MATH 217 - LINEAR ALGEBRA
HOMEWORK 5, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:
Section 3.3: 24, 28, 32
Section 4.1: 12, 28, 46

Solution.

3.3.24 We can row reduce the matrix to

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This shows that the first, third, fourth and fifth columns of the original matrix form a basis for the image. That is, the column vectors

$$\begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \end{bmatrix}$$

form a basis for the image. Since the kernel is not affected by elementary row operations, we can find a basis for the kernel of B and it will also be a basis for the kernel of the original matrix. $Bx = 0$ corresponds to the system

$$\begin{cases} x_1 + 2x_2 & = 0 \\ x_3 & = 0 \\ x_4 & = 0 \\ x_5 & = 0 \end{cases}$$

which has solution set parametrized by

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t,$$

so the vector $[-2, 1, 0, 0, 0]$ is a basis for the kernel.

3.3.28 We row-reduce the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & k \end{bmatrix} \xrightarrow{R_4 - 2R_1 - 3R_2 - 4R_4} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{bmatrix}$$

By Theorem 3.3.9, the given vectors form a basis of $\mathbb{R}^4$ if and only if $k \neq 29$.

3.3.32 We need to find a basis for the solution space V to

$$x_1 - x_3 + x_4 = 0$$
$$x_2 + 2x_3 + 3x_4 = 0,$$

or equivalently

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

The matrix is already in row-reduced form, so it is easy to solve. We have two free variables $x_3 = s$, $x_4 = t$ and we have two dependent variables

$$x_1 = s - t, \quad x_2 = -2s - 3t.$$

From this one sees that a basis for V is given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Of course, there are many other choices of a basis for V.

4.1.12 This is a subspace. Indeed if we take two arithmetic sequences given by $(a_1, a_1 + k_1, \dots)$ and $(a_2, a_2 + k_2, \dots)$ for some constants a_1, a_2, k_1, k_2 and add them as described in example 5, we get the arithmetic sequence

$$((a_1 + a_2), (a_1 + a_2) + (k_1 + k_2), (a_1 + a_2) + 2(k_1 + k_2), \dots)$$

which is the arithmetic sequence beginning with $(a_1 + a_2)$ where we add $(k_1 + k_2)$. Similarly, if we scale the first arithmetic sequence by a constant $c \in \mathbb{R}$, we get another arithmetic sequence $(ca_1, ca_1 + ck, ca_1 + 2ck, \dots)$ adding ck to ca_1.

4.1.28 If A commutes with B this implies that $AB = BA$, so $AB - BA = 0$. That is, we can realize the space of 2×2 matrices commuting with B as the kernel of the linear transformation $T(A) = AB - BA$. If we take a generic 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

notice that

$$AB = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

and

$$BA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}.$$

For these to be equal we must have

$$\begin{cases} a & = a + c \\ a + b & = b + d \\ c & = c \\ c + d & = d \end{cases}$$

Solving this we find $c = 0$, $a = d$ and b can be anything. So the matrices that commute with B are of the form

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

This has a basis given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Alternative Solution: Alternatively, we can find a basis for the kernel of the linear transformation T as we usually do. $\mathbb{R}^{n\times n}$ has a basis given by the matrices E_{ij} with (i, j) entry equal to 1 and every other entry 0. If we take $(E_{0,0}, E_{0,1}, E_{1,0}, E_{1,1})$ as our ordered basis for $\mathbb{R}^{n\times n}$ we can find a standard matrix for T

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We can row reduce this to

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The kernel of this matrix is parametrized by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} t_2,$$

so these two vectors form a basis for the kernel. However, recall what our coordinates for $\mathbb{R}^{2\times 2}$ are. The first vector corresponds to $E_{0,0} + E_{1,1}$ and the second corresponds to $E_{0,1}$. This is precisely the basis we found via the first method of solving this problem.

4.1.46 Each arithmetic sequence is specified by the choice of a and k. We claim that the arithmetic sequences given by $b_n = (1, 1, \dots)$ and $c_n = (0, 1, 2, \dots)$ form a basis for the space of arithmetic sequences. Indeed, given any arithmetic sequence $(a, a + k, a + 2k, \dots)$, we can realize this as $ab_n + kc_n$. So the span of $(1, 1, \dots)$ and $(0, 1, 2, \dots)$ is all arithmetic sequences. We can also see that these two arithmetic sequences are linearly independent. Indeed, suppose we have

$$a(1, 1, \dots) + b(0, 1, 2, \dots) = (0, 0, \dots)$$

for some constants $a, b \in \mathbb{R}$. From the first coordinate we conclude that $a = 0$ and from the second coordinate we conclude that $a + b = 0$, so $b = 0$. This proves these two arithmetic sequences are indeed linearly independent and that the space of arithmetic sequences is two dimensional.

