Math 217 – Final Exam

Fall 2018

Solutions

Name: ______________________________ Section: ____________

Question	Points	Score
1	12	
2	15	
3	10	
4	13	
5	13	
6	14	
7	11	
8	12	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) the *dimension* of the subspace V of $\mathbb{R}^n$

   Solution: The *dimension* of the subspace V of $\mathbb{R}^n$ is the number of vectors in any basis of V.

   (b) the list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   Solution: The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent* if for all $c_1, \ldots, c_n \in \mathbb{R}$, if $\displaystyle\sum_{i=1}^{n} c_i\vec{v}_i = \vec{0}$ then $c_i = 0$ for each $1 \leq i \leq n$.

   (c) the linear transformation $T : V \to V$ of the finite-dimensional vector space V is *diagonalizable*

   Solution: The linear transformation $T : V \to V$ of the finite-dimensional vector space V is *diagonalizable* if there is an ordered basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

   (d) for an inner product space $(V, \langle \cdot, \cdot \rangle)$ with subspace W, the *orthogonal complement* of W in V

   Solution: The *orthogonal complement* of W in V is the set

   $$W^{\perp} \;=\; \{\vec{v} \in V \;:\; \langle \vec{v}, \vec{w} \rangle = 0 \text{ for all } \vec{w} \in W\}.$$

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For every $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$, if $\det(A) = \det(-A)$ then A is not invertible.

Solution: FALSE. For instance, letting $A = -I_2$, we have $\det(A) = \det(-I_2) = 1$ and $\det(-A) = \det(I_2) = 1$, but I_2 *is* invertible.

(b) (3 points) For every linear transformation $T : \mathbb{R}^{2\times 2} \to \mathcal{P}_2$, if $\dim(\ker(T)) = 1$, then T is surjective. (Here $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2.).

Solution: TRUE. Let $T : \mathbb{R}^{2\times 2} \to \mathcal{P}_2$ be a linear transformation, and suppose $\dim(\ker(T)) = 1$. Then since $\dim(\mathbb{R}^{2\times 2}) = 4$, by Rank-Nullity we know $\dim(\operatorname{im}(T)) = 4 - 1 = 3$. Thus $\operatorname{im}(T)$ is a 3-dimensional subspace of $\mathcal{P}_2$. But $\mathcal{P}_2$ is also 3-dimensional, so $\operatorname{im}(T) = \mathcal{P}_2$, which implies that T is surjective.

(c) (3 points) There exists a 3×3 symmetric matrix $A \in \mathbb{R}^{3\times 3}$ such that $\ker(A)$ is spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $A\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}$.

Solution: FALSE. Let $A \in \mathbb{R}^{3\times 3}$ and suppose $\ker(A)$ is spanned by $\vec{v} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $A\vec{w} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}$ where $\vec{w} = \begin{bmatrix}2\\1\\0\end{bmatrix}$. Then $\vec{v}$ is an eigenvector of A corresponding to the eigenvalue 0, and $\vec{w}$ is an eigenvector of A corresponding to the eigenvalue -1. We also see that $\vec{v} \cdot \vec{w} = 2$, so $\vec{v}$ and $\vec{w}$ are not orthogonal to each other. Since eigenvectors of a *symmetric* matrix that correspond to distinct eigenvalues must be orthogonal to each other (8.1.2), we conclude that A is not symmetric.

(Problem 2, Continued).

(d) (3 points) For every 2×2 matrix $A \in \mathbb{R}^{2\times 2}$, if $\det(A) = -20$ and $\operatorname{tr}(A) = 1$ then A is diagonalizable.

Solution: TRUE. Let $A \in \mathbb{R}^{2\times 2}$ and suppose $\det(A) = -20$ and $\operatorname{tr}(A) = 1$. Then the characteristic polynomial of A is

$$x^2 - (\operatorname{tr}(A))x + \det A \;=\; x^2 - x - 20 \;=\; (x-5)(x+4),$$

so A has the two eigenvalues 5 and -4. Since A is 2×2 and every $n \times n$ matrix with n distinct real eigenvalues is diagonalizable, we conclude that A is diagonalizable.

(e) (3 points) For every 4×3 matrix A and for every vector $\vec{b} \in \mathbb{R}^4$, if the columns of A are linearly dependent then the linear system $A\vec{x} = \vec{b}$ has infinitely many least-squares solutions.

