^n - \{0\}) \cong \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

PROOF. The case $n = 2$ was shown in Example 5.4. The general case follows from induction on n, via Proposition 6.11. □

An invertible real $n \times n$ matrix A defines a linear isomorphism $\mathbb{R}^n \to \mathbb{R}$, and a diffeomorphism

$$f_A : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$$

Lemma 6.14 For each $n \geqslant 2$, the induced map $f_A^* : H^{n-1}(\mathbb{R}^n - \{0\}) \to H^{n-1}(\mathbb{R}^n - \{0\})$ operators by multiplication by $\det(A)/|\det A| \in \{\pm 1\}$.

PROOF. Let B be obtained from A by replacing the r-th row by the sum of the r-th row and e times the s-th row, where $r \neq s$ and $c \in \mathbb{R}$,

$$B = (I + cE_{r,s})A$$

where I is the identity matrix and $E_{r,s}$ is the matrix with entry 1 in its r-th row and s-th column and zeros elsewhere. A homotopy between f_A and f_B is defined by the matrices

$$(I + tcE_{r,s})A, \quad 0 \leqslant t \leqslant 1.$$

From Theorem 6.8 it follows that $f_A = f_B$. Furthermore $\det_A = \det_B$. By a sequence of elementary operations of this kind, A can be changed to $\operatorname{diag}(1, \cdots, 1, \pm 1)$, where $d = \det A$. Hence it suffices to prove the assertion for diagonal matrices. The matrices

$$\operatorname{diag}(1, \cdots, 1, \frac{|d|^t d}{|d|}), \quad 0 \leqslant t \leqslant 1$$

yield a homotopy, which reduces the problem to the two cases $A = \operatorname{diag}(1, \cdots, 1, \pm 1)$, so f_A is either the identity or the map R from Addendum 6.12. This proves the assertion □

From topological invariance (see Corollary 6.9) and the calculation in Theorem 6.13, supplemented with

$$H^p(\mathbb{R}^1 - \{0\}) \cong \begin{cases} \mathbb{R} \otimes \mathbb{R} & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

we get

Proposition 6.15 If $n \neq m$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

PROOF. A possible homeomorphism $\mathbb{R}^n \to \mathbb{R}^m$ may be assumed to map 0 to 0, and would induce a homeomorphism between $\mathbb{R}^n - \{0\}$ and $\mathbb{R}^m - \{0\}$. Hence

$$H^p(\mathbb{R}^n - \{0\}) \cong H^p(\mathbb{R}^m - \{0\})$$

for all p, in conflict with our calculations. □

Remark 6.16 We offer the following more conceptual proof of Addendum 6.12. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^* & \xrightarrow{f^*} & B^* & \xrightarrow{g^*} & C^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \beta^*} & & \downarrow{\scriptstyle \gamma^*} & & \\ 0 & \longrightarrow & A_1^* & \xrightarrow{f^*} & B_1^* & \xrightarrow{g^*} & C_1^* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain complexes with exact rows. It is not hard to prove that the diagram

$$\begin{array}{ccc} H^p(C^*) & \xrightarrow{\partial^*} & H^{p+1} \\ \downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle A^*} \\ H^p(C_1^*) & \xrightarrow{\partial_1^*} & H^{p+1}(A_1^*) \end{array}$$

is commutative. In the situation of Addendum 6.12 consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*(U) & \xrightarrow{I^*} & \Omega^*(U_1) \oplus \Omega^*(U_2) & \longrightarrow & \Omega^*(U_1 \cap U_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle R^*} & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle -R_0^*} & & \\ 0 & \longrightarrow & \Omega^*(U) & \xrightarrow{I^*} & \Omega^*(U_1) \oplus \Omega^*(U_2) & \longrightarrow & \Omega^*(U_1 \cap U_2) & \longrightarrow & 0 \end{array}$$

With $R(\omega_1, \omega_2) = (R_1^*\omega_2, R_2^*\omega_1)$. This gives equation (2) of the proof of the addendum.

7. APPLICATIONS OF DE RHAM COHOMOLOGY

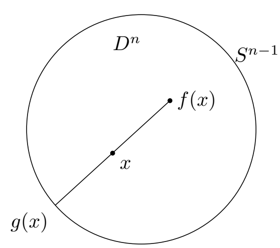
Let us introduce the standard notation

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\} \qquad \text{the } n\text{-ball}$$
$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \qquad \text{the } (n-1)\text{-sphere}$$

A fixed point for a map $f : x \to X$ is a point $x \in X$, such that $f(x) = x$.

Theorem 7.1 (Brouwer's fixed point theorem, 1912) Every continuous map $f : D^n \to D^n$ has a fixed point.

PROOF. Assume that $f(x) \neq x$ for all $x \in D^n$. For every $x \in D^n$ we can define the point $g(x) \in S^{n-1}$ as the point of the intersection between S^{n-1} ans the half-line from $f(x)$ to x.



We have that $g(x) = x + tu$, where $u = \frac{x-f(x)}{\|x-f(x)\|}$, and

$$t = -x \cdot u + \sqrt{1 - \|x\|^2 + (x \cdot u)^2}$$

Here $x \cdot u$ denotes the usual inner product. The expression for $g(x)$ is obtained by solving the equation $(x + tu) \cdot (x + tu) = 1$. There are two solutions since the line determined by $f(x)$ and x intersects S^{n-1} in two points. We are interested in the solution with $t \geqslant 0$. Since g is continuous with $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, the theorem follows from the lemma below. □

Lemma 7.2 There is no continuous map $g : D^n \to S^{n-1}$, with $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$.

PROOF. We may assume that $n \geqslant 2$. For the map $r : \mathbb{R} - \{0\} \to \mathbb{R}^n - \{0\}, r(x) = x/\|x\|$, we get that $\mathrm{id}_{\mathbb{R}^n - \{0\}} \simeq r$, because $\mathbb{R}^n - \{0\}$ always contains the line segment between x and $r(x)$ (see Example 6.5). If g is of the indicated type, then $g(t \cdot r(x)), 0 \leqslant t \leqslant 1$ defines a homotopy from a constant map to r. This shows that $\mathbb{R}^n - 0$ is contractible. Corollary 6.10 asserts that $H^{n-1}(\mathbb{R}^n - \{0\}) = 0$, which contradicts Theorem 6.13 □

The tangent space of S^n in the point $x \in S^n$ is $T_xS^n = \{x\}^\perp$, the orthogonal complement in $\mathbb{R}^{n+1}$ to the position vector. A tangent vector field on S^n is a continuous map $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x) \in T_xS^n$ for every $x \in S^n$.

Theorem 7.3 The sphere S^n has a tangent vector field v with $v(x) \neq 0$ for all $x \in S^n$ if and only if n is odd.

PROOF. Such a vector field v can be extended to a vector field w on $\mathbb{R}^n - \{0\}$ by setting

$$w(x) = v(\frac{x}{\|x\|})$$

We have that $w(x) \neq 0$ and $w(x) \cdot x = 0$. The expression

$$F(x,t) = (\cos \pi t)x + (\sin \pi t)w(x)$$

defines a homotopy from $f_0 = \mathrm{id}_{\mathbb{R}^{n+1}-\{0\}}$ to the antipodal map $f_1, f_1(x) = -x$. Theorem 6.8.(i) shows that f_i^* is the identity on $H^n\mathbb{R}^{n+1} - \{0\}$, which by Theorem 6.13 is 1-dimensional. On the other hand Lemma 6.14 evaluates f_i^* to be multiplication with $(-1)^{n+1}$. Hence n is odd. Conversely, for $n = 2m-1$, we can define a vector field v with

$$v(x_1, x_2, \cdots, x_{2m}) = (-x_2, x_1, -x_4, x_3, \cdots, -x_{2m}, x_{2m-1}).$$

□

In 1962 J. F. Adams solved the so-called "vector field problem": find the maximal number of linearly independent tangent vector fields one may have on S^n. (Tangent vector fields $v_1, \cdots, v_d$ on S^n are called linearly independent if for every $s \in S^n$ the vectors $v_1(x), \cdots, v_d(x)$ are linearly independent.)

Adams' theorem For $n = 2m-1$, let $2m = (2c+1)2^{4a+b}$, where $0 \leqslant b \leqslant 3$. The maximal number of linearly independent tangent vector field on S^n is equal to $2^b + 8a - 1$.

Lemma 7.4 (Urysohn-Tietze) If $A \subseteq \mathbb{R}^n$ is closed and $f : A \to \mathbb{R}^n$ continuous, then there exists a continuous map $g : \mathbb{R}^n \to \mathbb{R}^m$ with $g|_A = f$.

PROOF. We denote Euclidean distance in $\mathbb{R}^n$ by $d(x,y)$ and for $x \in \mathbb{R}^n$ we define

$$d(x,A) = \inf_{y \in A} d(x,y)$$

For $p \in \mathbb{R}^n - A$ we have an open neighborhood $U_p \subseteq \mathbb{R}^n - A$ of p given by

$$U_p = \left\{ x \in \mathbb{R}^n \middle| d(x,p) < \frac{1}{2} d(p,A) \right\}$$

These sets cover $\mathbb{R}^n - A$ and we can use Theorem A.1 to find a subordinate partition of unity ϕ_p. We have g by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ \displaystyle\sum_{p \in \mathbb{R}^n - A} \phi_p(x) f(a(p)) & \text{if } x \in \mathbb{R}^n - A \end{cases}$$

where $p \in \mathbb{R}^n - A$, $a(p) \in A$ is chosen such that

$$d(p, a(p)) < 2d(p, A)$$

Since the sum is locally finite on $\mathbb{R}^n - A$, g is smooth on $\mathbb{R}^n - A$ The only remaining problem is the continuity of g at a point x_0 on the boundary of A. Ifx $\in U_p$ then

$$d(x_0, p) \leqslant d(x_0, x) + d(x, p) < d(x_0, x) + \frac{1}{2} d(p, A) \leqslant d(x_0, x) + \frac{1}{2} d(p, x_0).$$

Hence $d(x_0, p) < 2d(x_0, x)$ for $x \in U_p$. Since $d(p, a(p)) < 2d(p, A) \leqslant d(x_0, p)$, we get for $x \in U_p$ that $d(x_0, a(p)) \leqslant d(x_0, p) + d(p, a(p)) < 3d(x_0, p) < 6d(x_0, x)$. For $x \in \mathbb{R}^n - A$ we have

$$g(x) - g(x_0) = \sum_{p \in \mathbb{R}^n - A} \phi_p(x)(f(a(p)) - f(x_0))$$

and

$$\|g(x) - g(x_0)\| \leqslant \sum_p \phi_p(x) \|f(a(p)) - f(x_0)\|, \tag{1}$$

where we sum over the points p with $x \in U_p$. For an arbitary $\epsilon > 0$ choose $\delta > 0$ such that $\|f(y) - f(x_0)\| < \epsilon$ for every $y \in A$ with $d(x_0, y) < 6\delta$. If $x \in \mathbb{R}^n - A$ and $d(x, x_0) < \delta$ then, for p with $x \in U_p$, we have that $d(x_0, a(p)) < 6\delta$ and $\|f(a(p)) - f(x_0)\| < \epsilon$. Then (1) yields

$$\|g(x) - g(x_0)\| \leqslant \sum_p \phi_p(x) \cdot \epsilon = \epsilon.$$

Continuity of g at x_0 follows. □

Remark 7.5 The proof above still holds, with marginal changes, when $\mathbb{R}^n$ is replaced by a metric space and $\mathbb{R}^m$ by a locally convex topological vector space.

Lemma 7.6 Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed sets and let $\phi : A \to B$ be a homeomorphism. There is a homeomorphism h of $\mathbb{R}^{n+m}$ to itself, such that

$$h(x, 0_m) = (0_m, \phi(x))$$

for all $x \in A$.

Proof. By Lemma 7.4 we can extend ϕ to a continuous map $f_1 : \mathbb{R}^n \to \mathbb{R}^m$. A homeomorphism $h_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is defined by

$$h_1(x, y) = (x, y + f_1(x)).$$

The inverse to h_1 is obtained by subtracting $f_1(x)$ instead. Analogously we can extend ϕ^{-1} to a continuous map $f_2 : \mathbb{R}^m \to \mathbb{R}^n$ and defines a homeomorphism $h_2 : \mathbb{R}^n \times \mathbb{R}^m$ by

$$h_2(x, y) = (x + f_2(y), y).$$

If h is defined to be $h = h_2^{-1} \circ h_1$, then we have for $x \in A$ that

$$h_1(x, 0_m) = h_2^{-1}(x, f_1(x)) = h_2^{-1}(x, \phi(x)) = (x - f_2(\phi(x)), \phi(x)) = (0_n, \phi(x)).$$

□

We identity $\mathbb{R}^n$ with the subspace of $\mathbb{R}^{n+m}$ consisting of vectors of the form

$$(x_1, \cdots, x_n, 0, \cdots, 0).$$

Corollary 7.7 If $\phi : A \to B$ is a homeomorphism between closed subsets A and B of $\mathbb{R}^n$, then ϕ can be extended to a homeomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

Proof. We merely have to compose the homeomorphism h from Lemma 7.6 with the homeomorphism of $\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ to itself that switches the two factors. □

Note that ϕ by restriction gives a homeomorphism between $\mathbb{R}^{2n} - A$ and $\mathbb{R}^{2n} - B$. In contrast it can occurs that $\mathbb{R}^n - A$ is not homeomorphism to $\mathbb{R}^n - B$. A well-known example is *Alexander's "horned sphere"* Σ in $\mathbb{R}^3$: Σ is homeomorphism to S^2, but $\mathbb{R}^3 0\Sigma$ is not homeomorphism to $\mathbb{R}^3 - S^2$. This and numerous examples are treated in [Rushing].

Theorem 7.8 Assume that $A \neq \mathbb{R}^n$ and $B \neq \mathbb{R}^n$ are closed subsets of $\mathbb{R}^n$. If A and B are homeomorphism, then

$$H^p(\mathbb{R}^n - A) \cong H^p(\mathbb{R}^n - B)$$

Proof. By induction on m Proposition 6.11 yields isomorphism

$$H^{p+m}(\mathbb{R}^{n+m} - A) \cong H^p(\mathbb{R}^n - A)$$
$$H^m(\mathbb{R}^{n+m} - A) \cong H^0(\mathbb{R}^n - A)/\mathbb{R} \cdot 1$$

for all $m \geqslant 1$. Analogously for B. From Corollary 7.7 we known that $\mathbb{R}^n - A$ and $\mathbb{R}^{2n} - B$ are homeomorphic. Topological invariance (Corollary 6.9) shows that they are isomorphic de Rham cohomologies. We thus have the isomorphism

$$H^p(\mathbb{R}^n - A) \cong H^{p+n}(\mathbb{R}^{2n} - A) \cong H^{p+n}(\mathbb{R}^{2n} - B) \cong H^p(\mathbb{R}^n - B)$$

for $p > 0$ and

$$H^p(\mathbb{R}^n - A)/\mathbb{R} \cdot 1 \cong H^{p+n}(\mathbb{R}^{2n} - A) \cong H^{p+n}(\mathbb{R}^{2n} - B) \cong H^p(\mathbb{R}^n - B)/\mathbb{R} \cdot 1$$

□

For a closed set $A \subseteq \mathbb{R}^n$ the open complement $U = \mathbb{R}^n - A$ will always be a disjoint union of at most countably many connected components, which all are open. If there are infinitely many, then $H^0(U)$ will have infinite dimension. Otherwise the number of connected components is equal to $\dim H^0(U)$.

Corollary 7.9 If A and B are two homeomorphic closed subsets of $\mathbb{R}^n$, then $\mathbb{R}^n - A$ and $\mathbb{R}^n - B$ have the same number of connected components.

Proof. If $A \neq \mathbb{R}^n$ and $B \neq \mathbb{R}^n$ the assertion follows from Theorem 7.8 and the remarks above. If $A = \mathbb{R}^n$ and $B \neq \mathbb{R}^n$ then $\mathbb{R}^{n+1} - A$ has precisely 2 connected components (the open half-spaces), while $\mathbb{R}^{n+l} - B$ is connected. Hence this case cannot occur. □

Theorem 7.10 (Jordan-Brouwer separation theorem) If $\Sigma \subseteq \mathbb{R}^n, (n \geqslant 2)$ is homeomorphic to S^{n-1} then

(i) $\mathbb{R}^n - \Sigma$ has precisely 2 connected components U_1 and U_2, where U_1 is bounded and U_2 is unbounded.

(ii) Σ is the set of boundary points for both U_1 and U_2.

We say U_1 is the domain inside Σ and U_2 the domain outside Σ.

Proof. Since Σ is compact, Σ is closed in $\mathbb{R}^n$. To show (i), it suffices, by Corollary 7.9, to verify it for $S^{n-1} \subseteq \mathbb{R}^n$. The two connected components of $\mathbb{R}^n - S^{n-1}$ are

$$D^n = \{x \in \mathbb{R}^n \big| \|x\| < 1\} \text{ and } W = \{x \in \mathbb{R}^n \big| \|x\| > 1\}$$

By choosing $r = \max_{x \in \Sigma} \|x\|$, the connected set

$$rW = \{x \in \mathbb{R}^n \big| \|x\| > r\}$$

will be contained in one of the two components in $\mathbb{R}^n - \Sigma$, and the other component must be bounded. This completes the proof of (i).
Let $p \in \Sigma$ be given and consider an open neighbourhood V of P in $\mathbb{R}^n$. The set $A = \Sigma - (\Sigma \cap V)$ is closed and homeomorphic to a corresponding proper closed subset B of S^{n-1}. It is obvious that $\mathbb{R}^n - B$ is connected, so by Corollary 7.9 the same is the case for $\mathbb{R}^n - A$. For $p_1 \in U_1$ and $p_2 \in U_2$, we can find a continuous curve $\gamma : [a, b] \to \mathbb{R}^n - A$ with $\gamma(a) = p_1$ and $\gamma(b) = p_2$. By (i) the curve must intersect Σ, i.e. $\gamma^{-1}(\Sigma)$ is nonempty. The closed set $\gamma^{-1}(\Sigma) \subseteq [a, b]$ has a first element c_1 and a last element c_2, which both belong to (a, b). Hence $\gamma(c_1) \in \Sigma \cap V$ and $\gamma(c_2) \in \Sigma \cap V$ are points of contact for $\gamma([a, c_1)) \subseteq U_1$ and $\gamma([a, c_2)) \subseteq U_2$ respectively. Therefore we can find $t_1 \in [a, c_1)$ and $t_2 \in [c_2, b)$, such that $\gamma(t_1) \in U_1 \cap V$ and $\gamma(t_2) \in U_2 \cap V$. This shows that p is a boundary point for both U_1 and U_2, and proves (ii). □

Theorem 7.11 If $A \subseteq \mathbb{R}^n$ is homeomorphic to D^k, with $k \leqslant n$, then $\mathbb{R}^n - A$ is connected.

PROOF. Since A is compact, A is closed. By Corollary 7.9 it is sufficient to prove the assertion for $D^k \subseteq \mathbb{R}^k \subseteq \mathbb{R}^n$. This is left to the reader. □

Theorem 7.12 (Brouwer) Let $U \subseteq \mathbb{R}^n$ be an arbitrary open set and $f : U \to \mathbb{R}^n$ an injective continuous map. The image $f(U)$ is open in $\mathbb{R}^n$, and f maps U homeomorphically to $f(U)$.

PROOF. It is sufficient to prove that $f(U)$ is open; the same will then hold for $f(W)$, where $W \subseteq U$ is an arbitrary open subset. This proves continuity of the inverse function from $f(U)$ to U. Consider a closed sphere.

$$D = \{x \in \mathbb{R}^n \big| \|x - x_0\| \leqslant \delta\}$$

contained in U with boundary S and interior $\mathring{D} = D - S$. It is sufficient to show that $f(\mathring{D})$ is open. The case $n = 1$ follows from elementary theorems about continuous functions of one variable, so we assume $n \geqslant 2$.
Both Sand $E = f(S)$ are homeomorphic to S^{n-1}. Let U_1 and U_2 be the two connected components of $\mathbb{R}^n - \Sigma$ from Theorem 7.10. They are open; U_1 is bounded and U_2 is unbounded. By Theorem 7.11, $\mathbb{R}^n - f(D)$ is connected. Since this set is disjoint from E, it must be contained in U_1 or U_2. As $f(D)$ is compact, $\mathbb{R}^n - f(D)$ is unbounded. We must have $\mathbb{R}^n - f(D) \subseteq U_2$. It follows that $\Sigma \cup U_1 = \mathbb{R}^n - U_2 \subseteq f(D)$. Hence

$$U_1 \subseteq f(\mathring{D}).$$

Since D is connected, $f(\mathring{D})$ will also be connected (even though it is not known whether or not $f(\mathring{D})$ is open). Since $f(\mathring{D}) \subseteq U_1 \cup U_2$, we must have that $U_1 = f(\mathring{D})$. This completes the proof. □

Corollary 7.13 (Invariance of domain) If $V \in \mathbb{R}^n$ has the topology induced by $\mathbb{R}^n$ and is homeomorphic to an open subset of $\mathbb{R}^n$ then V is open in $\mathbb{R}^n$.

PROOF. This follows immediately from Theorem 7.12. □

Corollary 7.14 (Dimension invariance) Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be non-empty open sets. If U and V are homeomorphic then $n = m$.

PROOF. Assume that $m < n$. From Corollary 7.13 applied to V, considered as a subset of $\mathbb{R}^n$ via the inclusion $\mathbb{R}^m \subseteq \mathbb{R}^n$, it follows that V is open in $\mathbb{R}^n$. This contradicts that V is contained in a proper subspace. □

Example 7.15 A knot in $\mathbb{R}^3$ is a subset $\Sigma \subseteq \mathbb{R}^3$ that is homeomorphic to S^1. The corresponding knot-complement is the open set $U = \Sigma - \mathbb{R}^3$. We show:

$$H^p(U) \cong \begin{cases} \mathbb{R} & \text{if } 0 \leqslant p \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

According to Theorem 7.8, it is sufficient to show this for the "trival knot" $S^1 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$. First we calculate

$$H^p(\mathbb{R}^2 - S^1) = H^p(\mathring{D}^2) \oplus H^p(\mathbb{R}^2 - D^2) \tag{2}$$

Here $\mathring{D}^2$ is star-shaped, while $\mathbb{R}^2 - \mathring{D}^2$ is diffeomorphic to $\mathbb{R}^2 - 0$. Using Theorem 3.15 and Example 5.4 it follows that (2) has dimension 2 for $p = 0$, dimension 1 for $p = 1$, and dimension 0 for $p \geqslant 2$. Apply Proposition 6.11.

An analogous calculation of $H^*(\mathbb{R}^n - E)$ can be done for a higher-dimensional knot $\Sigma \subseteq \mathbb{R}^n$, where Σ is homeomorphic to $S^k(1 \leqslant k \leqslant n-2)$. See Exercise 2.

