Math 217 – Final Exam

Fall 2019

Solutions

Student ID Number: ______________________ Section: __________

Question	Points	Score
1	12	
2	16	
3	10	
4	16	
5	13	
6	12	
7	10	
8	11	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The *dimension* of the subspace V of $\mathbb{R}^n$

   Solution: The *dimension* of the subspace V of $\mathbb{R}^n$ is the number of vectors in any basis of V.

   (b) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   Solution: The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent* if for all $c_1, \ldots, c_n \in \mathbb{R}$, if $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ then $c_i = 0$ for each $1 \leq i \leq n$.

   (c) The *rank* of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

   Solution: The *rank* of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the dimension of $\mathrm{im}(T)$.

   (d) For a subset X of the vector space V, the *span* of X in V

   Solution: The *span* of X in V is the set

$$\left\{ \sum_{i=1}^n c_i\vec{v}_i \;:\; n \in \mathbb{N},\, c_1, \ldots, c_n \in \mathbb{R}, \text{ and } \vec{v}_1, \ldots, \vec{v}_n \in X \right\}.$$

   Solution: The *span* of X in V is the set of all linear combinations of vectors in V.

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) For all $n \in \mathbb{N}$, the set W of all orthogonal $n \times n$ matrices is a subspace of the vector space $\mathbb{R}^{n\times n}$.

   Solution: FALSE. For any n, the $n \times n$ zero matrix is not orthogonal, but every subspace of $\mathbb{R}^{n\times n}$ contains the $n \times n$ zero matrix.

   (b) (3 points) For all integers $0 \leq k \leq n$, if the $n \times n$ matrix A has k distinct eigenvalues, then rank $A \geq k$.

   Solution: FALSE. For instance, the 1×1 zero matrix has one eigenvalue (namely 0), but its rank is zero.

   (c) (3 points) If $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is a linear transformation whose image is contained in its kernel, then $\operatorname{rank}(T) \leq 4$.

   Solution: TRUE. Let $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ be a linear transformation such that $\operatorname{im}(T) \subseteq \ker(T)$, so $\dim(\operatorname{im}(T)) \leq \dim(\ker(T))$. Then, using the Rank-Nullity Theorem for the second equality below, we have

   $$9 \;=\; \dim(\mathbb{R}^{3\times 3}) \;=\; \dim(\operatorname{im}(T)) + \dim(\ker(T)) \;\leq\; 2\dim(\ker(T)),$$

   which shows $\dim(\ker(T)) \geq 5$. Since $\operatorname{rank}(T) = 9 - \dim(\ker(T))$ by Rank-Nullity, this implies $\operatorname{rank}(T) \leq 4$.

(Problem 2, Continued).

(d) (3 points) For every matrix $A \in \mathbb{R}^{m\times n}$, if $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, then the columns of A are linearly independent.

Solution: TRUE. Let $A \in \mathbb{R}^{m\times n}$, and assume that $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then for each $1 \leq i, j \leq n$, we have

$$A\vec{e}_i \cdot A\vec{e}_j \ = \ \vec{e}_i \cdot \vec{e}_j \ = \ \delta_{ij},$$

which shows that the columns of A form an orthonormal list of vectors in $\mathbb{R}^m$. But we proved in class that orthonormal lists of vectors are linearly independent.

(e) (4 points) For every matrix $A \in \mathbb{R}^{n\times n}$, if $A^2 = A$ then A is diagonalizable over $\mathbb{R}$.

Solution: TRUE. Let $A \in \mathbb{R}^{n\times n}$ and suppose $A^2 = A$. Let $y \in \operatorname{im}(A)$ be arbitrary, and fix $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{y}$. Then $A\vec{y} = A(A\vec{x}) = A^2\vec{x} = A\vec{x} = \vec{y}$. Thus $\operatorname{im}(A)$ consists of eigenvector of A with eigenvalue 1. Furthermore, $\ker(A)$ consists of eigenvectors of A with eigenvalue 0. Let $\mathcal{B}$ be a basis of $\ker(A)$ and $\mathcal{C}$ a basis of $\operatorname{im}(A)$. Then $\mathcal{B} \cup \mathcal{C}$ consists of eigenvectors of A, has size n by Rank-Nullity, and is linearly independent since $0 \neq 1$. So $\mathcal{B} \cup \mathcal{C}$ is an eigenbasis for A, which means A is diagonalizable.

3. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable x. Let $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by

$$T(p)(x) = p'(x) + p''(x) \quad \text{for all } x \in \mathbb{R}.$$

(a) (6 points) *(No justification is necessary for this part of the problem.)*

(i) Find a basis of $\mathrm{im}(T)$.

Solution: $(1, x)$

(ii) Find a basis of $\ker(T)$.

Solution: (1)

(iii) Compute $\det(T)$.

Solution: $\det(T) = 0$

(b) (4 points) Find a polynomial p that is an eigenvector of T, and find the associated eigenvalue along with the geometric multiplicity of this eigenvalue. Justify your answer.

Solution: The constant polynomial $p(x) = 1$ is an eigenvector of T with corresponding eigenvalue 0. We have $\mathrm{gemu}(0) = \dim(E_0) = \dim(\ker(T)) = 1$ by part (a)(ii).

4. Consider the 3×3 matrix $A = \begin{bmatrix} a & 0 & 1 \\ 0 & b & 0 \\ -1 & 0 & 0 \end{bmatrix}$, where $a, b \in \mathbb{R}$. In parts (a) – (d) below, find *all* values of a and b for which the given condition holds, or else write "none" if there are no such values. *(No justification is needed for any part of this problem.)*

(a) (2 points) A is invertible.

Solution: Since $\det(A) = b$, we see that A is invertible iff $b \neq 0$.

(b) (2 points) A is orthogonal.

Solution: $a = 0$ and $b = \pm 1$.

(c) (2 points) A is orthogonally diagonalizable.

Solution: None, since A is not symmetric no matter what a and b are.

(d) (4 points) A has one eigenvalue with algebraic multiplicity 3.

Solution: ($a = 2$ and $b = 1$) or ($a = -2$ and $b = -1$). To see this, note that the characteristic polynomial of A is $f_A(t) = (t - b)(t^2 - at + 1)$, so for A to have a single eigenvalue with almu 3, the discriminant of $t^2 - at + 1$ must be zero, which means $a = \pm 2$. If $a = 2$ then $f_A(t) = (t - b)(t - 1)^2$, which gives us $b = 1$, and if $a = -2$ then $f_A(t) = (t - b)(t + 1)^2$, which gives us $b = -1$.

For parts (e) and (f) below, fix $b = 1$, and find all values of a for which the given condition holds or else write "none" if there are no such values.

(e) (3 points) A is diagonalizable over $\mathbb{R}$.

Solution: $a < -2$ or $a > 2$. (To see this, use the characteristic polynomial $f_A(t) = (t - 1)(t^2 - at + 1)$, and note that when $a = 2$ we have $1 = \text{gemu}(1) < \text{almu}(1) = 3$, and when $a = -2$ we have $1 = \text{gemu}(-1) < \text{almu}(-1) = 2$.)

(f) (3 points) A is diagonalizable over $\mathbb{C}$.

Solution: $a \neq \pm 2$. (If $b = 1$ and $a \neq \pm 2$, then A has three distinct complex eigenvalues and is therefore diagonalizable over $\mathbb{C}$. If $b = 1$ and $a = \pm 2$, then A fails to be diagonalizable over $\mathbb{C}$ for the reasons given in (e) above.)

5. Let V be a k-dimensional subspace of $\mathbb{R}^n$, where $0 < k < n$. Let $\operatorname{refl}_V : \mathbb{R}^n \to \mathbb{R}^n$ be reflection through the subspace V, so $\operatorname{refl}_V(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$ and $\operatorname{refl}_V(\vec{w}) = -\vec{w}$ for all $\vec{w} \in V^\perp$. Let A be the standard matrix of refl_V.

   (a) (3 points) Find $\det(A)$ in terms of n and k. *(No justification needed.)*

   Solution: $\det(A) = (-1)^{n-k}$.

   (b) (4 points) Is A symmetric? Answer yes or no, and briefly justify your answer.

