第二类曲线积分的对称性

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1 关于 x 对称

设积分弧段 $L=L_1+L_2$ 第二类曲线的积分对称性设 $L_1\begin{cases}x=x\\y=\phi(x)\end{cases}$, x 从 a 到 b 设 $L_2\begin{cases}x=x\\y=-\phi(x)\end{cases}$, x 从 b 到

(1) 若 $P(x,y)=P(x,-y)$, 则 $\int_L P(x,y)\ \mathrm{d}x=0$

证明: 因为 $P(x,y)=P(x,-y)$ 即 $\int_a^b P[x,\phi(x)]\,\mathrm{d}x=\int_a^b P[x,-\phi(x)]\,d$ 即 $\int_{L_1} P(x,y)\,\mathrm{d}x=-\int_b^a P[x,-\phi(x)]\ \mathrm{d}x$ 因为 $\int_{L_2} P(x,y)\ \mathrm{d}x=\int_b^a P[x,-\phi(x)]\,d\lambda$ 即 $\int_{L_1} P(x,y)\ \mathrm{d}x=-\int_{L_2} P(x,y)\ \mathrm{d}x$ 即 $\int_L P(x,y)\ \mathrm{d}x=0$ □

(2) 若 $Q(x,y)=Q(x,-y)$, 则 $\int_L Q(x,y)\,dy=2\int_{L_1} Q(x,y)\,dy=2\int_{L_2} Q(x,y)\,dy$

证明: $\int Q(x,y)\,dy=\int_a^b Q[x,\phi(x)]\,\phi'(x)\,\mathrm{d}x\int_a^b Q[x,\phi(x)]\,\phi'(x)\,\mathrm{d}x=\int_a^b Q[x,-\phi(x)]\,\phi'(x)\,d$ $\int_{L_2} Q(x,y)\,dy=-\int_b^a Q[x,-\phi(x)]\,\phi'(x)\,d$ 即 $\int_{L_1} Q(x,y)\,dy=\int_{L_2} Q(x,y)\,dy\int_L Q(x,y)\,dy=2\int_{L_1} Q(x,y)\,dy=2\int_{L_2} Q(x,y)\,dy$ □

(3) 若 $-P(x,y)=P(x,-y)$, 则 $\int_L P(x,y)\ \mathrm{d}x=2\int_{L_1} P(x,y)\ \mathrm{d}x=2\int_{L_2} P(x,y)\,dy$

证明: 因为 $-P(x,y)=P(x,-y)$ 即 $\int_a^b P[x,\phi(x)]\ \mathrm{d}x=-\int_a^b P[x,-\phi(x)]\ \mathrm{d}x$ 因为 $\int_{L_2} P(x,y)\,\mathrm{d}x=\int_b^a P[x,-\phi(x)]\,\mathrm{d}x$ 即 $\int_{L_1} P(x,y)\,\mathrm{d}x=\int_{L_2} P(x,y)\,\mathrm{d}x\int_L P(x,y)\,\mathrm{d}x=2\int_{L_1} P(x,y)\ \mathrm{d}x=2\int_{L_2} P(x,y)\,dy$ □

(4) 若 $-Q(x,y)=Q(x,-y)$, 则 $\int_L Q(x,y)\,dy=0$

证明: 因为 $-Q(x,y)=Q(x,-y)$ $\int_a^b Q[x,\phi(x)]\,\phi'(x)\ \mathrm{d}x=-\int_a^b Q[x,-\phi(x)]\,\phi'(x)\,d$ $\int_{L_2} Q(x,y)\,dy=-\int_b^a Q[x,-\phi(x)]\,\phi'(x)\ \mathrm{d}x$ 即 $\int_{L_1} Q(x,y)\,dy=-\int_{L_2} Q(x,y)\,dy$ 即 $\int_L Q(x,y)\,dy=0$ □

