Math 217 – Final Exam

Winter 2022

Solutions

Student ID Number: ______________________ Section: ________

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	16	12	12	14	12	10	12	100
Score:									

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The *kernel* of the linear transformation $T : V \to W$ from the vector space V to the vector space W

   Solution: The *kernel* of the linear transformation $T : V \to W$ is the set $\{\vec{v} \in V \ : \ T(\vec{v}) = \vec{0}_W\}$.

   (b) A *basis* of the vector space V

   Solution: A *basis* of the vector space V is a linearly independent subset of V that spans V.

   (c) The function $T : V \to W$ from the vector space V to the vector space W is a *linear transformation*

   Solution: The function $T : V \to W$ is a *linear transformation* if for all $v_1, v_2 \in V$ and $c \in \mathbb{R}$, we have $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv) = cT(v)$.

   (d) The vector $\vec{v}$ in the vector space V is an *eigenvector* of the linear transformation $T : V \to V$

   Solution: The vector $\vec{v}$ in the vector space V is an *eigenvector* of the linear transformation $T : V \to V$ if $\vec{v} \neq \vec{0}$ and there is $\lambda \in \mathbb{R}$ such that $T(\vec{v}) = \lambda\vec{v}$.

2. State whether each statement is True or False and provide a short proof of your claim. For each part, indicate your answer by clearly writing "T" or "F" in the box on the left.

(a) (4 points) If $(\vec{v}_1, \vec{v}_2)$ and $(\vec{w}_1, \vec{w}_2)$ are bases of the subspaces V and W of $\mathbb{R}^4$, respectively, where $V \neq W$, then $(\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2)$ is a basis of $\mathbb{R}^4$.

Solution: FALSE. For instance, we could let $\vec{v}_1 = \vec{e}_1$, $\vec{v}_2 = \vec{e}_2 = \vec{w}_1$, and $\vec{w}_2 = \vec{e}_3$. Then $V \neq W$ since $\vec{e}_3 \in W \setminus V$, but $(\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2) = (\vec{e}_1, \vec{e}_2, \vec{e}_2, \vec{e}_3)$ is linearly dependent since it has a repeated vector, hence is not a basis of $\mathbb{R}^4$.

(b) (4 points) For every finite-dimensional vector space V, every surjective linear transformation from V to V is injective.

Solution: TRUE. Let V be a vector space of finite dimension $n \in \mathbb{N}$, and let $T : V \to V$ be a surjective linear transformation. Then $\operatorname{im}(T) = V$, so $\dim \operatorname{im}(T) = n$. By Rank-Nullity, it follows that $\dim \ker(T) = 0$, which implies $\ker(T) = \{\vec{0}\}$ and therefore T is injective.

(c) (4 points) Every square matrix A has the same characteristic polynomial as its transpose, $A^\top$.

Solution: TRUE. Let A be an $n \times n$ matrix, and let f_A and $f_{A^\top}$ be the characteristic polynomials of A and $A^\top$, respectively. Then

$$\begin{aligned} f_{A^\top}(x) &= \det(xI_n - A^\top) = \det\big((xI_n)^\top - A^\top\big) \\ &= \det\big((xI_n - A)^\top\big) = \det(xI_n - A) = f_A(x) \end{aligned}$$

since every matrix has the same determinant as its transpose.

(d) (4 points) If the square matrix A is symmetric, then every matrix that is similar to A is diagonalizable.

Solution: TRUE. Let A be a symmetric matrix. Then A is orthogonally diagonalizable by the Spectral Theorem, so we can write $A = QDQ^\top$ where Q is orthogonal and D is diagonal. Now let B be any matrix that is similar to A, and fix an invertible matrix S such that $B = SAS^{-1}$. Then

$$B = SAS^{-1} = SQDQ^\top S^{-1} = SQDQ^{-1}S^{-1} = (SQ)D(SQ)^{-1},$$

so B is diagonalizable.

3. Let $M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and define the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by

$$T(A) \;=\; MA - AM \quad \text{for all } A \in \mathbb{R}^{2\times 2}.$$

(You do not have to prove that T is linear.)

