American Options on AAPL

FE620 Pricing and Hedging

May 4, 2022

1.Introduction

Options are derivative financial instruments typically acquired by purchase, based on the value of an underlying security such as a stock, as part of a complex financial transaction.

According to the news in WFE, global derivatives trading jumped in response to the market volatility touched off by the Covid-19 crisis. The trade group for the exchange industry reported that derivatives trading volume surged by 40.4% in 2020, which was more than triple the rise in volume the previous year, when trading activity rose by 11.4%.Both the options and the futures markets saw trading volumes rise in 2020, the report noted. Options trading increased by 44.1% and futures trading was up by 37.5%.Equity derivatives trading increased by 56.5% in 2020. Derivatives become a favorite of individual investors.

American options allow investors to make immediate profits when the price of the underlying stock moves favorably and are therefore usually exercised before expiration.

Being able to price options is an important tool for investors looking to use options as part of their investment strategy.

Derivatives trading soars in response to pandemic: WFE | Investment Executive

2.Description

For this project, a model is built to price three-month (3M) American Options for a large-cap equity with quarterly dividend payments. Apple Inc. (AAPL) stock was chosen due to the company's unique position in the technology sector and the group's interest in the company's offerings. The Black-Scholes pricing model was implemented with binomial trees in R over the duration of January 05, 2022, to April 04, 2022. To utilize the Black Scholes Merton (BSM) Model, estimates of the spot price, volatility, and risk-free rate are needed. The stock's spot price, *S0*, was taken as the closing price on the first day of observation, in this case January 05, 2022, and was found to be *$178.44*.

The volatility is an estimate of the future variability for the asset underlying the options contract. The BSM model assumes the price of the underlying asset follows a geometric Brownian motion with constant drift and volatility. The Itô processto model stock prices for the Black Scholes model is as follows:

$$dS(t) = rS(t)dt + S(t)dW(t)$$

The lookback period for the calculations in this project is one year, thus daily closing prices of the stock are used to compute the daily log-returns which yield the annual historical volatility. Lastly, the risk-free rate, *r*, is required, and for the BSM model, it is best to use the risk-free interest rate at the time of expiration for the option. The 3M Treasury Constant Maturity Rate (DGS1) on 2022-04-04 is used to model the risk-free rate.

2 Market Data Analysis

The market data required as input to the BSM model was obtained by using the R package "quantmod". AAPL's daily stock price is obtained through the Yahoo Finance endpoint using quantmod's "getSymbols" function.

2.1 Analyzing the stock price :

Figure 1 shows the daily closing prices of AAPL for the 3-month period starting on 2022-01-05.

Figure 2 using the daily closing price as Si, the daily log-returns were computed as follows:

We compute the daily log-returns as follows:

$$u_i=\log(S_i/S_{i-1})$$

Figure 3 shows the QQ plot of the log-returns. The QQ-plot of the returns represents a straight line with normal distribution. We observed a deviation of the tails from lognormality.

