Math 217 – Final Exam

Fall 2021

Solutions

Student ID Number: ______________________ Section: ________

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	16	12	13	12	12	11	12	100
Score:									

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

(a) The *image* of the linear transformation $T : V \to W$ from the vector space V to the vector space W

> **Solution:** The *image* of the linear transformation $T : V \to W$ from the vector space V to the vector space W is the subset $\{T(\vec{v}) \; : \; \vec{v} \in V\}$ of W.

(b) The vector $\vec{v}$ in the vector space V is an *eigenvector* of the linear transformation $T : V \to V$

> **Solution:** The vector $\vec{v}$ in the vector space V is an *eigenvector* of the linear transformation $T : V \to V$ if $\vec{v} \neq \vec{0}$ and $T(\vec{v}) = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$.

(c) The *geometric multiplicity* $\text{gemu}(\lambda)$ of the eigenvalue λ of the linear transformation $T : V \to V$, where V is a finite-dimensional vector space

> **Solution:** The *geometric multiplicity* $\text{gemu}(\lambda)$ of λ is the dimension of the λ-eigenspace $E_\lambda = \{\vec{v} \in V \; : \; T(\vec{v}) = \lambda\vec{v}\}$.

(d) The function $f : X \to Y$ is *injective*

> **Solution:** The function $f : X \to Y$ is *injective* if for all $x_1, x_2 \in X$, we have $f(x_1) \neq f(\vec{x}_2)$ whenever $x_1 \neq x_2$.

> **Solution:** The function $f : X \to Y$ is *injective* if for all $x_1, x_2 \in X$, we have $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

2. State whether each statement is True or False and provide a short proof of your claim. For each part, indicate your answer by clearly writing "T" or "F" in the box on the left.

(a) (4 points) For all matrices $A \in \mathbb{R}^{n\times n}$, $\det(A) = 0$ if and only if A is not diagonalizable.

Solution: FALSE. For instance, $\det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ but $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable since it is already diagonal. (This works for any square zero matrix, or indeed any diagonal matrix with a zero entry somewhere on the diagonal.)

(b) (4 points) For all matrices $A \in \mathbb{R}^{n\times m}$ and vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, if $AA^\top\vec{v} = \lambda_1\vec{v}$ and $AA^\top\vec{w} = \lambda_2\vec{w}$ where $\lambda_1 \neq \lambda_2$, then $\vec{v}\cdot\vec{w} = 0$.

Solution: TRUE. Let $A \in \mathbb{R}^{n\times m}$ and $\vec{v}, \vec{w} \in \mathbb{R}^n$, and suppose $AA^\top\vec{v} = \lambda_1\vec{v}$ and $AA^\top\vec{w} = \lambda_2\vec{w}$ where $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned} \lambda_1(\vec{v}\cdot\vec{w}) &= (\lambda_1\vec{v})\cdot\vec{w} = (AA^\top\vec{v})\cdot\vec{w} = \vec{v}^\top(AA^\top)^\top\vec{w} \\ &= \vec{v}^\top AA^\top\vec{w} = \vec{v}^\top\lambda_2\vec{w} = \vec{v}\cdot(\lambda_2\vec{w}) = \lambda_2(\vec{v}\cdot\vec{w}), \end{aligned}$$

which implies $\vec{v}\cdot\vec{w} = 0$ since $\lambda_1 \neq \lambda_2$.

Solution: TRUE. Assume the hypotheses. Since $(AA^\top)^\top = A^\top(A^\top)^\top = AA^\top$, we know that $AA^\top$ is orthogonally diagonalizable by the Spectral Theorem. This means there is an orthonormal eigenbasis of $\mathbb{R}^n$ for $AA^\top$, which means distinct eigenspaces of $AA^\top$ are orthogonal to each other. Thus if $\lambda_1 \neq \lambda_2$ then $\vec{v}$ and $\vec{w}$ are eigenvectors of $AA^\top$ belonging to distinct eigenspaces, which means $\vec{v}$ and $\vec{w}$ are orthogonal as desired.

