Worksheet 26: Complex Eigenvalues (§7.5)

Problem 1. For each $z \in \mathbb{C}$, let $\overline{z}$ be the complex conjugate of z, defined by $\overline{a+bi} = a - bi$. Determine which of the following are true:

(a) For all $z, w \in \mathbb{C}$, $\overline{z+w} = \overline{z} + \overline{w}$.

(b) For all $z, w \in \mathbb{C}$, $\overline{zw} = (\overline{z})(\overline{w})$.

(c) For $z \in \mathbb{C}$, $\overline{z} = z$ if and only if $z \in \mathbb{R}$.

Solution: All the statements are true.

(a) $\overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i = (a-bi)+(c-di) = \overline{a+bi} + \overline{c+di}$.

(b) $\overline{z_1 z_2} = \overline{r_1 e^{i\theta_1} r_2 e^{i\theta_2}} = \overline{r_1 r_2 e^{i(\theta_1+\theta_2)}} = r_1 r_2 e^{-i(\theta_1+\theta_2)} = r_1 e^{-i\theta_1} r_2 e^{-i(\theta_1+\theta_2)} = \overline{r_1 e^{i\theta_1}}\,\overline{r_2 e^{i\theta_2}} = \overline{z_1}\,\overline{z_2}$.

(c) $z = a+bi \in \mathbb{R} \iff b = 0 \iff a+bi = a-bi \iff \overline{z} = z$.

We can treat "vectors" in $\mathbb{C}^n$ just like we treated vectors in $\mathbb{R}^n$. An element $\vec{z}$ of $\mathbb{C}^n$ is just an ordered list of n complex numbers, $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ where each $z_i \in \mathbb{C}$. We can view $\mathbb{C}^n$ as a ($2n$-dimensional) vector space over $\mathbb{R}$, but it is natural to view $\mathbb{C}^n$ as a *complex vector space*, where we allow scalar multiplication by complex numbers. (Of course, as a complex vector space $\mathbb{C}^n$ is n-dimensional.) In many ways, complex vector spaces work just like real vector spaces.

Problem 2. If $\vec{z}$ is a vector in $\mathbb{C}^n$, we will use the (unfortunate) notation $\overline{\vec{z}}$ for the vector whose entries are the conjugates of the entries in $\vec{z}$; that is, $\overline{\vec{z}} = \begin{bmatrix} \overline{z}_1 \\ \vdots \\ \overline{z}_n \end{bmatrix}$. Similarly, if A is a matrix with entries in $\mathbb{C}$, we will write $\overline{A}$ for the matrix whose entries are the conjugates of the entries in A. Using these conventions, determine which of the following are true:

(a) For any $c \in \mathbb{C}$ and $\vec{z} \in \mathbb{C}^n$, $\overline{c\vec{z}} = \overline{c}\,\overline{\vec{z}}$.

(b) For any $c \in \mathbb{C}$ and "complex" matrix A in $\mathbb{C}^{m\times n}$, $\overline{cA} = \overline{c}\overline{A}$.

(c) For any complex vector $\vec{z} \in \mathbb{C}^n$ and matrix $A \in \mathbb{C}^{m\times n}$, $\overline{A\vec{z}} = \overline{A}\,\overline{\vec{z}}$.

(d) For all $\vec{z} \in \mathbb{C}^n$, $\overline{\vec{z}} = \vec{z}$ if and only if $\vec{z} \in \mathbb{R}^n$.

(e) For all $A \in \mathbb{C}^{m\times n}$, $\overline{A} = A$ if and only if $A \in \mathbb{R}^{m\times n}$.

Solution: All the statements are true, as can be shown using the results from Problem 1. For instance, given $A \in \mathbb{C}^{m\times n}$ with (i,j)-entry a_{ij}, we have $\overline{cA} = (\overline{ca_{ij}}) = (\overline{c}\,\overline{a_{ij}}) = \overline{c}\overline{A}$.

Definition If A is an $n \times n$ matrix with complex (so possibly real) entries, then $\lambda \in \mathbb{C}$ is called a *complex eigenvalue* of A if there is a vector $\vec{z} \in \mathbb{C}^n$ such that $A\vec{z} = \lambda\vec{z}$.

Problem 3. Suppose that A is an $n \times n$ matrix with real entries and that $\vec{z} \in \mathbb{C}^n$ is a complex eigenvector of A with corresponding complex eigenvalue λ. Show that $\overline{\vec{z}}$ is also a complex eigenvector of A. What is the corresponding complex eigenvalue?

Solution: If $A\vec{z} = \lambda\vec{z}$, then since A has real entries we have $A\overline{\vec{z}} = \overline{A\vec{z}} = \overline{\lambda\vec{z}} = \overline{\lambda}\,\overline{\vec{z}}$, showing that $\overline{\vec{z}}$ is also a complex eigenvalue of A, with corresponding eigenvalue $\overline{\lambda}$.