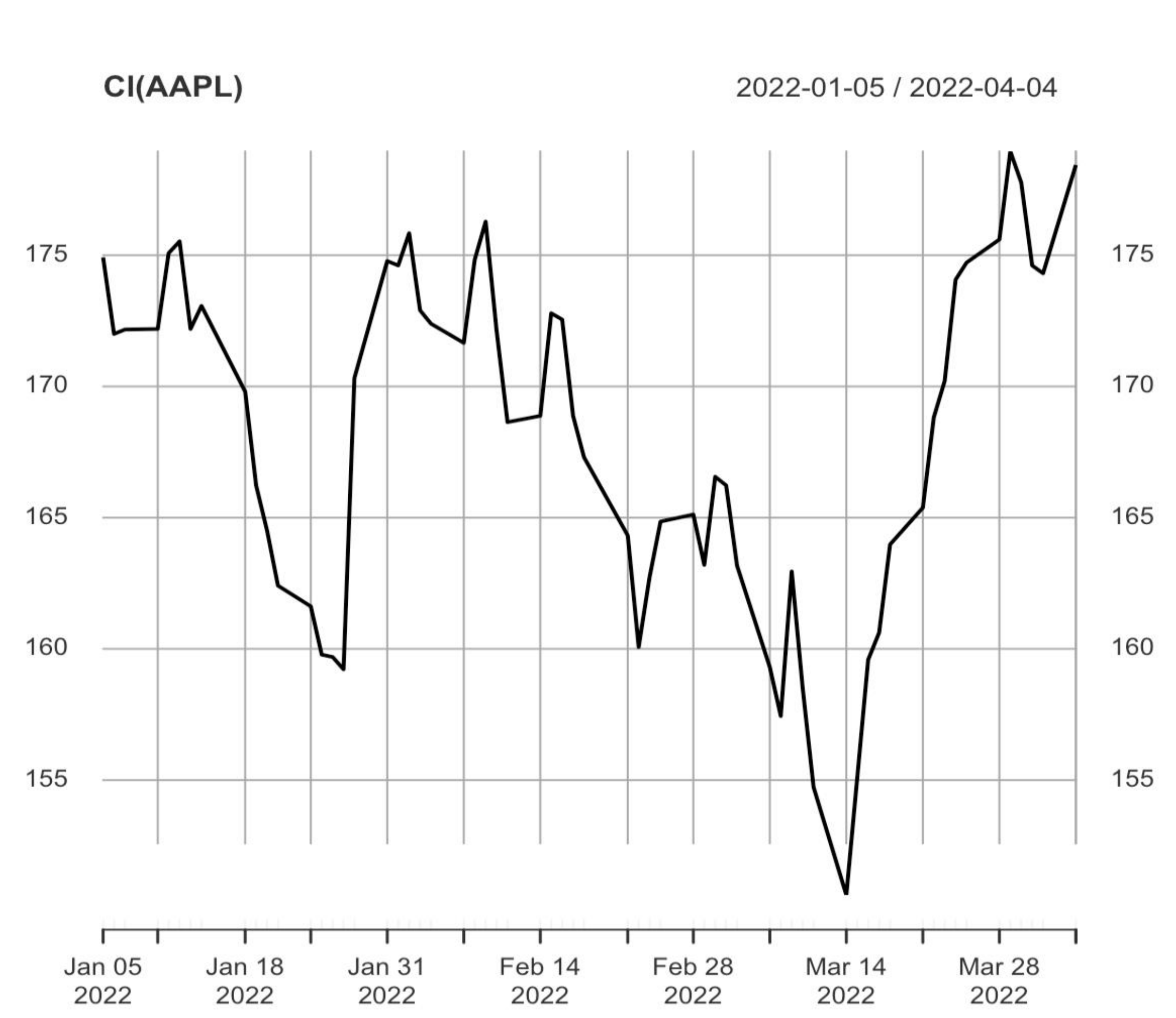


Figure 1: AAPL Daily Closing Prices

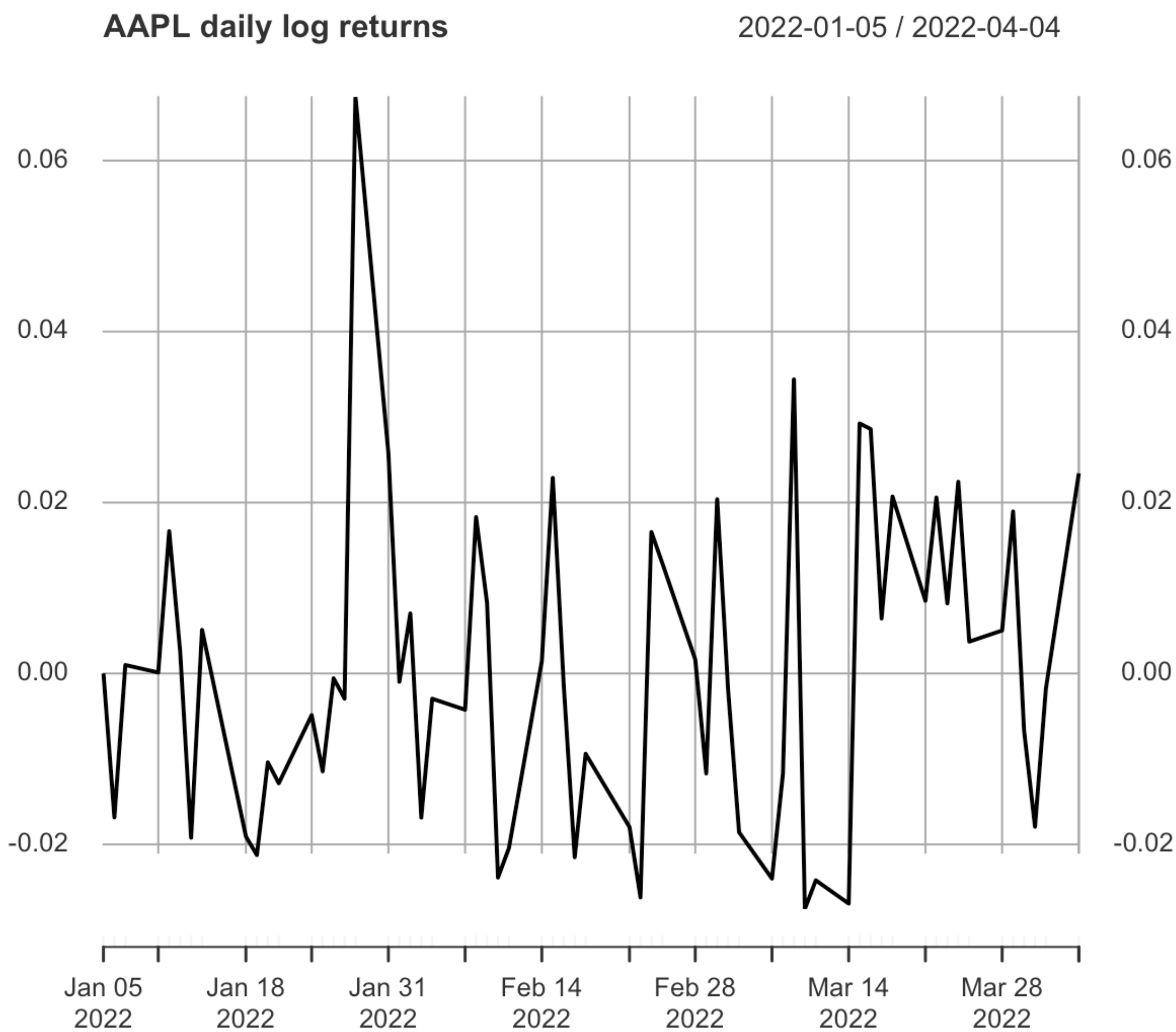


Figure 2: AAPL Daily Log Returns

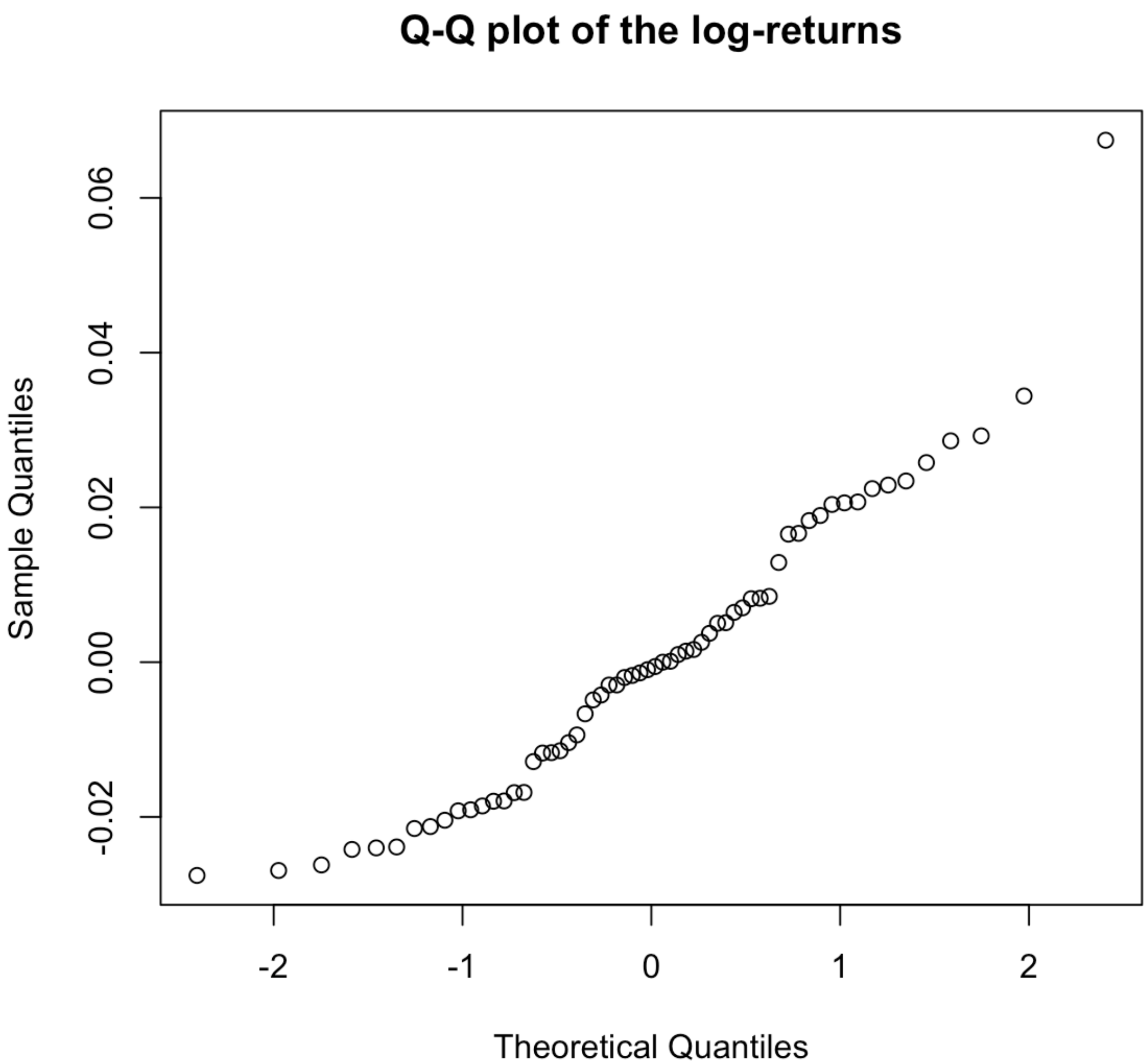


Figure 3: AAPL Log Return QQ Plot

2.2 Estimating Volatility:

After calculating the daily log returns for AAPL, the volatility of the log returns is calculated by finding their standard deviation and annualized by multiplying the value by 252 (square root of the average number of trading days per year)

Figure 4 shows 3M,6M,9M and 1Y volatilities

	Volatility(%)	Standard error
1Y	**24.10%**	**1.52%**
9M	**22.00%**	**1.39%**
6M	**20.96%**	**1.32%**
3M	**21.04%**	**1.33%**

Figure 4: Volatility and Error

2.3 Estimating Risk-Free Rate:

For use in the BSM model, the risk-free rate should be used at option expiration. The 3M Treasury Bill Fixed Maturity Rate was chosen to represent the calculated interest rate, as shown in Figure 5. The 3-month Treasury bill rate on April 4, 2022, is found to be r = 0.68%

