Math 217 – Final Exam

Fall 2018

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work, unless the instructions for a problem explicitly say otherwise, in which case correct justification for partially correct answers may still receive partial credit.
3. No calculators, notes, or other outside assistance allowed.
4. When appropriate, please circle your answers.
5. On this exam, *unless stated otherwise*, "eigenvalue" means *real* eigenvalue, and terms such as "similar" and "diagonalizable" mean over $\mathbb{R}$.

Name: ______________________________ Section: ____________

Question	Points	Score
1	12	
2	15	
3	10	
4	13	
5	13	
6	14	
7	11	
8	12	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) the *dimension* of the subspace V of $\mathbb{R}^n$

   (b) the list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   (c) the linear transformation $T : V \to V$ of the finite-dimensional vector space V is *diagonalizable*

   (d) for an inner product space $(V, \langle \cdot, \cdot \rangle)$ with subspace W, the *orthogonal complement* of W in V

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For every $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$, if $\det(A) = \det(-A)$ then A is not invertible.

(b) (3 points) For every linear transformation $T : \mathbb{R}^{2\times 2} \to \mathcal{P}_2$, if $\dim(\ker(T)) = 1$, then T is surjective. (Here $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2.).

(c) (3 points) There exists a 3×3 symmetric matrix $A \in \mathbb{R}^{3\times 3}$ such that $\ker(A)$ is spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $A\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}$.

(Problem 2, Continued).

(d) (3 points) For every 2×2 matrix $A \in \mathbb{R}^{2\times 2}$, if $\det(A) = -20$ and $\operatorname{tr}(A) = 1$ then A is diagonalizable.

(e) (3 points) For every 4×3 matrix A and for every vector $\vec{b} \in \mathbb{R}^4$, if the columns of A are linearly dependent then the linear system $A\vec{x} = \vec{b}$ has infinitely many least-squares solutions.

3. Let $A = \begin{bmatrix} 3 & 2 & a \\ 2 & 0 & b \\ 0 & 0 & c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$. In each part below, find all values of a, b, c for which the given condition holds, or write "none" if no such values exist.

   No justification is needed in this problem. If you fail to mention any of a, b, c in your answer, we will interpret this to mean that that variable could be any real number.

   (a) (2 points) $\det A = 4$.

   (b) (2 points) A is not invertible.

   (c) (2 points) A is orthogonal.

   (d) (2 points) A is orthogonally diagonalizable.

   (e) (2 points) $\vec{e}_3$ is an eigenvector of A.

4. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable x, and let

$$\mathcal{E} = (1, x, x^2) \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that $\mathcal{E}$ is an ordered basis of $\mathcal{P}_2$. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation whose $\mathcal{E}$-matrix is A.

(a) (3 points) Compute $T(p)$, where $p \in \mathcal{P}_2$ is defined by $p(x) = x$.

(b) (3 points) Find the characteristic polynomial of T. (You may leave your answer in factored form).

(c) (4 points) Find a basis $\mathcal{B}$ of the eigenspace E_1 corresponding to the eigenvalue $\lambda = 1$ of T.

(d) (3 points) Either find an ordered basis $\mathcal{C}$ of $\mathcal{P}_2$ such that $[T]_{\mathcal{C}}$ is diagonal, or else briefly explain why this is impossible.

5. Let U be the 3-dimensional inner product space of upper-triangular 2×2 matrices, with inner product
$$\langle A, B \rangle \;=\; \operatorname{tr}(A^\top B) \qquad \text{for all} \quad A, B \in U.$$
Let $V = \operatorname{Span}(P, Q) \subseteq U$, where $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and let $V^\perp$ be the orthogonal complement of V in U relative to the above inner product.

(a) (5 points) Find a basis of V that is orthonormal relative to the given inner product.

(b) (4 points) Let $T : U \to U$ be the orthogonal projection onto $V^\perp$. Find all eigenvalues of T, along with their algebraic and geometric multiplicities. *(No justification is necessary).*

(c) (4 points) Let $R \in U$ be a matrix such that $Q - R \in V^\perp$ but $Q \neq R$, so that $\mathcal{B} = (P, Q, R)$ is an ordered basis of U. Find the $\mathcal{B}$-matrix $[T]_\mathcal{B}$ of the orthogonal projection $T : U \to U$ onto $V^\perp$.

6. Let $A = \begin{bmatrix} | & 1 & 2 \\ & 2 & 0 \\ \vec{v} & 0 & 1 \\ | & a & b \end{bmatrix} \in \mathbb{R}^{4\times 3}$, and suppose that $A = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{4}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ is the QR-factorization of A. Let $\mathcal{U} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$, so that $\mathcal{U}$ is an ordered basis of $\operatorname{im}(A)$.

(a) (2 points) Find $\vec{v} \cdot \vec{v}$. *(No justification necessary).*

(b) (3 points) Find the $\mathcal{U}$-coordinates of $A\vec{e}_3$. *(No justification necessary).*

(c) (3 points) Find $\det(A^\top A)$.

(d) (2 points) Find the volume of the parallelepiped P that is determined by the columns of A. (Note that "volume" here means 3-volume, and that P is a 3-dimensional parallelepiped inside $\mathbb{R}^4$.) *(No justification necessary).*

(e) (4 points) Assuming that $\ker(AA^\top)$ is spanned by $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, find a and b.

7. Let $A \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix.

   (a) (5 points) Prove that if $\vec{x} \cdot (A\vec{x}) > 0$ for all nonzero vectors $\vec{x} \in \mathbb{R}^n$, then every real eigenvalue of A is positive.

   (b) (6 points) Prove that if A is symmetric and every real eigenvalue of A is positive, then $\vec{x} \cdot (A\vec{x}) > 0$ for all nonzero vectors $\vec{x} \in \mathbb{R}^n$.

8. Let $n \in \mathbb{N}$, let V be an n-dimensional vector space, let $I : V \to V$ be the identity transformation on V, and let $T : V \to V$ be a linear transformation.

   (a) (6 points) Prove that if every nonzero vector in V is an eigenvector of T, then $T = cI$ for some $c \in \mathbb{R}$.

   (b) (6 points) Suppose now that $T[W] \subseteq W$ for every $(n-1)$-dimensional subspace W of V. Prove that $T = cI$ for some $c \in \mathbb{R}$. (Recall that $T[W] = \{T(\vec{w}) \,:\, \vec{w} \in W\}$).

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