Worksheet 11: Abstract Vector Spaces (§§4.1, 4.2)

The first three chapters of the text focus on the "coordinate vector spaces" $\mathbb{R}^n$, but Chapter 4 is an introduction to abstract *vector spaces*, which we met on Worksheet 5. We will not repeat the lengthy definition here — suffice it to say that a *vector space* is a set of elements, called *vectors*, on which two operations are defined:

(i) vector addition, and

(ii) scalar multiplication;

where these operations have a bunch of familiar properties such as commutativity, associativity, the distributivity of scalar multiplication over vector addition, etc.

Almost all of the concepts and definitions we have developed in $\mathbb{R}^n$ extend without change to abstract vector spaces, and indeed many of them we have already stated in this more general setting. These include, for instance, concepts like *linear combination*, *span*, *linear dependence* and *independence*, *basis*, *dimension*, *linear transformation*, *kernel* and *image*, etc.

Problem 1: Vector spaces, subspaces, and bases.

In each of (a) – (h) below, try to find some bases of the given vector space. In fact, try first to find a "natural" or "standard" basis if this is possible (analogous to the way that $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_n)$ is a standard basis of $\mathbb{R}^n$), and then find a second basis that is different from the first one you found.

(a) the vector space $\mathbb{R}^{2\times 2}$ of all 2×2 matrices

(b) the vector space $\mathcal{P}_3$ of all polynomial functions of degree at most 3 in the variable t

(c) the vector space $\mathbb{C}$ of complex numbers

(d) the plane in $\mathbb{R}^3$ defined by $x + 2y - z = 0$

(e) the vector space U_3 of all 3×3 upper triangular matrices

(f) the kernel of the linear transformation $T : U_3 \to \mathbb{R}^{3\times 3}$ defined by $T(A) = A - A^\top$ for all $A \in U_3$, where U_3 is as in part (e)

(g) the image of the linear transformation T from part (f)

(h) the space $\mathbb{R}^{\mathbb{N}}$ of all infinite sequences of real numbers

Solution: In (a), (b), and (c) we write "standard" bases of the given space.

(a) $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$

(b) $(1, t, t^2, t^3)$

(c) $(1, i)$

(d) The given plane doesn't really have a "standard" basis, but $\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right)$ is a basis.

(e) $\left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)$.

(f) $\left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)$.

(g) $\left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\right)$.

Problem 2: Isomorphisms.

(a) Remind yourself of the definition: "given vector spaces V and W, an *isomorphism* from V to W is ..." ... an invertible linear transformation from V to W.

(b) Suppose $T : V \to W$ is an isomorphism. What are the domain and codomain of T^{-1}? Is T^{-1} also an isomorphism? The domain of T^{-1} is W, and the codomain is V. Yes, T^{-1} is also an isomorphism, since the inverse of an invertible linear transformation is itself linear (and invertible).

(c) Suppose $T : U \to V$ and $S : V \to W$ are isomorphisms. Is $S \circ T$ an isomorphism? Yes, a composition of isomorphisms is an isomorphism, since a composition of linear transformations is linear and a composition of bijections is bijective.

(d) The vector space V is *isomorphic* to the vector space W, written $V \cong W$, if there is an isomorphism from V to W. Intuitively, V is isomorphic to W if "they look and behave exactly the same way, as far as linear algebra is concerned." Prove that isomorphism is an *equivalence relation*, i.e., prove the following:

   (i) [Reflexivity] every vector space is isomorphic to itself;
   (ii) [Symmetry] if V is isomorphic to W, then W is isomorphic to V;
   (iii) [Transitivity] if U is isomorphic to V and V is isomorphic to W, then U is isomorphic to W.

Solution: Since the identity transformation on any vector space is an isomorphism, every vector space is isomorphic to itself, so the isomorphism relation is reflexive. For symmetry, if T is an isomorphism from V to W then T^{-1} is an isomorphism from W to V, so V being isomorphic to W implies that W is isomorphic to V. Finally, for transitivity, if $S : U \to V$ and $T : V \to W$ are isomorphisms, then also $T \circ S$ is an isomorphism, so U is isomorphic to W.

(e) Do you think any pairs of vector spaces from Problem 1 are isomorphic to each other?

Solution: The vector spaces in (a) and (b) are isomorphic to each other, the vector spaces in (c) and (d) are isomorphic to each other, and the vector spaces in (f) and (g) are isomorphic to each other, and there are no other pairs from the list that are isomorphic to each other.

Problem 3: Isomorphisms and Bases

Let V and W be vector spaces, suppose $T : V \to W$ is an isomorphism, and let $\mathcal{B} = (v_1, \ldots, v_n)$ be a list of vectors in V. Prove that $\mathcal{B}$ is a basis of V if and only if $\big(T(v_1), \ldots, T(v_n)\big)$ is a basis of W.

