From Calculus to Cohomology

de Rham cohomology and characteristic classes

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PREFACE

This text offers a self-contained exposition of the cohomology of differential forms, de Rham cohomology, and of its application to characteristic classes defined in terms of the curvature tensor. The only formal prerequisites are knowledge of standard calculus and linear algebra, but for the later part of the book some prior knowledge of the geometry of surfaces, Gaussian curvature, will not hurt the reader.
The first seven chapters present the cohomology of open sets in Euclidean spaces and give the standard applications usually covered in a first course in algebraic topology, such as Brouwer's fixed point theorem, the topological invariance of domains and the Jordan-Brouwer separation theorem. The next four chapters extend the definition of cohomology to smooth manifolds, present Stokes' theorem and give a treatment of degree and index of vector fields, from both the cohomological and geometric point of view.
The last ten chapters give the more advanced part of cohomology: the Poincaré-Hopf theorem, Poincare duality, Chern classes, the Euler class, and finally the general Gauss-Bonnet formula. As a novel point we prove the so called splitting principles for both complex and real oriented vector bundles. The text grew out of numerous versions of lecture notes for the beginning course in topology at Aarhus University. The inspiration to use de Rham cohomology as a first introduction to topology comes in part from a course given by G. Segal at Oxford many years ago, and the first few chapters owe a lot to his presentation of the subject. It is our hope that the text can also serve as an introduction to the modern theory of smooth four-manifolds and gauge theory.
The text has been used for third and fourth year students with no prior exposure to the concepts of homology or algebraic topology. We have striven to present all arguments and constructions in detail. Finally we sincerely thank the many students who have been subjected to earlier versions of this book. Their comments have substantially changed the presentation in many places.

Aarhus, January 1996

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1. INTRODUCTION

It is well-known that a continuous real function, that is defined on an open set of $\mathbb{R}$has a primitive function. How about multivariable functions? For the sake of simplicity we restrict ourselves to smooth (or C^∞-) functions, i.e. functions that have continuous partial derivatives of all orders. We begin with functions of two variables. Let $f : U \to \mathbb{R}^2$ be a smooth function .defined on an open set of $\mathbb{R}^2$.

Question 1.1 Is there a smooth function $F : U \to \mathbb{R}$, such that:

$$\frac{\partial F}{\partial x_1} = f_1 \text{ and } \frac{\partial F}{\partial x_2} = f_2, \text{ where } f = (f_1, f_2)? \tag{1}$$

Since

$$\frac{\partial^2 F}{\partial x_2 \partial x_1} = \frac{\partial^2 F}{\partial x_1 \partial x_2}$$

we must have

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1} \tag{2}$$

The correct question is therefore whether F exists, assuming $f = (f_1, f_2)$ satisfies (2). Is condition (2) also sufficient?

Example 1.2 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, \right)$$

It is easy to show that (2) is satisfied. However, there is no function $F : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ that satisfies (1). Assume there were; then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1, 0) - F(1, 0) = 0$$

On the other hand the chain rule gives

$$\begin{aligned} \frac{d}{d\theta} F(\cos\theta, \sin\theta) &= \frac{dF}{dx} \cdot (-\sin\theta) + \frac{dF}{dy} \cdot \cos\theta \\ &= -f_1(\cos\theta, \sin\theta) \cdot \sin\theta + f_2(\cos\theta, \sin\theta) \cdot \cos\theta \\ &= 1 \end{aligned}$$

This contradiction can only be explained by the non-existence of F.

Definition 1.3 A subset $X \subseteq \mathbb{R}^n$ is said to be star-shaped with respect to the point $x_0 \in X$ if the line segemtn $\{tx_0 + (1-t)x | t \in [0, 1]\}$ is contained in X for all $x \in X$.

Theorem 1.4 Let $U \subseteq \mathbb{R}^2$ be star-shaped. Then for any smooth function $f : U \to \mathbb{R}^2$ that satisfies (2), Question 1.1 has a solution.

PROOF. For the sake of simplicity we assume that $x_0 = 0 \in \mathbb{R}^2$. Consider the function $F : U \to \mathbb{R}$.

$$F(x_1, x_2) = \int_0^1 [x_1 f_1(tx_1, tx_2) + x_2 f_2(tx_1, tx_2)] \, dt.$$

Then one has

$$\frac{\partial F}{\partial x_1} = \int_0^1 \left[f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_2}{\partial x_1}(tx_1, tx_2) \right] \, dt$$

and

$$\frac{d}{dt} t f_1(tx_1, tx_2) = f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_2}{\partial x_1}(tx_1, tx_2)$$

Substituting this result into the formula, we get

$$\begin{aligned} \frac{\partial F}{\partial x_1}(x_1, x_2) &= \int_0^1 \left[\frac{d}{dt} t f_1(tx_1, tx_2) + tx_2 \left(\frac{\partial f_2}{\partial x_1}(tx_1, tx_2) - \frac{\partial f_1}{\partial x_2}(tx_1, tx_2) \right) \right] \, dt \\ &= f_1(tx_1, tx_2) \Big|_{t=0}^{1} \\ &= f_1(x_1, x_2) \end{aligned}$$

Analogously, $\frac{\partial F}{\partial x_2} = f_2(x_1, x_2)$. □

Example 1.2 and Theorem 1.4 suggest that the answer to Question 1.1 depends on the "shape" of "topology" of U. Instead of searching for a further examples or counterexamples of set U and fucntion f, we define an invariant of U, which tells us or not the question has an affirmative answer (for all f), assuming the necessary condition (2).
Give the open set $U \subseteq \mathbb{R}^2$, let $C^\infty(U, \mathbb{R}^k)$ denote the set of smooth functions $\phi : U \to \mathbb{R}^k$. This is a vecter space. If $k = 2$ one may consider $\phi : U \to \mathbb{R}^k$ as a vecter filed on U by plotting $\phi(u)$ from the point u. We define the *gradient* and *rotation*:

$$\text{grad} : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^2), \qquad \text{grad} : C^\infty(U, \mathbb{R}^2) \to C^\infty(U, \mathbb{R})$$

by

$$\text{grad}(\phi) = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right), \qquad \text{rot}(\phi) = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}$$

Note that $\text{rot} \circ \text{grad} = 0$. Hence the kernel of rot contains the image of grad,

$$\text{Ker(rot)} = \text{Kernel of rot}$$
$$\text{Im(grad)} = \text{Image of grad}$$

Since both rot and grad are linear operators, Im(grad) is a subspace of Ker(rot). Therefore we can consider the quotient vector space, i.e. the vector space of cosets 0: $\alpha + \mathrm{Im(grad)}$ where 0: $\alpha \in \mathrm{Ker(rot)}$:

$$H^1(U) = \mathrm{Ker(rot)}/\,\mathrm{Im(grad)}. \tag{3}$$

Both Ker(rot) and Im(grad) are infinite-dimensional vector spaces. It is remarkable that the quotient space $H^1(U)$ is usually finite-dimensional. We can now reformulate Theorem 1.4 as

$$H^1(U) = 0 \text{ where } U \subseteq \mathbb{R}^2 \text{ is star-shaped.} \tag{4}$$

On the other hand, Example 1.2 tells us that $H^1(\mathbb{R}^2 - \{0\}) \neq 0$. Later on we shall see that $H^1(\mathbb{R}^2 - \{0\})$ is 1-dimensional, and that $H^1(\mathbb{R}^2 - \cup_{i=1}^{k}\{x_i\}) \cong \mathbb{R}^k$. The dimension of $H^1(U)$ is the number of "holes" in U.
In analogy with (3) we introduce

$$H^0(U) = \mathrm{Ker(grad)} \tag{5}$$

This definition works for open sets U of $\mathbb{R}^k$ with $k \geqslant 1$, when we define

$$\mathrm{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

Theorem 1.5 An open set $U \subseteq \mathbb{R}^k$ is connected if and only if $H^0(U) = \mathbb{R}$.

PROOF. Assume that $\mathrm{grad}(f) = 0$. Then f is locally constant: each $x_0 \in U$ has a neighborhood $V(x_0)$ with $f(x) = f(x_0)$ when $x \in V(x_0)$. If U is connected, then every locally constant function is constant. Indeed, for $x_0 \in U$ the set

$$\{x \in U | f(x) = f(x_0) = f^{-1}(f(x_0))\}$$

is closed because f is continuous, and open since f is locally constant. Hence it is equal to U, and $H^0(U) = \mathbb{R}$. Conversely, if U is not connected, then there exists a smooth, surjective function $f : U \to \{0, 1\}$. Such a function is locally constant, so $\mathrm{grad}(f) = 0$. It follows that $\dim H^0(U) > 1$. □

The reader may easily extend the proof of Theorem 1.5 to show that $\dim H^0(U)$ is precisely the number of connected components of U.
We next consider functions of three variables. Let $U \subseteq \mathbb{R}^3$ be an open set. A real function on U has three partial derivatives and (2) is replaced by three equations. We introduce the notation

$$\begin{aligned} \mathrm{grad} &: C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^3) \\ \mathrm{rot} &: C^\infty(U, \mathbb{R}^3) \to C^\infty(U, \mathbb{R}^3) \\ \mathrm{div} &: C^\infty(U, \mathbb{R}^3) \to C^\infty(U, \mathbb{R}) \end{aligned}$$

for the linear operators defined by

$$\operatorname{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$
$$\operatorname{rot}(f) = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$
$$\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

Note that $\operatorname{rot} \circ \operatorname{grad} = 0$ and $\operatorname{div} \circ \operatorname{rot} = 0$. We define $H^0(U)$ and set $H^1(U)$ as in Equations (3) and (5) and

$$H^2(U) = \operatorname{Ker}(\operatorname{div}) / \operatorname{Im}(\operatorname{rot})$$

Theorem 1.6 For an open star-shaped set in $\mathbb{R}^3$ we have that $H^0(U) = R, H^1(U) = 0$ and $H^2(U) = 0$.

PROOF. The values of $H^0(U)$ and $H^1(U)$ are obtained as above, so we shall restrict ourselves to showing that $H^2(U) = 0$. It is convenient to assume that U is star-shaped with respect to 0. Consider a function $F : U \to \mathbb{R}^3$ with $\operatorname{div} F = 0$, and define $G : U \to \mathbb{R}^3$ by

$$G(x) = \int_0^1 (F(tx) \times tx)\, dt$$

where the $\times$ denotes the cross product.

$$(f_1, f_2, f_2) \times (x_1, x_2, x_3) = \begin{vmatrix} e_1 & f_1 & x_1 \\ e_2 & f_2 & x_2 \\ e_3 & f_3 & x_3 \end{vmatrix} = (f_2x_3 - f_3x_2, f_3x_1 - f_1x_3, f_1x_2 - f_2x_1)$$

Straightforward calculations give

$$\operatorname{rot}(F(tx) \times tx) = \frac{d}{dt}(t^2 F(tx))$$

Hence

$$\operatorname{rot} G(x) = \int_0^1 \frac{d}{dt}(t^2 F(tx))\, dt = F(x)$$

□

If $U \in \mathbb{R}^3$ is not star-shaped both $H^1(U)$ and $H^2(U)$ may be non-zero.

Example 1.7 Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 = 1, x_3 = 0\}$ be the unit circle in the (x_1, x_2)-plane. Consider the function

$$f(x_1, x_2, x_3) = \left(\frac{-2x_1x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{-2x_2x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{x_1^2 + x_2^2 - 1}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}\right)$$

on the open set $U = \mathbb{R}^3 - S$.

One finds that $\operatorname{rot}(f) = 0$. Hence f defines an element $[f] \in H^1(U)$. By integration along a curve γ in U, which is linked to S (as two links in a chain), we shall show that $[f] \neq 0$. The curve in question is

$$\gamma(t) = \left(\sqrt{1+\cos t}, 0, \sin t\right), -\pi \leqslant t \leqslant \pi$$

Assume $\operatorname{grad}(F) = f$ as a function on U. We can determine the integral of $\frac{d}{dt}F(\gamma(t))$ in two ways. On the hand we have

$$\int_{\pi-\epsilon}^{-\pi+\epsilon} \frac{d}{dt}F(\gamma(t))\, dt = F(\gamma(-\pi+\epsilon)) - F(\gamma(\pi-\epsilon)) \to 0 \qquad \text{for } \epsilon \to 0$$

and on the other hand the chain rule gives

$$\begin{aligned}\frac{d}{dt}F(\gamma(t)) &= f_1(\gamma(t)) \cdot \gamma_1'(t) + f_2(\gamma(t)) \cdot \gamma_2'(t) + f_3(\gamma(t)) \cdot \gamma_3'(t) \\ &= \sin^2 t + 0 + \cos^2 t = 1.\end{aligned}$$

Therefore the integral also converges to 2π, which is a contradiction.

Example 1.8 Let U be an open set in $\mathbb{R}^k$ and $X : U \to \mathbb{R}^k$ a smooth function (a smooth vector field). Recall that the *energy* $A_\gamma(X)$, of X along a smooth curve $\gamma : [a, b] \to U$ is defined by the integral

$$A_\gamma(X) = \int_a^b \langle X \circ \gamma(t), \gamma'(t)\rangle\, dt$$

where $\langle\cdot\rangle$ denotes the standard product. If $X = \operatorname{grad}(\Phi)$ and $\Phi_\gamma(a) = \Phi_\gamma(b)$, then the energy is zero, since

$$\langle X \circ \gamma(t), \gamma'(t)\rangle = \frac{d}{dt}\Phi(\gamma(t))$$

by the rule; compare Example 1.2.

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2. THE ALTERNATING ALGEBRA

Let V be a vector space over $\mathbb{R}$. A map

$$f : \underbrace{V \times V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

is called k-*linear* (or *multilinear*), if f is linear in each factor.

Definition 2.1 A k-linear map $\omega : V^k \to \mathbb{R}$ is said to be alternating if $\omega(\xi_1, \cdots, \xi_k) = 0$ whenever $\xi_i = \xi_j$ for some pair $i \neq j$. The vector space of alternating, k-linear maps is denoted by $\mathrm{Alt}^k(V)$.