(a) (4 points) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

Solution: Note that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \;=\; \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} c & b+d-a \\ -c & -c \end{bmatrix}.$$

Therefore,

$$[T]_{\mathcal{E}} \;=\; \begin{bmatrix} | & & | \\ [T(E_{11})]_{\mathcal{E}} & \cdots & [T(E_{22})]_{\mathcal{E}} \\ | & & | \end{bmatrix} \;=\; \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

(b) (4 points) Find the characteristic polynomial of T.

Solution: The characteristic polynomial f_T of T is given by

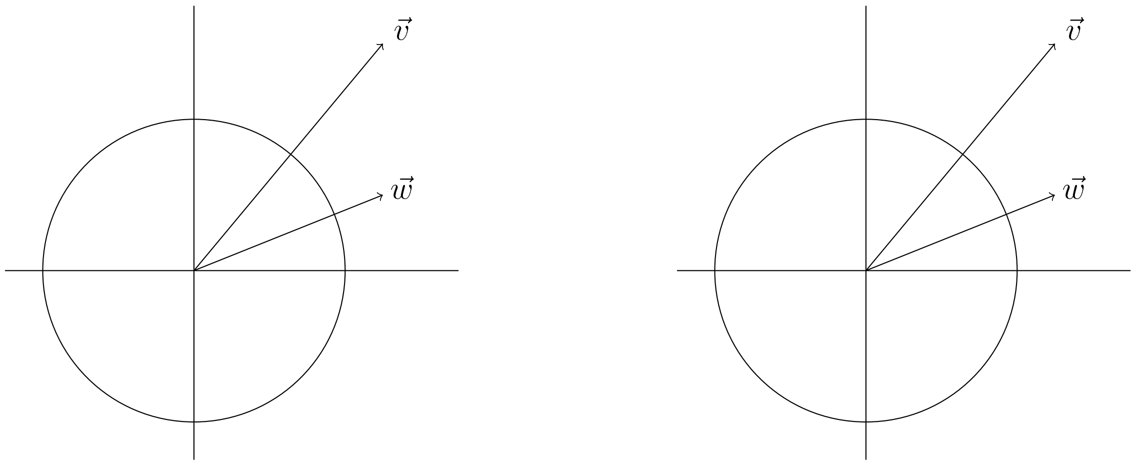
$$f_T(x) \;=\; \det\left(xI_4 - [T]_{\mathcal{E}}\right) \;=\; \det \begin{bmatrix} x & 0 & -1 & 0 \\ 1 & x-1 & 0 & -1 \\ 0 & 0 & x+1 & 0 \\ 0 & 0 & 1 & x \end{bmatrix} \;=\; x^2(x-1)(x+1).$$

(c) (4 points) Diagonalize T; that is, find a basis $\mathcal{B}$ of $\mathbb{R}^{2\times 2}$ and a diagonal matrix D such that $[T]_{\mathcal{B}} = D$.

Solution: From (b), we see that the eigenvalues of T are 0, 0, 1, and -1. From the definition of T we see that M and I_2 belong to $\ker(T)$, and by inspection we see that $\vec{e}_2$ is a 1-eigenvector of $[T]_{\mathcal{E}}$, so $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a 1-eigenvector of T. Finally, a calculation shows that $\ker\left(T + I_2\right)$ is spanned by $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. So we can take

$$\mathcal{B} \;=\; \left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right) \quad \text{and} \quad D = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

4. Below are two copies of the same picture of the unit circle in $\mathbb{R}^2$, along with vectors $\vec{v}$ and $\vec{w}$ lying in the first quadrant. Assume $\vec{v} = \vec{w} + \vec{e}_2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$.



(a) (3 points) Draw and clearly label the vector $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ in the first picture above.

Solution: $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ should be the vector $\vec{e}_2$ in the picture.

(b) (3 points) Draw and clearly label the vectors $\vec{u}_1$ and $\vec{u}_2$ in the second picture above, where $(\vec{u}_1, \vec{u}_2)$ is the orthonormal basis of $\mathbb{R}^2$ obtained by applying the Gram-Schmidt procedure to $(\vec{v}, \vec{w})$.

Solution: $\vec{u}_1$ is the unit vector $\frac{\vec{v}}{\|\vec{v}\|}$, and $\vec{u}_2$ is the unit vector perpendicular to $\vec{u}_1$ that lies in the fourth quadrant.

