# Worksheet 12: Introduction to Coordinates (§§3.4, 4.3)



Let $V$ be any vector space, and suppose that $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ is an ordered basis of $V$. Let $L_{\mathcal{B}} : V \to \mathbb{R}^n$ be the $\mathcal{B}$-*coordinate transformation* defined by

$$L_{\mathcal{B}}(\vec{v}) \;=\; \left[\vec{v}\right]_{\mathcal{B}} \;=\; \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad \text{where } \vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n.$$

Recall that $[\vec{v}]_{\mathcal{B}} \in \mathbb{R}^n$ is the $\mathcal{B}$-*coordinate vector* of $\vec{v}$, and that $L_{\mathcal{B}}$ is an isomorphism.

**Problem 1.** Let $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$, and consider the ordered bases

$$\mathcal{E} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad \mathcal{B} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

of the vector space $\mathbb{R}^{2\times 2}$ of $2 \times 2$ matrices.

(a) Find $[I_2]_{\mathcal{E}}$ and $[A]_{\mathcal{E}}$.

**Solution:** $[I_2]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $[A]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

(b) Find $[I_2]_{\mathcal{B}}$ and $[A]_{\mathcal{B}}$.

**Solution:** $[I_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $[A]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$.

(c) Find an ordered basis $\mathcal{C}$ of $\mathbb{R}^{2\times 2}$ such that $[A]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

**Solution:** $\mathcal{C} = \left( \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right)$.

**Problem 2.** Let $V$ be the subspace of $\mathbb{R}^3$ given by $x_1 + x_2 - 2x_3 = 0$.

(a) Find an ordered basis $\mathcal{B}$ of $V$ in which the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has coordinates $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Solution:** $\mathcal{B} = \left( \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$

(b) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Find the coordinates of $\vec{v}_1$ in the basis $(\vec{v}_2, \vec{v}_3)$, of $\vec{v}_2$ in the basis $(\vec{v}_1, \vec{v}_3)$, and of $\vec{v}_3$ in the basis $(\vec{v}_1, \vec{v}_2)$.

**Solution:** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

**Problem 3.** We have seen that if $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ is an ordered basis of the vector space $V$, then the assignment of $\mathcal{B}$-coordinates $[\vec{v}]_{\mathcal{B}} \in \mathbb{R}^n$ to each vector $\vec{v} \in V$ is an isomorphism from $V$ to $\mathbb{R}^n$. Conversely, suppose that

$$T : V \to \mathbb{R}^n$$

is an isomorphism from $V$ to $\mathbb{R}^n$. Can you use $T$ to pick a basis for $V$?

**Solution:** Yes. Given an isomorphism $T : V \to \mathbb{R}^n$, the list $\left(T^{-1}(\vec{e}_1), \ldots, T^{-1}(\vec{e}_n)\right)$ is an ordered basis of $V$.

**Remark:** *This shows that picking a basis for a vector space $V$ of dimension $n$ is exactly the same as picking an isomorphism from $V$ to $\mathbb{R}^n$.*

**Problem 4.** Recall that in the past, we have associated to every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ an $m \times n$ matrix $A$ (called the *standard matrix* of $T$) such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

(a) Take a moment to remind yourself how this is done. For instance, suppose $T$ is the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 2x + 3y \end{bmatrix},$$

and find a matrix $A$ such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

**Solution:** $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Note that if we write $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ for the *standard basis* of $\mathbb{R}^2$, then we could rewrite the equation $A\vec{x} = T(\vec{x})$ in the seemingly more complicated form

$$A[\vec{x}]_{\mathcal{E}} = \big[T(\vec{x})\big]_{\mathcal{E}},$$

which says the exact same thing since $\vec{v} = [\vec{v}]_{\mathcal{E}}$ for all $\vec{v} \in \mathbb{R}^2$. But this form suggests that $A$ is really the matrix of $T$ *relative to the standard basis*, $\mathcal{E}$. Now let's try to find the matrix of $T$ relative to some other basis of $\mathbb{R}^2$ ...

(b) Let $\mathcal{B}$ be the ordered basis

$$\mathcal{B} = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$$

of $\mathbb{R}^2$, and let $B$ (notice the font change) be the $2 \times 2$ matrix

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Given $\vec{x} \in \mathbb{R}^2$, how are $B$, $\vec{x}$, and $[\vec{x}]_{\mathcal{B}}$ related to each other?

**Solution:** $B[\vec{x}]_{\mathcal{B}} = \vec{x}$, and $B^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$.

(c) Let $T$ be as in part (a), and let $\mathcal{B}$ (and $B$) be as in part (b). Find *the matrix of $T$ relative to $\mathcal{B}$*. In other words, find a matrix (let's call it $C$, for now) such that for all $\vec{x} \in \mathbb{R}^2$,

$$C[\vec{x}]_{\mathcal{B}} = \big[T(\vec{x})\big]_{\mathcal{B}}.$$

(Hint: find the columns of $C$).

**Solution:** Write $B = [\vec{b}_1\ \vec{b}_2] = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, and note that for all $\vec{x} \in \mathbb{R}^2$, we have $B[\vec{x}]_{\mathcal{B}} = \vec{x}$ and $[\vec{x}]_{\mathcal{B}} = B^{-1}\vec{x}$. To find the columns of $C$, we have

$$C\vec{e}_1 \ = \ C[\vec{b}_1]_{\mathcal{B}} \ = \ [T(\vec{b}_1)]_{\mathcal{B}},$$

and similarly $C\vec{e}_2 = [T(\vec{b}_2)]_{\mathcal{B}}$. Therefore

$$C \ = \ \big[\, [T(\vec{b}_1)]_{\mathcal{B}} \ \ [T(\vec{b}_2)]_{\mathcal{B}} \,\big] \ = \ \big[\, [A\vec{b}_1]_{\mathcal{B}} \ \ [A\vec{b}_2]_{\mathcal{B}} \,\big] \ = \ \big[\, B^{-1}A\vec{b}_1 \ \ B^{-1}A\vec{b}_2 \,\big] \ = \ B^{-1}AB.$$

Explicitly, $C = B^{-1}AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \ = \ \begin{bmatrix} -35 & -62 \\ 22 & 39 \end{bmatrix}$.

(d) How is $C$ related to $A$? Can you write a simple equation relating them?

**Solution:** As we saw above, $C = B^{-1}AB$.

**Problem 5.** Let $\vec{v} = \begin{bmatrix} 12 \\ 5 \end{bmatrix} \in \mathbb{R}^2$, and let $\ell = \operatorname{Span}(\vec{v})$ be the line in $\mathbb{R}^2$ generated by $\vec{v}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of $\mathbb{R}^2$ over the line $\ell$.

(a) Find the standard matrix of $T$.

**Solution:** Using $[T]_{\mathcal{E}} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ where $\tan\theta = \frac{5}{12}$, along with the trig identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we see that $[T]_{\mathcal{E}} = \frac{1}{169}\begin{bmatrix} 119 & 120 \\ 120 & -119 \end{bmatrix}$.

(b) Find an ordered basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the matrix of $T$ relative to $\mathcal{B}$ is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

**Solution:** For instance, we can take $\mathcal{B} = \left( \begin{bmatrix} 12 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ 12 \end{bmatrix} \right)$.