Worksheet 21: Determinants and Elementary Row Operations (§§6.1,6.2)

Recall the following:

- an *elementary matrix* is any matrix that results from applying a single elementary row operation to an identity matrix.
- If E is the elementary matrix arising from applying the elementary row operation R to I_n, then EA is the matrix obtained by applying R to the $n \times n$ matrix A.

Problem 1. Let $0 \neq a \in \mathbb{R}$ and suppose $i \neq j$.

(a) If E is the elementary matrix obtained from I_n by scaling row i by a, what is $\det(E)$?

(b) If E is the elementary matrix obtained from I_n by interchanging rows i and j, what is $\det(E)$?

(c) If E is the elementary matrix obtained from I_n by adding a times row i to row j, what is $\det(E)$?

Solution:

(a) Using the definition of determinant, or alternatively multilinearity, $\det E = a \det I_n = a$.

(b) There is only one pattern in E that does not contain a zero entry. Every entry in the pattern is 1, and the sign of this pattern is -1 since it contains $|i-j|+|i-j|-1 = 2|i-j|-1$ inversions. Thus $\det E = -1$.

(c) By multilinearity of the determinant on rows, $\det E$ is the sum of $\det I_n$ with a times the determinant of a matrix (call it C) that has two rows both equal to $\vec{e}_i^{\top}$. Since any pattern in C must contain a zero entry in at least one of the two rows equal to $\vec{e}_i^{\top}$, we have $\det C = 0$, so $\det E = 1$.

Problem 2. Let A be an $n \times n$ matrix.

(a) If E is the elementary matrix obtained from I_n by scaling row i by a, what is $\det(EA)$ in terms of $\det(E)$ and $\det(A)$?

(b) If E is the elementary matrix obtained from I_n by interchanging rows i and j, what is $\det(EA)$ in terms of $\det(E)$ and $\det(A)$?

[Hint: This one is a little tricky. It might help to consider first the case where $j = i + 1$, and then treat the general case as successive applications of this simpler case.]

(c) What is $\det(A)$ if two different rows of A are identical to each other? What if two different columns are identical?

(d) If E is the elementary matrix obtained from I_n by adding a times row i to row j, what is $\det(EA)$ in terms of $\det(E)$ and $\det(A)$?

Solution:

(a) By multilinearity of det and Problem 1(a), $\det(EA) = a \det A = (\det E)(\det A)$.

(b) Suppose first that $j = i + 1$. Then EA is the matrix obtained from A by interchanging rows i and $i + 1$ of A. For each pattern P in A, let P' be the "corresponding" pattern in EA which chooses the same entries after those entries have changed locations. Then $\text{prod}_{EA}(P') = \text{prod}_A(P)$. Depending on the entries chosen by P in rows i and $i + i$, the pattern P' will have either one more or one fewer inversion than P, so that $\text{sgn}(P') = -\text{sgn}(P)$. Thus

$$\begin{aligned}\det(EA) &= \sum_{P'} \text{sgn}(P')\text{prod}_{EA}(P') = \sum_{P'} \text{sgn}(P')\text{prod}_A(P) \\ &= \sum_{P'} -\text{sgn}(P)\text{prod}_A(P) = -\sum_{P} \text{sgn}(P)\text{prod}_A(P) = -\det A,\end{aligned}$$

where for the second-last equality above we are using the fact that the association of P' to P defines a bijection from the set of patterns to itself.

Now, for the general case, note that swapping rows i and j can be accomplished by $2|i - j| - 1$ swaps of adjacent rows. Thus (using Problem 1(b)),

$$\det(EA) = (-1)^{2|i-j|-1} \det A = -\det A = (\det E)(\det A).$$

(c) Suppose rows i and j of the square matrix A are equal to each other (where $i \neq j$). Let E be the elementary matrix that swaps rows i and j, so $EA = A$. Then by part (b) we see that

$$\det(A) = \det(EA) = -\det A,$$

which implies that $\det A = 0$. Of course, the same holds when two columns of A are equal, since $\det A = \det A^\top$.

(d) By multilinearity, $\det(EA)$ is the sum of $\det(A)$ with a times the determinant of a matrix (call it C) that has two equal rows (namely rows i and j). Thus $\det C = 0$ by part (c), so we conclude that

$$\det(EA) = \det A = (\det E)(\det A).$$

Problem 3. Let A be an $n \times n$ matrix.