Proposition 7.16 Let $\Sigma \subseteq \mathbb{R}^n(n \geqslant 2)$ be homeomorphic to S^{n-1} and let U_1 and U_2 be the interior and exterior domains of Σ. Then

$$H^p(U_1) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{Otherwise} \end{cases} \qquad \text{and} \qquad H^p(U_2) \cong \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The case $p = 0$ follows from Theorem 7.10. Set $W = \mathbb{R}^n - D^n$. For $p > 0$ there are isomorphisms

$$\begin{aligned} H^p(U_1) \oplus H^p(U_2) &\cong H^p(\mathbb{R}^n - \Sigma) \cong H^p(\mathbb{R}^n - S^{n-1}) \\ &\cong H^p(\mathring{D}^n) \oplus H^p(W) \cong H^p(W) \end{aligned}$$

The inclusion map $i : W \to \mathbb{R}^n - \{0\}$ is a Homotopy equivalence with homotopy inverse defined by

$$g(x) = \frac{\|x\| + 1}{\|x\|} x.$$

The two required homotopies are given by Example 6.5. From Theorem 6.8.(iii) we have that $H^p(i)$ is an isomorphism. The calculation from Theorem 6.13 yields

$$H^p(W) \cong \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

We now have that $H^p(U_1) = 0$ and $H^p(U_2) = 0$ when $p \neq \{0, n-1\}$. On the other hand the dimension of $H^{n-1}(U_1)$ and $H^{n-1}(U_2)$ are 0 and 1, so it suffices to show that $H^{n-1}(U_2) \neq 0$.
Without loss of generality we may assume that $0 \in U_1$ and that the bounded set $U_1 \cup \Sigma$ is contained in D^n. We thus have a commutative diagram of inclusion maps

$$\begin{tikzcd} & \mathbb{R}^n - \{0\} \\ W \arrow[ur, "i"] \arrow[r] & U_2 \arrow[u] \end{tikzcd}$$

and apply H^{n-l} to get the commutative diagram

$$\begin{tikzcd} & H^{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{R} \arrow[dl, "H^{n-1}(i)"'] \arrow[d] \\ H^{n-1}(W) \arrow[r] & H^{n-1}(U_2) \end{tikzcd}$$

where $H^{n-1}(i)$ is an isomorphism. It follows that $H^{n-1}(U_2) \neq 0$. □

Remark 7.17 The above result about $H^*(U_1)$might suggest that U1 is contractible (cf. Corollary 6.10). In general, however, this is not the case. In *Topological Embeddings*, Rushing discusses several examples for $n = 3$, where U_1 is not simply connected (i.e. there exists a continuous map $S^1 \to U_1$, which is not homotopic to a constant map). Hence U_1 is not contractible either. Corresponding examples can be found for $n > 3$. If $n = 2$ a theorem by Schoenflies (cf. [Moise]) states that there exists a homeomorphism

$$h : U_1 \cup \Sigma \to D^2.$$

By Theorem 7.12, such a homeomorphism applied to $h|_{\nu_1}$ and $h|^{-1}_{\mathring{D}^2}$ will map U_1 homeomorphically to $\mathring{D}^2$.

A result by M. Brown from 1960 shows that the conclusion in Schoenflies' theorem is also valid if $n > 2$, provided it is additionally assumed that Σ is *flat* in $\mathbb{R}^n$, that is, there exists a $\delta > 0$ and a continuous injective map $\phi : S^{n-1} \times (-\delta, \delta) \to \mathbb{R}^n$ with $\Sigma = \phi(S^{n-1} \times \{0\})$.

Example 7.18 One can also calculate the cohomology of "$\mathbb{R}^n$ with m holes", i.e. the cohomology of

$$V = \mathbb{R}^n - \left(\bigcup_{j=1}^{m} K_j \right).$$

The "holes" K_j in $\mathbb{R}^n$ are disjoint compact sets with boundary Σ_j, homeomorphic to S^{n-1}. Hence the interiors $\mathring{K}_j = K_j - \Sigma_j$ are exactly the interior domains of Σ_j. One has

$$H^l(V) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \\ \mathbb{R}^m & \text{if } p = n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We use induction on m. The case $n = 1$ follows from Proposition 7.16. Assume the assertion is true for

$$V_1 = \mathbb{R}^n - \left(\bigcup_{j=1}^{m-1} K_j \right)$$

Let $V_2 = \mathbb{R}^n - K_m$. Then $V_1 \cup V_2 = \mathbb{R}^n$ and $V_1 \cap V_2 = V$. For $p \geqslant 0$ we have the exact Mayer-Vietoris sequence

$$H^p(\mathbb{R}^n) \xrightarrow{I^*} H^p(V_1) \oplus H^p(V_2) \xrightarrow{J^*} H^p(V) \to 0.$$

If $p = 0$ then $H^0(\mathbb{R}^n) \cong \mathbb{R}$ and I^* is injective. We get $H^0(V_1) \cong \mathbb{R}$ by induction and $H^0(V_2) \cong \mathbb{R}$ from Proposition 7.16. The exact sequence yields $H^0(V) \cong \mathbb{R}$. If $p > 0$ then $H^p(\mathbb{R}^n) = 0$ and the exact sequence gives the isomorphism

$$H^p(V_1) \oplus H^p(V_2) \cong H^p(V)$$

Now (3) follows by induction.

This page intentionally left blank.

8. SMOOTH MANIFOLDS

A topological space X has a countable topological base, when there exists a countable system of open sets $V = \{U_i \big| i \in \mathbb{N}\}$, such that every open set can be written in the form $\cup_{i \in I} U_i$, where $I \subseteq \mathbb{N}$.
For instance $\mathbb{R}^n$ has a countable basis for the topology given by

$$\mathcal{V} = \{\mathring{D} \big| x = (x_1, \cdots, x_n), x_i \in \mathbb{Q}; \epsilon \in \mathbb{Q}, \epsilon > 0\}$$

where $\mathring{D}(x;\epsilon)$ is the open ball with center at x and radius ϵ. A topological space X is a Hausdorff space when, for arbitrary distinct $x, y \in X$, there exist open neighborhoods U_x and U_y with $U_x \cap u_y = \varnothing$.

Definition 8.1 A topological manifold M is a topological Hausdorff space that has a countable basis for its topology and that is locally homeomorphic to $\mathbb{R}^n$. The number n is called the dimension of M.

Remark 8.2 Every open ball $\mathring{D}^n(0, \epsilon)$ in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$ via the map Φ, given by

$$\Phi(y) = \begin{cases} \tan(\pi\|y\|/2\epsilon) \cdot y/\|y\| & \text{if } y \neq 0, \|y\| < \epsilon \\ 0 & \text{if } y = 0 \end{cases}$$

(Smoothness of Φ and Φ^{-1} at 0 can be shown by means of the Taylor series at 0 for tan and arctan.) Thus in Definition 8.1 it does not matter whether we require that M^n is locally homeomorphic to $\mathbb{R}^n$ or to an open set in $\mathbb{R}^n$.

Definition 8.3

(i) A chart (U, h) on an n-dimensional manifold is a homeomorphism $h : U \to U'$, where U is an open set in M and U' is an open set in $\mathbb{R}^n$.

(ii) A system $A = \{h_i : U_i \to U'_i \big| i \in J\}$ of charts is called an atlas, provided $\{U_i \big| i \in J\}$ covers M.

(iii) An atlas is smooth when all of the maps

$$h_{ji} = h_j \circ h_i^{-1} : h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$$

are smooth. They are called chart transformations (or transition functions) for the given atlas.

Note in Definition 8.3.(iii) that $h_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$. Two smooth atlases A_1, A_2 are smoothly equivalent if $A_1 \cup A_2$ is a smooth atlas. This defines an equivalence relation on the set of atlases on M. A smooth structure on M is an equivalence class $\mathcal{A}$ of smooth atlases on M.

Definition 8.4 A smooth manifold is a pair $(M, \mathcal{A})$ consisting of a topological manifold M and a smooth structure $\mathcal{A}$ on M.

Usually $\mathcal{A}$ is suppressed from the notation and we write M instead of $(M, \mathcal{A})$.

Example 8.5 The n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} \big| \|x\| = 1\}$ is an n-dimensional smooth manifold. We define an atlas with $2(n+1)$ charts $(U_{\pm i}, h_{\pm i})$ where

$$U_{+i} = \{x \in S^n \big| x_i > 0\}, \qquad U_{-i} = \{x \in S^n \big| x_i < 0\}$$

and $h_{\pm i} : U_{\pm i} \to \mathring{D}^n$ is the map given by $h_{\pm i}(x) = (x_1, \cdots, \hat{x}_i, \cdots, x_{n+1})$. The circumflex over x_i denotes that x_i is omitted. The inverse map is

$$h_{\pm i}^{-1}(u) = \left(u_1, \cdots, u_{i-1}, \pm\sqrt{1 - \|u\|^2}, u_i, \cdots, u_n\right)$$

It is left to the reader to prove that the chart transformations are smooth.

Example 8.6 (The projective space $\mathbb{RP}^n$) On S^n we define an equivalence relation:

$$x \sim y \Longleftrightarrow x = y \text{ or } x = -y$$

The equivalence classes $[x] = \{x, -x\}$ define the set $\mathbb{RP}^n$. Alternatively one can consider $\mathbb{RP}^n$ as all lines in $\mathbb{R}^{n+1}$ through 0. Let 1r be the canonical projection

$$\pi : S^n \to \mathbb{RP}^n; \qquad \pi(x) = [x].$$

We give $\mathbb{RP}^n$ the quotient topology, i.e.

$$U \subseteq \mathbb{RP}^n \text{ open } \Longleftrightarrow \pi^{-1}(U) \subseteq S^n \text{ open }.$$

With the conventions of Example 8.5, $\pi(U_{-i}) = \pi(U_{+i})$. We define $U_i = \pi_{\pm i} \subseteq \mathbb{RP}^n$, and note that $\pi^{-1}(U_i) = U_{+i} \cup U_{-i}$ with $U_{+i} \cap U_{-i} = \varnothing$. An equivalence calss $[x] \in U_i$ has one representive in U_{+i} and one representive in U_{-i}. Hence $\pi : U_{+i} \to U_{-i}$ is a homeomorphism. We define

$$h_i = h_{+i} \circ \pi^{-1} : U_i \to \mathring{D}^n,$$

$i = 1, \cdots, n$. This gives a smooth atlas on $\mathbb{RP}^n$.

Definition 8.7 Consider smooth manifolds M_1 and M_2 and a continuous map $f : M_1 \to M_2$. The map f is called smooth at $x \in M_1$ if there exist charts $h_1 : U_1 \to U_1'$ and $h_2 : U_2 \to U_2'$ on M_1 and M_2 with $x \in U_1$ and $f(x) \in U_2$, such that

$$h_2 \circ f \circ h_1^{-1} : h_1(f^{-1}(U_2)) \to U_2'$$

is smooth in a neighborhood of $h_1(x)$. If f is smooth at all points of M_1 then f is said to be smooth.

Since chart transformations (by Definition 8.3.(iii)) are smooth, we have that Definition 8.7 is independent of the choice of charts in the given atlases for M_1 and M_2. A composition of two smooth maps is smooth.
A *diffeomorphism* $f : M_1 \to M_2$ between smooth manifolds is a smooth map that has a smooth inverse. In particular a diffeomorphism is a homeomorphism. As soon as we have chosen an atlas $\mathcal{A}$ on a manifold M, we know which functions on M are smooth. In particular we know when a homeomorphism $f : V \to V'$ between an open set $V \subset M$ and an open set $V' \subseteq \mathbb{R}^n$ is a diffeomorphism. We can therefore define a new *maximal atlas*, $\mathcal{A}_{\max}$, associated with the given smooth structure:

$$\mathcal{A}_{\max} = \{f : V \to V' \big| V \subseteq M^n \text{ open }, V' \subseteq \mathbb{R}^n \text{ open }, f \text{ diffeomorphism } \}.$$

The inverse diffeomorphisms $f^{-1} : V' \to V$ will be called local parametrizations. From Remark 8.2 it follows that every point in a smooth manifold M^n has an open neighborhood $V \subseteq M^n$ that is diffeomorphic to $\mathbb{R}^n$.
From now on chart will mean a chart in the maximal atlas.

Definition 8.8 A subset $N \subset M^n$ of a smooth manifold is said to be a smooth submanifold (of dimension k), if the following condition is satisfied: for every $x \in N$ there exists a chart $h : U \to U'$ on M such that

$$x \in U \text{ and } h(U \cap N) = U' \cap \mathbb{R}^k, \tag{1}$$

where $\mathbb{R}^k \subseteq \mathbb{R}^n$ is the standard subspace.

It is easy to see that a smooth submanifold N of a smooth manifold M is a smooth manifold again. A smooth atlas on N is given by all $h : U \cap N \to U'\mathbb{R}^k$, where (U, h) are charts on M satisfying (1).

Example 8.9 The n-sphere S^n is a smooth submanifold of $\mathbb{R}^{n+1}$. In fact the charts $(U_{\pm i}, h_{\pm i})$ from Example 8.5 can easily be extended to diffeomorphisms satisfying (1).

Definition 8.10 An *embedding* is a smooth map $f : N \to M$ such that $f(N) \subset M$ is a smooth submanifold and $f : N \to f(N)$ is a diffeomorphism.

Theorem 8.11 Let M^n be a smooth manifold of dimension n. There exists an embedding of M^n into a Euclidean space $\mathbb{R}^{n+k}$.

This result will be proved below for a compact M, but let us first note that $N^n = f(M^n)$ satisfies the following condition:
For every $p \in N^n$ there exists an open neighborhood $V \subseteq \mathbb{R}^{n+k}$, an open set $U' \subseteq \mathbb{R}^n$ and a homeomorphism

$$g : U' \to N \cap V,$$

such that g is smooth (consider as a map from U' to V) and such that $D_x g : \mathbb{R}^n \to \mathbb{R}^{n+k}$ is injective.

This is the usual definition of an embedded manifold (regular surface when $n = 2$). Theorem 8.11 tells us that every smooth manifold is diffeomorphic to an embedded manifold. Conversely, if $N \subseteq \mathbb{R}^{n+k}$ satisfies the above condition, then the implicit function theorem shows that it is a submanifold in the sense of Definition 8.8. A theorem by H. Whitney asserts that the codimension k in Theorem 8.11 can always be taken to be less or equal to $n+1$. On the other hand k cannot be arbitrarily small. $\mathbb{RP}^2$ cannot be embedded in $\mathbb{R}^3$.

Lemma 8.12 Let M^n be an n-dimensional smooth manifold. For $p \in M$ there exist smooth maps

$$\phi_p : M \to \mathbb{R}, \qquad\qquad f_p : M \to \mathbb{R}^n$$

such that $\phi_p(p) > 0$, and f_p maps open set $M \to \phi_p^{-1}(0)$ diffeomorphically onto an open subset of $\mathbb{R}^n$.

PROOF. Choose a chart $h : V \to V'$ with $p \in V$. By Lemma A.2 we can find a function $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with compact support $\text{supp}_{\mathbb{R}^n}(\psi) \subseteq V'$, such that ψ is constantly equal to 1 on an open neighborhood $U' \subset V'$ of $h(p)$.The smooth map f_p can now be defined by

$$f_p(q) = \begin{cases} \psi(h(q))h(q) & \text{if } q \in V \\ 0 & \text{otherwise} \end{cases}$$

On the open neighborhood $U = h^{-1}(U')$ the function f_p coincides with h and therefore maps U diffeomorphically onto U'. Choose $\psi_0 \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with compact support $\text{supp}_{\mathbb{R}^n}(\psi_0) \subseteq U'$ and $\psi_0(h(p)) > 0$, and let

$$\phi(q) = \begin{cases} \psi_0(h(q)) & \text{if } q \in V \\ 0 & \text{otherwise} \end{cases}$$

Since $M - \phi_p^{-1}(0) \subseteq U$, the final assertion holds. □

Proof of Theorem 8.11 (M compact)For every $p \in M$ choose ϕ_p and f_p as in Lemma 8.12. By compactness M can be covered by a finite number of the sets $M - \phi_p^{-1}(0)$. After a change of notation we have smooth functions

$$\phi_j : M \to \mathbb{R}, \qquad f_j : M \to \mathbb{R}^n \qquad (1 \leqslant j \leqslant d)$$

satisfying the following conditions:

(i) The open sets $U_j = M - \phi_j^{-1}(0)$ cover M.

(ii) $f_{j|U_j}$ maps U_j diffeomorphically onto an open set $U'_j \in \mathbb{R}^n$.

We define a smooth map $f : M \to \mathbb{R}^{nd+d}$ by setting

$$f(q) = (f_1(q), \cdots, f_d(q), \phi_1(q), \cdots, \phi_d(q)).$$

Assuming $f(q_1) = f(q_2)$, we can by (i) choose j such that $q_1 \in U_j$. Then $\phi_j(q_2) = \phi_j(q_1) \neq 0, q_2 \in U_j$, and by (ii), $q_1 = q_2$. Hence f is injective. Since M is compact, f is a homeomorphism from M to $f(M)$. Let

$$\pi_1 : \mathbb{R}^{nd+d} \to \mathbb{R}, \qquad \pi_2 : \mathbb{R}^{nd+d} \to \mathbb{R}^{n(d-1)+d}$$

be the projections on the first n coordinates and the last $n(d-1)+d$ coordinates, respectively. By (ii) $\pi_1 \circ f = f_1$ is a diffeomorphism from U_1 to U_1'. In particular π_1 maps $f(U_1)$ bijectively onto U_1'. Hence $f(U_1)$ is the graph of the smooth map

$$g_1 : U_1' \to \mathbb{R}^{n(d-1)+d}; \qquad g_1 = \pi_2 \circ f \circ (f_{1|U_1})^{-1}$$

Define a diffeomorphism h_1 from $\pi_1^{-1}(U_1')$ to itself by the formula

$$h_1(x, y) = (x, y - g_1(x)), \qquad x \in U_1', y \in \mathbb{R}^{n(d-1)+d}.$$

We see that h_1 maps $f(U_1)$ bijectively onto $U_1' \times \{0\}$. Since $f(U_1)$ is open in $f(M), f(U_1) = f(M) \cap W_1$ for an open set $W_1 \subseteq \mathbb{R}^{nd+d}$, which can be chosen to be contained in $\pi_1^{-1}(U_1')$. The restriction $h_{1|W_1}$ is a diffeomorphism from W_1 onto an open set W_1', and it maps $f(M) \cap W_1$ bijectively onto $W_1' \cap \mathbb{R}^n$, as required by Definition 8.8. The remaining $f(U_j)$ are treated analogously. Hence $f(M)$ is a smooth submanifold of $\mathbb{R}^{nd+d}$. Note also that $f_{|U_1} : U_1 \to f(U_1)$ is a diffeomorphism, namely the composite of $f_{1|U_1} : U_1 \to U_1'$ and the inverse to the diffeomorphism $f(U_) \to U_1'$ induced by π_1. The remaining U_j are treated analogously. Hence $f : M \to f(M)$ is a diffeomorphism. □

Remark 8.13 The general case of Theorem 8.11 is shown in standard text books on differential topology. To get $k = n + 1$ (*Whitney's embedding theorem*) one uses Theorem 11.6 below. In the proofs above one can change "smooth manifold" to "topological manifold", "smooth map" to "continuous map" and "diffeomorphism" to "homeomorphism". This will lead to the theorem below, where the concept (locally flat) topological submanifold is defined in analogy to Definition 8.8, but with a homeomorphism instead of the diffeomorphism h.

Theorem 8.14 Every compact topological n-dimensional manifold is homeomorphic to a (locally flat) topological submanifold of a Euclidean space $\mathbb{R}^{n+k}$

On a topological manifold M^n we have the $\mathbb{R}$-algebra $C^0(M, \mathbb{R})$ of continuous functions $M \to \mathbb{R}$. A smooth structure $\mathcal{A}$ on M^n gives a subalgebra

$$C^\infty((M, \mathcal{A}), \mathbb{R}) \subseteq C^0(M, \mathbb{R})$$

consisting of the maps $M \to \mathbb{R}$, that are smooth in the structure $\mathcal{A}$ on M (and the standard structure on $\mathbb{R}$). Usually $\mathcal{A}$ is suppressed from the notation, and the $\mathbb{R}$-algebra of smooth real-valued functions on M is denoted by $C^\infty(M, \mathbb{R})$. This subalgebra of $C^0(M, \mathbb{R})$ uniquely determines the smooth structure on M. This is a consequence of the following Proposition 8.15 applied to the identity maps id_M

$$(M, \mathcal{A}_1) \leftrightarrows (M, \mathcal{A}_2).$$

Proposition 8.15 If $g : N \to M$ is a continuous map between smooth manifolds N and M, then g is smooth if and only if the homomorphism

$$g^* : C^0(M, \mathbb{R}) \to C^0(N, \mathbb{R})$$

given by $g^*(\psi) = \psi \circ g$ maps $C^\infty(M, \mathbb{R})$ to $C^\infty(N, \mathbb{R})$.

Proof. "Only if" follows because a composition of two smooth maps is smooth. Conversely if the condition on g^* is satisfied, Lemma 8.12 applied to $p = g(q)$ yields a smooth map $f : M^n \to \mathbb{R}^n$ and an open neighborhood V of p in M, such that the restriction $f_{|V}$ is a diffeomorphism of V onto an open subset of $\mathbb{R}^n$. For the j-th coordinate function $f_j \in C^\infty(M, \mathbb{R})$ we have $f_j \circ g = g^*(f_j) \in C^\infty(N, \mathbb{R})$, so that $f \circ g : N \to \mathbb{R}$ is smooth. Using the chart $f_{|W}$ on M, g is seen to be smooth at q. □

Remark 8.16 There is a quite elaborate theory which attempts to classify n dimensional smooth and topological manifolds up to diffeomorphism and homeomorphism. Every connected 1-dimensional smooth or topological manifold is diffeomorphic or homeomorphic to $\mathbb{R}$ or S^1. For $n = 2$ there is a complete classification of the compact connected surfaces. There are two infinite families of them:

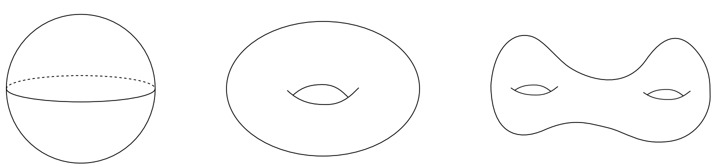


Figure 8.1: Orientable surfaces

Non-orientable surfaces: $\mathbb{RP}^2$, Klein's bottle, etc. See e.g. [Hirsch], [Massey]. In dimension 3, one meets a famous open problem: the Poincare conjecture, which asserts that every compact topological 3-manifold that is homotopy equivalent to 53 is homeomorphic to S^3. It is known that every topological 3-manifold M^3 has a smooth structure $\mathcal{A}$ and that two homeomorphic smooth 3-manifolds also are diffeomorphic. In the mid 1950s J. Milnor discovered smooth 7-manifolds that are homeomorphic to S^7, but not diffeomorphic to S^7. In collaboration with M. Kervaire he classified such exotic n-spheres. For example they showed that there are exactly 28 oriented diffeomorphism classes of exotic 7-spheres. In 1960 Kervaire described a topological 10-manifold that has no smooth structure. During the 1960s the so-called "surgery" technique was developed, which in principle classifies all manifolds of a specified homotopy type, but only for $n \geqslant 5$.