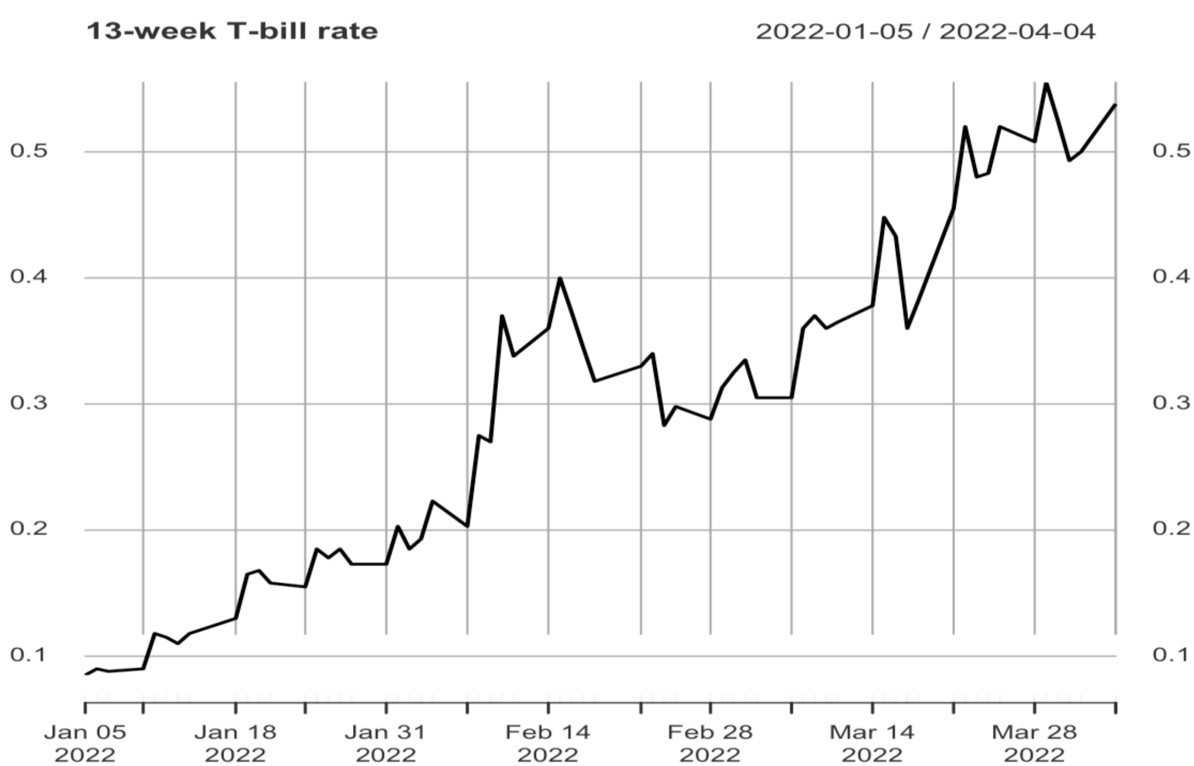


Figure 5: 13-week T-bill rate

OVDV APPLE INC Equity

Note: Given surface is more than 24hrs older than requested

AAPL US Equity | 90) Asset | 91) Actions | 92) Views | 93) Settings | Volatility Surface

APPLE INC 175.06 USD Bloomberg As of 05-Apr-2022 16:15

1) Vol Table | 2) 3D Surface | 3) Term | 4) Skew | 5) Dividends | 6) Prices

From 8 Apr 2022

Expiry	8 Apr 2022				14 Apr 2022				22 Apr 2022			
Put/Call	Call		Put		Call		Put		Call		Put	
Strike	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
157.5	16.45	18.9	0.01	0.03	17.2	18.4	0.17	0.2	17.15	18.65	0.33	0.43
160	14.75	15.45	0.02	0.03	15.15	16.25	0.24	0.26	14.95	15.8	0.48	0.53
162.5	11.95	13.2	0.04	0.06	12.3	13.3	0.33	0.38	12.8	13.8	0.67	0.72
165	9.8	10.8	0.09	0.1	10.4	10.95	0.48	0.52	10.4	11.5	0.93	0.98
167.5	6.55	9	0.16	0.18	7.9	8.65	0.73	0.8	8.65	9.35	1.29	1.36
170	5.3	5.65	0.36	0.4	6.2	6.4	1.15	1.2	6.7	7.3	1.81	1.88
172.5	3.3	3.5	0.8	0.85	4.3	4.5	1.74	1.83	5.05	5.3	2.52	2.62
175	1.72	1.78	1.65	1.71	2.75	2.83	2.68	2.75	3.55	3.75	3.45	3.55
177.5	0.71	0.75	3	3.25	1.61	1.66	3.95	4.1	2.42	2.45	4.7	4.85
180	0.25	0.27	4.75	5.45	0.86	0.88	5.65	5.85	1.47	1.52	5.65	7
182.5	0.09	0.1	6.95	8.35	0.43	0.46	7.65	8.2	0.89	0.96	7.7	8.55
185	0.04	0.05	9.45	10.9	0.22	0.25	9.5	10.9	0.51	0.58	9.85	10.85
187.5	0.02	0.03	11.95	13.45	0.13	0.14	11.9	13.15	0.3	0.36	11.35	13.8
190		0.01	14.1	16	0.08	0.09	14.45	15.9	0.2	0.22	14.25	16.25
192.5		0.01	15.35	19.8	0.05	0.09	15.4	19.9	0.12	0.26	15.55	19.95
195		0.01	18.75	21.4	0.04	0.08	19.25	21.05	0.08	0.11	18.95	21.2
197.5		0.01	20.35	24.85	0.03	0.07	21.25	23.6	0.07	0.09	20.4	24.95
200		0.01	24.15	26.1	0.02	0.04	23.6	26.15	0.05	0.08	23.95	26.15
202.5		0.01	25.4	29.9								
205		0.01	28.8	31.25	0.01	0.03	29.15	31.75		0.06	28.9	31.15
210		0.01	33.6	36		0.02	33.7	36.05	0.01	0.05	33.6	36.8
215		0.01	38	42.35		0.02	39.2	41.15		0.05	39.1	41.85
220		0.01	43	47.35		0.01	44.15	46.15		0.03	43.75	47.25
225		0.01	47.85	52.35		0.01	49.2	51.1		0.04	48.1	51.95
230		0.01	53	57.15		0.01	54.15	56.05		0.04	53	57.35

Figure 6: American option prices on AMZN as of 4-Apr-2022. (Bloomberg)

3 American options on AAPL

Several options are traded on AAPL. As of 4-Apr-2022, the first expiries are 8-Apr-2022, 14-Apr-2022, 22-Apr-2022. These dates are the Fridays of each week.

The prices of these options for several strikes are shown in Figure 3, taken from Bloomberg.

We would like to price these American options in the Black-Scholes model using a binomial tree approach. The first two maturities of the options are 8-Apr-2022, 14-Apr-2022 and we count only business days when determining the maturity of options on stocks. In each small interval of time the stock price is assumed to move up by a proportional amount u or to move down by a proportional amount d.The function computes the binomial tree using up and down moves as follows:

$$u = \exp((r-d) * h + h)$$
$$d = \exp((r-d)*h-h)$$

And when we are using Tree methods, we work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate. And we also need to know how many times steps n we should use in the tree. The result will depend very sensitively on this choice. We determine the number of steps by examining the convergence of the prices as n becomes very large.

3.1 Sensitivity study

Let us price one of the American options and study the dependence of its price on the number of time steps n, while keeping fixed the option maturity $T = nt$.

Consider the put option with expiry 14-Apr-2022 and strike K = 177.5 which is closest to the at-the-money point $K = S_0$. The binomial tree price with n = 9 time steps (daily time steps) gives

$$P(K = 177.5; n = 9) = 3.98$$

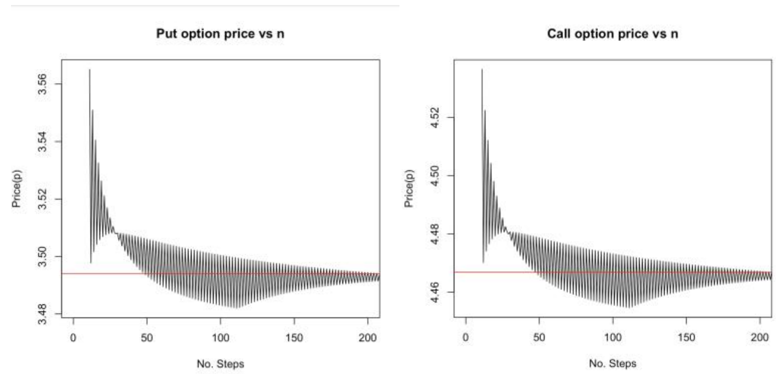


Figure7 : Convergence of the American option prices in the binomial tree model as the number of time steps *n* increases. Left: n = 200-time steps (red). Right: AAPL does not pay dividends so the American call price should converge to the European call price (red line).

