Math 217 – Final Exam

Winter 2018

Solutions

Name: ______________________________ Section: __________

Question	Points	Score
1	10	
2	15	
3	14	
4	12	
5	14	
6	12	
7	11	
8	12	
Total:	100	

1. (10 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) A *subspace* of the vector space V

   Solution: A *subspace* of the vector space V is a nonempty subset W of V that contains $\vec{0}$ and is closed under vector addition and scalar multiplication, meaning that for all $\vec{v}, \vec{w} \in W$ and $c \in \mathbb{R}$, we have $\vec{v} + \vec{w} \in W$ and $c\vec{v} \in W$.

   (b) The finite list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   Solution: The finite list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent* if for all scalars $c_1, \ldots, c_n \in \mathbb{R}$, if $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ then $c_i = 0$ for each integer i such that $1 \leq i \leq n$.

   (c) The *image* of the linear transformation $T : V \to W$ from the vector space V to the vector space W

   Solution: The *image* of the linear transformation $T : V \to W$ from the vector space V to the vector space W is the set $\operatorname{im}(T) = \{T(\vec{v}) \ : \ \vec{v} \in V\}$.

   (d) The *geometric multiplicity* of an eigenvalue λ of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$

   Solution: The *geometric multiplicity* of an eigenvalue λ of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is the dimension of the eigenspace associated to λ.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) There exists a 3×5 matrix of rank 4.

Solution: FALSE. Let A be a 3×5 matrix. Then $\mathrm{im}(A) \subseteq \mathbb{R}^3$, so $\mathrm{rank}(A) = \dim(\mathrm{im}(A)) \leq 3$. Thus there is no 3×5 matrix of rank 4.

(b) (3 points) For every 10×10 matrix A, if A is diagonalizable then so is $A + 7I_{10}$.

Solution: TRUE. Let A be a 10×10 diagonalizable matrix, say $A = PDP^{-1}$ where P is an invertible matrix and D is a diagonal matrix. Then

$$P^{-1}(A + 7I_{10})P \;=\; P^{-1}AP + P^{-1}7I_{10}P \;=\; D + 7P^{-1}P \;=\; D + 7I_{10}.$$

Since $D + 7I_{10}$ is diagonal, we see that $A + 7I_{10}$ is diagonalizable.

(c) (3 points) There exists a symmetric 2×2 matrix A with eigenvalues 3 and -2, and with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively.

Solution: FALSE. Let A be a symmetric 2×2 matrix with eigenvalues 3 and -2, say with corresponding eigenvectors $\vec{v}$ and $\vec{w}$, respectively. Then

$$3(\vec{v} \cdot \vec{w}) = A\vec{v} \cdot \vec{w} = (A\vec{v})^T\vec{w} = (\vec{v}^T A^T)\vec{w} = \vec{v}^T(A\vec{w}) = \vec{v}^T(-2\vec{w}) = -2(\vec{v} \cdot \vec{w}),$$

which shows that $\vec{v} \cdot \vec{w} = 0$. Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \neq 0$, it follows that there is no symmetric 2×2 matrix A with the stated eigenvalues and eigenvectors.

Solution: FALSE. Suppose A is a 2×2 matrix with eigenvectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, corresponding to eigenvalues 3 and -2, respectively. Then the only eigenvectors of A are nonzero scalar multiples of $\vec{v}$ and $\vec{w}$, which implies that A does not admit an orthonormal eigenbasis since $\vec{v} \cdot \vec{w} = 4 \neq 0$. Therefore A cannot be symmetric by the Spectral Theorem.

(Problem 2, Continued).

(d) (3 points) For any $n \times m$ matrix A, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of AA^T.

Solution: TRUE. Let A be an $n \times m$ matrix, so AA^T is an $n \times n$ matrix. Note that AA^T is symmetrix, since $(AA^T)^T = (A^T)^T A^T = AA^T$. Thus AA^T is orthogonally diagonalizable by the Spectral Theorem, meaning that there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of AA^T.

(e) (3 points) For any vector space V and linear transformation $T : V \to V$, if $\vec{v}$ and $\vec{w}$ are eigenvectors of T, then $\vec{v} + \vec{w}$ is also an eigenvector of T.

Solution: FALSE. For a counterexample, let $V = \mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(\vec{v}) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{v} \qquad \text{for all } \vec{v} \in \mathbb{R}^2.$$

Then $T(\vec{e}_1) = \vec{e}_1$ and $T(\vec{e}_2) = 2\vec{e}_2$, so both $\vec{e}_1$ and $\vec{e}_2$ are eigenvectors of T, but $T(\vec{e}_1 + \vec{e}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $\vec{e}_1 + \vec{e}_2$ is not an eigenvector of T.