We immediately note that $\mathrm{Alt}^k(V) = 0$ if $k > \dim V$. Indeed, let $e_1, \cdots, e_n$ be a basis of V, and let $\omega \in \mathrm{Alt}^k(V)$. Using multilinearity,

$$\omega(\xi_1, \ldots, \xi_k) = \omega\Big(\sum \lambda_{i,1} e_i, \ldots, \sum \lambda_{i,k} e_i\Big) = \sum \lambda_J \omega(e_{j_1}, \ldots, e_{j_k})$$

with $\lambda_J = \lambda_{j_1,1}, \cdots, \lambda_{j_k,k}$. Since $k > n$, there must be at least one repetition among the elments $e_{j_1}, \cdots, e_{j_k}$. Hence $\omega(e_{j_1}, \cdots, e_{j_k}) = 0$.
The symmetric group of permutations of the set $\{1, \cdots, k\}$ is denoted by $S(k)$. We remind the reader that any permutation can be written as a composition of transpositions. The transposition that interchanges i and j will be denoted by (i, j). Furthemiore, and this fact will be used below, any permutation can be written as a composition of transpositions of the type $(i, i+1)$, $(i, i+1) \circ (i+1, i+2) \circ (i, i+1) = (i, i+2)$ and so forth. The sign of a permutation:

$$\mathrm{sign} : S(k) \to \{\pm 1\} \tag{1}$$

is a homomorphism, $\mathrm{sign}(\sigma \circ \tau) = \mathrm{sign}(\sigma) \circ \mathrm{sign}(\tau)$, which maps every transposition to -1. Thus the sign of $\sigma \in S(k)$ is -1 precisely if σ decomposes into a product consisting of an odd number of transpositions.

Lemma 2.2 If $\omega \in \mathrm{Alt}^k(V)$ and $\sigma \in S(k)$, then

$$\omega\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}\big) = \mathrm{sign}(\sigma)\omega(\xi_1, \ldots, \xi_k).$$

Proof. It is sufficient to prove the formula when $\sigma = (i, j)$. Let

$$\omega_{i,j}(\xi, \xi') = \omega(\xi_1, \ldots, \xi, \ldots, \xi', \ldots, \xi_k),$$

with ξ and ξ' eoccurring at positions i and j respectively. The remaining $\xi_p \in V$ are arbitrary but fixed vectors. From the definition it follows that $\omega_{i,j} \in \mathrm{Alt}^2(V)$. Hence $\omega_{i,j}(\xi_i + \xi_j, \xi_i + \xi_j) = 0$. Bilinearity yields that $\omega_{i,j}(\xi_i + \xi_j) + \omega_{i,j}(\xi_j + \xi_i) = 0$ □

Example 2.3 Let $V = \mathbb{R}^k$ and $\xi_i = (\xi_{i1}, \cdots, \xi_{ik})$. The function $\omega(\xi_1, \cdots, \xi_k) = \det((\xi_{ij}))$ is alternating, by the calculational rules for determinants.

We want to define the *exterior product*

$$\wedge : \mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V) \to \mathrm{Alt}^{p+q}(V).$$

When $p = q = 1$, it is given by $(\omega_1 \wedge \omega_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_2(\xi_1)\omega_1(\xi_2)$.

Definition 2.4 A (p, q)-*shuffle* σ is a permutation of the set $\{1, \cdots, p+q\}$ satisfying

$$\sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

The set of all such permutations is denoted by $S(p, q)$. Since a (p, q)-shuffle is uniquely determined by the set $\{\sigma(1), \cdots, \sigma(p)\}$, the cardinality of $S(p, q)$ is $\binom{p+q}{p}$.

Definition 2.5 (Exterior product) For $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, we defined

$$\begin{aligned}(\omega_1 \wedge \omega_2)&(\xi_1, \ldots, \xi_{p+q}) \\ &= \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}) \cdot \omega_2(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}).\end{aligned}$$

It is obvious that $\omega_1 \wedge \omega_2$ is a $(p+q)$-linear map, but moreover.

Lemma 2.6 If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2\, \mathrm{Alt}^q(V)$ then $\omega_1 \wedge \omega_2 \in \mathrm{Alt}^{p+q}(V)$.

Proof. We first show that $\omega_1 \wedge \omega_2(\xi_1, , \xi_2, \cdots, \xi_{p+q}) = 0$ when $\xi_i = \xi_j$. We let

(i) $S_{12} = \{\sigma \in S(p, q) | \sigma(1) = 1, \sigma(p+1) = 2\}$

(ii) $S_{21} = \{\sigma \in S(p, q) | \sigma(1) = 2, \sigma(p+1) = 1\}$

(iii) $S_0 = S(p, q) - (S_{12} - S_{21})$

If $\sigma \in S_0$ then either $\omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}) = 0$ or $\omega_2(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}) = 0$, since $\xi_P\sigma(1) = \xi_{\sigma(2)}$ or $\xi_{\sigma(p+1)} = \xi_{\sigma(p+2)}$. Left composition with the transposition $\tau = (1, 2)$ is a bijecrtion $S_{12} \to S_{21}$. We therefore have

$$\begin{aligned}&(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \ldots, \xi_{p+q}) \\ &= \sum_{\sigma \in S_{12}} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}) \\ &- \sum_{\sigma \in S_{12}} \mathrm{sign}(\sigma)\omega_1(\xi_{r\sigma(1)}, \ldots, \xi_{\tau\sigma(p)}) \cdot \omega_2(\xi_{\tau\sigma(p+1)}, \ldots \xi_{\tau\sigma(p+q)}).\end{aligned}$$

Since $\sigma(1) = 1$ and $\sigma(p+1) = 2$, while $\tau\sigma(1) = 2$ and $\tau\sigma(p+1) = 1$, we see that $\tau\sigma(i) = \sigma(i)$ where $i \neq 1, p+1$. But $\xi_1 = \xi_2$ so the terms in the two sums cancel. The case $\xi_i = \xi_{i+1}$ is similar. Now $\omega_1 \wedge \omega_2$ is alternating according to Lemma 2.7 below. □

Lemma 2.7 A k-linear map ω is alternating if $\omega(\xi_1, \cdots, \xi_k) = 0$ for all k-tuples with $\xi_i = \xi_{i+1}$ for some $1 \leqslant i \leqslant k-1$.

Proof. $S(k)$ is generated by the transpositions $(i, i+1)$, and by the argument of Lemma 2.6,

$$\omega(\xi_1, \cdots, \xi_i, \xi_{i+1}, \cdots, \xi_k) = -\omega(\xi_1, \cdots, \xi_{i+1}, \xi_i, \cdots, \xi_k).$$

Hence Lemma 2.6 holds for all $\sigma \in S(k)$, and ω is alternating. □

It is clear from the definition that

$$\begin{aligned}
&(\omega_1 + \omega_1') \wedge \omega_2 = \omega_1 \wedge \omega_2 + \omega_1' \wedge w_2 \\
&(\lambda\omega_1) \wedge \omega_2 = \lambda(\omega_1 \wedge \omega_2) = \omega_1 \wedge \lambda\omega_2 \\
&\omega_1 \wedge (\omega_2 + \omega_2') = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_2'
\end{aligned}$$

for $\omega_1, \omega_1' \in \mathrm{Alt}^p(V)$ and $\omega_2, \omega_2' \in \mathrm{Alt}^q(V)$.

Lemma 2.8 If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, then $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$.

Proof. Let $\tau \in S(p+q)$ be the element with

$$\begin{array}{llll}
\tau(1) = p+1, & \tau(2) = p+2, & \cdots, & \tau(q) = p+q. \\
\tau(q+1) = 1, & \tau(q+2) = 2, & \cdots, & \tau(p+q) = p.
\end{array}$$

We have $\mathrm{sign}(\tau) = (-1)^{pq}$. Composition with τ defines bijiection

$$S(p,q) \xrightarrow{\cong} S(q,p), \quad \sigma \mapsto \tau \circ \sigma$$

Note that

$$\begin{aligned}
\omega_2(\xi_{\sigma\gamma(1)}, \cdots, \xi_{\sigma\gamma(q)}) &= \omega_2(\xi_{\tau\sigma(p+1)}, \cdots, \xi_{\tau\sigma(p+q)}). \\
\omega_1(\xi_{\sigma\gamma(q+1)}, \cdots, \xi_{\sigma\gamma(p+q)}) &= \omega_1(\xi_{\tau\sigma(1)}, \cdots, \xi_{\tau\sigma(p)}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\omega_2 \wedge \omega_1(\xi_1, \ldots, \xi_{p+q}) \\
&= \sum_{\sigma \in S(q,p)} \mathrm{sign}(\sigma)\omega_2\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(q)}\big)\omega_1\big(\xi_{\sigma(q+1)}, \ldots, \xi_{\sigma(p+q)}\big) \\
&= \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma\tau)\omega_2\big(\xi_{\sigma\tau(1)}, \ldots, \xi_{\sigma\tau(q)}\big)\omega_1\big(\xi_{\sigma\tau(q+1)}, \ldots, \xi_{\sigma\tau(p+q)}\big) \\
&= (-1)^{pq} \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma)\omega_1\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}\big)\omega_2\big(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}\big) \\
&= (-1)^{pq}\omega_1 \wedge \omega_2(\xi_1, \ldots, \xi_{p+q}).
\end{aligned}$$

□

Lemma 2.9 If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$ and $\omega_3 \in \mathrm{Alt}^r(V)$, then

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

PROOF. Let $S(p,q,r) \subseteq S(p+q+r)$ consist of the permutations σ with

$$\begin{gathered}\sigma(1) <\cdot< \sigma(p)\\ \sigma(p+1) <\cdot< \sigma(p+q)\\ \sigma(p+q+1) <\cdot< \sigma(p+q+r).\end{gathered}$$

We will need the subset $S(\tilde{p},q,r)$ and $S(p,q,\sim r)$ of $S(p,q,r)$ given by

$$\begin{aligned}\sigma \in S(\tilde{p},q,r) &\iff \sigma \text{ is the identity on } \{1,\cdots,p\} \text{ and } \sigma \in S(p,q,r)\\ \sigma \in S(p,q,\tilde{r}) &\iff \sigma \text{ is the identity on } \{p+q+1,\cdots,p+q+r\}\\ &\qquad\qquad\qquad\qquad \text{and } \sigma \in S(p,q,r)\end{aligned}$$

There are bijections

$$\begin{aligned}S(p,q,r) \times S(p,q,r) &\xrightarrow{\cong} S(p,q,r); \quad (\sigma,\tau) \mapsto \sigma \circ \tau\\ S(p,q,r) \times S(p,q,\tilde{r}) &\xrightarrow{\cong} S(p,q,r); \quad (\sigma,\tau) \mapsto \tau \circ \sigma.\end{aligned} \tag{2}$$

With these notations we have

$$\begin{aligned}&[\omega_1 \wedge (\omega_2 \wedge \omega_3)](\xi_1,\ldots,\xi_{p+q+r})\\ &= \sum_{\sigma \in S(p,q+r)} \operatorname{sign}(\sigma)\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})(\omega_2 \wedge \omega_3)(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q+r)})\\ &= \sum_{\sigma \in S(p,q+r)} \operatorname{sign}(\sigma) \sum_{\sigma \in S(p,q,r)} \operatorname{sign}(\tau)\Big[\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})\\ &\quad \omega_2(\xi_{\sigma\tau(p+1)},\ldots,\xi_{\sigma\tau(p+q)})\omega_3(\xi_{\sigma\tau(p+q+1)},\ldots,\xi_{\sigma\tau(p+q+r)})\Big]\\ &= \sum_{u \in S(p,q,r)} \Big[\operatorname{sign}(u)\omega_1(\xi_{u(1)},\ldots,\xi_{u(p)})\omega_2(\xi_{u(p+1)},\ldots,\xi_{u(p+q)})\\ &\quad \omega_3(\xi_{u(p+q+1)},\ldots,\xi_{u(p+q+r)})\Big]\end{aligned}$$

where the last equality follows from the first equation in (2). Quite analogously one can calculate $[(\omega_1 \wedge \omega_2) \wedge \omega_3](\xi_1,\cdots,\xi_{p+q+r})$, employing the second equation in (2). □

Remark 2.10 In other textbook on alternating functions one can often see the definition

$$\begin{aligned}&\omega_1 \bar{\wedge} \omega_2(\xi_1,\ldots,\xi_{p+q})\\ =&\frac{1}{p!q!} \sum_{\sigma \in S(p+q)} \operatorname{sign}(\sigma)\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)}).\end{aligned}$$

Note that in this formula $\{\sigma(1),\cdots,\sigma(p)\}$ and $\{\sigma(p+1),\cdots,\sigma(p+q)\}$ are not ordered. There are exactly $S(p) \times S(q)$ ways to come from an ordered set to the arbitrary sequence above; this causes the factor $\frac{1}{p!q!}$, so $\omega_1 \bar{\wedge} \omega_2 = \omega_1 \wedge \omega_2$.

An $\mathbb{R}$-algebra A consists of a vector space over $\mathbb{R}$ and a bilinear map $\mu : A \times A \to A$ which is associative, $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ for every $a, b, c \in A$. The algebra is called *unitary* if there exists a unit element for μ, $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$.

Definition 2.11

(i) A graded $\mathbb{R}$-algebra A_* is a sequence of vector spaces $A_k, k = 0, 1, \cdots,$ and bilinear maps $\mu : A_k \times A_l \to A_{k+l}$ which are associative.

(ii) The algebra A_* is called connected if there exists a unit element $1 \in A_0$ and if $\epsilon : \mathbb{R} \to A_0$, given by $\epsilon(r) = r \cdot 1$, is an isomorphism.

(iii) The algebra called (graded) commutative (or anti-commutative), if $\mu(a, b) = (-1)^{kl}\mu(b, a)$ for all $a \in A_k$ and $b \in A_l$.

The elements in A_k are said to have degree k. The set $\mathrm{Alt}^k(V)$ is a vector space over $\mathbb{R}$ in the usual manner:

$$\begin{aligned}(\omega_1 + \omega_2)(\xi_1, \ldots, \xi_k) &= \omega_1(\xi_1, \ldots, \xi_k) + \omega_2(\xi_1, \ldots, \xi_k)\\ (\lambda\omega)(\xi_1, \ldots, \xi_k) &= \lambda\omega(\xi_1, \ldots, \xi_k), \quad \lambda \in \mathbb{R}.\end{aligned}$$

The product from Definition 2.5 is a bilinear map from $\mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V)$ to $\mathrm{Alt}^{p+q}(V)$. We set $\mathrm{Alt}^0(V) = \mathbb{R}$ and expand the product to $\mathrm{Alt}^0(V) \times \mathrm{Alt}^p(V)$ by using the vector space structure. The basic formal properties of the alternating forms can now be summarized in.

Theorem 2.12 $\mathrm{Alt}^*(V)$ is an anti-commutative and connected graded algebra.

$\mathrm{Alt}^*(V)$ is called the exterior or alternating algebra associated to V.

