# Math 217 – Midterm 2

Fall 2018

**Time: 120 mins.**

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
3. No calculators, notes, or other outside assistance allowed.
4. On this exam, *unless stated otherwise*, terms such as "orthogonal" and "orthonormal" as applied to vectors in $\mathbb{R}^n$ mean with respect to the usual dot product on $\mathbb{R}^n$.

Name: ______________________________ Section: ____________

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 15 | |
| 3 | 12 | |
| 4 | 11 | |
| 5 | 15 | |
| 6 | 12 | |
| 7 | 12 | |
| 8 | 11 | |
| Total: | 100 | |

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The vector space $V$ is *isomorphic* to the vector space $W$

   (b) The *coordinates* of the vector $\vec{v}$ in the vector space $V$ relative to the ordered basis $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ of $V$

   (c) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the inner product space $V$ is *orthonormal* relative to the inner product $\langle \cdot, \cdot \rangle$ on $V$

   (d) The $n \times n$ matrix $A$ is *similar* to the $n \times n$ matrix $B$

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) For all square matrices $A$ and $B$, if $A$ is similar to $B$ and $A$ is invertible, then $B$ is also invertible.

   (b) (3 points) For every subspace $V$ of $\mathbb{R}^n$, the orthogonal projection $\mathrm{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ onto $V$ is an orthogonal transformation.

   (c) (3 points) For every inner product space $V$, if $\vec{x}$ is a vector in $V$ such that $\langle \vec{x}, \vec{y} \rangle = 0$ for every $\vec{y} \in V$, then $\vec{x} = \vec{0}$.

(Problem 2, Continued).

(d) (3 points) The polynomial functions $p(t) = 2t + 1$ and $q(t) = 2t - 1$ are orthogonal in the inner product space $\mathcal{P}_2$ of polynomials of degree at most 2 with inner product given by $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$ for all $f, g \in \mathcal{P}_2$.

(e) (3 points) For every $n \times k$ matrix $A$ and vector $\vec{b} \in \mathbb{R}^n$, if the columns of $A$ form an orthonormal list of vectors, then $A^\top \vec{b}$ is a least-squares solution of the linear system $A\vec{x} = \vec{b}$.

3. Let $W$ be the subspace of $\mathbb{R}^4$ consisting of all solutions of the linear system

$$\begin{aligned} x_1 - x_2 &= 0, \\ x_1 + 2x_3 - x_4 &= 0. \end{aligned}$$

(a) (4 points) Find a $4 \times 4$ matrix $A$ such that $\ker(A) = W$. *(No justification required).*

(b) (4 points) Find a basis of $W^{\perp}$. *(No justification required).*

(c) (4 points) Find an ordered basis $\mathcal{B}$ of $\mathbb{R}^4$ such that the $\mathcal{B}$-matrix of the orthogonal projection onto $W$ in $\mathbb{R}^4$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

4. Let $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation defined by $T(A) = MA$ for all $A \in \mathbb{R}^{2\times 2}$. Also let

$$\mathcal{E} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad \mathcal{B} = \left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right),$$

so that $\mathcal{E}$ and $\mathcal{B}$ are ordered bases of $\mathbb{R}^{2\times 2}$ (you do not have to prove that they are bases).

(a) (3 points) Find the change-of-coordinates matrix $S_{\mathcal{B}\to\mathcal{E}}$ which changes from $\mathcal{B}$-coordinates to $\mathcal{E}$-coordinates. *(No justification required).*

(b) (5 points) Find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of $T$.

(c) (3 points) Find the $\mathcal{B}$-coordinates of $T\left( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$.

5. Let

$$A = \begin{bmatrix} | & | & 1 \\ \vec{a}_1 & \vec{a}_2 & -1 \\ & & 0 \\ | & | & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & | \\ 0 & 0 & \\ 1/2 & 0 & \vec{u}_3 \\ \sqrt{3}/2 & 0 & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & q & r \\ p & 3 & s \\ 0 & 0 & t \end{bmatrix}}_{R}$$

be the QR-factorization of the $4 \times 3$ matrix $A$, where $\vec{a}_1, \vec{a}_2, \vec{u}_3 \in \mathbb{R}^4$ and $p, q, r, s, t \in \mathbb{R}$.

(a) (2 points) What are all the possible values of $p$ and $q$ that are consistent with the given information? *(No justification necessary).*

(b) (4 points) Find $\vec{u}_3$.

(c) (5 points) Assuming that $\vec{a}_1 \cdot \vec{a}_2 = 0$, find the values of $p$, $q$, $r$, $s$, and $t$.

(d) (4 points) Letting $C = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$, find a vector $\vec{b} \in \mathbb{R}^4$ such that the solution set of $C\vec{x} = \vec{b}$ is the set of least-squares solutions of $C\vec{x} = \vec{e}_3$.

6. Let $V$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B} = (b_1, b_2, b_3)$ be an ordered basis of $V$ such that for each $1 \leq i, j \leq 3$, the number in the $i$th row and $j$th column of the table below is $\langle b_i, b_j \rangle$.

| | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|
| $b_1$ | 2 | $-2$ | 0 |
| $b_2$ | $-2$ | 6 | 3 |
| $b_3$ | 0 | 3 | 9 |

Also let $(u_1, u_2, u_3)$ be the orthonormal list obtained by applying the Gram-Schmidt process to $\mathcal{B}$, and let $W = \text{span}(b_1, b_3)$ be the subspace of $V$ spanned by $b_1$ and $b_3$.

(a) (5 points) Find $u_1$ and $u_2$ as linear combinations of $b_1$ and $b_2$.

(b) (4 points) Find the orthogonal projection of $b_2$ onto $W$ as a linear combination of $b_1$ and $b_3$.

(c) (3 points) Find the $\mathcal{B}$-coordinates of the vector in $W$ that is closest to $b_2$.

7. Let $B$ be an invertible $n \times n$ matrix, and let $A = B^\top B$. For each $1 \leq i, j \leq n$, let $a_{ij}$ denote the $(i, j)$-entry of $A$.

   (a) (2 points) Show that $A$ is symmetric.

   (b) (3 points) Show that the diagonal entries of $A$ are positive.

   (c) (5 points) Prove that $a_{ij}^2 \leq a_{ii}a_{jj}$ for all integers $i, j \in \{1, \ldots, n\}$.

   (d) (2 points) Give an example of an invertible symmetric matrix $S$ that is *not* of the form $C^\top C$ for any invertible matrix $C$.

8. Let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_k)$ be an orthonormal list of vectors in $\mathbb{R}^n$, let $B = \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{bmatrix}$ be the $n \times k$ matrix whose columns are the vectors in $\mathcal{B}$, and let $V = \mathrm{Span}(\mathcal{B})$. Also let $M$ be an orthogonal $k \times k$ matrix, and let $C = BM$.

   (a) (7 points) Prove that the list $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_k)$ of column vectors of $C$ is an orthonormal basis of $V$.

   (b) (4 points) With $\mathcal{B}$ and $\mathcal{C}$ as above, prove that $M$ is the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$; that is, prove that $M = S_{\mathcal{C} \to \mathcal{B}}$.

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