Solution: TRUE. Let $A \in \mathbb{R}^{4\times 3}$, let $\vec{b} \in \mathbb{R}^4$, and suppose the columns of A are linearly dependent, which implies $\ker(A) \neq \{\vec{0}\}$. The least-squares solutions of $A\vec{x} = \vec{b}$ are the solutions of the linear system $A\vec{x} = \operatorname{proj}_{\operatorname{im}(A)}(\vec{b})$, which is consistent since $\operatorname{proj}_{\operatorname{im}(A)}(\vec{b}) \in \operatorname{im}(A)$. If we let $\vec{v} \in \mathbb{R}^4$ be a vector such that $A\vec{v} = \operatorname{proj}_{\operatorname{im}(A)}(\vec{b})$, then the set of least-squares solutions of $A\vec{x} = \vec{b}$ is $\{\vec{v} + \vec{x} \,:\, \vec{x} \in \ker(A)\}$, which is infinite since $\ker(A) \neq \{\vec{0}\}$.

3. Let $A = \begin{bmatrix} 3 & 2 & a \\ 2 & 0 & b \\ 0 & 0 & c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$. In each part below, find all values of a, b, c for which the given condition holds, or write "none" if no such values exist.

No justification is needed in this problem. If you fail to mention any of a, b, c in your answer, we will interpret this to mean that that variable could be any real number.

(a) (2 points) $\det A = 4$.

Solution: $\det A = -\det \begin{bmatrix} 2 & 3 & a \\ 0 & 2 & b \\ 0 & 0 & c \end{bmatrix} = -4c$. Thus $\det A = 4$ iff $c = -1$.

(b) (2 points) A is not invertible.

Solution: A is not invertible iff $\det A = 0$, so from part (a) we see that A is not invertible iff $c = 0$.

(c) (2 points) A is orthogonal.

Solution: An orthogonal matrix has unit vectors as columns, but neither of the first two columns of A is a unit vector, so there are *no* values of a, b, c that will make A orthogonal.

(d) (2 points) A is orthogonally diagonalizable.

Solution: By the Spectral Theorem, A is orthogonally diagonalizable iff A is symmetric, which will happen iff $a = b = 0$.

(e) (2 points) $\vec{e}_3$ is an eigenvector of A.

Solution: $\vec{e}_3$ is an eigenvector of A iff there is $k \in \mathbb{R}$ such that $A\vec{e}_3 = k\vec{e}_3$, i.e., iff $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}$. Thus $\vec{e}_3$ is an eigenvector of A iff $a = b = 0$.

4. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable x, and let

$$\mathcal{E} \ = \ (1, x, x^2) \qquad \text{and} \qquad A \ = \ \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that $\mathcal{E}$ is an ordered basis of $\mathcal{P}_2$. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation whose $\mathcal{E}$-matrix is A.

(a) (3 points) Compute $T(p)$, where $p \in \mathcal{P}_2$ is defined by $p(x) = x$.

Solution: $[T(p)]_{\mathcal{E}} \ = \ [T]_{\mathcal{E}}[p]_{\mathcal{E}} \ = \ \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ = \ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, so $T(p) = -p$.

(b) (3 points) Find the characteristic polynomial of T. (You may leave your answer in factored form).

Solution: The characteristic polynomial f_T of T is

$$f_T(\lambda) \ = \ \det\left([T]_{\mathcal{E}} - \lambda I_3\right) \ = \ \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \ = \ -(1-\lambda)^2(1+\lambda).$$

Remark: If you compute $\det(\lambda I_3 - [T]_{\mathcal{E}})$ instead, that would be fine too. You can also use a different variable, such as x or t instead of λ (though x is not ideal given that it's the variable in the polynomials), and you could use $[T]_{\mathcal{B}}$ where $\mathcal{B}$ is *any* ordered basis of $\mathcal{P}_2$.

(c) (4 points) Find a basis $\mathcal{B}$ of the eigenspace E_1 corresponding to the eigenvalue $\lambda = 1$ of T.

Solution: We know $E_1 = \ker(T - I)$. Using $\mathcal{E}$-coordinates, we have

$$[T]_{\mathcal{E}} - I_3 \ = \ \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose kernel is spanned by $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Converting this back into an element of $\mathcal{P}_2$, we obtain $\mathcal{B} = (2 - x)$ as a basis of E_1.

(d) (3 points) Either find an ordered basis $\mathcal{C}$ of $\mathcal{P}_2$ such that $[T]_{\mathcal{C}}$ is diagonal, or else briefly explain why this is impossible.

Solution: This is impossible. T is not diagonalizable since there is an eigenvalue λ of T, namely $\lambda = 1$, whose geometric multiplicity is strictly less than its algebraic multiplicity. Specifically, from part (b) we see that $\text{almu}(1) = 2$, while from part (c) we see that $\text{gemu}(1) = 1$.

