Math 217 – Final Exam

Winter 2018

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work.
3. No calculators, notes, or other outside assistance allowed.
4. On this exam, *unless stated otherwise*, terms such as "orthogonal" and "orthonormal" mean with respect to the usual dot product on $\mathbb{R}^n$.
5. On this exam, *unless stated otherwise*, "eigenvalue" means *real* eigenvalue, and terms such as "similar" and "diagonalizable" mean over $\mathbb{R}$.

Name: ________________________________ Section: ____________

Question	Points	Score
1	10	
2	15	
3	14	
4	12	
5	14	
6	12	
7	11	
8	12	
Total:	100	

1. (10 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) A *subspace* of the vector space V

   (b) The finite list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   (c) The *image* of the linear transformation $T : V \to W$ from the vector space V to the vector space W

   (d) The *geometric multiplicity* of an eigenvalue λ of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) There exists a 3×5 matrix of rank 4.

   (b) (3 points) For every 10×10 matrix A, if A is diagonalizable then so is $A + 7I_{10}$.

   (c) (3 points) There exists a symmetric 2×2 matrix A with eigenvalues 3 and -2, and with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively.

(Problem 2, Continued).

(d) (3 points) For any $n \times m$ matrix A, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of AA^T.

(e) (3 points) For any vector space V and linear transformation $T : V \to V$, if $\vec{v}$ and $\vec{w}$ are eigenvectors of T, then $\vec{v} + \vec{w}$ is also an eigenvector of T.

3. Let $A \in \mathbb{R}^{3\times 3}$ be a non-invertible matrix such that $A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in \ker(A + I_3)$.

   (a) (3 points) Find all the eigenvalues of A, along with their algebraic multiplicities.

   (b) (4 points) Is A diagonalizable? Justify your answer.

   (c) (3 points) Compute $A^2 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$.

   (d) (4 points) Assuming that $\vec{e}_1 \in \ker(A)$, find A. (You may leave your answer as an unsimplified expression for A, if you wish).

4. Let V be the plane in $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Let $A \in \mathbb{R}^{3\times 3}$ be the standard matrix of the orthogonal projection onto V in $\mathbb{R}^3$.

   (a) (4 points) Find an orthonormal basis $(\vec{u}_1, \vec{u}_2)$ of V.

   (b) (3 points) Let B and C be the 3×2 matrices $B = [\vec{v}_1\ \vec{v}_2]$ and $C = [\vec{u}_1\ \vec{u}_2]$, where $\vec{u}_1$, $\vec{u}_2$ are as in part (a). Of the twelve matrices below, circle those that equal A.

   $B^TB \qquad BB^T \qquad B^T(B^TB)^{-1}B \qquad B(B^TB)^{-1}B^T \qquad B^T(BB^T)^{-1}B \qquad B(BB^T)^{-1}B^T$

   $C^TC \qquad CC^T \qquad C^T(C^TC)^{-1}C \qquad C(C^TC)^{-1}C^T \qquad C^T(CC^T)^{-1}C \qquad C(CC^T)^{-1}C^T$

   (c) (5 points) If possible, find an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$. If this is not possible, explain why.

5. Let V be the vector space of 2×2 upper triangular matrices, and let $T : V \to V$ be the linear map defined by $T(A) = MAM$, where $M = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$. Note that $M = M^{-1}$.

   (a) (4 points) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

   (b) (3 points) Is T invertible? Justify your answer.

   (c) (3 points) Find an eigenvector of T corresponding to the eigenvalue $\lambda = 1$.

   (d) (4 points) Is T diagonalizable? Justify your answer.

6. Let $B = \begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ -4 & 2 & c \end{bmatrix}$ be a 3×3 matrix whose third column is unknown. *(Note: the additional assumptions stated below do NOT carry over from one part to the next).*

(a) (4 points) Assuming that B is invertible, find the first column of B^{-1}.

(b) (4 points) Assuming that $\det B = 12$, determine as many of the values a, b, and c as possible. In your answer, clearly indicate which of these values, if any, cannot be determined from the assumption that $\det B = 12$.

(c) (4 points) Assuming that the linear system $B\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is inconsistent, find a least-squares solution of it.

7. Let $Q \in \mathbb{R}^{3\times 3}$ be an orthogonal 3×3 matrix.

   (a) (4 points) Prove that for all $\lambda \in \mathbb{R}$, if λ is an eigenvalue of Q then $|\lambda| = 1$.

   (b) (7 points) Prove that if $\det Q = 1$, then there is a nonzero vector $\vec{v} \in \mathbb{R}^3$ such that $Q\vec{v} = \vec{v}$.

8. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Recall that for any $X \subseteq V$, we define $T[X] = \{T(\vec{x}) \,:\, \vec{x} \in X\}$.

   (a) (6 points) Suppose that V_1 and V_2 are subspaces of V that contain $\ker(T)$. Prove that if $T[V_1] = T[V_2]$, then $V_1 = V_2$.

   (b) (6 points) Prove that T is injective if and only if for all subspaces V_1 and V_2 of V, if $T[V_1] = T[V_2]$ then $V_1 = V_2$.

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