Math 217 – Midterm 2

Winter 2018

Time: 120 mins.

1. Answer each question in the space provided, circling your answer where appropriate. If you require more space, you may use the blank page at the end of the exam. **You must clearly indicate, in the provided answer space, if you do this.** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work.
3. No calculators, notes, or other outside assistance is allowed.

Name: ______________________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	12	
4	15	
5	12	
6	12	
7	12	
8	10	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) An *isomorphism* from the vector space V to the vector space W

   (b) The matrix $A \in \mathbb{R}^{n\times n}$ is *orthogonal*

   (c) A *least-squares solution* of the system of linear equations $A\vec{x} = \vec{b}$

   (d) The *norm* (or *magnitude*, or *length*) of the vector $\vec{v}$ in the inner product space $(V, \langle \cdot, \cdot \rangle)$

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) For any matrix $A \in \mathbb{R}^{m\times n}$, if the columns of A form an orthonormal list of vectors in $\mathbb{R}^m$, then $AA^T = I_m$, where I_m is the $m \times m$ identity matrix.

   (b) (3 points) If A is an invertible 2018×2018 matrix, then $\det(-A) = -\det(A)$.

   (c) (3 points) For every orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$, $|\det(S_{\mathcal{E}\to\mathcal{B}})| = 1$, where $\mathcal{E}$ is the standard basis of $\mathbb{R}^n$ and $S_{\mathcal{E}\to\mathcal{B}}$ is the change-of-coordinates matrix from $\mathcal{E}$ to $\mathcal{B}$.

(Problem 2, Continued).

(d) (3 points) For any matrix $A \in \mathbb{R}^{m\times n}$, if the columns of A are linearly independent, then there is an $n \times m$ matrix B such that $BA = I_n$, where I_n is the $n \times n$ identity matrix.

(e) (3 points) For any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$,

$$\det \begin{bmatrix} | & | \\ \vec{u}-\vec{v} & \vec{v}-\vec{w} \\ | & | \end{bmatrix} = \det \begin{bmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{bmatrix} - \det \begin{bmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{bmatrix}.$$

3. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$. The rule

$$\langle \vec{x}, \vec{y} \rangle \;=\; \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \;=\; \vec{x}^\top \begin{bmatrix} 8 & -6 \\ -6 & 5 \end{bmatrix} \vec{y} \;=\; 8x_1y_1 - 6(x_1y_2 + x_2y_1) + 5x_2y_2$$

defines an inner product on $\mathbb{R}^2$. (You may assume this without proof).

(a) (4 points) Compute the length $\vec{v}$ with respect to $\langle \cdot, \cdot \rangle$.

(b) (4 points) Find a nonzero vector $\vec{w} \in \mathbb{R}^2$ that is orthogonal (relative to $\langle \cdot, \cdot \rangle$) to $\vec{v}$.

(c) (4 points) Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ that is orthonormal with respect to $\langle \cdot, \cdot \rangle$.

4. Let $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ and $\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$, so that $\mathcal{E}$ and $\mathcal{B}$ are ordered bases of the vector space V of 2×2 upper-triangular matrices.

(a) (3 points) Find an ordered basis $\mathcal{C}$ of V such that $S_{\mathcal{C} \to \mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

(b) (3 points) Find the $\mathcal{B}$-coordinates $[A]_{\mathcal{B}}$ of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(c) (5 points) Find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of the linear transformation $T : V \to V$, where T is defined so that for each $A \in V$, $T(A) = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} A$.

(d) (4 points) Find $\det(T)$, where T is the linear transformation from part (c).

5. Let A be an $m \times n$ matrix with linearly independent columns, let $A = QR$ be the QR-factorization of A, and let $\vec{b} \in \mathbb{R}^m$.

   (a) (3 points) Write the normal equation of the linear system $A\vec{x} = \vec{b}$.

   (b) (6 points) Show that the vector $\vec{x}^* = R^{-1}Q^\top\vec{b}$ is a least-squares solution of the linear system $A\vec{x} = \vec{b}$.

   (c) (3 points) Is $R^{-1}Q^\top\vec{b}$ the *unique* least-squares solution of the linear system $A\vec{x} = \vec{b}$? Explain.

6. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable t, and consider $\mathcal{P}_2$ as an inner product space with inner product

$$\langle p, q \rangle \; = \; p(0)q(0) + p'(0)q'(0) + \frac{1}{2}p''(0)q''(0).$$

Also let $f(t) = 1 + t$ and $g(t) = 2 - t^2$ be polynomials in $\mathcal{P}_2$, and let $W = \text{span}(f, g)$ be the subspace of $\mathcal{P}_2$ spanned by f and g.

(a) (6 points) Find a basis of W that is orthonormal with respect to $\langle \cdot, \cdot \rangle$.

(b) (6 points) Let $\text{proj}_W : \mathcal{P}_2 \to \mathcal{P}_2$ be the orthogonal projection onto W in $\mathcal{P}_2$. Find a polynomial $h \in \mathcal{P}_2$ such that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the $\mathcal{B}$-matrix of proj_W, where $\mathcal{B}$ is the ordered basis of $\mathcal{P}_2$ given by $\mathcal{B} = (f, g, h)$.

7. (a) (5 points) Prove that for every $n \times n$ matrix A, if $A^\top A = AA^\top$ then $\|Ax\| = \|A^\top x\|$ for all $x \in \mathbb{R}^n$. (Here length is defined with respect to the dot product on $\mathbb{R}^n$).

(b) (7 points) Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, and let $T : V \to V$ be a linear transformation. Prove that if $\langle T(v), w\rangle = \langle v, T(w)\rangle$ for all $v, w \in V$, then $\ker(T) = (\mathrm{im}(T))^\perp$.

8. (10 points) Let $n \in \mathbb{N}$, let V be a subspace of $\mathbb{R}^n$, and let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto V. Prove that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, $P \circ T = T \circ P$ if and only if $T[V] \subseteq V$ and $T[V^\perp] \subseteq V^\perp$.

   $\big($Recall that, by definition, $T[X] = \{T(x) \,:\, x \in X\}$ for any subset $X \subseteq \mathbb{R}^n\big)$.

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