How accurate is this? The left plot in Fig. 5 shows the price of this option as n increases from n = 9 to 200. We observe that prices converge for sufficiently large n. For example, with n = 200 steps we have

$$P(K = 177.5;\ n = 200) = 4.636$$

Clearly the n = 200 value is more accurate.

Another test for the binomial tree pricing can be done if we recall that American call options on a stock which does not pay dividends have the same price as the European call options with the same maturity.

The table below shows the binomial tree price of the American call option with strike K = 177.5 for several values of n, the time steps of the tree. We observe that as n increases, the binomial tree prices approach the European option price, as expected.

n	9	30	50	80	100	BS
C(K=177.5)	4.72	4.644	4.642	4.638	4.636	4.642

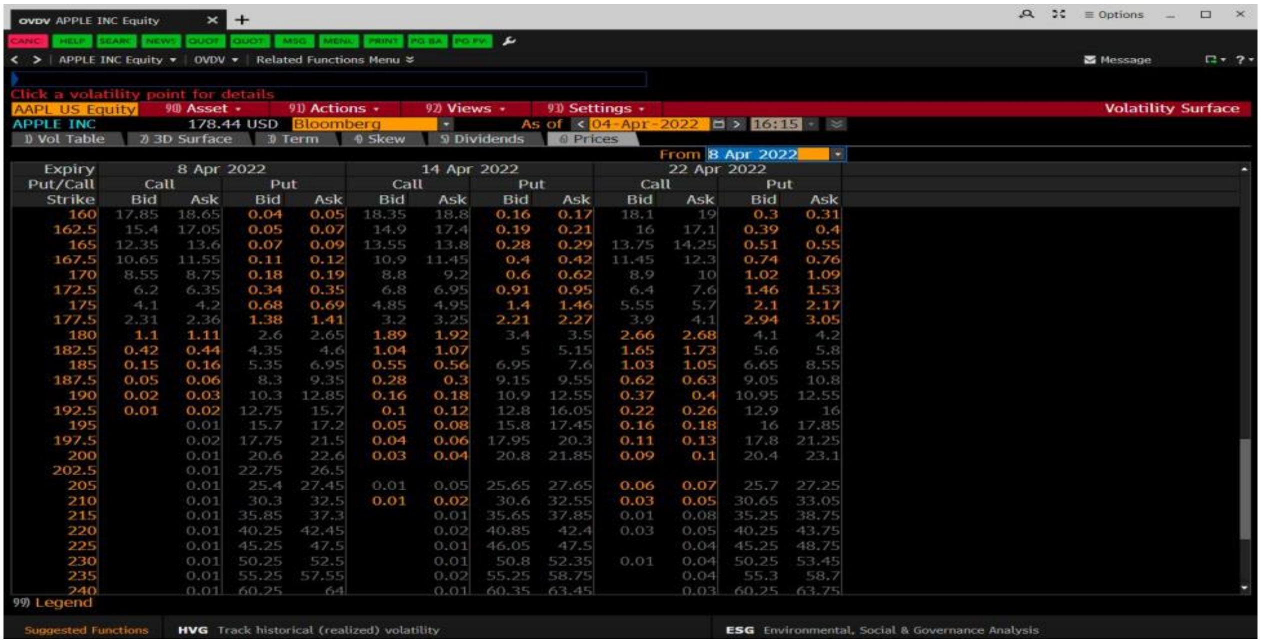
3.2 Estimated Prices vs Market Prices:

Figure 9 displays the market prices of American options for AAPL in April of 2022 with various strikes and expiration dates. Some of the variation between the calculated and observed option prices can be attributed to the volatility smile effect which implies that market options cannot be accurately calculated with a single volatility value.

Furthermore, the volatility smile suggests that as the option becomes more Figure 10 shows the implied volatility, the volatility value which when used in the BSM model produces market option prices, which depend on strike price. Because the model used in this project uses a singular volatility value from the end of the option's life, some variation from the observed values can be expected.

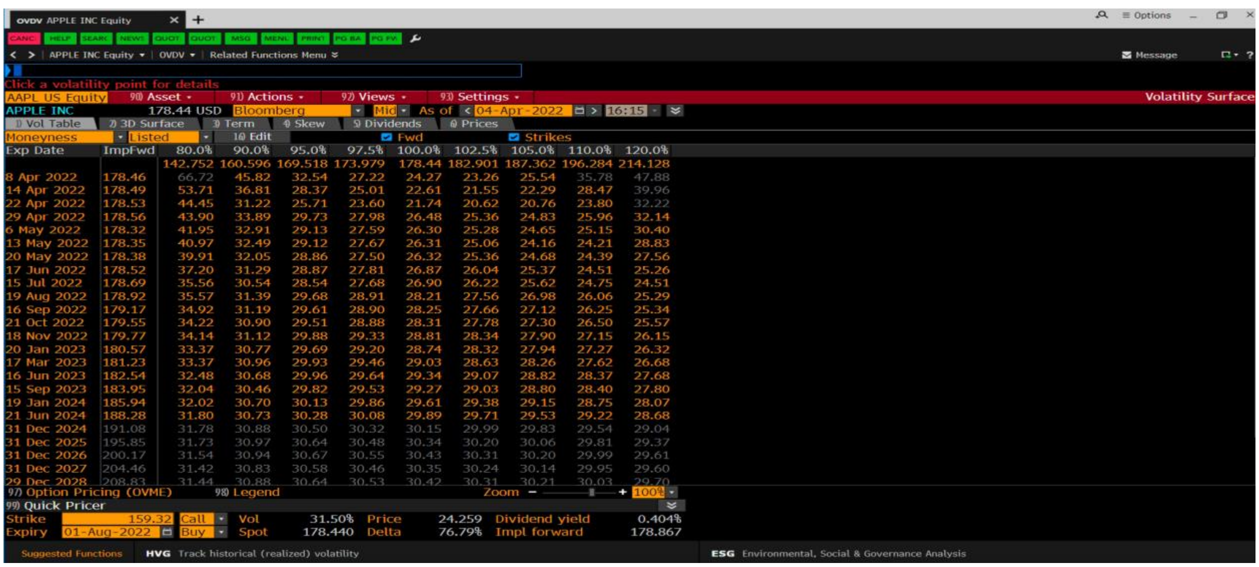
Strike	Call Option		Put Option	
	Num	[Bid,Ask]	Num	[Bid,Ask]
177.5	4.47	[3.2,3.25]	3.50	[2.21,2.27]
180	3.27	[1.89,1.92]	4.80	[3.4,3.5]
182.5	2.30	[1.04,1.07]	6.34	[5,5.15]
185	1.59	[0.55,0.56]	8.12	[6.95,7.6]
187.5	1.057	[0.28,0.3]	10.09	[9.15,9.55]
190	0.675	[0.16,0.18]	12.21	[10.9,12.55]
192.5	0.414	[0.1,0.12]	14.45	[12.8,16.05]