Lemma 2.13 For 1-forms $\omega_1, \cdots, \omega_p \in \mathrm{Alt}^1(V)$, we have

$$(\omega_1 \wedge \cdots \wedge \omega_p)(\xi_1, \ldots, \xi_p) = \det \begin{pmatrix} \omega_1(\xi_1) & \omega_1(\xi_2) & \cdots & \omega_1(\xi_p) \\ \omega_2(\xi_1) & \omega_2(\xi_2) & \cdots & \omega_2(\xi_p) \\ \vdots & \vdots & & \vdots \\ \omega_p(\xi_1) & \omega_p(\xi_2) & \cdots & \omega_p(\xi_p) \end{pmatrix}$$

Proof. The case $p = 2$ is abvious. We procced by induction on p. According to Definition 2.5,

$$\begin{aligned}&\omega_1 \wedge (\omega_2 \wedge \ldots \wedge \omega_p)(\xi_1, \ldots, \xi_p)\\ &= \sum_{j=1}^{p} (-1)^{j+1} \omega_1(\xi_j)(\omega_2 \wedge \ldots \wedge \omega_p)\Big(\xi_1, \quad, \dot{\xi}_j, \ldots, \xi_p\Big)\end{aligned}$$

where $(\xi_1, \cdots, \hat{\xi_j}, \cdots, \xi_p)$ denotes the $p - 1$-tuple where ξ_j has been omitted. The lemma follows by expanding the determinant by the first row. $\square$

Note, from Lemma 2.13, that if the 1-forms $\omega_1, \cdot, \omega_p \in \mathrm{Alt}^1(V)$ are linearly independent then $\omega_1 \wedge \cdots \wedge \omega_p \neq 0$. Indeed, we can choose elements $\xi \in V$ with $\omega_i(\xi_i) = 0$ for $i \neq j$ and $\omega_j(\xi_j) = 0$, so that $\det(\omega_i(\xi_j)) = 1$. Conversely, if $\omega_1, \cdots, \omega_p$ are linearly dependent, we can express one of them, say ω_p, as a linear combination of the others. If $\omega_p = \sum_{i=1}^{p-1} r_i \omega_i$, then

$$\omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_p = \sum_{i=1}^{p-1} r_i \omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_i = 0,$$

as the determinant in Lemma 2.13 has two rows. We have proved.

Lemma 2.14 For 1-form $\omega_1, \cdots, \omega_p$ on V, $\omega_1 \wedge \cdots \wedge \omega_p \neq 0$ if and only if they are linearly independent.

Theorem 2.15 Let $e_1, \cdots, e_n$ be a basis of V and $\epsilon_1, \cdots, \epsilon_n$ the dual basis of V^*. Then

$$\left\{\epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(k)}\right\}_{\sigma \in S(p,n-p)}$$

is a basis of $\mathrm{Alt}^p(V^*)$. In particular

$$\dim \mathrm{Alt}^p(V^*) = \binom{\dim V}{p}.$$

Proof. Since $\epsilon_i e_j = 0$ when $i \neq j$, and $\epsilon_i e_j = 1$, Lemma 2.13 gives

$$\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_p}(e_{j_1}, \ldots, e_{j_p}) = \begin{cases} 0 & \text{if } \{i_1, \cdots, i_p\} \neq \{j_1, \cdots, j_p\}, \\ \mathrm{sign}(\sigma) & \text{if } \{i_1, \cdots, i_p\} = \{j_{\sigma(1)}, \cdots, j_{\sigma(p)}\}. \end{cases} \tag{3}$$

Here σ is the permutation $\epsilon(i_k) = j_k$. From Lemma 2.13 and (3) we get

$$\omega = \sum_{\sigma \in S(p,n-p)} \omega\big(e_{\sigma(1)}, \ldots, e_{\sigma(p)}\big) \epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)}$$

for any alternating p-form. Thus $\epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(p)}$ generates the vector space $\mathrm{Alt}^p(V)$. Linear independence follows from (3), since a relation

$$\sum_{\sigma \in S(p,n-p)} \lambda_\sigma \epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)} = 0, \quad \lambda_\sigma \in \mathbb{R}$$

evaluated on $(e_{\sigma(1)}, \cdots, e_{\sigma(p)})$ gives $\lambda_\sigma = 0$. □

Note from Theorem 2.15 that $\mathrm{Alt}^n(V) \xrightarrow{\cong} \mathbb{R}$ if $n = \dim V$ and, as mentioned earlier, that $\mathrm{Alt}^p(V) = 0$ if $p > n$. A basis of $\mathrm{Alt}^n(V)$ is given by $\epsilon_1 \wedge \cdots \wedge \epsilon_n$. In particular every alternating n-fonn on $\mathbb{R}^n$ is proportional to the form in Example 2.3.

A linear map $f : V \to W$ induces the linear map

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^p(W) \to \mathrm{Alt}^p(V) \tag{4}$$

by setting $\mathrm{Alt}^p(f)(W)(\xi_1, \cdots, \xi_p) = \omega(f(\xi_1), \cdots, f(\xi_p))$. For the composition of maps we have $\mathrm{Alt}^p(g \circ f) = \mathrm{Alt}^p(f) \circ \mathrm{Alt}^p(g)$, and $\mathrm{Alt}^P(\mathrm{id}) = \mathrm{id}$. These two properties are summarized by saying that $\mathrm{Alt}^p(-)$ is a *contravariant functor*. If $\dim V = n$ and $f : V \to V$ is a linear map then

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^n(V) \to \mathrm{Alt}^n(V)$$

is a linear endomorphism of 1-dimensional vector space and thus multiplication by a number d. From Theorem 2.16 below it follows that $d = \det(f)$. We shall also be using other maps

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^p(V) \to \mathrm{Alt}^p(V)$$

Let $\mathrm{tr}(g)$ denotes the trace of a linear endomorphism g.

Theorem 2.16 The characteristic polynomial ofa linear endomorphism $f : V \to V$ is given by

$$\det(f - t) = \sum_{i=0}^{n} (-1)^i \mathrm{tr}\Big(\mathrm{Alt}^{n-i}(f)\Big) t^i,$$

when $n = \dim V$.

Proof. Choose a basis $e_1, \cdots, e_n$ of V Assume first that $e_1, \cdots, e_n$ are eigenvectors of f,

$$f(e_i) = \lambda_i e_i, i = 1, \cdots, n.$$

Let $\epsilon_1, \cdots, \epsilon_n$ be the dual basis of $\mathrm{Alt}^1(V)$. Then

$$\mathrm{Alt}^p(f)\big(\epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(p)}\big) = \lambda_{\sigma(1)} \cdot \cdots \cdot \lambda_{\sigma(p)} \epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(p)}$$

and

$$\mathrm{tr}\,\mathrm{Alt}^p(f) = \sum_{\sigma \in S(p, n-p)} \lambda_{\sigma(1)} \cdot \cdots \cdot \lambda_{\sigma(p)}.$$

On the other hand

$$\det(f - t) = \prod_{i=1}^{n} (\lambda_i - t) = \sum (-1)^{n-p} \left(\sum \lambda_{\sigma(1)} \cdot \cdots \cdot \lambda_{\sigma(p)}\right) t^{n-p}.$$

This proves the formula when f is diagonal.

If f is replaced by gfg^{-l}, with g an isomorphism on V, then both sides of the equation of Theorem 2.16 remain unchanged. This is obvious for the left-hand side and follows for the right-hand side since

$$\mathrm{Alt}^p(gfg^{-1}) = \mathrm{Alt}^p(g)^{-1} \circ \mathrm{Alt}^p(f) \circ \mathrm{Alt}^p(g).$$

by the functor property. Hence $\mathrm{tr}\,\mathrm{Alt}^p(g \circ f \circ g^{-l}) = \mathrm{tr}\,\mathrm{Alt}^p(f)$. Consider the set

$$D = \{gf^{-1}g^{-1} | f \text{ diagonal }, g \in GL(V)\}.$$

If V is a vector space over $\mathbb{C}$ and all maps are complex linear, then D is dense in the set of linear endomorphisms on V. We shall not give a formal proof of this, but it follows since every matrix with complex entries can be approximated arbitrarily closely by a matrix for which all roots of the characteristic polynomial are distinct. Since eigenvectors belonging to different eigenvalues are linearly independent, V has a basis consisting of eigenvectors for such a matrix, which then belongs to D.
For general $f \in \mathrm{End}(V)$ we can choose a sequence $d_n \in D$ with $d_n \to f$ (i.e. the (i, j)-th element in dn converges to the (i, j)-th element in f). Since both sides in the equation we want to prove are continuous, and since the equation holds for d_n , it follows for f. $\square$

It is not true that the set of diagonalizable matrices over $\mathbb{R}$ is dense in the set of matrices over $\mathbb{R}$ – a matrix with imaginary eigenvalues cannot be approximated by a matrix of the form gfg^{-1}, with f a real diagonal matrix. Therefore in the proof of Theorem 2.16 we. must pass to complex linear maps, even if we are mainly interested in real ones.

3. DE RHAM COHOMOLOG

In this chapter U will denote an open set in $\mathbb{R}^n, e_1, \cdots, e_n$ the standard basis and $E_1, \cdots, E_n$ the dual basis of $\mathrm{Alt}^1(\mathbb{R}^n)$.

Definition 3.1 A differential p-form on U is a smooth map $w : U \to \mathrm{Alt}^1(\mathbb{R}^n)$. The vector space of all such maps is denoted by $\Omega^p(U)$.

If $p = 0$ then $\mathrm{Alt}^1(\mathbb{R}^n) = \mathbb{R}$ and $\Omega^0(U)$ is just the vector space of all smooth real-valued functions on U, $\Omega^0(U) = \Omega^0(U, \mathbb{R})$.
The usual derivative of a smooth map $\omega : U \to \mathrm{Alt}^p(\mathbb{R}^n)$ is denoted $D\omega$ and its value at x by $D_x\omega$. It is the linear map

$$D_x\omega : \mathbb{R}^n \to \mathrm{Alt}^p(\mathbb{R}^n)$$

with

$$(D_x\omega)(e_i) = \frac{d}{dt}\omega(x + te_i)\Big|_{t=0} = \frac{\partial \omega}{\partial x_i}(x)$$

In $\mathrm{Alt}^p(\mathbb{R}^n)$ we have the basis $\epsilon_1 \wedge \cdots \wedge \epsilon_p$ where I runs over all sequences with $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$. Hence every $\omega \in \Omega^p(U)$ can be written in the form $\omega(x) = \sum \omega_I(x)\epsilon_I$, with $\omega_I(x)$ smooth real-valued functions of $x \in U$. The differential $D_x\omega$ is the linear map

$$D_x\omega(e_j) = \sum_I \frac{\partial \omega_I}{\partial x_j}(x)\epsilon_I, j = 1, \cdots, n. \tag{1}$$

The function $x \mapsto D_x\omega$ is a smooth map from U to the vector space of linear maps from $\mathbb{R}^n$ to $\mathrm{Alt}^p(\mathbb{R}^n)$

Definition 3.2 The exterior differential $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is the linear operator

$$d_x\omega(\xi_1, \cdots, \xi_{p+1}) = \sum_{l=1}^{p+1} (-1)^{l-1} D_x\omega(\xi_l)(\xi_1, \cdots, \hat{\xi}_l, \cdots, \xi_{p+1})$$

with $(\xi_1, \cdots, \hat{\xi}_l, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \xi_{l-1}, \xi_{l+1}, \cdots, \xi_{p+1})$.

It follows from Lemma 2.7 that $d_x\omega \in \mathrm{Alt}^{p+1}\mathbb{R}^n$. Indeed, if $\xi_i = \xi_{i+1}$, then

$$\begin{aligned}
&\sum_{l=1}^{p+1} (-1)^{l-1} D_x\omega(\xi_l)(\xi_1, \cdots, \hat{\xi}_l, \cdots, \xi_{p+1}) \\
=&(-1)^{i-1} D_x\omega(\xi_i)(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1}) \\
&+ (-1)^i D_x\omega(\xi_{i+1})(\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1})
\end{aligned}$$

because $(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1})$.

Example 3.3 Let $x_i : U \to \mathbb{R}$ be the i-th projection. Then $dx_i \Omega^1(U)$ is the constant map $dx_i : x \to \epsilon_i$. This follows from (1). In general, for $f \in \Omega^0(U)$, (1) shows that

$$d_x f(\zeta) = \frac{\partial f}{\partial x_i}\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}\zeta^n \tag{2}$$

with $(\zeta^1, \cdots, \zeta^n) = \zeta$. In other words, $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

Lemma 3.4 If $\omega(x) = f(x)\epsilon_I$ then $d_x\omega = d_x f \wedge \epsilon_I$.

PROOF. By (1) we have

$$D_x\omega(\zeta) = (D_x f)(\zeta)\epsilon_I = \left(\frac{\partial f}{\partial x_1}\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}\zeta^n\right)\epsilon_I = d_x f(\zeta)\epsilon_I$$

and Definition 3.2 gives

$$\begin{aligned} d_x\omega(\xi_1, \cdots, \xi_{p+1}) &= \sum_{k=1}^{p+1} (-1)^{k-1}\, d_x f(\xi_k)\epsilon_I\left(\xi_1, \cdots, \hat{\xi}_k, \cdots \xi_{p+1}\right) \\ &= [d_x f \wedge \epsilon_I](\xi_1, \cdots, \xi_{p+1}). \end{aligned}$$

$\square$

Note for $\epsilon_I \in \mathrm{Alt}^p(\mathbb{R}^n)$ that

$$\epsilon_k \wedge \epsilon_I = \begin{cases} 0 & \text{if } k \in I \\ (-1)^r \epsilon_J & \text{if } k \notin I \end{cases}$$

with r the number determined by $i_r < k < i_{r+l}$ and $J = (i_1, \cdots, i_r, k, \cdots, i_p)$.

Lemma 3.5 For $p \geqslant 0$ the composition $\Omega^p(U) \to \Omega^{p+1}(U) \to \Omega^{p+2}(U)$ is indentity zero.

PROOF. Let $\omega = f\epsilon_I$. Then

$$d\omega = df \wedge \epsilon_I = \frac{\partial f}{\partial x_1}\epsilon_1 \wedge \epsilon_I + \cdots + \frac{\partial f}{\partial x_n}\epsilon_n \wedge \epsilon_I$$

Now use $\epsilon_i \wedge \epsilon_i = 0$ and $\epsilon_i \wedge \epsilon_j = -\epsilon_j \wedge \epsilon_i$ to obtain that

$$\begin{aligned} d^2\omega &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\epsilon_i \wedge (\epsilon_j \wedge \epsilon_I) \\ &= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)\epsilon_i \wedge \epsilon_j \wedge \epsilon_I \\ &= 0. \end{aligned}$$

$\square$

The exterior product in $\mathrm{Alt}^* \mathbb{R}^n$, induces an exterior product on $\Omega^*(U)$ upon defining

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x)$$

The exterior product of a differential p-form and a differential q-fonn is a differential $(p+q)$-form, so we get a bilinear map

$$\wedge : \Omega^p(U) \times \Omega^q(U) \to \Omega^{p+q}(U)$$

For a smooth function $f \in C^\infty(U, \mathbb{R})$, we have that

$$(f\omega_1) \wedge \omega_2 = f(\omega_1 \wedge \omega_2) = \omega_1 \wedge (f\omega_2)$$

This just expresses the bilinearity of the product in $\mathrm{Alt}^* \mathbb{R}^n$. Also note that $f \wedge \omega = f\omega$ when $f \in \Omega^0 U$ and $\omega \in \Omega^p(U)$.

