Worksheet 14: Coordinates and $\mathcal{B}$-Matrices (Computations) (§§3.4, 4.3)

Problem 1: Two Bases of $\mathbb{R}^2$. Let $\vec{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, let $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$, and consider the ordered bases $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B} = (\vec{b}_1, \vec{b}_2)$ of $\mathbb{R}^2$. Find the following coordinate vectors:

(a) $[\vec{b}_1]_\mathcal{B}$ (c) $[\vec{e}_1]_\mathcal{B}$ (e) $[\vec{e}_1]_\mathcal{E}$ (g) $[\vec{b}_1]_\mathcal{E}$ (i) $[2\vec{b}_1 - 1\vec{b}_2]_\mathcal{B}$

(b) $[\vec{b}_2]_\mathcal{B}$ (d) $[\vec{e}_2]_\mathcal{B}$ (f) $[\vec{e}_2]_\mathcal{E}$ (h) $[\vec{b}_2]_\mathcal{E}$ (j) $[2\vec{e}_1 - 1\vec{e}_2]_\mathcal{B}$

Problem 2: $\mathcal{B}$-Matrices.

Solution:

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (g) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (i) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$ (f) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (h) $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ (j) $\begin{bmatrix} 9 \\ -5 \end{bmatrix}$

Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. Given $P \in \mathbb{R}^{2\times 2}$, define the function $T_P : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T_P(A) = PA$ for all $A \in \mathbb{R}^{2\times 2}$.

(a) Is T_P always linear? If so, is T_P ever an isomorphism?

Solution: Yes, T_P is linear for every P, and T_P is an isomorphism if and only if P is invertible.

(b) Find $[T_P]_\mathcal{E}$ if $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

Solution: $[T_P]_\mathcal{E} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$.

(c) Find a basis of $\operatorname{im}(T_P)$ and a basis of $\ker(T_P)$, if $P = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Solution: In this case $T_P\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-c & b-d \\ 0 & 0 \end{bmatrix}$, so a basis of $\operatorname{im}(T_P)$ is $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$, and a basis of $\ker(T_P)$ is $\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$.

Problem 3: Changing Bases.

Let $\mathcal{P}_2$ be the vector space of polynomial functions of degree less than or equal to 2 in the variable t. Let $\mathcal{E}$ be the (ordered) basis of $\mathcal{P}_2$ given by $\mathcal{E} = (1, t, t^2)$. To help you save time with computations in parts (a) and (b) below, you may use:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

(a) Verify that $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ is a basis of $\mathcal{P}_2$, where

$$(\vec{b}_1, \vec{b}_2, \vec{b}_3) \ = \ (1+t,\ t^2 - t,\ 1 - t + t^2).$$

Describe explicitly the coordinate isomorphism $L_{\mathcal{B}} : \mathcal{P}_2 \to \mathbb{R}^3$ given by $\mathcal{B}$. Where does this isomorphism send the element $a + bt + ct^2$ of $\mathcal{P}_2$? (Here a, b, c are scalars.)

Solution: We know that $L_{\mathcal{E}} : \mathcal{P}_2 \to \mathbb{R}^3$ is an isomorphism, so $\mathcal{B}$ will be a basis of $\mathcal{P}_2$ if and only if $L_{\mathcal{E}}[\mathcal{B}] = (L_{\mathcal{E}}(\vec{b}_1), L_{\mathcal{E}}(\vec{b}_2), L_{\mathcal{E}}(\vec{b}_3))$ is a basis of $\mathbb{R}^3$. Thus we have to show that the matrix

$$\begin{bmatrix} | & | & | \\ L_{\mathcal{E}}(\vec{b}_1) & L_{\mathcal{E}}(\vec{b}_2) & L_{\mathcal{E}}(\vec{b}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ [\vec{b}_1]_{\mathcal{E}} & [\vec{b}_2]_{\mathcal{E}} & [\vec{b}_3]_{\mathcal{E}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

is invertible, which we are given. In fact, this matrix is just $S_{\mathcal{B}\to\mathcal{E}}$, with inverse $S_{\mathcal{E}\to\mathcal{B}}$. Thus, for any vector $a + bt + ct^2 \in \mathcal{P}_2$, we have

$$L_{\mathcal{B}}(a + bt + ct^2) \ = \ S_{\mathcal{E}\to\mathcal{B}}[a + bt + ct^2]_{\mathcal{E}} \ = \ S_{\mathcal{E}\to\mathcal{B}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \ = \ \begin{bmatrix} b + c \\ -a + b + 2c \\ a - b - c \end{bmatrix} \ \in \ \mathbb{R}^3.$$

