Worksheet 20: Determinants (§§6.1,6.2)

Let A be an $n \times n$ matrix, with (i, j)-entry a_{ij} for each $1 \leq i, j \leq n$.

- A *pattern* P in A is a list $(a_{1j_1}, \ldots, a_{nj_n})$ of n entries of A that contains exactly one entry from each row of A and one entry from each column of A. If $P = (a_{1j_1}, \ldots, a_{nj_n})$ is a pattern in A, write $\text{prod}_A(P)$ for the product

$$\text{prod}_A(P) = a_{1j_1} \cdots a_{nj_n}$$

  of the entries in P.

- If $P = (a_{1j_1}, \ldots, a_{nj_n})$ is a pattern in A, then an *inversion* in P is a pair $(a_{i_1j_1}, a_{i_2j_2})$ of entries in P such that $i_1 < i_2$ but $j_1 > j_2$.

- If P is a pattern in A, then the *sign* of P, written $\text{sgn}(P)$, is defined to be 1 if P has an even number of inversions, and -1 if P has an odd number of inversions.

Definition: If A is an $n \times n$ matrix, the *determinant* of A is defined to be the number

$$\det(A) \; = \; \sum_P \text{sgn}(P)\text{prod}_A(P),$$

where the sum is taken over all possible patterns P in the matrix A.

Example: Here is a pattern in a 3×3 matrix with lines representing the inversions:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

Problem 1.

(a) Illustrate all possible patterns (along with their inversions) of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as in the example above, and compute the sign of each; then give a formula for $\det(A)$.

(b) Illustrate all possible patterns (along with their inversions) of the 3×3 matrix $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ as in the example above, and compute the sign of each; then give a formula for $\det(B)$.

(c) How many patterns does an $n \times n$ matrix have?

(d) What is the determinant of the matrix given in the example above?

Solution:

(a) $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

(b) $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei - afh - bdi + bfg + cdh - ceg$

(c) $n!$

(d) -9

Problem 2. What is the determinant of a triangular matrix? Justify your answer.

Solution: The determinant of a triangular matrix A is the product of the diagonal entries of A. To see this, let A be an $n \times n$ upper triangular matrix. If P is any pattern in A that does not simply consist of the diagonal entries in A, then in the first column of A for which P does *not* choose the diagonal entry, P must choose an entry *below* the main diagonal, which will be 0, making $\text{prod}_A(P) = 0$. Thus the only pattern through A that contributes a nonzero term to $\det A$ will be the diagonal pattern, which contains no inversions so its sign is 1. It follows that $\det A$ is indeed the product of the diagonal entries in A. The argument for lower triangular A is similar.

Problem 3. Let A be a square matrix. What is $\det(A^\top)$ in terms of $\det(A)$? Justify your answer.

Solution: For any square matrix A, $\det A^\top = \det A$. To see this, let A be a square matrix, and for any pattern P in A, let $P^\top$ be the "transpose" pattern which contains the entry (i, j) if and only if P contains the entry (j, i). It is easy to see that $P^\top$ is indeed a pattern, and that changing from P to $P^\top$ defines a bijection of the set of all patterns with itself. Moreover, it is easy to check that $\text{sgn}(P) = \text{sgn}(P^\top)$ and $\text{prod}_{A^\top}(P^\top) = \text{prod}_A(P)$ for every pattern P. Thus

$$\begin{aligned} \det A^\top &= \sum_P \text{sgn}(P)\text{prod}_{A^\top}(P) = \sum_{P^\top} \text{sgn}(P^\top)\text{prod}_{A^\top}(P^\top) \\ &= \sum_{P^\top} \text{sgn}(P^\top)\text{prod}_A P = \sum_P \text{sgn}(P)\text{prod}_A(P) = \det A. \end{aligned}$$

Problem 4. Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$ be an $n \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and let $c \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^n$.

(a) What is $\det \begin{bmatrix} | & | & & | \\ c\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$?

Solution: Let $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$ and let $B = [c\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$. Then since every pattern P in B contains exactly one entry from the first column of B, we have

$$\begin{aligned} \det B &= \sum_P \operatorname{sgn}(P)\operatorname{prod}_B(P) = \sum_P \operatorname{sgn}(P) c \cdot \operatorname{prod}_A(P) \\ &= c \cdot \sum_P \operatorname{sgn}(P)\operatorname{prod}_A(P) = c \cdot \det A. \end{aligned}$$

(b) What is $\det \begin{bmatrix} | & | & & | \\ \vec{a}_1 + \vec{b} & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$?

Solution: Let $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$, let $B = [\vec{b} \ \ \vec{a}_2 \ \cdots \ \vec{a}_n]$ and let $C = [\vec{a}_1 + \vec{b} \ \ \vec{a}_2 \ \cdots \ \vec{a}_n]$. We claim that $\det C = \det A + \det B$. To see this, observe that

$$\begin{aligned} \det C &= \sum_P \operatorname{sgn}(P)\operatorname{prod}_C(P) = \sum_P \operatorname{sgn}(P)\Big(\operatorname{prod}_A(P) + \operatorname{prod}_B(P)\Big) \\ &= \sum_P \operatorname{sgn}(P)\operatorname{prod}_A(P) + \sum_P \operatorname{sgn}(P)\operatorname{prod}_B(P) = \det A + \det B. \end{aligned}$$

(c) Do your claims in (a) and (b) above carry over to any column of A, and not just its first? How about to any row?

Note: the fancy way of expressing this is to say that the determinant function is a multilinear *function of both the columns and the rows of a matrix.*

Solution: Yes, the claims in (a) and (b) hold also for any column, with similar proofs. Any claim about the determinant that holds for columns must hold also for rows, by Problem 3. Thus the determinant function is multilinear on both the columns and on the rows of square matrices.

(d) If some column of A is $\vec{0}$, what is $\det(A)$? What if some row of A is $\vec{0}$?

Solution: The determinant of a square matrix with a zero row or a zero column is zero. This is easy to see directly from the definition of determinant, since every pattern must contain exactly one entry in every row and exactly one in every column, but it also follows from the multilinearity property treated in parts (a) – (c).

(e) Is the determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ a linear transformation from the vector space of $n \times n$ matrices to $\mathbb{R}$?

Solution: No, if $n > 1$ then the determinant map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ is *not* a linear transformation. In fact, if A is $n \times n$ then for any $c \in \mathbb{R}$ we have

$$\det(cA) \; = \; c^n \cdot \det A.$$