Lemma 3.6 For $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^q(U)$,

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2$$

Proof. It is sufficient to show the formula when $\omega_1 = f\epsilon_I$ and $\omega_2 = g\epsilon_J$. But then $\omega_1 \wedge \omega_2 = fg\epsilon_I \wedge \epsilon_J$, and

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= d(fg) \wedge \epsilon_I \wedge \epsilon_J = ((df)g + fdg) \wedge \epsilon_I \wedge \epsilon_J \\ &= dfg \wedge \epsilon_I \wedge \epsilon_J + fdg \wedge \epsilon_I \wedge \epsilon_J \\ &= df \wedge \epsilon_I \wedge g\epsilon_J + (-1)^p f\epsilon_I \wedge dg \wedge \epsilon_J \\ &= d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2. \end{aligned}$$

□

Summing up, we have introduced an anti-commutative algebra $\Omega^*(U)$ with a *differential*,

$$\mathrm{d} : \Omega^*(U) \to \Omega^{*+1}(U), \qquad \mathrm{d} \circ \mathrm{d} = 0$$

and d is a *derivation* (satisfies Lemma 3.6): $(\Omega^*(U), \mathrm{d})$ is a commutative *DGA* (differential graded algebra). It is called the *de Rham complex* of U.

Theorem 3.7 There is precisely one linear operator $\mathrm{d} : \Omega^0(U) \to \Omega^{p+1}(U), p = 0, 1, \cdots$, such that

(i) $f \in \Omega^*(U), \mathrm{d}f = \frac{\partial f}{\partial x_i}\epsilon_1 + \cdots + \frac{\partial f}{\partial x_n}\epsilon_n$

(ii) $\mathrm{d} \circ \mathrm{d} = 0$

(iii) $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \mathrm{d}\omega_1 \wedge \mathrm{d}\omega_1$ if $\omega_1 \in \Omega^p(U)$.

PROOF. We have already defined d with the asserted properties. Conversely assume that d' is a linear operator satisfying (i), (ii) and (iii). We will show that d' is the exterior differential.
The first property tells us that $d = d'$ on $\Omega^0(V)$. In particular $d'x_i = dx_i$ for the i-th projection $x_i : U \to \mathbb{R}$. It follows from Example 3.3 that $d'x_i = \epsilon_i$, the constant function. Since $d' \circ d' = 0$ we have that $d'\epsilon_i$. Then (iii) gives $d'\epsilon_I = 0$. Now let $\omega = f\epsilon_I = f \wedge \epsilon_I$. Again by using (iii),

$$d'\omega = d'f \wedge \epsilon_I + f \wedge d'\epsilon_I = d'f \wedge \epsilon_I = df \wedge \epsilon_I = d\omega.$$

Since every p-form is the sum of such special p-forms, $d = d'$ on all of $\Omega^p(U)$. □

For an open set V in $\mathbb{R}^3$, $d : \Omega^1(U) \to \Omega^2(U)$ is given as

$$\begin{aligned} &d(f_1\epsilon_1 + f_2\epsilon_2 + f_3\epsilon_3) = df_1 \wedge \epsilon_1 + df_2 \wedge \epsilon_2 + df_3 \wedge \epsilon_3 \\ &= \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\epsilon_1 \wedge \epsilon_2 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\epsilon_2 \wedge \epsilon_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\epsilon_3 \wedge \epsilon_1. \end{aligned}$$

The first equality follows from Theorem 3.7.(iii), as $\epsilon_i : U \to \mathrm{Alt}^1(\mathbb{R}^3)$ is the constant map, and hence $d\epsilon_i = 0$, by (1). Alternatively, we have already noted that the 1-forms ϵ_i and dx_i agree, and hence $d\epsilon_i = d \circ d(x_i) = 0$ by Theorem 3.7.(ii). The second equality comes from the anti-commutativity, $\epsilon_i \wedge \epsilon_j = -\epsilon_j \wedge \epsilon_i$. and Theorem 3.7.(i). Quite analogously we can calculate that

$$d(g_3\epsilon_1 \wedge \epsilon_2 + g_1\epsilon_2 \wedge \epsilon_3 + g_2\epsilon_3 \wedge \epsilon_1) = \left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3}\right)\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3.$$

Definition 3.8 The p-th (de Rham) cohomology group is the quotient vector space

$$H^p(U) = \frac{\mathrm{Ker}\left(d\colon \Omega^p(U) \to \Omega^{p+1}(U)\right)}{\mathrm{Im}(d\colon \Omega^{p-1}(U) \to \Omega^p(U))}.$$

In particular $H^p(U) = 0$ for $p < 0$, and $H^0(U)$ is the kernel of

$$d : C^\infty(U) \to \Omega^1(U).$$

and therefore is the vector space of maps $f \in C^\infty(U, \mathbb{R})$ with vanishing derivatives. This is precisely the space of locally constant maps.
Let $\sim$ be the equivalence relation on the open set V such that $q_1 \sim q_2$ if there exists a continuous curve $\alpha : [a, b] \to V$ with $\alpha(a) = q_1$ and $\alpha(b) = q_2$. The equivalence classes partition V into disjoint open subsets, namely the connected components of U. A connected component of U is a maximal non-empty subset W of U that cannot be written as the disjoint union of two non-empty open subsets of W (in the topology induced by $\mathbb{R}^n$). An open set $U \subseteq \mathbb{R}^n$ has at most countably many connected components (in each of them one can choose a point with rational coordinates.)

Lemma 3.9 $H^0(U)$ is the vector space of maps $U \to \mathbb{R}$ that are constant on each connected component of U.

Proof. A locally constant function $f : U \to \mathbb{R}$ gives a partition of U into the mutually disjoint open sets $f^{-1}(c), c \in \mathbb{R}$. Consequently $f : U \to \mathbb{R}$ is locally constant precisely when f is constant on each connected component of U. □

It follows that $\dim_{\mathbb{R}} H^0(U)$ (considered as a non-negative integer or ∞) is precisely the number of connected components of U.
The elements in $\Omega^p(U)$ with $d\omega = 0$ are called the closed p-*forms*. The elements of the image $\Omega^{p-1}(U) \subset \Omega^p(U)$ are the *exact* p-*forms*. The p-th cohomology group thus measures whether every closed p-form is exact. This condition is satisfied precisely when $H^p(u) = 0$. A closed p-form $\omega \subset \Omega^p(U)$ gives a cohomology class, denoted by

$$[\omega] = \omega + d\Omega^{p-1}(U) \in H^p(U),$$

and $[\omega] = [\omega']$ if and only if $\omega - \omega'$ is exact. In general the vector space of *closed* p-*form* and the vector space of exact p-forms are infinite-dimensional. In contrast $H^p(U)$ usually has finite dimension.
We can define a bilinear, associative and anti-commutative product

$$H^p(U) \times H^q(U) \to H^{p+q}(U) \tag{3}$$

by setting $[\omega_1][\omega_2] = [\omega_1 \wedge \omega_2]$. This is well-defined because

$$\begin{aligned}(\omega_1 + d\eta_1) \wedge (\omega_2 + d\eta_2) &= \omega_1 \wedge \omega_2 + d\eta_1 \wedge \omega_2 + d\eta_1 \wedge \omega_2 + d\eta_1 \wedge d\eta_2 \\ &= \omega_1 \wedge \omega_2 + d(\eta_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \omega_2 + \eta_1 \wedge d\eta_2)\end{aligned}$$

We want to make $U \to H^p(U)$ into a *contravariant functor*. Thus to a smooth map $\phi : U_1 \to U_2$ between open sets $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$, we shall define a linear map

$$H^p(\phi) : H^p(U_2) \to H^p(U_1)$$

such that

$$\begin{aligned} H^p(\phi_2 \circ \phi_1) &= H^p(\phi_1) \circ H^p(\phi_2) \\ H^p(\mathrm{id}) &= \mathrm{id} \end{aligned} \tag{4}$$

We first make $\Omega^*(-)$ into a contravariant functor.

Definition 3.10 Let $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ be open sets and $\phi : U_1 \to U_2$ a smooth map. The induced morphism $\Omega^p(\phi) : \Omega^p(U_2) \to \Omega^p(U_1)$ is defines by

$$\Omega^p(\phi)(\omega)_x = \mathrm{Alt}^p(D_x\phi) \circ \omega(\phi(x)), \quad \Omega^0(\phi)(\omega)_x = \omega_{\phi(x)}.$$

Frequently one writes ϕ^* instead of $\Omega^p(\phi)$. We note that the analogue of (4) is satisfied. Indeed,

$$\phi^*(\omega)_x(\xi_1, \cdots, \xi_p) = \omega_{\phi(x)}(D_x\phi(\xi_1), \cdots, D_x\phi(\xi_p)),$$

and using the chain rule $D_x(\phi \circ \phi) = D_{\phi(x)}\phi \circ D_x\phi$, for $\phi : U_1 \to U_2$, $\psi : U_2 \to U_3$, it is easy to see that

$$\Omega^p(\psi \circ \phi) = \Omega^p(\phi) \circ \Omega^p(\psi), \qquad \Omega^p(\mathrm{id}_U) = \mathrm{id}_{\Omega^p(U)}\,.$$

It should be noteed that $\Omega^p(i)(\omega) = \omega \circ i$ when $i : U_1 \hookrightarrow U_2$ is an inclusion, since then $D_x i = \mathrm{id}$.

Example 3.11 For the constant 1-form $\epsilon_i \in \Omega^1(U_2)$, we have that

$$\phi^*(\epsilon_i) = \sum_{k=1}^{n} \frac{\partial \phi_i}{\partial x_k}\epsilon_k = d\phi_i$$

With ϕ_i the i-th coordinate function. To see this, let $\zeta \in \mathbb{R}^n$. Then

$$\begin{aligned}\phi^*(\epsilon_i)(\zeta) = \epsilon_i(D_x\phi(\zeta)) &= \epsilon_i\left(\sum_{k=1}^{m}\Big(\sum_{l=1}^{n}\frac{\partial \phi_k}{\partial x_l}\zeta^l\Big)e_k\right)\\ &= \sum_{l=1}^{n}\frac{\partial \phi_i}{\partial x_l}\zeta^l = \sum_{l=1}^{n}\frac{\partial \phi_i}{\partial x_l}\epsilon_l(\zeta) = d\phi_i(\zeta).\end{aligned}$$

Theorem 3.12 With Definition 3.10 we have the relations

(i) $\phi^*(\omega \wedge \tau) = \phi^*(\omega) \wedge \phi^*(\tau)$

(ii) $\phi^*(f) = f \circ \phi$ if $f \in \Omega^0(U_2)$

(iii) $d\phi^*(\omega) = \phi^*(d\omega)$

Conversely, if $\phi' : \Omega^*(U_2) \to \Omega^*(U_1)$ is a linear map satisfying three conditions, then $\phi' = \phi$.

Proof. Let $x \in U_1$ and let $\xi_1, \cdots, \xi_{p+q}$ be vectors in $\mathbb{R}^n$. Then

$$\begin{aligned}&\phi^*(\omega \wedge \tau)_x(\xi_1, \cdots, \xi_{p+q}) = (\omega \wedge \tau)_{\phi(x)}(D_x\phi(\xi_1), \cdots, D_x\phi(\xi_{p+q}))\\ &= \sum \mathrm{sign}(\sigma)\Big[\omega_{\phi(x)}\big(D_x\phi\big(\xi_{\sigma(1)}\big), \cdots, D_x\phi\big(\xi_{\sigma(p)}\big)\big)\\ &\qquad\qquad \tau_{\phi(x)}(D_x\phi(\xi_{\sigma(p+1)}), \cdots, D_x\phi(\xi_{\sigma(p+q)}))\Big]\\ &= \sum \mathrm{sign}(\sigma)\phi^*(\omega)_x\big(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}\big)\phi^*(\tau)_x\big(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}\big)\\ &= (\phi^*(\omega)_x \wedge \phi^*(\tau)_x)(\xi_1, \cdots, \xi_{p+q}).\end{aligned}$$

This shows (i) when $p > 0$ and $q > O$. If $p = 0$ or $q = 0$ the proof is quite analogous, but easier. Property (ii) is contained in the definition of ϕ^* for degree 0. So we are left with (iii). We shall first show that $d\phi^*(f) = \phi^*(df)$ when $f \in \Omega^0(U_2)$. We have that

$$df = \sum_{i=k}^{n} \frac{\partial f}{\partial x_k}\epsilon_k = \sum_{i=k}^{n} \frac{\partial f}{\partial x_k} \wedge \epsilon_k,$$

when ϵ_k is considered as the element in $\Omega^1(U_2)$ with constant value ϵ_k. From (i) and (ii) we obtain

$$\begin{aligned}
\phi^*(df) &= \sum_{k=1}^{m} \phi^*\left(\frac{\partial f}{\partial x_k}\right) \wedge \phi^*(\epsilon_k) = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \circ \phi\right) \wedge \left(\sum_{l=1}^{n} \frac{\partial \phi_k}{\partial x_l} \epsilon_l\right) \\
&= \sum_{k=1}^{m} \sum_{l=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ \phi\right) \left(\frac{\partial \phi_k}{\partial x_l}\right) \epsilon_l = \sum_{l=1}^{n} \left(\sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \circ \phi\right) \frac{\partial \phi_k}{\partial x_l}\right) \epsilon_l \\
&= \sum_{l=1}^{n} \frac{\partial (f \circ \phi)}{\partial x_l} \epsilon_l = d(f \circ \phi) = d(\phi^*(f)).
\end{aligned}$$

In more general case $\omega = f\epsilon_I = f \wedge \epsilon_I$, Lemma 3.6 gives $d\omega = df \wedge \epsilon_I$, because $d\epsilon_I = 0$. Hence

$$\begin{aligned}
\phi^*(d\omega) &= \phi^*(df) \wedge \phi^*(\epsilon_I) = d(\phi^*(f)) \wedge \phi^*(\epsilon_I) \\
&= d(\phi^*(f) \wedge \phi^*(\epsilon_I)) = d(\phi^*\omega)
\end{aligned}$$

The second last equality uses Lemma 3.6 and the fact that $d\epsilon_I = 0$:

$$\begin{aligned}
d\phi^*(\epsilon_I) &= d\big(\phi^*(\epsilon_{i_1}) \wedge \ldots \wedge \phi^*(\epsilon_{i_p})\big) \\
&= \sum (-1)^{k-1} \phi^*(\epsilon_{i_1}) \wedge \ldots \wedge d\phi^*(\epsilon_{i_k}) \wedge \ldots \wedge \phi^*(\epsilon_{i_p}) \\
&= 0
\end{aligned}$$

since $d\phi^*(\epsilon_{i_k}) = 0$ by Example 3.11 and Lemma 3.5. □

In the following it will be convenient to use the notation of Example 3.3 and write