Problem 4. In (3) you showed that if A is an $n \times n$ matrix with real entries and if λ is a complex eigenvalue of A, then also $\overline{\lambda}$ is a complex eigenvalue of A. Give a second explanation of why this must be true using the characteristic polynomial of A.

Solution: If A is an $n \times n$ matrix with real entries, then the characteristic polynomial of A has real coefficients, so its complex roots (which are the eigenvalues of A) occur in conjugate pairs. Therefore if λ is an eigenvalue of A, so is $\overline{\lambda}$.

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Problem 5. Let a and b be real numbers, not both zero, and let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(a) Find the complex eigenvalues of A.

Solution: The eigenvalues are $\lambda = a \pm bi$.

(b) Factor A as a product of a scalar matrix rI_2 and a rotation matrix R_θ. How are r and θ related to the complex eigenvalues of A?

Solution: $A = rI_2R_\theta$ where $r = \sqrt{a^2+b^2}$ and $\tan\theta = b/a$. Thus $re^{i\theta}$ is the polar form of $a + bi$.

(c) Diagonalize A *over* $\mathbb{C}$; that is, find complex matrices P and D such that $A = PDP^{-1}$ where D is diagonal.

Solution: $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$.

(d) Describe geometrically the effect of applying the transformation T_A repeatedly to a given point in $\mathbb{R}^2$. What is the difference between the cases $r > 1$, $r = 1$, and $0 < r < 1$?

Solution: Repeated applications of T_A move a point $\vec{x}$ in $\mathbb{R}^2$ around the origin in a spiral pattern, making a jump of angle θ with each iteration. If $r = 1$ then the spiral is a circle, whereas the path spirals inward toward the origin if $r < 1$ and outward to infinity if $r > 1$.

Problem 6. In this problem we consider the matrix $A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$.

(a) Find the characteristic polynomial of A. Does A have any real eigenvalues?

Solution: $t^2 - 1.6t + 1$

(b) Find the complex eigenvalues of A.

Solution: $\lambda = \frac{4}{5} \pm \frac{3}{5}i$.

(c) Find complex eigenvectors corresponding to the complex eigenvalues you found in (b).

Solution: We have $A - (0.8 + 0.6i)I_2 = \begin{bmatrix} -0.3 - 0.6i & -0.6 \\ 0.75 & 0.3 - 0.6i \end{bmatrix}$. The second column of this matrix is $0.4 - 0.8i$ times the first, so an eigenvector corresponding to $0.8 + 0.6i$ is

$$\begin{bmatrix} 0.4 - 0.8i \\ -1 \end{bmatrix}, \qquad \text{or (after scaling),} \qquad \begin{bmatrix} -2 + 4i \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} i.$$

So two linearly independent (complex) eigenvectors are

$$\begin{bmatrix} -2 \\ 5 \end{bmatrix} \pm \begin{bmatrix} 4 \\ 0 \end{bmatrix} i.$$

These are equivalent to $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \pm \begin{bmatrix} 0 \\ 2 \end{bmatrix} i$, since $(1 - 2i) \begin{bmatrix} -2 \\ 1 + 2i \end{bmatrix} = \begin{bmatrix} -2 + 4i \\ 5 \end{bmatrix}$.

(d) Choose one of the complex eigenvectors $\vec{z}$ that you found in (c), and write $\vec{z}$ as $\vec{z} = \vec{v} + i\vec{w}$ where $\vec{v}, \vec{w} \in \mathbb{R}^2$. Find $P^{-1}AP$ where $P = [\vec{w} \quad \vec{v}]$. What kind of matrix is $P^{-1}AP$?

Solution: $P^{-1}AP = \begin{bmatrix} -2 & 4 \\ 5 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 5 & 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$, a rotation! So A is similar to a rotation, which means that A *is* a rotation "relative to a suitable basis," such as the basis of $\mathbb{R}^2$ given by the columns of P.

(e) Can you describe geometrically the action of the transformation T_A on $\mathbb{R}^2$?

Solution: It moves each point in $\mathbb{R}^2$ in an elliptical orbit around the origin.

Problem 7. Let A be any 2×2 matrix with real entries that has a pair of (non-real) complex eigenvalues $a \pm bi$. Show that A is similar (over $\mathbb{R}$) to the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Hint: consider the fact that both A and R are similar *over* $\mathbb{C}$ to the complex diagonal matrix

$$\begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}.$$

Solution: Suppose $A(\vec{v} + i\vec{w}) = (a+bi)(\vec{v} + i\vec{w})$, so also $A(\vec{v} - i\vec{w}) = (a-bi)(\vec{v} - i\vec{w})$. Then, diagonalizing A over $\mathbb{C}$, we have

$$\begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix}^{-1} A \begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix} = \begin{bmatrix} a+bi & \\ & a-bi \end{bmatrix},$$

which by Problem 5(c) implies

$$\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix}^{-1} A \begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

By direct computation we have $\begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = [\vec{w} \quad \vec{v}]$, so

$$[\vec{w} \quad \vec{v}]^{-1} A \, [\vec{w} \quad \vec{v}] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$