(c) (4 points) For every symmetric matrix $A \in \mathbb{R}^{n\times n}$, the equation $\langle\vec{x},\vec{y}\rangle = \vec{x}^\top A\vec{y}$ defines an inner product on $\mathbb{R}^n$.

Solution: FALSE. For instance, let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ be the 2×2 zero matrix, so A is symmetric. Then the assignment $\langle\vec{x},\vec{y}\rangle = \vec{x}^\top A\vec{y}$ is symmetric and bilinear but not postive-definite, since for instance $\vec{e}_1^\top A\vec{e}_1 = 0$ even though $\vec{e}_1 \neq \vec{0}$.

(d) (4 points) For every $n\times n$ matrix A, $\det A = 0$ if and only if $\det(\text{rref}(A)) = 0$.

Solution: TRUE. Let A be an $n\times n$ matrix, and write $R = \text{rref}(A)$. If $\det(A) \neq 0$ then A is invertible which means $R = I_n$, so $\det(R) = 1 \neq 0$. Conversely, if $\det(A) = 0$ then A is *not* invertible, which means $R \neq I_n$ and therefore the last row of R is a zero row, so $\det(R) = 0$.

Solution: TRUE. Let A be an $n \times n$ matrix, and write $R = \text{rref}(A)$. We know there is an invertible matrix P such that $PA = R$ (namely, let P be the product of the elementary matrices corresponding to the elementary row operations that transform A into R). Then $\det R = \det(PA) = (\det P)(\det A)$ where $\det P \neq 0$, which implies $\det A = 0$ if and only if $\det R = 0$.

3. Consider the 3×3 matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & a & 0 \\ 0 & b & 3 \end{bmatrix}$, where $a, b \in \mathbb{R}$. In parts (a) and (b) below, find *all* values of a and b in $\mathbb{R}$ for which the given condition holds, or else write "none" if there are no such values. Justify your answers.

   (a) (4 points) A is invertible.

   Solution: Using the fact that A is invertible iff $\det(A) \neq 0$, we compute, using a Laplace expansion along the third column,

$$\det A \;=\; 3\det\begin{bmatrix} 2 & 1 \\ 1 & a \end{bmatrix} \;=\; 3(2a-1) \;=\; 6a-3.$$

   Thus A is invertible iff $a \neq \frac{1}{2}$. (b can be any real number.)

   (b) (3 points) There exists an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

   Solution: There exists an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A if and only if A is orthogonally diagonalizable, which by the Spectral Theorem happens if and only if A is symmetric. Thus a can be any real number, but we must have $b = 0$.

   (c) (5 points) Assuming $a = 2$, find all $b \in \mathbb{R}$ for which A is diagonalizable (over $\mathbb{R}$).

   Solution: Letting $a = 2$, we calculate the characteristic polynomial of A:

$$\det\begin{bmatrix} 2-x & 1 & 0 \\ 1 & 2-x & 0 \\ 0 & b & 3-x \end{bmatrix} \;=\; (3-x)\big[(2-x)^2 - 1\big] \;=\; (3-x)(3-4x+x^2)$$
$$=\; (3-x)(x-3)(x-1) \;=\; -(x-3)^2(x-1).$$

   Thus the eigenvalues of A are 3 and 1, with $\text{almu}(3) = 2$ and $\text{almu}(1) = 1$. So A will be diagonalizable iff $\text{gemu}(3) = 2$. But

$$A - 3I_3 \;=\; \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & b & 0 \end{bmatrix},$$

   which has rank 1 (and thus $\text{gemu}(3) = \dim\ker(A - 3I_3) = 2$) iff $b = 0$. It follows that A is diagonalizable iff $b = 0$.

4. Let $\mathbb{R}^{2\times 2}$ be the vector space of 2×2 real matrices, and let $\mathcal{E}$ be the ordered basis of $\mathbb{R}^{2\times 2}$ given by

$$\mathcal{E} \;=\; (E_{11}, E_{12}, E_{21}, E_{22}) \;=\; \left(\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right).$$

Let $M = \begin{bmatrix}1&4\\0&1\end{bmatrix} \in \mathbb{R}^{2\times 2}$, and let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation defined by

$$T(A) = MA - A^\top$$

for every $A \in \mathbb{R}^{2\times 2}$. (You do *not* need to prove T is linear, or that $\mathcal{E}$ is a basis of $\mathbb{R}^{2\times 2}$.)