In the early 1980s M. Freedman completely classified the simply-connected compact topological 4-manifolds; see [Freedman-Quinn]. At the same time S. Donaldson proved some very surprising results about smooth compact 4-manifolds, which showed that there is a tremendous difference between smooth and topological 4-manifolds. Donaldson used methods originating in mathematical physics(Yang-Mills theory). This has led to a wealth of new results on smooth 4-manifolds; see [Donaldson-Kronheimer].

One of the most bizarre conclusions of the work of Donaldson and Freedman is that there exists a smooth structure on $\mathbb{R}^4$ such that the resulting smooth 4-manifold "$\mathbb{R}^4$" is not diffeomorphic to the usual $\mathbb{R}^4$. It was proved earlier by S. Smale that every smooth structure on $\mathbb{R}^n$ for $n \neq 4$ is diffeomorphic to the standard $\mathbb{R}^n$.

This page intentionally left blank.

9. DIFFERENTIAL FORMS ON SMOOTH MANIFOLDS

In this chapter we define the de Rham complex $\Omega^*(M)$ of a smooth manifold M^m and generalize the material of earlier chapters to the manifold case. For a given point $p \in M^m$ we shall construct an m-dimensional real vector space T_pM called the *tangent space* at p. Moreover we want a smooth map $j : M \to N$ to induce a linear map $D_pf : T_pM \to T_{f(p)}M$ known as the tangent map of j at p.

Remark 9.1

(i) In the case $p \in U \subseteq \mathbb{R}^m$, where U is open, one usually identifies the tangent space to U at p with $\mathbb{R}^m$. Better suited for generalization is the following description: Consider the set of smooth parametrized curves $\gamma : I \to U$ with $\gamma(0) = p$, defined on open intervals around 0. An equivalence relation on this set is given by the condition $\gamma_1'(0) = \gamma_2'(0)$. There is a 1-1 correspondence between equivalence classes and $\mathbb{R}^m$, which to the class $[\gamma]$ of, associates the velocity vector $\gamma'(0) \in \mathbb{R}^m$.

(ii) Consider a further open set $V \subseteq \mathbb{R}^n$ and a smooth map $F : U \to V$. The Jacobi matrix of F evaluated at $p \in U$ defines a linear map $D_pF : \mathbb{R}^m \to \mathbb{R}^n$. For $\gamma : I \to U, \gamma(0) = p$ as in (i) the *chain rule* implies that $D_pF(\gamma'(0)) = (F \circ \gamma'(0))$. Interpreting tangent spaces as given by equivalence classes of curves we have

$$D_pF([\gamma]) = [F \circ \gamma] \tag{1}$$

In particular the equivalence class of $F \circ \gamma$, depends only on $[\gamma]$.

Let (U, h) be a smooth chart around $p \in M^m$. On the set of smooth curves $\alpha : I \to M$ with $\alpha(0) = p$ defined on open intervals around 0 we have an equivalence relation

$$\alpha_1 \sim \alpha_2 \iff (h \circ \alpha_1)'(0) = (h \circ \alpha_2)'(0) \tag{2}$$

This equivalence relation is independent of the choice of (U, h). In fact, if $(\tilde{U}, \tilde{h})$ is another smooth chart around p, one finds that

$$(h \circ \alpha_1)'(0) = (h \circ \alpha_2)'(0) \iff (\tilde{h} \circ \alpha_1)'(0) = (\tilde{h} \circ \alpha_2)'(0)$$

by applying the last statement of Remark 9.1.(ii) to the transition diffeomorphism $F = h \circ \tilde{h}^{-1}$ and its inverse.

Definition 9.2 The *tangent space* T_pM^m is the set of equivalence classes with respect to (2) of smooth curves $\alpha : I \to M, \alpha(0) = p$.

We give T_pM the structure of an m-dimensional vector space defined by the following condition: if (U, h) is a smooth chart in M with $p \in U$, then

$$\Phi_h : T_pM \to \mathbb{R}^m, \qquad \Phi_h([\alpha]) = (h \circ \alpha)'(0),$$

is a linear isomorphism; here $[\alpha] \in T_pM$ is the equivalence class of α.
By definition Φ_h is a bijection. The linear structure on T_pM is well-defined. This can be seen from the following commutative diagram, where $F = h \circ \tilde{h}^{-1}$, $q = \tilde{h}(p)$

$$\begin{tikzcd}
 & \mathbb{R}^m \arrow[dd, "D_qF", "\cong"'] \\
T_pM \arrow[ur, "\Phi_{\tilde{h}}"] \arrow[dr, "\Phi_h"'] & \\
 & \mathbb{R}^m
\end{tikzcd}$$

Lemma 9.3 Let $f : M^m \to N^n$ be a smooth map and $p \in M$.

(i) There is a linear map $D_pf : T_pM \to T_{f(p)}N$ given in terms of representing curves by

$$D_pf([\alpha]) = [f \circ \alpha]$$

(ii) If (U, h) is a chart around p in M and (V, g) a chart around $f(p)$ in N, then we have the following commutative diagram:

$$\begin{tikzcd}
T_pM \arrow[r, "D_pf"] \arrow[d, "\Phi_h", "\cong"'] & T_{f(p)}M \arrow[d, "\Phi_g", "\cong"'] \\
\mathbb{R}^m \arrow[r, "D_{h(p)}(g \circ f \circ h^{-1})"] & \mathbb{R}^n
\end{tikzcd}$$

Proof. Remark 9.1.(ii) applied to $F = g \circ f \circ h^{-1}$, defined $h(U \cap f^{-1}(V))$, shows that the bottom map in the diagram is linear and given on representing curves as stated there. Since Φ_h and Φ_g are linear isomorphisms, there exists a linear map D_pf making the diagram commutative. The formula in (i) follows by chasing around the diagram. $\square$

Note that the linear isomorphism Φ_h in Definition 9.2 can be identified with D_ph through the identification discussed in Remark 9.1.(ii). From now on we write $D_ph : T_pM \to \mathbb{R}^m$ for this linear isomorphism, and similarly $D_{h(p)}h^{-1} : \mathbb{R}^m \to T_pM$ for its inverse.
Suppose $M^m \subseteq \mathbb{R}^l$ is a smooth submanifold with inclusion map $i : M \to \mathbb{R}^l$. Definition 8.8 implies that $D_pi : T_pM^m \to T_p\mathbb{R}^l \cong \mathbb{R}^l$ is injective. In this case we usually identify T_pM with the image in $\mathbb{R}^l$, consisting of all vectors $\alpha'(0)$ where $\alpha : I \to M \subseteq \mathbb{R}^l$ is a smooth parametrized curve with $\alpha(0) = p$.

For composite $\varphi \circ f$ of smooth maps $f : M \to N, \varphi : N \to P$ and a point $p \in M$ we have the chain rule, immediately from Lemma 9.3(i),

$$D_p(\varphi \circ f) = D_{f(p)}\varphi \circ D_p(\varphi). \tag{3}$$

Remark 9.4 Given a smooth chart (U, h) around the point $p \in M$ m we obtain a basis for T_pM

$$\left(\frac{\partial}{\partial x_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_m}\right)_p,$$

where $(\frac{\partial}{\partial x_i})_p$ is the image under $D_{h(p)}h^{-1} : \mathbb{R}^m \to T_pM$ of the i-th standard basis vector $e_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^m$. A tangent vector $X_p \in T_pM$ can be written uniquely as

$$X_p = \sum_{i=1}^m \left(\frac{\partial}{\partial x_i}\right)_p \tag{4}$$

where $\mathbf{a} = (a_1, \cdots, a_m) \in \mathbb{R}^m$. If $X_p = [\alpha]$, where $\alpha : I \to U$ with $\alpha(0) = p$ is a representing smooth curve, we have

$$\mathbf{a} = (h \circ \alpha)'(0)$$

Given $f \in C^\infty(M, \mathbb{R})$ we have the tangent map

$$D_pf : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R} \tag{5}$$

The *directional derivative* $X_pf \in \mathbb{R}$ is defined to be the image in $\mathbb{R}$ of X_p under (5), i.e. $X_pf = (f \circ \alpha)'(0)$. In terms of $f \circ h^{-1}$ we have by the chain rule

$$X_pf = \frac{d}{dt}\left(fh^{-1} \circ h\alpha(t)\right)_{|t=0} = \sum_{i=1}^m \frac{\partial fh^{-1}}{\partial x_i}(h(p))_{a_i}$$

In particular

$$\left(\frac{\partial}{\partial x_j}\right)_p f = \frac{\partial fh^{-1}}{\partial x_j}(h(p)) \tag{6}$$

Under the assumptions of Lemma 9.3.(ii) there is a similar basis $(\partial/\partial_{y_i})_{f(p)}$, $1 \leqslant j \leqslant n$, for $T_{f(p)}N$ and the D_pf with respect to our bases for T_pM and $T_{f(p)}M$ is the Jacobian matrix at $h(p)$ of $g \circ f \circ h^{-1}$. Specializing this to the case $N = M$, $f = \mathrm{id}_M$ we find that

$$\left(\frac{\partial}{\partial x_i}\right)_p = \sum_{j=1}^m \frac{\partial \varphi_j}{\partial x_j}(h(p))\left(\frac{\partial}{\partial y_j}\right)_p. \tag{7}$$

This holds when (U, h) and (V, g) are two charts around p with transition function

$$(\varphi_1, \cdots, \varphi_m) = \varphi = g \circ h^{-1}$$

expressing the y_j-coordinates in terms of the x_i-coordinates.

Suppose X is a function that to each $p \in M$ assigns a tangent vector $X_p \in T_pM$. Given the chart (U, h), formula (4) holds for $p \in U$ with certain coefficient functions $a_i : U \to \mathbb{R}$. If these are smooth in a neighborhood of $p \in U$, X is said to be smooth at p. From (7), this condition is independent of the choice of smooth chart around p. If X is smooth at every point $p \in M$, X is a a smooth vector field on M.
Let us next consider families $\omega = \{\omega_p\}_{p\in M}$ of alternating k-forms on T_pM, where $\omega_p \in \mathrm{Alt}^k(T_pM)$. We need a notion of w being smooth as function of p. Let $g : W \to M$ be a local parametrization, i.e. the inverse of a smooth chart, where W is an open set in $\mathbb{R}^m$. For $x \in W$,

$$D_x g : \mathbb{R}^m \to T_{g(x)}M$$

is an isomorphism, and induces an isomorphism

$$\mathrm{Alt}^k(D_x g) : \mathrm{Alt}^k(T_{g(x)}M) \to \mathrm{Alt}^k(\mathbb{R}^m).$$

We define $g^*(\omega) : W \to \mathrm{Alt}^k(\mathbb{R}^m)$ to be the function whom value at x is

$$g^*(\omega)_x = \mathrm{Alt}^k(D_x g)(\omega_{g(x)}) \qquad (g^*(\omega)_x = \omega_g(x) \text{ for } k = 0)$$

Definition 9.5 A family $\omega = \{\omega_p\}_{p\in M}$ of alternating k-forms on T_pM is said to be smooth if $g^*(\omega)$ is a smooth function for every local parametrization. The set of such smooth families is a vector space $\Omega^k(M)$. In particular, $\Omega^0(M) = C^\infty(M, \mathbb{R})$.

Lemma 9.6 Let $g_i : W_i \to N$ be a family of local parametrizations with $N = \cup_{g_i}(\omega)$ is smooth for all i, then ω is smooth.

Proof. Let$g : W \to N$ be any local parametrization. and $z \in W$. We show that $g^*(w)$ is smooth close to z. Choose an index i with $g(z) \in g_i(W_i)$. Close to z we can write $g = g_i \circ g = g_i \circ h$, where $h = g_i^{-1} \circ g(g_i(W_i)) \to W_i$ is a smooth map between open sets in Euclidean space. Thus

$$g^*(\omega) = (g_i \circ h)^*(\omega) = h^*(g_i^*(\omega))$$

in a neighborhood of z, and the right-hand side is a smooth k-form by assumption. □

The exterior differential

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

can be defined via local parametrizations $g : W \to M$ as follows. If $\omega = \{\omega_p\}_{p\in M}$ is a smooth k-form on M then

$$d_p\omega = \mathrm{Alt}^{k+1}((D_x g)^{-1}) \circ d_x(g^*\omega), \qquad p = g(x) \tag{8}$$

It is not immediately obvious that $d_p\omega$ is independent of the choice of local parametrization, but this is indeed the case: Given a local parametrization g, then any other locally has the form $g \circ \phi$, with $\phi : U \to W$ a diffeomorphism. Let $\xi_1, \cdots, \xi_{k+1} \in T_pM$. We choose $v_1, \cdots, v_{k+1} \in \mathbb{R}^n$ so that $D_x(g \circ \phi)(v_i) = \xi_i$. We must show that

$$d_y g^*(\omega)(w_1, \cdots, w_{k+1}) = d_x(g \circ \phi)^*(\omega)(v_1, \cdots, v_{k+1})$$

where $\phi(x) = y$ and $D_x\phi(v_i) = w_i$. This follows from the equations

$$\begin{aligned}(g \circ \phi)^* &= \phi^*(g^*(\omega)) \\ d\phi^*(\tau) &= \phi^* \, d(\tau)\end{aligned}$$

where $\tau = g^*(\omega)$; see Theorem 3.12. It is obviously that $d \circ d = 0$. Hence we have defined a chain complex

$$\cdots \to \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots,$$

We have $\Omega^k(M) = 0$ if $k > \dim(M)$, since $\mathrm{Alt}^k(T_pM) = 0$ when $k > \dim T_p(M)$. A smooth map $\phi : M \to N$ induces a chain map $\phi^* : \Omega^*(N) \to \Omega^*(M)$,

$$\phi^*(\tau)_p = \mathrm{Alt}^k(D_p\phi)(\tau_{\phi(p)}), \tau \in \Omega^k(N); \qquad \phi^*(\tau)_p = \tau_{\phi(p)} \text{ if } k = 0. \tag{9}$$

One defines a bilinear product $\omega \wedge \tau$ by $(\omega \wedge \tau)_p = \tau_{\phi(p)}$,

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+1}(M). \tag{10}$$

One shows by choosing local parametrizations that $\phi^*\omega$ and $\omega \wedge \tau$ are smooth. It is equally easy to see that

$$\begin{aligned} d(\omega \wedge \tau) &= d\omega + (-1)^k \omega \wedge d\tau \\ \omega \wedge \tau &= (-1)^{kl} \tau \wedge \omega \end{aligned} \tag{11}$$

for $\omega \in \Omega^k(M), \tau \in \Omega^l(M)$.

Definition 9.7 The p-th (de Rham) cohomology of the manifold M, denoted $H^p(M)$, is the p-th cohomology vector space of $\Omega^*(M)$.

The exterior product induces a product $H^p(M)xH^q(M) \to H^{p+q}(M)$ which makes $H^*(M)$ into a graded algebra. Note that $H^p(M) = 0$ for $p > n = \dim M$ or $p < 0$.
The chain map ϕ^* induced by a smooth map $\phi : M \to N$ induces linear maps

$$H^p(\phi) : H^p(N) \to H^p(M)$$

and the de Rham cohomology becomes a contravariant functor from the category of smooth manifolds and smooth maps to the category of graded anti-commutative $\mathbb{R}$-algebras.

Definition 9.8

(i) A smooth manifold M^n of dimension n is called orientable, if there exists an $\omega \in \Omega^n(M^n)$ with $\omega_p \neq 0$ for all $p \in M$. Such an ω is called an orientation form on M.

(ii) Two orientation forms ω, τ on M are equivalent if $\tau = f \cdot \omega$, for some $f \in \Omega^0(M)$ with $f(p) > 0$ for all $p \in M$. An orientation of M is an equivalence class of orientation forms on M.

On the Euclidean space IR n we have the orientation form $dx_1 \wedge \cdots \wedge dx_n$, which represents the standard orientation of $\mathbb{R}^n$.
Let M^n be oriented by the orientation form ω. A basis $b_1, \cdots, b_n$ of T_pM is said to be positively or *negatively oriented* with respect to ω depending on whether the number

$$\omega_p(b_1, \cdots, b_n) \in \mathbb{R}$$

is positive or negative. (It cannot be 0, because $\omega_p \neq 0$.) The sign depends only on the orientation determined by ω. If ω and τ are two orientation forms on M^n, then $\tau = f \cdot \omega$ for a uniquely determined function $f \in \Omega^0(M)$ with $f(p) \neq 0$ for all $p \in M$. We say that wand τ determine the same orientation at p, if $f(p) > 0$. Equivalently, ω and τ induce the same positively oriented bases of T_pM. If M is connected, then f has constant sign on M, so we have:

Lemma 9.9 On a connected orientable smooth manifold there are precisely 2 orientations.

If V is an open subset of an oriented manifold M^n, then an orientation of V is induced by using the restriction of an orientation form on M. Conversely we have:

Lemma 9.10 Let $\mathcal{V} = (V_i)_{i \in I}$ be an open cover of the smooth submanifold M^n. Suppose that all V_i have orientations and that the restrictions of the orientations from V_i and V_j to $V_i \cap V_j$ coincide for all $i \neq j$. Then M has a uniquely determined orientation with the given restriction to V_i for all $i \in I$.

The proof is a typical application of a smooth partition of unity in the following form:

Theorem 9.11 Let $\mathcal{V} = (V_i)_{i \in I}$ be an open cover of the smooth manifold $M^n \subseteq \mathbb{R}^n$. Then there exist smooth functions $\phi_i : M \to [0, 1] (i \in I)$ that satisfy

(i) $\mathrm{Supp}_M(\phi_i) \subseteq V_i$ for all $i \in I$.

(ii) Every $p \in M$ has an open neighborhood where only finitely many of the functions $\phi_i (i \in I)$ do not vanish.

(iii) For every $p \in M$ we have $\sum_{i \in I} \phi_i(p) = 1$.

PROOF. Since M has the topology induced by $\mathbb{R}^n$, we can choose an open set $U_i \subseteq \mathbb{R}^l$ with $U_i \cap M = Vi$ for each $i \in I$. By applying Theorem A.1 to $U = \cup_{i \in I} U_i$ we get smooth functions $\psi_i : U \to [0, 1]$ with

(i) $\mathrm{Supp}_U(\psi_i) \subseteq U_i$.

(ii) Local finiteness.

(iii) For every $x \in U$, $\sum_{i \in I} \psi_i(x) = 1$.

Let $\phi_i : M \to [0,1]$ be the restriction of ψ_i; conditions (i), (ii) and (iii) of the theorem follow immediately. □

Proof of Lemma 9.10. Let the orientation of V_i be given by the orientation form $\omega_i \in \Omega^n(V_i)$, and choose smooth functions $\phi_i : M \to [0,1]$ as in Theorem 9.11. We can define $\omega \in \Omega^n(M)$ by

$$\omega = \sum_{i \in I} \phi_i \omega_i$$

where $\phi_i \omega_i$ is extended to an n-form on all of M by letting it vanish on $M - \mathrm{Supp}_M(\phi_i)$. This is an orientation form, because if $p \in V_i \subseteq M$ and $b_1, \cdots, b_n$ is a basis of T_pM, with $\omega_{i,p}(b_1, \cdots, b_n) > 0$, then $\omega_{i',p}(b_1, \cdots, b_n) > 0$ for every other if with $p \in V_i$, and in the formula

$$\omega_p(b_1, \cdots, b_n) = \sum_i \phi_i(p)\omega_{i,p}(b_1, \cdots, b_n)$$

all terms are positive (or zero). Thus ω is an orientation form on M, and $b_1, \cdots, b_n$ are positively oriented with respect to ω. The orientation of M determined by ω has the desired property.
If $\tau \in \Omega^n(M^n)$ is another orientation form that gives the orientation of the required type, then $\tau = f \cdot \omega$ and $\tau_p(b_1, \cdots, b_n) = f(p)\omega_p(b_1, \cdots, b_n)$. Since both τ_p and wp are positive on $b_1, \cdots, b_n$ we have $f(p) > 0$. Hence ω and τ detennine the same orientation. □

Definition 9.12 Let $\phi : M_1^n \to M_2^n$ be a diffeomorphism between manifolds that are oriented by the orientation forms $\omega_j \in \Omega^n(M_j^n)$. Then $\phi^*(\omega_2)$ is an orientation form on M_1^n. We say ϕ is orientation-preserving (resp. orientation-reversing), when $\phi^*(\omega_2)$ detennines the same orientation of M_1^n as ω_1 (resp. $-\omega_1$).

Example 9.13 Consider a diffeomorphism $\phi : U_1 \to U_2$ between open subsets U_1, U_2 of $\mathbb{R}^n$, both equipped with the standard orientation of $\mathbb{R}^n$. It follows from Example 3.13.(ii) that ϕ is orientation-preserving if and only if $\det(D_x\phi) > 0$ for all $x \in U_1$. Analogously ϕ is orientation-reversing if and only if all Jacobi detenninants are negative.

Around any point on an oriented smooth manifold M^n we can find a chart $h : U \to U'$ such that h is an orientation-preserving diffeomorphism when U is given the orientation of M and U' the orientation of $\mathbb{R}^n$. We call h an oriented chart of M. The transition function associated with two oriented charts of M is an orientation-preserving diffeomorphism. For any atlas of M consisting of oriented charts, all Jacobi detenninants of the transition functions will be positive. Such an atlas is called positive.

Proposition 9.14 If $\{h_1 : U_1 \to U_i'\}$ is a *positive atlas* on M^n, then M^n has a uniquely determined orientation, so all hi are oriented charts.

PROOF. For $i \in I$ we orient U_i so that h_i is an orientation-preserving diffeomorphism. By Example 9.13, the two orientations on $U_i \cap U_j$ defined by the restriction from U_i and U_j coincide. The assertion follows from Lemma 9.10. $\square$

Definition 9.15 A *Riemannian structure* (or Riemannian metric) on a smooth manifold M^n is a family of inner products $\langle , \rangle_p$ on T_pM, for all $p \in M$, that satisfy the following condition: for any local parametrization $f : W \to M$ and any pair $v_1, v_2 \in \mathbb{R}^n$,

$$x \to \langle D_x f(v_1), D_x f(v_2) \rangle_{f(x)}$$

is a smooth function on W.