Remark: You could also prove that the map $(a, k) \mapsto (a, a+k, a+2k, \dots)$ is actually an isomorphism of vector spaces between $\mathbb{R}^2$ and the space of arithmetic sequences.

Part B (25 points)

Problem 1. Let V and W be subspaces of $\mathbb{R}^n$.

(a) Prove that if $W \subseteq V$, then $\dim(W) \leq \dim(V)$.
(b) Prove that if $W \subseteq V$, then we have $\dim(W) = \dim(V)$ if and only if $W = V$.

Solution.

(a) Let $(\vec{w}_1, \dots, \vec{w}_d)$ be a basis for W and $(\vec{v}_1, \dots, \vec{v}_r)$ a basis for V so that $\dim(W) = d$ and $\dim(V) = r$. Then the vectors $\vec{w}_1, \dots, \vec{w}_d$ lie in V and are linearly independent, while the vectors $\vec{v}_1, \dots, \vec{v}_r$ lie in V and span V. By Theorem 3.3.1 of the textbook, we see that $d \leq r$, so that $\dim(W) \leq \dim(V)$.

(b) It is clear that if $W = V$, then $\dim(W) = \dim(V)$. Conversely, suppose $\dim(W) = \dim(V)$. Then $\dim(V) = d$ and $\vec{w}_1, \dots, \vec{w}_d$ are d linearly independent vectors in V. By Theorem 3.3.4 of the textbook, $(\vec{w}_1, \dots, \vec{w}_d)$ must be a basis of V and thus span V. This shows that $V \subseteq W$ and thus $W = V$.

Problem 2. Let $m, n \in \mathbb{N}$, let $\mathcal{V} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a nonempty subset of $\mathbb{R}^n$, and let $\mathcal{U}$ be a nonempty subset of $\mathcal{V}$. Determine whether the following statements are true or false. Be sure to justify each of your answers with a proof or a counterexample.

(a) If $\mathcal{U}$ spans $\mathbb{R}^n$, then $\mathcal{V}$ spans $\mathbb{R}^n$.[1]
(b) If $\mathcal{U}$ is linearly independent, then $\mathcal{V}$ is linearly independent.
(c) If $\mathcal{V}$ spans $\mathbb{R}^n$, then $\mathcal{U}$ spans $\mathbb{R}^n$.
(d) If $\mathcal{V}$ is a basis of $\mathbb{R}^n$, then $\mathcal{U}$ is linearly independent.

Solution.

(a) True. Assume, renumbering the vectors if necessary, that $\mathcal{U} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ where $k \leq n$ and $\mathcal{U}$ spans $\mathbb{R}^n$. We will show that $\mathcal{V}$ spans $\mathbb{R}^n$. Let $\vec{x} \in \mathbb{R}^n$. Since $\mathcal{U}$ spans $\mathbb{R}^n$, there exists $c_1, \dots, c_k \in \mathbb{R}$ such that $\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$. Thus $\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \cdots + 0\vec{v}_m$. Hence $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ span $\mathbb{R}^n$.

(b) False. For $n = 2$, $\mathcal{V} = \{(0\ 1)^T, (1\ 0)^T, (1\ 1)^T\} \subseteq \mathbb{R}^2$, $U = \{(0\ 1)^T, (1\ 0)^T\}$. Then $\mathcal{U}$ is linearly independent but $\mathcal{V}$ is linearly dependent since $(1\ 1)^T$ is a redundant vector.

(c) False. For $n = 2$, $\mathcal{V} = \{(0\ 1)^T, (1\ 0)^T\} \subseteq \mathbb{R}^2$, $U = \{(0\ 1)^T\}$. Then $\mathcal{V}$ spans $\mathbb{R}^2$ but $\mathcal{U}$ does not span $\mathbb{R}^2$.

