Measuring Systemic Risk : Copula CoVaR

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Although copula modeling has been applied in a growing number of financial applications, high-dimensional copula modeling is still in its early stages. Vine copula modeling not only has the advantage of extending to higher dimensions easily, but also provides a more flexible measure to capture an asymmetric dependence among assets. CoVaR, the Value-at-Risk of institutions conditional on other institutions being in distress, is introduced by Adrian and Brunnermeier (2011). $\Delta CoVaR$ *is the risk contribution that the institution adds to the entire system. Combined with the modified CoVaR methodology and estimation of the dependence structures through vine copula modeling, we empirically investigate the evolution of dependence structure and systemic risk in 10 S&P 500 sector indices in the U.S. stock market by estimating daily Copula* $\Delta CoVaR$ *and Copula* $\Delta CoES$ *from January 1, 1995 to July 31, 2013. Our model (Copula CoVaR) reveals a real-time and efficient tool that can be used to analyze systemic risk. Furthermore, this approach could offer a systemic risk index for those countries which do not have an instrument like VIX, and can be tailored to any underlying sector, country or financial market easily.*

1. Introduction

Over the past few decades, the global financial system has become more complicated due to increased financial system complexity. The challenge still lies in providing practical methods for measuring systemic risk and supervising the financial system stability. In this paper, the emphasis is placed on empirical systemic risk measurement that can provide the macroprudential regulator with reasons for better decision-making, and can improve the decision for the regulator on the appropriate tools to deal with the level of systemic risk

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as measured.

Kaufman and Scott (2003) defined systemic risk as "*the risk or probability of breakdowns (losses) in an entire system as opposed to breakdowns in individual parts or components and is evidenced by comovements (correlation) among most or all the parts* ". The collapse of an entire system results from the contagious effect of the shock of a single entity. Although there is no consensus on how to define systemic risk, we quote the definition of systemic risk from the Report to G20 Finance Ministers and Governors (2009) agreed upon among the International Monetary Fund (IMF), Bank for International Settlements (BIS) and Financial Stability Board (FSB) that is "*(i) caused by an impairment of all or parts of the financial system and (ii) has the potential to have serious negative consequences for the real economy*". Furthermore, "*G-20 members consider an institution, market or instrument as systemic if its failure or malfunction causes widespread distress, either as a direct impact or as a trigger for broader contagion."* A common factor so far in the various definitions of systemic risk is that a trigger event, such as an economic shock or institutional failure, causes a chain of bad economic consequences, referred to as "domino effect". Understanding the interdependencies of the components that constitute the financial system is critical for modeling systemic risk. Since companies today are able to obtain their financing through the capital markets without going through banks or other financial intermediaries, estimating systemic risk becomes much more complicated and difficult than in the past. Therefore, we should consider systemic risk resulting not only from the financial sector but also from other sectors because of disintermediation (see Schwarcz (2008)).

The recent financial crisis of 2007-2008 was a showcase of large risk spillovers from one bank to another heightening systemic risk, which was reflected in all markets simultaneously due to systemic information efficiency and financial tight coupling. Khashanah and Miao (2011) empirically investigated the structural evolution of the financial system from the U.S. system to the global system. They described a simplified U.S. financial system composed of the SPX, three-month T-bill, USDX, VIX, and gold and oil, and the study showed quantitatively that the U.S. financial system, in the sense of a minimum spanning tree distance metric, contracts during an economic recession and expands during economic expansions. This paper provides the important conclusion on how financial structural factors transform from the expansion phase to the contraction phase. It provides an insight into market behavior that is not easily observed from the correlation matrix. The analysis also lets us obtain consistencies as well as evolutions in the patterns of market interactions over time.

Sklar (1959) introduced copula, but copula modeling was developed in the financial field between 1987 and 1991 (see Mackenzie and Spears (2012)). Fermanian and Scaillet (2004) observed that copula modeling is becoming more and more popular in academics and the industry because it is well known that the dependence structures of the returns of financial assets are non-Gaussian and exhibit strong nonlinearities. Joe (1997) and Nelsen (1999) provided comprehensive studies of copulas. Li (2000) was the first to apply copulas to a credit risk analysis, and Rodriguez (2007) applied copulas to financial contagions with a switch-parameter copula model. In that work, the author showed evidence of changing dependence over time in global financial markets. Cherubini *et al* (2004) presented descriptions and applications of copulas in mathematical finance and risk management. Patton (2009) introduced the symmetrized Joe-Clayton (SJC) copula and parameterized the upper and lower tail dependence coefficients and defined a "conditional copula" as a multivariate distribution of variables that are distributed as a conditional uniform distribution.

As for financial rare events applications, a flexible modeling of tail probabilities is essential. To that end, Gaussian copula has had their share of criticism pertaining principally to their asymptotic independent tails (see MacKenzie and Spears (2012)). Notwithstanding the criticism directed at Gaussian copula when widely applied to model default correlations in credit risk analysis (see Li (2000)), non-Gaussian copulas improve those shortcomings (see Cherubini *et al* (2004)). The reason is that using non-Gaussian copulas provides a more flexible measure for combining marginal distributions into multivariate distributions and can show the dependence structure of two or more distributions observed in the tail dependence.

Even though copula modeling has been used in research for different purposes in the fields of finance and economics (see Cherubini *et al* (2004)), the challenge remains to extend copula-based models to high dimensions. Archimedean copulas are not well-designed for high-dimensional applications since they have only one or two parameters to capture the dependence structure. When it is necessary to have copula models with flexible asymmetry and tail dependence, then vine copulas modeling may be the best choice (see Joe *et al* (2010)). Meanwhile, Kurowicha and Joe (2011) show that regular vine (R-vine) copulas provide a better model to capture high-dimensional asymmetric characteristics in financial data. Formally, Joe (1996) was the first to give a probabilistic construction of multivariate distribution functions based on pair-copula construction (PCC), while Aas *et al* (see Kurowicha and Joe (2011)) were the first to recognize that the pair-copula construction (PCC) principal can be used with arbitrary pair-copulas, the graphical structure of R-vines, via the maximum likelihood estimation. To help in organizing them, Bedford and Cooke (2001) introduced graphical models denoting regular vines (R-vines) copulas. Kurowicha and Joe (2011) described the comprehensive features of R-vine Copulas. R-vines include two special-tree structures: line trees and star trees. They refer to line trees as D-vines copulas, and refer to star trees as C-vines copulas. A D-vines is an R-vines for which the first tree has nodes with a degree of two or less (line structure), and a C-vines is an R-vines which contains a node with maximum degree in each tree (star structure). Aas el al (2009) and Czado (2010) showed that C- and D-vines can be constructed by simple recursive conditioning, which is used in time series very often. Min and Czado (2010) extended their methodology to D-vines and specialized R-vines only using the bivariate student's t copula as pair-copula. Even though the student's t copula modeling can explain the behavior of fat tails in financial markets, it cannot describe the behavior of asymmetric tails such as a Clayton or Gumbel copula did. Czado *et al* (2012) showed that large simulations present good small-sample performances in the U.S. exchange rate market with the mixed C-vines selection procedure by using the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). However, the class of R-vines is much larger than the class of C-vines and D-vines and currently there are few applications of R-vines. One of the reasons is that there are an enormous number of possible R-vine tree sequences. Dissmann *et al* (2013) developed an automated algorithm of jointly searching for an appropriate R-vines tree structures, the pair-copula families and their parameters. For each tree selection, they used the maximum spanning tree algorithm, where edge weights are chosen appropriately to reflect large dependencies and pair-copulas are chosen independently via AIC. Afterwards, the corresponding pair-copula parameter estimation follows the same sequential estimation approach as suggested for C-vines and D-vines copulas. Moreover, they implemented a 16-dimensional R-vines copula modeling application with financial data to show that R-vines copula modeling provides a better fit than both C-vines and D-vines copula modeling. Accordingly, the selection of a sequential tree-by-tree chosen from the maximum spanning tree, appropriately reflects suitable dependencies. In a nutshell, this methodology allows us to extend the implementation to higher dimensions easily, which are especially needed for the risk assessment of larger financial portfolios. Kurowicha and Cooke (2006) and Joe *et al* (2010) discussed tail dependencies of vine distributions. Similarly, Kurowicha and Joe (2011) summarized recent developments and applications in vine copulas.

Bisias *et al* (2012) described an overview of systemic risk measures. Similarly, Benoit *et al* (2013) summarized a comprehensive comparison of the major systemic risk methodologies – marginal expected shortfall (MES) (see Acharya *et al* (2012)), systemic risk (SRISK) (see Brownlees and Engle (2010)), and $\Delta\mathrm{CoVaR}$ (see Adrain and Brunnermeier (2011)).

Brownless and Engle (2010) defined marginal expected shortfall (MES) as the expected loss of a bank's equity value if the entire market dropped sharply. They extended the concept of the Marginal Expected Shortfall (MES) by Acharya *et al*. (2012) to construct an indicator of systemic risk (SRISK) based on the degree of leverage and the equity loss in each institution. They concluded that the company with the highest SRISK is the company that contributes the most to the market undercapitalization during a financial distress and is the most systemically risky institution.