It is sufficient to have the smoothness condition satisfied for the functions

$$g_{ij}(x) = \langle D_x f(e_i), D_x f(e_j) \rangle_{f(x)}, \qquad 1 \leqslant i, j \leqslant n.$$

where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$. These functions are called the coefficients of the first fundamental form*fundamental form, first*. For $x \in W$ the $n \times n$ matrix $g_{ij}(x)$ is symmetric and positive definite.
A smooth manifold equipped with a Riemannian structure is called a Riemannian manifold. A smooth submanifold $M^n \subseteq \mathbb{R}^l$ has a Riemannian structure defined by letting $\langle , \rangle_p$ be the restriction to the subspace $T_pM \subseteq \mathbb{R}^l$ of the usual inner product on $\mathbb{R}^l$.

Proposition 9.16 If M^n is an oriented *Riemannian manifold* then M^n has a uniquely determined orientation form vol_M with

$$\mathrm{vol}_M(b_1, \cdots, b_n) = 1$$

for every positively oriented orthonormal basis of a *tangent space* T_pM. We call vol_M the volume form on M.

PROOF. Let the orientation be given by the orientation form $\omega \in \Omega^n(M^n)$. Consider two positively oriented orthonormal bases $b_1, \cdots, b_n$ and $b_1', \cdots, b_n'$ in the same tangent space T_pM. There exists an orthogonal $n \times n$ matrix $C = (C_{ij})$ such that

$$b_i' = \sum_{j=1}^{n} e_{ij} b_j,$$

and $\omega_p \in \mathrm{Alt}^n\, T_pM$ satisfies

$$\omega_(b_1', \cdots, b_n') = (\det C)\omega_p(b_1, \cdots, b_n). \tag{12}$$

Positivity ensures that det $C > 0$; but then $\det C = 1$. Hence there exists a function $\rho : M \to (0, \infty)$ such that $\rho(p) = \omega_p(b_1, \cdots, b_n)$ for every positively oriented orthonormal basis $b_1, \cdots, b_n$ of T_pM. We must show that ρ is smooth; then $\mathrm{vol}_M = \rho^{-1}\omega$ will be the volume form.
Consider an orientation-preserving local parametrization $f : W \to M^n$ ans set

$$X_j(q) = \left(\frac{\partial}{\partial x_j}\right)_q = D_q f(e_j) \in T_{f(q)}M \qquad \text{for } 1 \leqslant j \leqslant n \text{ and } q \in W.$$

These form a *positively oriented basis* of $T_{f(q)}M$. An application of the Gram-Schmidt orthonormalization process gives an upper triangular matrix $A(q) = (a_{ij}(q))$ of smooth functions on W with $a_{ii}(q) > 0$, such that

$$b_i(q) = \sum_{j=1}^{n} a_{ij}(q)X_j(q), \qquad i = 1, \cdots, n. \tag{13}$$

is a positively oriented orthonormal basis of $T_{f(q)}M$. Then

$$\begin{aligned} \rho \circ f(q) &= \omega_{f(q)}(b_1(1), \cdots, b_n(q)) = (\det A(q))\omega_{f(q)}(X_1(q), \cdots, X_n(q)) \\ &= (\det A(q))(f^*\omega)_q(e_1, \cdots, e_n). \end{aligned} \tag{14}$$

This shows that ρ is smooth. □

Addendum 9.17 There is the following formula for vol_M in local coordinates:

$$f^*(\mathrm{vol}_M) = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n. \tag{15}$$

PROOF. Repeat the proof above starting with $W = \mathrm{vol}_M$, so that $\rho(p) = 1$ for all $p \in M$. Formula (14) becomes

$$f^*(\mathrm{vol}_M) = (\det A(x))^{-1}\, dx_1 \wedge \cdots \wedge dx_n. \tag{16}$$

The inner product of (13) with the corresponding formula for $b_k(q)$ yields (with a Kronecker delta notation)

$$\delta_{ik} = \langle b_i(q), b_k(q)\rangle_{f(q)} = \sum_{j=1}^{n}\sum_{l=1}^{n} a_{ij}(q)g_{jl}(q)a_{kl}(q)$$

This is the matrix identity, $I = A(q)G(q)A(q)^t$, where $G(q) = (g_{jl}(q))$. In particular $(\det A(q))^2 \det G(q) = 1$. Since $\det A(q) = \prod_i a_{ii}(q) > 0$, we obtain

$$(\det A(q))^{-1} = \sqrt{\det G(q)}.$$

□

Example 9.18 Define an $(n-1)$-form $\omega_0 \in \Omega^{n-1}(\mathbb{R}^n)$ by

$$\omega_{0x}(w_1, \cdots, w_{n-1}) = \det(x, w_1, \cdots, w_{n-1}) \in \mathrm{Alt}^{n-1}(\mathbb{R}^n) \tag{17}$$

for $x \in \mathbb{R}$. Since $\omega_{0x}(e_1, \cdots, \hat{e}_i, \cdots, e_n) = (-1)^{i-1}x_i$, we have

$$\omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \, dx_n. \tag{18}$$

If $x \in S^{n-1}$ and $w_1, \cdots, w_{n-1}$ is a basis of $T_x S^{n-l}$ then $w, w_1, \cdots, w_{n-1}$ becomes a basis for $\mathbb{R}^n$ and (17) shows that $\omega_{0x} \neq 0$. Hence $\omega_{0|S^{n-1}} = i^*(\omega_0)$ is an orientation form on S^{n-1}. For the orientation of S^{n-1} given by ω_0, the basis $w_1, \cdots, w_{n-1}$ of $T_x S^{n-1}$ is positively oriented if and only if the basis $w, w_1, \cdots, w_{n-1}$ for $\mathbb{R}^n$ is positively oriented.

We give S^{n-l} the Riemannian structure induced by $\mathbb{R}^n$. Then (17) implies that $\mathrm{vol}_{S^{n-1}} = \omega_{0|S^{n-1}}$.
We may construct a closed $(n-1)$-form on $\mathbb{R}^n - \{0\}$ with $\omega_{|S^{n-1}} = \mathrm{vol}_{S^{n-1}}$ by setting $\omega = r^*(\mathrm{vol}_{S^{n-1}})$, where $r : \mathbb{R}^n - \{0\} \to S^{n-1}$ is the map $r(x) = x/\|x\|$. For $x \in \mathbb{R}^n - \{0\}$, $\omega_x \in \mathrm{Alt}^{n-1}(\mathbb{R}^n)$ is given by

$$\begin{aligned}\omega_x(v_1, \cdots, v_{n-1}) &= \omega_{0r(x)}(D_x r(v_1), \cdots, D_x r(v_{n-1})) \\ &= \|x\|^{-1} \det(x, \underset{x}{\det}\, r(v_1), \cdots, D_x r(v_{n-1})).\end{aligned}$$

Now we have

$$D_x(r(v)) = \begin{cases} 0 & \text{if } v \in \mathbb{R}x \\ \|x\|^{-1} & \text{if } v \in (\mathbb{R}x)^{\perp} \end{cases}$$

so that $D_x r(v) = \|x\|^{-1} w$, where w is the orthogonal projection of v on $(\mathbb{R}x)^{\perp}$. Letting w_i be the orthogonal projection of v_i on $(\mathbb{R}x)^{\perp}$ we have

$$\begin{aligned}\omega_x(v_1, \cdots, v_{n-1}) &= \|x\|^{-n} \det(x, w_1, \cdots, w_{n-1}) = \|x\|^{-n} \det(x, v_1, \cdots, v_{n-1}) \\ &= \|x\|^{-n} \omega_{0x}(v_1, \cdots, v_{n-1}).\end{aligned}$$

Hence the closed form ω is given by

$$\omega = \frac{1}{\|x\|^n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \, dx_n \tag{19}$$

Example 9.19 For the *antipodal map*

$$A : S^{n-1} \to S^{n-1}; \qquad Ax = -x$$

we have

$$A^*(\mathrm{vol}_{S^{n-1}}) = (-1)^n \mathrm{vol}_{S^{n-1}}$$

and A is orientation-preserving if and only if n is even. In this case we get an orientation form τ on $\mathbb{RP}^{n-1}$ such that $\pi^*(\tau) = \mathrm{vol}_{S^{n-1}}$, where π is the canonical map $\pi : S^{n-1} \to \mathbb{RP}^{n-1}$. For $x \in S^{n-1}$,

$$T_x S^{n-1} \xrightarrow{D_x A} T_{Ax} S^{n-1}$$

is a linear isometry. Hence there exists a Riemannian structure on $\mathbb{RP}^{n-1}$ characterized by the requirement that the isomorphism

$$T_x S^{n-1} \xrightarrow{D_x \pi} T_{\pi(x)} \mathbb{RP}^{n-1}$$

is an isometry for every $x \in S^{n-1}$. If n is even and $\mathbb{RP}^{n-1}$ is oriented as before, one gets $\pi^*(\mathrm{vol}_{\mathbb{RP}^{n-1}}) = \mathrm{vol}_{S^{n-1}}$. Conversely suppose that $\mathbb{RP}^{n-1}$ is orientable, $n \geqslant 2$. Choose an orientation and let $\mathrm{vol}_{\mathbb{RP}^{n-1}}$ be the resulting volume form. Since $D_x\pi$ is an isometry, $\pi^*(\mathrm{vol}_{\mathbb{RP}^{n-1}})$ must coincide with $\pm\mathrm{vol}_{S^{n-1}}$ in all points, and by continuity the sign is locally constant. Since S^{n-1} is connected the sign is constant on all of S^{n-1}. We thus have that

$$\pi^*(\mathrm{vol}_{\mathbb{RP}^{n-1}}) = \delta \mathrm{vol}_{S^{n-1}}$$

where $\delta = \pm 1$. We can apply A^* and use the equation $\pi \circ A = \pi$ to get

$$\begin{aligned}(-1)^n \delta \mathrm{vol}_{S^{n-1}} &= \delta A^*(\mathrm{vol}_{S^{n-1}}) = A^*\pi^*(\mathrm{vol}_{\mathbb{RP}^{n-1}}) \\ &= (\pi \circ A)(\mathrm{vol}_{\mathbb{RP}^{n-1}}) = \pi^*(\mathrm{vol}_{\mathbb{RP}^{n-1}}) = \delta \mathrm{vol}_{S^{n-1}}.\end{aligned}$$

This requires that n is even and implies that $\mathbb{RP}^{n-1}$ is orientable if and only if n is even.

Remark 9.20 For two smooth manifolds M^m and N^n the Cartesian product $M^m \times N^n$ is a smooth manifold of dimension $m+n$. For a pair of charts $h : U \to U'$ and $K : V \to V'$ of M and N, respectively, we can use $h \times k : U \times V \to U \times V'$ as a chart of $M \times N$. These product charts form a smooth atlas on $M \times N$. For $p \in M$ and $q \in N$ there is a natural isomorphism

$$T_{(p,q)}(M \times N) \cong T_p M \oplus T_q N.$$

If M and N are oriented, one can use oriented charts (V, h) and (V, k). The transition diffeomorphisms between the charts $(V \times V, h \times k)$ satisfy the condition of Proposition 9.14. Hence we obtain a *product orientation* of $M \times N$. If the orientations are specified by orientation forms $\omega \in \Omega^m(M)$ and $\sigma \in \Omega^n(N)$, the product orientation is given by the orientation form $\mathrm{pr}_M^*(\omega) \wedge \mathrm{pr}_N^*(\sigma)$, where pr_M and pr_N are the projections of $M \times N$ on M and N.

In the following we shall consider a smooth submanifold $M^n \subseteq \mathbb{R}^{n+k}$ of dimension n. At every point $P \in M$ we have a normal vector space $T_p M^\perp$ of dimension k. A smooth *normal vector field* Y on an open set $W \subseteq M$ is a smooth map $Y : W \to \mathbb{R}^{n+k}$ with $Y(p) \sin T_p M^\perp$ for every $p \in W$. In the case $k = 1$, Y is called a *Gauss map* on W when all $Y(p)$ have length 1. Such a map always exists locally since we have the following:

Lemma 9.21 For every $p_0 \in M^n \subseteq \mathbb{R}^{n+k}$ there exists an open neighborhood W of p_0 on M and smooth normal vector fields $Y_j (1 \leqslant j \leqslant k)$ on W such that $Y_1(p), \cdots, Y_k(p)$ form an orthonormal basis of $T_pM^\perp$ for every $p \in W$.

PROOF. On a coordinate patch around $p_0 \in M$, there exist smooth tangent vector fields $X_1, \cdots, X_n$, which at every point p yield a basis of T_pM, cf. Remark 9.4. Choose a basis $V_1, \cdots, V_k$ of $T_{p_0}M^\perp$. Since the $(n+k) \times (n+k)$ determinant

$$\det(X_1(p), \cdots, X_n(p), V_1, \cdots, V_k)$$

is non-zero at p_0, it also non-zero for all p in some open neighborhood W of p_0 on M. Gram-Schmidt orthonormalization applied to the basis

$$X_1(p), \cdots, X_n(p), V_1, \cdots, V_k \qquad (p \in W)$$

of $\mathbb{R}^{n+k}$ gives an orthonormal basis

$$\tilde{X}_1(p), \cdots, \tilde{X}_n(p), Y_1(p), \cdots, Y_k(p),$$

where the first n vectors span T_pM. The formulas of the Gram-Schmidt orthonormalization show that all X_i and Y_j are smooth on W, so that $Y_1, \cdots, Y_k$ have the desired properties. □

Proposition 9.22 Let $M^n \subseteq \mathbb{R}^{n+1}$ be a smooth submanifold of codimension 1.

(i) There is a 1-1 correspondence between smooth normal vector fields Y on M and n-forms in $\Omega^n M$. It associates to Y the n-form $\omega = \omega_Y$ given by

$$\omega_p(W_1, \cdots, W_n) = \det(Y(p), W_1, \cdots, W_n)$$

for $p \in M, W_i \in T_pM$.

(ii) This induces a 1-1 correspondence between Gauss maps $Y : M \to S^n$ and orientations of M.

PROOF. If $p \in M$ then $Y(p) = 0$ if and only if $\omega_p = 0$. Since ω_Y depends linearly on Y, the map $Y \to \omega_Y$ must be injective. If Y is a Gauss map, then ω_Y is an orientation form and it can be seen that ω_Y is exactly the *volume form* associated to the orientation determined by ω_Y and the Riemannian structure on M induced from $\mathbb{R}^{n+1}$. If M has a Gauss map Y then (i) follows, since every element in $\Omega^n(M)$ has the form $f \cdot \omega_Y = \omega_{fY}$ for some $f \in C^\infty(M, \mathbb{R})$. Now M can be covered by open sets, for which there exist Gauss maps. For each of these (i) holds, but then the global case of (i) automatically follows.
An orientation of M determines a volume form vol_M and from (i) one gets a Y with $\omega_Y = \mathrm{vol}_M$. This Y is a Gauss map. □

Theorem 9.23 (Tubular neighborhoods) Let $M^n \subseteq \mathbb{R}^{n+k}$ be a smooth submanifold. There exists an open set $V \subseteq \mathbb{R}^{n+k}$ with $M \subseteq V$ and an extension of id_M to a smooth map $r : V \to M$, such that

(i) For $x \in V$ and $y \in M$, $\|x - r(x)\| \leqslant \|x - y\|$, with equality if and only if $y = r(x)$.

(ii) For every $p \in M$ the fiber $r^{-1}(p)$ is an open ball in the affine subspace $p + T_pM^\perp$ with center at p and radius $\rho(p)$, where ρ is a positive smooth function on M. If M is compact then ρ can be taken to be constant.

(iii) If $\epsilon : M \to \mathbb{R}$ is smooth and $0 < \epsilon(p) < \rho(p)$ for all $p \in M$ then

$$S_\epsilon = \{x \in V \big| \|x - r(x)\| = \epsilon(r(x))\}$$

is a smooth submanifold of codimension 1 in $\mathbb{R}^{n+k}$.

We call $V(= V_\rho)$ the open tubular neighborhood of M of radius ρ.

PROOF. We first give a local construction around a point $p_0 \in M$. Choose normal vector fields $Y_1, \cdots, Y_k$ as in Lemma 9.21, defined on an open neighborhood W of p_0 in M for which we have a diffeomorphism $f : \mathbb{R}^n \to W$ with $f(0) = p_0$. Let us define $\Phi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ by

$$\Phi(x, t) = f(x) + \sum_{j=1}^{k} t_j Y_j(f(x)) \qquad (x \in \mathbb{R}^n, t \in \mathbb{R}^k).$$

The Jacobi matrix of Φ at 0 has the columns

$$\frac{\partial f}{\partial x_1}(0), \cdots, \frac{\partial f}{\partial x_n}(0), Y_1(p_0), \cdots, Y_k(p_0)$$

The first n form a basis of $T_{p_0}M$ and the last k a basis of $T_{p_0}M^\perp$. By the inverse function theorem, Φ is a local diffeomorphism around 0. There exists a (possibly) smaller open neighbourhood W_0 of p_0 in M and an $\epsilon_0 > 0$, such that

$$\Phi_0(p, t) = p + \sum_{j=1}^{k} t_j Y_j(p)$$

defines a diffeomorphism from $W_0 \times \epsilon_0 \mathring{D}^k$ to an open set $V_0 \subseteq \mathbb{R}^{n+k}$. The map $r_0 = \mathrm{pr}_{W_0} \circ \Phi_0^{-1}$ defines a smooth map $r_0 : V_0 \to W_0$, which extends id_{W_0} so that the fiber $r_0^{-1}(p)$ is the open ball in $p + T_pM^\perp$ with center at p and radius ϵ_0 for every $p \in W_0$ By shrinking ϵ_0 and cutting W_0 down we can arrange that the following condition holds:

$$\text{For } x \in V_0 \text{ and } y \in M \text{ we have } \|x - r_0(x)\| \leqslant \|x - y\| \tag{20}$$

with equality if and only if $y = r_0(x)$. This can be done as follows. By Definition 8.8 there exists an open neighborhood $\hat{W}$ of p_0 in $\mathbb{R}^{n+k}$ such that $M^n \cap \hat{W}$ is closed in $\hat{W}$. In the above we can ensure that $V_0 \subseteq \hat{W}$ where $M \cap V_0$ remains closed in V_0. Choose compact subsets $K_1 \subseteq K_2$ of W_0 so that (in the induced topology on M) $p_0 \in \mathrm{int} K_1 \subseteq K_1 \subseteq \mathrm{int} K_2$, where $\mathrm{int} K_i$ denotes the interior of K_i. The set

$$B = (\mathbb{R}^{n+k} - V_0) \cup (M \cap V_0 - \mathrm{int} K_2)$$

is closed in $\mathbb{R}^{n+k}$ and disjoint from K_1. There exists an $\epsilon \in (0, \epsilon_0]$ such that $\|b - y\| \geqslant 2\epsilon$ for all $b \in B$, $y \in K_1$. If we introduce the open set

$$V_0' = \{x \in V_0 \big| r_0(x) \in \mathrm{int} K_1 \text{ and } \|x - r_0(x)\| < \epsilon\},$$

we get for $x \in V_0'$ and $b \in M - K_1 \subseteq B$ that

$$\|x - b\| \geqslant \|b - r_0(x)\| - \|x - r_0(x)\| > \epsilon$$

Since $\|x - r_0(x)\| < \epsilon$, the function $y \to \|x - y\|$, defined on M, attains a minimum less that ϵ somewhere on the compact set K_2. Consider such a $y_0 \in K_2$ with

$$\|x - y_0\| = \min_{y \in M} \|x - y\| \leqslant \|x - r_0(x)\| < \epsilon \leqslant \epsilon_0.$$

Hence $x - y_0$ is a normal vector to M at y_0 (see Exercise 1), but $x \in V_0$ and $y_0 = r_0(x)$. This shows that Condition 20 can be satisfied by replacing (W_0, V_0, ϵ_0) by $(\mathrm{int} K_1, V_0', \epsilon)$.
Now all of M can be covered with open sets of the type V_0 with the associated smooth maps r_0 which satisfy Condition (20). If (V_1, r_1) is a different pair then r_0 and r_1 will coincide on $V_0 \cap V_1$. On the union V' of all such open sets (of the type above) we can now define a smooth map $r : V' \to M$, which extends id_M, such that part (i) of the theorem is satisfied. We have $r_{W_0} = r_0$ for every local (V_0, r_0). If in the above we always choose $\epsilon_0 \leqslant 1$, then the fiber $r^{-1}(p)$ over a $p \in M$ will be an open ball in $p + T_p M^{\perp}$ with radius $\hat{\rho}(p)$ for which $0 < \hat{\rho}(p) \leqslant 1$. Thus we have satisfied (i) and (ii), except that $\hat{\rho}$ might be discontinuous.
The distance function from M,

$$d_M(x) = \inf_{y \in M} \|x - y\|,$$

is continuous on all of $\mathbb{R}^{n+k}$. For $x \in V'$, (i) shows that

$$d_m(x) = \|x - r(x)\| \qquad (x \in V')$$

If $p \in M$ and $r \in T_p M^{\perp}$ ahs distance $\hat{\rho}(p)$ from p, we can conclude that $d_m(x) = \hat{\rho}(p)$. In this case (i) exclusion that $x \in V'$, so x lies on the boundary of V'. Hence the distance function

$$d : M \to \mathbb{R}; \qquad d(p) = \inf_{x \notin V'} \|p - z\|$$

satisfies $0 < d(p) \leqslant \hat{\rho}(p)$ and is continuous.

By Lemma A.3 the function $\frac{1}{2}d \circ r : V' \to \mathbb{R}$ can be approximated by a smooth function $\psi : V' \to \mathbb{R}$ such that

$$\|\psi(x) - \frac{1}{2}\, d \circ r(x)\| \leqslant \frac{1}{4}\, d \circ r(x)$$

for all $x \in V'$. In particular

$$\frac{1}{4}\, d(x) \leqslant \psi(x) \leqslant \frac{3}{4}\, d(x) \qquad \text{when } x \in M$$

Hence the restriction $\rho = \psi_{|M} : M \to \mathbb{R}$ is a positive smooth function with $\rho(p) < \hat{\rho}(p)$ for all $p \in M$. When M is compact, the same can be archieved for the constant function which takes the value $\rho = \frac{1}{2}d(p)$. If we define

$$V = \{x \in V' \big| \|x - r(x)\| < \rho(r(x))\|\}$$

both (i) hold for the restriction of r to V.
It ramians to prove (iii). It is sufficient to show that $S_\epsilon \cap V_0$ is empty or a smooth submanifold of V_0 of codimension 1. The image under the diffeomorphism $\Phi_0^{-1} : V_0 \to W_) \times \epsilon_0 \mathring{D}^k$ of $S_\epsilon \cap V_0$ is the set

$$S = \{(\rho, t) \in W_0 \times \mathbb{R}^k \big| \|t\| = \epsilon(p) < \epsilon_0\}.$$

The projection of S on W_0 is the open set

$$U = \{p \in W_0 \big| \epsilon(p) < \epsilon_0\} \subseteq M.$$

The diffeomorphism $\phi : U \times \mathbb{R}^k \to U \times \mathbb{R}^k$ given by $\phi(p, t) = (p, \epsilon(p)t)$ maps $U \times S^{k-1}$ to S. This yields (iii). $\square$

Remark 9.24 In Chapter 11 we will need additional information in the case where $M^n \subseteq \mathbb{R}^{n+k}$ is compact and $\rho > 0$ is constant. To any number $\epsilon, 0 < \epsilon < \rho$, we define the closed tubular neighborhood of radius ϵ: around M by

$$N_\epsilon = \{x \in V \big| \|x - r(x)\| \leqslant \epsilon\}.$$

This set is the disjoint union of the closed balls in $p + T_p M^\perp$ with centers at p and radius ϵ. Note that N_ϵ is compact and that S_ϵ is the set of boundary points of N_ϵ in $\mathbb{R}^{n+k}$. By Theorem 9.23.(i) we see for $p \in M$ that the real-valued function on $S_\epsilon : x \to \|x - p\|$, attains its minimum value ϵ: at all points $x \in S_\epsilon$ with $r(x) = p$. It follows that

$$x - r(x) \in T_x S_\epsilon^\perp. \tag{21}$$

We end this chapter with a few applications of the existence of tubular neighborhoods. Let (V, i, r) be a tubular neighborhood of M with $i : M \to V$ the inclusion map and $r : V \to M$ the smooth retraction map such that $r \circ i = \mathrm{id}_M$.