(d) True. Assume, renumbering the vectors if necessary, that $\mathcal{U} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ where $k \leq n$. We will show that $\mathcal{U}$ is a linearly independent set of vectors. Assume that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for $c_1, \dots, c_k \in \mathbb{R}$. Then $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \cdots + 0\vec{v}_n = \vec{0}$. Since $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis then it is linearly independent, we have $c_1 = c_2 = \cdots = c_k = 0$. This implies that $\mathcal{U} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ are linearly independent.

Problem 3. Let A and B be $n \times n$ matrices. Prove or disprove each of the following.

(a) $\text{rank}(AB) = \text{rank}(BA)$
(b) $\text{rank}(AB) \leq \text{rank}(A)$
(c) $\text{rank}(AB) \leq \text{rank}(B)$

[1]Recall that for a subset S of the vector space V, "S spans V" just means that $\text{Span}(S) = V$.

(d) $\operatorname{rank}(AB) = \operatorname{rank}(A) - \dim(\ker(B))$

Solution.

(a) False. Consider the matrices A and B

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

BA is the zero matrix, but $AB = B$ is not. So $\operatorname{rank}(BA) = 0$ and $\operatorname{rank}(AB) = \operatorname{rank}(B) = 1$.

(b) True. Notice that $\operatorname{im}(AB) \subseteq \operatorname{im}(A)$. From Problem 1(a) we have $\dim(\operatorname{im} AB) \leq \dim(\operatorname{im} A)$, but $\dim(\operatorname{im} AB) = \operatorname{rank}(AB)$ and $\dim(\operatorname{im} A) = \operatorname{rank}(A)$.

(c) True. This time notice that $\ker(B) \subseteq \ker(AB)$. Indeed if $Bx = 0$, then $ABx = 0$ as well. By rank-nullity $\operatorname{rank}(AB) = n - \dim(\ker(AB))$. But since $\ker(B) \subseteq \ker(AB)$ this means by problem 1(b) again that $\dim(\ker(B)) \leq \dim(\ker(AB))$ or equivalently $-\dim(\ker(AB)) \leq -\dim(\ker(B))$. So

$$\operatorname{rank}(AB) \leq n - \dim(\ker(B)) = \dim(\operatorname{im} B) = \operatorname{rank}(B),$$

where the first equality follows by rank-nullity.

(d) False. Take the same A and B as in part (a). Then $\operatorname{rank}(AB) = 1$ since $AB = B$, but $\operatorname{rank}(A) = 1$ and $\dim(\ker(B)) = 1$.

Definition. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be *symmetric* if $A^\top = A$, and *skew-symmetric* if $A^\top = -A$.

Problem 4. Let Sym_n be the set of all symmetric $n \times n$ matrices, and let Skew_n be the set of all skew-symmetric $n \times n$ matrices. Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the function defined by $T(A) = A + A^\top$ for all $A \in \mathbb{R}^{n\times n}$.

(a) Prove that T is linear.
(b) Prove that $\ker(T) = \mathrm{Skew}_n$ and $\operatorname{im}(T) = \mathrm{Sym}_n$.
(c) Prove that Sym_n and Skew_n are subspaces of $\mathbb{R}^{n\times n}$.
(d) Find $\dim(\mathrm{Sym}_n)$ and $\dim(\mathrm{Skew}_n)$.
(e) **(Recreational)** What is $\mathrm{Sym}_n \cap \mathrm{Skew}_n$? What is $\mathrm{Sym}_n + \mathrm{Skew}_n$?

Solution.

(a) Given two matrices $A, B \in \mathbb{R}^{n\times n}$ and a scalar $c \in \mathbb{R}$,

$$T(A+B) = (A+B) + (A+B)^\top = A + B + A^\top + B^\top = (A + A^\top) + (B + B^\top) = T(A) + T(B).$$

Similarly $T(cA) = (cA)^\top = cA^\top = cT(A)$. So T is linear.

(b) By definition $\ker(T) = \{A \in \mathbb{R}^{n\times n} \mid T(A) = 0\}$. So we have the following equivalences: $A \in \ker(T)$ if and only if $T(A) = 0$ if and only if $A + A^\top = 0$ if and only if $A = -A^\top$. This is exactly the definition of being a skew-symmetric matrix.