3. Let $A \in \mathbb{R}^{3\times 3}$ be a non-invertible matrix such that $A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\2\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\3\end{bmatrix} \in \ker(A+I_3)$.

(a) (4 points) Find all the eigenvalues of A, along with their algebraic multiplicities.

Solution: Since A is not invertible, we know that 0 is an eigenvalue of A. Since $A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\2\\2\end{bmatrix}$, we know that 2 is an eigenvalue of A. And finally, since $\begin{bmatrix}2\\1\\3\end{bmatrix} \in \ker(A+I_3)$, we know that -1 is an eigenvalue of A. Thus A has three distinct eigenvalues, namely 0, 2, and -1, and since A is 3×3 it follows that each eigenvalue has algebraic (and geometric) multiplicity 1.

(b) (3 points) Is A diagonalizable? Justify your answer.

Solution: Yes, A is diagonalizable. Since $1 \le \text{gemu}(\lambda) \le \text{almu}(\lambda)$ for every eigenvalue λ, from part (a) we see that $\text{gemu}(\lambda) = 1$ for each eigenvalue λ of A, and thus the sum of the geometric multiplicities of the eigenvalues of A is 3. Since A is 3×3, this implies that there is an eigenbasis of $\mathbb{R}^3$ for A, which means that A is diagonalizable.

(Another way to see that the sum of the geometric multiplicities of A is 3 is to note that A has three *distinct* eigenvalues.)

(c) (3 points) Compute $A^2\begin{bmatrix}2\\3\\5\end{bmatrix}$.

Solution: $A^2\begin{bmatrix}2\\3\\5\end{bmatrix} = A^2\left(2\begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix}\right) = 2A^2\begin{bmatrix}0\\1\\1\end{bmatrix} + A^2\begin{bmatrix}2\\1\\3\end{bmatrix} = 8\begin{bmatrix}0\\1\\1\end{bmatrix} + (-1)^2\begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}0\\8\\8\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}2\\9\\11\end{bmatrix}$.

(d) (4 points) Assuming that $\vec{e}_1 \in \ker(A)$, find A. (You may leave your answer as an unsimplified expression for A, if you wish).

Solution: $A = \begin{bmatrix}0&2&1\\1&1&0\\1&3&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&2&1\\1&1&0\\1&3&0\end{bmatrix}^{-1} = \begin{bmatrix}0&1&-1\\0&7/2&-3/2\\0&9/2&-5/2\end{bmatrix}$.

4. Let V be the plane in $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}$. Let $A \in \mathbb{R}^{3\times 3}$ be the standard matrix of the orthogonal projection onto V in $\mathbb{R}^3$.

(a) (4 points) Find an orthonormal basis $(\vec{u}_1, \vec{u}_2)$ of V.

Solution: Let $\vec{w}_1 = \vec{v}_1$ and

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix}2\\1\\0\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}.$$

Then an orthonormal basis of V is

$$\mathcal{U} = (\vec{u}_1, \vec{u}_2) = \left(\frac{\vec{w}_1}{\|\vec{w}_1\|}, \frac{\vec{w}_2}{\|\vec{w}_2\|}\right) = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}\right).$$

(b) (3 points) Let B and C be the 3×2 matrices $B = [\vec{v}_1\ \vec{v}_2]$ and $C = [\vec{u}_1\ \vec{u}_2]$, where $\vec{u}_1$, $\vec{u}_2$ are as in part (a). Of the twelve matrices below, circle those that equal A.

B^TB $\quad BB^T$ $\quad B^T(B^TB)^{-1}B$ $\quad B(B^TB)^{-1}B^T$ $\quad B^T(BB^T)^{-1}B$ $\quad B(BB^T)^{-1}B^T$

C^TC $\quad CC^T$ $\quad C^T(C^TC)^{-1}C$ $\quad C(C^TC)^{-1}C^T$ $\quad C^T(CC^T)^{-1}C$ $\quad C(CC^T)^{-1}C^T$

Solution: The three matrices $B(B^TB)^{-1}B^T$, CC^T, and $C(C^TC)^{-1}C^T$ equal A.

(c) (5 points) If possible, find an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$. If this is not possible, explain why.

Solution: Using the notation from part (a), let $\vec{w}_3 = \vec{w}_1 \times \vec{w}_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix}$, so that $\vec{w}_3$ is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$. Then we let $\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$ and

$$Q = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3] = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}, \quad \text{so that} \quad D = Q^TAQ = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

is diagonal.

5. Let V be the vector space of 2×2 upper triangular matrices, and let $T : V \to V$ be the linear map defined by $T(A) = MAM$, where $M = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$. Note that $M = M^{-1}$.

