Worksheet 4: Linear Functions (§§2.1,2.2)

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Problem 1: Functions and Linearity.

On the previous worksheet, we called a function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ *linear* if for all vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ and for all scalars $c \in \mathbb{R}$, the following hold:

$$f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w}); \tag{1}$$
$$f(c\vec{v}) = cf(\vec{v}). \tag{2}$$

Furthermore, we associated to each $m \times n$ matrix A the linear function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\vec{v}) = A\vec{v}. \tag{3}$$

In (a) – (h) below, you are given a function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ for some particular n and m. Determine, in each case, whether or not the given function is linear, and *if it is linear* see if you can find a matrix A such that $f = T_A$.

(a) f is the *identity function* on $\mathbb{R}^4$, defined by $f(\vec{v}) = \vec{v}$ for all $\vec{v} \in \mathbb{R}^4$.

(b) $f : \mathbb{R}^3 \to \mathbb{R}^3$ is *dilation by* 2, defined by $f(\vec{v}) = 2\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

(c) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x + 1$ for all $x \in \mathbb{R}$.

(d) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection over the line $y = x$, defined by $f(x, y) = (y, x)$ for all $(x, y) \in \mathbb{R}^2$.

(e) $f : \mathbb{R}^2 \to \mathbb{R}$ assigns to every point in the plane its distance from the origin, so that $f(x, y) = \sqrt{x^2 + y^2}$ for all $(x, y) \in \mathbb{R}^2$.

(f) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the *shear* transformation defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

for all $x, y \in \mathbb{R}$.

(g) $f : \mathbb{R}^n \to \mathbb{R}^m$ is the *zero transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined by $f(\vec{v}) = \vec{0}$ for all $\vec{v} \in \mathbb{R}^n$.

(h) $f : \mathbb{R}^3 \to \mathbb{R}^3$ is counterclockwise rotation by 90° about the z-axis.

Solution:

(a) Yes, f is linear. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Yes, f is linear. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) No, f is not linear.

(d) Yes, f is linear. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(e) No, f is not linear (although it does preserve scalar multiplication by positive scalars).

(f) Yes, f is linear. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(g) Yes, f is linear. A is the $m \times n$ zero matrix.

(h) Yes, f is linear. $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 2: "Linear transformations† preserve linear combinations."

Prove that for all $n \in \mathbb{N}$, for any function $f : \mathbb{R}^m \to \mathbb{R}^p$, and for any vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$, if f is linear then

$$f\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i f(\vec{v}_i).$$

Solution: For the induction base $n = 1$, we have that for any linear transformation $f : \mathbb{R}^m \to \mathbb{R}^p$ and for any vector $\vec{v}_1 \in \mathbb{R}^m$ and scalar $c_1 \in \mathbb{R}$, $f(c_1\vec{v}_1) = c_1 f(\vec{v}_1)$ by linearity of f.

For the inductive step, let $n \in \mathbb{N}$ and assume for inductive hypothesis that for every linear transformation $f : \mathbb{R}^m \to \mathbb{R}^p$ and for all vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$,

$$f\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i f(\vec{v}_i).$$

Let $f : \mathbb{R}^m \to \mathbb{R}^p$ be linear, and let $\vec{v}_1, \ldots, \vec{v}_{n+1} \in \mathbb{R}^m$ and $c_1, \ldots, c_{n+1} \in \mathbb{R}$. Then, using linearity of f and the inductive hypothesis, we have

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} c_i \vec{v}_i\right) &= f\left(c_{n+1}\vec{v}_{n+1} + \sum_{i=1}^{n} c_i \vec{v}_i\right) = f(c_{n+1}\vec{v}_{n+1}) + f\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) \\ &= c_{n+1} f(\vec{v}_{n+1}) + \sum_{i=1}^{n} c_i f(\vec{v}_i) = \sum_{i=1}^{n+1} c_i f(\vec{v}_i). \end{aligned}$$

This completes the proof by induction.

†When functions are linear, we often call them *linear transformations* rather than *linear functions*, although this is just a convention and it is not wrong to refer to them still as functions.

Problem 3: Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Your answers in Problem 1 might lead you to believe that *every* linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ corresponds to multiplication by some matrix. Amazingly this turns out to be true! Consider a simple case first:

(a) Prove that for every linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$, there is a 2×2 matrix A such that $f(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

Now consider the general case:

(b) Prove that for every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, there is a matrix A such that $f(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. (Be sure to give the size of A)!

(c) Is the matrix A from part (b) unique? That is, is it true that for every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, there is a *unique* matrix A such that $f(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$?

Solution:

(a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Then for all $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we have

$$f(\vec{x}) = f\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_1 f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} \vec{x}.$$

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Generalizing the proof from part (a), for each $1 \le k \le n$ let $\vec{e}_k$ be the *kth standard basis vector*, i.e., the vector in $\mathbb{R}^n$ whose kth entry is 1 and whose other entires are all 0. Then for all $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, we have

$$f(\vec{x}) = f\left(\sum_{k=1}^{n} x_k \vec{e}_k\right) = \sum_{k=1}^{n} x_k f(\vec{e}_k) = \begin{bmatrix} | & & | \\ f(\vec{e}_1) & \cdots & f(\vec{e}_n) \\ | & & | \end{bmatrix} \vec{x}.$$

Thus $f(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$, where A is the $m \times n$ matrix whose kth column is $f(\vec{e}_k)$.

(c) Yes, the matrix A is unique. To see this, suppose there is another $m \times n$ matrix B such that $f(\vec{x}) = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Then, in particular, $A\vec{e}_i = f(\vec{e}_i) = B\vec{e}_i$ for each $1 \le i \le n$, which shows that A and B have all the same columns and are therefore equal to each other.