(a) Prove that if A is not invertible, then $\det(A) = 0$.

(b) Prove that if A *is* invertible, then $\det(A) \neq 0$ and $\det(A^{-1}) = (\det A)^{-1}$.

Solution:

(a) Suppose that A is not invertible. Then $\text{rref}(A)$ has a zero row, so $\det(\text{rref}(A)) = 0$. Let $E_1, \ldots, E_k$ be a sequence of elementary matrices such that

$$(E_k \cdots E_1)A = \text{rref}(A).$$

By repeatedly applying the results of Problem 2, we see that

$$(\det E_k) \cdots (\det E_1) \det A \;=\; \det(\operatorname{rref}(A)) \;=\; 0.$$

Since the determinant of an elementary matrix is never zero (by Problem 1), this shows $\det A = 0$.

Alternate Proof: If $A = [\vec{a}_1 \;\cdots\; \vec{a}_n]$ is not invertible, then its columns are linearly dependent; suppose $\vec{a}_k = \sum_{i=1}^{k-1} c_i \vec{a}_i$. Then by multilinearity we have

$$\det A \;=\; \sum_{i=1}^{k-1} c_i \det \left[\vec{a}_1 \;\cdots\; \vec{a}_{k-1} \; \vec{a}_i \; \vec{a}_{k+1} \;\cdots\; \vec{a}_n\right] \;=\; 0.$$

(b) Now suppose that A is invertible, so $\operatorname{rref}(A) = I_n$. Again choose elementary matrices $E_1, \ldots, E_k$ so that

$$(E_k \cdots E_1)A \;=\; \operatorname{rref}(A) \;=\; I_n,$$

and thus

$$(\det E_k) \cdots (\det E_1) \det A \;=\; \det I_n \;=\; 1.$$

This shows that A is invertible (with inverse $E_k \cdots E_1$), and (applying Problem 2 again)

$$\det A^{-1} \;=\; \det(E_k \cdots E_1) \;=\; (\det E_k) \cdots (\det E_1) \;=\; (\det A)^{-1}.$$

Problem 4. Prove that for any $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

[Hint: show that every invertible matrix is a product of elementary matrices, and use Problem 2.]

Solution: Let A and B be arbitrary $n \times n$ matrices. It follows from the Invertible Matrix Theorem that AB is invertible if and only if both A and B are invertible. Thus if AB is *not* invertible, we have by Problem 3 that both $\det(AB) = 0$ and $(\det A)(\det B) = 0$. So we may assume A and B (and hence AB) are invertible, which means none of their determinants are zero.

Since A and B are invertible, we can write them as products of elementary matrices, say

$$A \;=\; E_1 \cdots E_k \qquad \text{and} \qquad B \;=\; F_1 \cdots F_\ell.$$

Then by repeatedly applying the results of Problem 2, we have

$$\begin{aligned} \det(AB) &= \det(E_1 \cdots E_k F_1 \cdots F_\ell) \\ &= \det(E_1) \cdots \det(E_k) \det(F_1) \cdots \det(F_\ell) \\ &= \det(E_1 \cdots E_k) \det(F_1 \cdots F_\ell) \;=\; (\det A)(\det B). \end{aligned}$$

Problem 5. If A and B are similar $n \times n$ matrices, how are $\det(A)$ and $\det(B)$ related to each other? What is $\det(A^n)$ where $n \in \mathbb{N}$?

Solution: If A and B are similar, meaning there is an invertible matrix P such that $B = PAP^{-1}$, then

$$\det B \;=\; \det(PAP^{-1}) \;=\; (\det P)(\det A)(\det P^{-1}) \;=\; (\det P)(\det A)(\det P)^{-1} \;=\; \det A.$$

Thus similar matrices have the same determinant. By repeatedly applying the multiplicative property from Problem 4, we see that for any $n \in N$, $\det(A^n) \;=\; (\det A)^n$. In fact, if A is invertible then we have $\det(A^n) = (\det A)^n$ for any integer $n \in \mathbb{Z}$, as long as we remember that $A^0 = I_n$ by definition.