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

instead of the (constant) p-form $\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_p}$. An arbitrary p-forrn can then be written as

$$\omega(x) = \sum \omega_I(x) dx_I$$

and Example 3.11 becomes $\phi^*(dy_i) = d\phi_i$ when $y_i : U_2 \to \mathbb{R}$ is the i-th coordinate function and $\phi_i = y_i \circ \phi$ the i-th coordinate of ϕ; cf. Theorem 3.12.(ii),(iii)

Example 3.13

(i) Let $\gamma : (a, b) \to U$ be an smooth curve in U, $\gamma = (\gamma_1, \cdots, \gamma_n)$, and that

$$\omega = f_1 dx_1 + \cdots + f_n dx_n$$

be a 1-form on U. Then we have that

$$\begin{aligned}\gamma^\star(\omega) &= \gamma^*(f_1)\wedge\gamma^*(dx_1)+\cdots+\gamma^*(f_n)\wedge\gamma^*(dx_n)\\ &= \gamma^*(f_1)d(\gamma^*(x_1))+\cdots+\gamma^*(f_n)d(\gamma^*(x_n))\\ &= (f_1\circ\gamma)d\gamma_1+\cdots+(f_n\circ\gamma)d\gamma_n\\ &= [(f_1\circ\gamma)\gamma_1'+\cdots+(f_n\circ\gamma)\gamma_n']\,dt\\ &= \langle f(\gamma(t)),\gamma'(t)\rangle\,dt.\end{aligned}$$

Here $\langle\cdot\rangle$ is the usual inner product. Compare Example 1.8

(ii) Let $\phi : U_1 \to U_2$ be a smooth map between open sets in $\mathbb{R}^n$. Then

$$\phi^*(dx_1\wedge\cdots\wedge dx_n) = \det(D_x\phi)dx_1\wedge\cdots\wedge dx_n.$$

indeed, from Theorem 3.12,

$$\begin{aligned}\phi^*(dx_1\wedge\ldots\wedge dx_n) &= \phi^*(dx_1)\wedge\ldots\wedge\phi^*(dx_n) = d\phi^*(x_1)\wedge\ldots\wedge d\phi^*(x_n)\\ &= d\phi_1\wedge\ldots\wedge d\phi_n = \det(D_x\phi)dx_1\wedge\ldots\wedge dx_n.\end{aligned}$$

The last equality is a consequence of Lemma 2.13.

Example 3.14 If $\phi : \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ is given by $\phi(x,t) = \psi(t)x$, where $\phi(t)$ is a smooth real valued function, Then

$$\phi^*(dx_i) = x_i\psi'(t)dt + \psi(t)dx_i.$$

To a smooth map $\phi : U_1 \to U_2$ we can now associate a linear map

$$H^p(\phi) : H^p(U_2)\to H^p(U_1)$$

by setting $H^p(\phi)[\omega] = [\Omega^p(\phi)(\omega)](= \phi^*(\omega))$. The definition is independent of the choice of representative, since $\phi^*(\omega + d\nu) = \phi^*(\omega) + \phi^*(\omega) + d\phi^*(\nu)$.
Furthermore,

$$H^{p+q}(\phi)([\omega_1][\omega_2]) = (H^p(\phi)[\omega_1])(H^q(\phi)[\omega_2])$$

such that $H^*(\phi) : H^*(U_2)\to H^*(U_1)$ is a homomorphism of graded algebras.

Theorem 3.15 (Poincaré's Lemma) If U is a star-shaped open set then $H^p(U) = 0$ for $p > 0$, and $H^0(U) = \mathbb{R}$.

Proof. We may assume U to be star-shaped with respect to the origin $0\in\mathbb{R}^n$, and wish to construct a linear operator

$$S_p : \Omega^p(U)\to\Omega^{p-1}(U)$$

such that $dS_p + S_{p+1}d = \mathrm{id}$ when $p > 0$ and $S_1d = \mathrm{id} - e$, where $e(\omega) = \omega(0)$ for $\omega\in\Omega^0(U)$. Such an operator immediately implies our theorem, since $dS_p(\omega) = \omega$

for a closed p-form, $p > 0$, and hence $[\omega] = 0$. If $p = 0$ we have $\omega - \omega(0) = S_1 d\omega = 0$, and ω must be constant.
First we construct

$$\hat{S}_p : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U).$$

Every $\omega \in \Omega^p(U \times \mathbb{R})$ can be written in the form

$$\omega = \sum f_I(x,t)dx_I + \sum g_J(x,t)dx_J \wedge dt$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_{p-1})$. We define

$$\hat{S}_p(\omega) = \sum \left(\int_0^1 g_J(0,t)dt \right) dx_J$$

Then we have that

$$\begin{aligned}
d\hat{S}_p(\omega) + \hat{S}_{p+1}d(\omega) &= \sum_{J,i} \left(\int_0^1 \frac{\partial g_J(x,t)}{\partial x_i} dt \right) dx_i \wedge dx_J \\
&+ \sum_I \left(\int_0^1 \frac{\partial f_I(x,t)}{\partial t} dt \right) dx_I - \sum_{J,i} \left(\int_0^1 \frac{\partial g_J}{\partial x_i} dt \right) dx_i \wedge dx_J \\
&= \sum \left(\int_0^1 \frac{\partial f_I(x,t)}{\partial t} dt \right) dx_I \\
&= \sum f_I(x,1)dx_I - \sum f_I(x,0)dx_I.
\end{aligned}$$

We apply this result to $\phi^*(\omega)$, where

$$\phi : U \times \mathbb{R} \to U, \quad \phi(x,t) = \psi(t)x.$$

and $\psi(t)$ is a smooth function for which

$$\begin{cases} \psi(t) = 0, & \text{if } t \leqslant 0 \\ \psi(t) = 1, & \text{if } t \geqslant 1 \\ 0 \leqslant \psi(t) \leqslant 1, & \text{otherwise} \end{cases} \tag{5}$$

Define $S_p(\omega) = \hat{S}_p(\phi^*(\omega))$ with $\hat{S}_p : \Omega(U \times \mathbb{R}) \to \Omega^{p-1}(U)$ as above. Assume that $\omega = \sum h_I(x)dx_I$. Form Example 3.14 we have

$$\phi^*(\omega) = \sum h_I(\psi(t)x)(d\psi(t)x_{i_1} + \psi(t)dx_{i_1}) \wedge \cdots \wedge \left(d\psi(t)x_{i_p} + \psi(t)dx_{i_p}\right)$$

In the notation used above we then get that

$$\sum f_I(x,t)dx_I = \sum h_I(\psi(t)x)\psi(t)^p dx_I$$

This implies that

$$dS_p(\omega) + S_{p+1}d\omega = \begin{cases} \sum h_I(x)dx_I = \omega & p > 0 \\ \omega(x) - \omega(0) & p = 0 \end{cases}$$

$\square$

4. CHAIN COMPLEXES AND THEIR COHOMOLOGY

In this chapter we present some general algebraic definitions and viewpoints, which should illuminate some of the constructions of Chapter 3. The algebraic results will be applied later to de Rham cohomology in Chapters 5 and 6.
A sequence of vector spaces and linear maps

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1}$$

is said to be *exact* when $\operatorname{Im} f = \operatorname{Ker} g$, where as above

$$\begin{aligned} \operatorname{Ker} g &= \{b \in B | g(b) = 0\} \qquad \text{the kernel of } g \\ \operatorname{Im} f &= \{f(a) | a \in A\} \qquad \text{the image of } f \end{aligned}$$

Note that $A \xrightarrow{f} B \to 0$ is exact precisely when f is surjective and that $0 \to B \xrightarrow{g} C$ is exact precisely when g is injective. A sequence $A^* = \{A^i, d^i\}$,

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots \tag{2}$$

of vector spaces and linear maps is called a *chain complex* provided $d^{i+1} \circ d^i = 0$ for all i. It is exact if

$$\operatorname{Ker} d^i = \operatorname{Im} d^{i-1}$$

for all i. An exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{3}$$

is called *short exact*. This is equivalent to requiring that

$$f \text{ is injective, } g \text{ is surjective and } \operatorname{Im} f = \operatorname{Ker} g$$

The *cokernel* of a linear map $f : A \to B$ is

$$\operatorname{Cok}(f) = B / \operatorname{Im}(f).$$

For a short exact sequence, g induces an isomorphism

$$g : \operatorname{Cok}(f) \xrightarrow{\cong} C.$$

Every (long) exact sequence, as in (2), induces short *exact sequences* (which can be used to calculate A^i)

$$0 \to \operatorname{Im} d^{i-1} \to \operatorname{Im} d^i \to 0$$

Furthennore the isomorphisms

$$A^{i-1}/\operatorname{Im} d^{i-1} \cong A^{i-1}/\operatorname{Ker} d^{i-1} \xrightarrow[\cong]{d^{i-1}} \operatorname{Im} d^{i-1}$$

are frequently applied in concrete calculations.
The *direct sum* of vector spaces A and B is the vector space

$$\begin{aligned}
&A \oplus B = \{(a,b) | a \in A, b \in B\} \\
&\lambda(a,b) = (\lambda a, \lambda b), \qquad \lambda \in \mathbb{R} \\
&(a_1,b_1) + (a_2,b_2) = (a_1 + a_2, b_1 + b_2)
\end{aligned}$$

If $\{a_i\}$ and $\{b_j\}$ are bases of A and B, respectively, then $\{(a_i, 0), (0, b_j)\}$ is a basis of $A \oplus B$. In particular

$$\dim(A \oplus B) = \dim A + \dim B$$

Lemma 4.1 Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence ofvector spaces. Then B is finite-dimensional if both A and C are, and $B \cong A \oplus C$.

PROOF. Choose a basis $\{a_i\}$ of A and $\{c_j\}$ of C. Since g is surjective there exist $b_j \in B$ with $g(b_j) = c_j$. Then $\{f(a_i), b_j\}$ is a basis of B: For $b \in B$ we have $g(b) = \sum \lambda_i c_j$. Hence $b - \sum \lambda_i b_i \in \operatorname{Ker} g$. Since $\operatorname{Ker} g = \operatorname{Im} f$, $b - \sum \lambda_i b_i = f(a)$, so

$$b - \sum \lambda_j b_j = f\left(\sum \mu_i a_i\right) = \sum \mu_i f(a_i).$$

This shows that b can be written as a linear combination of $\{b_j\}$ and $\{f(a_i)\}$. It is left to the reader to show that $\{b_j, f(a_i)\}$ are linearly independent. □

Definition 4.2 For a chain complex $A^* = \{\cdots \to A^{p-1} \xrightarrow{d^{p-1}} A^p \xrightarrow{d^p} A^{p+1} \to \cdots\}$, we define the p-th cohomology vector space to be

$$H^p(A^*) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

The elements of Ker d^p are called p-cycles (or are said to be closed) and the elements of $\operatorname{Im} d^{p-1}$ are called p-boundaries (or said to be exact). The elements of $H^p(A^*)$ are called *cohomology classes*.
A chain map $f : A^* \to B^*$ between chain complexes consists of a family $f^p : A^p \to B^p$ of linear maps, satisfying $d_B^p \circ f^p = f^{p+1} \circ d_A^p$. A chain map is illustrated as the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p-1} \arrow[r, "d^{p-1}"] \arrow[d, "f^{p-1}"] & A^p \arrow[r, "d^p"] \arrow[d, "f^p"] & A^{p+1} \arrow[r] \arrow[d, "f^{p+1}"] & \cdots \\
\cdots \arrow[r] & B^{p-1} \arrow[r, "d^{p-1}"] & B^p \arrow[r, "d^p"] & B^{p+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 4.3 A chain map $f : A^* \to B^*$ induces a linear map

$$f^* = H^*(f) : H^p(A^*) \to H^p(B^*), \text{ for all } p$$

PROOF. Let $\alpha \in A^p$ be a cycle ($d^p a = 0$) and $[a] = a + \operatorname{Im} d^{p-l}$ its corresponding cohomology class in $H^p(A^*)$. We define $f^*([a]) = [f^p(a)]$. Two remarks are needed. First, we have $d_B^p f^p(a) = f^{p+1} d_A^p(a) = f^{p+1}(0) = 0$. Hence $f^p(a)$ is a cycle. Second, $[f^p(a)]$ is independent of which cycle a we choose in the class $[a]$. If $[a_1] = [a_2]$ then $a_1 - a_2 \in \operatorname{Im} d_A^{p-1}$, and $f^p(a_1 - a_2) = f^p d_A^{p-1}(x) = d_B^{p-1} f^{p-1}(x)$. Hence $f^p(a_1) - f^p(a_2) \in \operatorname{Im} d_B^{p-1}$, and $f^p(a_1), f^p(a_2)$ define the same cohomology class. $\square$

A *category* C consists of "objects" and "morphisms" between them, such that "composition" is defined. If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morphisms, then there exists a morphism $g \circ f : C_1 \to C3$. Furthermore it is to be assumed that $\mathrm{id}_C : C \to C$ is a morphism for every object C of C. The concept is best illustrated by examples:

- The category of open sets in Euclidean spaces, where the morphisms are the smooth maps.
- The category of vector spaces, where the morphisms are the linear maps.
- The category of abelian groups, where the morphisms are homomor phisms.
- The category of chain complexes, where the morphisms are the chain maps.
- A category with just one object is the same as a semigroup, namely the semigroup of morphisms of the object.
- Every partially ordered set is a category with one morphism from c to d, when $c \leqslant d$.

A contravariant functor $F : \mathcal{C} \to \mathcal{V}$ between two categories maps every object $C \in \operatorname{ob} \mathcal{C}$ to an object $F(C) \in \operatorname{ob} \mathcal{V}$, and every morphism $f : C_1 \to C_2$ in $\mathcal{C}$ to a morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathcal{V}$, such that

$$F(g \circ f) = F(f) \circ F(g), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}.$$

A *covariant functor* $F : \mathcal{C} \to \mathcal{V}$ is an assignment in which $F(f) : F(C_1) \to F(C_2)$, and

$$F(g \circ f) = F(g) \circ F(f), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}.$$

Functors thus are the "structure-preserving" assignments between categories. The contravariant ones change the direction of the arrows, the covariant ones preserve directions. We give a few examples:

- Let A be a vector space and $F(C) = \operatorname{Hom}(C, A)$, the linear maps from C to A. For $\phi : C_1 \to C_2$, $\operatorname{Hom}(\phi, A) : \operatorname{Hom}(C_2, A) \to \operatorname{Hom}(C_2, A)$ is given by $\operatorname{Hom}(\phi, A)(\psi) = \psi \circ \phi$. This is a contravariant functor from the category of vector spaces to itself.