   Solution: Yes, A is symmetric by the Spectral Theorem since A is orthgonally diagonalizable. (To see this, note that if $\mathcal{B}$ is an orthonormal basis of $V = E_1$ and $\mathcal{C}$ is an orthonormal basis of $V^\perp = E_{-1}$, then $\mathcal{B} \cup \mathcal{C}$ is an orthonormal eigenbasis of V.)

   (c) (4 points) Assuming $n \geq 3$, find the area of the parallelogram P in $\mathbb{R}^n$ determined by the vectors $\vec{v}_1 = \vec{e}_1 + \vec{e}_2$ and $\vec{v}_2 = \vec{e}_1 + \vec{e}_3$ in $\mathbb{R}^n$.

   Solution: Let $A = [\vec{v}_1\ \vec{v}_2] \in \mathbb{R}^{n\times 2}$. Then the area of P is $\sqrt{\det(A^\top A)}$, where

$$A^\top A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

   Thus the area of P is $\sqrt{3}$.

   (d) (2 points) With P as in part (c), find the area of $\operatorname{refl}_V[P]$. You may give your answer in terms of the area of P, if you wish. *(No justification needed.)*

   Solution: Since refl_V is an orthogonal transformation, the area of $\operatorname{refl}_V[P]$ is the same as the area of P, namely $\sqrt{3}$.

6. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Let $\mathcal{B} = (\vec{x}, \vec{y}, \vec{z})$ be a basis of $\mathbb{R}^3$, and assume that

$$T(\vec{x}) = \vec{y}, \qquad T(\vec{y}) = \vec{z}, \qquad T(\vec{z}) = \vec{x}.$$

Let A be the standard matrix of T, so that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

(a) (4 points) Compute $\det(A)$. Justify your answer.

Solution: If $\mathcal{B} = (\vec{x}, \vec{y}, \vec{z})$, then $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, so $\det(A) = \det[T]_{\mathcal{B}} = 1$ since $[T]_{\mathcal{B}}$ can be converted to I_3 be performing two row swaps.

(b) (4 points) Find an eigenvector of T and the corresponding eigenvalue. Justify your answer.

Solution: Let $\vec{v} = \vec{x} + \vec{y} + \vec{z}$. Then $\vec{v} \neq \vec{0}$ since $\mathcal{B}$ is a basis of $\mathbb{R}^3$, and $T(\vec{v}) = T(\vec{x}) + T(\vec{y}) + T(\vec{z}) = \vec{y} + \vec{z} + \vec{x} = \vec{v}$. Thus $\vec{v}$ is an eigenvector of T with corresponding eigenvalue 1.

(c) (4 points) Determine whether T is diagonalizable, and justify your answer.

Solution: The characteristic polynomial of T is

$$\det\left(tI_3 - [T]_{\mathcal{B}}\right) \;=\; \det \begin{bmatrix} t & 0 & -1 \\ -1 & t & 0 \\ 0 & -1 & t \end{bmatrix} \;=\; t^3 - 1 \;=\; (t-1)(t^2+t+1).$$

Since $t^2 + t + 1$ has no real roots, it follows that T has just one real eigenvalue (even counting multiplicities), so T is not diagonalizable.

7. Fix an $n \times n$ matrix M, and let $T\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the linear transformation defined by $T(A) = MA$ for all $A \in \mathbb{R}^{n\times n}$.

   (a) (4 points) Prove that if $\vec{v}$ is an eigenvector of M with eigenvalue λ, then the matrix $A = \begin{bmatrix} | & & | \\ \vec{v} & \cdots & \vec{v} \\ | & & | \end{bmatrix}$ with all columns equal to $\vec{v}$ is an eigenvector of T.

   Solution: Suppose $\vec{v}$ is an eigenvector of M with eigenvalue λ. Then $\vec{v} \neq \vec{0}$, so $A \neq 0$, and we have

$$T(A) = MA = M \begin{bmatrix} | & & | \\ \vec{v} & \cdots & \vec{v} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ M\vec{v} & \cdots & M\vec{v} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda\vec{v} & \cdots & \lambda\vec{v} \\ | & & | \end{bmatrix} = \lambda A.$$

   Thus A is an eigenvector of T with corresponding eigenvalue λ.

   (b) (6 points) Prove that if M has n distinct real eigenvalues, then T is diagonalizable.

