Worksheet 16: Gram-Schmidt and QR-Factorization (§5.2)

Recall that a set $(\vec{v}_1, \ldots, \vec{v}_r)$ of vectors in $\mathbb{R}^n$ is *orthonormal* if for each i, j,

$$\vec{v}_i \cdot \vec{v}_j \;=\; \delta_{ij} \;=\; \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Orthonormal sets are always linearly independent, so any orthonormal set of n vectors in $\mathbb{R}^n$ is an *orthonormal basis* of $\mathbb{R}^n$.

Problem 1.

(a) Let V_1 be the subspace of $\mathbb{R}^4$ given by

$$\begin{array}{rrrrcl} x_1 & -x_2 & -2x_3 & & = & 0, \\ & x_2 & +x_3 & -2x_4 & = & 0. \end{array}$$

Find an orthonormal basis of V_1.

Solution: The standard procedure for finding a basis of the kernel of a matrix produces

$$(\vec{u}, \vec{v}) \;=\; \left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right),$$

which is already orthogonal, so normalizing gives the orthonormal basis $\left(\frac{1}{\sqrt{3}}\vec{u}, \frac{1}{3}\vec{v}\right)$.

(b) Let V_2 be the subspace of $\mathbb{R}^4$ given by

$$x_1 + x_2 - x_3 - 2x_4 \;=\; 0.$$

Find an orthonormal basis of V_2.

Solution: The standard procedure for finding a basis of the kernel of a matrix produces

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) \;=\; \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Now let $\vec{w}_1 = \vec{v}_1$, $\vec{w}_2 = \vec{v}_2 - \mathrm{proj}_{\vec{w}_1}(\vec{v}_2)$, and $\vec{w}_3 = \vec{v}_3 - \mathrm{proj}_{\vec{w}_1}(\vec{v}_3) - \mathrm{proj}_{\vec{w}_2}(\vec{v}_3)$, so that

$$(\vec{w}_1, \vec{w}_2, \vec{w}_3) \;=\; \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ 3 \end{bmatrix} \right).$$

Normalizing the vectors in $(\vec{w}_1, \vec{w}_2, \vec{w}_3)$ gives an orthonormal basis of V_2.

Problem 2. Let $\vec{x}$ be the vector

$$\vec{x} \;=\; \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} \;\in\; \mathbb{R}^4.$$

Find the orthogonal projections of $\vec{x}$ onto the subspaces V_1, V_2 from Problem 1.

Solution: Letting $\vec{u}, \vec{v}$ be as in the solution to Problem 1(a), we have

$$\mathrm{proj}_{V_1}(\vec{x}) \;=\; \left(\vec{x}\cdot\frac{1}{\sqrt{3}}\vec{u}\right)\frac{1}{\sqrt{3}}\vec{u} + \left(\vec{x}\cdot\frac{1}{3}\vec{v}\right)\frac{1}{3}\vec{v} \;=\; \frac{1}{9}\begin{bmatrix} 10 \\ 22 \\ -6 \\ 8 \end{bmatrix}.$$

By a similar computation, (or by noting that $\vec{x} \in V_2$),

$$\mathrm{proj}_{V_2}(\vec{x}) \;=\; \vec{x} \;=\; \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}.$$

Problem 3. Suppose $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_r)$ is a basis of the subspace V of $\mathbb{R}^n$. Let

$$\vec{w}_1 = \vec{v}_1 \qquad \text{and} \qquad \vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|},$$

and then for each $1 \le k < r$ let

$$\vec{w}_{k+1} \;=\; \vec{v}_{k+1} - \sum_{i=1}^{k}(\vec{v}_{k+1}\cdot\vec{u}_i)\vec{u}_i \qquad \text{and} \qquad \vec{u}_{k+1} = \frac{\vec{w}_{k+1}}{\|\vec{w}_{k+1}\|}.$$

Then $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_r)$ is the orthonormal basis of V obtained by applying the Gram-Schmidt algorithm to $\mathcal{B}$.

(a) Find the change-of-coordinates matrix $S_{\mathcal{B}\to\mathcal{U}}$ in terms of the vectors $\vec{v}_i$, $\vec{w}_i$, and $\vec{u}_i$.

Solution: $S_{\mathcal{B}\to\mathcal{U}} =$

$$\begin{bmatrix} \|\vec{w}_1\| & \vec{u}_1\cdot\vec{v}_2 & \vec{u}_1\cdot\vec{v}_3 & \cdots & \cdots & \vec{u}_1\cdot\vec{v}_r \\ 0 & \|\vec{w}_2\| & \vec{u}_2\cdot\vec{v}_3 & \cdots & \cdots & \vec{u}_2\cdot\vec{v}_r \\ \vdots & 0 & \|\vec{w}_3\| & \cdots & \cdots & \vec{u}_3\cdot\vec{v}_r \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \|\vec{w}_r\| \end{bmatrix}, \text{ i.e., } S_{\mathcal{B}\to\mathcal{U}}(i,j) \;=\; \begin{cases} \vec{u}_i\cdot\vec{v}_j & \text{if } i<j; \\ \|\vec{w}_i\| & \text{if } i=j; \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find the QR-factorization of the matrix $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 1 & -2 \\ \frac{1}{2} & 0 & 4 \\ 0 & -1 & 0 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 1 & -2 \\ \frac{1}{2} & 0 & 4 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{-\sqrt{3}}{\sqrt{14}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{14}} \\ \frac{1}{2} & 0 & \frac{3}{\sqrt{14}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{14} \end{bmatrix}.$$

Problem 4. Assuming that V is a subspace of $\mathbb{R}^n$, find the following:

(a) $V \cap V^{\perp}$

(b) $\dim(V) + \dim(V^{\perp})$

(c) $(V^{\perp})^{\perp}$

Draw a picture in $\mathbb{R}^3$ to illustrate these facts!

Solution:

(a) $V \cap V^{\perp} = \{\vec{0}\}$

(b) $\dim(V) + \dim(V^{\perp}) = n$

(c) $(V^{\perp})^{\perp} = V$

Problem 5. Let V and W be subspaces of coordinate vector spaces (i.e., some $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), and suppose that $T : V \to W$ is an isomorphism.

(a) If $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_r\}$ is a basis of V, is the set $\{T(\vec{b}_1), \ldots, T(\vec{b}_r)\}$ necessarily a basis of W?

Solution: Yes!

(b) If $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_r\}$ is an orthonormal basis of V, is the set $\{T(\vec{b}_1), \ldots, T(\vec{b}_r)\}$ necessarily an orthonormal basis of W?

Solution: No! For instance, dilating $\mathbb{R}^2$ by a factor of 2 converts the orthonormal basis $(\vec{e}_1, \vec{e}_2)$ of $\mathbb{R}^2$ into the non-orthonormal basis $(2\vec{e}_1, 2\vec{e}_2)$.

(c) Which $n \times n$ matrices A have the property that for every orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_n\}$ of $\mathbb{R}^n$, the set $\{A\vec{u}_1, \ldots, A\vec{u}_n\}$ is also an orthonormal basis of $\mathbb{R}^n$?

Solution: By taking $(\vec{u}_1, \ldots, \vec{u}_n)$ to be the standard basis of $\mathbb{R}^n$, we see that the columns of A must form an orthonormal basis of $\mathbb{R}^n$. In fact, this is also a sufficient condition, since a product of orthogonal $n \times n$ matrices is itself an orthogonal matrix. We will be better equipped to understand this after defining *orthogonal* transformations and matrices on the next worksheet.