(b) Verify that $\mathcal{C} = (\vec{c}_1, \vec{c}_2, \vec{c}_3)$ is also a basis of $\mathcal{P}_2$, where

$$(\vec{c}_1, \vec{c}_2, \vec{c}_3) \ = \ (1 - t,\ 1 + t^2,\ t).$$

As in part (a), explicitly describe the coordinate isomorphism $L_{\mathcal{C}} : \mathcal{P}_2 \to \mathbb{R}^3$ by computing its effect on a generic element $a + bt + ct^2$ of $\mathcal{P}_2$.

Solution: As in part (a), $\mathcal{C}$ is a basis of $\mathcal{P}_2$ if and only if $L_{\mathcal{E}}[\mathcal{C}]$ is a basis of $\mathbb{R}^3$, which it is since

$$S_{\mathcal{C}\to\mathcal{E}} \ = \ \begin{bmatrix} | & | & | \\ [\vec{c}_1]_{\mathcal{E}} & [\vec{c}_2]_{\mathcal{E}} & [\vec{c}_3]_{\mathcal{E}} \\ | & | & | \end{bmatrix} \ = \ \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is invertible with inverse $S_{\mathcal{E}\to\mathcal{C}}$. Then we have

$$L_{\mathcal{C}}(a + bt + ct^2) \ = \ S_{\mathcal{E}\to\mathcal{C}}[a + bt + ct^2]_{\mathcal{E}} \ = \ S_{\mathcal{E}\to\mathcal{C}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \ = \ \begin{bmatrix} a - c \\ c \\ a + b - c \end{bmatrix} \ \in \ \mathbb{R}^3.$$

(c) Let $p, q \in \mathcal{P}_2$. If $[p]_{\mathcal{C}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, what is $[p]_{\mathcal{B}}$? Conversely, if $[q]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, what is $[q]_{\mathcal{C}}$? (You may leave your answers as unsimplified products of matrices and vectors, or else use technology to help obtain simplified answers).

Solution: If $[p]_{\mathcal{C}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then

$$\begin{aligned} [p]_{\mathcal{B}} &= S_{\mathcal{C}\to\mathcal{B}}[p]_{\mathcal{C}} = S_{\mathcal{E}\to\mathcal{B}}S_{\mathcal{C}\to\mathcal{E}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a+b+c \\ -2a+b+c \\ 2a-c \end{bmatrix}. \end{aligned}$$

Similarly, if $[q]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then we have

$$[q]_{\mathcal{C}} = S^{-1}_{\mathcal{C}\to\mathcal{B}}[q]_{\mathcal{B}} = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b \\ b+c \\ 2a-2b-c \end{bmatrix}.$$

(d) Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map

$$T(f) = 2f + f' - f'' + f(0) - 6t\int_0^1 f.$$

Find the matrix of T in

(i) The basis $\mathcal{E}$;

(ii) The basis $\mathcal{B}$;

(iii) The basis $\mathcal{C}$.

How are these matrices related to each other?

Solution: $[T]_{\mathcal{E}} = \begin{bmatrix} 3 & 1 & -2 \\ -6 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $[T]_{\mathcal{B}} = \begin{bmatrix} -7 & 3 & -3 \\ -11 & 8 & -1 \\ 11 & -6 & 3 \end{bmatrix}$, and $[T]_{\mathcal{C}} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ -3 & -7 & 0 \end{bmatrix}$.
Furthermore, $[T]_{\mathcal{B}} = S_{\mathcal{E}\to\mathcal{B}}[T]_{\mathcal{E}}S_{\mathcal{B}\to\mathcal{E}}$ and $[T]_{\mathcal{C}} = S_{\mathcal{E}\to\mathcal{C}}[T]_{\mathcal{E}}S_{\mathcal{C}\to\mathcal{E}}$.

Problem 4: Rotations in $\mathbb{R}^2$.

For each angle θ, let R_θ be the counterclockwise rotation of $\mathbb{R}^2$ through an angle θ. Let $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ be the standard basis of $\mathbb{R}^2$.