Suppose $A \in \mathrm{Sym}_n$, notice that

$$T(\tfrac{1}{2}A) = \tfrac{1}{2}A + \tfrac{1}{2}A^\top = \tfrac{1}{2}A + \tfrac{1}{2}A = A$$

where the second equality follows because $A \in \mathrm{Sym}_n$ so $A = A^\top$. This shows that $\mathrm{Sym}_\mathrm{n} \subseteq \operatorname{im}(T)$. For the converse, suppose $A \in \operatorname{im}(T)$, this means $A = B + B^\top$ for some $B \in \mathbb{R}^{n\times n}$.

But then

$$A^\top = (B + B^\top)^\top = B^\top + B = A$$

so A is a symmetric matrix. This proves $\mathrm{im}(T) = \mathrm{Sym}_n$.

(c) We showed in class that the kernel and image of a linear transformation are both subspaces, so part (b) implies part (c).

(d) Rank-Nullity theorem in combination with part (b) shows $\dim(\mathrm{Sym}_n) + \dim(\mathrm{Skew}_n) = n^2$. We will argue that $\dim(\mathrm{Sym}_n) = \frac{n(n+1)}{2}$ so that $\dim(\mathrm{Skew}_n) = \frac{n(n-1)}{2}$. From the first sentence, we only need to argue $\dim(\mathrm{Sym}_n) = \frac{n(n+1)}{2}$. A basis for Sym_n is given by the matrices A_{ij} with $i \geq j$ whose $(i,j) = (j,i)$ entry is equal to 1. There are exactly $\frac{n(n+1)}{2}$ such matrices. To see they form a basis, first notice that any symmetric matrix is a linear combination of these A_{ij}. Indeed if B is any other symmetric matrix with entries (b_{ij}) then $B = \sum_{i \geq j} b_{ij} A_{ij}$. Furthermore, the A_{ij} are linearly independent because their non-zero entries are always in different positions.

(e) In order for a matrix to be both symmetric and skew it must be true that

$$A = A^\top = -A^\top.$$

Entry by entry this implies that $a_{ij} = a_{ji} = -a_{ij}$ for every $1 \leq i, j \leq n$. But this means that $2a_{ij} = 0$, so $a_{ij} = 0$, i.e. A is the zero matrix.

For the second part, we will show that $\mathrm{Sym}_n + \mathrm{Skew}_n = \mathbb{R}^{n \times n}$. One way to do this is by dimension count. Since both Sym_n and Skew_n are subspaces of $\mathbb{R}^{n \times n}$, so is $\mathrm{Sym}_n + \mathrm{Skew}_n$. $\dim(\mathbb{R}^{n \times n}) = n^2$, so if we can show $\dim(\mathrm{Sym}_n + \mathrm{Skew}_n) = n^2$ they must be equal. But $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$. This is exercise 69 in Section 3.3. We just showed that $\dim(V \cap W) = 0$, so $\dim(V + W) = \dim(V) + \dim(W) = \frac{n^2}{2} + \frac{n^2}{2}$ from part (d). This proves $\dim(V + W) = \dim(\mathbb{R}^{n \times n})$. So $\mathrm{Sym}_n + \mathrm{Skew}_n = \mathbb{R}^{n \times n}$.

Alternative Solution: Alternatively, $\mathbb{R}^{n \times n}$ has a basis given by the matrices E_{ij} with (i,j) entry equal to 1 and every other entry 0. We will show that each $E_{ij} \in \mathrm{Sym}_n + \mathrm{Skew}_n$. Indeed, it is the sum of the symmetric matrix with $\frac{1}{2}$ in entry (i,j) and (j,i) and the skew symmetric matrix with $\frac{1}{2}$ in entry (i,j) and $-\frac{1}{2}$ in entry (j,i) for $i \neq j$. If $i = j$, E_{ij} is a symmetric matrix. This proves that $\mathrm{Sym}_n + \mathrm{Skew}_n$ contains every element of a basis for $\mathbb{R}^{n \times n}$ and since it is a subspace it contains the span of this basis as well, so $\mathbb{R}^{n \times n} \subset \mathrm{Sym}_n + \mathrm{Skew}_n$. But we also have $\mathrm{Sym}_n + \mathrm{Skew}_n \subset \mathbb{R}^{n \times n}$ so they are equal.

