Worksheet 7: Matrices, Linear Transformations, and Invertibility (§§2.3, 2.4)

Problem 1: "Canceling matrices."

Suppose that A is a 2×3 matrix and B is a 3×2 matrix, so that it makes sense to consider both AB and BA. Consider the following proof of the statement: "if $AB = I_2$, then $BA = I_3$." (Recall that for each $n \geq 1$, I_n is the $n \times n$ identity matrix).

Proof. Suppose $AB = I_2$.

$\Rightarrow B(AB) = BI_2$
$\Rightarrow B(AB) = B$
$\Rightarrow (BA)B = B$
$\Rightarrow (BA)B = I_3B$
$\Rightarrow BA = I_3$ (canceling B). □

Is this proof correct? Is the statement even true? What can you conclude about "canceling matrices?"

Solution: The statement is not true, as can be seen using the counterexample

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So the proof cannot be correct! The invalid step is the last one, where B is "canceled" from both sides to obtain $BA = I_3$ from $(BA)B = I_3B$. The problem is that you can only cancel a matrix from both sides of an equation *if* that matrix is invertible, which B is not; what you are really doing when you "cancel" a matrix is multiplying by its inverse and then simplifying.

Problem 2: —jective functions.

Determine whether each the following linear transformations is injective, surjective, bijective, or none of these; for those that are bijective, find the inverse transformation.

(a) The counterclockwise rotation $\mathrm{Rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ through the angle θ about the origin.

(b) The reflection $\mathrm{Ref}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ about the line ℓ through the origin that makes an angle of θ with the positive x-axis.

(c) The orthogonal projection $\mathrm{proj}_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ onto the line ℓ passing through the origin.

(d) The vertical shear $S : \mathbb{R}^2 \to \mathbb{R}^2$ determined by $S(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$ and $S(\vec{e}_2) = \vec{e}_2$.

(e) The matrix-transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

(f) The matrix-transformation $T_B : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T_B(\vec{x}) = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$, where

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(g) The matrix-transformation $T_C : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T_C(\vec{x}) = C\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$, where

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Solution: The maps in (a), (b), (d), and (e) are bijective, with the following inverses:

$$(a)\ R_\theta^{-1} = R_{-\theta}, \quad (b)\ I_\theta^{-1} = I_\theta, \quad (d)\ S^{-1}(\vec{x}) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\vec{x}, \quad (e)\ T_A^{-1}(\vec{x}) = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\vec{x}.$$

The map in (c) is neither injective nor surjective. The map in (f) is injective but not surjective, and the map in (g) is surjective but not injective.

Problem 3: Invertible matrices

Let A be an $n \times n$ matrix. Recall that A is *invertible*, by definition, if there exists an $n \times n$ matrix B such that $AB = BA = I_n$.

(a) Prove that if A is invertible, then there is a *unique* matrix B such that $AB = BA = I_n$. (This allows us to call B the *inverse* of A, and write $B = A^{-1}$).

Solution: Suppose $AB = BA = I_n$ and also $AC = CA = I_n$. Then $B = I_nB = (CA)B = C(AB) = CI_n = C$.

(b) Prove that if A is invertible, then for all $\vec{b} \in \mathbb{R}^n$ the linear system $A\vec{x} = \vec{b}$ has a unique solution.

Solution: If A is invertible, then the linear system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

(c) Characterize the invertibility of A in terms of its *rank*. In other words, complete the statement: "A is invertible if and only if rank(A) . . . " (No proof necessary, though you should think about why your statement is true!)

Solution: The $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$.

(d) Now characterize the invertibility of A in terms of its reduced row echelon form.

Solution: The $n \times n$ matrix A is invertible if and only if $\text{rref}(A) = I_n$.

(e) Find the inverse of the invertible matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -4 \\ 0 & -1 & 1 \end{bmatrix}$. Does your technique generalize to work for *any* square matrix?

Solution: $A^{-1} = \begin{bmatrix} -3 & -2 & -2 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$.