(a) (3 points) Find the $\mathcal{E}$-matrix $[T]_\mathcal{E}$ of T.

Solution: For all $A = \begin{bmatrix}a&b\\c&d\end{bmatrix} \in \mathbb{R}^{2\times 2}$, we have

$$T(A) \;=\; \begin{bmatrix}1&4\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} - \begin{bmatrix}a&c\\b&d\end{bmatrix} \;=\; \begin{bmatrix}4c & b+4d-c\\ c-b & 0\end{bmatrix}.$$

From this we see that

$$T(E_{11}) = \begin{bmatrix}0&0\\0&0\end{bmatrix},\; T(E_{12}) = \begin{bmatrix}0&1\\-1&0\end{bmatrix},\; T(E_{21}) = \begin{bmatrix}4&-1\\1&0\end{bmatrix},\; T(E_{22}) = \begin{bmatrix}0&4\\0&0\end{bmatrix},$$

so

$$[T]_\mathcal{E} \;=\; \begin{bmatrix}0&0&4&0\\0&1&-1&4\\0&-1&1&0\\0&0&0&0\end{bmatrix}.$$

(b) (3 points) Find the characteristic polynomial of T.

Solution: The characteristic polynomial of T is

$$\begin{aligned}\det\big(xI_4 - [T]_\mathcal{E}\big) \;&=\; \det\begin{bmatrix}x&0&-4&0\\0&x-1&1&-4\\0&1&x-1&0\\0&0&0&x\end{bmatrix}\\ &=\; x^2\Big((x-1)^2-1\Big) \;=\; x^2(x^2-2x+1-x) \;=\; x^3(x-2).\end{aligned}$$

(Problem 4, Continued).

(c) (4 points) For each eigenvalue of T, find a basis of the corresponding eigenspace. Clearly indicate your eigenvalues and which basis goes with which eigenvalue.

Solution: For the eigenvalue $\lambda = 0$, we first find a basis of $\ker\big([T]_{\mathcal{E}}\big)$, which by inspection is $(\vec{e}_1)$. Thus, transferring back to $\mathbb{R}^{2\times 2}$,

$$\text{a basis of the eigenspace } E_0 \text{ is: } \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

For the eigenvalue $\lambda = 2$, we first find a basis of

$$\ker\big([T]_{\mathcal{E}} - 2I_4\big) \;=\; \ker \begin{bmatrix} -2 & 0 & 4 & 0 \\ 0 & -1 & -1 & 4 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

and we obtain $\begin{bmatrix} 2 & -1 & 1 & 0 \end{bmatrix}^\top$. It follows that

$$\text{a basis of the eigenspace } E_2 \text{ is: } \left(\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\right).$$

(d) (3 points) Is T diagonalizable? Briefly explain your answer.

Solution: No, T is not diagonalizable since $1 = \text{gemu}(0) < \text{almu}(0) = 3$, so there are not enough eigenvectors to form an eigenbasis of $\mathbb{R}^{2\times 2}$ for T.

5. (12 points) In each part, find the smallest positive integer n such that there is a matrix $A \in \mathbb{R}^{n\times n}$ with the indicated property.

No justification is required for this problem. In particular, you do not have to provide an example matrix; just say what n is, and write your answer clearly in the box.

(a) A has at least 5 distinct eigenvalues.

Solution: $n = 5$

(b) A has no real eigenvalues.

Solution: $n = 2$

(c) A has at least 3 distinct real eigenvalues and is not diagonalizable (over $\mathbb{R}$).

Solution: $n = 4$

(d) The complex numbers i, $1+i$, $1-i$, $2+i$, and 7 are some of the complex eigenvalues of A.

Solution: $n = 7$

(e) A has at least one real eigenvalue and is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$.