2 关于 y 对称

设积分弧段 $L=L_1+L_2$ 设 $L_2\begin{cases}x=-\phi(y)\\y=v\end{cases}$, y 从 b 到

1. 若 $P(x,y)=P(-x,y)$，则 $\int_{L}P(x,y)\ \mathrm{d}x=2\int_{L_1}P(x,y)\ \mathrm{d}x=2\int_{L_2}P(x,y)\ \mathrm{d}x$ $\int_{L_1}P(x,y)\ \mathrm{d}x=\int_{a}^{b}P[\phi(y),y]\phi'(y)\,dy$

   证明: 因为 $P(x,y)=P(-x,y)$ $\int_{L_1}P(x,y)\mathrm{d}x=-\int_{b}^{a}P[-\phi(y),y]\phi'(y)\,dy$ $\int P(x,y)\mathrm{d}x=-\int^{a}P[-\phi(y),y]\phi'(y)\,dy$ 即 $\int_{L_1}P(x,y)\mathrm{d}x=\int_{L_2}P(x,y)\mathrm{d}x$ $\int_{L}P(x,y)\mathrm{d}x=2\int_{L_1}P(x,y)\mathrm{d}x=2\int_{L_2}P(x,y)\mathrm{d}x$ $\int_{L}Q(x,y)\,dy=0$. 因为 $\int_{L_1}Q(x,y)\,dy=\int_{a}^{b}Q[\phi(y),y]\,dy$ 即 $\int_{a}^{b}Q[\phi(y),y]\,dy=\int_{a}^{b}Q[-\phi(y),y]\,dy$ 即 $\int_{L_1}Q(x,y)\,dy=-\int_{b}^{a}Q[-\phi(y),y]\,dy$ 因为 $\int_{L_2}Q(x,y)\,dy=\int_{b}^{a}Q[-\phi(y),y]\,dy$ 即 $\int_{L}Q(x,y)\,dy=0$ □

2. 若 $-P(x,y)=P(-x,y)$，则 7. 若 $-P(x,y)=P(-x,y)$，则 $\int_{L_1}P(x,y)\ \mathrm{d}x=\int_{a}^{b}P[\phi(y),y]\phi'(y)\,dy$

   证明: 因为 $-P(x,y)=P(-x,y)$ $\int_{L_1}P(x,y)\mathrm{d}x=\int_{b}^{a}P[-\phi(y),y]\phi'(y)\,dy$ 即 $\int_{L_1}P(x,y)\mathrm{d}x=-\int_{L_2}P(x,y)\mathrm{d}x$ 即 $\int_{L}P(x,y)\mathrm{d}x=0$ $\int_{L}Q(x,y)\,dy=2\int_{L_1}Q(x,y)\,dy=2\int_{L_2}Q(x,y)\,dy$ 因为 $\int_{L_1}Q(x,y)\,dy=\int_{a}^{b}Q[\phi(y),y]\,dy$ 因为 $-Q(x,y)=Q(-x,y)$ $\int_{a}^{b}Q[\phi(y),y]\,dy=-\int_{a}^{b}Q[-\phi(y),y]\,dy$ 即 $\int_{L_1}Q(x,y)\,dy=\int_{b}^{a}Q[-\phi(y),y]\,dy$ 因为 $\int_{L_2}Q(x,y)\,dy=\int_{b}^{a}Q[-\phi(y),y]\,d$ $\int_{L}Q(x,y)\,dy=2\int_{L_1}Q(x,y)\,dy=2\int_{L_2}Q(x,y)\,dy$ □

3 关于原点对称

设 $L_1\begin{cases}x=x\\y=\phi(x)\end{cases}$，$x$ 从 a 到 b. 设 $L_2\begin{cases}x=x\\y=-\phi(-x)\end{cases}$，$x$ 从 $-b$ 到 $-c$.