Figure8: Estimated Prices vs Market Prices



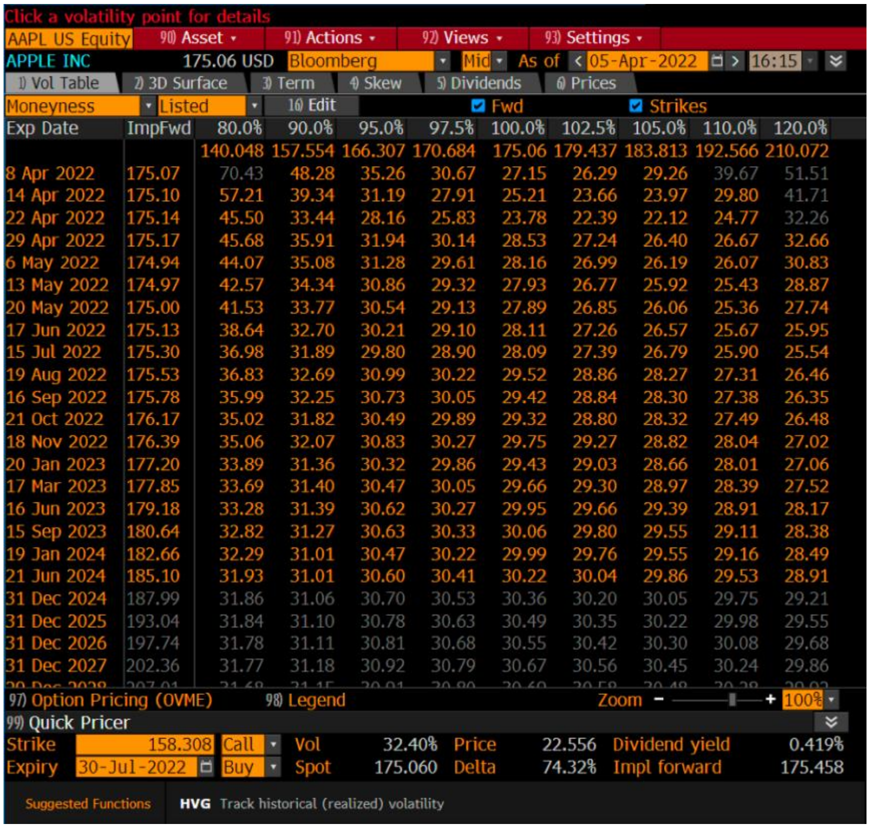
Expiry	8 Apr 2022				14 Apr 2022				22 Apr 2022			
Put/Call	Call		Put		Call		Put		Call		Put	
Strike	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
160	17.85	18.65	0.04	0.05	18.35	18.8	0.16	0.17	18.1	19	0.3	0.31
162.5	15.4	17.05	0.05	0.07	14.9	17.4	0.19	0.21	16	17.1	0.39	0.4
165	12.35	13.6	0.07	0.09	13.55	13.8	0.28	0.29	13.75	14.25	0.51	0.55
167.5	10.65	11.55	0.11	0.12	10.9	11.45	0.4	0.42	11.45	12.3	0.74	0.76
170	8.55	8.75	0.18	0.19	8.8	9.2	0.6	0.62	8.9	10	1.02	1.09
172.5	6.2	6.35	0.34	0.35	6.8	6.95	0.91	0.95	6.4	7.6	1.46	1.53
175	4.1	4.2	0.68	0.69	4.85	4.95	1.4	1.46	5.55	5.7	2.1	2.17
177.5	2.31	2.36	1.38	1.41	3.2	3.25	2.21	2.27	3.9	4.1	2.94	3.05
180	1.1	1.11	2.6	2.65	1.89	1.92	3.4	3.5	2.66	2.68	4.1	4.2
182.5	0.42	0.44	4.35	4.6	1.04	1.07	5	5.15	1.65	1.73	5.6	5.8
185	0.15	0.16	5.35	6.95	0.55	0.56	6.95	7.6	1.03	1.05	6.65	8.55
187.5	0.05	0.06	8.3	9.35	0.28	0.3	9.15	9.55	0.62	0.63	9.05	10.8
190	0.02	0.03	10.3	12.85	0.16	0.18	10.9	12.55	0.37	0.4	10.95	12.55
192.5	0.01	0.02	12.75	15.7	0.1	0.12	12.8	16.05	0.22	0.26	12.9	16
195		0.01	15.7	17.2	0.05	0.08	15.8	17.45	0.16	0.18	16	17.85
197.5		0.02	17.75	21.5	0.04	0.06	17.95	20.3	0.11	0.13	17.8	21.25
200		0.01	20.6	22.6	0.03	0.04	20.8	21.85	0.09	0.1	20.4	23.1
202.5		0.01	22.75	26.5								
205		0.01	25.4	27.45	0.01	0.05	25.65	27.65	0.06	0.07	25.7	27.25
210		0.01	30.3	32.5	0.01	0.02	30.6	32.55	0.03	0.05	30.65	33.05
215		0.01	35.85	37.3		0.01	35.65	37.85	0.01	0.08	35.25	38.75
220		0.01	40.25	42.45		0.02	40.85	42.4	0.03	0.05	40.25	43.75
225		0.01	45.25	47.5		0.01	46.05	47.5		0.04	45.25	48.75
230		0.01	50.25	52.5		0.01	50.8	52.35	0.01	0.04	50.25	53.45
235		0.01	55.25	57.55		0.02	55.25	58.75		0.04	55.3	58.7
240		0.01	60.25	64		0.01	60.35	63.45		0.03	60.25	63.75

Figure9:Market prices of AAPL options



Exp Date	ImpFwd	80.0%	90.0%	95.0%	97.5%	100.0%	102.5%	105.0%	110.0%	120.0%
		142.752	160.596	169.518	173.979	178.44	182.901	187.362	196.284	214.128
8 Apr 2022	178.46	66.72	45.82	32.54	27.22	24.27	23.26	25.54	35.78	47.88
14 Apr 2022	178.49	53.71	36.81	28.37	25.01	22.61	21.55	22.29	28.47	39.96
22 Apr 2022	178.53	44.45	31.22	25.71	23.60	21.74	20.62	20.76	23.80	32.22
29 Apr 2022	178.56	43.90	33.89	29.73	27.98	26.48	25.36	24.83	25.96	32.14
6 May 2022	178.32	41.95	32.91	29.13	27.59	26.30	25.28	24.65	25.15	30.40
13 May 2022	178.35	40.97	32.49	29.12	27.67	26.31	25.06	24.16	24.21	28.83
20 May 2022	178.38	39.91	32.05	28.86	27.50	26.32	25.36	24.68	24.39	27.56
17 Jun 2022	178.52	37.20	31.29	28.87	27.81	26.87	26.04	25.37	24.51	25.26
15 Jul 2022	178.69	35.56	30.54	28.54	27.68	26.90	26.22	25.62	24.75	24.51
19 Aug 2022	178.92	35.57	31.39	29.68	28.91	28.21	27.56	26.98	26.06	25.29
16 Sep 2022	179.17	34.92	31.19	29.61	28.90	28.25	27.66	27.12	26.25	25.34
21 Oct 2022	179.55	34.22	30.90	29.51	28.88	28.31	27.78	27.30	26.50	25.57
18 Nov 2022	179.77	34.14	31.12	29.88	29.33	28.81	28.34	27.90	27.15	26.15
20 Jan 2023	180.57	33.37	30.77	29.69	29.20	28.74	28.32	27.94	27.27	26.32
17 Mar 2023	181.23	33.37	30.96	29.93	29.46	29.03	28.63	28.26	27.62	26.68
16 Jun 2023	182.54	32.48	30.68	29.96	29.64	29.34	29.07	28.82	28.37	27.68
15 Sep 2023	183.95	32.04	30.46	29.82	29.53	29.27	29.03	28.80	28.40	27.80
19 Jan 2024	185.94	32.02	30.70	30.13	29.86	29.61	29.38	29.15	28.75	28.07
21 Jun 2024	188.28	31.80	30.73	30.28	30.08	29.89	29.71	29.53	29.22	28.68
31 Dec 2024	191.08	31.78	30.88	30.50	30.32	30.15	29.99	29.83	29.54	29.04
31 Dec 2025	195.85	31.73	30.97	30.64	30.48	30.34	30.20	30.06	29.81	29.37
31 Dec 2026	200.17	31.54	30.94	30.67	30.55	30.43	30.31	30.20	29.99	29.61
31 Dec 2027	204.46	31.42	30.83	30.58	30.46	30.35	30.24	30.14	29.95	29.60
29 Dec 2028	208.83	31.44	30.88	30.64	30.53	30.42	30.31	30.21	30.03	29.70

Figure 10: Implied volatility of AAPL



Click a volatility point for details

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APPLE INC 175.06 USD Bloomberg Mid As of 05-Apr-2022 16:15