CoVaR is one of the earlier quantitative measures of systemic risk according to which it is posited that CoVaR is the value at risk (VaR) of the banking system conditional on an individual bank being distressed. Gauthier *et al* (2010) used it for Canadian institutions, whereas Adams *et al* (2012) studied risk spillovers among financial institutions, including hedge funds. White *et al* (2010) analyzed the influence of spillover effects between the VaR of a financial entity and the VaR of the market. In addition, Hautsch *et al* (2010) defined systemic risk beta as "the marginal effect of a bank's VaR on the VaR of the entire financial system", estimated by two-stage quantile regressions. Individual VaRs are calculated by bank characteristics of leverage, maturity mismatch and market capitalization and macroeconomic state variables of liquidity spreads, credit spreads, yield curve slope and VIX. Afterwards, the predicted VaRs are estimated by quantile regression of the system VaR on individual VaRs and macroeconomic state variables, and they empirically estimated systemic risk in the U.S. financial sector.

However, the three major methodologies (marginal expected shortfall (MES), indicator of systemic risk (SRISK), and $\Delta\mathrm{CoVaR}$) have their own drawbacks even though they are popular and dominant concepts. Benoit *et al* (2013) concluded that the systemic risk rankings of financial institutions based on their MES are the same as the rankings obtained by sorting firms on beta. Therefore, it would be curious to know whether MES and beta provide similar results. If MES and beta provide the same ordering of risk, we should be skeptical of possible confusion between systemic risk and systematic risk. Although the SRISK is constructed by MES, it is much less sensitive to beta. However, the SRISK tends to identify the same rankings as the leverage during normal periods and as the liabilities under distressed periods. Furthermore, Benoit *et al* (2013) show that the SRISK is much more highly correlated with the leverage and liability than with the beta of the company. Due to the definition of systemic risk given in the quote above, measuring systemic risk is to estimate the probability of failure of an institute that is the cause of distress for the financial system. We only consider the $\Delta CoVaR$ measure proposed by Adrian and Brunnermeier (2011), the difference between the VaR that the institution adds to the entire system conditional on the distress of a particular institution and the unconditional VaR of the financial system. As for $\Delta\mathrm{CoVaR}$, Benoit *et al* (2013) described that it is a function of the linear dependence between returns. In addition, $\Delta\mathrm{CoVaR}$ is proportional to an institution VaR, which renders the forecasting risk contribution of an institution to systemic risk as forecasting its risk in isolation. Most measures of systemic risk (except CoVaR) depend on the dependence structure between the marginal distributions of the firms. Because CoVaR does not take the dependence structure into account, Hakwa (2011) and Hakwa *et al* (2012) derived a modified CoVaR methodology based on bivariate copula modeling. We extend their methodology and apply vine copulas modeling provided by Schepsmeier (2012) into a high dimensional analyses in systemic risk. As we known, this paper is the first to empirically investigate systemic risk with the methodology based on CoVaR and high dimension vine copulas modeling in the U.S. market, and it provides the important conclusion that it is a real-time and efficient tool to analyze systemic risk. Furthermore, this approach could offer a systemic risk index for those countries which do not have an instrument like VIX, and can be tailored to any underlying sector, country or financial market easily.

Why do we use indices price instead of other financial instruments or financial accounting numbers? One of the main reasons is that an index price could reflect a timely financial environment in contrast to financial accounting numbers that are published quarterly. Furthermore, indices can easily be constructed and tell us which sector contributes more risk to the entire market. In addition, simply knowing a firm's CDS exposure is only of limited use if one does not have access to the firm's positions in the underlying bond. Therefore, we use ten S&P500 sector indices to analyze how much systemic risk a sector provides rather than only concentrating on a specific sector such as the financial sector. Furthermore, although the VIX index is usually referred to as the fear index, it only represents how big the panic exists in the entire financial market. Our indicator provides not only systemic risk from each sector, but a systemic risk index for those markets without a VIX index.

This paper has four sections. The first section briefly introduces existing research regarding to systemic risk. The second section describes the definition of the Copula $\Delta CoVaR$ and Copula $\Delta CoES$, and outlines the methodology of vine copulas modeling used to evaluate the dynamic dependence structure among each other. The third section describes the data and explains the empirical results of Copula $\Delta CoVaR/\Delta CoES$ and the systemic risk indicator. The fourth section concludes our findings.

2. Methodology

2.1 Risk Methodology

One of the major concerns in risk management is the idea of Value-at-Risk (VaR). The definition of Value-at-Risk is the biggest loss by which an asset or portfolio could lose at a given probability over a particular time period. People usually use a probability of 95, 99, or 99.9 percent for their situation. "*VaR answers the question: how much can I lose with x% probability over a pre-set horizon*" (see Riskmetrics: technical document (1996)). The general form, VaR can be derived from the probability distribution of an asset or portfolio value $f(x)$. At a given confidence level α, we try to find the worst possible realization x^* such that the probability of exceeding this value is α.

$$\alpha = \int_{x^*}^{\infty} f(x)dx$$

or such that the probability of a value lower than x^*, $\mathrm{P}(\mathrm{x} \le x^*)$, is $1-\alpha$.

$$1-\alpha = \int_{-\infty}^{x^*} f(x)dx = P(x \le x^*)$$

Where the number x^* is called the quantile of the distribution, which is the cutoff value with a fixed probability of being exceeded.

Conditional Value-at-Risk (CVaR) is also called expected shortfall (ES), which is more sensitive to the tail of the loss distribution because CVaR is the expected value of the $(1-\alpha)\ \mathrm{tail}$ distribution.

$$\mathrm{CVaR}_{1-\alpha} = \frac{1}{1-\alpha}\int_0^{1-\alpha} VaR_s(x)ds$$

where $\mathrm{VaR}_{1-\alpha}$ is the Value-at-Risk at a given confidence level α. Both VaR and ES have been widely used to risk assess in finance. Even though VaR is widely used in applications, Artzner *et al* (1999) advocated ES as preferred in practice due to its better properties. ES is a coherent risk measure satisfying homogeneity, monotonicity, translation invariance, and subadditivity, while VaR is not coherent because it does not satisfy subadditivity.

2.2 Definition of $\Delta \boldsymbol{CoVa}\mathbf{R}$ and $\Delta \boldsymbol{CoES}$

Adrian and Brunnermeier (2011) defined that $CoVaR_{1-\alpha}^{j|i}$ denotes the VaR of institution j (or the system) conditional on $X^i = VaR_{1-\alpha}^i(X^*)$ of institution i. Mainik and Schaanning (2012) found that conditioning on $\mathrm{X} \le \mathrm{Va}R_{1-\alpha}(X^*)$ gives a much better response to dependence between X and Y than conditioning on $\mathrm{X} = \mathrm{Va}R_{1-\alpha}(X^*)$. They provided counter-examples showing that CoVaR based on the stress event $\mathrm{X} = \mathrm{Va}R_{1-\alpha}(X^*)$ is not dependence consistent. Meanwhile, the article concluded that the alternative definition of CoES gives a much more consistent response to dependence than the original definition (see Adrian and Brunnermeier (2011)).

Adrian and Brunnermeier (2011) defined prefix “Co” as conditional, tail correlation, contagion, or co-movement, which concentrates on the nature of systemic risk. Meanwhile, they defined $\Delta CoVaR$ as the difference between the VaR if the financial company adds to the entire system j conditional on the distress of a particular institution i, ${CoVaR_{1-\alpha}}^{j|i}$, and the unconditional VaR of the financial system j, $VaR_{1-\alpha}^j$. They called this $\Delta CoVa\mathrm{R}$ the “exposure $CoVa\mathrm{R}$” since it measures how much systemic risk an institution provides. They denote institution $i's$ risk contribution to j by:

$$\Delta CoVaR_{1-\alpha}^{j|i} = CoVaR_{1-\alpha}^{j|i} - VaR_{1-\alpha}^j$$

Expected shortfall has a number of advantages relative to VaR and can be calculated as the expected value of the $(1-\alpha)\ \mathrm{tail}$ distribution. Adrian and Brunnermeier (2011) described that the methodology can be extended from $CoVaR_{1-\alpha}^{j|i}$ to $\mathrm{CoES}_{1-\alpha}^{j|i}$. They denote the $\mathrm{CoES}_{1-\alpha}^{j|i}$ is the Expected Shortfall of the financial system j conditional on the expected value of $X^i \leq VaR_{1-\alpha}^i$ of institution i. Similarly, they also denote institution $i's$ risk contribution to j by:

$$\Delta\mathrm{CoES}_{1-\alpha}^{j|i} = CoES_{1-\alpha}^{j|i} - ES_{1-\alpha}^{j}$$