(c) (3 points) Assuming $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$, find $\det(T)$ in terms of a and b.

Solution: If $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ then $\vec{v} = \begin{bmatrix} a \\ b+1 \end{bmatrix}$, so $A = \begin{bmatrix} a & a \\ b+1 & b \end{bmatrix}$ and thus

$$\det(T) \;=\; \det(A) \;=\; ab - a(b+1) \;=\; -a.$$

(d) (3 points) Assuming $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$, solve the linear system $A\vec{x} = \text{proj}_{\vec{e}_1}(\vec{v})$. (Your answer may involve a or b.)

Solution: Assuming $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$, we have $A = \begin{bmatrix} a & a \\ b+1 & b \end{bmatrix}$ and $\text{proj}_{\vec{e}_1}(\vec{v}) = a\vec{e}_1$, so we must solve the linear system with augmented matrix $\begin{bmatrix} a & a & a \\ b+1 & b & 0 \end{bmatrix}$. Noting from the picture that $a, b > 0$, we find after row reducing that

$$\text{rref} \begin{bmatrix} a & a & a \\ b+1 & b & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -b \\ 0 & 1 & b+1 \end{bmatrix},$$

so the unique solution is $\vec{x} = \begin{bmatrix} -b \\ b+1 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$, where $a, b, c \in \mathbb{R}$. In each part below, determine the set of all real numbers a, b, c that make the given statement true. *(No justification required.)*

   (a) (2 points) A is invertible.

   > **Solution:** $ab(c+1) \neq 0$.
   > Justification: A is invertible iff $\det(A) \neq 0$ iff $ab(c+1) \neq 0$.

   (b) (2 points) Multiplication by A preserves length; that is, for all $\vec{x} \in \mathbb{R}^4$, $\|A\vec{x}\| = \|\vec{x}\|$.

   > **Solution:** None.
   > Justification: A preserves length iff A is orthogonal, which is impossible since the final column of A is not a unit vector, no matter what a, b, c are.

   (c) (2 points) Multiplication by A preserves (4-dimensional) volume; that is, for every parallelepiped P in $\mathbb{R}^4$, the 4-volume of $\{A\vec{x} \,:\, \vec{x} \in P\}$ equals the 4-volume of P.

   > **Solution:** $|ab(c+1)| = 1$.
   > Justification: A preserves volumes iff $|\det(A)| = 1$ iff $|ab(c+1)| = 1$.

   (d) (4 points) A is diagonalizable over $\mathbb{R}$.

   > **Solution:** $a \neq b$ and ($c > 3$ or $c < -1$).
   > Justification: using block matrices, we see that A is diagonalizable over $\mathbb{R}$ iff both 2×2 blocks $B = \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}$ and $C = \begin{bmatrix} c & -1 \\ 1 & 1 \end{bmatrix}$ are diagonalizable over $\mathbb{R}$. But B is diagonalizable over $\mathbb{R}$ iff B is diagonalizable over $\mathbb{C}$ iff $a \neq b$, so our solution follows from the following observations about C: if $c < -1$ or $3 < c$, then f_C has two distinct real roots; if $-1 < c < 3$, then f_C has two non-real complex roots; if $c = -1$ then $\text{gemu}(0) < \text{almu}(0)$; and if $c = 3$ then $\text{gemu}(2) < \text{almu}(2)$.

   (e) (4 points) A is diagonalizable over $\mathbb{C}$.

   > **Solution:** $a \neq b$ and $c \neq 3$ and $c \neq -1$.
   > Justification: essentially the same as that given in part (d). The only difference here is that A is diagonalizable over $\mathbb{C}$ (but not over $\mathbb{R}$) whenever $a \neq b$ and $-1 < c < 3$, since then A has four distinct complex roots, two of them nonreal.

6. Let $u, v \in \mathbb{R}^4$ and let $A = \begin{bmatrix} u & v & u+v & u-v \end{bmatrix} \in \mathbb{R}^{4\times 4}$. Suppose that the vectors $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ are eigenvectors of A, with eigenvalues 1 and 2, respectively.

(a) (4 points) Find a basis of $\ker(A)$, and justify your answer.