1) Vol Table | 2) 3D Surface | 3) Term | 4) Skew | 5) Dividends | 6) Prices

Moneyness | Listed | 16) Edit | Fwd | Strikes

Exp Date	ImpFwd	80.0%	90.0%	95.0%	97.5%	100.0%	102.5%	105.0%	110.0%	120.0%
		140.048	157.554	166.307	170.684	175.06	179.437	183.813	192.566	210.072
8 Apr 2022	175.07	70.43	48.28	35.26	30.67	27.15	26.29	29.26	39.67	51.51
14 Apr 2022	175.10	57.21	39.34	31.19	27.91	25.21	23.66	23.97	29.80	41.71
22 Apr 2022	175.14	45.50	33.44	28.16	25.83	23.78	22.39	22.12	24.77	32.26
29 Apr 2022	175.17	45.68	35.91	31.94	30.14	28.53	27.24	26.40	26.67	32.66
6 May 2022	174.94	44.07	35.08	31.28	29.61	28.16	26.99	26.19	26.07	30.83
13 May 2022	174.97	42.57	34.34	30.86	29.32	27.93	26.77	25.92	25.43	28.87
20 May 2022	175.00	41.53	33.77	30.54	29.13	27.89	26.85	26.06	25.36	27.74
17 Jun 2022	175.13	38.64	32.70	30.21	29.10	28.11	27.26	26.57	25.67	25.95
15 Jul 2022	175.30	36.98	31.89	29.80	28.90	28.09	27.39	26.79	25.90	25.54
19 Aug 2022	175.53	36.83	32.69	30.99	30.22	29.52	28.86	28.27	27.31	26.46
16 Sep 2022	175.78	35.99	32.25	30.73	30.05	29.42	28.84	28.30	27.38	26.35
21 Oct 2022	176.17	35.02	31.82	30.49	29.89	29.32	28.80	28.32	27.49	26.48
18 Nov 2022	176.39	35.06	32.07	30.83	30.27	29.75	29.27	28.82	28.04	27.02
20 Jan 2023	177.20	33.89	31.36	30.32	29.86	29.43	29.03	28.66	28.01	27.06
17 Mar 2023	177.85	33.69	31.40	30.47	30.05	29.66	29.30	28.97	28.39	27.52
16 Jun 2023	179.18	33.28	31.39	30.62	30.27	29.95	29.66	29.39	28.91	28.17
15 Sep 2023	180.64	32.82	31.27	30.63	30.33	30.06	29.80	29.55	29.11	28.38
19 Jan 2024	182.66	32.29	31.01	30.47	30.22	29.99	29.76	29.55	29.16	28.49
21 Jun 2024	185.10	31.93	31.01	30.60	30.41	30.22	30.04	29.86	29.53	28.91
31 Dec 2024	187.99	31.86	31.06	30.70	30.53	30.36	30.20	30.05	29.75	29.21
31 Dec 2025	193.04	31.84	31.10	30.78	30.63	30.49	30.35	30.22	29.98	29.55
31 Dec 2026	197.74	31.78	31.11	30.81	30.68	30.55	30.42	30.30	30.08	29.68
31 Dec 2027	202.36	31.77	31.18	30.92	30.79	30.67	30.56	30.45	30.24	29.86

97) Option Pricing (OVME) 98) Legend Zoom 100%

99) Quick Pricer

Strike	158.308	Call	Vol 32.40%	Price	22.556	Dividend yield	0.419%
Expiry	30-Jul-2022	Buy	Spot 175.060	Delta	74.32%	Impl forward	175.458

Suggested Functions HVG Track historical (realized) volatility

Figure 11: Implied volatilities of the American option prices on AAPL as of 05-Apr-2022 vs moneyness K/S_0. (Bloomberg)

Greeks Analysis

The Greek letters are different measures which quantify different aspects of the risk in an option position. The delta and gamma of an option represent the rate of change of the options price with respect to the underlying stock and the change in delta with respect to the underlying asset, respectively.

Figures 12, respectively, show the numerical results for the price of 3M American put and call options expiring 14-Apr-2022, along with the price, delta, and gamma (columns Price, Delta, and Gamma) found using the binomial tree model based on their respective strike prices.

putData

kStrikes	putPrice	putDelta	putGamma
162.5	0.1812065	-0.04145959	7.030658e-06
165.0	0.3392642	-0.07062732	1.586209e-01
167.5	0.6146617	-0.12891908	2.243196e-06
170.0	1.0286343	-0.17589540	7.355161e-05
172.5	1.6143766	-0.24209147	1.904306e-01
175.0	2.4399656	-0.37030451	6.236382e-05
177.5	3.5004661	-0.44785748	1.310261e-04
180.0	4.7997936	-0.52757796	5.043488e-04
182.5	6.3368545	-0.65727990	4.612555e-01

callData

kStrikes	callPrice	callDelta	callGamma
162.5	16.152378	0.9585423	7.105427e-13
165.0	13.810873	0.9295696	1.556107e-01
167.5	11.586640	0.8710883	7.105427e-13
170.0	9.500974	0.8241318	3.552714e-13
172.5	7.586979	0.7585921	1.808657e-01
175.0	5.912638	0.6297407	1.065814e-12
177.5	4.473199	0.5522373	0.000000e+00
180.0	3.272395	0.4726871	4.440892e-13
182.5	2.308817	0.3438597	4.814109e-01

Figure 12 : Prices and Greeks (Delta and Gamma) of the American put options with maturity 14-Apr-2022 vs strike K obtained using the binomial tree method with $n = 100$ time steps. The Gamma is very noisy.

Delta is computed using central finite differences with step $S_0 \to S_0 \pm 0.1$.

$$\Delta = \frac{P(S_0 + 0.1) + P(S_0 - 0.1)}{0.2}$$

Gamma is computed on the same grid of points $(S_0 - 0.1,\ S_0\ , S_0 + 0.1)$

$$\Gamma\ = \frac{P(S_0 + 0.1) - 2P(S_0) + P(S_0 - 0.1)}{0.1^2}$$

For this test we choose the American options with maturity 14-Apr-2022 ($T = \frac{7}{252}\ years$). The Delta and Gamma of these options are shown in the tables of Fig. 2 plots the Delta and Gamma for the put options.

The Greeks are quite noisy due to the discontinuity inherent in the binomial tree as the stock price crosses certain discrete values imposed by tree structure.

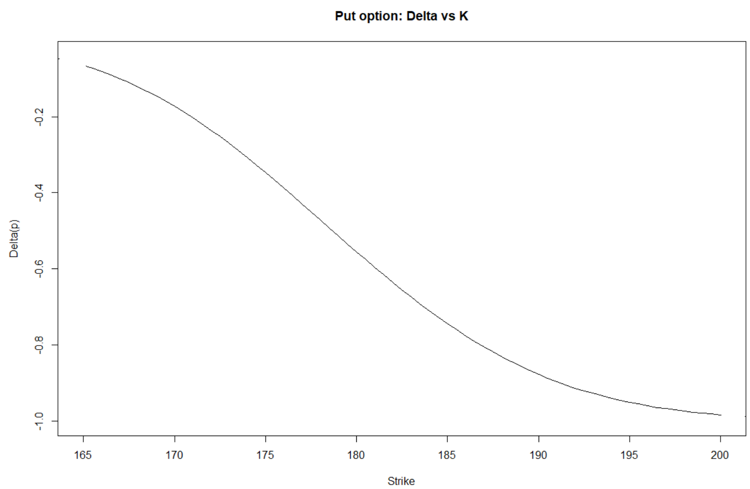

Figure 13 : Put Option Delta vs Strike (K)

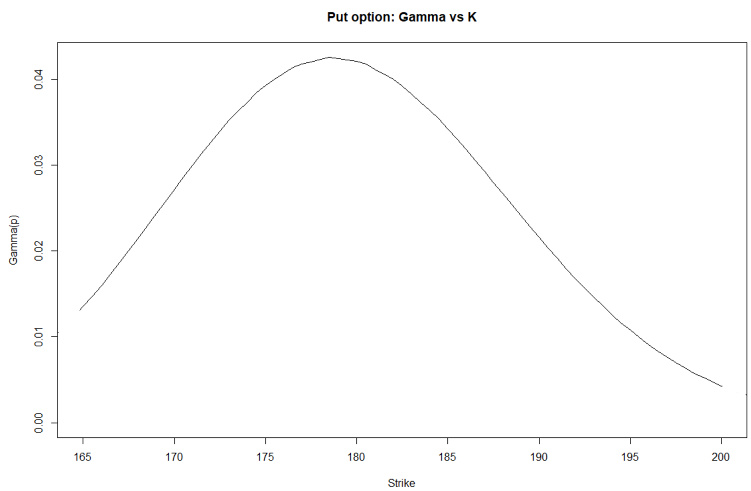

Figure 14 : Put Option Gamma vs Strike (K)

The greeks in American Options have a unique composition and pattern compared to those of European Options. This is because American Options can be exercised early (before expiration). As seen in Figures 13 and 14, the Greeks of American Options vs Strike price, delta levels out to -1 at the critical strike of 200, after which it will remain constant, and Gamma is discontinuous at a certain critical strike price K. The difference between gamma and delta is that gamma becomes discontinuous after the critical strike value, in this case K is about 200. Similar to the Put Option, a Call Option will have the delta level out, but at 1 instead of -1. And, just as in European Options the gamma will be the same for Calls and Puts.

