Worksheet 13: Coordinates and $\mathcal{B}$-Matrices (Theory) (§§3.4, 4.3)

Let V be any vector space, and suppose that $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is an ordered basis of V. Let $L_\mathcal{B} : V \to \mathbb{R}^n$ be the $\mathcal{B}$-*coordinate transformation* defined by

$$L_\mathcal{B}(\vec{v}) \;=\; \big[\vec{v}\big]_\mathcal{B} \;=\; \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad \text{where } \vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n.$$

Recall that $[\vec{v}]_\mathcal{B} \in \mathbb{R}^n$ is the $\mathcal{B}$-*coordinate vector* of $\vec{v}$, and that $L_\mathcal{B}$ is an isomorphism.

Problem 1: Warm-Up. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Recall how to find the *standard matrix* $[T]_\mathcal{E}$ of T, and how to show that your formula for $[T]_\mathcal{E}$ is correct. (This is important to remember now, since we will generalize this fact along with its proof below).

Solution: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then for all $\vec{x} \in \mathbb{R}^n$ we have

$$T(\vec{x}) \;=\; T\left(\sum_{i=1}^n x_i\vec{e}_i\right) \;=\; \sum_{i=1}^n x_i T(\vec{e}_i) \;=\; \big[T(\vec{e}_1) \;\cdots\; T(\vec{e}_n)\big]\vec{x},$$

so $[T]_\mathcal{E} = \big[T(\vec{e}_1) \;\cdots\; T(\vec{e}_n)\big]$.

Problem 2: Change-of-Coordinates. Let V be a vector space. If we choose a pair of ordered bases $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ and $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_n)$ of V, then $L_\mathcal{C} \circ L_\mathcal{B}^{-1}$ is an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, so its standard matrix S will be an invertible $n \times n$ matrix which is called the *change-of-coordinates* matrix from $\mathcal{B}$ to $\mathcal{C}$, denoted $S = S_{\mathcal{B}\to\mathcal{C}}$.

(a) Given $1 \le i \le n$, what is the ith column of $S_{\mathcal{B}\to\mathcal{C}}$? Explain why your claim is correct.

(b) Show that $S_{\mathcal{B}\to\mathcal{C}}[\vec{v}]_\mathcal{B} = [\vec{v}]_\mathcal{C}$ for all $\vec{v} \in V$.

(c) How are $S_{\mathcal{B}\to\mathcal{C}}$ and $S_{\mathcal{C}\to\mathcal{B}}$ related to each other? Can you prove your claim?

Solution:

(a) For each $1 \le i \le n$, the ith column of $S_{\mathcal{B}\to\mathcal{C}}$ is $S_{\mathcal{B}\to\mathcal{C}}\vec{e}_i = L_\mathcal{C}\big(L_\mathcal{B}^{-1}(\vec{e}_i)\big) = L_\mathcal{C}(\vec{b}_i) = [\vec{b}_i]_\mathcal{C}$.

(b) $S_{\mathcal{B}\to\mathcal{C}}$ is the standard matrix of $L_\mathcal{C} \circ L_\mathcal{B}^{-1}$, so for all $\vec{v} \in V$ we have

$$S_{\mathcal{B}\to\mathcal{C}}[\vec{v}]_\mathcal{B} \;=\; L_\mathcal{C}\big(L_\mathcal{B}^{-1}([\vec{v}]_\mathcal{B})\big) \;=\; L_\mathcal{C}(\vec{v}) \;=\; [\vec{v}]_\mathcal{C}.$$

(c) For all $\vec{v} \in V$, $S_{\mathcal{C}\to\mathcal{B}}S_{\mathcal{B}\to\mathcal{C}}[\vec{v}]_\mathcal{B} = S_{\mathcal{C}\to\mathcal{B}}[\vec{v}]_\mathcal{C} = [\vec{v}]_\mathcal{B}$ and $S_{\mathcal{B}\to\mathcal{C}}S_{\mathcal{C}\to\mathcal{B}}[\vec{v}]_\mathcal{C} = S_{\mathcal{B}\to\mathcal{C}}[\vec{v}]_\mathcal{B} = [\vec{v}]_\mathcal{C}$, which implies that $S_{\mathcal{B}\to\mathcal{C}}^{-1} = S_{\mathcal{C}\to\mathcal{B}}$ and $S_{\mathcal{C}\to\mathcal{B}}^{-1} = S_{\mathcal{B}\to\mathcal{C}}$.

Problem 3: What if $V = \mathbb{R}^n$? Let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ and $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_n)$ be ordered bases of $\mathbb{R}^n$, and write B and C for the invertible $n \times n$ matrices whose columns are the $\mathcal{B}$-basis vectors and the $\mathcal{C}$-basis vectors, respectively.

(a) What are the standard matrices of $L_{\mathcal{B}}$ and $L_{\mathcal{C}}$?

