Worksheet 6: Geometric Transformations (§§2.2, 2.3)

We have seen that for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, there is a unique $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$; we will call this matrix A the *standard matrix* of T. On this worksheet we will find the standard matrices of various natural geometric transformations.

Problem 1: Rotations.
For each $\theta \in \mathbb{R}$, let $\mathrm{Rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be counter-clockwise rotation about the origin through an angle of θ (measured in radians).

(a) Give an intuitive *geometric* argument that Rot_θ is linear. (You should draw some pictures, but do not write any equations or try to write out a formal proof).

(b) Confirm that Rot_θ is indeed linear by finding a matrix A_θ such that $\mathrm{Rot}_\theta(\vec{x}) = A_\theta\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. For what values of θ does $A_\theta = I_2$?

Solution: $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. $A_\theta = I_2$ when θ is a multiple of 2π.

(c) Given a pair of rotations Rot_θ and Rot_ϕ, what sort of transformation (geometrically speaking) is the composite function $\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta$? In general, are the functions $\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta$ and $\mathrm{Rot}_\theta \circ \mathrm{Rot}_\phi$ equal, or different? Can you give a simple description of each?

Solution: If Rot_θ and Rot_ϕ are rotations, then $\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta$ is also a rotation, namely

$$\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta = \mathrm{Rot}_{\phi+\theta}.$$

Since addition is commutative, it follows that $\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta = \mathrm{Rot}_\theta \circ \mathrm{Rot}_\phi$.

(d) How could you compute $(\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta)(\vec{x})$ using matrix-vector products?

Solution: $(\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta)(\vec{x}) = A_\phi(A_\theta\vec{x})$.

Problem 2: A brief digression on matrix multiplication.

We have defined matrix-vector products of the form $A\vec{x}$, where the number of columns of A is equal to the number of components in $\vec{x}$, and we can use this to define general matrix-matrix products. If A and B are matrices with $B = [\vec{b}_1 \cdots \vec{b}_n]$, so that $\vec{b}_i$ is the ith column of B, we define

$$AB \;=\; \left[A\vec{b}_1 \cdots A\vec{b}_n\right] \qquad (*)$$

provided the expression on the right makes sense. There are various ways of defining matrix multiplication, and you may be aware of other ways from past classes, but all are equivalent to the definition just given. You may use whatever definition is familiar to you for computing purposes, but you should also remember equation $(*)$ for future use.

(a) If A is an $m \times n$ matrix and B is a $p \times q$ matrix, what must be true in order for AB to make sense? If AB *is* defined, what size is it?

Solution: If A is $m \times n$ and B is $p \times q$, then n must equal p in order for the product AB to be defined; if $n = p$, then AB is an $m \times q$ matrix.

(b) Give an example of two matrices A and B such that the product AB is defined, but the product BA is not defined.

Solution: For instance, we could let $A = [1]$ and $B = [1\ 1]$.

(c) Suppose A and B are two matrices such that both AB and BA are defined. Must AB and BA be the same size? If AB and BA *are* the same size, must they be equal? Either prove your claims or give counterexamples!

Solution: Just because AB and BA are both defined, they need not be the same size; for instance, A could be any non-square matrix and B its transpose. However, even if AB and BA are both defined and the same size, they need not be equal! For instance,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(d) Suppose $f : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations with standard matrices $B \in \mathbb{R}^{k \times n}$ and $A \in \mathbb{R}^{m \times k}$, respectively. Then the function $g \circ f : \mathbb{R}^n \to \mathbb{R}^m$ is also linear, say with standard matrix $C \in \mathbb{R}^{m \times n}$. Explain how to find C in terms of A and B. Can you prove your claim?

Solution: $C = AB$. To prove this, note that for each $1 \leq j \leq n$ we have

$$C\vec{e}_j \;=\; (g \circ f)(\vec{e}_j) \;=\; g\big(f(\vec{e}_j)\big) \;=\; A(B\vec{e}_j),$$

and thus by our definition of matrix multiplication, $C = \begin{bmatrix} | & & | \\ A(B\vec{e}_1) & \cdots & A(B\vec{e}_n) \\ | & & | \end{bmatrix} = AB.$

Problem 3: Orthogonal projections.

For each line ℓ through the origin in $\mathbb{R}^2$, let $\mathrm{proj}_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto ℓ.

(a) Verify that if $\vec{w}$ is a vector parallel to ℓ, then

$$\text{proj}_\ell(\vec{x}) \;=\; \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

for all $\vec{x} \in \mathbb{R}^2$. (Remember Calc 3, and draw a picture!)

Solution: $\text{proj}_\ell(\vec{x}) = c\vec{w}$ for some $c \in \mathbb{R}$. To find c, we use the fact that $\vec{x} - c\vec{w}$ is perpendicular to $\vec{w}$, so $(\vec{x} - c\vec{w}) \cdot \vec{w} = 0$. Solving this equation for c, we find $c = \dfrac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$.

(b) What happens to the formula above if $\vec{w}$ is a *unit* vector?

Solution: It simplifies to $\text{proj}_\ell(\vec{x}) = (\vec{x}\cdot\vec{w})\vec{w}$.

(c) Show that proj_ℓ is linear by finding a matrix A_ℓ such that $A_\ell\vec{x} = \text{proj}_\ell(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$. [Hint: let $\vec{u}$ be a unit vector parallel to ℓ, and write A_ℓ in terms of the components of $\vec{u}$].

Solution: For any unit vector $\vec{u} = [u_1\ u_2]^T$ parallel to ℓ, we have

$$A_\ell = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \;=\; \vec{u}\vec{u}^T.$$

Note that this matrix is *symmetric*, i.e., equal to its own transpose.

(d) Find the matrix A_ℓ where ℓ is the line through the origin in $\mathbb{R}^2$ with slope 2.

Solution: $A_\ell = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(e) The textbook writes $\vec{x}^{\parallel}$ for $\text{proj}_\ell(\vec{x})$. Why? It also writes $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$. Draw a picture to explain what this means.

Problem 4: Reflections.

For each $\theta \in \mathbb{R}$, let $\text{Ref}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of points in the plane over the line ℓ through the origin that makes an angle θ with the positive x-axis.

(a) Give an intuitive *geometric* argument (using pictures) that Ref_θ is linear.

(b) Try to find a matrix B_0 such that $\text{Ref}_0(\vec{x}) = B_0\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

Solution: $B_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(c) In general, can you find a matrix B_θ such that $\mathrm{Ref}_\theta(\vec{x}) = B_\theta \vec{x}$ for all $\vec{x} \in \mathbb{R}^2$?

[Hint: there are several ways to go about doing this. For instance, you might write B_θ using the angle θ, or using a unit vector $\vec{u}$ that makes an angle θ with the positive x-axis, or by combining B_0 with R_θ in the right way. It is interesting to compare what you get using each of these methods!]

Solution: In terms of θ, $B_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$. Note that $\mathrm{Ref}_\theta = R_\theta \circ \mathrm{Ref}_0 \circ R_\theta^{-1}$, so (writing A_θ as in Problem 1) we have $B_\theta = A_\theta B_0 A_{-\theta}$. If $\vec{u}$ is a unit vector that makes an angle θ with the positive x-axis, then for all $\vec{x} \in \mathbb{R}^2$ we have

$$\mathrm{Ref}_\theta(\vec{x}) \;=\; 2\mathrm{proj}_\ell(\vec{x}) - \vec{x} \;=\; 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x},$$

which means that $B_\theta = \begin{bmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{bmatrix}$.