5 Hedging exercise

The closing prices of AAPL for the first 9 trading days of 2022-04-04 are shown below. For the hedging exercise, consider the put option with strike = 177.5 for each of the last 4 days of its existence: 4-Apr to 8-Apr.

For each day compute the option price and its Delta using the binomial tree model. The results are shown in the table below.

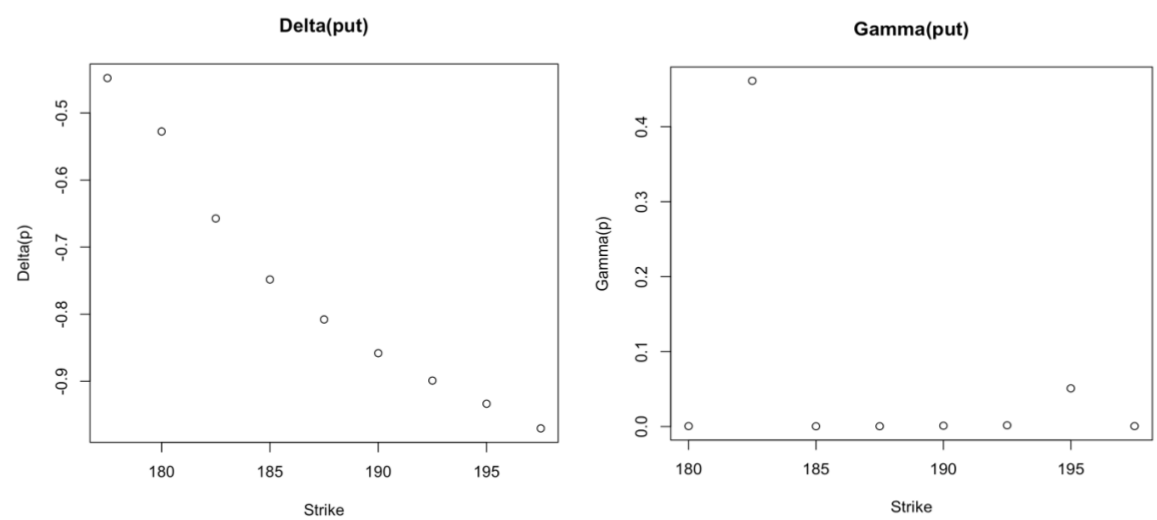

Figure15 : The Delta and Gamma of the American put options with maturity 14-Apr-2022 vs strike K.

```
> Cl(AAPL)
           AAPL.Close
2022-04-04     178.44
2022-04-05     175.06
2022-04-06     171.83
2022-04-07     172.14
2022-04-08     170.09
2022-04-11     165.75
2022-04-12     167.66
2022-04-13     170.40
2022-04-14     165.29
```

Figure16 : The closing prices of APPL.

The put option with strike K = 177.5, ends up on 8-Apr, when the stock price is 170.09. How can we hedge it?

Using the Delta values in the table, we construct a dynamically hedged portfolio: for one put option, we purchase Δ shares of stock. Denote the price of the option + stock hedged portfolio.

```
> putPrice
[1] 2.193213 3.682586 5.903238 5.424074 7.410000
> # 13.28089 17.82806 22.38536 39.58822 28.40000
> putDelta
[1] -0.4519412 -0.6850065 -0.9009339 -0.9548679
> # -0.4562717 -0.6145443 -0.7557467 -0.9893727
> dPrice # daily PnL of unhedged option
[1]  1.4893729  2.2206527 -0.4791649  1.9859264
> #  4.547176   4.557296  17.202862 -11.188219
> hedgedPrice # daily PnL of hedged option
[1] -0.038188163  0.008081577 -0.199875364  0.028447146
```

Figure17 : the value of the put price, put delta, daily PNL, hedge price.

The daily price change of the hedged portfolio is：

$$\Pi(t) - \Pi(t-1) = P(t) - P(t-1) + \Delta(t-1)(S(t) - S(t-1))$$

The P(t) - P(t-1) and $\pi(t) - \pi(t-1)$ columns in Table 3 show the daily price changes of the unhedged put option, and of the Delta hedged position of option plus stock. The hedge is adjusted daily according to the Delta of the put option on each day, computed using the binomial tree model. We see that the hedged position has much smaller daily price volatility than the “naked” put option.

Here is the hedging strategy:

Sell one share of put option and one share of stock in 4-Apr and buy one share of put option in 8-Apr-22. With the hedging stocks’ net income, you will finally get 5.487 dollars per share.

sell one put option	1.252
sell one stock	178.44

Day	Maturity	S_0	Put Price	Delta	P(t) - P(t-1)	$\Pi(t) - \Pi(t-1)$	share
4-Apr-22	4/252	178.44	2.193	-0.452	-	-	-0.452
5-Apr-22	3/252	175.06	3.683	-0.685	1.489	-0.038	-0.685
6-Apr-22	2/252	171.83	5.903	-0.901	2.221	0.008	-0.901
7-Apr-22	1/252	172.14	5.42	-0.955	-0.479	-0.2	-0.955
8-Apr-22	-	170.09	7.41	-1	1.986	0.028	-1

sell earn money	earn	stock total sell
80.65488	80.65488	0.452
119.9161	200.57098	1.137
154.81883	355.38981	2.038
164.3937	519.78351	2.993
170.09	689.87351	3.993

sell stock +	689.87351
buy stock -	679.16937
Net hedge	10.70414
Sell 1 put you get	2.193
Buy 1 put you cost	7.41
Final	5.48714

Table 1 Hedging strategy for the put option on AAPL with expiry 8-Apr-22 and strike K=177.5

The hedge is not perfect because of Theta (time change Greek) and Gamma contributions, and because of time discretization errors (Delta hedging is perfect only when performed continuously in time).

6 Other topics

If there is extra time we could explore further along different directions.

1. Improve the tree pricing using the European option as control variate

2. Use Bloomberg and R language to automatically price American options, and design arbitrage strategies， even multiple options trading strategies. We could calculate the Sharpe ratio and maximum drawdown.