(a) (4 points) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

Solution: Write $\mathcal{E} = (E_{11}, E_{12}, E_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$. Then

$$T(E_{11}) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad T(E_{12}) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad T(E_{22}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

so

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | \\ [T(E_{11})]_{\mathcal{E}} & [T(E_{12})]_{\mathcal{E}} & [T(E_{22})]_{\mathcal{E}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) (3 points) Is T invertible? Justify your answer.

Solution: Yes, T is invertible, since $\det T = \det[T]_{\mathcal{E}} = -1 \neq 0$.

Solution: Yes, T is invertible, since $\operatorname{rank}(T) = \operatorname{rank}([T]_{\mathcal{E}}) = 3 = \dim(V)$.

Solution: Yes, T is invertible, since $M^2 = I_2$, so $T^2(A) = M^2AM^2 = A$ for all $A \in V$, which shows that T is its own inverse.

(c) (3 points) Find an eigenvector of T corresponding to the eigenvalue $\lambda = 1$.

Solution: Note that $T(M) = MMM = M$, so M is an eigenvector of T corresponding to the eigenvalue 1.

(d) (4 points) Is T diagonalizable? Justify your answer.

Solution: The characteristic polynomial of T is

$$\det(xI_V - T) = \det(xI_3 - [T]_{\mathcal{E}}) = \det \begin{bmatrix} x-1 & 0 & 0 \\ 1 & x+1 & -1 \\ 0 & 0 & x-1 \end{bmatrix} = (x-1)^2(x+1).$$

We know that T is diagonalizable if the sum of the geometric multiplicities of its eigenvalues is $\dim(V) = 3$. We have $\operatorname{gemu}(-1) = \operatorname{almu}(-1) = 1$, so to show that T is diagonalizable it will suffice to show that $\text{gemu}(1) = 2$. Note that $T(I_2) = MI_2M = MM = I_2$, so that both M and I_2 are eigenvectors of T corresponding to the eigenvalue 1. Since M and I_2 are linearly independent, we see that $\text{gemu}(1) = 2$, so T is indeed diagonalizable.

6. Let $B = \begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ -4 & 2 & c \end{bmatrix}$ be a 3×3 matrix whose third column is unknown. *(Note: the additional assumptions stated below do NOT carry over from one part to the next).*

(a) (4 points) Assuming that B is invertible, find the first column of B^{-1}.

Solution: To find (the first column of) B^{-1}, we row reduce $[B \mid I_3]$:

$$\begin{bmatrix} 1 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 1 & 0 \\ -4 & 2 & c & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 1 & 0 \\ 0 & 2 & c+4a & 4 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & a & 1 & 0 & 0 \\ 0 & 1 & \frac{c+4a}{2} & 2 & 0 & \frac{1}{2} \\ 0 & 0 & b & 0 & 1 & 0 \end{bmatrix}.$$

Note that the additional row operations that are required to transform B into I_3 will not alter the fourth column of the last matrix given above, so the first column of B^{-1} is $[1\ 2\ 0]^\top$.

(b) (4 points) Assuming that $\det B = 12$, determine as many of the values a, b, and c as possible. In your answer, clearly indicate which of these values, if any, cannot be determined from the assumption that $\det B = 12$.

Solution: Using either the definition of determinant or Laplace expansions, we see that $\det B = -2b$. Therefore $b = -6$, but $\det B$ does not depend on a or c so we cannot determine these values from the assumption that $\det B = 12$.

(c) (4 points) Assuming that the linear system $B\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is inconsistent, find a least-squares solution of it.

Solution: Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and assume that $B\vec{x} = \vec{v}$ is inconsistent. This implies that $\operatorname{rank}(B) = 2$. Since the first two columns of B are linearly independent, this means that the third column of B must be a linear combination of the first two, so $\operatorname{im}(B) = \operatorname{Span}(B\vec{e}_1, B\vec{e}_2)$, which is the xz-plane. The least-squares solutions of $B\vec{x} = \vec{v}$ are the solutions of the consistent linear system $B\vec{x} = \operatorname{proj}_{\operatorname{im} B}(\vec{v})$, ie,

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ -4 & 2 & c \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Solving this linear system gives $\vec{x}^* = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ as a least-squares solution of $B\vec{x} = \vec{v}$.

7. Let $Q \in \mathbb{R}^{3\times 3}$ be an orthogonal 3×3 matrix.

   (a) (4 points) Prove that for all $\lambda \in \mathbb{R}$, if λ is an eigenvalue of Q then $|\lambda| = 1$.