(a) Remind yourself what the standard matrix $[R_\theta]_{\mathcal{E}}$ of R_θ is.

(b) Let $\mathcal{B} = (2\vec{e}_1, 2\vec{e}_2)$. Guess what $[R_\theta]_{\mathcal{B}}$ is, then check your guess by finding it.

(c) Let $\mathcal{C} = (2\vec{e}_1, 3\vec{e}_2)$. Guess what $[R_\theta]_{\mathcal{C}}$ is, then check your guess by finding it.

(d) Let $\mathcal{D} = (\vec{e}_1, \vec{e}_1 + \vec{e}_2)$. Find $[R_\theta]_{\mathcal{D}}$.

(e) Can you find all ordered bases $\mathcal{U} = (\vec{u}, \vec{v})$ such that $[R_\theta]_{\mathcal{U}} = [R_\theta]_{\mathcal{E}}$ for each θ?

Solution:

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(b) $[R_\theta]_{\mathcal{B}} = S_{\mathcal{B}\to\mathcal{E}}^{-1}[R_\theta]_{\mathcal{E}}S_{\mathcal{B}\to\mathcal{E}} = (2I_2)^{-1}[R_\theta]_{\mathcal{E}}(2I_2) = [R_\theta]_{\mathcal{E}}$.

(c) $[R_\theta]_{\mathcal{C}} = S_{\mathcal{C}\to\mathcal{E}}^{-1}[R_\theta]_{\mathcal{E}}S_{\mathcal{C}\to\mathcal{E}} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\frac{3}{2}\sin\theta \\ \frac{2}{3}\sin\theta & \cos\theta \end{bmatrix}$.

(d) $[R_\theta]_{\mathcal{D}} = S_{\mathcal{D}\to\mathcal{E}}^{-1}[R_\theta]_{\mathcal{E}}S_{\mathcal{D}\to\mathcal{E}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} [R_\theta]_{\mathcal{E}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta - \sin\theta & -2\sin\theta \\ \sin\theta & \sin\theta + \cos\theta \end{bmatrix}$.

(e) All bases of the form $(\vec{u}, \vec{v}) = \left(\begin{bmatrix} a\cos\theta \\ a\sin\theta \end{bmatrix}, \begin{bmatrix} -a\sin\theta \\ a\cos\theta \end{bmatrix} \right)$ where $0 \neq a \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

Problem 5: Rotations in $\mathbb{R}^3$.

Let

$$\vec{w} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \quad \text{and} \quad \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

so that $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \times \vec{v} = 3\vec{w}$. Let $V = \operatorname{Span}(\vec{u}, \vec{v})$.

(a) Find the standard matrix of the counterclockwise rotation of $\mathbb{R}^3$ about the z-axis through an angle θ.

Solution: $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Let S_θ be the counterclockwise rotation of $\mathbb{R}^3$ about $\vec{w}$ through an angle θ, and notice that $S_\theta(\vec{x}) \in V$ for every vector $\vec{x} \in V$, so that it makes sense to view S_θ as a linear transformation of V. Can you find the matrix of S_θ as a transformation of V relative to the basis $(\vec{u}, \vec{v})$?

Solution: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(c) Define S_θ as in part (b), but now think of it again as a transformation of all of $\mathbb{R}^3$. Can you find the standard matrix of S_θ? (You may leave your answer as a product of matrices).

Solution: We have

$$\begin{aligned}
[S_\theta]_{\mathcal{E}} &= S_{\mathcal{B}\to\mathcal{E}}[S_\theta]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}} = \begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}[S_\theta]_{\mathcal{B}}\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & -2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{9} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{bmatrix} \\
&= \frac{1}{9} \begin{bmatrix} 5\cos\theta + 4 & 2(\cos\theta + 3\sin\theta - 1) & 2(\cos\theta + \sin\theta) \\ 2(\cos\theta - 3\sin\theta - 1) & 8\cos\theta + 1 & -2(2\cos\theta + 2\sin\theta - 3) \\ 4\cos\theta + 3\sin\theta - 4 & -2(\cos\theta - 3\sin\theta - 1) & 2(2\cos\theta + 2\sin\theta + 3) \end{bmatrix}
\end{aligned}$$