5. Let U be the 3-dimensional inner product space of upper-triangular 2×2 matrices, with inner product
$$\langle A, B\rangle = \operatorname{tr}(A^\top B) \qquad \text{for all} \quad A, B \in U.$$
Let $V = \operatorname{Span}(P, Q) \subseteq U$, where $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and let $V^\perp$ be the orthogonal complement of V in U relative to the above inner product.

(a) (5 points) Find a basis of V that is orthonormal relative to the given inner product.

Solution: We apply the Gram-Schmidt algorithm to (P, Q) in order to obtain an orthonormal basis (v_1, v_2) of V. Note that

$$\langle P, P\rangle = \operatorname{tr}(P^\top P) = \operatorname{tr}(I_2) = 2 \quad \text{and} \quad \langle P, Q\rangle = \operatorname{tr}(P^\top Q) = \operatorname{tr}(Q) = 2,$$

so $\|P\| = \sqrt{2}$, which implies $v_1 = \frac{1}{\sqrt{2}}P$, while v_2 is the normalization of

$$w = Q - \frac{\langle P, Q\rangle}{\langle P, P\rangle}P = Q - \frac{2}{2}P = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Since $\|w\|^2 = \langle w, w\rangle = \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\right) = 4$, we conclude that $(v_1, v_2) = \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ is an orthonormal basis of V.

(b) (4 points) Let $T : U \to U$ be the orthogonal projection onto $V^\perp$. Find all eigenvalues of T, along with their algebraic and geometric multiplicities. *(No justification is necessary).*

Solution: Since T is a projection, T is diagonalizable and its eigenvalues are 0 and 1. Since $\dim V = 2$ and $\dim(U) = 3$, we know $\dim V^\perp = 1$. Finally, since $E_0 = \ker(T) = V$ and $E_1 = \operatorname{im}(T) = V^\perp$, we conclude that $\operatorname{almu}(0) = \operatorname{gemu}(0) = 2$ and $\operatorname{almu}(1) = \operatorname{gemu}(1) = 1$.

(c) (4 points) Let $R \in U$ be a matrix such that $Q - R \in V^\perp$ but $Q \neq R$, so that $\mathcal{B} = (P, Q, R)$ is an ordered basis of U. Find the $\mathcal{B}$-matrix $[T]_\mathcal{B}$ of the orthogonal projection $T : U \to U$ onto $V^\perp$.

Solution: Since T is orthogonal projection onto $V^\perp$ and $\{P, Q\} \subseteq V$ while $Q - R \in V^\perp$, we know $T(P) = T(Q) = 0$ and $T(Q - R) = Q - R$. Since T is linear, we also know $T(Q - R) = T(Q) - T(R) = -T(R)$. Thus $T(R) = R - Q$,

$$\text{so} \qquad [T]_\mathcal{B} = \begin{bmatrix} | & | & | \\ [T(P)]_\mathcal{B} & [T(Q)]_\mathcal{B} & [T(R)]_\mathcal{B} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} | & 1 & 2 \\ & 2 & 0 \\ \vec{v} & 0 & 1 \\ | & a & b \end{bmatrix} \in \mathbb{R}^{4\times 3}$, and suppose that $A = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{4}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ is the QR-factorization of A. Let $\mathcal{U} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$, so that $\mathcal{U}$ is an ordered basis of $\operatorname{im}(A)$.

(a) (2 points) Find $\vec{v}\cdot\vec{v}$. *(No justification necessary).*

Solution: $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2 = 3$.

(b) (3 points) Find the $\mathcal{U}$-coordinates of $A\vec{e}_3$. *(No justification necessary).*

Solution: $[A\vec{e}_3]_{\mathcal{U}} = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ \sqrt{3} \end{bmatrix}$.

(c) (3 points) Find $\det(A^\top A)$.

Solution: Since the columns of Q are orthonormal, $Q^\top Q = I_3$, so

$$\begin{aligned} \det(A^\top A) &= \det\big((QR)^\top (QR)\big) = \det(R^\top Q^\top Q R) = \det(R^\top R) \\ &= (\det R^\top)(\det R) = (\det R)^2 = 3\cdot\frac{2}{3}\cdot 3 = 6. \end{aligned}$$

(d) (2 points) Find the volume of the parallelepiped P that is determined by the columns of A. (Note that "volume" here means 3-volume, and that P is a 3-dimensional parallelepiped inside $\mathbb{R}^4$.) *(No justification necessary).*

Solution: Using part (c), we see that the volume of the parallelepiped P determined by the columns of A is

$$\sqrt{\det(A^\top A)} = \sqrt{6}.$$

(e) (4 points) Assuming that $\ker(AA^\top)$ is spanned by $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, find a and b.