1. $\int_{L}P(x,y)\ \mathrm{d}x=2\int_{L_1}P(x,y)\ \mathrm{d}x=2\int_{L_2}P(x,y)\,d$

   证明: 因为 $\int_{L_1}P(x,y)\ \mathrm{d}x=\int_{a}^{b}P[x,\phi(x)]\ \mathrm{d}x$ 因为 $P(x,y)=P(-x,-y)$ 根据区间再现公式 $\int_{b}^{a}f(x)\ \mathrm{d}x=\int_{-a}^{-b}f(-x)\ \mathrm{d}x$ 即 $\int_{L_1}P(x,y)\ \mathrm{d}x=\int_{-b}^{-a}P[x,-\phi(-x)]\ \mathrm{d}x$ 因为 $\int_{L_2}P(x,y)\ \mathrm{d}x=\int_{-b}^{-a}P[x,-\phi(-x)]\ \mathrm{d}x$ 即 $\int_{L_1}P(x,y)\ \mathrm{d}x=\int_{L_2}P(x,y)\ \mathrm{d}x$ $\int_{L}P(x,y)\ \mathrm{d}x=2\int_{L_1}P(x,y)\ \mathrm{d}x=2\int_{L_2}P(x,y)\,dy$ □

2. 若 $Q(x,y) = Q(-x,-y)$，则 $\int_L Q(x,y)\,dy = 2\int_{L_1} Q(x,y)\,dy = 2\int_{L_2} Q(x,y)\,dy$
$\int_L Q(x,y)\,dy = \int_a^b Q[x,\phi(x)]\,\phi'(x)\ \mathrm{d}x$

证明： 因为 $Q(x,y) = Q(-x,-y)$ 根据区间再现公式 $\int_b^a f(x)\ \mathrm{d}x = \int_{-a}^{-b} f(-x)\,d$
$\int_L Q(x,y)\,dy = \int_{-b}^{-a} Q[x,-\phi(-x)]\,\phi'(-x)\,\mathrm{d}x$ $\int_{L_2} Q(x,y)\,dy = \int_{-b}^{-a} Q[x,-\phi(-x)]\,\phi'(-x)\,dt$
即 $\int_{L_1} Q(x,y)\,dy = \int_{L_2} Q(x,y)\,dy$ $\int_L Q(x,y)\,dy = 2\int_{L_1} Q(x,y)\,dy = 2\int_{L_2} Q(x,y)\,d$
证毕 □

3. $\int_L P(x,y)\ \mathrm{d}x = 0$

证明： 因为 $\int P(x,y)\ \mathrm{d}x = \int_a^b P[x,\phi(x)]\ \mathrm{d}x$ 根据区间再现公式 $\int_a^b P[x,\phi(x)]\ \mathrm{d}x =$ $\int_{-b}^{-a} P[-x,\phi(-x)]\,d\lambda$ 因为 $-P(x,y) = P(-x,-y)$ 即 $\int_{L_1} P(x,y)\,\mathrm{d}x = -\int_{-b}^{-a} P[x,-\phi(-x)]\,\mathrm{d}x$
因为 $\int_{L_2} P(x,y)\ \mathrm{d}x = \int_{-b}^{-a} P[x,-\phi(-x)]\ \mathrm{d}x$ 即 $\int_L P(x,y)\ \mathrm{d}x = 0$ □

4. 若 $-Q(x,y) = Q(-x,-y)$，则 $\int_L Q(x,y)\,dy = 0$ $\int_L Q(x,y)\,dy = \int_a^b Q[x,\phi(x)]\,\phi'(x)\,\mathrm{d}x$

证明： 根据区间再现公式 $\int_a^b Q[x,\phi(x)]\,\phi'(x)\ \mathrm{d}x = \int_{-b}^{-a} Q[-x,\phi(-x)]\,\phi'(-x)\,d$ 因为
$-Q(x,y) = Q(-x,-y)$ $\int_1^{-a} Q[-x,\phi(-x)]\,\phi'(-x)\,\mathrm{d}x = -\int_1^{-a} Q[x,-\phi(-x)]\,\phi'(-x)\,\mathrm{d}x$
$\int_L Q(x,y)\,dy = -\int_{-b}^{-a} Q[x,-\phi(-x)]\,\phi'(-x)\,dt$ $\int_{L_2} Q(x,y)\,dy = \int_{-b}^{-a} Q[x,-\phi(-x)]\,\phi'(-x)\,\mathrm{d}x$
即 $\int_L Q(x,y)\,dy = 0$ □