Solution: From the equations $A\vec{x} = \vec{x}$ and $A\vec{y} = 2\vec{y}$ we get

$$\vec{x} \;=\; u + u + v + u - v \;=\; 3u \qquad \text{and} \qquad 2\vec{y} \;=\; v + u + v - (u - v) \;=\; 3v,$$

which implies $u = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $v = \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$, so $A = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 1 & 2 & 3 & -1 \\ 1 & -2 & -1 & 3 \end{bmatrix}$. Observing that (u, v) is linearly independent while $u + v$ and $u - v$ are redundant in the list of columns of A, we see that $\text{rank}(A) = 2$, so $\dim(\ker A) = 2$ by Rank-Nullity. It follows that the vectors $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, which belong to $\ker(A)$ by inspection, form a basis of $\ker A$. (Alternatively, we could row reduce A and use the usual procedure to find a basis of $\ker A$.)

Solution: By inspection we see that the vectors $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ belong to $\ker(A)$. Since $\vec{x}$ and $\vec{y}$ are eigenvectors of A corresponding to distinct nonzero eigenvalues, $\{\vec{x}, \vec{y}\}$ is a linearly independent subset of $\text{im}(A)$, so $\text{rank}(A) \geq 2$. This implies $\dim\ker(A) \leq 2$ by Rank-Nullity, so we conclude that $\dim\ker(A) = 2$ and $(\vec{a}, \vec{b})$ is in fact a basis of $\ker(A)$.

Solution: By inspection we see that the vectors $\vec{a}$ and $\vec{b}$ (as above) belong to $\ker(A)$, and are therefore eigenvectors of A with eigenvalue 0. Since unions of linearly independent subsets of distint eigenspaces are still linearly independent (by 7.3.3, or a result from the worksheets), the set $\{\vec{a}, \vec{b}, \vec{x}, \vec{y}\}$ is a linearly independent subset of $\mathbb{R}^4$, and is thus a basis of $\mathbb{R}^4$. This implies that $(\vec{a}, \vec{b})$ spans $\ker(A)$, and is therefore a basis of $\ker(A)$.

(b) (4 points) Orthogonally diagonalize A. That is, explicitly find an orthogonal matrix

Q and a diagonal matrix D such that $Q^\top AQ = D$. *(No justification required.)*

Solution:

$$Q = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}.$$

(c) (4 points) Either write down a triangular matrix that has the same characteristic polynomial as A but is *not* similar to A, if this is possible, or else state that this is impossible. Briefly justify your answer.

Solution: This is possible. For instance, we can let $B = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}$. Then B is not diagonalizable, since $\operatorname{gemu}(0) = 1 < 2 = \operatorname{almu}(0)$, so B cannot be similar to A since A is diagonalizable. However, the characteristic polynomial of B is $x^2(x-1)(x+1)$, just like A.

7. Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation.

   (a) (4 points) Prove that every eigenvector of T belongs to $\ker(T)$ or $\operatorname{im}(T)$.

   Solution: Let $\vec{v} \in V$ be an eigenvector of T, say with corresponding eigenvalue λ. If $\lambda = 0$, then $T(\vec{v}) = 0\vec{v} = \vec{0}$, so $\vec{v} \in \ker(T)$. If $\lambda \neq 0$, then $T(\lambda^{-1}\vec{v}) = \lambda^{-1}T(\vec{v}) = \lambda^{-1}\lambda\vec{v} = \vec{v}$, so $\vec{v} \in \operatorname{im}(T)$. Either way, we see that $\vec{v} \in \ker(T) \cup \operatorname{im}(T)$, completing the proof.

   (b) (6 points) Prove that if T is diagonalizable, then $\ker(T) \cap \operatorname{im}(T) = \{\vec{0}\}$.