- $F(C) = \mathrm{Hom}(C, A), F(\phi) : \psi \to \phi \circ \psi$. This is a covariant functor from the category of vector spaces to itself.
- Let $\mathcal{U}$ be the category of open sets in Euclidean spaces and smooth maps, and *Vect* the category of vector spaces. The vector space of differential p-forms on $U \in \mathcal{U}$ defines a contravariant functor

$$\Omega^p(U) : \mathcal{U} \to \mathrm{Vect}.$$

- Let Vect* be the category of chain complexes. The de Rham complex defines a contravariant functor $\Omega^* : \mathcal{U} \to \mathrm{Vect}^*$.
- For every p the homology $H^p : \mathrm{Vect}^* \to \mathrm{Vect}$ is a covariant functor.
- The composition of the two functors above is exactly the de Rham cohomology functor $H^p : \mathcal{U} \to \mathrm{Vect}$. It is contravariant.

A short exact sequence of chain complexes

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$$

consists of chain maps f and g such that $0 \to A^p \xrightarrow{f} B^p \xrightarrow{g} C^p \to 0$ is exact for every p.

Lemma 4.4 For a short exact sequence of chain complexes the sequence

$$H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*)$$

is exact.

PROOF. Since $g^p \circ f^p = 0$, we have

$$g^* \circ f^*([a]) = g^*([f^p(a)]) = [g^p(f^p(a))] = 0$$

for every cohomology class $[a] \in H^p(A^*)$. Conversely, assume for $[b] \in H^p(B)$ that $g^*[b] = 0$. Then $g^p(b) = d_C^{p-1}(c)$. Since g^{p-1} is surjective, there exists $b_1 \in B^{p-1}$ with $g^{p-1}(b_1) = c$. It follows that $g^p(b - d_B^{p-1}(b_1)) = 0$. Hence there exists $a \in A^p$ with $f^p(a) = b - d_B^{p-1}(b_1)$. We will show that a ia a p-cycle. Since f^{p+1} is injective, it is sufficient to note that $f^{p+1}(d_A^p(a)) = 0$. But

$$f^{p+1}\big(d_A^p(a)\big) = d_B^p\big(f^p(a)\big) = d_B^p\big(b - d_B^{p-1}(b_1)\big) = 0$$

since b is a p-cycle and $d^p \circ d^{p-1} = 0$. We have thus found a cohomology class $[a] \in H^p(A)$, and $f^*([a]) = [b - d_B^{p-1}(b_1)]$. □

One might expect that the sequence of Lemma 4.4 could be extended to a short exact sequence, but this is not so. The problem is that, even though $g^p : B^p \to C^p$ is surjective, the pre-image $(g^P)^{-1}(c)$ of a p-cycle with $c \in C^p$ need not contain a cycle. We shall measure when this is the case by introducing.

Definition 4.5 For a short exact sequence of chain complexes $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$, we define

$$\partial^* : H^p(C^*) \to H^{p+1}(A^*)$$

to be the linear map given by

$$\partial^*([c]) = \left[\left(f^{p+1}\right)^{-1}\left(d_B^p\left((g^p)^{-1}(c)\right)\right)\right]$$

There are several things to be noted. The definition expresses that for every $b \in (g^p)^{-1}(c)$ we have $d_B^p(b) \in \mathrm{Im}(f^{p+1})$, and that the uniquely determined $a \in A^{p+1}$ with $f^{p+1}(a) = d_B^p(b)$ is a $p+1$-cycle. Finally it is postulated that $[a] \in H^{p+1}(A^*)$ is independent of the choice of $b \in (g^b)^{-1}(c)$.
In order to prove these assertions it is convenient to write the given short exact sequence in a diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & A^{p-1} \arrow[r, "f^{p-1}"] \arrow[d, "d_A^{p-1}"] & B^{p-1} \arrow[r, "g^{p-1}"] \arrow[d, "d_B^{p-1}"] & C^{p-1} \arrow[r] \arrow[d, "d_C^{p-1}"] & 0 \\
0 \arrow[r] & A^p \arrow[r, "f^p"] \arrow[d, "d_A^p"] & B^p \arrow[r, "g^p"] \arrow[d, "d_B^p"] & C^p \arrow[r] \arrow[d, "d_C^p"] \arrow[dll] & 0 \\
0 \arrow[r] & A^{p+1} \arrow[r, "f^{p+1}"] \arrow[d] & B^{p+1} \arrow[r, "g^{p+1}"] \arrow[d] & C^{p+1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

The slanted arrow indicates the definition of ∂^*. We shall now prove the necessary assertions which, when combined, make ∂^* well-defined. Namely:

(i) If $g^p(b) = c$ and $d_C^p(c) = 0$, then $d_B^p(b) \in f^{p+1}$.

(ii) If $f^{p+1}(a) = d_B^p(b)$, then $d_A^{p+1}(a) = 0$.

(iii) If $g^p(b_1) g^p(b) = c$ and $f^{p+1}(a_i) = d_B^p(b_i)$, then $[a_1] = [a_2] \in H^{p+1}(A^*)$.

The first assertion follows, because $g^{p+1} d_B^p(b) = d_C^p(c) = 0$, and $\mathrm{Ker}\, g^{p+1} = \mathrm{Im}\, f^{p+1}$; (ii) uses the injectivity of f^{p+2} and that $f^{p+2} d_A^{p+1}(a) = d_B^{p+1} f^{p+1}(a) = d_B^{p+1} d_B^p(b) = 0$; (iii) follows since $b_1 - b_2 = f^p(a)$ so that $d_B^p(b_1) - d_B^p(b_2) = d_B^p f^p(a) = f^{p+1} d_A^p(a)$, and therefore $(f^{p+1})^{-1}(d_B^p(b_1)) = (f^{p+1})^{-1}(d_B^p(b_2)) + d_A^p(a)$.

Example 4.6 Here is a short exact sequence of chain complexes (the dots indicate that the chain groups are zero) with $\partial^* \neq 0$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathbb{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

One can easily verify that $\partial^* : \mathbb{R} \to \mathbb{R}$ is an isomorphism.

Lemma 4.7 The sequence $H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*)$ is exact.

Proof. We have $\partial^* g^*([b]) = \partial^* g^p([b]) = [(f^{p+1})^{-1}(d_B(b))] = 0$. Conversely assume that $\partial^*([c]) = 0$. Choose $b \in B^p$ with $g^p(b) = c$ and $a \in A^p$, such that

$$d_B^p(b) = f^{p+1}(d_A^p a).$$

Now we have $d_B^p(b - f^p(a)) = 0$ and $g^p(b - f^p(a)) = c$. Hence $g^*[b - f^p(a)] = [c]$. $\square$

Lemma 4.8 The sequence $H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*)$ is exact.

Proof. We have $f^*\partial^*([c]) = [d_B^p(b)] = 0$, where $g^p(b) = c$. Conversely assume that $f^*([a]) = 0$. i.e., $f^{p+1}(a) = d_B^p(b)$. Then $d_C^p(g^p(b)) = g^{p+1}f^{p+1}(a) = 0$, and $\partial^*[g^p(b)] = [a]$. $\square$

We can sum up Lemmas 4.4, 4.7 and 4.8 in the important.

Theorem 4.9 (Long exact homology sequence) Let $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$ be a short exact sequence of chain complexes. Then the sequence

$$\cdots \to H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*) \to \cdots$$

is exact.

Definition 4.10 Two chain maps $f, g : A^* \to B^*$ are said to be *chain homotopic* if there exists a linear map $s : A^p \to B^{p-1}$ satisfying

$$d_B s + s d_A = f - g : A^p \to B^p$$

for every p.

In the form of a diagram, a chain homotopy is given by the slanted arrows.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p-1} \arrow[r] \arrow[d, "f-g"] & A^p \arrow[r] \arrow[d, "f-g"] \arrow[dl] & A^{p+1} \arrow[r] \arrow[d, "f-g"] \arrow[dl] & A^{p+2} \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & B^{p-1} \arrow[r] & B^p \arrow[r] & B^{p+1} \arrow[r] & B^{p+2} \arrow[r] & \cdots
\end{tikzcd}$$

The name *chain homotopy* will be explained in Chapter 6.

Lemma 4.11 For two chain-homotopic chain maps $f, g : A^* \to B^*$ we have that

$$f^* = g^* : H^p(A^*) \to H^p(B^*).$$

PROOF. If $[a] \in H^p(A^*)$, then

$$(f^* - g^*)[a] = [f^p(a) - g^p(a)] = [(d_B^{p-1}s(a) + sd_A^p(a)] = [d_B^{p-1}s(a)] = 0.$$

□

Remark 4.12 In the proof of the Poincare lemma in Chapter 3 we constructed linear maps

$$S^p : \Omega^p(U) \to \Omega^{p-1}(U)$$

with $d^{p-1}S^p + S^{p+1}d^p = \mathrm{id}$ for $p > 0$. Hence $\mathrm{id} = 0$ on $H^p(U)$, and $H^p(U) = 0$ when $p > 0$.

Lemma 4.13 If A^* and B^* are chain complexes then

$$H^p(A^* \otimes B^*) = H^p(A^*) \otimes H^p(B^*).$$

PROOF. It is obvious that

$$\begin{aligned} \mathrm{Ker}(d^p_{A\otimes B}) &= \mathrm{Ker}\, d^p_A \otimes \mathrm{Ker}\, d^p_B \\ \mathrm{Im}(d^{p-1}_{A\otimes B}) &= \mathrm{Im}\, d^{p-1}_A \otimes \mathrm{Im}\, d^{p-1}_B. \end{aligned}$$

and the lemma follows. □

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5. THE MAYER-VIETORIS SEQUENCE

This chapter introduces a fundamental calculational technique for de Rham cohomology, namely the so-called Mayer-Vietoris sequence, which calculates $H^*(U_1 \cup U_2)$ as a "function" of $H^*(U_1), H^*(U_2)$ and $H^*(U_1 \cap U_2)$. Here UI and U_2 are open sets in $\mathbb{R}^n$. By iteration we get a calculation of $H^*(U_1 \cap \cdots \cap U_n)$ as a "function" of $H^*(u_\alpha)$, where 0: runs over the subsets of $1, \cdots, n$ and $U_{i_1} \cap \cdots \cap U_{i_r}$ when $\alpha = i_1, \cdots, i_r$. Combined with the Poincaré lemma, this yields a *principal* calculation of $H^*(U)$ for quite general open sets in $\mathbb{R}^n$. If, for instance, U can be covered by a finite number of convex open sets U_i, then every U_α will also be convex and $H^*(U_\alpha)$ thus known from the Poincaré lemma

Theorem 5.1 Let U_1 and U_2 be open sets in $\mathbb{R}^n$ with union $U = U_1 \cup U_2$. For $\nu = 1, 2$, let $i_\nu : U_\nu \to U$ and $j_\nu : U_1 \cap U_2 \to U_\nu$ be the corresponding inclusions. Then the sequence

$$0 \to \Omega^p(U) \xrightarrow{I^p} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{J^p} \Omega^p(U_1 \cap U_2) \to\to 0$$

is exact, where $I^p(\omega) = (i_1^*(\omega), i_2^*(\omega)), J^p(\omega_1, \omega_2) = j_1^*(\omega_1) - j_2^*(\omega_2)$.

PROOF. For a smooth map $\phi : V \to W$ and a p-form $\omega = \sum f_I dx_I \in \Omega^p(W)$,

$$\Omega^p(\phi)(\omega) = \phi^*(\omega) = \sum (f_I \circ \phi) d\phi_{i_1} \wedge \ldots \wedge d\phi_{i_p}.$$

In particular, if ϕ is an inclusion of open sets in $\mathbb{R}^n$, i.e., $\phi_i(x) = x_i$, then

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_p} = dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Hence

$$\phi^*(\omega) = \sum f_I \circ \phi dx_I. \tag{1}$$

This will be used for $\phi = i_\nu, j_\nu, \nu = 1, 2$. It follows from (1) that I^p is injective. If namely $I^p(\omega) = 0$ then $i_1^*(\omega) = 0 = i_2^*(\omega)$, and

$$i^*(\nu) = \sum (f_I \circ i_\nu) dx_I = 0$$

If and only if $f_I \circ i_\nu = 0$ for all I. However $f_I \circ i_1 = 0$ and $f_I \circ i_2 = 0$ imply that $f_I = 0$ on all of U, since U_1 and U_2 cover U. Similarly, we show that $\operatorname{Ker} J^p = \operatorname{Im} I^p$. First

$$J^p \circ I^p(\omega) = j_2^* i_2^*(\omega) - j_1^* i_1^*(\omega) = j^*(\omega) - j_1^*(\omega) = 0$$

where $j : U_1 \cap U_2 \to U$ is the inclusion. Hence $\operatorname{Im} I^p \subseteq \operatorname{Ker} J^p$. To show the converse inclusion we start with two p-form $\omega_\nu \in \Omega^p(U_\nu)$.

$$\omega_1 = \sum f_I dx_I, \quad \omega_2 = \sum g_I dx_I.$$

Since $J^p(\sum h_I dx_I) = (\omega_1, \omega_2)$, we have that $j_1^*(\omega_1) = j_2^*(\omega_2)$, which by (1) translates into $f_I \circ j_1 = g_I \circ j_2$ or $f_I(x) = g_I(x)$ for $x \in U_1 \cap U_2$. We define a smooth function $h_I : U \to \mathbb{R}^n$ by

$$h_I(x) = \begin{cases} f_I(x), & x \in U_1, \\ g_I(x), & x \in U_2. \end{cases}$$

Then $I^p(\sum h_I dx_I) = (\omega_1, \omega_2)$. Finally we show that J^p is surjective. To this end we use a partition of unity $\{p-1, p_2\}$ with support in $\{U_1, U_2\}$. i.e., smooth functions $h_I : U \to \mathbb{R}^n$ by

$$p_\nu : U \to \{0, 1\}, \quad \nu = 1, 2$$

for which $\mathrm{supp}_U(p_\nu) \subset U_\nu$, and such that $p_1(x) + p_2(x) = 1$ for $x \in U$ (cf. Appendix A).
Let $f : U_1 \cap U_2 \to \mathbb{R}$ be a smooth function. We use $\{p_1, p_2\}$ to extend f to U_1 and U_2. Since $\mathrm{supp}_U(p_1) \cap U_2 \subset U_1 \cap U_2$, we can define a smooth function by

$$f_2(x) = \begin{cases} -f(x)p_1(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_2 - \mathrm{supp}_U(p_1) \end{cases}$$

Analogously we define

$$f_1(x) = \begin{cases} f(x)p_2(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_1 - \mathrm{supp}_U(p_2) \end{cases}$$

Note that $f_1(x) - f_2(x) = f(x)$ when $x \in U_1 \cap U_2$, because $p_1(x) + p_2(x) = 1$. For a differential form $\omega \in \Omega^p(U_1 \cap U_2)$, $\omega = \sum f_I dx_I$, we can apply the above to each of the functions $f_I : U_1 \cap U_2 \to \mathbb{R}$. This yields the functions $f_{I\nu} : U_\nu \to \mathbb{R}$, and thus the differential form $\omega_\nu = \sum f_{I,\nu} dx_I \in \Omega^p(U_\nu)$. With this choice $J^p(\omega_1, \omega_2) = \omega$. □

It is clear that

$$\begin{aligned} I{:}\,&\Omega^*(U) \to \Omega^*(U_1) \oplus \Omega^*(U_2) \\ J{:}\,&\Omega^*(U_1) \oplus \Omega^*(U_2) \to \Omega^*(U_1 \cap U_2) \end{aligned}$$

are chain maps, so that Theorem 5.1 yields a short exact sequence of chain complexes. From Theorem 4.9 one thus obtains a long exact sequence of cohomology vector spaces. Finally Lemma 4.13 tells us that

$$H^p(U)(\Omega^*(U_1) \oplus \Omega^*(U_2)) = H^p(U_1) \oplus H^p(U_2)$$

We have proved:

Theorem 5.2 (Mayer-Vietoris) Let U_1 and U_2 be open sets in $\mathbb{R}^n$ and $U = U_1 \cup U_2$. There exists an exact sequence of cohomology vector spaces

$$\cdots \to H^p(U) \xrightarrow{I^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}(U) \to \cdots$$

Here $I^*(\omega) = (i_1^*([\omega]), i_2^*([\omega]))$ and $J^*([\omega_1], [\omega_2]) = [j_1^*(\omega_1) - j_2^*(\omega_2)]$ in the notation of Theorem 5.1.

