MATH 217 - LINEAR ALGEBRA
HOMEWORK 4, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:
Section 3.1: #8, 20, 32, 34;
Section 3.2: #26, 32, 46.

Solution.

3.1.8. Recall that the kernel of A is exactly the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$. The reduced row echelon form of our matrix A is given by:

$$\text{rref}(A) = \text{rref}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so if we solve the homogeneous system $A\vec{x} = \vec{0}$ we see that the solutions are exactly all vectors of the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

for $s, t \in \mathbb{R}$. Hence, the kernel of A is spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

3.1.20. Recall that the image of the transformation T_A is exactly the span of the columns of A (Theorem 3.1.3). In this case, the reduced row echelon form of A is given by:

$$\text{rref}(A) = \text{rref}\begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\text{rank}(A) = 2 < 3$, so $A\vec{x} = \vec{0}$ has nontrivial solutions and one of the columns of A is in the span of the other two (e.g., the third column is twice the first, minus the the second). Hence, the image of T_A is a **plane** in $\mathbb{R}^3$.

3.1.32. Since the image of a linear transformation T_A is exactly the span of the columns of A (Theorem 3.1.3), the 3×1 matrix

$$A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

is a linear transformation T_A whose image is the line spanned by this vector.

3.1.34. The line spanned by the vector

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

consists of all vectors of the form $t\vec{v}$, for $t \in \mathbb{R}$. Notice that any 3×3 matrix A whose kernel is this line must have rank 2, since there should only be one free variable occurring in the solution to $A\vec{x} = \vec{0}$. Taking it to be in reduced row echelon form (to make things easier), we may assume that A looks like e.g.,

$$A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix},$$

and satisfies

$$A\vec{v} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+2a \\ 1+2b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, if we take $a = \frac{1}{2}$ and $b = -\frac{1}{2}$, then

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

is as desired. (Note: There are infinitely many correct answers here.)

3.2.26. In the matrix

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$$

observe that the third column is equal to 3 times the first column plus the second:

$$\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Equivalently, we have the nontrivial linear relation

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

among the columns of A, and so

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

lies in the kernel of A.

3.2.32. Given the matrix A

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in reduced row echelon form, we can use Theorem 3.2.5 to conclude that the columns containing pivots form a basis for the image (i.e., span of the columns) of A:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

3.2.46. To find a basis for the kernel of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix},$$

we can solve the homogeneous linear system $A\vec{x} = \vec{0}$. Note that A is already in reduced row echelon form, so solving for $\vec{x} \in \mathbb{R}^5$ we get:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r - 3s - 5t \\ r \\ -4s - 6t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

for $r, s, t \in \mathbb{R}$ arbitrary. Thus, the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

span the kernel of A. Moreover, if we consider the linear combination

$$r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2r - 3s - 5t \\ r \\ -4s - 6t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

then we see that we must have $r = s = t = 0$, so these vectors are linearly independent by Theorem 3.2.7. Thus, they form a basis for the kernel of A.

Part B (25 points)

Problem 1. In each of the following, a vector space V is given (you do **not** have to prove that it is a vector space) along with a subset S of V. Determine whether S is a subspace of V or not, and prove your claim.

(a) Let $V = \mathbb{R}^3$ and S the set of all vectors perpendicular to $\vec{v} = (1, 2, 3)$.
(b) Let $V = \mathbb{R}^2$ and S the upper-half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$.
(c) Let $V = M_{2\times 2}(\mathbb{R})$, the vector space of all 2×2 matrices with real entries, and S the set of all **non**-invertible 2×2 matrices.
(d) Let $V = M_{2\times 2}(\mathbb{R})$ and S the set of all upper-triangular[1] 2×2 matrices.
(e) $V = C([0, 1])$, the vector space[2] of all continuous real-valued functions on $[0, 1]$, and S the set of all functions $f \in V$ such that $f(0) = 0$.
(f) $V = C([0, 1])$ and S the set of all differentiable functions on $[0, 1]$.

Solution.

(a) S is a subspace of $\mathbb{R}^3$. To see this, recall that a vector $\vec{w}$ in $\mathbb{R}^3$ is perpendicular to $\vec{v}$ if and only if $\vec{v} \cdot \vec{w} = 0$. Since $\vec{v} \cdot \vec{0} = 0$, we have that $\vec{0} \in S$. Suppose that $\vec{w}_1, \vec{w}_2 \in S$ and $k \in \mathbb{R}$. Then,
$$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2 = 0 + 0 = 0,$$
so $\vec{w}_1 + \vec{w}_2 \in S$, and
$$\vec{v} \cdot (k\vec{w}_1) = k(\vec{v} \cdot \vec{w}_1) = k0 = 0,$$
so $k\vec{w}_1 \in S$. Thus, S is a subspace of $\mathbb{R}^3$.