Solution: Let $Q \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix, and let $T_Q : \mathbb{R}^3 \to \mathbb{R}^3$ be the associated linear transformation. Then T_Q is an orthogonal transformation, so it preserves length, meaning that $\|Q\vec{v}\| = \|\vec{v}\|$ for all $\vec{v} \in \mathbb{R}^3$. Now let $\lambda \in \mathbb{R}$ be an eigenvalue of Q, with associated eigenvalue $\vec{v} \neq \vec{0}$. Then

$$\|\vec{v}\| \;=\; \|Q\vec{v}\| \;=\; \|\lambda\vec{v}\| \;=\; |\lambda|\|\vec{v}\|,$$

which implies $|\lambda| = 1$ since $\vec{v} \neq 0$.

   (b) (7 points) Prove that if $\det Q = 1$, then there is a nonzero vector $\vec{v} \in \mathbb{R}^3$ such that $Q\vec{v} = \vec{v}$.

Solution: Let $Q \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix with determinant 1. Suppose first that Q has no nonreal complex eigenvalues. Then by part (a), all three eigenvalues of Q are ± 1. Since $\det Q$ is the product of the three eigenvalues of Q, but $(-1)^3 = -1 \neq 1$, at least one of the eigenvalues of Q must be 1, which means there is nonzero $\vec{v} \in \mathbb{R}^3$ such that $Q\vec{v} = \vec{v}$.

To complete the proof, suppose now that Q has some nonreal complex eigenvalues. Since the characteristic polynomial of Q is cubic, and thus has at least one real root, we know that Q has at least one real eigenvalue, say λ. Then since nonreal complex eigenvalues of matrices with real entries come in conjugate pairs, we know that the other two eigenvalues of Q have the form $a \pm bi \in \mathbb{C}$, where $b \neq 0$. Then

$$1 \;=\; \det Q \;=\; \lambda(a+bi)(a-bi) \;=\; \lambda(a^2+b^2).$$

Since $|\lambda| = 1$ by part (a) and since $a^2 + b^2$ is a positive real number, we know that in fact $a^2 + b^2 = 1$ and thus also $\lambda = 1$, so again we find that there is nonzero $\vec{v} \in \mathbb{R}^3$ such that $Q\vec{v} = \vec{v}$.

8. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Recall that for any $X \subseteq V$, we define $T[X] = \{T(\vec{x}) : \vec{x} \in X\}$.

   (a) (6 points) Suppose that V_1 and V_2 are subspaces of V that contain $\ker(T)$. Prove that if $T[V_1] = T[V_2]$, then $V_1 = V_2$.

   Solution: Let V_1 and V_2 be subspaces of V that contain $\ker(T)$, and suppose that $T[V_1] = T[V_2]$. Let $\vec{v} \in V_1$, so $T(\vec{v}) \in T[V_1] = T[V_2]$. Choose $\vec{w} \in V_2$ such that $T(\vec{w}) = T(\vec{v})$. Then $T(\vec{w} - \vec{v}) = T(\vec{w}) - T(\vec{v}) = \vec{0}$, so $\vec{w} - \vec{v} \in \ker(T) \subseteq V_2$. Thus $\vec{v} = \vec{w} - (\vec{w} - \vec{v}) \in V_2$. This shows $V_1 \subseteq V_2$, and the reverse inclusion follows by a similar argument. Explicitly, since $T[V_1] = T[V_2]$, for any $\vec{w} \in V_2$ we can choose $\vec{v} \in V_1$ such that $T(\vec{w}) = T(\vec{v})$, which means $\vec{w} - \vec{v} \in \ker(T) \subseteq V_1$ and therefore $\vec{w} = \vec{v} + (\vec{w} - \vec{v}) \in V_1$.

   (b) (6 points) Prove that T is injective if and only if for all subspaces V_1 and V_2 of V, if $T[V_1] = T[V_2]$ then $V_1 = V_2$.

   Solution: For the forward direction, suppose T is injective, so $\ker(T) = \{\vec{0}\}$. Then since every subspace of V contains $\vec{0}$, for any subspaces V_1 and V_2 of V we have $\ker(T) \subseteq V_1$ and $\ker(T) \subseteq V_2$, so the fact that $T[V_1] = T[V_2]$ implies $V_1 = V_2$ follows from part (a).

   For the backward direction, we prove the contrapositive. Suppose T is *not* injective, so $\ker(T) \neq \{\vec{0}\}$. Then $\ker(T)$ and $\{\vec{0}\}$ are subspaces of V such that $T[\ker(T)] = T[\{\vec{0}\}] = \{\vec{0}\}$ even though $\ker(T) \neq \{\vec{0}\}$.