FE 621: HW3

Due date: March 24th at 11:59 pm

Problem 1

Use Monte Carlo simulation to construct an estimator and a 95% confidence interval for the constant π.

- Clearly explain your simulation procedure and how you construct the confidence interval.
- Plot the evolution of the estimator and the confidence interval as the sample size n grows (use n ranging from 10 to $10,000$ in increments of 10).
- Report the values of the estimator and the margin of error (half-width of the confidence interval) for $n = 10,000$.

Hint: π is the area of the unit circle and it follows that

$$\frac{\pi}{4} = \mathbb{P}(U_1^2 + U_2^2 \leq 1) = \mathbb{E}[\mathbf{1}\{U_1^2 + U_2^2 \leq 1\}],$$

where U_1, U_2 are independent $U[-1, 1]$ random variables.

Problem 2 (Delta hedging)

In this problem we use simulation to estimate the hedging error resulting from discrete portfolio rebalancing. We sell a 3-month European call option and hedge our position by holding "delta" shares of the underlying stock. Assume that we can borrow and deposit money at a constant risk-free interest rate.

- At initiation of the contract, we receive the option premium from the client and buy delta shares of the stock. We may need to borrow money to set up the hedging portfolio.
- At each time step, the stock price has evolved from the previous step and the hedge must be adjusted. Depending on how the delta has changed, we need either to buy or sell shares. We also pay or earn interest on any money borrowed or deposited over the previous period.
- At maturity of the contract, we close our positions. This means selling our shares of the stock, closing our cash account, and paying $(S_T - K)^+$ to the client. What is left after that is our profit or loss (PnL).

Assume the underlying asset to follow a geometric Brownian motion with the parameters specified below. Black-Scholes theory says that if we sell the option at the Black-Scholes price and continuously hedge using the Black-Scholes delta, then the hedging error (PnL) goes to zero as the rebalancing frequency increases. However, continuous trading is impossible in practice, and any discrete hedging strategy results in a nonzero hedging error.

$$\begin{array}{rr} \text{Initial price: } S_0 = 50, & \text{Rate of return: } \mu = 10\% \\ \text{Volatility: } \sigma = 30\%, & \text{Interest rate: } r = 5\% \\ \text{Strike: } K = 50, & \text{Expiration: } T = 0.25 \end{array}$$

(a) Simulate $n = 10,000$ paths of the stock price and create a histogram of the discounted hedging error as a fraction of the price of the option for both weekly ($m = 13$) and daily ($m = 63$) rebalancing. Also report the means and the standard deviations. Comment on your findings; are they in line with theory?

Note: For each simulated path, there is one hedging error (PnL). Each histograms will thus based on n *values* $e^{-rT}HE(k)/C$, $k = 1, \ldots, n$ *where* $HE(k)$ *is the hedging error and* C *is the price of the option.*

(b) Consider values of μ ranging from r to 1 and plot the mean and standard deviation of the hedging error as a function of μ. Use $n = 10,000$ and consider weekly ($m = 13$) and daily ($m = 63$) rebalancing. How does the value of μ impact the hedging errors? Comment on your findings; are they in line with theory?

(c) Let Δt denote the rebalancing interval. According to theory, the hedging error goes to zero as $\Delta t \to 0$. What do your simulation results indicate about the order of this convergence? Does the hedging error appear to be of order $(\Delta t)^\alpha$ for some $\alpha > 0$, and if so, what α?

Hint: Let $m = 13$, $m = 63$, $m = 252$, $m = 1008$ *and create a log-log plot of the hedging error as a function of* $\Delta t = T/m$. *Then note that*

$$error \sim (\Delta t)^\alpha \quad \Longrightarrow \quad \log(error) \sim \alpha \log(\Delta t),$$

so the order of convergence can be estimated by the slope of the log-log curve. For the hedging error you can either use the standard deviation or the root-mean-squared-error $RMSE = \sqrt{Bias^2 + Var}$.

Problem 3 (Trading volatility)

In Problem 1 we assumed future realized volatility σ_R to be equal to the implied volatility σ (i.e., the volatility that the option is sold and traded at). In this problem we examine what happens if that is not the case.

Specifically, assume that the option is traded at implied volatility σ, but the stock evolves according to a geometric Brownian motion with volatility σ_R. How should you then hedge the option position in Problem 1? Should you compute delta using the market implied volatility σ or the actual volatility σ_R?

(a) Assume $\sigma = 0.3$ and $\sigma_R = 0.4$. Repeat Problem 1(a) using (i) hedging with implied volatility and (ii) hedging with realized volatility. What does the hedging error look like? What is the impact of higher hedging frequency?

(b) Assume $\sigma = 0.3$ and $\sigma_R = 0.2$ and repeat part (a). How do the results change?

(c) You are a trader and have a reason to believe that future realized volatility will be lower than the implied volatility indicated by the option price. Based on your results in parts (a) and (b), what is a viable trading strategy to benefit from this situation?

What if you believe that future realized volatility will be higher than the implied volatility of the option?

Food for thought: In Problems 1-2 we assumed the Black-Scholes dynamic and computed delta using the Black-Scholes formula. In practice, market makers and financial institutions generally use the Black-Scholes implied volatility to hedge option positions. This is despite the fact that actual markets are not in accordance with the Black-Scholes model, i.e., they do not follow a geometric Brownian motion. In other words, risky option positions are hedged using the "wrong" delta. One reason for this practice is that market dynamics and future volatility are unknown, so coming up with a hedging strategy that consistently works better than the Black-Scholes delta is difficult.

How do you think that using the wrong delta impacts the PnL of hedged option positions in reality? To examine this you can download historical option data and examine the PnL resulting from selling an option and then hedging the market risk using the Black-Scholes delta.

Problem 4 (Effect of transaction costs)

In this problem we consider how the hedging strategy is impacted by transaction costs. The setup is the same as in Problem 1 except that we must also pay a fee per transaction that is a fraction k of the transaction dollar amount.[1] For simplicity, this only applies at rehedging times, i.e., it does not apply at initiation or at expiry.

(a) Repeat Problem 1(a) with transaction cost $k = 2\%$. How do the results change?

(b) Does higher rebalancing frequency help? What are the pros and cons of more frequent hedging?

(c) Given your simulation results, would the presence of transaction costs impact the price at which you were willing to sell the option to a client?

[1] For example, if the stock price is 50 and we need to adjust delta by buying or selling 0.1 shares of the stock, then the transaction fee is $k \times (0.1 \times 50)$.