(b) S is **not** a subspace of $\mathbb{R}^3$. To see this, note that the zero vector $\vec{0} = (0, 0)$ in $\mathbb{R}^2$ is **not** in S, so S is not a subspace of $\mathbb{R}^2$.

(c) S is **not** a subspace of $M_{2\times 2}(\mathbb{R})$. To see this, observe that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both in S, being non-invertible, but their sum, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not (it is its own inverse). Hence, S is not closed under addition and thus is not a subspace of $M_{2\times 2}(\mathbb{R})$.

(d) S is a subspace of $M_{2\times 2}(\mathbb{R})$. To see this, first note that the "zero vector" in the vector space $M_{2\times 2}(\mathbb{R})$, namely $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, is upper-triangular (all of the entries below the diagonal are certainly 0) and thus in S. Suppose that $A, B \in S$ and $k \in \mathbb{R}$. Then, we can write
$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$
for some constants $a, b, c, d, e, f \in \mathbb{R}$. Then,
$$A + B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a + d & b + e \\ 0 & c + f \end{bmatrix},$$

[1] The square matrix A with (i, j)-entry a_{ij} is *upper-triangular* if $a_{ij} = 0$ whenever $i > j$.

[2] The vector space operations in $C([0, 1])$ are pointwise addition of functions, $(f + g)(x) = f(x) + g(x)$, and pointwise scalar multiplication, $(k \cdot f)(x) = k \cdot f(x)$, for $f, g \in C([0, 1])$ and $k \in \mathbb{R}$.

so $A + B \in S$, and

$$kA = k \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix},$$

so $kA \in S$. Thus, S is a subspace.

(e) S is a subspace of $C([0,1])$. To see this, first note that the "zero vector" in the vector space $C([0,1])$ is the function $\mathbf{0}$ on $[0,1]$ which has constant value 0. In particular, $\mathbf{0}(0) = 0$, so $\mathbf{0} \in S$. Suppose that $f, g \in S$ and $k \in \mathbb{R}$. Then,

$$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0,$$

so $f + g \in S$, and

$$(kf)(0) = kf(0) = k0 = 0,$$

so $kf \in S$. Thus, S is a subspace.

(f) S is a subspace of $C([0,1])$. (Note that S is a sub*set* of $C([0,1])$, since every differentiable function is necessarily continuous, by a theorem from calculus.) As in part (e), the "zero vector" here is the constantly 0 function, which is certainly differentiable. Suppose that $f, g \in S$ and $k \in S$. Then, again by theorems from calculus, $f + g$ and kf are differentiable (with derivatives $f' + g'$ and kf', respectively), and thus in S. Hence, S is a subspace.

Problem 2. Let W be a vector space, and let $U, V \subseteq W$ be subspaces of W.

(a) Prove that $U \cap V$ is a subspace of W.
(b) Show by example that $U \cup V$ need **not** be a subspace of W.
(c) We define $U + V$ to be the subset of W given by

$$U + V = \{u + v \in W \ : \ u \in U \text{ and } v \in V\}.$$

Prove that $U + V$ is a subspace of W.
(d) Prove that if X is any subspace of W that contains both U and V, then $U + V \subseteq X$. (We describe this by saying that $U + V$ is the *smallest* subspace of W containing both U and V.)
(e) **(Recreational)** Prove that $U \cup V$ is a subspace of W if and only if $U \subseteq V$ or $V \subseteq U$.

Solution.

(a) Since U and V are both subspaces of W, the zero vector 0 of W lies in both U and V, hence $0 \in U \cap V$. Suppose that $w_1, w_2 \in U \cap V$ and $k \in \mathbb{R}$. Then, since U is a subspace and $w_1, w_2 \in U$, we have that $w_1 + w_2 \in U$. Similarly, $w_1 + w_2 \in V$, showing that $w_1 + w_2 \in U \cap V$. Likewise, since U is a subspace and $w_1 \in U$, $kw_1 \in U$, and similarly, $kw_1 \in V$, showing that $kw_1 \in U \cap V$. Thus, $U \cap V$ is a subspace of W.

(b) Consider $W = \mathbb{R}^2$ with $U = \{(x, 0) : x \in \mathbb{R}\}$ and $V = \{(0, y) : y \in \mathbb{R}\}$, the x and y-axes, respectively. Note that both U and V are subspaces of W, being lines through the origin (cf. Example 2 on page 123). But, if we consider the vectors $(1, 0)$ and $(0, 1)$, the former is in U, while the latter is in V, hence both are in $U \cup V$, but their sum $(1, 1)$ is in neither U nor V, and thus not in $U \cup V$. Therefore, $U \cup V$ is not closed under addition and fails to be a subspace.