Solution: Suppose $\ker(AA^\top)$ is spanned by $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$. Since $\ker(AA^\top) = \ker(A^\top)$, this means

$$\begin{bmatrix} - & & \vec{v} & - \\ 1 & 2 & 0 & a \\ 2 & 0 & 1 & b \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = 0,$$

so $a = 1$ and $b = 0$.

7. Let $A \in \mathbb{R}^{n\times n}$ be an $n \times n$ matrix.

(a) (5 points) Prove that if $\vec{x} \cdot (A\vec{x}) > 0$ for all nonzero vectors $\vec{x} \in \mathbb{R}^n$, then every real eigenvalue of A is positive.

Solution: Let $A \in \mathbb{R}^{n\times n}$, and suppose $\vec{x} \cdot (A\vec{x}) > 0$ for all nonzero vectors $\vec{x} \in \mathbb{R}^n$. Suppose λ is a real eigenvalue of A with corresponding eigenvector $\vec{v} \in \mathbb{R}^n$, so $A\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq \vec{0}$. Then by assumption

$$0 \;<\; \vec{v} \cdot (A\vec{v}) \;=\; \vec{v} \cdot (\lambda\vec{v}) \;=\; \lambda(\vec{v} \cdot \vec{v}).$$

Since $\vec{v} \cdot \vec{v} > 0$ by positive-definiteness of the dot product, it follows that $\lambda > 0$.

(b) (6 points) Prove that if A is symmetric and every real eigenvalue of A is positive, then $\vec{x} \cdot (A\vec{x}) > 0$ for all nonzero vectors $\vec{x} \in \mathbb{R}^n$.

Solution: Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and suppose that every real eigenvalue of A is positive. By the Spectral Theorem, A has an orthonormal eigenbasis $(\vec{v}_1, \ldots, \vec{v}_n)$, and by assumption the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are all positive.

Let $\vec{x}$ be any nonzero vector in $\mathbb{R}^n$. We can write $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ for some scalars $c_1, \ldots, c_n$ which are not all zero. Then

$$\begin{aligned}
\vec{x} \cdot (A\vec{x}) &= (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \cdot (A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)) \\
&= (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \cdot (\lambda_1 c_1\vec{v}_1 + \cdots + \lambda_n c_n\vec{v}_n) \\
&= \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2 \qquad \text{(since } (\vec{v}_1, \ldots, \vec{v}_n) \text{ is orthonormal)} \\
&> 0 \qquad \text{(since } \lambda_1, \ldots, \lambda_n > 0 \text{ and } c_1, \ldots, c_n \text{ are not all zero).}
\end{aligned}$$

Solution: Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, suppose that every real eigenvalue of A is positive. By the Spectral Theorem, A is orthogonally diagonalizable, i.e. there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ such that $A = QDQ^\top$. Note that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, so $\lambda_i > 0$ for each $1 \leq i \leq n$. Therefore we can define the diagonal matrix $E := \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$, so that $E^\top = E$ and $E^2 = D$. Now let $\vec{x} \in \mathbb{R}^n$ be any nonzero vector. Since $Q^\top$ and E are both invertible, $EQ^\top\vec{x} \neq \vec{0}$. Therefore by positive-definiteness, we obtain

$$\vec{x} \cdot (A\vec{x}) = \vec{x}^\top A\vec{x} = \vec{x}^\top QE^2Q^\top\vec{x} = (EQ^\top\vec{x})^\top(EQ^\top\vec{x}) = (EQ^\top\vec{x}) \cdot (EQ^\top\vec{x}) > 0.$$

8. Let $n \in \mathbb{N}$, let V be an n-dimensional vector space, let $I : V \to V$ be the identity transformation on V, and let $T : V \to V$ be a linear transformation.

   (a) (6 points) Prove that if every nonzero vector in V is an eigenvector of T, then $T = cI$ for some $c \in \mathbb{R}$.