3. Employ Least Square Monte Carlo Simulation Method to compute the American options.

4. Consider transact action cost and Corporate Risk.

5. Try to consider the extra dividend

R code for data analysis

```
source("AmericanOptionPricer.R")
library(quantmod)
# get historical stock prices, and compute the historical volatility
getSymbols("AAPL", src="yahoo", from="2022-04-04", to="2022-04-14")
# visualise the data downloaded
Cl(AAPL)
names(AAPL)
head(Cl(AAPL))
tail(Cl(AAPL))   # S0
#print the closing prices
plot(Cl(AAPL))
# estimate the historical volatility
prices <- Cl(AAPL)
head(prices)
tail(prices)
ndays <- length(prices)
#compute daily log-returns
logret <- periodReturn(prices, period="daily", type="log")

plot(logret, main="AAPL daily log returns")
# check the normality of the log-returns
qqnorm(logret, main="Q-Q plot of the log-returns")
# compute annualized volatility
yearvol = sqrt(252.0)*sd(logret)  # stdev(log-ret) STDEV
print(yearvol)   #sigma = 15.75% +- 1.39% (3m)
histvol <- yearvol
histvolerr <- yearvol/sqrt(2*ndays)
# Risk-free interest rate ^IRX = 13-week T-bill rate
getSymbols("^IRX", src="yahoo", from="2022-01-05", to="2022-04-05")
names(IRX)
rfrate <- na.omit(Cl(IRX))
nrf <- length(rfrate)
plot(rfrate, main="13-week T-bill rate")
tail(rfrate)
rf <- 0.538/100
rf
#########################################################
### Price the AMZN options
#########################################################
S0 <- 178.44
```

```
Texp <- 9/252
rf <- 0.538/100
histvol <- 0.295
kStrikes <- numeric()
callPrice <- numeric()
putPrice <- numeric()
callDelta <- numeric()
callGamma <- numeric()
putDelta <- numeric()
putGamma <- numeric()
for (i in 1:9) {
  k <- 177.5+ 2.5*(i-1)
  S0up <- 178.44+ 0.1
  S0down <-178.44 - 0.1
#  copt <- binomial_option("call", histvol, 4/252, rf, k, S0, 40, TRUE)
  copt <- binomial_option("call", histvol, Texp, rf, k, S0, 100, TRUE)
  coptup <- binomial_option("call", histvol, Texp, rf, k, S0up, 100, TRUE)
  coptdown <- binomial_option("call", histvol, Texp, rf, k, S0down, 100, TRUE)
  popt <- binomial_option("put", histvol, Texp, rf, k, S0, 100, TRUE)
  poptup <- binomial_option("put", histvol, Texp, rf, k, S0up, 100, TRUE)
  poptdown <- binomial_option("put", histvol, Texp, rf, k, S0down, 100, TRUE)
  kStrikes <- c(kStrikes,k)
  callPrice <- c(callPrice,copt$price)
  callDelta <- c(callDelta,(coptup$price - coptdown$price)/0.2)
  callGamma <- c(callGamma,(coptup$price + coptdown$price - 2*copt$price)/0.01)
  putPrice <- c(putPrice,popt$price)
  putDelta <- c(putDelta,(poptup$price - poptdown$price)/0.2)
  putGamma <- c(putGamma,(poptup$price + poptdown$price - 2*popt$price)/0.01)
}
callData <- data.frame(kStrikes,callPrice, callDelta,callGamma)
putData <- data.frame(kStrikes,putPrice, putDelta,putGamma)
putData
callData
# plot the call option price, delta and gamma vs K
plot(kStrikes, callPrice,
    xlab="Strike", ylab="Price(c)", type='p', main="Price(c)")
plot(kStrikes, callDelta,xlab="Strike", ylab="Delta(c)", type='p',
    main="Delta(c)")
plot(kStrikes[2:9], callGamma[2:9],xlab="Strike", ylab="Gamma(c)", type='p',
    main="Gamma(c)")
```

```
# plot the put option price, delta and gamma vs K
plot(kStrikes, putPrice,
    xlab="Strike", ylab="Price(p)", type='p')
plot(kStrikes, putDelta,xlab="Strike", ylab="Delta(p)", type='p',
    main="Delta(put)")
plot(kStrikes[2:9], putGamma[2:9],xlab="Strike", ylab="Gamma(p)", type='p',
    main="Gamma(put)")
# convergence study
nSteps <- numeric()
putPriceVsN <- numeric()
callPriceVsN <- numeric()
for (i in 1:200) {
  n <- i + 10
  coptn <- binomial_option("call", histvol, 9/252, rf, 177.5, S0, n, TRUE)
  nSteps <- c(nSteps,n)
  callPriceVsN <- c(callPriceVsN,coptn$price)
}
callBStest <- BSoptionprice("call",S0,162.5,9/252,histvol,rf)
plot(nSteps,callPriceVsN,xlim=c(0,200), #ylim=c(2.2, 2.3),
    xlab="No. Steps", ylab="Price(p)", type='l', main="Call option price vs n")
abline(h=15.14, col="red")
nSteps <- numeric()
putPriceVsN <- numeric()
for (i in 1:200) {
  n <- i + 10
  poptn <- binomial_option("put", histvol, 9/252, rf, 177.5, S0, n, TRUE)
  nSteps <- c(nSteps,n)
  putPriceVsN <- c(putPriceVsN,poptn$price)
}
putPriceVsN[100]
plot(nSteps,putPriceVsN,xlim=c(0,200), #ylim=c(2.2, 2.3),
    xlab="No. Steps", ylab="Price(p)", type='l', main="Put option price vs n")
abline(h=putPriceVsN[200], col="red")
############################################
# hedging study
############################################
#American put K=1780.0, 4 days to expiry
getSymbols("AAPL", src="yahoo", from="2022-04-04", to="2022-04-16")
# visualise the data downloaded
Cl(AAPL)
```

```
tail(Cl(AAPL))   # S0
putPrice <- numeric()
putDelta <- numeric()
dPrice <- numeric()
hedgedPrice <- numeric()
#4-4 start
k <- 177.5
#day 0: 2-Dec-2019
S0 <- 178.44
S0up <- S0+ 0.1
Texp <- 4/252

popt <- binomial_option("put", histvol, Texp, rf, k, S0, 100, TRUE)
poptup <- binomial_option("put", histvol, Texp, rf, k, S0up, 100, TRUE)

price0 <- popt$price
delta0 <- (poptup$price - popt$price)/0.1
putPrice <- c(putPrice,price0)
putDelta <- c(putDelta,delta0)

#day 1: 3-Dec-2019
S1 <- 175.06
S1up <- S1 + 0.1
Texp <- 3/252

popt <- binomial_option("put", histvol, Texp, rf, k, S1, 100, TRUE)
poptup <- binomial_option("put", histvol, Texp, rf, k, S1up, 100, TRUE)

price1 <- popt$price
delta1 <- (poptup$price - popt$price)/0.1
putPrice <- c(putPrice,price1)
putDelta <- c(putDelta,delta1)

dp1 <- price1 - price0
hedgeddp1 <- dp1 - delta0*(S1 - S0)
dPrice <- c(dPrice,dp1)
hedgedPrice <- c(hedgedPrice,hedgeddp1)

#day 2: 4-Dec-2019
S2 <-  171.83
```

```
S2up <- S2 + 0.1
Texp <- 2/252

popt <- binomial_option("put", histvol, Texp, rf, k, S2, 100, TRUE)
poptup <- binomial_option("put", histvol, Texp, rf, k, S2up, 100, TRUE)

price2 <- popt$price
delta2 <- (poptup$price - popt$price)/0.1
putPrice <- c(putPrice,price2)
putDelta <- c(putDelta,delta2)

dp2 <- price2 - price1
hedgeddp2 <- dp2 - delta1*(S2 - S1)
dPrice <- c(dPrice,dp2)
hedgedPrice <- c(hedgedPrice,hedgeddp2)

#day 3: 5-Dec-2019
S3 <- 172.14
S3up <- S3 + 0.1
Texp <- 1/252

popt <- binomial_option("put", histvol, Texp, rf, k, S3, 100, TRUE)
poptup <- binomial_option("put", histvol, Texp, rf, k, S3up, 100, TRUE)

price3 <- popt$price
delta3 <- (poptup$price - popt$price)/0.1
putPrice <- c(putPrice,price3)
putDelta <- c(putDelta,delta3)

dp3 <- price3 - price2
hedgeddp3 <- dp3 - delta2*(S3 - S2)
dPrice <- c(dPrice,dp3)
hedgedPrice <- c(hedgedPrice,hedgeddp3)
#day 4: 6-Dec-2019
S4 <-  170.09
price4 <- max(k - S4,0)
putPrice <- c(putPrice,price4)
dp4 <- price4 - price3
hedgeddp4 <- dp4 - delta3*(S4 - S3)
dPrice <- c(dPrice,dp4)
```

```
hedgedPrice <- c(hedgedPrice,hedgeddp4)
putPrice
# 13.28089 17.82806 22.38536 39.58822 28.40000
putDelta
# -0.4562717 -0.6145443 -0.7557467 -0.9893727
dPrice # daily PnL of unhedged option
#  4.547176   4.557296  17.202862 -11.188219
hedgedPrice # daily PnL of hedged option
# -0.7638266 -1.1395306  1.9292216 -0.1863942
#####################################################
```

References

[1] J. Hull, Options, Futures and Other Derivatives, Pearson 10th Edition