In cohomology this gives

$$H^d(i) \circ H^d(r) = \mathrm{id}_{H^d(M)},$$

so that $H^d(i) : H^d(V) \to H^d(M)$ is surjective and $H^d(r) : H^d(M) \to H^d(V)$ is injective.

Proposition 9.25 For any compact differentiable manifold M^n all cohomology spaces $H^d(M)$ are finite-dimensional.

Proof. We may assume that M^n is a smooth submanifold of $\mathbb{R}^{n+k}$ by Theorem 8.11, and that (V, i, r) is a tubular neighborhood. Since M is compact we can find finitely many open balls $U_1, \cdots, U_r$ in $\mathbb{R}^{n+k}$ such that their union $U = U_1 \cup \cdots \cup U_r$ satisfies $M \subseteq U \subseteq V$. Now we have a smooth inclusion $i : M \to U$ and a smooth map $r_{|U} : U \to W$ with $r_{|U} \circ i = \mathrm{id}_M$. The argument above shows that

$$H^d(M)(i) : H^d(U) \to H^d(M)$$

is surjective, and the assertion now follows from Theorem 5.5. □

Proposition 9.26 Let M_1 and M_2 be smooth submanifolds of Euclidean spaces.

(i) If $f_0, f_1 : M_1 \to M_2$ are two homotopic smooth maps, then

$$H^d(f_0) = H^d(f_1) : H^d(M_2) \to H^d(M_1).$$

(ii) Every continuous map $M_1 \to M_2$ is homotopic to a smooth map.

Proof. Choose tubular neighborhoods (V_ν, i_ν, r_ν) of $M_\nu, \nu = 1, 2$. Lemma 6.3 implies that $i_2 \circ f_0 \circ r_1 \backsimeq i_2 \circ f_2 \circ r_1$. Hence $H^d(i_2 \circ f_0 \circ r_1) = H^d(i_2 \circ f_2 \circ r_1)$, so that

$$H^d(r_1) \circ H^d(f_0) \circ H^d(i_2) = H^d(r_1) \circ H^d(f_1) \circ H^d(i_2)$$

Since $H^d(r_1)$ is injective and $H^d(i_2)$ is surjective, we conclude that $H^d(f_0) = H^d(f_1)$. If $\phi : M_1 \to M_2$ is continuous, we can use Lemma 6.6.(i) to find a smooth map $g : V_1 \to V_2$ with $g \backsimeq i_2 \circ \phi \circ r_1$. For ther smooth map $f = r_2 \circ g \circ i_1 : M_1 \to M_2$, Lemma 6.3 shows that $f \backsimeq r_2 \circ (i_2 \circ \phi \circ r_1) \circ i_1 = \phi$. □

Remark 9.27 As in the discussion preceeding Theorem 6.8, the de Rham cohomology can now be made functorial on the category of smooth submanifolds of Euclidean space and continuous maps. Theorem 6.8 and Corollary 6.9 are valid (with the same proofs) with open sets in Euclidean space replaced by smooth submanifolds. By Theorem 8.11 the same can be done for differentiable manifolds in general.

Corollary 9.28 If $M^n \subseteq \mathbb{R}^{n+k}$ is a smooth submanifold and (V, i, r) an open inbular neighborhood, then $H^d(i) : H^d(V) \to H^d(M)$ is an isomorphism with $H^d(r)$ as its inverse.

Proof. We have $r \circ i = \mathrm{id}_M$ and $i \circ r \simeq \mathrm{id}_V$, as V contains the line segment between x and $r(x)$ for all $x \in V$. By Proposition 9.26.(i) we can conclude that $H^d(r)$ and $H^d(i)$ are inverses. $\square$

Example 9.29 For $n \geqslant 1$, we have

$$H^d(S^m) \cong \begin{cases} \mathbb{R} & \text{if } d = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Let $i : S^n \to \mathbb{R}^{n+1} - \{0\}$ be the inclusion and define $\mathbb{R}^{n+1} - \{0\} \to S^n$ by $r(x) = \frac{x}{\|x\|}$. Then $r \circ i = \mathrm{id}_{S^n}, i \circ r \simeq \mathrm{id}_{\mathbb{R}^{n+1}-\{0\}}$ and $H^d(i)$ is an isomorphism. The result follows Theorem 6.13.

Remark 9.30 Let U_1 and U_2 be open subsets of a smooth submanifold $M^n \subseteq S^n$. Using Theorem 9.11, the proof of Theorem 5.1 can be carried through without any significant changes. As in Chapter 5 this gives rise to the Mayer-Vietoris sequence

$$\to H^p(U_1 \cup U_2) \xrightarrow{I^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cup U_2) \xrightarrow{\partial^*} H^p(U_1 \cup U_2) \to$$

Example 9.31 We shall compute the de Rham cohomology of $\mathbb{RP}^{n-1}$ $(n \geqslant 2)$. With the notation od Example 9.19 we see that

$$\mathrm{Alt}^p(D_x A) : \mathrm{Alt}^p(T_{\pi(x)}\mathbb{RP}^{n-1}) \to \mathrm{Alt}^p(T_x S^{n-1})$$

is an isomorphism for every $x \in S^{n-1}$. Therefore

$$\Omega^p(\pi) : \Omega^p(\mathbb{RP}^{n-1}) \to \Omega(S^{n-1})$$

is an monomorphism, and we find that the image of $\Omega^p(\pi)$ is equal to the set of p-form ω on S^{n-1} such that $A^*\omega = \omega$. Since $A^* = \Omega^p(A) : \Omega^p(S^{n-1}) \to \Omega^p(S^{n-1})$ has order 2 we can decompose it into (± 1)-eigenspaces

$$\Omega^p(S^{n-1}) = \Omega^p_+(S^{n-1}) \oplus \Omega^p_-(S^{n-1})$$

where

$$\Omega^p_\pm(S^{n-1}) = \mathrm{Im}(\frac{1}{2}(\mathrm{id} \pm \Omega^p(A)))$$

This in fact decomposes the de Rham complex of S^{n-1} into a direct sum of two subcomplexes

$$\Omega^*(S^{n-1}) = \Omega^*_+(S^{n-1}) \oplus \Omega^*_-(S^{n-1}). \tag{22}$$

There is an isomorphism of chain complexes

$$\Omega^*(\mathbb{RP}^{n-1}) \cong \Omega^*_+(S^{n-1}) \tag{23}$$

induced by $\pi : S^{n-1} \to \mathbb{RP}^{n-1}$. From (22) we get isomorphisms

$$\begin{aligned} H^p(S^{n-1}) &\cong H^p(\Omega^*_+(S^{n-1})) \oplus H^p(\Omega^p_-(S^{n-1})) \\ &\cong H^p(\Omega^p_+(S^{n-1})) \oplus H^p(\Omega^p_-(S^{n-1})) \end{aligned}$$

where $H^p_\pm(S^{n-1})$ is the (± 1)-eigenspaces of A^* on $H^p(S^{n-1})$. Combining with (23) we find that

$$H^p(\mathbb{RP}^{n-1}) \cong H^p_+(S^{n-1}). \tag{24}$$

There is a commutative diagram with vertical isomorphisms (See Example 9.29)

$$\begin{tikzcd} H^{n-1}(\mathbb{R}^n - \{0\}) \arrow[r] \arrow[d, "\cong"', "i^*"] & H^{n-1}(\mathbb{R}^n - \{0\}) \arrow[d, "\cong"', "i^*"] \\ H^{n-1}(S^{n-1}) \arrow[r, "A^*"] & H^{n-1}(S^{n-1}) \end{tikzcd}$$

where the top map is induced by the linear map $x \to -x$ of $\mathbb{R}^n$ into itself. Lemma 6.14 shows that the bottom map is multiplication by $(-1)^n$. Using (24) and Example 9.29 we finally get

$$H^p(\mathbb{RP}^{n-1}) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } p = n-1 \text{ with } n \text{ even} \\ 0 & \text{otherwise} . \end{cases} \tag{25}$$

10. INTEGRATION ON MANIFOLDS

Let M^n be an oriented n-dimensional smooth manifold. We define an integral

$$\int_M : \Omega_c^n(M^n) \to \mathbb{R}$$

on the vector space of differential n-forms with compact support. Next we shall consider integration on subsets of M^n, and Stokes' theorem will be proved. Finally we calculate $H^n(M^n)$ for an arbitrary orientable compact connected smooth manifold M^n.
In the special case where $M^n = \mathbb{R}^n$ (with the standard orientation) we can write $\omega \in \Omega_c^n(\mathbb{R}^n)$ in the form

$$\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n,$$

where $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ has compact support. We then define

$$\int_{\mathbb{R}^n} f(x)\, dx_1 \wedge \cdots \wedge dx_n = \int_{\mathbb{R}^n} f(x)\, d\mu_n$$

where $d\mu_n$ is the usual Lebesgue measure on $\mathbb{R}^n$. The same definition can be used when $\omega \in \Omega_c^n(V)$ for an open set $V \subseteq \mathbb{R}^n$, since ω and f are smoothly extendable to the whole of $\mathbb{R}^n$ by setting them equal to 0 on $\mathbb{R}^n - \mathrm{supp}_V(\omega)$.

Lemma 10.1 Let $\phi : V \to W$ be a diffeomorphism between open subsets V and W of $\mathbb{R}^n$, and assume that we the Jacobi determinant $\det(D_x\phi)$ is of constant sign $\delta = \pm 1$ for $x \in V$. For $\omega \in \Omega_c^n(W)$ we have that

$$\int_V \phi^*(\omega) = \delta \cdot \int_W \omega.$$

Proof. If ω is written in the form

$$\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$$

with $f \in C_c^\infty(W, \mathbb{R})$, it follows from Example 3.13.(ii) that

$$\begin{aligned} \phi^*(\omega) &= f(\phi(x)) \det(D_x\phi)\, dx_1 \wedge \cdots \wedge dx_n \\ &= \delta f(\phi(x)) |\det(D_x\phi)|\, dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

The assertion follows from the transformation theorem for integrals which states that

$$\int_W f(x)\, d\mu_n = \int_V f(\phi(x)) |\det(D_x\phi)|\, d\mu_n.$$

□

Proposition 10.2 For an arbitrary oriented n-dimensional smooth manifold M^n there exists a unique linear map

$$\int_M : \Omega_c^n(M^n) \to \mathbb{R}$$

with the following properties: If $\omega \in \Omega_c^n(M^n)$ has support contained in U, where (U, h) is a positively oriented C^∞-chart, then

$$\int_M \omega = \int_{h(U)} h^{-1}(\omega)^*. \tag{1}$$

PROOF. First consider $\omega \in \Omega_c^n(M^n)$ with "small" support, i.e. such that $\mathrm{supp}_M(\omega)$ is contained in a coordinate patch. Then (U, h) can be chosen as above and the integral is determined by (1). We must show that the right-hand side is independent of the choice of chart. Assume that $(\tilde{U}, \tilde{h})$ is another positively oriented C^∞-chart with $\mathrm{supp}_M(\omega) \subseteq \tilde{U}$.
The diffeomorphism $\phi : V \to W$ from $V = h(U \cap \tilde{U})$ to $W = \tilde{h}(U \cap \tilde{U})$ given by $\phi = \tilde{h} \circ h^{-1}$ has everywhere positive Jacobi determinant. Since

$$\mathrm{supp}_{h(U)}((h^{-1})^*(\omega)) \subseteq V, \qquad \mathrm{supp}_{\tilde{h}(\tilde{U})}((\tilde{h}^{-1})^*(\omega)) \subseteq W,$$

and $\phi^*(\tilde{h}^{-1})^*(\omega) = (h^{-1})^*(\omega)$, Lemma 10.1 shows that

$$\int_{h(u)} (h^{-1})^*(\omega) = \int_{\tilde{h}(\tilde{U})} (\tilde{h}^{-1})^*(\omega).$$

So for $\omega \in \Omega_c^n(M)$ with "small" support the integral defined by (1) is independent of the chart.
Now choose a smooth partition of unity $(\rho_\alpha)_{\alpha \in A}$ on M subordinate to an oriented C^∞-atlas on M. For $\omega \in \Omega_c^n(M)$ we have that

$$\omega = \sum_{\alpha \in A} \rho_\alpha \omega,$$

where every term $\rho_\alpha \omega \in \Omega_c^n(M)$ has "small" support, and where only finitly many terms are non-zero. We define

$$I(\omega) = \sum_{\alpha \in A} \int_M \rho_\alpha \omega,$$

where the term associated to $\alpha \in A$ is given by (1), applied to a U_α with $\mathrm{supp}_M(\rho_\alpha) \subseteq U_\alpha$. It is obvious that I is a linear operator on $\Omega_c^n(M)$. If, in particular, $\mathrm{supp}_M(\omega) \subseteq U$, where (U, h) is a positively oriented C^∞-chart, the terms of the sum can be calculated by (1), applied to (U, h). This yields

$$I(\omega) = \int_M \omega,$$

which shows that I is a linear operator with the desired properties. Uniqueness follows analogously. $\square$

Lemma 10.3

(i) $\int_M \omega$ changes sign when the orientation of M^n is reversed.

(ii) If $\omega \in \Omega_c^n(M^n)$ has support contained in an open set $W =\subset M^n$, then

$$\int_M \omega = \int_W \omega,$$

where W is the orientation induced by M.

(iii) If $\phi : N^n \to M^n$ is an orientation persevering diffeomorphism, then we have that

$$\int_M \omega = \int_N \phi^*(\omega)$$

for $\omega \in \Omega_c^n(M)$.

Proof. By a partition of unity, we can restrict ourselves to the case where $\mathrm{supp}_M(\omega)$ is contained in a coordinate patch. All three properties are now easy consequences of Lemma 10.1 and (1). □

Remark 10.4 In the above we could have considered integrals of continuous n-forms with compact support on M^n. If the orientation of M is given by the orientation form $\sigma \in \Omega^n(M)$, a continuous n-form can be written uniquely as $f\sigma$, where $f \in C^0(M, \mathbb{R})$. The support of $f\sigma$ is equal to the support of f. The integral of (1) extended to continuous n-forms gives rise to a linear operator

$$I_\sigma : C_c^n(M, \mathbb{R}) \to \mathbb{R}; \qquad I_\sigma(f) = \int_M f\sigma.$$

This linear operator is positive, i.e. $I_\sigma(f) \geqslant 0$ for $f \geqslant 0$. By a partition of unity it is sufficient to show this when $\mathrm{supp}(f) \subseteq U$, where (U, h) is a positive oriented C^∞-chart. Then we have that

$$I_\sigma(f) = \int_{h(U)} f \circ h^{-1}(x)\phi(x)\, d\mu_n,$$

where ϕ is determined by $(h^{-1})^*(\sigma) = \phi(x)\, dx_1 \wedge \cdots \wedge dx_n$. Since ϕ is positive we get $I_\sigma(f) \geqslant 0$.
According to Riesz's representation theorem (see for instance chapter 2 of [Rudin]) I_σ determines a positive measure μ_σ on M which satisfies

$$\int_M f(x)\, d\mu_\sigma = \int_M f\sigma, \qquad f \in C_c^n(M, \mathbb{R}).$$

The entire Lebesgue integration machinery now becomes available, but we shall use only very little of it in the following.

If M^n is an oriented Riemannian manifold, the volume form vol_M will determine a measure μ_M on M^n analogous to the Lebesgue measure on $\mathbb{R}^N$. For a compact set K the volume of K can be defined by

$$\mathrm{vol}(K) = \int_K \mathrm{vol}_M \in \mathbb{R}.$$

Definition 10.5 Let M^n be a smooth manifold. A subset $N \subseteq M^n$ is called a domain with smooth boundary or a codimension zero submanifold with boundary, if for every $p \in M$ there exists a C^∞-chart (U, h) around p, such that

$$h(U \cap N) = h(U) \cap \mathbb{R}^n_-. \tag{2}$$

where $\mathbb{R}^n_- = \{(x_1, \cdots x_n) \in \mathbb{R}^n \mid x_1 \leqslant 0\}$.

Note that (2) is automatically satisfied when p is an interior or an exterior point of N (one can choose (U, h) with $p \in U$, such that $h(U)$ is contained in an open half-space in IRn defined by either $x_1 < 0$ or $x_1 > 0$). If p is a boundary point of N then $h(p)$ has first coordinate equal to zero. Let (U, h) and (V, k) be smooth charts around a boundary point $p \in \partial N$. The resulting transition diffeomorphism

$$\phi = k \circ h^{-1} : h(U \cap V) \to k(U \cap V)$$

induces a map

$$h(U \cap V) \cap \mathbb{R}^n_- \to k(U \cap V) \cap \mathbb{R}^n_-,$$

which restricts to a diffeomorphism

$$\Psi : h(U \cap V) \cap \partial\mathbb{R}^n_- \to k(U \cap V) \cap \partial\mathbb{R}^n_-.$$

The Jacobi matrix at the point $q = h(p) \in \partial\mathbb{R}^n_-$ for $\phi = (\phi_1, \cdots, \phi_n)$ has the form

$$D_q\phi = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1}(q) & 0 & \cdots & 0 \\ * & & & \\ \vdots & & D_q\Psi & \\ * & & & \end{pmatrix}$$

We must have $\partial\phi_1/\partial x_1(q) \neq 0$, as $D_q\phi$ is invertible. Since ϕ maps $\mathbb{R}^n_-$ into $\mathbb{R}^n_-$, we have that $\partial\phi_1/\partial x_1(q) > 0$.
A tangent vector $w \in T_pM$ at a boundary point $p \in \partial N$ is said to be outward directed, if there exists a C^∞-chart (U, h) around p with $h(U \cap N) = h(U) \cap \mathbb{R}^n_-$ and such that $D_p h(w) \in \mathbb{R}^n$ has a positive first coordinate. This will then also be the case for any other smooth chart around p.

Lemma 10.6 Let $N \subseteq M^n$ be a domain with smooth boundary. Then ∂N is an $(n-1)$-dimensional smooth submanifold of M^n.

Suppose $M^n (n \geqslant 2)$ is oriented. There is an induced orientation of ∂N with the following property: if $p \in \partial N$ and $v_1 \in T_pM$ is an outward directed tangent vector then a basis $v_2, \cdots .v_n$ for $T_p\partial N$ is positively oriented if and only if the basis $v_1, v_2, \cdots, v_n$ for T_pM is positively oriented.

PROOF. Every smooth chart (U, h) in M that satisfies $h(U \cap N) = h(U) \cap \mathbb{R}^n_-$ can be restricted to a chart $(U \cap \partial N, h_|)$ on ∂N:

$$h_| : U \cap \partial N \to h(U) \cap \partial\mathbb{R}^n_-.$$

These charts have mutual smooth overlap according to the above. This yields a smooth atlas on ∂N.
Suppose M^n is oriented. Then, possibly changing the sign of x_2, we can choose a positively oriented chart (U, h) of the considered type around any $p \in \partial N$. The resulting smooth charts $(U \cap \partial N, h_|)$ on ∂N have positively oriented transformation diffeomorphisms, and they determine an orientation of ∂N that satisfies the stated property. □

Remark 10.7

(i) We want to integrate n-forms $\omega \in \Omega^n_c(M)$ over domains N with smooth boundary. In view of Remark 10.4 we can set

$$\int_N \omega = \int_M 1_N \omega$$

where 1_N is the function with value I on N and zero outside N. Alternatively, one can prove an extension of Lemma 10.1 which uses the following version of the transformation theorem: Let $\phi : V \to W$ be a diffeomorphism of open sets in $\mathbb{R}^n$ that maps $\mathbb{R}^n_- \cap V$ to $\mathbb{R}^n_- \cap W$, and let f be a smooth function on W with compact support. Then

$$\int_{\mathbb{R}^n_- \cap W} d(x)\, d\mu_n = \int_{\mathbb{R}^n_- \cap V} f\phi(x) |\det(D_x\phi)|\, d\mu_n.$$

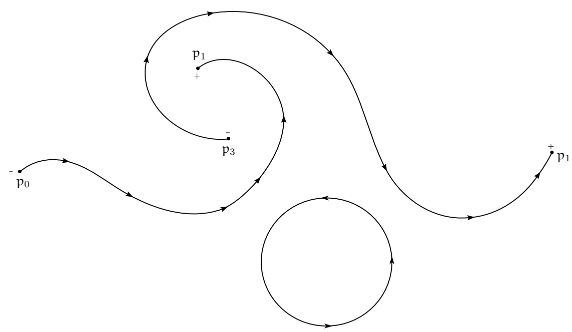
(One could approximate both integrals by integrals over W and V upon multiplying f by a sequence of smooth functions ψ_i with values in $[0, 1]$ and converging to $1_{\mathbb{R}^n_-}$)

(ii) In the case $n = 1$, Lemma 10.6 holds in the following modified form. An orientation of ∂N consists of a choice of sign, + or -, for every point $p \in \partial N$. Let $v_1 \in T_pM$ be outward directed. Then p is assigned the sign + if v_1 is a positively oriented basis of T_pM, otherwise the sign is -.

A 0-form on ∂N is a function $f : \partial N \to \mathbb{R}$. When f has compact support we define

$$\int_{\partial N} f = \sum_{p \in \partial N} \operatorname{sgn}(p) f(p).$$

These conventions are used in the case $n = 1$ of Stokes' theorem below.



Theorem 10.8 (Stokes' theorem) Let $N \subseteq M^n$ be a domain with smooth boundary in an oriented smooth manifold. Let ∂N have induced orientation. For every $\omega \in \Omega^{n-1}(M)$ with $M \cap \operatorname{supp}_M(\omega)$ compact we have

$$\int_{\partial N} i^*(\omega) = \int_N d\omega,$$

where $i : \partial N \to M$ is the inclusion map.