Corollary 5.3 If U_1 and U_2 are disjoint open sets in $\mathbb{R}^n$ then

$$I^* : H^p(U_1 \cup U_2) \to H^p(U_1) \oplus H^p(U_2)$$

is an isomorphism.

PROOF. It follows from the Theorem 5.1 that

$$I^p : \Omega^p(U_1 \cup U_2) \to \Omega^p(U_1) \oplus \Omega^p(U_2)$$

is an isomorphism, and Lemma 4.13 gives that corresponding map on cohomology is also an isomorphism. □

Example 5.4 We use Theorem 5.2 to calculate the de Rham cohomology vector spaces of the punctured plane $\mathbb{R}^n - \{0\}$. Let

$$\begin{aligned} U_1 &= \mathbb{R}^2 - \{(x_1, x_2) \mid x_1 \geqslant 0, x_2 = 0\} \\ U_2 &= \mathbb{R}^2 - \{(x_1, x_2) \mid x_1 \leqslant 0, x_2 = 0\}. \end{aligned}$$

These are star-shaped open sets, such that $H^p(U_1) = H^p(U_2) = 0$ for $p > 0$ and $H^0(U_1) = H^0(U_2) = \mathbb{R}$. Their intersection

$$U_1 \cap U_2 = \mathbb{R}^2 - \mathbb{R} = \mathbb{R}^2_+ \cup \mathbb{R}^2_-$$

is disjoint union of the open half-planes $x_2 > 0$ and $x_2 < 0$. Hence

$$H^p(U_1 \cap U_2) = \begin{cases}0 & \text{if } p > 0. \quad \mathbb{R} \oplus \mathbb{R} \text{ if } \quad p = 0\end{cases} \tag{2}$$

by the Poincaré lemma and Corollary 5.3. From the *Mayer-Vietoris sequence* we have

$$\begin{aligned} \cdots \to H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cap U_2) \xrightarrow{\partial^*} \\ H^{p+1}(\mathbb{R}^2 - \{0\}) \xrightarrow{I^*} H^{p+1}(U_1) \oplus H^{p+1}(U_2) \to \cdots \end{aligned}$$

For $p > 0$,

$$0 \to H^p(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}(\mathbb{R}^2 - \{0\}) \to 0$$

is exact, i.e., ∂^* is an isomorphism and $H * q(\mathbb{R}^2 - \{0\}) = 0$ for $q > 0$ according to (2). If $p = 0$, one gets the exact sequence

$$\begin{aligned} H^{-1}(U_1 \cap U_2) \to H^0(\mathbb{R}^2 - \{0\}) \xrightarrow{I^0} H^0(U_1) \oplus H^0(U_2) \xrightarrow{J^0} \\ H^0(U_1 \cap U_2) \xrightarrow{\partial^*} H^1(\mathbb{R}^2 - \{0\}) \xrightarrow{I^1} H^1(U_1) \oplus H^1(U_2) \end{aligned} \tag{3}$$

Since $H^{-1}(U) = 0$ for all open sets, and in particular $H^{-1}(U_\nu)$, I^0 is injective. Since $H^{-1}(U_\nu) = 0$, ∂^* is surjective, and the sequence (3) reduces to the exact sequence

$$
\begin{array}{ccccccc}
 & & \mathbb{R} \otimes \mathbb{R} & & \mathbb{R} \otimes \mathbb{R} & & \\
 & & \| & & \| & & \\
0 \to H^0(\mathbb{R}^2 - \{0\}) & \xrightarrow{I^0} & H^0(U_1) \oplus H^0(U_2) & \xrightarrow{J^0} & H^0(U_1 \cap U_2) & \xrightarrow{\partial^*} & H^1(\mathbb{R}^2 - \{0\}) \to 0
\end{array}
$$

However, $\mathbb{R}^2 - \{0\}$ is connected. Hence $H^0(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}$, and since I^0 is injective we must have that $\operatorname{Im} J^0 \cong \mathbb{R}$. Exactness gives $\operatorname{Ker} J^0 \cong \mathbb{R}$, so that J^0 has rank 1. Therefore $\operatorname{Im} J^0 \cong \mathbb{R}$ and, once again, by exactness

$$\partial^* : H^0(U_1 \cap U_2)/\operatorname{Im} J^0 \to H^1(\mathbb{R}^2 - \{0\})$$

Since $H^0(U_1 \cap U_2)/\operatorname{Im} J^0 \cong \mathbb{R}$, we have shown

$$H^p(\mathbb{R} - \{0\}) = \begin{cases} 0 & \text{if } p > 2, \\ \mathbb{R} & \text{if } p = 1 \\ \mathbb{R} & \text{if } p = 0 \end{cases}$$

In the proof above we could alternatively have calculated

$$J^0 : H^0(U_1) \oplus H^0(U_2) \to H^0(U_1 \cap U_2)$$

by using Lemma 3.9: $H^0(U)$ consists of locally constant functions. If f_i is a constant function on U_i, then

$$J^0(f_1) = f_{1|U_1 \cap U_2} \text{ and } J^0(f_2) = f_{2|U_1 \cap U_2}$$

so that $J^0(a, b) = a - b$.

Theorem 5.5 Assume that the open set U is covered by convex open sets $U_1, \cdots, U_r$. Then $H^p(U)$ is finitely generated.

Proof. We use induction on the number of open sets. If $r = 1$ the assertion follows from the Poincaré lemma. Assume the assertion is proved for $r - 1$ and let $V = U_1 \cup \cdots \cup U_{r-1}$, such that $U = V \cup U_r$. From Theorem 5.2 we have the exact sequence

$$H^{p-1}(V \cup U_r) \xrightarrow{\partial^*} H^p(U) \xrightarrow{I^*} H^p(V) \oplus H^p(U_r)$$

which by Lemma 4.1 yields

$$H^p(U) \backsimeq \operatorname{Im} \partial^* \oplus \operatorname{Ker} I^*.$$

Now both V and $V \cap U_r = (U_1 \cap U_r) \cup \cdots \cup (U_{r-1} \cap U_r)$ are unions by $r - 1$ convex open sets. Therefore Theorem 5.5 holds for $H^*(V \cap U_r), H^*(V)$ and $H^*(U_r)$, and hence also for $H^*(U)$. $\square$

6. HOMOTOPY

In this chapter we show that de Rham cohomology is functorial on the category of continuous maps between open sets in Euclidean spaces and calculate $H^*(\mathbb{R}^n - \{0\})$.

Definition 6.1 Two continuous maps $f_\nu : X \to Y, \nu = 0, 1$ between topological spaces are said to be homotopic, if there exists a continuous map

$$F : X \times [0,1] \to Y$$

such that $F(x, \nu) = f_\nu(x)$ for $\nu = 0, 1$ and all $x \in X$.

This is denoted by $f_0 \simeq f_1$, and F is called a *Homotopy* from f_0 to f_1. It is convenient to think of F as a family of continuous maps $f_t : X \to Y (0 \leqslant t \leqslant 1)$, given by $f_t(x) = F(x, t)$, which deform f_0 to f_1.

Lemma 6.2 Homotopy is an equivalence relation.

PROOF. If F is a homotopy from f_0 to f_1, a homotopy from f_1 to f_0 is defined by $G(x, t) = F(x, 1 - t)$. If $f_0 \simeq f_1$ via F and $f_1 \simeq f_2$ via G, then $f_0 \simeq f_2$ via

$$H(x,t) = \begin{cases} F(x, 2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ G(x, 2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Finally we have that $f \simeq f$ via $F(x, t) = f(x)$. $\square$

Lemma 6.3 Let X, Y and Z be topological spaces and let $f_\nu : X \to Y$ and $g_\nu : Y \to Z$ be continuous maps for $\nu = 0, 1$. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

PROOF. Given homotopies F from f_0 to f_1 and G from g_0 to g_1, the homotopy H from $g_0 \circ f_0$ to $g_1 \circ f_1$ can be defined by $H(x, t) = G(F(x, t), t)$. $\square$

Definition 6.4 A continuous map $f : X \to Y$ is called a *homotopy equivalence*, if there exists a continuous map $g : Y \to X$, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Such a map g is said to be a *homotopy inverse* to f.

Two topological spaces X and Y are called *homotopy equivalent* if there exists a homotopy equivalence between them. We say that X is contractible, when X is homotopy equivalent to a single-point space. This is the same as saying that id_X is homotopic to a constant map. The equivalence classes of topological spaces defined by the relation homotopy equivalence are called *homotopy types*.

Example 6.5 Let $Y \subseteq \mathbb{R}^m$ have the topology induced by $\mathbb{R}^m$. If, for the continuous maps $f_\nu : X \to Y, \nu = 0, 1$, the line segement in $\mathbb{R}^m$ from $f_0(x)$ to $f_1(x)$ is contained in Y for all $x \in X$, we can define a homotopy $F : X \times [0,1] \to Y$ from f_0 to f_1 by

$$F(x, t) = (1 - t)f_0(x) + tf_1(x).$$

In particular this shows that a star-shaped set in $\mathbb{R}^m$ is contractible.

Lemma 6.6 If U, V are open sets in Euclidean spaces, then

(i) Every continuous map $f : U \to V$ is homotopic to a smooth map.

(ii) If two smooth maps $f_\nu : U \to V, \nu = 0, 1$ are homotopic, then there exists a smooth map $F : U \times \mathbb{R} \to U$ with $F(x, \nu) = f_\nu(x)$ for $\nu = 0, 1$ and all $x \in U$. (F is called a *smooth homotopy* from f_0 to f_1).

PROOF. We use Lemma A.9 to approximate h by a smooth map $f : U \to V$. We can choose f such that V contains the line segment from $h(x)$ to $f(x)$ for every $x \in U$. Then $h \simeq f$ by Example 6.5.
Let G be a homotopy from f_0 to f_1. Use continuous function $\psi : \mathbb{R} \to [0, 1]$ with $\psi(t) = 0$ for $t \leqslant \frac{1}{3}$ and $\psi(t) = 1$ for $t \geqslant \frac{2}{3}$ to construct

$$H : U \times \mathbb{R} \to V, \quad H(x, t) = G(x, \psi(t)).$$

Since $H(x, t) = f_0(x)$ for $t \leqslant \frac{1}{3}$ and $H(x, t) = f_1(x)$ for $t \geqslant \frac{2}{3}$, H is smooth on $U \times (-\infty) \cup U \times (\frac{2}{3}, \infty)$. Lemma A.9 allows us to approximate H by a smooth map $F : U \times \mathbb{R} \to V$ such that F and H have the same restiction on $U \times \{0, 1\}$. For $\nu = 0, 1$ and $x \in U$ we have that $F(x, \nu) = H(x, \nu) = f_\nu(x)$. □

Theorem 6.7 If $f, g : U \to V$ are smooth maps and $f \simeq g$ then the induced chain maps

$$f^*, g^* : \Omega^*(V) \to \Omega^*(U)$$

are chain-homotopic (see Definition 4.10).

PROOF. Recall, from the proof of Theorem 3.15, that every p-form ω on $U \times \mathbb{R}$ can be written as

$$\omega = \sum f_I(x, t)dx_I + \sum g_J(x, t)dt \wedge dx_J$$

If $\phi : U \to U \times \mathbb{R}$ is the inlclusion map $\phi(x) = \phi_0(x) = (x, 0)$, then

$$\phi^*(\omega) = \sum f_I(x, 0)d\phi_I = \sum F_I(x, 0)dx_I.$$

Indeed, $\phi^*(dt \wedge dx_J) = 0$ since the last component (the t-component) of ϕ is constant; see Example 3.11. Analogously, for $\phi_1(x) = (x, 1)$, we have that

$$\phi_1^*(\omega) = \sum F_I(x, 1)dx_I.$$

In the proof of Theorem 3.15 we constructed

$$\hat{S}_p^* : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$$

such that

$$(d\hat{S}_p + \hat{S}_{p+1}d)(\omega) = \phi_1^*(\omega) - \phi_0^*(\omega). \tag{1}$$

Consider the composition $U \xrightarrow{\phi_v} U \times \mathbb{R} \xrightarrow{F} V$, where F is a smooth homotopy between f and g. The we have that $F \circ \phi_0 = f$ and $F \circ \phi_1 = g$. We define

$$S_p : \Omega^p(V) \to \Omega^{p-1}(U)$$

to be $S_p = \hat{S}_p \circ F$, and assert that

$$\begin{aligned} d\hat{S}_p(F^*(\omega)) + \hat{S}_{p+1}dF^*(\omega) &= \phi_1^*(\omega) - \phi_0^*(\omega) \\ &= (F \circ \phi_0)^*(\omega) - (F \circ \phi_0)^*(\omega) \\ &= g^*(\omega) - f^*(\omega). \end{aligned}$$

Furthermore $\hat{S}_{p+1}F^*(\omega) = \hat{S}_{p+1}F^*d(\omega) = S_{p+1}d(\omega)$, since F^* is a chain map. □

In the situation of Theorem 6.7, Lemma 4.11 states that $f^* = g^* : H^p(V) \to H^(U)$. For a continuous map $\phi : U \to V$ we can find a smooth map $f : U \to V$ with $\phi \backsimeq f$ by (i) of Lemma 6.6, and by Lemma 6.2 and the result above we see that $f^* : H^p(V) \to H^(U)$ is independent of the choice of f. Hence we can define

$$\phi^* = H^p(\phi) : H^p(V) \to H^p(U).$$

be setting $\phi^* = f^*$, where $f : U \to V$ is a smooth map homotopy to ϕ.