(c) Since U and V are both subspaces, $0 \in U$ and $0 \in V$, so $0 = 0 + 0 \in U + V$. Suppose that $w_1, w_2 \in U + V$ and $k \in U + V$. By definition of $U + V$, there must be vectors $u_1, u_2 \in U$ and $v_1, v_2 \in V$ such that $w_1 = u_1 + v_1$ and $w_2 = u_2 + v_2$. Then,

$$w_1 + w_2 = (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2),$$

which is in $U + V$, since $u_1 + u_2 \in U$ and $v_1 + v_2 \in V$, as U and V are subspaces. Likewise,

$$kw_1 = k(u_1 + v_1) = ku_1 + kv_1,$$

which is also in $U + V$, since $ku_1 \in U$ and $kv_1 \in V$. Thus, $U + V$ is a subspace of W.

(d) Suppose that X is a subspace of W that contains both U and V. Let $w \in U + V$. Then, there is some $u \in U$ and $v \in V$ such that $w = u + v$. But since X contains both U and V, $u, v \in W$, and since W is a subspace, $w = u + v \in X$. Thus, $U + V \subseteq X$.

(e) ($\Leftarrow$) If $U \subseteq V$, then $U \cup V = V$, while if $V \subseteq U$, then $U \cup V = U$, and we already know that U and V are both subspaces of W.

($\Rightarrow$) Suppose that $U \cup V$ is a subspace of W. In order to prove the disjunction "$U \subseteq V$ or $V \subseteq U$", we will assume that one of the disjuncts is false and show that the other must be true. To this end, suppose that $U \not\subseteq V$. Then, there must be some vector $u_0 \in U$ which is not in V. Let $v \in V$ be arbitrary. Then, $u_0 \in U \cup V$ and $v \in U \cup V$, so $u_0 + v \in U \cup V$, since $U \cup V$ is a subspace. This means that either $u_0 + v \in U$ or $u_0 + v \in V$. In the latter case, since V is a subspace and $v \in V$, $u_0 = (u_0 + v) + (-v)$ must be in V, but this contradicts that u_0 is not in V. Thus, we must have that $u_0 + v \in U$. But then, since U is a subspace and $u_0 \in U$, $v = (u_0 + v) + (-u_0)$ must also be in U. We have thus shown that every arbitrary element $v \in V$ must be in U, and so $V \subseteq U$, proving the claim.

Problem 3. Let $T : V \to W$ be a linear transformation between vector spaces V and W.

(a) Prove that if T is injective and $(v_1, \ldots, v_k)$ is a linearly independent list of vectors in V, then $(T(v_1), \ldots, T(v_k))$ is a linearly independent list of vectors in W.
(b) Show by example that the assumption of injectivity is necessary[3] in part (a).
(c) Prove that if T is surjective and the list of vectors $(v_1, \ldots, v_k)$ spans V, then the list of vectors $(T(v_1), \ldots, T(v_k))$ spans W.
(d) Show by example that the assumption of surjectivity is necessary in part (c).

Solution.

(a) Suppose that T is injective and $(v_1, \ldots, v_k)$ is a linearly independent list of vectors in V. Towards a contradiction, suppose that $(T(v_1), \ldots, T(v_k))$ is **not** a linearly independent list of vectors in W. Then, by the definition of linear independence (from the Definitions handout on the Canvas page, or Theorem 3.2.7), there are constants $c_1, \ldots, c_k \in \mathbb{R}$ not all 0 such that

$$c_1 T(v_1) + \cdots + c_k T(v_k) = 0_W,$$

[3] That is, give an example to show that the statement in part (a) could be false if we do not assume that T is injective.

where 0_W is the zero vector of W. Since T is linear, we have that

$$T(c_1v_1 + \cdots + c_kv_k) = 0_W.$$

T is injective and $T(0_V) = 0_W$, where 0_V is the zero vector of V, so we must have that

$$c_1v_1 + \cdots + c_kv_k = 0_V.$$

But since the $c_1, \ldots, c_k$ are not all 0, this shows that $(v_1, \ldots, v_k)$ is not linearly independent in V, contrary to our assumptions. Thus, $(T(v_1), \ldots, T(v_k))$ must be linearly independent in W.

(b) Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{v}) = \vec{0}$ for all $\vec{v} \in \mathbb{R}^2$. Then, T is linear (it is multiplication by the 2×2 zero matrix), not injective, but if we consider the standard basis vectors $(\vec{e}_1, \vec{e}_2)$, which are linearly independent, then $(T(\vec{e}_1), T(\vec{e}_2)) = (\vec{0}, \vec{0})$ is not linearly independent (since, e.g., $1 \cdot \vec{0} + 2 \cdot \vec{0} = \vec{0}$).