2.3 Definition of Copula $\Delta CoVa\mathbf{R}$ and Copula $\Delta CoES$

In our methodology, we extend the concept of the modified CoVaR (see Hakwa (2011) and Hakwa *et al* (2012)), replacing the panel regression methodology in the original paper (see Adrian and Brunnermeier (2011)) with vine copulas methodology to obtain the $\mathrm{Co}VaR_{t,1-\alpha}^{j|i}$ and $\mathrm{CoES}_{t,1-\alpha}^{j|i}$, named Copula $CoVaR_{t,1-\alpha}^{j|i}$ and Copula $\mathrm{CoES}_{t,1-\alpha}^{j|i}$, and then obtain the following Copula $\Delta CoVaR_{t,1-\alpha}^{j|i}$ and Copula $\Delta\mathrm{CoES}_{t,1-\alpha}^{j|i}$. Copula $CoVaR_{t,1-\alpha}^{j|i}$ and Copula $\mathrm{CoES}_{t,1-\alpha}^{j|i}$ at time t are dynamically calculated by previous rolling dataset. We follow the same notations from Adrian and Brunnermeier (2011). We denote the sector $i's$ risk contribution to the market j by:

$$\text{Copula } \Delta CoVaR_{t,1-\alpha}^{j|i} = \text{Copula } CoVaR_{t,1-\alpha}^{j|i} - \text{Copula } VaR_{t,1-\alpha}^{j}$$

Similarly, Copula $\mathrm{CoES}_{t,1-\alpha}^{j|i}$ is defined as the expectation value over the $(1-\alpha)$ tail of the conditional probability distribution. Therefore, the Copula $\Delta\mathrm{CoES}_{t,1-\alpha}^{j|i}$ is given by

$$\text{Copula } \Delta\mathrm{CoES}_{t,1-\alpha}^{j|i} = \text{Copula } CoES_{t,1-\alpha}^{j|i} - \text{Copula } ES_{t,1-\alpha}^{j}$$

2.4 Copula theory

A wide-spread consensus in the empirical literature is that the dependence structure between the returns of financial assets is usually asymmetric, nonlinear, and time-varying. One of the advantages to using copula modeling is that copulas provide a more flexible measure for combining marginal distributions into multivariate distributions and show the dependence structure of two or mode distributions, which is observed in the tail dependence. In addition, copula modeling is a suitable method for an application of VaR or CVaR. The copula approach permits us to avoid calculating VaR or CVaR under the usual assumption of marginal and joint normality. For multivariate distributions, the univariate margins and the dependence structure can be separated in order to obtain suitable marginals and copulas. Formally Sklar's Theorem (see Sklar (1959)) states that given random variables $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}$ with continuous distribution functions $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_\mathrm{n}$ and joint distribution function H, there exists a unique copula C such that for all $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}) \in R^n$

$$\mathrm{H}(\mathrm{x}) = \mathrm{C}\big(\mathrm{F}_1(\mathrm{x}_1), \mathrm{F}_2(\mathrm{x}_2), \ldots, \mathrm{F}_\mathrm{n}(\mathrm{x}_\mathrm{n})\big)$$

where cumulative joint distribution function H evaluated at point $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n})$. Conversely, given any distribution functions $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_\mathrm{n}$ and cpula C, then joint distribution function H is an n-variate distribution function with marginals $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_\mathrm{n}$.

$$\mathrm{C}(u_1, u_2, \ldots, u_n) = \mathrm{H}\big(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n)\big) \text{ with } u_i \in [0,1]^n$$

where $F_i^{-1}(u_i)$ is the inverse of the cumulative distribution function. If the joint distribution function is n-times differentiable, then taking the nth cross-partial derivative of the equation:

$$f(x) \equiv \frac{\partial^n}{\partial x_1 \dots \partial x_n} H(x)$$
$$= \prod_{i=1}^{n} f_i(x_i) \frac{\partial^n}{\partial u_1 \dots \partial u_n} C\big(F_1(x_1), \dots, F_n(x_n)\big)$$
$$\equiv \prod_{i=1}^{n} f_i(x_i) \times c(F_1(x_1), \dots, F_n(x_n))$$

where u_i is the probability integral transform of x_i. It implies that the joint log-likelihood is simply the sum of univariate log-likelihoods and the copula log-likelihood. Generally, the two-step separation procedure is called the inference functions for margin method (IFM) (see Joe (1997)).

$$\log f(x) = \sum_{i=1}^{n} log f_i(x_i) + log c(F_1(x_1), \dots, F_n(x_n))$$

In our study, we use this two-step procedure to do copulas modeling, where marginal distributions and copulas are estimated separately.

2.4.1 Parametric bivariate Copulas

Joe (1997) and Nelson (1999) gave comprehensive copula definitions for each family and wrote down its corresponding density. Here are the copula families we used in pair-copula construction. The Gaussian copula is defined as

$$C^{Gaussian}(u_1, u_2) = \Phi_\rho\big(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\big)$$

where Φ_ρ is the bivariate joint normal distribution with linear correlation coefficient ρ and Φ is the standard normal marginal distribution. Therefore, the Gaussian copula is

$$C^{Gaussian}(u_1, u_2) = \Phi_\rho\big(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\big) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{(\frac{2\rho st - s^2 - t^2}{2(1-\rho^2)})} dsdt$$

The bivariate student's t copula is

$$C^{student's\,t}(u_1, u_2) = \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{s^2 + t^2 - 2\rho st}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} dsdt$$

where ρ is the linear correlation coefficient and ν is the degree of freedom.

The Clayton generator is given by $\varphi(u) = u^{-\theta} - 1$ and is also known as Cook and Johnson's (1981) family. Its copula is given by

$$C(u_1, u_2) = \max([u_1^{-\theta} + u_2^{-\theta} - 1]^{-\frac{1}{\theta}}, 0) \text{ with } \theta > 0$$

The Gumbel generator is given by $\varphi(u) = (-\ln u)^{\theta}$, and the bivariate Gumbel copula is given by

$$C(u_1,u_2)=\exp(-[(-\ln u_1)^{\theta}+(-\ln u_2)^{\theta}]^{-\frac{1}{\theta}}) \text{ with } \theta>1$$

The Frank generator is given by $\varphi(u)=\ln(\frac{e^{-\theta u}-1}{e^{-\theta}-1})$, and the corresponding bivariate Frank is given by

$$C(u_1,u_2)=-\frac{1}{\theta}\ln(1+\frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}) \text{ with } \theta>0$$

The Joe generator is $\varphi(u)=u^{-\theta}-1$, and the Joe copula is given by

$$C(u_1,u_2)=1-(\overline{u_1}^{\theta}+\overline{u_2}^{\theta}-\overline{u_1}^{\theta}\overline{u_2}^{\theta})^{\frac{1}{\theta}} \text{ with } 1\le\theta<\infty$$

Two-parameter families might be used to capture more than one type of dependence such as one parameter for upper tail dependence and one for concordance, or one parameter for upper tail dependence and one for lower tail dependence. The BB1 copula is given by

$$C(u_1,u_2)=(1+[(u_1{}^{-\theta}-1)^{\delta}+(u_2{}^{-\theta}-1)^{\delta}]^{\frac{1}{\delta}})^{\frac{-1}{\theta}} \text{ with } \theta>0,\delta\ge1$$

The BB6 copula is

$$C(u_1,u_2)=1-(1-\exp\{-[(-\log(1-\overline{u_1}^{\theta}))^{\delta}+\left(-\log(1-\overline{u_2}^{\theta})\right)^{\delta}]^{\frac{1}{\delta}}\})^{\frac{1}{\theta}} \text{ with } \theta\ge1,\delta\ge1$$

The BB7 copula is given by

$$C(u_1,u_2)=1-(1-[(1-\overline{u_1}^{\theta})^{-\delta}+(1-\overline{u_2}^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}})^{\frac{1}{\theta}} \text{ with } \theta\ge1,\delta\ge0$$

The BB8 copula is

$$C(u_1,u_2)=\frac{1}{\delta}(1-[1-\frac{1}{1-(1-\delta)^{\theta}}(1-(1-\delta u_1)^{\theta})(1-(1-\delta u_2)^{\theta})]^{\frac{1}{\theta}})$$
$$\text{with } \theta\ge1,0\le\delta\le1$$

2.4.2 Tail dependence

Tail dependence looks at the concordance and discordance in the tail, or extreme values of u_1 and u_2. It concentrates on the upper and lower quadrant tails of the joint distribution function. Given two random variables $u_1{\sim}F_1$ and $u_2{\sim}F_2$ with copula C, the coefficients of tail dependency are given by (see Joe (1997), Nelson (1999), and Cherubini *et al* (2004))

$$\lambda_L\equiv\lim_{u\to0+}P[F_1(u_1)<u|F_2(u_2)<u]=\lim_{u\to0+}\frac{C(u,u)}{u}$$

$$\lambda_U\equiv\lim_{u\to1-}P[F_1(u_1)>u|F_2(u_2)>u]=\lim_{u\to1-}\frac{1-2u+C(u,u)}{1-u}$$

where C is said to have lower (upper) tail dependency $iff\ \lambda_L\neq0\ (\lambda_U\neq0)$. The interpretation of the tail dependency is that it measures the probability of two random variables both taking extreme values shown as table 1 (see Joe (1997), Nelson (1999), and Cherubini *et al* (2004)).