   Solution: Suppose that M has n distinct real eigenvalues $\lambda_1, \ldots \lambda_n$, with corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. By part (a), each λ_i is also an eigenvalue of T. Since $\dim(\mathbb{R}^{n\times n}) = n^2$, in order to show that T is diagonalizable it will suffice to show that $\sum_{i=1}^n \operatorname{gemu}(\lambda_i) = n^2$. Fix $1 \leq i \leq n$. For each $1 \leq j \leq n$, let A_{ij} be the matrix whose jth column is $\vec{v}_i$ and whose other columns are all $\vec{0}$. Then $A_{ij} \neq 0$ and $T(A_{ij}) = MA_{ij} = \lambda_i A_{ij}$, so A_{ij} is an eigenvector of T. This is true for each $1 \leq j \leq n$, so $\operatorname{gemu}(\lambda_i) = n$. Thus $\sum_{i=1}^n \operatorname{gemu}(\lambda_i) = \sum_{i=1}^n j = n^2$, showing that T is diagonalizable as desired.

8. Let V be an inner product space of dimension n with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ be an orthonormal basis of V with respect to this inner product. Let $T \colon V \to V$ be a linear transformation and assume for all $\vec{x}, \vec{y} \in V$ that

$$\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T(\vec{y}) \rangle.$$

(a) (5 points) Prove that $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix of T, is a symmetric matrix.

Solution: Let $1 \leq i, j \leq n$. Then the jth column of $[T]_{\mathcal{B}}$ is $[T(\vec{b}_j)]_{\mathcal{B}}$. Since $\mathcal{B}$ is orthonormal, we have

$$T(\vec{b}_j) \ = \ \sum_{k=1}^{n} \langle T(\vec{b}_j), \vec{b}_k \rangle \vec{b}_k,$$

so the (i, j)-entry of $[T]_{\mathcal{B}}$ is the ith component of $[T(\vec{b}_j)]_{\mathcal{B}}$, which is $\langle T(\vec{b}_j), \vec{b}_i \rangle$. By hypothesis we have $\langle T(\vec{b}_j), \vec{b}_i \rangle = \langle \vec{b}_j, T(\vec{b}_i) \rangle$, and $\langle \vec{b}_j, T(\vec{b}_i) \rangle = \langle T(\vec{b}_i), \vec{b}_j \rangle$ is the (j, i)-entry of $[T]_{\mathcal{B}}$ by the argument above. Thus $[T]_{\mathcal{B}}$ is symmetric.

(b) (6 points) Prove that there exists an orthonormal basis $\mathcal{U}$ of V which is an eigenbasis for the linear transformation T.

Solution: Using part (a) and the Spectral Theorem, fix an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that $Q^\top [T]_{\mathcal{B}} Q$ is diagonal. For each $1 \leq i \leq n$, let $\vec{u}_i = L_{\mathcal{B}}^{-1}(Q\vec{e}_i)$. Then $Q\vec{e}_i = [\vec{u}_i]_{\mathcal{B}}$ for each i, so $Q = S_{\mathcal{U} \to \mathcal{B}}$. We claim that $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ is an orthonormal eigenbasis for T. To see that $\mathcal{U}$ is orthonormal, observe that

$$\langle \vec{u}_i, \vec{u}_j \rangle \ = \ [\vec{u}_i]_{\mathcal{B}} \cdot [\vec{u}_j]_{\mathcal{B}} \ = \ Q[\vec{u}_i]_{\mathcal{U}} \cdot Q[\vec{u}_j]_{\mathcal{U}} \ = \ \vec{e}_i^{\,\top} Q^\top Q \vec{e}_j \ = \ \vec{e}_i \cdot \vec{e}_j \ = \ \delta_{ij}$$

for each $1 \leq i, j \leq n$. Finally, note that the matrix

$$[T]_{\mathcal{U}} \ = \ S_{\mathcal{B} \to \mathcal{U}} [T]_{\mathcal{B}} S_{\mathcal{U} \to \mathcal{B}} \ = \ Q^\top [T]_{\mathcal{B}} Q$$

is diagonal, se each $\vec{u}_i$ is an eigenvector of T.