Worksheet 22: More on Determinants (§§6.1,6.2,6.3)

Problem 1: Laplace expansions.

Let A be an $n \times n$ matrix, with (i,j)-entry a_{ij}. For each $1 \leq i,j \leq n$, let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j. Then for any $1 \leq i \leq n$ and $1 \leq j \leq n$,

$$\begin{aligned}\det A &= \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\cdot\det A_{ij} \qquad \longleftarrow \quad \text{(Laplace expansion along column } j\text{)}\\ &= \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\cdot\det A_{ij} \qquad \longleftarrow \quad \text{(Laplace expansion along row } i\text{)}\end{aligned}$$

Find the determinants of the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

using Laplace expansions along various rows and columns.

Solution: Using a Laplace expansion along the first row, we have

$$\begin{aligned}\det\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix} &= 1\det\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} - 0\det\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 2\det\begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}\\ &= 1(3\cdot 0 - 2\cdot 1) - 0(0\cdot 0 - 1\cdot(-1)) + 2(0\cdot 2 - 3\cdot(-1))\\ &= -2 + 2(3) \;=\; 4.\end{aligned}$$

Using a Laplace expansion along the second column, we have

$$\det\begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix} = -2\det\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = (-2)(3) = -6.$$

Problem 2. Use elementary row operations to compute the determinant of the following matrix:

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \\ 2 & -4 & -2 & 8 \\ -1 & 3 & 6 & -1 \\ 1 & -3 & -1 & 2 \end{bmatrix}.$$

Solution: Scaling the first row of A by 2 and then applying three successive row addition operations reduces A to the upper triangular matrix

$$R = \begin{bmatrix} 1 & -3 & -1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Thus $-30 = \det R = 2 \det A$, so $\det A = -15$.

Problem 3: Volumes of parallelepipeds.

(a) If Q is an orthogonal $n \times n$ matrix, what are the possible values of $\det Q$?

Solution: Suppose Q is orthogonal, so $Q^\top Q = I_n$. Then $1 = \det I_n = \det(Q^\top Q) = \det(Q^\top) \det Q = (\det Q)^2$. Thus $\det Q = \pm 1$.

(b) Draw a picture showing that if $\vec{v}_1, \vec{v}_2$ are linearly independent vectors in $\mathbb{R}^2$, the area of the parallelogram $\mathcal{P}(\vec{v}_1, \vec{v}_2)$ determined by them is $\|\vec{v}_1\| \|\vec{v}_2 - \text{proj}_{\vec{v}_1}(\vec{v}_2)\|$.

(c) Higher-dimensional analogues of parallelograms are called *parallelepipeds*. Using the usual "volume = base $\times$ height" formula, show that the volume of the parallelepiped $\mathcal{P}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ determined by the linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ is

$$\|\vec{v}_1\| \|\vec{v}_2 - \text{proj}_{\vec{v}_1}(\vec{v}_2)\| \|\vec{v}_3 - \text{proj}_{\vec{v}_1, \vec{v}_2}(\vec{v}_3)\|.$$

(d) In higher dimensions, the "n-dimensional volume" of the n-dimensional parallelepiped $\mathcal{P}(\vec{v}_1, \ldots, \vec{v}_n)$ determined by the linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^n$ is given by

$$\prod_{k=1}^{n} \|\vec{v}_k - \text{proj}_{V_k}(\vec{v}_k)\|, \qquad \text{where} \quad V_k = \text{Span}(\vec{v}_1, \ldots, \vec{v}_{k-1}).$$

Show that the volume of $\mathcal{P}(\vec{v}_1, \ldots, \vec{v}_n)$ is $\big|\det[\vec{v}_1 \ \cdots \ \vec{v}_n]\big|$.

$\big[$HINT: consider the QR-factorization of $[\vec{v}_1 \ \cdots \ \vec{v}_n]\big]$.

Solution: Let $\vec{v}_1, \ldots, \vec{v}_n$ be linearly independent vectors in $\mathbb{R}^n$, and let $A = [\vec{v}_1 \ \cdots \ \vec{v}_n]$, so that A is invertible. Let $A = QR$ be the QR-factorization of A, so Q is orthogonal and R is upper triangular. Then $\det A = (\det Q)(\det R)$, which implies $|\det A| = |\det R|$ since $\det Q = \pm 1$. But $\det R$ is just the product of the diagonal entries of R, which by the definition of QR-factorization are the magnitudes $\|\vec{v}_k^\perp\| = \|\vec{v}_k - \text{proj}_{V_k}(\vec{v}_k)\|$.

Problem 4: Determinant as expansion factor.