No.	Family	Lower tail dependence	Upper tail dependence
1	Gaussian	*	*
2	Student's t	$2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{\frac{1-\theta}{1+\theta}})$	$2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{\frac{1-\theta}{1+\theta}})$
3	Clayton	$2^{-\frac{1}{\theta}}$	*
4	Gumbel	*	$2-2^{\frac{1}{\theta}}$
5	Frank	*	*
6	Joe	*	$2-2^{\frac{1}{\theta}}$
7	BB1 (Clayton-Gumbel)	$2^{-\frac{1}{\theta\delta}}$	$2-2^{\frac{1}{\delta}}$
8	BB6 (Joe-Gumbel)	*	$2-2^{\frac{1}{\theta\delta}}$
9	BB7 (Joe-Clayton)	$2^{-\frac{1}{\delta}}$	$2-2^{\frac{1}{\theta}}$
10	BB8 (Frank-Joe)	*	$2-2^{\frac{1}{\theta}}$ if $\delta=1,\ otherwise\ 0$

Table 1 The coefficients of tail dependency

2.5 Vine Copulas

Even though it is simple to generate multivariate Archimedean copulas, they are limited in that there are only one or two parameters to capture the dependence structure. Take the Archimedean copula for example, the θ or δ parameter in the Archimedean copula is the only driver of the dependence. The vine copulas method allows a joint distribution to be built from bivariate and conditional bivariate copulas arranged together according to the graphical structure of a regular vine. This avoids problems of compatibility and leverages a bivariate copula to enable extensions to arbitrary dimensions. For n-dimensional models, we can independently specify n(n − 1)/2 copulas, of which n − 1 are unconditional. Vines can be sampled and inferred from data. Since high-dimensional vine copulas can be factored into bivariate copulas, vine copulas cover a larger range of dependence than only using one or two parameters to fit a high-dimensional Archimedean copula. Aas and Berg (2009) and Fischer *et al* (2009) studied the performances, comparing PCCs with other multivariate models, and summarized that PCCs outperformed other models. Brechmann and Czado (2012) showed that a multivariate density can be constructed as a product of pair-copulas and its marginal density. The edges in an R-vine tree can be uniquely identified by two nodes, the conditioned nodes, and a set of conditioning nodes. The edges are denoted by e=j(e),k(e)|D(e) where D(e) is the conditioning set. The n-dimensional multivariate copula can be separated into trees $T_1, \dots, T_{n-1}$ with the corresponding edge $\mathrm{e} = \mathrm{j(e)}, \mathrm{k(e)}|\mathrm{D(e)}$ and the corresponding bivariate copula $c_{j(e),k(e)|D(e)}$. According to research (see Brechmann and Czado (2011)), the n-dimensional R-vines copula density distribution is given by

$$c(F_1(x_1), \dots, F_n(x_n)) = \prod_{i=1}^{n-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)} \left(F\left(x_{j(e)} \middle| x_{D(e)}\right), F\left(x_{k(e)} \middle| x_{D(e)}\right) \right)$$

where $x_{D(e)}$ denotes the subvector of $x = (x_1, \dots, x_n)$ indicated by the indices contained in D(e). Depending on the types of trees, various vine copulas can be constructed. Two boundary types are C-vines as in figure 2.1 and D-vines as in figure 2.2. This set of pairwise conditional probabilities can provide important insights into interlinkages and the likelihood of contagions between the sectors in the U.S. financial system.

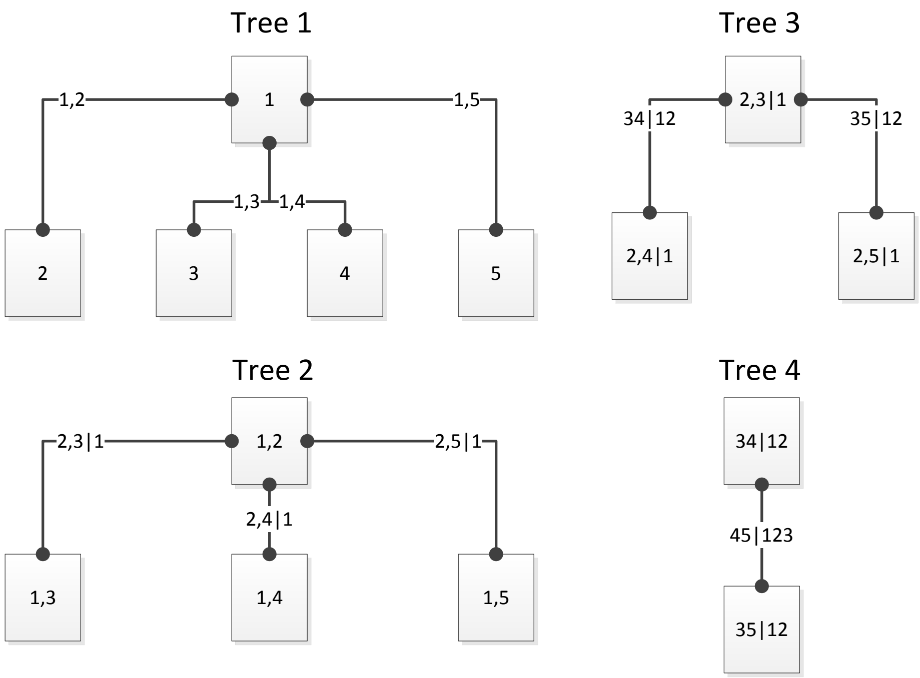

Figure 2.1 C-vines

For n=5, we obtain the C-vine decomposition by pairwise copulas using iterative levels $T_1, \dots, T_4$

$$f(x_1,\dots,x_5)=c_{12}\times c_{13}\times c_{14}\times c_{15}\times c_{23|1}\times c_{24|1}\times c_{25|1}\times c_{34|12}\times c_{35|12}\times c_{45|123}\times\prod_{k=1}^{5}f_k(x_k)$$

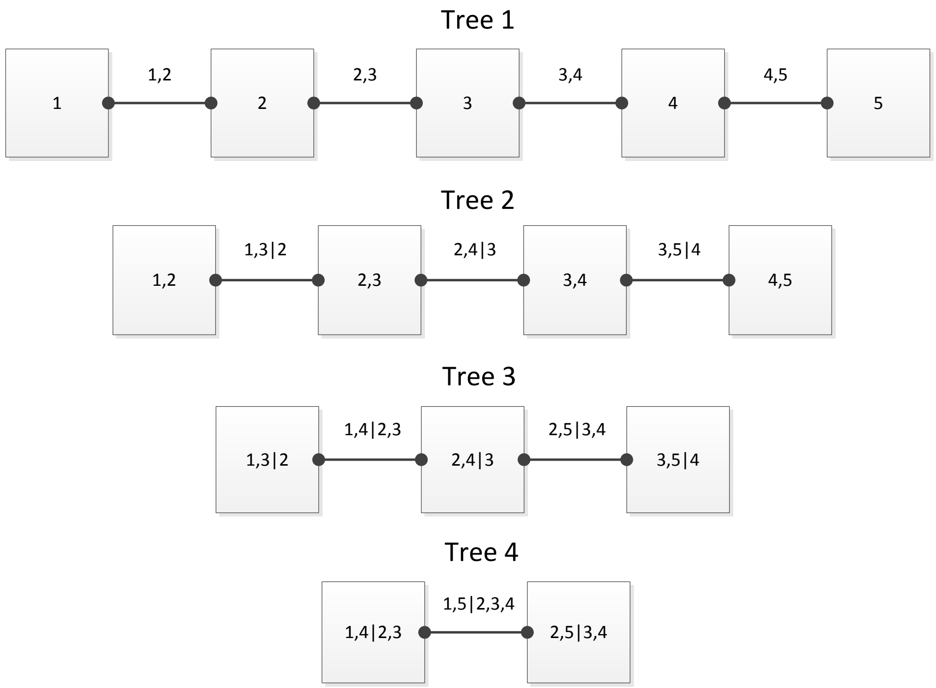

Figure 2.2 D-vines

For n=5, we obtain the D-vine decomposition by pairwise copulas using iterative levels $T_1,\dots,T_4$

$$f(x_1,\dots,x_5)=c_{12}\times c_{23}\times c_{34}\times c_{45}\times c_{13|2}\times c_{24|3}\times c_{35|4}\times c_{14|23}\times c_{25|34}\times c_{15|234}\times\prod_{k=1}^{5}f_k(x_k)$$

Dissmann *et al* (2013) proposed that the automated strategy involves searching for an appropriate R-vine tree structure, the pair-copula families, and the parameter values of the chosen pair-copula families shown as in figure 2.3. They refer to this method as the sequential method, starting by identifying the first tree, its pair-copula families, and estimating their parameters and then identifying the specification of the next tree depending on the determination made in the previous tree.

For each tree selection, they use a maximum spanning tree algorithm that maximizes the sum of absolute empirical Kendall's tau, where edge weighs are chosen appropriately to reflect large dependencies. Once an R-vine tree, either C-vine or D-vine structure, is found, they use AIC, which corrects the log likelihood of a copula for the number of parameters to choose an appropriate pair-copula, and pair-copulas are chosen independently. They use the maximum spanning tree algorithm to determine only their ranks of data instead using the actual values of the edges. Therefore, the algorithm leads to the same results if we transform the edge values by a monotone increasing function.

Figure 2.3 R-vine Sequential Algorithm (Automated Model Selection)

3. Data and Empirical Findings

3.1 Data Representation

How did we choose these 10 different sector indices? Standard and Poor separates the 500 members in the S&P 500 index into 10 different sector indices based on the Global Industrial Classification Standard (GICS) shown in figure 3.1.