Problem 4: Inverting compositions, matrix products, and linear transformations

(a) If $f : X \to Y$ and $g : Y \to Z$ are invertible functions, is their composition $g \circ f$ invertible? If so, what is its inverse?

Solution: If $f : X \to Y$ and $g : Y \to Z$ are invertible, then so is $g \circ f$, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(b) If A and B are invertible $n \times n$ matrices, is AB invertible? If so, what is its inverse?

Solution: If A and B are invertible $n \times n$ matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If $A_1, \ldots, A_k$ are invertible $n \times n$ matrices, what is $(A_1 \cdots A_k)^{-1}$?

Solution: If $A_1, \ldots, a_k$ are invertible $n \times n$ matrices, then $A_1 \cdots A_k$ is invertible and

$$(A_1 \cdots A_k)^{-1} = A_k^{-1} \cdots A_1^{-1}.$$

(d) Let V and W be vector spaces, and suppose that $T : V \to W$ is a bijective linear transformation[†] from V to W. Prove that T^{-1} is linear.[*]

[†]It will come up soon anyway, so we may as well start using the word now: a bijective linear transformation between vector spaces is called an *isomorphism*. Furthermore, we say that V is *isomorphic* to W, written $V \cong W$, if there exists an isomorphism from V to W.

[*]Be careful here. The proof is not long, but it can be a little tricky. Note that by this problem, a linear transformation is bijective if and only if it is invertible, so that an *isomorphism* could alternatively be defined as an *invertible linear transformation*.

Solution: Let $w, z \in W$, and write $x = T^{-1}(w)$ and $y = T^{-1}(z)$, so that $T(x) = w$ and $T(y) = z$. Since T is linear, we have $T(x+y) = T(x) + T(y) = w + z$, which, after applying T^{-1} to each side, gives us

$$T^{-1}(w+z) \;=\; T^{-1}(T(x+y)) \;=\; x+y \;=\; T^{-1}(w) + T^{-1}(z).$$

Similarly, given $w \in W$ and $c \in \mathbb{R}$, write $x = T^{-1}(w)$ so that $T(x) = w$, and observe that since $T(cx) = cT(x) = cw$ by linearity of T, we have

$$T^{-1}(cw) \;=\; T^{-1}(T(cx)) \;=\; cx \;=\; cT^{-1}(w).$$

Problem 5: Matrices and linear maps.

Let A be an $m \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear transformation.

In parts (a) and (b), circle the appropriate words to make TRUE statements, and then justify your claims using rref(A).

(a) If $m < n$, then T_A (is / is not) (injective / surjective).

(b) If $m > n$, then T_A (is / is not) (injective / surjective).

(c) Using (a) and (b), prove that if A is not square, then T_A is not an isomorphism.

Solution:

(a) If $m < n$, then T_A is not injective. To see this, note that if $m < n$ then rref(A) will have some columns that are not pivot columns, which will correspond to free variables in the homogeneous linear system $A\vec{x} = \vec{0}$, which means that $A\vec{x} = \vec{0}$ will have infinitely many solutions. Thus there are infinitely many vectors $\vec{x} \in \mathbb{R}^n$ such that $T_A(\vec{x}) = \vec{0}$, showing that T_A is not injective.

(b) If $m > n$, then T_A is not surjective. To see this, note that if $m > n$ then rref(A) will have rows consisting entirely of zeros. So if we perform in reverse the sequence of elementary row operations that transforms A into rref(A) upon the vector $\vec{e}_m$, we will obtain a vector $\vec{b} \in \mathbb{R}^m$ that is not in the image of T_A, showing that T_A is not surjective.

(c) Suppose A is not square, so $m \neq n$. If $m < n$, then by part (a) T_A is not injective, hence not bijective, hence not invertible. On the other hand if $m > n$, then by part (b) T_A is not surjective, hence not bijective, hence not invertible. So we see that if A is not square, then T_A is not invertible, and therefore is not an isomorphism.