(b) Let $\vec{x} \in \mathbb{R}^n$. Using the matrices B and C, write equations relating $\vec{x}$ to $[\vec{x}]_{\mathcal{B}}$ and to $[\vec{x}]_{\mathcal{C}}$.

(c) Using the matrices B and C, write formulas for the change-of-coordinates matrices $S_{\mathcal{B}\to\mathcal{C}}$ and $S_{\mathcal{C}\to\mathcal{B}}$.

Solution:

(a) Since $L_{\mathcal{B}}^{-1}(\vec{e}_i) = \vec{b}_i$ for each i, the standard matrix of $L_{\mathcal{B}}^{-1}$ is B, which means the standard matrix of $L_{\mathcal{B}}$ is B^{-1}. Similarly, the standard matrix of $L_{\mathcal{C}}$ is C^{-1}.

(b) $B[\vec{x}]_{\mathcal{B}} = \vec{x}$ and $B^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$. Similarly, $C[\vec{x}]_{\mathcal{C}} = \vec{x}$ and $C^{-1}\vec{x} = [\vec{x}]_{\mathcal{C}}$.

(c) $S_{\mathcal{B}\to\mathcal{C}}$ is the standard matrix of $L_{\mathcal{C}} \circ L_{\mathcal{B}}^{-1}$, so from part (a) we see that $S_{\mathcal{B}\to\mathcal{C}} = C^{-1}B$. Likewise, $S_{\mathcal{C}\to\mathcal{B}} = B^{-1}C$.

Problem 4: The $\mathcal{B}$-matrix of a Linear Transformation. Let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ be an ordered basis of the vector space V, and let $T : V \to V$ be a linear transformation of V. Prove that there exists a unique $n \times n$ matrix $[T]_{\mathcal{B}}$, called the $\mathcal{B}$-*matrix* of T, such that

$$[T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} \;=\; \big[T(\vec{v})\big]_{\mathcal{B}}$$

for all $\vec{v} \in V$. Your proof should include finding an expression for $[T]_{\mathcal{B}}$ in terms of T and $\mathcal{B}$.

Solution: Consider the linear transformation $U = L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. We know that U has a standard matrix $[U]_{\mathcal{E}}$, which is unique, and has the property that $[U]_{\mathcal{E}}\vec{x} = U(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$. Let $\vec{v} \in V$ be arbitrary. Then

$$[U]_{\mathcal{E}}[\vec{v}]_{\mathcal{B}} \;=\; [U]_{\mathcal{E}}L_{\mathcal{B}}(\vec{v}) \;=\; U(L_{\mathcal{B}}(\vec{v})) \;=\; \big(L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1} \circ L_{\mathcal{B}}\big)(\vec{v}) \;=\; L_{\mathcal{B}}\big(T(\vec{v})\big) \;=\; [T(\vec{v})]_{\mathcal{B}}.$$

Thus we see that $[T]_{\mathcal{B}} = [U]_{\mathcal{E}} = \big[U(\vec{e}_1) \;\cdots\; U(\vec{e}_n)\big] = \big[[T(\vec{b}_1)]_{\mathcal{B}} \;\cdots\; [T(\vec{b}_n)]_{\mathcal{B}}\big]$.

Problem 5: Similarity. Let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ and $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_n)$ be ordered bases of V, and let $T : V \to V$ be a linear transformation.

(a) Find a simple equation relating $[T]_{\mathcal{B}}$, $[T]_{\mathcal{C}}$, and $S_{\mathcal{B}\to\mathcal{C}}$.

Solution: $[T]_{\mathcal{C}} = S_{\mathcal{B}\to\mathcal{C}}[T]_{\mathcal{B}}S_{\mathcal{B}\to\mathcal{C}}^{-1}$, or equivalently $[T]_{\mathcal{B}} = S_{\mathcal{B}\to\mathcal{C}}^{-1}[T]_{\mathcal{C}}S_{\mathcal{B}\to\mathcal{C}}$.

(b) The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B if there is an invertible $n \times n$ matrix S such that $B = S^{-1}AS$. Prove that the relation "A is similar to B" is an *equivalence relation* on the set of $n \times n$ matrices; i.e., prove that it is:

- *reflexive*: A is similar to A;
- *symmetric*: if A is similar to B, then B is similar to A; and
- *transitive*: if A is similar to B and B is similar to C, then A is similar to C.

Solution:

- For reflexivity, note that $A = I_n^{-1} A I_n$.
- For symmetry, if $B = S^{-1}AS$, then $(S^{-1})^{-1}BS^{-1} = A$.
- For transitivity, if $B = S^{-1}AS$ and $C = P^{-1}BP$, then $C = (SP)^{-1}A(SP)$.

(c) Find all matrices that are similar to I_n.

Solution: For any invertible $n \times n$ matrix S, we have $S^{-1}I_nS = S^{-1}S = I_n$, so the only matrix similar to I_n is I_n itself.