Index Name:	S&P 500
Last Updated:	Mar 28 2014
Sector Name	Index Weight %
Information Technology	18.6
Financials	16.4
Health Care	13.3
Consumer Discretionary	12.1
Industrials	10.6
Energy	10.2
Consumer Staples	9.7
Materials	3.5
Utilities	3.1

Telecommunication Services	2.5

Source: S&P Dow Jones Indices LLC.

Figure 3.1 The 10 S&P 500 sector indices

The daily data is taken from Bloomberg from January 1, 1995 to July 31, 2013, covering 4678 observations. We investigated 10 S&P 500 sector indices and the S&P 500 Index's cumulative returns, specifying the same starting value 1, since January 1, 1995 shown as figure 3.2. The statistics for the 10 S&P 500 sector indices and the S&P 500 Index are listed in table 2. As shown in table 2, the Jarque-Bera test showed that each series returns strongly reject the null hypothesis of normality, indicating the non-normality of the unconditional distribution of each series. The result is 1 if the test rejects the null hypothesis at the 5% significance level, or 0 otherwise for all three tests in our case. That is one of the reasons why the multivariate normal distribution would not be appropriate since we realized that all 10 S&P 500 sector indices and the S&P 500 index return distributions do not fit into a normal distribution. We performed the Lagrangian Multiplier (LM) test to examine whether the squared return is serially correlated in lag 1. This statistic clearly indicates that the Autoregressive Conditional Heteroscedastic (ARCH) effect is likely to be found in all market returns. The Ljung-Box autocorrelation test with correlation for heteroscedasticity is also implemented at lags 1, implying all return series are serially correlated. Therefore, we apply extreme value distribution and the non-parametric kernel smoothing function into the simulations of marginal distributions.

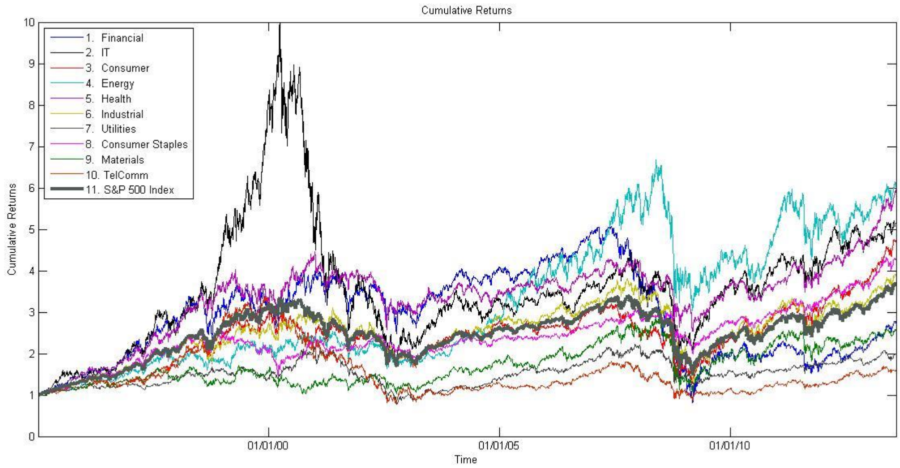

Figure 3.2 10 S&P 500 sector indices and the S&P 500 Index's cumulative returns

	Mean	Sigma	Skew	Kurt	Jarque-Bera test (p value)	ARCH LM test (p value)	Ljung-Box Q test (p value)

S5FINL Index S&P 500 Financials	0.022%	1.992%	-0.097	16.986	1 (0.001)	1 (0)	1 (4.072E-13)
S5INFT Index S&P 500 Information Technology	0.035%	1.891%	0.151	7.273	1 (0.001)	1 (0)	1 (2.339E-07)
S5COND Index S&P 500 Consumer Discretionary	0.033%	1.413%	-0.102	9.584	1 (0.001)	1 (0)	1 (8.721E-07)
S5ENRS Index S&P 500 Energy	0.039%	1.647%	-0.296	13.123	1 (0.001)	1 (0)	1 (0)
S5HLTH Index S&P 500 Health Care	0.038%	1.226%	-0.132	9.152	1 (0.001)	1 (0)	1 (9.314E-08)
S5INDU Index S&P 500 Industrials	0.029%	1.371%	-0.333	8.306	1 (0.001)	1 (0)	1 (2.387E-09)
S5UTIL Index S&P 500 Utilities	0.015%	1.196%	-0.017	13.000	1 (0.001)	1 (0)	1 (0.00074)
S5CONS Index S&P 500 Consumer Staples	0.031%	0.995%	-0.128	11.125	1 (0.001)	1 (0)	1 (1.220E-09)
S5MATR Index S&P 500 Materials	0.020%	1.563%	-0.236	9.165	1 (0.001)	1 (0)	1 (0.00027)
S5TELS Index S&P 500 Telecommunication Services	0.010%	1.473%	0.060	9.103	1 (0.001)	1 (0)	1 (0.00202)
S&P 500 Index	0.028%	1.251%	-0.238	10.776	1 (0.001)	1 (0)	1 (1.0335E-12)

Table 2 The statistics for the 10 S&P 500 sector indices and S&P 500 Index

3.2 Empirical Findings

For the selection of the rolling time window, we considered that a portfolio manager usually evaluates and adjusts his portfolio performance quarterly or annually (a.k.a. 60 or 252 workdays). In our study, we estimate systemic risk using vine copulas modeling by performing three easier tasks. The first step starts with modeling each univariate marginal distribution either parametrically or non-parametrically. As for the marginal distributions, we use the extreme value distribution for parametrical estimation and the (Normal) kernel smoothing function for non-parametrical estimation (see Bowman and Azzalini (1997)). The alternative hypothesis is that the nonparametric marginal distribution and the uniform distribution are from different continuous distributions. The result h is 1 if the test rejects the null hypothesis at the 5% significance level, and 0 otherwise. The results of Kolmogorov-Smirnov test of nonparametric cumulative distribution function estimation are listed in table 3.

Index Name	Kolmogorov-Smirnov test (p value)	Index Name	Kolmogorov-Smirnov test (p value)
S5FINL Index S&P 500 Financials	0 (0.986)	S5UTIL Index S&P 500 Utilities	0 (0.987)
S5INFT Index S&P 500 Information Technology	0 (0.986)	S5CONS Index S&P 500 Consumer Staples	0 (0.981)
S5COND Index S&P 500 Consumer Discretionary	0 (0.999)	S5MATR Index S&P 500 Materials	0 (0.999)
S5ENRS Index S&P 500 Energy	0 (0.895)	S5TELS Index S&P 500 Telecommunication Services	0 (0.987)
S5HLTH Index S&P 500 Health Care	0 (0.994)	S&P 500 Index	0 (0.940)
S5INDU Index S&P 500 Industrials	0 (0.999)		

Table 3 Kolmogorov-Smirnov test of nonparametric cumulative distribution function estimation

The second step is to select the appropriate R-vine tree structure, the pair-copula families, and the parameter values of the chosen pair-copula families. Our catalogue of pair-copula families includes elliptical copulas such as Gaussian and Student's t, single parameter Archimedean copulas such as Clayton, and Frank and Gumbel, as well as two parameter families such as BB1, BB6, BB7 and BB8. All various copulas we used are listed in table 4 (see Schepsmeier (2012)).

No.	**Short name**	**Long name**
0	I	Independence

1	N	Gaussian
2	t	t
3	C	Clayton
4	G	Gumbel
5	F	Frank
6	J	Joe
7	BB1	Clayton-Gumbel
8	BB6	Joe-Gumbel
9	BB7	Joe-Clayton
10	BB8	Frank-Joe
13	SC	Survival Clayton
14	SG	Survival Gumbel
16	SJ	Survival Joe
17	SBB1	Survival Clayton-Gumbel
18	SBB6	Survival Joe-Gumbel
19	SBB7	Survival Joe-Clayton
20	SBB8	Survival Joe-Frank
23	C90	Rotated Clayton 90 degrees
24	G90	Rotated Gumbel 90 degrees
26	J90	Rotated Joe 90 degrees
27	BB1_90	Rotated Clayton-Gumbel 90 degrees
28	BB6_90	Rotated Joe-Gumbel 90 degrees
29	BB7_90	Rotated Joe-Clayton 90 degrees
30	BB8_90	Rotated Frank-Joe 90 degrees
33	C270	Rotated Clayton 270 degrees
34	G270	Rotated Gumbel 270 degrees
36	J270	Rotated Joe 270 degrees
37	BB1_270	Rotated Clayton-Gumbel 270 degrees
38	BB6_270	Rotated Joe-Gumbel 270 degrees
39	BB7_270	Rotated Joe-Clayton 270 degrees
40	BB8_270	Rotated Frank-Joe 270 degrees

Table 4 Copula Families

The third step is to do the Monte Carlo simulation 10,000 times in each time interval, and then to calculate the Copula CoVaR/CoES and the following Copula $\Delta\mathrm{CoVaR}/\Delta\mathrm{CoES}$.