PROOF. We assume that $n \geqslant 2$ and leave it to the reader to make the necessary changes for the case $n = 1$.
It is clear that $i^*(\omega)$ has compact support. We can choose $f \in \Omega^0_c(M)$ with value constantly equal to 1 on $N \cap \operatorname{supp}_M(\omega)$. Since $f\omega$ coincides with ω on N, both integrals are unchanged when ω is replaced by $f(\omega)$, so we may assume that $omega$ has compact support.
Choose a smooth atlas on M consisting of charts of the type of Definition 10.5 and a subordinate smooth partition of unity $(\rho\alpha)_{\alpha \in A}$. The formulas

$$\int_{\partial N} \omega = \sum_\alpha \int_{\partial N} \qquad\qquad \int_N d\omega = \sum_\alpha \int_N d(\rho_\alpha \omega)$$

reduce the problem to the case where $\omega \in \Omega^n_c(M)$. $\operatorname{supp}_M(\omega) \subseteq U$ and (U, h) is a smooth chart $h(U \cap N) = h(U) \cap \mathbb{R}^n_-$. Furthermore the chart (U, h) is assumed to be positively oriented.
Let $\kappa \in \Omega^{n-1}_c(\mathbb{R}^n)$ be the $(n-1)$-form that is $(h^{-1})^*(\omega)$ on $h(U)$ and 0 on the rest of $\mathbb{R}^n$. By diffeomorphism invariance we then have that

$$\int_{\partial N} \omega = \int_{h(U) \cap \partial \mathbb{R}^n_-} (h^{-1})^*(\omega) = \int_{\mathbb{R}^n_-} \kappa$$

and

$$\int_N d\omega = \int_{h(U) \cap \mathbb{R}^n_-} (h^{-1})^* \, d(\omega) = \int_{\mathbb{R}^n_-} d\kappa.$$

Hence the proof reduces to the special case where $M = \mathbb{R}^n, N = \mathbb{R}^n_-$ and $\omega \in \Omega^{n-1}_c(\mathbb{R}^n)$. This case treated by direct calculation. We define

$$\omega = \sum_{i=1}^{n} f_i(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

and choose $b > 0$ such that $\operatorname{supp}_{\mathbb{R}^n} f_i \subseteq [-b, b]^n, 1 \leqslant i \leqslant n$. Using Theorem 3.12,

$$\omega_{|\partial \mathbb{R}^n_-} = f_1(0, x_2, \cdots, x_n)\, dx_2 \wedge \cdots \wedge dx_n.$$

Hence

$$\int_{\partial \mathbb{R}^n_-} \omega = \int f_1(0, x_2, \cdots, x_n)\, d\mu_{n-1}. \tag{3}$$

By Theorem 3.7 we have

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n.$$

Hence

$$\int_{\mathbb{R}^n_-} = \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i}\, d\mu_n. \tag{4}$$

For $2 \leqslant i \leqslant n$ we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}(x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n)\, dt \\
=&f_i(x_1, \cdots, x_{i-1}, b, x_{i+1}, \cdots, x_n) - f_i(x_1, \cdots, x_{i-1}, -b, x_{i+1}, \cdots, x_n). \\
=&0
\end{aligned}$$

and then by Fubini's theorem

$$\int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i}\, d\mu_n = 0 \qquad (2 \leqslant i \leqslant n). \tag{5}$$

When $i = 1$, one gets

$$\begin{aligned}
\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}(t, x_2, \cdots, x_n)\, dt &= f_1(0, x_2, \cdots, x_n) - f_1(-b, x_2, \cdots, x_n) \\
&= f_1(0, x_2, \cdots, x_n),
\end{aligned}$$

and by Fubini's theorem

$$\int_{\mathbb{R}^n_-} \frac{\partial f_1}{\partial x_1}\, d\mu_n = f_1(0, x_2, \cdots, x_n)\, d\mu_{n-1}. \tag{6}$$

By combining Equations (3)–(6) the desired formula follows. □

Taking $N = M$ in Theorem 10.8 we have

Corollary 10.9 If M^n is an oriented smooth manifold and $\omega \in \Omega_c^{n-1}(M)$ then $\int_M d\omega = 0$.

Remark 10.10 Let ω be a closed d-form on M^n. One way of showing that the Cohomology class $[\omega] \in H^d(M)$ is non-zero is to show that

$$\int_Q f^*(\omega) \neq 0 \tag{7}$$

for a suitably chosen smooth map $f : Q^d \to M$ from a d-dimensional compact oriented smooth manifold Q^d. If $\omega = d\tau$ for some $\tau \in \Omega^{d-1}(M)$, then Corollary 10.9 yields

$$\int_Q f^*(\omega) = \int_Q d(f^*(\tau)) = 0.$$

This, in essence, was the strategy from Examples 1.2 and 1.7. It can be shown (albeit in a very indirect way via cobordism theory) that $[\omega] = 0$ if and only if all integrals of the form of 7 vanish.

Example 10.11 In Example 9.18 we considered the closed $(n-1)$-form on $\mathbb{R}^n - \{0\}$,

$$\omega = \frac{1}{\|x\|^n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Since the pre-image of ω under the inclusion of S^{n-1} is the volume form $\mathrm{vol}_{S^{n-1}}$, which has positive integral over S^{n-1}, we can conclude from Remark 10.10 that $[\omega] \neq 0$ in $H^{n-1}(\mathbb{R}^n - \{0\})$. If $n \geqslant 2$ then, by Theorem 6.13, $[\omega]$ is a basis of $H^{n-1}(\mathbb{R}^n - \{0\})$. We thus have an isomorphism

$$H^{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{R} \qquad (n \geqslant 2)$$

defined by integration over S^{n-1}. The image of $[\omega]$ under this isomorphism is the volume

$$\mathrm{Vol}(S^{n-1}) = \int_{S^{n-1}} \mathrm{vol}_{S^{n-1}}.$$

Example 10.12 The volume of S^{n-1} can be calculated by applying Stokes' theorem to D^n with the standard orientation of IRn and the $(n-1)$-fonn on $\mathbb{R}^n$ given by

$$\omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Since $\omega_{0|S^{n-1}} = \mathrm{vol}_{S^{n-1}}$ and $d\omega_0 = n\,dx_1 \wedge \cdots \wedge dx_n$ we have that

$$\mathrm{Vol}(S^{n-1}) = \int_{S^{n-1}} \omega_0 = \int_{D^n} d\omega_0 = n\mathrm{Vol}(D^n).$$

By induction on m and Fubini's theorem, it can be shown that

$$\mathrm{Vol}(D^{2m}) = \frac{\pi^m}{m!}, \qquad \mathrm{Vol}(D^{2m+1}) = \frac{2^{2m+1} m! \pi^m}{(2m+1)!}.$$

This yields

$$\mathrm{Vol}(S^{2m-1}) = \frac{2\pi^m}{(m-1)!}, \qquad \mathrm{Vol}(S^{2m}) = \frac{2^{2m+1} m! \pi^m}{(2m)!}.$$

We conclude this chapter with a proof of the following:

Theorem 10.13 If M^n is a connected oriented smooth manifold, then the sequence

$$\Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c^n(M) \xrightarrow{\int_M} \mathbb{R} \to 0 \tag{8}$$

is exact.

Corollary 10.14 For a connected compact smooth manifold M^n, integration over M induces an isomorphism

$$\int_M : H^n(M^n) \xrightarrow{\cong} \mathbb{R}. \qquad \square$$

In (8) it is obvious that the integral is non-zero and hence surjective. It follows from Corollary 10.9 that the image of d is contained in the kernel of the integral. We show the converse inclusion.

Lemma 10.15 Theorem 10.13 holds for $M = \mathbb{R}^n, n \geqslant 1$.

Proof. Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be a diffeomorphism n-form with $\int_{\mathbb{R}^n} \omega = 0$. We must find that $\kappa \in \Omega_c^{n-1}(\mathbb{R}^n)$, such that $d\kappa = \omega$. We can write $\omega = f(\mathbf{x})\,dx_1 \wedge \cdots \wedge dx_n$, and let

$$\kappa = \sum_{j=1}^{n} (-1)^{i-1} f_j(\mathbf{x})\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

A simple calculation gives

$$d\kappa = \left(\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \right) dx_1 \wedge \cdots \wedge dx_n.$$

Hence we need to prove the following assertion:
(P_n): Let $f \in C_c^\infty(\mathbb{R}^n)$ be a function with $\int f(x)\,d\mu_n = 0$. There exsits functions $f_1, \cdots, f_n$ in $C_c^\infty(\mathbb{R}^n)$ such that

$$\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} = f.$$

We prove (P_n) by induction. For $n = 1$ we are given a smooth function $x \in C_c^\infty(\mathbb{R})$ with $\int_{-\infty}^{\infty} f(t)\,dt = 0$. The problem is solved by setting

$$f_1(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Assume that (P_{n-1}) for $n \geqslant 2$, and let $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ be a function with $\int f(x)\,d\mu_n = 0$. We choose $C > 0$ with $\mathrm{supp}(f) \subseteq [-C, C]^n$ and define

$$g(x_1, \cdots, x_{n-1}) = \int_{-\infty}^{\infty} f(x_1, \cdots, x_{n-1}, x_n)\,dx_n. \tag{9}$$

(The limits can be be replaced by $-C$ and C, respectively). The function g is smooth, since we can differentiate under the integral sign. Furthennore $\mathrm{supp}(g) \subseteq [-C, C]^{n-1}$. Fubini's theorem yields $\int g\,d\mu_{n-1} = \int f\,d\mu_n = 0$. Using (P_{n-1}) we get functions $g_1, \cdots, g_{n-1}$ in $C_c^\infty(\mathbb{R}^{n-1}, \mathbb{R})$ with

$$\sum_{j=1}^{n-1} \frac{\partial g_j}{\partial x_j} = g. \tag{10}$$

We choose a function $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$, and define $f_j \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$,

$$f_j(x_1, \cdots, x_n) = g_j(x_1, \cdots, x_{n-1})\rho(x_n), 1 \leqslant j \leqslant n-1. \tag{11}$$

Let $h \in C_c^\infty(\mathbb{R}, \mathbb{R})$ be the function

$$h = f - \sum_{j=1}^{n-1} \frac{\partial f_j}{\partial x_j}. \tag{12}$$

A function $f_n \in C_c^\infty(\mathbb{R}^n)$ with $\partial f_n/\partial x_n = h$ is given by

$$f_n(x_1, \cdots, x_{n-1}, x_n) = \int_{-\infty}^{x_n} h(x_1, \cdots, x_{n-1}, t)\,dt. \tag{13}$$

It is obvious that f_n is smooth, but we must show that it has compact support. To this end it is sufficient to show that the integral of (13) vanishes when the upper limit X n is replaced by 00. Now (10), (11) and (12) yield that

$$\begin{aligned} h(x_1, \cdots, x_{n-1}, x_n) &= f(x_1, \cdots, x_{n-1}, t) - \sum_{j=1}^{n-1} \frac{\partial g_j}{\partial x_j}\rho(t) \\ &= f(x_1, \cdots, x_{n-1}, t) - g(x_1, \cdots, x_{n-1})\rho(t). \end{aligned}$$

Finally from (10) it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty} h(x_1,\cdots,x_{n-1},t)\,dt\\ =&\int_{-\infty}^{\infty} f(x_1,\cdots,x_{n-1},t)\,dt - g(x_1,\cdots,x_{n-1})\int_{-\infty}^{\infty}\rho(t)\,dt\\ =&0.\end{aligned}$$

□

Lemma 10.16 Let $(U_\alpha)_{\alpha\in A}$ be an open cover of the connected manifold M, and let $p, q \in M$. There exists indices $\alpha_1,\cdots,\alpha_k$ such that

(i) $p \in U_\alpha$ and $q \in U_{\alpha_k}$.

(ii) $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \varnothing$ for $1 \leqslant i \leqslant k-1$.

PROOF. For a fixed p we define V to be the set of $q \in M$, for which there exists a finite sequence of indices $\alpha_1,\cdots,\alpha_k$ from A, such that (i) and (ii) are satisfied. It is obvious that V is both open and closed in M and that V contains p. Since M is connected, we must have $V = M$. □

Lemma 10.17 Let $U \subseteq M$ be an open set diffeomorphic to $\mathbb{R}^n$ and let $W \subseteq U$ be non-empty and open. For every $\omega \in \Omega_c^n(M)$ with $\operatorname{supp}_M(\omega) \subseteq U$, there exists a $\kappa \in \Omega_c^{n-1}(M)$ such that $\operatorname{supp}\kappa \subseteq U$ and $\operatorname{supp}(\omega - d\kappa) \subseteq W$.

PROOF. It suffices to prove the lemma when $M = U$, and by diffeomorphism invariance it is enough to consider the case where $M = U = \mathbb{R}^n$. Choose $\omega_1 \in \Omega_c^n(\mathbb{R}^n)$ with $\operatorname{supp}(\omega_1) \subseteq W$ and $\int_{\mathbb{R}^n}\omega_1 = 1$. Then

$$\int_{\mathbb{R}^n}(\omega - a\omega_1) = 0, \qquad \text{where } a = \int_{\mathbb{R}^n}\omega.$$

By Lemma 10.15 we can find a $\kappa \in \Omega_c^{n-1}(\mathbb{R}^n)$ with

$$\omega - a\omega_1 = d\kappa.$$

Hence $\omega - d\kappa = d\omega_1$ has its support in W. □

Lemma 10.18 Assume that M^n is connected and let $W \subseteq M$ be non-empty and open. For every $\omega \in \Omega_c^n(M)$ there exists a $\kappa \in \Omega_c^{n-1}(M)$ with $\operatorname{supp}(\omega - d\kappa) \subseteq W$.

PROOF. Suppose that $\operatorname{supp}\omega \subseteq U_1$ for some open set $U_1 \subseteq M$ diffeomorphic to $\mathbb{R}^n$. We apply Lemma 10.16 to find open sets $U_2,\cdots,U_k$, diffeomorphic to $\mathbb{R}^n$, such that $U_{i-1} \cap U_i \neq \varnothing$ for $2 \leqslant i \leqslant k$ and $U_k \subseteq W$. We use Lemma 10.17 to successively choose $\kappa_1,\cdots,\kappa_{k-1}$ in $\Omega_c^{n-1}(M)$ such that

$$\operatorname{supp}\left(\omega - \sum_{i=1}^{j} d\kappa_i\right) \subseteq U_j \cap U_{j+1} \qquad (1 \leqslant j \leqslant k-1).$$

The lemma holds for $\kappa = \sum_{i=1}^{k-1} \kappa_i$.
In the general case we use a partition of unity to write

$$\omega = \sum_{j=1}^{m} \omega_j,$$

where $\omega_j \in \Omega_c^n(M)$ has support contained in a open set diffeomorphic to $\mathbb{R}^n$. The above gives $\tilde{\kappa}_j \in \Omega_c^{n-1}(M), 1 \leqslant j \leqslant m$, such that $\operatorname{supp}(\omega_j - d\tilde{\kappa}_j) \subseteq W$. For

$$\tilde{\kappa} = \sum_{j=1}^{m} \tilde{\kappa}_j \in \Omega_c^{n-1}(M)$$

we have that

$$\omega - d\tilde{\kappa} = \sum_{j=1}^{m} (\omega_j - d\tilde{\kappa}_j)$$

Hence $\operatorname{supp}(\omega - d\tilde{\kappa}) \subseteq \cup_{j=1}^{m} \subseteq W$. $\square$

Proof of Theorem 10.13. Suppose given $\omega \in \Omega_c^n(M)$ with $\int_M \omega = 0$. Choose an open set $W \subseteq M$ diffeomorphic to $\mathbb{R}^n$. By Lemma 10.18 we can find a $\kappa \in \Omega_c^{n-1} \subseteq W$. But then by Corollary 10.9,

$$\int_W (\omega - d\kappa) = \int_M (\omega - d\kappa) = -\int_M d\kappa = 0$$

Lemma 10.15 implies that Theorem 10.13 holds for W, i.e. there exists a $\tau_0 \in \Omega_c^{n-1}(W)$ that satisfies

$$(\omega - d\kappa)_{|W} = d\tau_0.$$

Let $\tau \in \Omega_c^{n-1}(M)$ be the extension of τ_0 which vanishes outside $\operatorname{supp}_W(\tau_0)$. Then $\omega - d\kappa = d\tau$, so $\tau + \kappa$ maps ω under d. $\square$

11. DEGREE, LINKING NUMBERS AND INDEX OF VECTOR FIELDS

Let $f : N^n \to M^n$ be a smooth map between compact connected oriented manifolds of the same dimension n. We have the commutative diagram

$$\begin{array}{ccc} H^n(M) & \xrightarrow{H^n(f)} & H^n(N) \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{R} & \xrightarrow{\deg(f)} & \mathbb{R} \end{array} \tag{1}$$

where the vertical isomorphisms are given by integration over M and N respectively; cf. Corollary 10.14. The lower horizontal arrow is multiplication by the real number $\deg(f)$ that makes the diagram commutative. Thus for $\omega \in \Omega^n(M)$,

$$\int_N f^*\omega = \deg(f) \int_M \omega. \tag{2}$$

This formulation can be generalized to the case where N is not connected:

Proposition 11.1 Let $f : N^n \to M^n$ be a smooth map between compact n-dimensional oriented manifolds with M connected. There exists a unique $\deg(f) \in \mathbb{R}$ such that (2) holds for all $\omega \in \Omega^n(M)$. We call $\deg(f)$ the degree of f.

Proof. We write N as a disjoint union of its connected components $N_1, \cdots, N_k$ and and denote the restriction of f to N_j by f_j. We have already defined $\deg(f_i)$; we set

$$\deg(f) = \sum_{j=1}^{k} \deg(f_j). \tag{3}$$

Thus for $\omega \in \Omega^n(M)$, we have,

$$\int_N f^*\omega = \sum_{j=1}^{k} \int_{N_j} f_j^*\omega = \sum_{j=1}^{k} \deg(f_j) \int_M \omega = \deg(f) \int_M \omega.$$

□

Corollary 11.2 $\deg(f)$ depends only on the homotopy class of $f : N \to M$.

Proof. By (3) we can restrict ourselves to the case where N is connected. The assertion then follows from diagram (1), since $H^n(f)$ depends only on the homotopy class of f. □

Corollary 11.3 Suppose $N^n \xrightarrow{f} M^n \xrightarrow{g} P^n$ are smooth maps between n-dimensional compact oriented manifolds and that M and P are connected. Then

$$\deg(gf) = \deg(g)\deg(f).$$

PROOF. For $\omega \in \Omega^n(P)$,

$$\begin{aligned}\deg(gf)\int_P \omega &= \int_N (gf)^*(\omega) = \int_N f^*(g^*(\omega)) \\ &= \deg(f)\int_M g^*(\omega) = \deg(f)\deg(g)\int_P \omega.\end{aligned}$$

□

Remark 11.4 If $f : M^n \to N^n$ is a smooth map of a connected compact orientable manifold to itself then $\deg(f)$ can be defined by chosing an orientation of M and using it at both the domain and range. Change of orientation leaves $\deg(f)$. unaffected.

We will show that $\deg(f)$ takes only integer values. This follows from an important geometric interpretation of $\deg(f)$ which uses the concept of regular value. In general $p \in M$ is said to be a *regular value* for the smooth map $f : N^n \to M^n$ if

$$D_q f : T_p N \to T_p M$$

is surjective for all $q \in f^{-1}(p)$. In particular, points in the complement of $f(N^n)$ are regular values. Regular values are in rich supply:

Theorem 11.5 (Brown-Sard) For every smooth map $f : N^n \to M^m$ the set of regular values is dense in M^m.

When proving Theorem 11.5 one may replace M^m by an open subset $W \subseteq M$ diffeomorphic to $\mathbb{R}^n$, and replace N^n by $f^{-1}(W)$. This reduces Theorem 11.5 to the special case where $M^m = \mathbb{R}^m$.
In this case one shows, that almost all points in $\mathbb{R}^m$ (in the Lebesgue sense) are regular values. By covering N^n with countably many coordinate patches and using the fact that the union of countably many Lebesgue null-sets is again a null-set, Theorem 11.5 therefore reduces to the following result:

Theorem 11.6 (Sard, 1942) Let $f : U \to \mathbb{R}^m$ be a smooth map defined on an open set $U \subseteq \mathbb{R}^n$ and let

$$S = \{x \in U \big| \operatorname{rank} D_x f < m\}.$$

Then $f(S)$ is a Lebesgue null-set in $\mathbb{R}^m$.

Note that $x \in U$ belongs to S if and only if every $m \times m$ submatrix of the Jacobi matrix of f, evaluated at x, has determinant zero. Therefore S is closed in U and we can write S as a union of at most countably many compact subsets $K \subseteq S$. Theorem 11.6 thus follows if $f(K)$ is a Lebesgue null-set for every compact subset K of S. We shall only use and prove these theorems in the case $m = n$, where they follow from

Proposition 11.7 Let $f : U \to \mathbb{R}^n$ be a C^1-map defined on an open set $U \subseteq \mathbb{R}^n$, and let $K \subseteq S$ be a compact set such that $\det(D_x f) = 0$ for all $x \in K$. Then $f(K)$ is a Lebesgue null-set in $\mathbb{R}^n$.

