Math 217 – Midterm 1

Fall 2019

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
3. No calculators, notes, or other outside assistance allowed.

Student ID Number: ______________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	12	
4	12	
5	13	
6	12	
7	12	
8	12	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The function $f : X \to Y$ is *injective*

   (b) The list $(\vec{v}_1, \ldots, \vec{v}_m)$ of vectors in the subspace V of $\mathbb{R}^n$ is a *basis* of V

   (c) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly dependent*

   (d) For vector spaces V and W, the function $T : V \to W$ is an *isomorphism* from V to W

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) For all 2×2 matrices A and B, if $AB = 0$ then $BA = 0$.

   (b) (3 points) For every 2×2 matrix A, if $A^2 = I_2$ then $A = I_2$ or $A = -I_2$.

   (c) (3 points) For all $n \times n$ matrices A and B, we have $\mathrm{rref}(A+B) = \mathrm{rref}(A) + \mathrm{rref}(B)$.

(Problem 2, Continued).

(d) (3 points) For all finite-dimensional vectors spaces U, V, and W and for all linear transformations $T : U \to V$ and $S : V \to W$, if T is injective and S is surjective and $\mathrm{im}(T) = \ker(S)$, then $\dim(V) = \dim(U) + \dim(W)$.

(e) (3 points) For every $m \times n$ matrix A, if the columns of A span $\mathbb{R}^m$ then the rows of A span $\mathbb{R}^n$.

3. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & b \\ 0 & 1 & 1 \end{bmatrix}$$

with $a, b \in \mathbb{R}$.

(*No justification is required on any part of this problem. Your answers may be written in terms of a and b.*)

(a) (3 points) Assuming that T_A is *not* invertible, what, if anything, can be said about the values of a and b?

(b) (3 points) Assuming that T_A is *not* invertible, find a basis of $\ker(T_A)$.

(c) (3 points) Assuming that T_A is *not* invertible, find a basis of $\operatorname{im}(T_A)$.

(d) (3 points) Assuming now that T_A *is* invertible, what is the second column of A^{-1}?

4. Let $a, b, c, d \in \mathbb{R}$, and consider the subset V of $\mathbb{R}^4$ defined by

$$V = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ b \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} d \\ 0 \\ c \\ 1 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

In each of (a) – (d) below, find values of a, b, c, d which make the given claim true, or else write *none* if no such values exist. (*No justification is required on this problem.*)

(a) (2 points) $V = \mathbb{R}^4$.

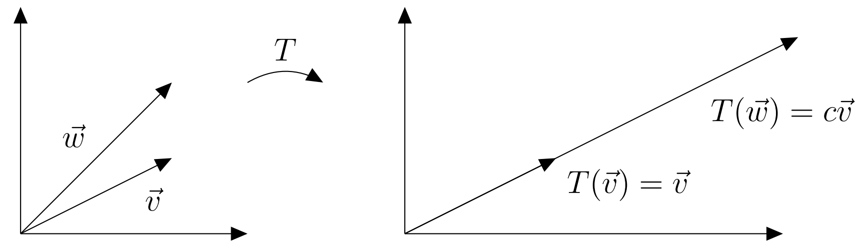
(b) (2 points) V is the kernel of some linear transformation.

(c) (2 points) V is a subspace of $\mathbb{R}^4$ and $\dim V = 1$.

(d) (3 points) V is the solution set of $A\vec{x} = \vec{v}$, where $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(e) (3 points) Assuming that $a = b = c = d = 0$, find a 4×5 matrix such that $V = \text{im}(A)$ or else explain why this is impossible.

5. Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ be the vectors and $T : \mathbb{R}^2 \to \mathbb{R}^2$ the linear transformation shown below, where you may assume that the pictures are drawn accurately and to scale, so in particular c is a positive scalar. Your answers in (b) and (c) may involve c, $\vec{v}$, or $\vec{w}$.



(a) (3 points) Answer each question *yes*, *no*, or *not enough information.*

Is T injective? ____________________

Is T surjective? ____________________

Is T invertible? ____________________

(*No justification needed*)

(b) (4 points) Find a basis of $\ker(T)$, and justify your answer.

(c) (3 points) Assuming $\vec{w} - \vec{v} = 2\vec{e}_2$, find the second column of the standard matrix of T. Be sure to show your work.

(d) (3 points) Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ and $R : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto ℓ and reflection over ℓ, respectively, where $\ell = \mathrm{Span}(\vec{v})$. Find an expression for $T(R(\vec{w}))$ that does not involve T or R, but can include other parameters used in this problem such as c, $\vec{v}$, $\vec{w}$, or P. Be sure to show some work, and circle your final answer.

6. Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. Let

$$V \;=\; \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in \mathbb{R}^{2\times 2} \;:\; a, b \in \mathbb{R} \right\},$$

and let $T : V \to V$ be the map defined by $T(A) = A + A^\top$ for all $A \in V$.

(a) (3 points) Show that V is a subspace of $\mathbb{R}^{2\times 2}$.

(b) (3 points) Find a basis of V. What is $\dim(V)$? (*No justification necessary.*)

(c) (3 points) Show that T is a linear transformation.

(d) (3 points) Find a basis of $\operatorname{im}(T)$ and a basis of $\ker(T)$. (*No justification necessary.*)

7. Let $n \in \mathbb{N}$, let V be an n-dimensional vector space, let $T : V \to V$ and $S : V \to V$ be linear transformations, and suppose $S \circ T$ is the identity transformation on V, so $S(T(v)) = v$ for all $v \in V$.

   (a) (4 points) Prove that $\ker(T) = \{0_V\}$.

   (b) (4 points) Prove that T is surjective.

   (c) (4 points) Does T have to be invertible? Justify your answer.

8. Let V be a vector space, and let $X = (\vec{x}_1, \ldots, \vec{x}_n)$ and $Y = (\vec{y}_1, \ldots, \vec{y}_m)$ be linearly independent lists of vectors in V of lengths n and m, respectively.

   (a) (6 points) Prove that if the list $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent, then $\mathrm{Span}(X) \cap \mathrm{Span}(Y) = \{\vec{0}\}$.

   (b) (6 points) Prove the converse of what you proved in part (a); that is, prove that if $\mathrm{Span}(X) \cap \mathrm{Span}(Y) = \{\vec{0}\}$, then the list $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent.

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