We investigated the differences among two different marginal distributions (the extreme value distribution and the (Normal) kernel smoothing function) and three different copula functions (Gaussian copula, Student's t copula and vine copulas). Afterwards, the simulation assesses Copula CoVaR and Copula CoES, and we obtain the following Copula $\Delta\mathrm{CoVaR}$ and Copula $\Delta\mathrm{CoES}$ from January 1, 1995 to July 31, 2013. We empirically examine which sector dominates more risk contributions on systemic risk given by a specific period with a Monte Carlo simulation technique using dynamic copula modeling.

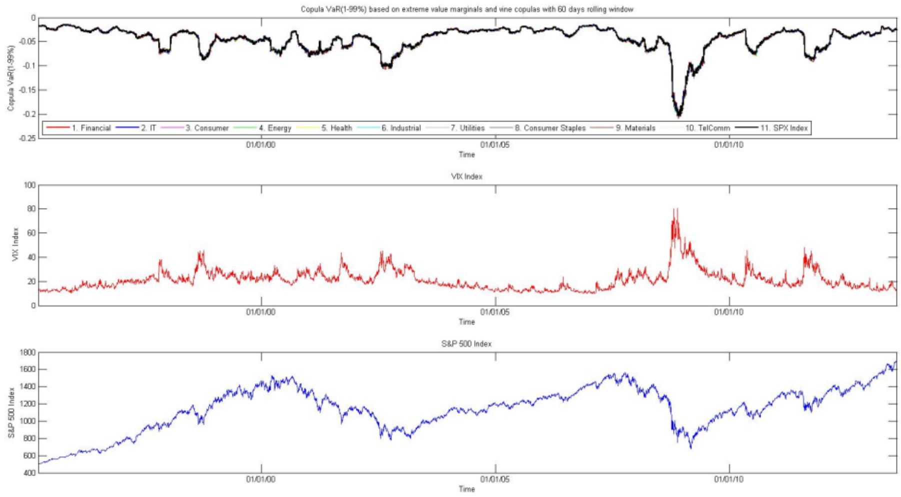

Figure 3.3 Copula $\mathrm{VaR}(1-99\%)$ based on extreme value theory and Vine Copula with 60 day rolling window

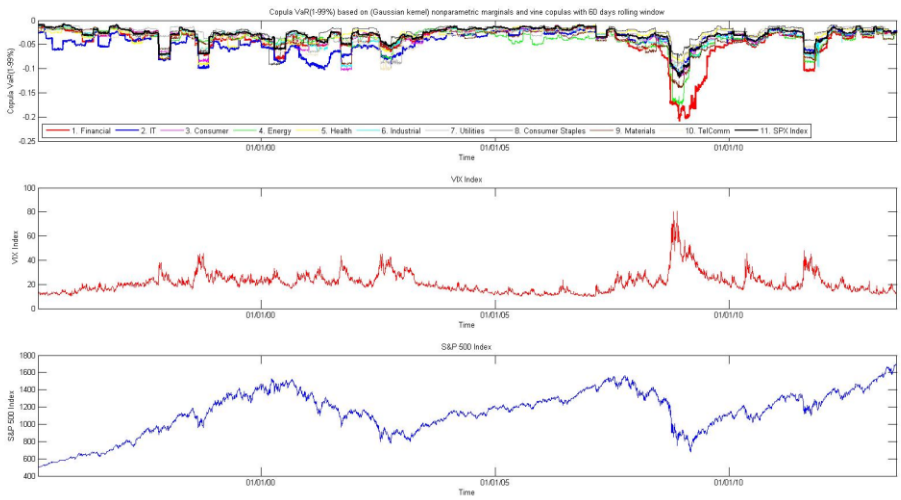

Figure 3.4 Copula $\mathrm{VaR}(1-99\%)$ based on nonparametric marginals and Vine Copula with 60 day rolling window

Extreme value theory has recently been an area of much theoretical and practical work. Univariate theory is a well-documented area, whereas multivariate extreme value theory has received significant attention recently. However, as the results show above (Figure 3.3 and Figure 3.4), we recognize that the performance of the nonparametric (Normal) kernel smoothing estimation distribution outperformed the performance of the extreme value distribution because the extreme value distribution concentrates highly on the tail parts and because the indices returns are too highly correlated. Therefore, we use nonparametric (Normal) kernel smoothing estimation to analyze systemic risk since it is more sensitive than others. In addition, the evidence also proves that choosing a suitable marginal distribution is more important than a suitable copula, and that the first tree of the R-vines often has the greatest influence on the model fit (see Dissmann *et al* (2013)). Here is the first tree of the R-vine tree structure in the first 60 days shown as figure 3.5, and the values of the log likelihood among Gaussian copula, student's t copula, and vine copulas in the first 60 days are 276.1964, 287.2379, and 331.2572, respectively.

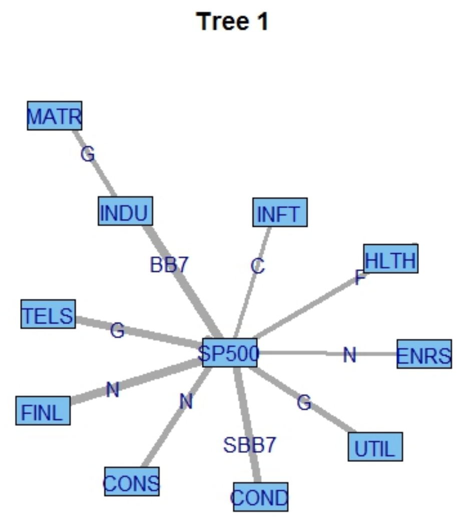

Figure 3.5 The first tree in the Vine Copulas based on nonparametric marginals with 60 day rolling window

The results are not surprising in the $Copula\,VaR$. As seen in figure 3.4 above, we realized that the IT sector caused more risk contribution during the Internet bubble from 2000 to 2002, and the financial sector caused more risk distribution during the subprime crisis from 2007 to 2010. In addition, the Energy sector reached the bubble in 2008 from 2004. The crude oil price increased from \$33/barrel in 2004 to \$145/barrel in 2008 due to economic growth. The results prove that this measure is a simplified and efficient methodology to analyze systemic risk.

3.3 Systemic Risk Indicator

The VaR of an index should not just sum up the VaRs of its individual component; it depends significantly on the correlations of the multivariate returns distributions. Therefore, we constructed an alternated VIX index (Sum of the weighted Copula VaR/ES) by combining each sector's Copula VaR/ES and its current market capitalization. As for the 60 day rolling window, the evidence shows that the Copula VaR index ($1-99\%$) based on nonparametric marginals and vine Copulas provided a vey high correlation ($\rho = -0.8636$) with the VIX index from 1995 to 2013. Similarly, the Copula VaR index ($1-99\%$) based on nonparametric marginals and the Gaussian copula provided a correlation ($\rho = -0.8626$) with the VIX index, and the Copula VaR index ($1-99\%$) based on nonparametric marginals and student's t copula provided a similarly high correlation ($\rho = -0.863$) with the VIX index as well.

As for the 252 day rolling window, the evidence shows that the Copula VaR index ($1-99\%$) based on nonparametric marginals and vine copulas provided a correlation ($\rho = -0.6912$) with the VIX index from 1995 to 2013. That is to say, we could easily simulate an alternate VIX (Sum of the weighted Copula VaR/ES) indicator for those markets or countries without the VIX index by using this methodology.

$$w_{t_i} = \frac{Market\ Capitalization_{t_i}}{\sum_i^{10} Market\ Capitalization_{t_i}}$$

$$Sum\ of\ the\ weighted\ Copula\ VaR\ Index_{t_i}\ (1-\alpha\%) = \sum_i^{10} w_{t_i} \times Copula\ VaR_{t_i}^{i}(1-\alpha\%)$$

$$Sum\ of\ the\ weighted\ Copula\ ES\ Index_{t_i}\ (1-\alpha\%) = \sum_i^{10} w_{t_i} \times Copula\ ES_{t_i}^{i}(1-\alpha\%)$$

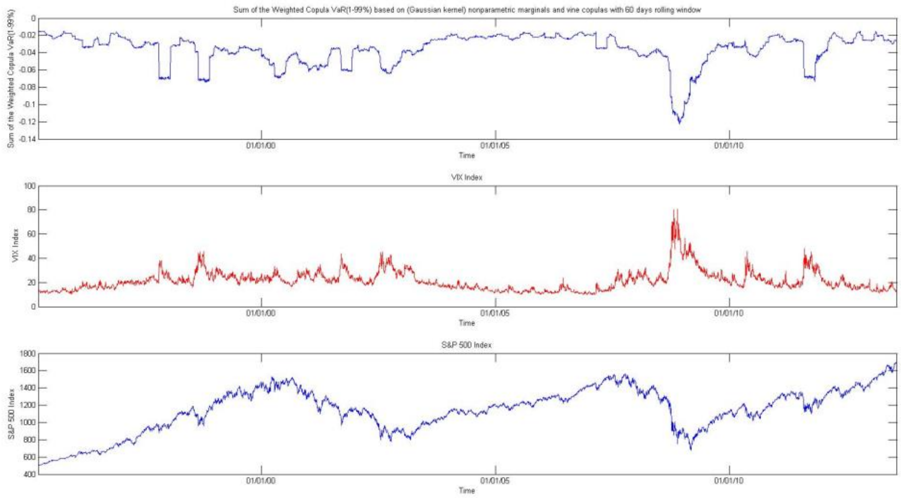

Figure 3.6 Sum of the weighted Copula VaR Index (1-99%) based on nonparametric marginals and