Proof. Choose a compact set $L \subseteq U$ which contains K in the interior, $K \subseteq L$. Let $C > 0$ be a constant such that

$$\sup_{\xi \in L} \| \operatorname{grad}_\xi f_j \| \leqslant C \qquad (1 \leqslant j \leqslant n). \tag{4}$$

Here f_j is the i-th coordinate function of f, and $\| \cdot \|$ denotes the Euclidean norm. Let

$$T = \prod_{i=1}^{n} [t_i, t_i + a]$$

be a cube such that $K \subseteq S$, and let $\epsilon > 0$. Since the functions $\partial f_j / \partial x_i$ are uniformly continuous on L, there exists a $\delta > 0$ such that

$$\|x - y\| \leqslant \delta \implies \left| \frac{\partial f_j}{\partial x_i}(x) - \frac{\partial f_j}{\partial x_i}(y) \right| \leqslant \epsilon, (1 \leqslant i, j \leqslant n \text{ and } x, y \in L). \tag{5}$$

We subdivide T into a union of N^n closed small cubes T_l with side length $\frac{a}{N}$, and choose N so that

$$\operatorname{diam}(T_1) = \frac{a\sqrt{n}}{N} \leqslant \delta, \qquad T_1 \cap K \neq 0 \implies T_1 \subseteq L. \tag{6}$$

For a small cube T_l with $T_1 \cap K \neq \varnothing$ we pick $x \in T_l \cap K$. If $y \in T_l$ the mean value theorem yields points ξ_j on the line segment between x and y for which

$$f_j(y) - f_j(x) = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\xi_j)(y_i - x_i). \tag{7}$$

Since $\xi_j \in T_l \subseteq L$, the Cauchy-Schwarz inequality and (4) give

$$|f_j(y) - f_j(x)| \leqslant C\|y - x\|$$

and by (6),

$$\|f(y) - f(x)\| \leqslant C\|y - x\| \leqslant C\sqrt{n}\operatorname{diam}(T_1) = \frac{anC}{N}. \tag{8}$$

Formula (7) can be rewritten as

$$f(y) = f(x) + D_x f(y - x) + z, \tag{9}$$

where $z = (z_1, \cdots, z_n)$ is given by

$$z_j = \sum_{i=1}^{n} \left(\frac{\partial f_j}{\partial x_i}(\xi_j) - \frac{\partial f_j}{\partial x_i}(x_i) \right)(y_i - x_i).$$

By (6), $\|\xi_j - x\| \leqslant \delta$, so that $|z_j| \leqslant \epsilon n \frac{a}{N}$. Hence

$$\|z\| \leqslant \epsilon \frac{an\sqrt{n}}{N}. \tag{10}$$

Since the image of $D_x f$ is a proper subspace of $\mathbb{R}^n$, we may choose an affine hyperplane $H \subseteq \mathbb{R}^n$ with

$$f(x) + \mathrm{Im}(D_x f) \subseteq H.$$

By (9) and (10) the distance from $f(y)$ to H is less than $\epsilon \frac{an\sqrt{n}}{N}$. Then (8) implies that $f(T_l)$ is contained in the set D_l consisting of all points $q \in \mathbb{R}^n$ whose orthogonal projection $\mathrm{pr}(q)$ on H lies in the closed ball in H with radius $\epsilon \frac{anC}{N}$ and centre $f(x)$ and $\|q - \mathrm{pr}(q)\| \leqslant \epsilon \frac{an\sqrt{n}}{N}$. For the Lebesgue measure μ_n on $\mathbb{R}^n$ we have

$$\mu_n(D_i) = 2\epsilon \frac{an\sqrt{n}}{N} \left(\frac{anC}{N} \right)^{n-1} \mathrm{vol}(D^{n-1}) = \epsilon \frac{c}{N^n}$$

where $c = 2a^n n^{n+\frac{1}{2}} C^{n-1} \mathrm{Vol}(D^{n-1})$. For evry small cube T_l with $T_l \cap K \neq \varnothing$ we now have $\mu_n(f(T_l)) \leqslant \epsilon \frac{c}{N^n}$. Since there are at most N^n such small cubes T_l, $\mu_n(f(K)) \leqslant c\epsilon$. This holds for every $\epsilon > 0$ and proves the assertion. □

Lemma 11.8 Let $p \in M^n$ be a regular value for the smooth map $f : N^n \to M^n$, with N^n compact. Then $f^{-1}(p)$ consists offinitely many points $q_1, \cdots, q_k$. Moreover, there exist disjoint open neighborhoods V_i of q_i in N^n, and an open neighborhood U of p in M^n, such that

(i) $f^{-1}(U) = \cup_{i=1}^{k} V_i$.

(ii) f_i maps V_i diffeomorphically onto U for $1 \leqslant i \leqslant k$.

PROOF. For each $q \in F^{-1}(U)$, $D_q f : T_q N \to T_q M$ is an isomorphism. From the inverse function theorem we know that f is a local diffeomorphism around q. In particular q is an isolated point in $f^{-1}(p)$. Compactness of N implies that $f^{-1}(p)$ consists of finitely many points $q_1, \cdots, q_k$. We can choose mutually disjoint open neighborhoods W_i of q_i in N, such that f maps W_i diffeomorphically onto an open neighborhood $f(W_i)$ of p in M. Let

$$U = \left(\bigcap_{i=1}^{k} f(W_i) \right) - f\left(N - \bigcup_{i=1}^{k} W_i \right)$$

Since $N - \bigcup_{i=1}^{k} W_i$ is closed in N and therefore compact $f(N - \bigcup_{i=1}^{k} W_i)$ is also compact. Hence U is an open neighborhood of p in M. We then set $V_i = W_i \cap f^{-1}(U)$. □

Consider a smooth map $f : N^n \to M^n$ between compact n-dimensional oriented manifolds, with M connected. For a regular value $p \in M$ and $q \in f^{-1}(p)$, define the local index

$$\operatorname{Ind}(f, g) = \begin{cases} 1 & \text{if } D_q f : T_p N \to T_p M \text{ preserves orientation} \\ -1 & \text{otherwise.} \end{cases} \tag{11}$$

Theorem 11.9 In the situation above, and for every regular value p,

$$\deg(f) = \sum_{q \in f^{-1}(p)} \operatorname{Ind}(f; q).$$

In particular $\deg(f)$ is an integer.

PROOF. Let q_i, V_i, and U be as in Lemma 11.8. We may assume that U and hence V_i connected. The diffeomorphism $f_{|V_i} : V_i \to U$ is positively or negatively oriented, depending on whether $\operatorname{Ind}(f; q)$ is 1 or -1. Let $\omega \in \Omega^n(M)$ be an n-form with

$$\operatorname{supp}_M(\omega) \subseteq U, \qquad \int_M \omega = 1.$$

Then $\operatorname{supp}_N(f^*(\omega)) \subseteq f^{-1}(U) = V_1 \cup \cdots \cup V_k$, and we can write

$$f^*(\omega) = \sum_{i=1}^k \omega_i$$

where $\omega_i \in \Omega^n(N)$ and $\operatorname{supp}(\omega_1) \subseteq U$. Here $\omega_{i|V_i} = (f_{|V_i})^*(\omega_{|U})$. The formula is a consequence of the following calculation:

$$\begin{aligned} \deg(f) = \deg(f) \int_M \omega = \int_M f^*(\omega) = \sum_{i=1}^k \int_N \omega_i = \sum_{i=1}^k \int_{V_i} (f_{|V_i})^*(\omega_{|U}) \\ = \sum_{i=1}^k \operatorname{Ind}(f; q_i) \int_U \omega_{|U} = \sum_{i=1}^k \operatorname{Ind}(f; q_i). \end{aligned}$$

In the special case where $f^{-l}(p) = 0$ the theorem shows that $\deg(f) = 0$ (in the proof above we get $f^*(\omega) = 0$). Thus we have □

Corollary 11.10 If $\deg(f) \neq 0$, then f is surjective.

Proposition 11.11 Let $F : P^{n+1} \to M^n$ be a smooth map between oriented smooth manifolds, with M^n compact and connected. Let $X \subseteq P$ be a compact domain with smooth boundary $N^n = \partial x$, and suppose N is the disjoint union ofsubmanifolds $N_i^n, \cdots, N_k^n$. If $f_i = F_{|N_i}$, then

$$\sum_{i=1}^k \deg(f_i) = 0.$$

Proof. Let $f = F_{|N}$ so that

$$\deg(f) = \sum_{i=1}^{k} \deg(f_i).$$

On the other hand, if $\omega \in \Omega^n(M)$ has $\int_M \omega = 1$, then

$$\deg(f) = \int_N f^*(\omega) = \int_X dF^*(\omega) = \int_X F^*(d\omega) = 0$$

where the second equation is from Theorem 10.8. □

We shall give two applications of degree. We first consider linking numbers, and then treat indices of vector fields.

Definition 11.12 Let J^d and K^l be two disjoint compact oriented connected smooth submanifolds of $\mathbb{R}^{n+1}$, whose dimensions $d \geqslant 1, l \geqslant 1$ satisfy $d + l = n$. Their linking number is the integer

$$\mathrm{lk}(J, K) = \deg(\Psi_{J,K})$$

where

$$\Psi = \Psi_{J,K} : J \times K \to S^n; \qquad \Psi(x, y) = \frac{y - x}{\|y - x\|}.$$

Here $J \times K$ is equipped with the product orientation (cf. Remark 9.20) and S^n is oriented as the boundary of D^{n+1} with the standard orientation of $\mathbb{R}^{n+1}$. We note that $\mathrm{lk}(J, K)$ changes sign when the orientation of either J or K is reversed.

Proposition 11.13

(i) $\mathrm{lk}(K^l, J^d) = (-1)^{(d+1)(l+1)}\mathrm{lk}(J^d, K^l)$.

(ii) If J and K can be separated by a hyperplane $H \subset \mathbb{R}^{n+1}$ then $\mathrm{lk}(J, K) = 0$.

(iii) Let g_t and h_t be homotopies of the inclusions $g_0 : J \to \mathbb{R}^{n+1}$ and $h_0 : K \to \mathbb{R}^{n+1}$ to smooth embeddings g_1 and h_1, such that $g_t(J) \cap h_t(K) = \varnothing$ for all $t \in [0, 1]$. Then $\mathrm{lk}(J, K) = \mathrm{lk}(g_1(J), h_1(K))$.

(iv) Let $\Phi : P^{n+1} \to \mathbb{R}^{n+1} - J$ be a smooth map with P oriented. Given a compact domain $R \subseteq P$ with smooth boundary ∂R, let $Q_1, \cdots, Q_k$ be the connected components of ∂R. Suppose each $\Phi_{|Q_j}$ is a smooth embedding. If $K_i = \Phi(Q_i)$, then

$$\sum_{i=1}^{k} \mathrm{lk}(J; K_i) = 0.$$

Proof. We look at the commutative diagram

$$\begin{tikzcd} J \times K \arrow[r, "\Psi_{J,K}"] \arrow[d, "T"] & S^n \arrow[d, "A"] \\ K \times J \arrow[r, "\Psi_{K,J}"] & S^n \end{tikzcd}$$

where T interchanges factors and A is the antipodal map $Av = -v$. Then (i) follows from Corollary 11.3 upon using that

$$\deg(T) = (-1)^{dl}, \qquad \deg(A) = (-1)^{n+1} = (-1)^{d+l+1}.$$

In the situation of (ii) the image of Ψ will not contain vectors parallel to H, and the assertion follows from Corollary 11.10.
Assertion (iii) is a consequence of the homotopy property, Corollary 11.2. Indeed, a homotopy $J \times K \times [0,1] \to S^n$ is given by

$$(h_t(y) - g_t(x))/\|h_t(y) - g_t(x)\|.$$

Finally (iv) follows from Proposition 11.11 applied to the map $F : J \times P \to S^n$ with

$$F(x,y) = (\Phi(y) - x)/\|\Phi(y) - x\|.$$

and to the domain $X = J \times R$ with boundary components $I \times Q_i$. Indeed, $f_i = E_{|J \times Q_i}$ has degree $\deg(f_i) = \mathrm{lk}(J, K_i)$. $\square$

Here is a picture to illustrate (iv):

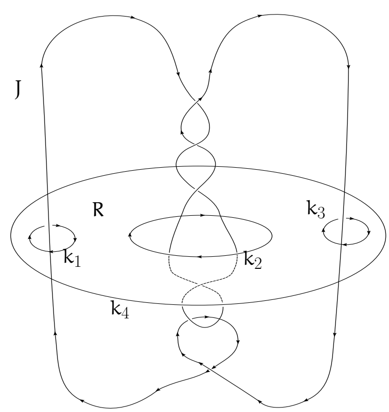


Figure 11.1: Figure 1

If $\mathrm{lk}(J,K) \neq 0$ then (ii) and (iii) of Proposition 11.13 imply that J and K cannot be deformed to manifolds separated by a hyperplane.
We shall now specialize to the classical case of knots in $\mathbb{R}^3$ where J and K are disjoint oriented submanifolds of $\mathbb{R}^3$ diffeomorphic to S^1. Let us choose smooth regular parametrizations

$$\alpha : \mathbb{R} \to J, \qquad\qquad \beta : \mathbb{R} \to K$$

with periods a and b, respectively, corresponding to a single traversing of J and K, respectively, agreeing with the orientation. For $p \in S^2$, consider the set

$$I(p)\{(q_1, q_2) \in J \times K \big| q_2 - q_1 = \lambda p, \lambda > 0\}.$$

Let $v(q_1)$ and $w(q_2)$ denote the positively oriented unit tangent vectors to J and K in q_1 and q_2, respectively.

Theorem 11.14 With the notation above we have:

(i) (Gauss)

$$\mathrm{lk}(J, K) = \frac{1}{4\pi} \int_0^a \int_0^b \frac{\det(\alpha(u) - \beta(v), \alpha'(u), \beta'(v))}{\|\alpha(u) - \beta(v)\|^3}$$

(ii) There exists a dense set of points $p \in S^2$ such that

$$\det(q_1 - q_2, v(q_1), w(q_2)) \neq 0 \text{ for } (q_1, q_2) \in I(p).$$

(iii) For such points p, $\mathrm{lk}(J, K) = \sum_{(q_1,q_2)\in I(p)} \delta(q_1, q_2)$, where $\delta(q_1, q_2)$ is the sign of the determinant in (ii).

PROOF. We apply formula (2) to the map $\Psi = \Psi_{J,k}$ and the volume form $\omega = \mathrm{vol}_{S^2}$ (with integral 4π) to get

$$\mathrm{lk}(J, k) = \deg(\Psi) = \frac{1}{4\pi} \int_{J\times K} \Psi^*(\mathrm{vol}_{S^2}). \tag{12}$$

We write $\Psi = r \circ f$ with

$$\begin{aligned} f : J \times K &\to \mathbb{R}^3 - \{0\}; & f(q_1, q_2) &= q_2 - q_1, \\ r : \mathbb{R}^3 - \{0\} &\to S^2; & r(v) &= \frac{x}{\|x\|}. \end{aligned}$$

For $x \in \mathbb{R}^3 - \{0\}, r^*(\mathrm{vol}_{S^2}) \in \mathrm{Alt}^2(\mathbb{R}^3)$ is given by

$$r^*(\mathrm{vol}_{S^2})_x(v, w) = \det(x, v, w)/\|x\|^3.$$

(cf. Example 9.18). The tangent space $T_{(q_1,q_2)}(J \times K)$ has a basis $\{v(q_1), w(q_2)\}$, and

$$Df_{(q_1,q_2)}(v(q_1)) = -v(q_1), \qquad Df_{(q_1,q_2)}(w(q_2)) = w(q_2).$$

Therefore

$$\begin{aligned} \Psi^*(\mathrm{vol}_{S^2})_{(q_1,q_2)}(v(q_1)w(q_2)) &= r^*(\mathrm{vol}_{S^2})_{q_2-q_1}(-v(q_1), w(q_2)) \\ &= \|q_1 - q_2\|^{-3} \det(q_1 - q_2, v(q_1), w(q_2)). \end{aligned} \tag{13}$$

The integral of (12) can be calculated by integrating $(\alpha \times \beta)^*\Psi^*(\mathrm{vol}_{S^2})$ over the period rectangle $[0, a] \times [0, b]$. This yields Gauss's integral.
For $p \in S^2$, $I(p)$ is exactly the pre-image under Ψ. Thus p is a regular value of Ψ if and only if the determinant in (13) is non-zero for all $(q_1, q_2) \in I(p)$, and the sign $\delta(q_1, q_2)$ is determined by whether $D_{(q_1,q_2)}\Psi$ preserves or reverses orientation. Assertions (ii) and (iii) now follow from Theorems 11.5 and 11.9. □

Remark 11.15 In Theorem 11.14.(ii), after a rotation of $\mathbb{R}^3$, the regular value p can be assumed to be the north pole $(0, 0, 1)$. The projections of J and K on the x_1, x_2-plane may be drawn indicating over- and undercrossings and orientations, e.g.

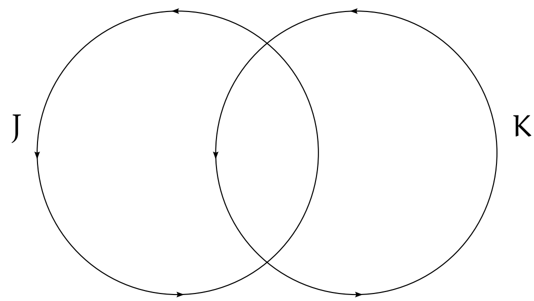


Figure 11.2: Figure 2

There is one element in $I(p)$ for every place where K crosses over (and not under) J. The corresponding sign δ is determined by the orientation of the curves and of the standard orientation of the plane as shown in the picture

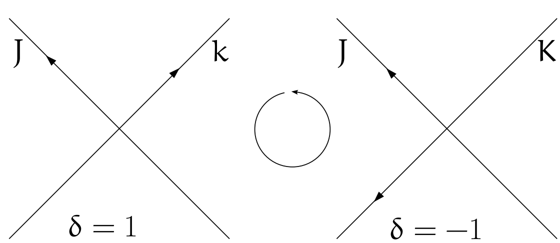


Figure 11.3: Figure 3

In Fig. 11.2 $\mathrm{lk}(J, K) = -1$. In Fig. 11.1,

$$\mathrm{lk}(J, K_2) = \mathrm{lk}(J, k_4), \qquad \mathrm{lk}(J, K_3) = 1, \qquad \mathrm{lk}(J, K_1) = -1. \tag{14}$$

We now apply the concept of degree to study singularities of vector fields. Consider a vector field $F \in C^\infty(U, \mathbb{R}^n)$ on the open set $U \subseteq \mathbb{R}^n, n \geqslant 2$, and let us assume that $0 \in U$ is an isolated zero for F. A zero for F is also called a *singularity* for the vector field. We can choose a $p > 0$ with

$$\rho D^n = \{x \in \mathbb{R}^n \big| \|x\| < \rho\} \subseteq U$$

and such that 0 is the only zero for F in ρD^n. Define a smooth map $F_\rho : S^{n-1} \to S^{n-1}$ by

$$F_\rho(x) = \frac{F(\rho x)}{\|F(\rho x)\|}.$$

The homotopy class of F_ρ is independent of the choice of ρ, and by Corollary 11.2 and Theorem 11.9, $\deg F_\rho \in \mathbb{Z}$ is independent of ρ.

Definition 11.16 The degree of F_ρ is called local index of F at 0, and is denoted $\iota(F;0)$.

Lemma 11.17 Suppose $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has the origin as its only zero. Then

$$F : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$$

induces multiplication by $\iota(F;0)$ on $H^{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{R}$.

Proof. Let $i : S^{n-1} \to \mathbb{R}^n - \{0\}$ be thw inclusion map and $r : \mathbb{R}^{n-1} - \{0\} \to S^{n-1}$ the restriction $r(x) = x/\|x\|$. We have $\iota(F;0) = \deg F_1$, where $F_1 = r \circ F \circ i$. The lemma follows from the commutative diagram below, where $H^{n-1}(i)$ and $H^{n-1}(r)$ are inverse isomorphisms:

$$\begin{tikzcd}
H^{n-1}(\mathbb{R}^n - \{0\}) \arrow[r, "H^{n-1}(F)"] \arrow[d, "H^{n-1}(r)"] & H^{n-1}(\mathbb{R}^n - \{0\}) \arrow[d, shift left, "H^{n-1}(i)"] \\
H^{n-1}(S^{n-1}) \arrow[r, "H^{n-1}(F_1)"] & H^{n-1}(S^{n-1}) \arrow[u, shift left, "H^{n-1}(r)"]
\end{tikzcd}$$

□

Given a diffeomorphism $\phi : U \to V$ to an open set $V \subseteq \mathbb{R}^n$ and a vector field on U, we can define the direct imgae $\phi_* F \in C^\infty(V, \mathbb{R}^n)$ by

$$\phi_* F(q) = D_q\phi(F(q)), p = \phi^{-1}(q).$$

Lemma 11.18 If $F \in C^\infty(U, \mathbb{R}^n)$ has 0 as an isolated singularity and $\phi : U \to V$ is a diffeomorphism to an open set $V \subseteq \mathbb{R}^n$ with $\phi(0) = 0$, then

$$\iota(\phi_* F; 0) = \iota(F, 0).$$

Proof. By shrinking U and V we can restrict ourselves to considering the case where 0 is the only zero for F in U, and where there exists a diffeomorphism $\psi : V \to \mathbb{R}^n$. The assertion about ϕ will follow from the corresponding assertions about ψ and $\psi \circ \phi$, since

$$\psi_*(\phi_* F) = (\psi \circ \phi)_* F.$$

Thus it suffices to treat the case where $\phi : U \to \mathbb{R}^n$ is a diffeomorphism and where $Y = \phi_* F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has the origin as its only singularity. Let $U_0 \subseteq U$ be open and star-shaped around 0. We define a homotopy

$$\Phi : U_0 \times [0,1] \to \mathbb{R}^n; \qquad \Phi_t(x) = \Phi(x,t) = \begin{cases} (D_0\phi)x & \text{if } t = 0 \\ \phi(tx)/t & \text{if } t \neq 0. \end{cases}$$

For $x \in U_0$,

$$\phi(x) = \int_0^1 \frac{d}{dt}\phi(tx)\,dt = \int_0^1 \left(\sum_{i=1}^n x_i \frac{\partial \phi_i}{\partial x_i}(tx)\right) dt = \sum_{i=1}^n x_i \phi_i(x).$$

where $\phi_i \in C^\infty(U_0, \mathbb{R}^n)$ is given by

$$\phi_i(x) = \int_0^1 \frac{\partial \phi}{\partial x_i}(tx)\,dt.$$

It follows that

$$\Phi(x,t) = \sum_{i=1}^n x_i \phi_i(tx)$$

and in particular that Φ has a smooth extension to an open set W with $U_0 \times [0,1] \subseteq W \subseteq U_0 \times \mathbb{R}$.
For each $t \in [0,1], \Phi_t$ is a diffeomorphism from U_0 to an open subset of $\mathbb{R}^n$. Consider the direct image under Φ_t^{-1} of Y restricted to $\Phi_t(U_0)$:

$$X_t = (\Phi_t^{-1})_* Y \in C^\infty(U_0, \mathbb{R}^n) \qquad X_t(x) = (D_x\Phi_t)^{-1} Y(\Phi_t(x)).$$

The function $X_t(x)$ is smooth on W. Now $X_1 = F_{|U_0}$ and $X_0 = (A^{-1})_* Y$, where $A = D_0\phi$.
Choose $\rho > 0$ such that $\rho D^n \subseteq U_0$. The homotopy $S^{n-1} \times [0,1] \to S^{n-1}$ given by

$$X_t(\rho x)/\|X_t(\rho x)\|, \qquad 0 \leqslant t \leqslant 1,$$

and Corollary 11.2 shows that

$$\imath(F;0) = \imath(X_1;0) = \imath(X_0;0) = \imath((A^{-1})_* Y; 0).$$

Since $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear we have $(A^{-1})_* Y = A^{-1} \circ Y \circ A : \mathbb{R}^n \to \mathbb{R}^n$. This yields the commutative diagram

$$\begin{tikzcd} \mathbb{R}^n - \{0\} \arrow[r, "(A^{-1})_* Y"] \arrow[d, "A"] & \mathbb{R}^n - \{0\} \arrow[d, "A"] \\ \mathbb{R}^n - \{0\} \arrow[r, "Y"] & \mathbb{R}^n - \{0\} \end{tikzcd}$$

Now use the function H^{n-1} and apply Lemma 11.17 to both Y and $(A^{-1})_*Y$ to get $\iota((A^{-1})_*Y;0) = \iota(Y;0)$. Hence $\iota(F;0) = \iota(Y;0)$. $\square$

Definition 11.19 Let X be a smooth tangent vector field on the manifold $M^n, n \geqslant 2$ with $p_0 \in M$ as an isolated zero. The local index $\iota(X;p_0) \in \mathbb{Z}$ of X is defined by

$$\iota(X;p_0) = \iota(h_* X_{|U};0),$$

where (U,h) is an arbitrary chart around p_0 with $h(p_0) = 0$.