Solution: $n = 3$

(f) A is not invertible and ± 1 are eigenvalues of A with

$$\text{gemu}(1) \ < \ \text{gemu}(-1) \ < \ \text{almu}(1).$$

Solution: $n = 6$

6. Let $V = C^\infty([-1,1])$ be the inner product space of smooth functions from $[-1,1]$ to $\mathbb{R}$ with inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$, and let W be the subspace of V given by $W = \text{Span}(1, t)$.

   (a) (4 points) Find an orthonormal basis $\mathcal{U}$ of the subspace W.

   Solution: Note that
$$\langle 1, t\rangle \;=\; \int_{-1}^{1} t\,dt \;=\; 0,$$
so 1 and t are already orthogonal and therefore $\left(\frac{1}{\|1\|}, \frac{t}{\|t\|}\right)$ is an orthonormal basis of W. Since
$$\langle 1, 1\rangle \;=\; \int_{-1}^{1} 1\,dt \;=\; 2 \quad\text{and}\quad \langle t, t\rangle \;=\; \int_{-1}^{1} t^2 \;=\; \frac{t^3}{3}\bigg]_{-1}^{1} \;=\; \frac{2}{3},$$
we have that $\mathcal{U} = \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,t\right)$ is an orthonormal basis of W.

   (b) (4 points) Find the orthogonal projection of t^2 onto $W^\perp$; that is, find $\text{proj}_{W^\perp}(t^2)$.

   Solution: Let $\mathcal{U} = (u_1, u_2)$ be our orthonormal basis from (a), so for any $v \in V$ we have $\text{proj}_W(v) = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2$. Using the fact that $t^2 = \text{proj}_W(t^2) + \text{proj}_{W^\perp}(t^2)$, we have
$$\text{proj}_{W^\perp}(t^2) \;=\; t^2 - \text{proj}_W(t^2) \;=\; t^2 - \langle t^2, u_1\rangle u_1 - \langle t^2, u_2\rangle u_2.$$
Since t^2 is an even function and u_2 is odd, their product is odd, so $\langle t^2, u_2\rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}} t^3 dt = 0$. Thus
$$\text{proj}_{W^\perp}(t^2) \;=\; t^2 - \langle t^2, u_1\rangle u_1 \;=\; t^2 - \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{t^2}{\sqrt{2}}\,dt \;=\; t^2 - \frac{1}{2}\int_{-1}^{1} t^2\,dt \;=\; t^2 - \frac{1}{3}.$$

   (c) (4 points) Let g be the cosine function on $[-1,1]$, so $g(t) = \cos t$ for all $t \in [-1,1]$. Find the function p in W that is *closest* to the cosine function in V, in the sense that $\|p - g\| \le \|q - g\|$ for all $q \in W$.

   Solution: The function p in W that is closest to g is
$$p \;=\; \text{proj}_W(g) \;=\; \langle g, u_1\rangle u_1 + \langle g, u_2\rangle u_2.$$
Since g is a even and u_2 is odd, their product is odd, so $\langle g, u_2\rangle = 0$ and thus
$$p \;=\; \langle g, u_1\rangle u_1 \;=\; \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{\cos t}{\sqrt{2}}\,dt \;=\; \frac{1}{2}\sin t\bigg]_{-1}^{1} \;=\; \frac{1}{2}\big(\sin 1 - \sin(-1)\big) \;=\; \sin 1.$$
So p is the constant function on $[-1,1]$ with constant value $\sin 1$.

7. Let U and A be $n \times n$ matrices with real entries, and suppose U is upper-triangular and A is invertible with QR-factorization $A = QR$.

   (a) (3 points) Prove that all the complex eigenvalues of U are real.

   Solution: Since U is upper-triangular, the eigenvalues of U are just the diagonal entries of U, so the fact that U has real entries guarantees that all the eigenvalues of U are real.

   (b) (3 points) Prove that if U is orthogonally diagonalizable, then U is already diagonal.

   Solution: Suppose U is orthogonally diagonalizable. Then U is symmetric by the Spectral Theorem, but upper-triangular matrices that are symmetric must be diagonal.