Vine Copula with 60 day rolling window

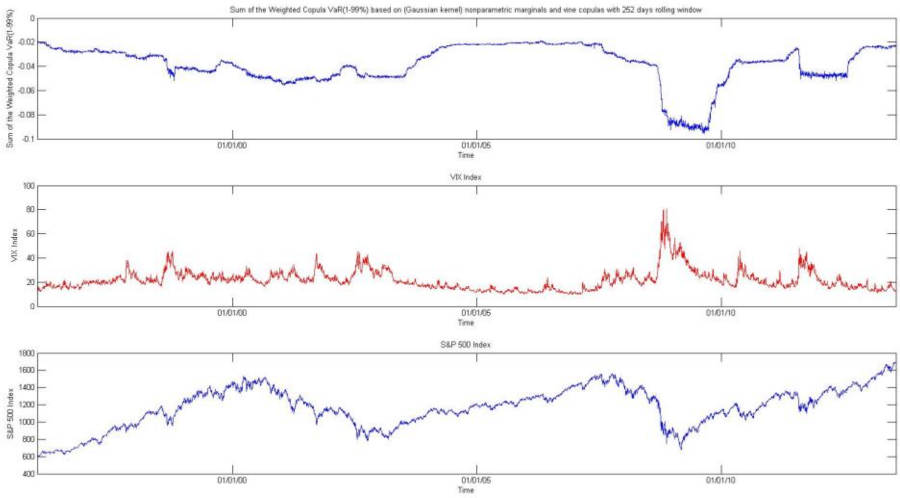

Figure 3.7 Sum of the weighted Copula VaR Index (1-99%) based on nonparametric marginals and Vine Copula with 252 day rolling window

Since each sectors' Copula $\Delta CoVaR/\Delta CoES$ represents how much risk contribution is to the financial system, the larger the absolute number, the much more systemic the risk provided to the market. Taking examples where linear correlations ρ are 0.8, 0 and -0.8 with Gaussian copula and standard normal marginals, we have three cases shown as in figure 3.8, figure 3.9 and figure 3.10. The red line in each figure displays the unconditional VaR of the market, and the black line displays the $CoVaR_{1-99\%}^{market|sector}$. The difference between these two lines are the $\Delta CoVaR^{market|sector}$. The sum of Copula $\Delta CoVaR/\Delta CoES$ can be eliminated if two assets have different signs of risk. It is similar with the portfolio diversification. Figure 3.11 is the empirical $\Delta CoVaR^{market|sector}$ result between ten different S&P 500 sector indices and the S&P 500 index based on nonparametric marginals and vine copulas with the first 60 days data, and figure 3.12 and figure 3.13 present $\Delta CoVaR^{market|sector}$ and $\Delta CoES^{market|sector}$ for the whole time period.

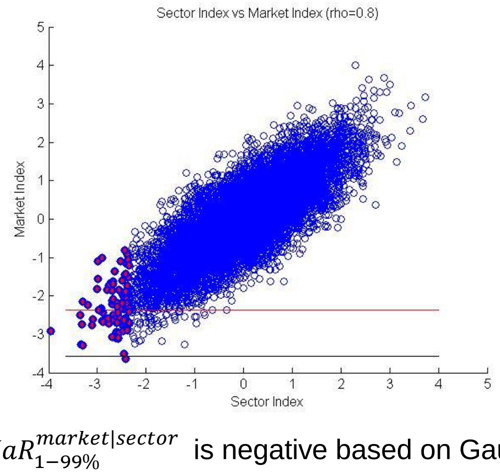

Figure 3.8 The $\Delta CoVaR_{1-99\%}^{market|sector}$ is negative based on Gaussian copula ($\rho = 0.8$) and standard normal marginals

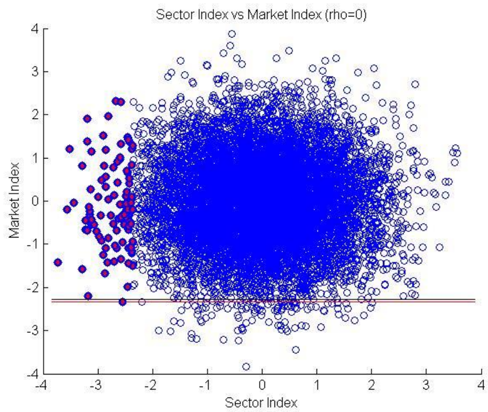

Figure 3.9 The $\Delta CoVaR_{1-99\%}^{market|sector}$ is around 0 based on Gaussian copula ($\rho = 0$) and standard normal marginals

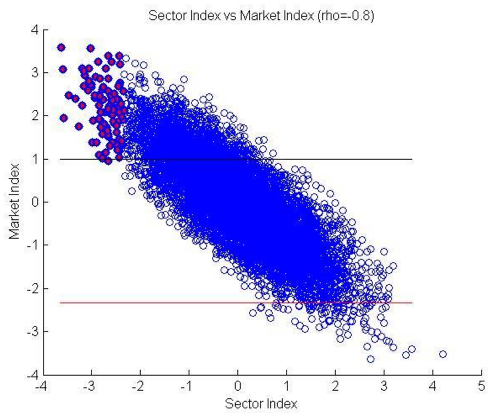

Figure 3.10 The $\Delta CoVaR_{1-99\%}^{market|sector}$ is positive based on Gaussian copula ($\rho = -0.8$) and standard normal marginal

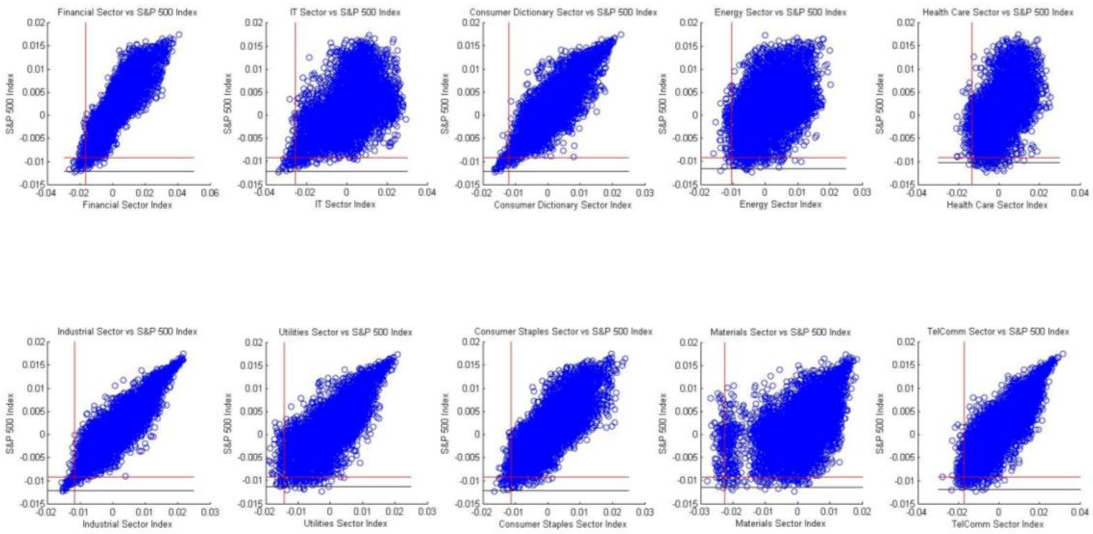

Figure 3.11 The empirical $\Delta CoVaR_{1-99\%}^{market|sector}$ results from ten S&P 500 sector indices and the S&P 500 index based on nonparametric marginals and vine copulas with

the first 60 days data

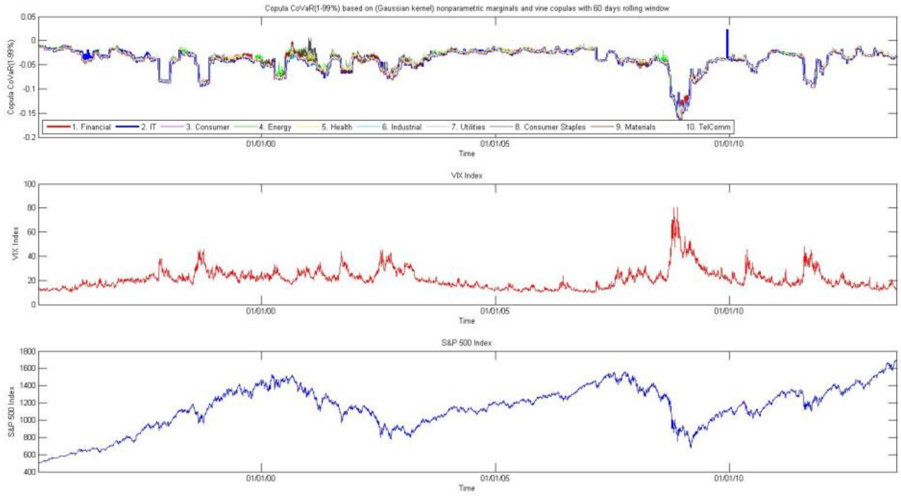

Figure 3.12 Copula $\Delta\mathrm{CoVaR}(1-99\%)$ based on nonparametric marginals and Vine Copula with 60 day rolling window