(c) Suppose that T is surjective and $(v_1, \ldots, v_k)$ spans V. Let $w \in W$ be arbitrary, we will show that it is contained in span of the vectors $(T(v_1), \ldots, T(v_k))$. Since T is surjective, there is some $v \in V$ such that $T(v) = w$. Since $(v_1, \ldots, v_k)$ spans V, we can express v as a linear combination of $v_1, \ldots, v_k$, that is, there are constants $c_1, \ldots, c_k \in \mathbb{R}$ such that

$$v = c_1v_1 + \cdots + c_kv_k.$$

Then, using the linearity of T, we have that

$$w = T(v) = T(c_1v_1 + \cdots + c_kv_k) = c_1T(v_1) + \cdots + c_kT(v_k),$$

showing that w is in the span of $(T(v_1), \ldots, T(v_k))$, as claimed. Thus, $(T(v_1), \ldots, T(v_k))$ spans W.

(d) Consider the same example as in (b), $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{v}) = \vec{0}$ for all $\vec{v} \in \mathbb{R}^2$. Then, T is linear, not surjective, the standard basis vectors $(\vec{e}_1, \vec{e}_2)$ span $\mathbb{R}^2$, but $(T(\vec{e}_1), T(\vec{e}_2)) = (\vec{0}, \vec{0})$ does not (it just spans $\{\vec{0}\}$).

Problem 4. Let $T : V \to W$ be a linear transformation between vector spaces V and W. Let $b \in \operatorname{im}(T)$ be a fixed vector, and let $S = \{x \in V : T(x) = b\}$.

(a) Prove that S is a subspace of V if and only if $b = \vec{0}$.
(b) Prove that for every vector $x_0 \in S$, we have $S = x_0 + \ker(T)$, where

$$x_0 + \ker(T) = \{x_0 + u \,:\, u \in \ker(T)\}.$$

(c) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and vector $\vec{b} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$.

(i) Find one particular solution $\vec{x}_0$ to the linear system $A\vec{x} = \vec{b}$.
(ii) Use part (b) to show that the set S of *all* solutions to the linear system $A\vec{x} = \vec{b}$ can be expressed as $\vec{x}_0 + \ker(A)$. What sort of geometric objects are S and $\ker(A)$, and what is the relationship between them?

Solution.

(a) ($\Leftarrow$) If $b = \vec{0}$, then $S = \{x \in V : T(x) = \vec{0}\} = \ker(T)$, which is a subspace of V by Theorem 3.2.2 (this is just in the case of matrix transformations, but the case for general vector spaces was proved on a Worksheet in class).

($\Rightarrow$) Suppose that S is a subspace of V. Then, in particular, S contains the zero vector $\vec{0}$, and so $T(\vec{0}) = b$, but T is linear, so

$$b = T(\vec{0}) = T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0}) = \vec{0}.$$

(b) Fix $x_0 \in S$. Take $x \in S$ arbitrary. So, $T(x_0) = b = T(x)$. Then, $x = x_0 + (x - x_0)$ and

$$T(x - x_0) = T(x) - T(x_0) = b - b = \vec{0},$$

since T is linear, showing that $u = x - x_0 \in \ker(T)$. Thus, $x \in x_0 + \ker(T)$. This shows that $S \subseteq x_0 + \ker(T)$.

For the reverse containment, take $x \in x_0 + \ker(T)$, so $x = x_0 + u$ for some $u \in \ker(T)$. Then,

$$T(x) = T(x_0 + u) = T(x_0) + T(u) = b + \vec{0} = b,$$

since T is linear and $x_0 \in S$, showing that $x \in S$. Thus, $x_0 + \ker(T) \subseteq S$, and so, $S = x_0 + \ker(T)$ as claimed.

(c) (i) The reduced row echelon form of the augmented matrix $\left[A \,\middle|\, \vec{b}\right]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & -\frac{1}{4} & -\frac{9}{4} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This system has general solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -\frac{9}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

for $t \in \mathbb{R}$, and so for example,

$$\vec{x}_0 = \begin{bmatrix} \frac{5}{2} \\ -\frac{9}{4} \\ 0 \end{bmatrix}$$

is a particular solution (obtained by setting $t = 0$).

(ii) If we take $T = T_A : \mathbb{R}^3 \to \mathbb{R}^3$, the linear transformation corresponding to A, then the set S of all solutions to $A\vec{x} = \vec{b}$ is exactly $S = \{\vec{x} \in \mathbb{R}^3 : T(\vec{x}) = \vec{b}\}$, which by part (b), is equal to $\vec{x}_0 + \ker(A)$. Since the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix},$$

the kernel of A, i.e., the solutions the homogeneous system $A\vec{x} = \vec{0}$, consists of all vectors of the form

$$t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

for $t \in \mathbb{R}$. Thus, geometrically, $\ker(A)$ is a line through the origin, while $S = \vec{x}_0 + \ker(T)$ is the parallel line resulting from translating that line by the vector $\vec{x}_0$.