We note that Lemma 11.18 shows that the local index does not depend on the choice of (U,h). One can picture vector fields in the plane by drawing their integral curves, e.g.

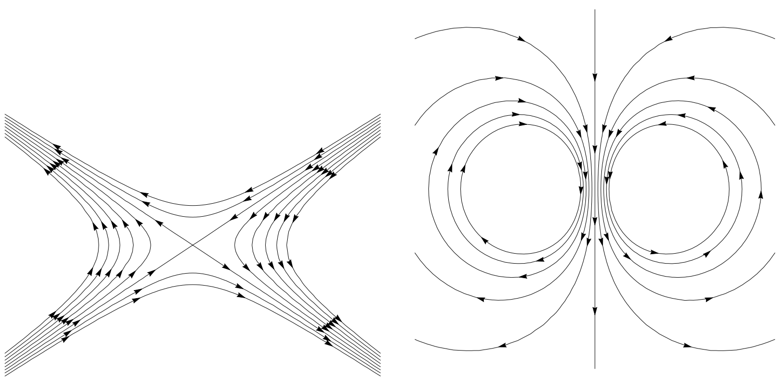


Figure 11.4: Figure 4

Let X be a smooth tangent vector field on Mn and let $p_0 \in M^n$ be a zero. Let

$$F = h_*(X_{|U}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$$

for a chart (U,h) with $h(p_0) = 0$. If $D_0F : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then p_0 is said to be a non-degenerate singularity or zero. Note that by the inverse function theorem F is a local diffeomorphism around 0, such that 0 is an isolated zero for F. Hence $p_0 \in M^n$ is also an isolated zero for X.

Lemma 11.20 If p_0 is a non-degenerate singularity, then

$$\iota(X,p_0) = \operatorname{sign}(\det D_0F) \in \{\pm 1\}.$$

Proof. By shrinking U we may assume that h maps U diffeomorphically onto an open set $U_0 \in \mathbb{R}^n$, which is star-shaped around 0, and that F is a diffeomorphism from U_0 to an open set. As in the proof of Lemma 11.18 we can define a homotopy

$$G : U_0 \times [0,1] \to \mathbb{R}^n; \qquad G(x,t) = \begin{cases} D_0F & \text{if } t = 0 \\ F(tx)/t & \text{if } t \neq 0. \end{cases}$$

where G can be extended smoothly to an open set W in $U_0 \times \mathbb{R}$ that contains $U_0 \times [0,1]$. Choose $p > 0$ so that $\rho D^n \subseteq U_0$. We get a homotopy $\tilde{G} : S^{n-1} \times [0,1] \to S^{n-1}$,

$$\tilde{G}(x,t) = G(\rho x, t)/\|G(\rho x, t)\|.$$

between the map F_ρ in Definition 11.16 and the analogous map A_ρ with $A = D_0 F$. It follows from Corollary 11.2 that

$$\iota(X; p_0) = \iota(F; 0) = \deg(F_\rho) = \deg A_\rho = \iota(A; 0).$$

The map $f_A : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ induced by A operates on $H^{n-1}(\mathbb{R}^n - \{0\})$ by multiplication by $\iota(X; p_0)$; cf. Lemma 11.17. The result follows from Lemma 6.14. □

Definition 11.21 Let X be a smooth vector field on M^n, with only isolated singularities. For a compact set $R \subseteq M$ we define the total index of X over R to be

$$\text{Index}(X; R) = \sum \iota(X; p)$$

where the summation runs over the finite number of zeros $p \in R$ for X. If M is compact we write Index (X) instead of Index $(X; M)$.

Theorem 11.22 Let $F \in C^\infty(U, \mathbb{R})$ be a vector field on an open set $U \subseteq \mathbb{R}^n$, with only isolated zeros. Let $R \subseteq U$ be a compact domain with smooth boundary ∂R, and assume that $F(p) \neq 0$ for $p \in \partial R$. Then

$$\text{Index}(F; R) = \deg(f),$$

where $f : \partial R \to S^{n-1}$ is the map $f(x) = F(x)/\|F(x)\|$.

PROOF. Let $p_1, \cdots, p_k$ be the zeros in R for F, and choose disjoint closed balls $D_j \subseteq R - \partial R$, with centers p_j. Define

$$f_j : \partial D_j \to S^{n-1}; \qquad f_j(x) = F(x)/\|F(x)\|.$$

We apply Proposition 11.11 with $X = R - \cup_j \mathring{D}_j$. The boundary ∂x is the disjoint union of aR and the $(n-1)$-spheres $\partial D_1, \cdots, \partial D_k$. Here ∂D_j, considered as boundary component of X, has the opposite orientation to the one induced from D_j. Thus

$$\deg(f) + \sum_{j=1}^{k} -\deg(f_j) = 0.$$

Finally $\deg(f_j) = \iota(F; p_j)$ by the definition of local index and Corollary 11.3. □

Corollary 11.23 In the situation of Theorem 11.22, Index$(F; R)$ depends only on the restriction of F to ∂R.

Corollary 11.24 In the situation of Theorem 11.22, suppose for every $p \in \partial R$ that the vector $F(p)$ points outward. Let $g : \partial R \to S^{n-1}$ be the Gauss map which to $p \in \partial R$ associates the outward pointing unit normal vector to ∂R. Then

$$\mathrm{Index}(F; R) = \deg(g).$$

Proof. By Corollary 11.2 it sufficies to show that f and g are homotopic. Since $f(p)$ and $g(p)$ belong to the same open half-space of $\mathbb{R}^n$, the desired homotopy can be defined by

$$\frac{(1-t)f(p) + tg(p)}{\|(1-t)f(p) + tg(p)\|}. \qquad (0 \leqslant t \leqslant 1).$$

□

□

Lemma 11.25 Suppose $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ has the origin as its only zero. Then there exists an $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$, with only non-degenerate zeros, that coincides with F outside a compact set.

Proof. We choose a function $\phi \in C^\infty(\mathbb{R}^n, [0, 1])$ with

$$\phi(x) = \begin{cases} 1 & \text{if } \|x\| \leqslant 1 \\ 0 & \text{if } \|x\| \geqslant 2. \end{cases}$$

We want to define $\tilde{F}(x) = F(x) - \phi(x)w$ for a suitable $w \in \mathbb{R}^n$. For $\|x\| > 2$ we have $\tilde{F}(x) = F(x)$. Set

$$c = \inf_{1 \leqslant \|x\| \leqslant 2} \|F(x)\| > 0$$

and choose $w < c$. For $1 \leqslant \|x\| \leqslant 2, \|\tilde{F}(x)\| \geqslant c - \|w\| > 0$. Thus all zeros of $\tilde{F}$ belong to the open unit ball D^n. Since $\tilde{F}$ coincides with $F - w$ on $\mathring{D}^n$

$$\tilde{F}^{-1}(0) = \mathring{D}^n \cap F^{-1}(w).$$

We can pick w as a regular value of F with $\|w\| < c$ by Sard's theorem. Then $D_p\tilde{F} = D_pF$ will be invertible for all $p \in \tilde{F}^{-1}(0)$, and P has the desired properties. □

Note, by Corollary 11.23, that

$$\iota(F; 0) = \sum_{p \in \tilde{F}^{-1}(p)} \iota(\tilde{F}, p) \tag{15}$$

Here is a picture of F and $\tilde{F}$ in a simple case:
The zero for F of index -2 has been replaced by two non-degenerate zeros for $\tilde{F}$, both of index -1.

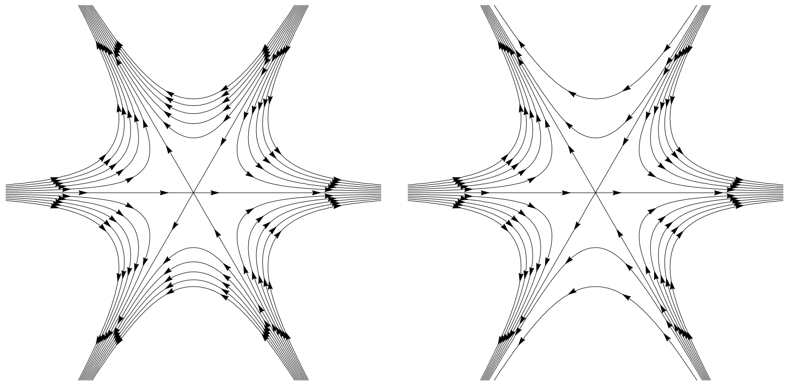


Figure 11.5: Figure 5(Left–F; Right–$\tilde{F}$)

Corollary 11.26 Let X be a smooth vector field on the compact manifold M^n with isolated singularities. Then there exists a smooth vector field $\tilde{X}$ on M having only non-degenerate zeros and with

$$\mathrm{Index}(X) = \mathrm{Index}(\tilde{X}). \tag{16}$$

Proof. We choose disjoint coordinate patches which are diffeomorphic to $\mathbb{R}^n$ around the finitely many zeros of X, and apply Lemma 11.25 on the interior of each of them to obtain $\tilde{X}$. The formula then follows from (16). $\square$

Theorem 11.27 Let $M^n \subseteq \mathbb{R}^n$ be a compact smooth submanifold and let N_ϵ be a tubular neighborhood of radius $\epsilon > 0$ around M. Denote by $g : \partial N_\epsilon \to S^{n+k-1}$ the outward pointing Gauss map. If X is an arbitrary smooth vector field on M^n with isolated singularities, then

$$\mathrm{Index}(X) = \deg(g).$$

Proof. By Corollary 11.26 one may assume that X only has non-degenerate zeros. From the construction of the tubular neighborhood we have a smooth projection $\pi : N \to M$ from an open tubular neighborhood N with $N_\epsilon \subseteq N \subseteq \mathbb{R}^{n+k}$, and can define a smooth vector field F on N by

$$F(q) = X(\pi(q)) + (q - \pi(q)). \tag{17}$$

ince the two summands are orthogonal, $F(q) = 0$ if and only if $q \in M$ and $X(q) = 0$. For $q \in \partial N_\epsilon$, $q - \pi(q)$ is a vector normal to $T_q\partial N_\epsilon$ pointing outwards. Hence $X(\pi(q)) \in T_q\partial N_\epsilon$ and $F(q)$ points outwards. By Corollary 11.24

$$\mathrm{Index}(F; N) = \deg(g).$$

and it suffices to show that $\imath(X;p) = \imath(F;p)$ for an arbitrary zero of X. In local coordinates around p in M, with p corresponding to $0 \in \mathbb{R}^n$, X can be written in the form

$$X = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}, \tag{18}$$

where $f_i(0) = 0$, and by Lemma 11.20 $\iota(X; p)$ is the sign of

$$\det \left(\frac{\partial f_i}{\partial x_j}(0) \right). \tag{19}$$

By differentiating (18) and substituting 0 one gets

$$\frac{\partial X}{\partial x_j}(0) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(0) \frac{\partial}{\partial x_i}. \tag{20}$$

It follows from (17) that $D_p F : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is the identity on $T_p M^\perp$ and by (20) $D_p F$ maps $T_p M$ into itself by the linear map with matrix $((\partial f_i / \partial x_i)(0))$ (with respect to the basis $(\partial/\partial x_i)_0$). It follows that p is a non-degenerate zero for F and that $\det D_p F$ has the same sign as the Jacobian in (19). $\square$

12. THE POINCARÉ-HOPF THEOREM

This page intentionally left blank.

13. POINCARÉ DUALITY

This page intentionally left blank.

14. THE COMPLEX PROJECTIVE SPACE $\mathbb{CP}^n$

This page intentionally left blank.

15. FIBER BUNDLES AND VECTOR BUNDLE

This page intentionally left blank.

16. OPERATIONS ON VECTOR BUNDLES AND THEIR SECTIONS

This page intentionally left blank.

17. CONNECTIONS AND CURVATURE

This page intentionally left blank.

18. CHARACTERISTIC CLASSES OF COMPLEX VECTOR BUNDLES

This page intentionally left blank.

19. THE EULER CLASS

This page intentionally left blank.

20. COHOMOLOGY OF PROJECTIVE AND GRASSMANNIAN BUNDLES

This page intentionally left blank.

21. THORN ISOMORPHISM AND THE GENERAL GAUSS-BONNET FORMULA

This page intentionally left blank.

A. SMOOTH PARTITION OF UNIT

The following technical theorem is a much used tool when working with smooth maps and smooth manifolds. For a function $f : U \to \mathbb{R}$ with domain $U \subseteq \mathbb{R}$ the support of f in U is the set

$$\mathrm{supp}_U(f) = \overline{\{x \in U \mid f(x) \neq 0\}}$$

where the bar denotes the closure of the set in the induced topology on U. If U is open in $\mathbb{R}^n$ then $U - \mathrm{supp}_U(d)$ is the largest open subset of U on which f vanishes.

Theorem A.1 If $U \subseteq \mathbb{R}^n$ is open and $\mathcal{V} = (V_i)_{i\in I}$ is a cover of U by open sets V_i, then there exists smooth functions $\phi_i : U \to [0, 1]$ $(i \in I)$, satisfying

(i) $\mathrm{supp}_U(\phi_i) \subseteq V_i$ for all $i \in I$.

(ii) Every $x \in U$ has a neighborhood W on which only finitely many ϕ_i do not vanish.

(iii) For every $x \in U$ we have $\sum_{i\in I} \phi_i(x) = 1$.

We say that $(\phi_i)_{i\in I}$ is a (smooth) partition of unity, which only is subordinate to the cover $\mathcal{V}$.

A family of functions $\phi_i : U \to \mathbb{R}$ that satisfy (ii) is called locally finite. Note that the sum $\sum_{i\in I} \phi_i$ in this case becomes a well-defined function $U \to \mathbb{R}$. Moreover, it is smooth when all the ϕ_i are smooth. The proof of Theorem A.1 requires some preparations.

Lemma A.2 If $A \in \mathbb{R}^n$ is closed and $U \subseteq \mathbb{R}^n$ is open with $A \subseteq U$, then there exists a smooth function $\psi : \mathbb{R}^n \to [0, 1]$ with $\mathrm{supp}_{\mathbb{R}^n}(\psi) \subseteq U$ and $\psi(x) = 1$ for $x \in A$.

Proof. Apply Theorem A.1 to the cover of $\mathbb{R}^n$ consisting of the open sets $V_1 = U$ and $V_2 = \mathbb{R}^n - A$. Now $\psi = \phi_1$ has the desired properties. $\square$

Lemma A.3 Suppose that $A \subseteq U_0 \subseteq U \subseteq \mathbb{R}^n$, where U_0 and U are open in $\mathbb{R}^n$ and A is closed in U (in the induced topology from $\mathbb{R}^n$). Let $h : U \to W$ be a continuous map to an open set $W \subseteq \mathbb{R}^m$ with smooth restriction to U_0. For any continuous function $\epsilon : U \to (0, \infty)$ there exists a smooth map $f : U \to W$ that satisfies

(i) $\|f(x) - h(x)\| \leqslant \epsilon(x)$ for all $x \in U$.

(ii) $f(x) = h(x)$ for all $x \in A$.

PROOF. If $W \neq \mathbb{R}^m$ then $\epsilon(x)$ can be replaced by

$$\epsilon_1(x) = \min(\epsilon(x), \frac{1}{2}\,d(h(x), \mathbb{R}^n - W))$$

where $d(y, \mathbb{R}^n - W) = \inf\{\|y - z\| \big| z \in \mathbb{R}^n - W\}$. If $f : U \to W$ satisfies (i) with ϵ_1 instead of ϵ, we will automatically get $f(U) \subseteq W$. Hence, without loas of generality, we may assume that $W = \mathbb{R}^m$.
Using the continuity of h and ϵ, we can find for each $p \in U - A$ an open set U_p with $p \in U_p \subseteq U - A$, such that $\|h(x) - h(p)\| \leqslant \epsilon(x)$ for all $x \in U_p$. Apply Theorem A.1 to the open cover of U consisting of the sets U_0 and $U_p, p \in U - A$. This yields smooth functions ϕ_0 and ϕ_p from U into $[0, 1]$, which satisfy Theorem A.1.(i), (ii) and (iii). By local finiteness, smoothness of h on U_0 and the property $\mathrm{supp}_U(\phi_0) \subseteq U_0$, we can define a smooth function $f : U \to \mathbb{R}^m$ by

$$f(x) = \phi_0(x)h(x) + \sum_{p \in U - A} \phi_p(x)h(p).$$

From Theorem A.1.(iii) one obtain $h(x) = \phi_0(x)h(x) + \sum_{p\in U-A} \phi_p(x)h(p)$ and thus

$$f(x) - h(x) = \sum_{p \in U - A} \phi_p(x)(h(p) - h(x)).$$

Now (ii) of the lemma follows because $\mathrm{Supp}(\phi_p) \subseteq U_p \subseteq U - A$, and (i) follows from the calculation

$$\begin{aligned}\|f(x) - h(x)\| &\leqslant \sum_{p \in U - A} \phi_p(x)\|h(p) - h(x)\| = \sum_{p \in U - A, x \in U_p} \phi_p(x)\|h(p) - h(x)\| \\ &\leqslant \sum \phi_p(x)\epsilon(x) = \left(\sum \phi_p(x)\right) \cdot \epsilon(x) \leqslant \epsilon(x).\end{aligned}$$

□

B. INVARIANT POLYNOMIALS

This page intentionally left blank.

C. PROOF OF LEMMAS 12.12 AND 12.13

This page intentionally left blank.

D. EXERCISES

1.1. Perform the calculations of Theorem 1.6.

1.2. Let $W \subseteq \mathbb{R}^3$ be the open set

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \big| \text{ either } x_3 \neq 0 \text{ or } x_1^2 + x_2^2 < 1\}.$$

Prove the existence and uniqueness of a function $f \in C^\infty(W, \mathbb{R})$ such that $\text{grad}(F)$ is the vector field considered in Example 1.8 and $F(0) = 0$. Find a simple expression for F valid when $x_1^2 + x_2^2 < 1$. (Hint: First note that F is constant on the open disc in the x_1, x_2-plane bounded by the unit circle S. Then integrate along lines parallel to the x_3-axis.)

2.1. Prov the formula in Remark 2.10.

2.2. Find an $\omega \in \text{Alt}^2 \mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.

2.3. Show that there exist isomorphisms

$$\mathbb{R}^3 \xrightarrow{i} \text{Alt}^1 \mathbb{R}^3, \qquad \mathbb{R}^3 \xrightarrow{j} \text{Alt}^2 \mathbb{R}^3$$

given by

$$i(v)(w) = \langle v, w \rangle, \qquad j(v)(w_1, w_2) = \det(v, w_1, w_2)$$

where $\langle , \rangle$ is the usual inner product. Show that for $v_1, v_2 \in \mathbb{R}^2$, we have

$$i(v_1) \wedge i(v_2) = j(v_1 \times v_2).$$

2.4. ...

7.1. Show that $\mathbb{R}^n$ does not contain a subset homeomorphic to D^m when $m > n$.

7.2. Let $\Sigma \subseteq \mathbb{R}^n$ be homeomorphic to S^k $(1 \leqslant k \leqslant n-2)$. Show that

$$H^p(\mathbb{R}^n - \Sigma) \cong \begin{cases} \mathbb{R} & \text{for } p = 0, n-k-1, n-1 \\ 0 & \text{otherwise .} \end{cases}$$

7.3. Show that there is no continuous map $g : D^n \to S^{n-1}$ with $g|_{S^{n-1}} \simeq \text{id}|_{S^{n-1}}$.

7.4. ...

9.1. Let $M \subseteq \mathbb{R}^l$ be a differentiable submanifold and assume the points $p \in \mathbb{R}^l$ and $p_0 \in M$ are such that $\|p - p_0\| \leqslant \|p - q\|$ for all $q \in M$. Show that $p - p_0 \in T_{p_0}M^\perp$.

9.2. A smooth map $\varphi : M^m \to N^n$ between smooth manifolds is called immersive at $p \in M$, when

$$D_p\varphi : T_pM \to T_qN, \qquad q \in \varphi(p)$$

is injective. Show that there exists smooth charts (U, h) in M with $p \in U$, $h(p) = 0$, and (V, k) in N with $q \in V$, $k(q) = 0$ such that

$$k \circ \varphi \circ h^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0).$$

in a neighborhood of 0.

(Hint: Reduce the problem to the case where $\varphi : W \to \mathbb{R}^n$ is on an open neighborhood W in $\mathbb{R}^n$ of 0 with $\varphi(0) = 0$, and

$$\left(\frac{\partial\varphi_i(0)}{\partial x_j}\right)_{1 \leqslant i,j \leqslant m}$$

is an invertiable $m \times m$ matrix. Apply the inverse function theorem to

$$F : W \times \mathbb{R}^{n-m} \to \mathbb{R}^n;$$
$$F(x_1, \cdots, x_n) = (\varphi_1(x_1, \cdots, x_m), \cdots, \varphi_m(x_1, \cdots, x_m), x_{m+1}, \cdots, x_n).)$$

9.3. ...

BIBLIOGRAPHY

[1] G. E. Bredon. *Topology and Geometry.* New York: Springer Verlag, 1993.

[2] M. P. do Carmo. *Differential Geometry of Curves and Surfaces.* New Jersey: Prentice-Hall Inc., 1976.

[3] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds.* Oxford: Oxford University Press, 1990.

[4] M. H. Freedman and F. Quinn. *Topology of 4-Manifolds.* New Jersey: Princeton University Press, 1990.

[5] M. W. Hirsch. *Differential Topology.* New York: Springer-Verlag, 1976.

[6] S. Lang. *Algebra.* Massachusetts: Addison-Wesley, 1965.

[7] W. S. Massey. *Algebraic Topology: An Introduction.* Hartcourt, Brace World Inc., 1967.

[8] J. Milnor. *Morse Theory.* New Jersey: Princeton University Press, 1963.

[9] J. Milnor and J. Stasheff. *Characteristic Classes.* Annals of Math. Studies, No 76. New Jersey: Princeton University Press, 1974.

[10] E. E. Moise. *Geometric Topology in Dimensions 2 and 3.* New York: Springer-Verlag, 1977.

[11] W. Rudin. *Real and Complex Analysis.* New York: McGraw-Hill, 1966.

[12] T. B. Rushing. *Topological Embeddings.* New York: Academic Press, 1973.

[13] H. Whitney. *Geometric Integration Theory.* New York: Princeton University Press, 1957.

INDEX