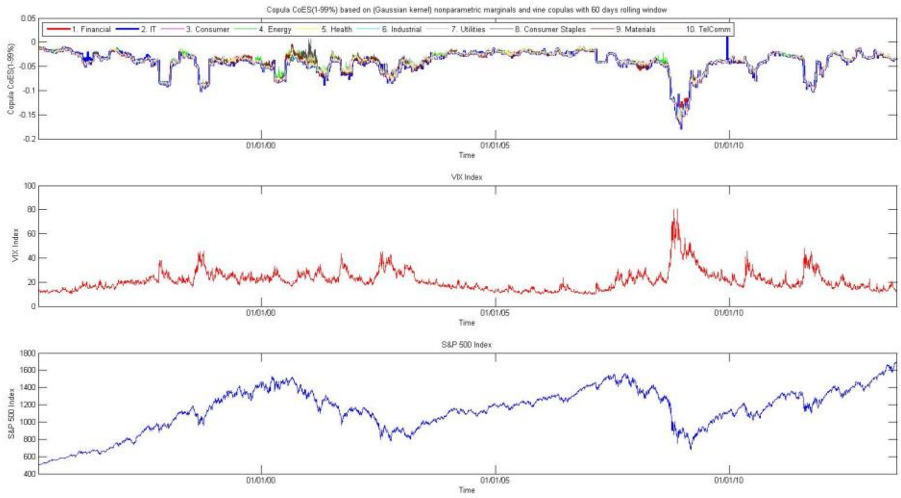

Figure 3.13 Copula $\Delta\mathrm{CoES}(1-99\%)$ based on nonparametric marginals and Vine Copula with 60 day rolling window

Therefore, we construct the systemic risk $(\Delta CoVaR/\Delta CoES)$ indicator, summing up the weighted copula $\Delta CoVaR/\Delta CoES$. The formula is shown as below. Meanwhile, the statistics for the systemic risk $(\Delta CoVaR/\Delta CoES)$ indicator are listed in table 5.

$$w_{t_i} = \frac{Market\ Capitalization_{t_i}}{\sum_i^{10} Market\ Capitalization_{t_i}}$$

$$Systemic\ Risk\ (\Delta CoVaR)\ Indicator_{t_i}\ (1-\alpha\%) = \sum_i^{10} w_{t_i} \times Copula\ \Delta CoVaR_{t_i}^{m|i}(1-\alpha\%)$$

$$Systemic\ Risk\ (\Delta CoES)\ Indicator_{t_i}\ (1-\alpha\%) = \sum_{i}^{10} w_{t_i} \times Copula\ \Delta CoES_{t_i}^{m|i}(1-\alpha\%)$$

	Mean	Sigma	Skewness	Kurtosis
$Systemic\ Risk\ (\Delta CoVaR)\ Indicator_{t_i}\ (1-\alpha\%)$	-0.0095	0.0058	-2.2776	11.4021
$Systemic\ Risk\ (\Delta CoES)\ Indicator_{t_i}\ (1-\alpha\%)$	-0.0074	0.0048	-2.4229	12.5438

Table 5 The statistics for the systemic risk $(\Delta CoVaR/\Delta CoES)$ indicator based on nonparametric marginal and Vine copula

In our study, we set the confidence level α equal to 99, and then the threshold of the systemic risk $(\Delta CoVaR/\Delta CoES)$ indicator will be the lowest 1 percentile of the systemic risk $(\Delta CoVaR/\Delta CoES)$ indicator as the red lines shown in figure 3.14 with 60 day rolling window and figure 3.15 with 252 day rolling window. The results show that the indicator reflects that the LTCM company and Russian government defaulted in 1998, the dot-com bubble in 2002, the financial subprime crisis in 2008 and the European Sovereign debt during 2010 and 2012. It shows that this systemic risk $(\Delta CoVaR/\Delta CoES)$ indicator is a useful and efficient tool to estimate systemic risk in the U.S. Equity market.

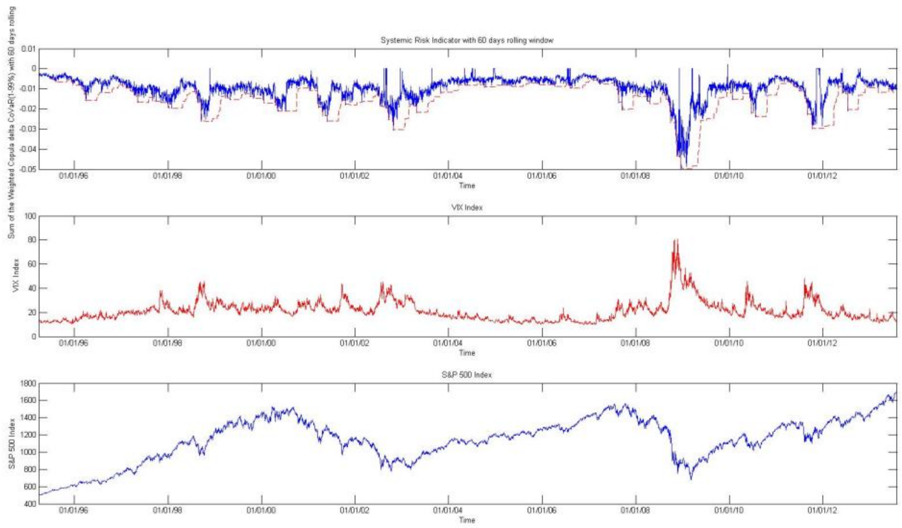

Figure 3.14 The systemic risk $(\Delta CoVaR_{1-99\%})$ indicator with 60 day rolling window

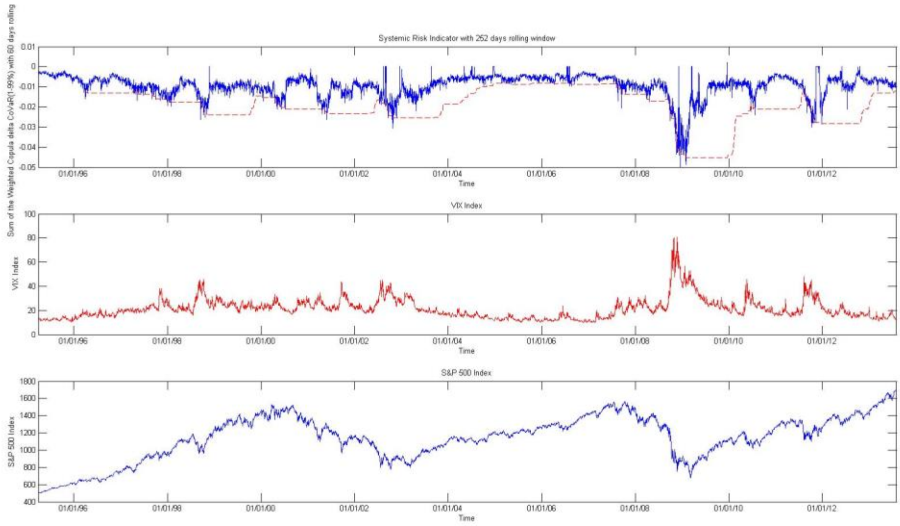

Figure 3.15 The systemic risk ($\Delta CoVaR_{1-99\%}$) indicator with 252 day rolling window

4. **Conclusion**

Even though many studies in the literature have applied copulas in financial fields, the greatest challenge is to extend copula-based models to high dimensions because existing models are not well-designed for high-dimension applications while it is simple to generate multivariate Archimedean copulas. Multivariate Archimedean copulas have are limited in that there are only one or two parameters to capture the dependence structure. The evidence in our paper shows that vine copulas modeling is an appropriate method to apply to high-dimensional modeling and captures the asymmetric characteristics in financial fields. Meanwhile, the evidence also proves that choosing a suitable marginal distribution is more important than a suitable copula, and that the first tree of the R-vine often has the greatest influence on the model fit (see Dissmann *et al* (2013)). That is why it is not obvious that vine copulas modeling outperforms Gaussian or Student's t copulas modeling in our study. In addition, our model implies that higher order multivariate distributions and more simulation paths could reveal much greater differences. We have shown that not only our Copula CoVaR/CoES measure could simulate an alternated VIX, but also the systemic risk ($\Delta CoVaR$ /$\Delta CoES$) indicator is a useful and efficient way to estimate systemic risk, no matter which entity, sector, or market is being looked at.

In addition, using Copula CoVaR/CoES methodology, we developed a real-time and flexible resolution without lagging financial accounting data. Moreover, this approach is very general and can be tailored to any underlying country and financial market easily. Our contributions not only provide an early systemic risk indicator, but also offer an alternative VIX indicator for those countries which do not offer the volatility Index. In further research, we would like to investigate the contagions in different regions and how to standardize systemic risk.

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