Solution: Suppose that every nonzero vector in V is an eigenvector of T. Fix a nonzero vector $\vec{v} \in V$, along with $c \in \mathbb{R}$ such that $T(\vec{v}) = c\vec{v}$. Now let $\vec{w} \in V$ be arbitrary. If $\vec{w}$ is a scalar multiple of $\vec{v}$, say $\vec{w} = k\vec{v}$, then $T(\vec{w}) = T(k\vec{v}) = kT(\vec{v}) = kc\vec{v} = c\vec{w}$. Otherwise, $(\vec{v}, \vec{w})$ is linearly independent and in particular $\vec{w} \neq \vec{0}$ and $\vec{v} + \vec{w} \neq \vec{0}$, so $\vec{w}$ and $\vec{v} + \vec{w}$ are eigenvectors of T, say $T(\vec{w}) = d\vec{w}$ and $T(\vec{v} + \vec{w}) = \lambda(\vec{v} + \vec{w})$ where $d, \lambda \in \mathbb{R}$. But then

$$c\vec{v} + d\vec{w} \;=\; T(\vec{v}) + T(\vec{w}) \;=\; T(\vec{v} + \vec{w}) \;=\; \lambda\vec{v} + \lambda\vec{w},$$

so $(c - \lambda)\vec{v} + (d - \lambda)\vec{w} = \vec{0}$, which forces $c = d = \lambda$ since $(\vec{v}, \vec{w})$ is linearly independent. We have shown $T(\vec{x}) = c\vec{x}$ for all $\vec{x} \in V$, so $T = cI$.

Solution: Suppose that every nonzero vector in V is an eigenvector of T. Fix any ordered basis $\mathcal{B}$ of V, and let A be the $\mathcal{B}$-matrix of T. Then every nonzero vector of $\mathbb{R}^n$ is an eigenvector of A. In particular, $\vec{e}_1, \ldots, \vec{e}_n$ are all eigenvectors of A, so A is a diagonal matrix, say with diagonal entries $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Let $\vec{w} := \vec{e}_1 + \cdots + \vec{e}_n \in \mathbb{R}^n$ be the all-ones vector, and let $c \in \mathbb{R}$ be the eigenvalue of $\vec{w}$. Then $A\vec{w} = c\vec{w}$ implies

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = c \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

so $\lambda_1, \ldots, \lambda_n$ are all equal to c. Therefore $A = cI_n$, and so $T = cI$.

(b) (6 points) Suppose now that $T[W] \subseteq W$ for every $(n-1)$-dimensional subspace W of V. Prove that $T = cI$ for some $c \in \mathbb{R}$. (Recall that $T[W] = \{T(\vec{w}) \,:\, \vec{w} \in W\}$).

Solution: We will show by contradiction that every nonzero vector in V is an eigenvector of T, from which it follows by part (a) that $T = cI$ for some $c \in \mathbb{R}$. So assume for contradiction that the nonzero vector $\vec{v} \in V$ is not an eigenvector of T, which means $(\vec{v}, T(\vec{v}))$ is linearly independent. Extend $(\vec{v}, T(\vec{v}))$ to a basis $\mathcal{B} = (\vec{v}, T(\vec{v}), \vec{w}_3, \ldots, \vec{w}_n)$ of V. Then $W = \operatorname{Span}(\vec{v}, \vec{w}_3, \ldots, \vec{w}_n)$ is an $(n-1)$-dimensional subspace of V, so $T[W] \subseteq W$ by assumption. In particular, $T(\vec{v}) \in W$, say $T(\vec{v}) = c_1\vec{v} + c_3\vec{w}_3 + \cdots + c_n\vec{w}_n$ where $c_1, c_3, \ldots, c_n \in \mathbb{R}$. But then we have

$$c_1\vec{v} - T(\vec{v}) + \sum_{i=3}^{n} c_i\vec{w}_i \;=\; \vec{0},$$

contradicting the fact that $\mathcal{B}$ is linearly independent. This completes the proof.

Solution: First we show that every ordered basis $\mathcal{B}$ of V is an eigenbasis for T. Write $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$, and let A be the $\mathcal{B}$-matrix of T. For each $1 \leq i \leq n$, let W_i be the $(n-1)$-dimensional subspace of V whose basis is $\mathcal{B}$ with the vector $\vec{v}_i$ deleted. Then by assumption, we have $T[W_i] \subseteq W_i$. In particular, for any $1 \leq j \leq n$ with $j \neq i$, since $\vec{v}_j \in W_i$, we get $T(\vec{v}_j) \in W_i$. That is, when $T(\vec{v}_j)$ is expressed as a linear combination of $\mathcal{B}$, the vector $\vec{v}_i$ does not appear. This means precisely that the (i,j)-entry of A is zero. Since this holds for all $1 \leq i,j \leq n$ with $i \neq j$, this implies that A is a diagonal matrix, i.e. $\mathcal{B}$ is an eigenbasis for T.

Since every nonzero vector of V can be extended to a basis (which is necessarily an eigenbasis for T), every nonzero vector of V is an eigenvector of T. Therefore by part (a), $T = cI$ for some $c \in \mathbb{R}$.