   (c) (5 points) Prove or disprove the following statement: A is orthogonally diagonalizable if and only if R is orthogonally diagonalizable.

   Solution: The statement is false. To show that the forward implication fails, consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, which has QR-factorization $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Note that A is orthogonally diagonalizable by the Spectral Theorem, since A is symmetric, but $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not orthogonally diagonalizable since it is not symmetric.

   Solution: The statement is false. To show that the backward implication fails, consider the matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, which has QR-factorization $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Note that $R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is orthogonally diagonalizable by the Spectral Theorem, since R is symmetric, but A is not orthogonally diagonalizable since it is not symmetric.

8. Suppose $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ are finite-dimensional inner product spaces.

   (a) (7 points) Prove that if $\dim V \le \dim W$, then there exists a linear transformation $T : V \to W$ such that $\langle x, y\rangle_V = \langle T(x), T(y)\rangle_W$ for all $x, y \in V$.

Solution: Suppose $\dim V \le \dim W$, and let $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{C} = (w_1, \ldots, w_m)$ be orthonormal bases of V and W, respectively, so $n \le m$. Given $x \in V$, let

$$T(x) = \sum_{i=1}^{n} c_i w_i \quad \text{where} \quad \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [x]_{\mathcal{B}}.$$

This defines a map $T : V \to W$, and we show that T is as desired. Let $x, y \in V$ and $k \in \mathbb{R}$, with $x = \sum_{i=1}^{n} c_i v_i$ and $y = \sum_{i=1}^{n} d_i v_i$. Then $[x+y]_{\mathcal{B}} = [x]_{\mathcal{B}} + [y]_{\mathcal{B}}$ and $[kx]_{\mathcal{B}} = k[x]_{\mathcal{B}}$, so

$$T(x+y) \;=\; \sum_{i=1}^{n}(c_i + d_i)w_i \;=\; \sum_{i=1}^{n} c_i v_i + \sum_{i=1}^{n} d_i v_i \;=\; T(x) + T(y)$$

and

$$T(kx) \;=\; \sum_{i=1}^{n} k c_i w_i \;=\; k\sum_{i=1}^{n} c_i w_i \;=\; kT(x).$$

This shows T is linear. Finally, we have

$$\begin{aligned}
\langle T(x), T(y)\rangle_W &= \left\langle T\left(\textstyle\sum c_i v_i\right), T\left(\textstyle\sum d_i v_i\right)\right\rangle_W \\
&= \left\langle \textstyle\sum c_i w_i, \sum d_i w_i\right\rangle_W \\
&= \textstyle\sum_i \sum_j c_i d_j \langle w_i, w_j\rangle_W \\
&= \textstyle\sum_i \sum_j c_i d_j \langle v_i, v_j\rangle_V \\
&= \left\langle \textstyle\sum c_i v_i, \sum d_j v_j\right\rangle_V \;=\; \langle x, y\rangle_V.
\end{aligned}$$

   (b) (5 points) Prove that if $\dim V > \dim W$, then there does *not* exist a linear transformation $T : V \to W$ such that $\langle x, y\rangle_V = \langle T(x), T(y)\rangle_W$ for all $x, y \in V$.

Solution: We prove the contrapositive. Suppose there *does* exist a linear transformation $T : V \to W$ such that $\langle x, y\rangle_V = \langle T(x), T(y)\rangle_W$ for all $x, y \in V$, and let T be such a map. Let $v \in \ker(T)$. Then

$$\langle v, v\rangle_V \;=\; \langle T(v), T(v)\rangle_W \;=\; \langle \vec{0}_W, \vec{0}_W\rangle_W \;=\; 0,$$

which implies $v = \vec{0}_V$ by positive-definiteness of $\langle\cdot,\cdot\rangle_V$. Thus $\ker(T) = \{\vec{0}_V\}$, so $\dim\ker(T) = 0$. But then $\dim\operatorname{im}(T) = \dim V$ by Rank-Nullity, and since $\operatorname{im}(T) \subseteq W$ this implies $\dim V \le \dim W$.