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation, the *direct image* of a subset $A \subseteq \mathbb{R}^n$ under T is the set

$$T[A] \;=\; \{T(\vec{x}) \;:\; \vec{x} \in A\} \;\subseteq\; \mathbb{R}^n.$$

If T is a *linear* transformation, and if the set $A \subseteq \mathbb{R}^n$ can be assigned a volume $\mathrm{Vol}(A)$, then the relation between $\mathrm{Vol}(A)$ and the volume $\mathrm{Vol}(T[A])$ of the direct image of A under T is given by the determinant of T, namely:

$$\mathrm{Vol}\big(T[A]\big) \;=\; |\det T|\,\mathrm{Vol}(A).$$

This enables us to think of (the absolute value of) the determinant as an *expansion factor*. Of course, for this to make sense we first have to define what is meant by the determinant of a linear transformation.

(a) Define the determinant $\det T$ of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. More generally, how could you define the determinant of a linear transformation $T : V \to V$ on a finite-dimensional vector space V? Explain why your definitions make sense!

Solution: In general, we define $\det T = \det[T]_{\mathcal{B}}$ where $\mathcal{B}$ is any basis of V. This definition makes sense because if $\mathcal{C}$ is any other basis of V, then $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are similar to each other, and we already know that similar matrices have the same determinant. Thus our definition of $\det T$ does not depend on the particular basis $\mathcal{B}$ that is chosen.

(b) The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is just the direct image of the unit circle under the linear transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax \\ by \end{bmatrix}$. Use this fact to derive a formula for the area of this ellipse in terms of a and b.

Solution: The matrix of T in the standard basis is $[T]_{\mathcal{E}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Thus the area of the region contained inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is just $\det T$ times the area of the unit circle, or πab.

(c) What is the volume of the region contained inside the (3-dimensional) ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

Solution: The region $\mathcal{R}$ contained inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is just the direct image of the unit sphere under the linear transformation whose matrix in standard coordinates is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

which has determinant abc. Thus the volume of $\mathcal{R}$ is abc times the volume of the unit sphere, or $\frac{4}{3}\pi abc$.

Problem 5. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -3 & c & 0 \\ c & -1 & 1 \end{bmatrix}$.

(a) Find all values of c, if any, for which A is not invertible.

(b) Find all values of c, if any, for which the mapping $T_A(\vec{x}) = A\vec{x}$ preserves volumes.

Solution:

(a) $\det A = \det \begin{bmatrix} -3 & c \\ c & -1 \end{bmatrix} + \det \begin{bmatrix} 2 & -1 \\ -3 & c \end{bmatrix} = (3 - c^2) + (2c - 3) = c(2 - c)$. Thus A is not invertible if and only if $c = 0$ or $c = 2$.

(b) The mapping T_A preserves volumes if $|\det A| = 1$, so we must solve $c(2 - c) = \pm 1$. These equations have solutions $c = 1$ and $c = 1 \pm \sqrt{2}$.

Problem 6. Let $n > 1$, and suppose that $T : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a multilinear function (of the columns of the matrices in $\mathbb{R}^{n\times n}$) with the property that $T(A) = 0$ whenever A has two identical columns.† Prove that for any $n \times n$ matrix A, if B is obtained from A by interchanging two columns of A, then $T(B) = -T(A)$.

Solution: We will use the following notation: for any $n \times n$ matrix C and $1 \le i, j \le n$, let $C(i/j)$ be the $n \times n$ matrix obtained from C by replacing the ith column of C with the jth column of C, and let $C(i/i + j)$ be the $n \times n$ matrix obtained from C by replacing the ith column of C with the sum of the ith and jth columns of C.

†A multilinear function with this property is usually said to be *alternating*.

Now, let $A \in \mathbb{R}^{n\times n}$, and suppose that B is obtained from A by interchanging columns i and j of A where $i \neq j$, so $B = A(i/j)(j/i)$ in our notation. Then

$$\begin{aligned}
A &= A(i/i+j) - A(i/j) \\
&= A(i/i+j)(j/i+j) - A(i/i+j)(j/i) - A(i/j) \\
&= A(i/i+j)(j/i+j) - A(i/j)(j/i) - A(j/i) - A(i/j).
\end{aligned}$$

Since T is alternating, we have

$$T(A(i/j)) \;=\; T(A(j/i)) \;=\; T\big(A(i/i+j)(j/i+j)\big) \;=\; 0.$$

Therefore, using multilinearity of T we obtain

$$\begin{aligned}
T(A) &= T(A(i/i+j)) - T(A(i/j)) \\
&= T(A(i/i+j)(j/i+j)) - T(A(i/i+j)(j/i)) \\
&= -T(A(i/j)(j/i) - T(A(j/i)) \;=\; -T(B).
\end{aligned}$$