Theorem 6.8 For $p \in \mathbb{Z}$ and open sets U, V, W in Euclidean spaces we have

(i) If $\phi_0, \phi_1 : U \to V$ are homotopic continuous maps, then

$$\phi_0^* = \phi_1^* : H^p(V) \to H^p(U).$$

(ii) If $\phi : U \to V$ and $\psi : V \to W$ are continuous, then $(\phi \circ \psi)^* = \psi^* \circ \phi^* : H^p(W) \to H^p(U)$.

(iii) If the continuous map $\phi : U \to V$ is a homotopy equivalence, then

$$\phi^* : H^p(V) \to H^p(U)$$

is an isomorphism.

Proof. Choose a smooth map $f : U \to V$ with $\phi \backsimeq f$. Lemma 6.2 gives that $\phi_1 \backsimeq f$ and (i) immediately follows. Part (ii), with smooth ϕ and ψ, follows from the formula

$$\Omega^p(\phi \circ \psi) = \Omega^p(\phi) \circ \Omega^p(\psi).$$

In the general case, choose smooth maps $f : U \to V$ and $g : V \to W$ with $\phi \backsimeq f$ and $\psi \backsimeq g$. Lemma 6.3 shows that $\phi \circ \psi \backsimeq g \circ f$, and we get

$$(\phi \circ \psi)^* = (g \circ f)^* = f^* \circ g^* = \psi^* \circ \phi^*.$$

If $\psi : U \to V$ is a homotopy inverse to ϕ, i.e.,

$$\psi \circ \phi \simeq \mathrm{id}_U, \text{ and } \phi \circ \psi \simeq \mathrm{id}_V,$$

then it follows from (ii) that $\psi^* : H^p(U) \to H^p(V)$ is inverse to ϕ^*. □

This result shows that $H^p(U)$ depends only on the homotopy type of U. In particular we have:

Corollary 6.9 (Topological invariance) A homeomorphism $h : U \to V$ between open sets in Euclidean spaces induces isomorphisms $h^* : H^p(U) \to H^p(V)$ for all p.

PROOF. The corollary follows from Theorem 6.8.(iii), as $h^{-1} : V \to U$ is a homotopy inverse to h. □

Corollary 6.10 If $U \subseteq \mathbb{R}$ is an open contractible set, then $H^p(U) = 0$ when $p > 0$ and $H^0(U) = \mathbb{R}$.

PROOF. Let $F : U \times [0,1] \to U$ be a homotopy from $f_0 = \mathrm{id}_U$ to a constant map f_1 with value $x_0 \in U$. For $x \in U, F(x,t)$ defines a continuous curve in U, which connects x to x_0. Hence U is connected and $H^0(U) = \mathbb{R}$ by Lemma 3.9. If $p > 0$ then $\Omega^p(f_1) : \Omega^p(U) \to \Omega^p(U)$ is the zero map. Hence by Theorem 6.8.(i) we get that

$$\mathrm{id}_{H^p(U)} = f_0^* = f_1^* = 0.$$

and thus $H^p(U) = 0$. □

In the proposition below, $\mathbb{R}^n$ is identified with the subspace $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^{n+1}$ and $\mathbb{R} \cdot 1$ denotes the 1-dimensional subspace consisting of constant functions.

Proposition 6.11 For an arbitrary closed subset A of $\mathbb{R}$ with $A \neq \mathbb{R}^n$ we have isomorphisms

$$\begin{aligned} H^p(\mathbb{R}^n - A) &\cong H^p(\mathbb{R}^n - A) \qquad \text{for} p \geqslant 1 \\ H^1(\mathbb{R}^{n+1} - A) &\cong H^0(\mathbb{R}^n - A)/\mathbb{R} \cdot 1 \\ H^0(\mathbb{R}^{n+1} - A) &\cong \mathbb{R}. \end{aligned}$$

PROOF. Define open subsets of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$,

$$\begin{aligned} U_1 &= \mathbb{R}^n \times (0,\infty) \cup (\mathbb{R}^n - A) \times (-1,\infty) \\ U_2 &= \mathbb{R}^n \times (-\infty,0) \cup (\mathbb{R}^n - A) \times (-\infty,1). \end{aligned}$$

Then $U_1 \cup U_2 = \mathbb{R}_{n+l} - A$ and $U_1 \cap U_2 = (\mathbb{R}_n - A)x(-1,1)$. Let $\phi : U_1 \to U_1$ be given by adding 1 to the $(n+l)$-st coordinate. For $x \in U_1$, U_1 contains the line segments from x to $\phi(x)$ and from $\phi(x)$ to a fixed point in $\mathbb{R}_n \times (0,\infty)$. As in Example 6.5 we get homotopies from id_{U_1} to ϕ and from ϕ to a constant map. It follows that U_1 is contractible. Analogously U_2 is contractible, and $H^p(U_v)$ is described in Corollary 6.10.

Let pr be the projection of $U_1 \cap U_2 = (\mathbb{R}^n - A) \times (-1, 1)$ on $\mathbb{R}^n - A$. Define $i : \mathbb{R}^n - A \to U_1 \cap U_2$ by $i(y) = (y, 0)$. We have $\mathrm{pr} \circ i = \mathrm{id}_{\mathbb{R}^n - A}$ and $i \circ \mathrm{pr} \simeq \mathrm{id}_{U_1 \cap U_2}$. From Theorem 6.8 (iii) we conclude that

$$\mathrm{pr}^* : H^p(\mathbb{R}^n - A) \to H^p(U_1 \cap U_2)$$

is an isomorphism for every p. Theorem 5.2 gives isomorphism

$$\partial^* : H^p(U_1 \cap U_2) \to H^{p+1}(\mathbb{R}^{n+1} - A)$$

for $p \geqslant 1$. By composition with pr^* one obtains the first part of Proposition 6.11. Consider the exact suquence

$$0 \to H^0(\mathbb{R}^{n+1} - A) \xrightarrow{I^*} H^0(U_1) \oplus H^0(U_2) \xrightarrow{J^*} H^0(U_1 \cap U_2) \xrightarrow{\partial^*} H^1(\mathbb{R}^{n+1} - A) \to 0.$$

An element of $H^0(U_1) \oplus H^0(U_2)$ is given by a pair of constant functions on U_1 and U_2 with values a_1 and a_2. Their images under J^* is by Theorem 5.2 the constant function on $U_1 \cap U_2$ with value $a_1 - a_2$. This shows that

$$\mathrm{Ker}\, \partial^* = \mathrm{Im}\, J^* = \mathbb{R} \cdot 1,$$

and we obtain the isomorphism

$$H^1(\mathbb{R}^{n+1} - A) \simeq H^0(U_1 \cap U_2)/\mathbb{R} \cdot 1 \cong H^0(\mathbb{R}^n - A)/\mathbb{R} \cdot 1.$$

We also have that $\dim(\mathrm{Im}(I^*)) = \dim(\mathrm{Ker}(J^*)) = 1$, so $H^0(\mathbb{R}^{n+1} - A) = \mathbb{R}$. □

Addendum 6.12 In the situation of Proposition 6.11 we have a *diffeomorphism*

$$R : \mathbb{R}^{n+1} - A \to \mathbb{R}^{n+1} - A$$

defined by $R(x_1, \cdots, x_n, x_{n+1}) = (x_1, \cdots, x_n, -x_{n+1})$. The induced linear map

$$R^* : H^{p+1}(\mathbb{R}^{n+1} - A) \to H^{p+1}(\mathbb{R}^{n+1} - A)$$

is multiplication by (-1) for $p \geqslant 0$.

Proof. In the notation of the proof above we have commutative diagrams, in which the horizontal diffeomorphisms are restrictions of R:

$$\begin{tikzcd}
\mathbb{R}^{n+1} - A \arrow[r, "R"] & \mathbb{R}^{n+1} - A \\
U_1 \arrow[u, "i_1"] \arrow[r, "R_1"] & U_2 \arrow[u, "i_2"'] \\
U_1 \cap U_2 \arrow[u, "j_1"] \arrow[r, "R_0"] & U_1 \cap U_2 \arrow[u, "j_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{R}^{n+1} - A \arrow[r, "R"] & \mathbb{R}^{n+1} - A \\
U_1 \arrow[u, "i_1"] \arrow[r, "R_2"] & U_2 \arrow[u, "i_2"'] \\
U_1 \cap U_2 \arrow[u, "j_1"] \arrow[r, "R_0"] & U_1 \cap U_2 \arrow[u, "j_2"']
\end{tikzcd}$$

In the proof of Proposition 6.11 we saw that

$$\partial^* : H^p(U_1 \cap U_2) \to H^{p+1}(\mathbb{R}^{n+1} - A)$$

is surjective. Therefore it is sufficient to show that $R^* \circ \partial^*([\omega]) = -\partial^*([\omega])$ for arbitary closed p-form ω on $U_1 \cap U_2$.
Using Theorem 5.1 we can find $\omega_\nu \in \Omega^p(U_\nu), \nu = 0, 1$, with $\omega = j_1^*(\omega_1) - j_2^*(\omega_2)$. The definition of ∂^* (see Definition 4.5) show that $\partial^*([\omega]) = [\tau]$ where $\tau \in \Omega^{p+1}(\mathbb{R}^{n+1} - A)$ is determined by $i_\nu^*(\tau) = d\omega_\nu$ for $\nu = 1, 2$. Furthermore we get

$$\begin{aligned}
-R_0^*\omega &= R_0^* \circ j_2^*(\omega_2) - R_0^* \circ j_1^*(\omega_1) = j_1^* \circ R_1^*(\omega_2) - j_2^* \circ R_2^*(\omega_1)\\
i_1^*(R^*\tau) &= R_1^*(i_2^*\tau) = R_1^*(d\omega_2) = d(R_1^*\omega_2)\\
i_2^*(R^*\tau) &= R_2^*(i_1^*\tau) = R_2^*(d\omega_1) = d(R_2^*\omega_1)
\end{aligned}$$

These equations and the definition of ∂^* give $\partial^*(-[R_0^*\omega]) = [\partial^*\tau]$. Hence

$$\partial^* \circ R_0^*([\omega]) = -R_0^* \circ \partial^*([\omega]). \tag{2}$$

For the projection $\mathrm{pr} : U_1 \times U_2 \to \mathbb{R}^n - A$ we have that $\mathrm{pr} \circ R_0 = \mathrm{pr}$ and therefore

$$H^p(\mathbb{R}^n - A) \xrightarrow{\mathrm{pr}^*} H^p(U_1 \cap U_2) \xrightarrow{R_0^*} H^p(U_1 \cap U_2)$$

is identical with pr^*. Since pr^* is an isomorphism, R_0^* is forced to be the identity map on $H^p(U_1 \cap U_2)$, and the left-hand side in (2) is $\partial^*[\omega]$. This completes the proof □

Theorem 6.13 For $n \geqslant 2$ we have the isomorphisms

$$H^p(\mathbb{R}^n - \{0\}) \cong \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The case $n = 2$ was shown in Example 5.4. The general case follows from induction on n, via Proposition 6.11. □

An invertible real $n \times n$ matrix A defines a linear isomorphism $\mathbb{R}^n \to \mathbb{R}$, and a diffeomorphism

$$f_A : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$$

Lemma 6.14 For each $n \geqslant 2$, the induced map $f_A^* : H^{n-1}(\mathbb{R}^n - \{0\}) \to H^{n-1}(\mathbb{R}^n - \{0\})$ operators by multiplication by $\det(A)/|\det A| \in \{\pm 1\}$.

Proof. Let B be obtained from A by replacing the r-th row by the sum of the r-th row and e times the s-th row, where $r \neq s$ and $c \in \mathbb{R}$,

$$B = (I + cE_{r,s})A$$

where I is the identity matrix and $E_{r,s}$ is the matrix with entry 1 in its r-th row and s-th column and zeros elsewhere. A homotopy between f_A and f_B is defined by the matrices

$$(I + tcE_{r,s})A, \quad 0 \leqslant t \leqslant 1.$$

From Theorem 6.8 it follows that $f_A = f_B$. Furthermore $\det_A = \det_B$. By a sequence of elementary operations of this kind, A can be changed to $\mathrm{diag}(1, \cdots, 1, \pm 1)$, where $d = \det A$. Hence it suffices to prove the assertion for diagonal matrices. The matrices

$$\mathrm{diag}(1, \cdots, 1, \frac{|d|^t d}{|d|}), \quad 0 \leqslant t \leqslant 1$$

yield a homotopy, which reduces the problem to the two cases $A = \mathrm{diag}(1, \cdots, 1, \pm 1)$, so f_A is either the identity or the map R from Addendum 6.12. This proves the assertion □

From topological invariance (see Corollary 6.9) and the calculation in Theorem 6.13, supplemented with

$$H^p(\mathbb{R}^1 - \{0\}) \cong \begin{cases} \mathbb{R} \otimes \mathbb{R} & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

we get

Proposition 6.15 If $n \neq m$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

Proof. A possible homeomorphism $\mathbb{R}^n \to \mathbb{R}^m$ may be assumed to map 0 to 0, and would induce a homeomorphism between $\mathbb{R}^n - \{0\}$ and $\mathbb{R}^m - \{0\}$. Hence

$$H^p(\mathbb{R}^n - \{0\}) \cong H^p(\mathbb{R}^m - \{0\})$$

for all p, in conflict with our calculations. □

Remark 6.16 We offer the following more conceptual proof of Addendum 6.12. Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^* & \xrightarrow{f^*} & B^* & \xrightarrow{g^*} & C^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \beta^*} & & \downarrow{\scriptstyle \gamma^*} & & \\
0 & \longrightarrow & A_1^* & \xrightarrow{f^*} & B_1^* & \xrightarrow{g^*} & C_1^* & \longrightarrow & 0
\end{array}$$

be a commutative diagram of chain complexes with exact rows. It is not hard to prove that the diagram

$$\begin{tikzcd} H^p(C^*) \arrow[r, "\partial^*"] \arrow[d, "\gamma^*"] & H^{p+1} \arrow[d, "A^*"] \\ H^p(C_1^*) \arrow[r, "\partial_1^*"] & H^{p+1}(A_1^*) \end{tikzcd}$$

is commutative. In the situation of Addendum 6.12 consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega^*(U) \arrow[r, "I^*"] \arrow[d, "R^*"] & \Omega^*(U_1) \oplus \Omega^*(U_2) \arrow[r] \arrow[d, "R"] & \Omega^*(U_1 \cap U_2) \arrow[r] \arrow[d, "-R_0^*"] & 0 \\ 0 \arrow[r] & \Omega^*(U) \arrow[r, "I^*"] & \Omega^*(U_1) \oplus \Omega^*(U_2) \arrow[r] & \Omega^*(U_1 \cap U_2) \arrow[r] & 0 \end{tikzcd}$$

With $R(\omega_1, \omega_2) = (R_1^*\omega_2, R_2^*\omega_1)$. This gives equation (2) of the proof of the addendum.

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