

Worksheet 2: Linear Equations and Matrices (§§1.1–1.2)

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Problem 1. Consider the linear system

$$\begin{aligned}4z - 9 &= 3x + 6y \\3z - 1 &= w + 2x + y \\3z - 3x &= 2w - 1 \\w + 4x + 5y + 7 &= 9z\end{aligned}$$

(In answering the questions below, write the variables w, x, y, z in their natural order).

- (a) Write the coefficient matrix of this linear system.
- (b) Write the augmented matrix of this linear system.
- (c) Find the reduced row echelon form (rref) of the augmented matrix of this linear system.
- (d) Express the solution set of this linear system as a set of vectors. Your solution should have the form

$$\{\vec{c} + t_1\vec{v}_1 + \cdots + t_k\vec{v}_k : t_1, \dots, t_k \in \mathbb{R}\},$$

where $\vec{c} \in \mathbb{R}^4$ and t_1, \dots, t_k are parameters. (You'll have to figure out what k should be).

Solution:

(a)
$$\begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \\ 1 & 4 & 5 & -9 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d)
$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} : y \in \mathbb{R} \right\}.$$

Problem 2. Consider a matrix A , which we transform into the matrix

$$B = \begin{bmatrix} 0 & a & 0 & 0 & b \\ c & 0 & d & 0 & e \\ 0 & 0 & 0 & 1 & f \end{bmatrix}$$

by a sequence of elementary row operations. Assume $B = \text{rref}(A)$.

- (a) What can we say about the constants a through f ? What is the first column of A ?

Solution: $a = 1$, $c = 0$, $d = 1$, and b, e, f can be anything. The first column of A is $\vec{0}$.

- (b) We (temporarily!*) define the *rank* of a matrix to be the number of leading 1s in its reduced row echelon form. What is the rank of the matrix A ?

Solution: 3

- (c) If $[A \mid \vec{0}]$ is the augmented matrix of a linear system, is the system consistent? If so, how many solutions does it have?

Solution: Yes, the system would be consistent, with infinitely many solutions.

- (d) Now, going further, suppose that $[A \mid \vec{v}]$ (where $\vec{v} \neq \vec{0}$) is the augmented matrix for a linear system. Is the system consistent? If so, how many solutions are there?

Solution: Again, the system would be consistent with infinitely many solutions.

- (e) Continue to suppose that $[A \mid \vec{v}]$ (where $\vec{v} \neq \vec{0}$) is the augmented matrix of a linear system. Can you change the last row of B so that the resulting linear system has no solutions? How does your answer depend on \vec{v} ? Can you change the last row of B to ensure that there is a *unique* solution?

Solution: By changing the last row of B to have all zeros, we could make the system have no solution, as long as the row operations that change A into B do not also change \vec{v} into a vector whose third component is zero. But there is no way to make the system have a unique solution, since B has more columns than rows.

Mathematical Proofs

Problem 3. DeMorgan's Law states that if A and B are sets, then $(A \cup B)^C = A^C \cap B^C$.

*Eventually we will define $\text{rank}(A) = \dim \text{im}(A)$, but we don't know what "dim" or "im" mean yet (although you're welcome to guess, if you can!).

- (a) Draw a Venn diagram to understand DeMorgan's Law and why it should be true.
- (b) Start to think about how you might *prove* DeMorgan's Law. What definitions would you need to understand? What kind of logical structure does the theorem have? How does this help you structure your proof? How would the proof begin? — and how should it end?
- (c) A useful strategy is to think of all the different ways that we can restate the conclusion we are trying to prove. For example, one technique for showing that two sets X and Y are equal is to show, separately, that both $X \subseteq Y$ and $Y \subseteq X$.[†] Using only this technique, outline a proof of DeMorgan's Law.
- (d) Refine your outline of the proof of DeMorgan's Law by incorporating the following proof techniques:
 - (i) The standard way to prove that $X \subseteq Y$ is to say “Let x be an element in X ” and then somehow show that x also belongs to Y .
 - (ii) The standard way to prove that an element x belongs to the intersection $Z \cap W$ is to show separately that $x \in Z$ and that $x \in W$.
- (e) Prove DeMorgan's Law by filling in this outline to obtain a complete argument.

Solution: Let A and B be sets, and let $x \in (A \cup B)^C$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$, which implies $x \in A^C$ and $x \in B^C$ and therefore $x \in A^C \cap B^C$. This shows $(A \cup B)^C \subseteq A^C \cap B^C$. Conversely, suppose $x \in A^C \cap B^C$. This means that $x \in A^C$ and $x \in B^C$, which implies $x \notin A$ and $x \notin B$, so $x \notin A \cup B$, and therefore $x \in (A \cup B)^C$. Thus $(A \cup B)^C \subseteq A^C \cap B^C$ and $(A \cup B)^C \supseteq A^C \cap B^C$, which means that $(A \cup B)^C = A^C \cap B^C$ as desired.

Problem 4. Define a *linear function*[‡] from \mathbb{R} to \mathbb{R} to be a function

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

for which $f(t) = mt$ for some constant m . *Using only this definition*, prove that for any linear function $f : \mathbb{R} \rightarrow \mathbb{R}$ and any real constants a and b , $f(ax + by) = af(x) + bf(y)$ for all $x, y \in \mathbb{R}$.

Solution: Let f be a linear function, and let $m \in \mathbb{R}$ be such that $f(t) = mt$ for all $t \in \mathbb{R}$. Suppose a, b, x, y are any real numbers. Then

$$f(ax + by) = m \cdot (ax + by) = a \cdot (mx) + b \cdot (my) = af(x) + bf(y).$$

Problem 5. It is a famous theorem of Pythagoras that $\sqrt{2}$ is an irrational number.

[†]From our **Unreasonably Useful Result** in the *Joy of Sets*.

[‡]Careful! In the past you may have used *linear* to mean “polynomial of degree one” (as in *linear*, *quadratic*, *cubic*, etc.) In this class we will use a different definition that vastly generalizes the one given in this problem. You will become thoroughly familiar with this more general definition of *linear* on the next few worksheets.

- (a) One way to prove this theorem is by *contradiction*: in this method of proof, we *assume* that the theorem is false, and from that assumption derive a logical absurdity (such as $0 = 1$ or “ P and not P ” where P is some statement). A usual first line in such a proof is “Suppose for contradiction that [the theorem] is false.” What is the first line here?

Solution: Suppose for contradiction that $\sqrt{2}$ is rational.

- (b) Complete the proof that $\sqrt{2}$ is irrational. You may use the following facts without proof:

Fact 1: Every positive rational number can be written in the form m/n where m and n are positive integers with no common divisors.

Fact 2: The square of any odd integer is odd.

Solution: Suppose for contradiction that $\sqrt{2}$ is rational. Then by Fact 1 we can write $\sqrt{2} = m/n$ where m and n are positive integers having no common divisors. Then $2 = m^2/n^2$, so $2n^2 = m^2$, which implies that m is even, say $m = 2k$ where k is a positive integer. Then $2n^2 = (2k)^2 = 4k^2$, so $n^2 = 2k^2$, which implies that n is even by Fact 2. But now m and n are both multiples of 2, contradicting our assumption that m and n have no common divisors. Therefore $\sqrt{2}$ is irrational.