Problem 5. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Suppose V is finite-dimensional, with $\dim V = n$.[2]

(a) Let $\mathcal{B}_0 = (\vec{b}_1, \ldots, \vec{b}_k)$ be a basis of $\ker(T)$, and extend $\mathcal{B}_0$ to a basis

$$\mathcal{B} \ = \ (\vec{b}_1, \ldots, \vec{b}_k, \vec{b}_{k+1}, \ldots, \vec{b}_n)$$

of V, so we have $0 \leq k \leq n$. (Note that we can always do this; i.e., such $\mathcal{B}_0$ and $\mathcal{B}$ do exist.[3]) Prove that $(T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n))$ is a basis of $\mathrm{im}(T)$.

[2]Note that n could be 0 here, in which case V is the trivial vector space that just contains the zero vector, i.e., $V = \{\vec{0}\}$. Recall that the empty set, $\emptyset$, is a basis of the trivial vector space $\{\vec{0}\}$.

[3]Note also that having $k = 0$ here just means that $\mathcal{B}_0 = \emptyset$, having $n = 0$ means that $\mathcal{B}_0 = \mathcal{B} = \emptyset$, and having $k = n$ implies that $(T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n))$ is the empty list. Each of these cases still makes sense, even if the notation may seem a little funny.

(b) Prove the *Rank-Nullity Theorem* for the finite dimensional vector space V; that is, prove that
$$\dim(V) = \dim(\operatorname{im} T) + \dim(\ker T).$$

Solution.

(a) We first note that if T is injective, we proved this result on the previous homework. This corresponds to the case where $k = 0$. Next we note that if T is the zero map, i.e. $k = n$, this result is true because $\{\emptyset\}$ is a basis for the zero vector space.

For the rest of this proof we will assume $0 < k < n$. First, we prove that $\operatorname{im}(T) = \operatorname{span}(T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n))$. Let $\vec{y} \in \operatorname{im}(T)$, then there exists at least one vector $\vec{x} \in V$ such that $T(\vec{x}) = \vec{y}$. Since the set $\{\vec{b}_1, \ldots, \vec{b}_k, \vec{b}_{k+1}, \ldots, \vec{b}_n\}$ is a basis of V, there exist a unique set $\{a_1, \ldots, a_n\}$ such that $\vec{x} = \sum_{i=1}^n a_i \vec{b}_i$. After applying the linear transformation T, one has that
$$\vec{y} = T(\vec{x}) = T(\sum_{i=1}^n a_i \vec{b}_i) = \sum_{i=1}^n a_i T(\vec{b}_i).$$
Since $\{\vec{b}_1, \ldots, \vec{b}_k\}$ is a basis of $\ker(T)$, $T(\vec{b}_1) = \cdots = T(\vec{b}_k) = \vec{0}$. Thus, we can rewrite $\vec{y}$ as
$$\vec{y} = \sum_{i=k+1}^n a_i T(\vec{b}_i).$$
Therefore, we conclude that $\operatorname{im}(T) = \operatorname{span}(T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n))$.

Next, we show that $\{T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n)\}$ is a linearly independent set. Consider the linear relation $\sum_{i=k+1}^n x_i T(\vec{b}_i) = \vec{0}$. Since T is a linear transformation, this equation can be rewritten as $T(\sum_{i=k+1}^n x_i \vec{b}_i) = \vec{0}$. In other words, $\sum_{i=k+1}^n x_i \vec{b}_i \in \ker(T)$. Since $\{\vec{b}_1, \ldots, \vec{b}_k\}$ is a basis of $\ker(T)$, there exist constants $x_1, \ldots, x_k$ such that
$$\sum_{i=k+1}^n x_i \vec{b}_i = \sum_{i=1}^k x_i \vec{b}_i.$$
Since the set $\{\vec{b}_1, \ldots, \vec{b}_k, \vec{b}_{k+1}, \ldots, \vec{b}_n\}$ is linearly independent (because it is a basis), all the coefficients x_i must be zero. In other words, the linear relation $\sum_{i=k+1}^n x_i T(\vec{b}_i) = \vec{0}$ implies $x_i = 0$ for all $i = k+1, \ldots, n$, so $\{T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n)\}$ is linearly independent.

Combining the two facts, we conclude that $\{T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n)\}$ is a basis of $\operatorname{im}(T)$.

(b) From part (b), we know that $\dim(V) = n$, $\dim(\ker(T)) = k$ and $\dim(\operatorname{im}(T)) = n - k$, so we conclude that $\dim(V) = \dim(\operatorname{im} T) + \dim(\ker T)$.