   Solution: Suppose T is diagonalizable, which means there is an eigenbasis $\mathcal{B}$ of V for T, say $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\vec{y} \in \ker(T) \cap \operatorname{im}(T)$, and fix $\vec{x} \in V$ such that $T(\vec{x}) = \vec{y}$. Let $c_1, \ldots, c_n$ be the unique scalars such that $\vec{x} = \sum_{i=1}^n c_i\vec{v}_i$. Then

$$\vec{y} \;=\; T(\vec{x}) \;=\; T\left(\sum_{i=1}^n c_i\vec{v}_i\right) \;=\; \sum_{i=1}^n c_i T(\vec{v}_i) \;=\; \sum_{i=1}^n c_i\lambda_i\vec{v}_i.$$

   Now, let $I = \{i \;:\; \lambda_i \neq 0\}$, and note that since $\vec{y} \in \ker(T)$, we must have $c_i\lambda_i = 0$ for each $i \in I$, which implies $c_i = 0$ for each $i \in I$. But this means $\vec{x} \in \ker(T)$, so $\vec{y} = T(\vec{x}) = \vec{0}$, and we conclude that $\ker(T) \cap \operatorname{im}(T) = \{\vec{0}\}$.

8. Let A be an $m \times n$ matrix. Let $V = (\ker A)^\perp = \operatorname{im} A^\top$ and $W = \operatorname{im} A = (\ker A^\top)^\perp$, and let $T : V \to W$ and $S : W \to V$ be the linear transformations defined by

$$T(\vec{x}) = A\vec{x} \quad \text{and} \quad S(\vec{y}) = A^\top \vec{y} \quad \text{for all } \vec{x} \in V \text{ and } \vec{y} \in W.$$

(a) (6 points) Prove that T is an isomorphism.

Solution: Since T is given by matrix multiplication, it is linear, so it will be enough to show that T is injective and surjective. For injectivity, let $\vec{x} \in V$ and suppose $T(\vec{x}) = \vec{0}$. Then $A\vec{x} = \vec{0}$, so $\vec{x} \in \ker A$, but also $\vec{x} \in V = (\ker A)^\perp$, so we must have $\vec{x} = \vec{0}$ since $\ker(A) \cap \ker(A)^\perp = \{\vec{0}\}$. This shows $\ker(T) = \{\vec{0}\}$, so T is injective. To show T is surjective, let $\vec{y} \in W = \operatorname{im} A$ be arbitrary, and fix $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{y}$. Let $\vec{z} = \operatorname{proj}_V(\vec{x}) \in V$, so $\vec{z} - \vec{x} \in V^\perp = \ker A$. Then

$$T(\vec{z}) \;=\; A\vec{z} \;=\; A(\vec{x} + (\vec{z} - \vec{x})) \;=\; A\vec{x} + A(\vec{z} - \vec{x}) \;=\; \vec{y} + \vec{0} \;=\; \vec{y}.$$

Since $\vec{y} \in W$ was arbitrary, this shows T is surjective.

Alternatively: Having shown just one of injectivity or surjectivity, one could use Rank-Nullity to show $\dim V = \dim W$, and then argue from there that T must be bijective.

Since part (a) holds for arbitrary $A \in \mathbb{R}^{m\times n}$, it follows that S is also an isomorphism, so $S \circ T$ is an isomorphism from V to V. (You do not have to prove this.)

(b) (6 points) Prove that $\det(S \circ T)$ is the product of all the (possibly repeated) nonzero eigenvalues of $A^\top A$.

Solution: Since $A^\top A$ is symmetric, it is orthogonally diagonalizable by the Spectral Theorem, so fix an orthonormal basis $(\vec{u}_1, \ldots, \vec{u}_k, \vec{u}_{k+1}, \ldots, \vec{u}_n)$ of $\mathbb{R}^n$ consisting of eigenvectors of $A^\top A$ and with $(\vec{u}_{k+1}, \ldots \vec{u}_n)$ a basis of $\ker(A^\top A) = \ker A$, so that $\mathcal{B} = (\vec{u}_1, \ldots, \vec{u}_k)$ is a basis of V. For each $1 \le i \le n$, fix λ_i such that $A^\top A\vec{u}_i = \lambda_i \vec{u}_i$, so $\lambda_1, \ldots, \lambda_k$ are the nonzero eigenvalues of $A^\top A$. Then $(S \circ T)(\vec{u}_i) = A^\top A \vec{u}_i = \lambda_i \vec{u}_i$ for each $1 \le i \le k$, so

$$\det(S \circ T) \;=\; \det[S \circ T]_{\mathcal{B}} \;=\; \det \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \;=\; \prod_{i=1}^{k} \lambda_i.$$