Solution: Let V and W be vector spaces, let $T : V \to W$ be an isomorphism, and let $\mathcal{B} = (v_1, \ldots, v_n) \subseteq V$. For the forward direction, suppose that $\mathcal{B}$ is a basis of V, and let $\vec{w} \in W$. Then since $\mathcal{B}$ spans V, there exist scalars $c_1, \ldots, c_n$ such that $T^{-1}(\vec{w}) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Applying T to both sides of this equation, we obtain

$$\vec{w} \;=\; T\big(T^{-1}(\vec{w})\big) \;=\; T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right) \;=\; \sum_{i=1}^{n} c_i T(\vec{v}_i).$$

Thus $(T(\vec{v}_1), \ldots, T(\vec{v}_n))$ spans W. To see that $(T(\vec{v}_1), \ldots, T(\vec{v}_n))$ is linearly independent, let $c_1, \ldots, c_n \in \mathbb{R}$ and suppose $\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$. Again applying T^{-1} to both sides, we get

$$\vec{0} \;=\; T^{-1}(\vec{0}) \;=\; T^{-1}\left(\sum_{i=1}^{n} c_i T(\vec{v}_i)\right) \;=\; (T^{-1} \circ T)\left(\sum_{i=1}^{n} c_i\vec{v}_i\right) \;=\; \sum_{i=1}^{n} c_i\vec{v}_i.$$

Since $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent, it follows that $c_i = 0$ for each i. This shows that $(T(\vec{v}_1), \ldots, T(\vec{v}_n))$ is linearly independent, and is therefore a basis of W. Finally, the reverse implication follows by symmetry since T^{-1} is also an isomorphism.

Problem 4: Bases and Coordinates.

Let V be any vector space, and suppose that $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is an ordered basis of V. Let $L_{\mathcal{B}} : V \to \mathbb{R}^n$ be the $\mathcal{B}$*-coordinate transformation* defined by

$$L_{\mathcal{B}}(\vec{v}) \;=\; \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \;\in\; \mathbb{R}^n, \quad \text{where} \quad \vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n.$$

We usually write $[\vec{v}]_{\mathcal{B}}$ for $L_{\mathcal{B}}(\vec{v})$, and call it the $\mathcal{B}$*-coordinate vector* of $\vec{v}$.

(a) Explain why the definition of $L_{\mathcal{B}}$ even makes sense. What property of bases is being used here?

(b) Show that $L_{\mathcal{B}}$ is an isomorphism.*

*Hint: first show linearity; then show $L_{\mathcal{B}}$ is surjective and has trivial kernel.

(c) A vector space V is said to be *finite-dimensional* if it has a finite spanning set. Prove that if V is finite-dimensional, then V has a finite basis.[†]

(d) Prove that if V is a finite-dimensional vector space, then any two bases of V contain the same number of vectors.[‡]

(e) What you proved in (d) justifies our definition of *dimension*. Namely, if V is a finite-dimensional vector space, then its *dimension*, written $\dim V$ or $\dim(V)$, is the number of vectors in any basis of V. Prove that if V and W are finite-dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Remark: *Parts (d) and (e) extend to the infinite-dimensional case, though we will not worry about proving this, since it would require an understanding of infinite cardinal numbers.*

Solution:

(a) $L_{\mathcal{B}}$ makes sense because the scalars $c_1, \ldots, c_n$ exist and are uniquely determined; this is because since $\mathcal{B}$ is a basis, every vector in V can be written in a unique way as a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$.

(b) To see that $L_{\mathcal{B}}$ is linear, let $\vec{x}, \vec{y} \in V$ and $c \in \mathbb{R}$. If

$$\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i \qquad \text{and} \qquad \vec{y} = \sum_{i=1}^{n} b_i \vec{v}_i,$$

then

$$c\vec{x} = \sum_{i=1}^{n} (ca_i)\vec{v}_i \qquad \text{and} \qquad \vec{x} + \vec{y} = \sum_{i=1}^{n} (a_i + b_i)\vec{v}_i,$$

so $L_{\mathcal{B}}(c\vec{x}) = cL_{\mathcal{B}}(\vec{x})$ and $L_{\mathcal{B}}(\vec{x} + \vec{y}) = L_{\mathcal{B}}(\vec{x}) + L_{\mathcal{B}}(\vec{y})$.

To see that $L_{\mathcal{B}}$ is surjective, note that for any $c_1, \ldots, c_n \in \mathbb{R}$,

$$L_{\mathcal{B}}(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Finally, if $L_{\mathcal{B}}$ were not injective, then it would be possible to write $\vec{0}$ (or any other vector in V) as a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ in more than one way, which is impossible since $\mathcal{B}$ is a basis of V. Thus $L_{\mathcal{B}}$ is a bijective linear transformation, i.e., an isomorphism.

(c) Suppose V is finite-dimensional, and let $\mathcal{S} \subseteq V$ be a finite spanning set for V. Then $\mathcal{S}$ must contain a *minimal* spanning set $\mathcal{B} \subseteq \mathcal{S}$, which will be a finite basis of V.

(d) Let V be a finite-dimensional vector space with finite ordered basis $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$. Then $L_{\mathcal{B}}$ is an isomorphism from V to $\mathbb{R}^n$. Now let $\mathcal{C}$ be any basis of V. Then $L_{\mathcal{B}}[\mathcal{C}]$ is a basis of $\mathbb{R}^n$ by Problem 3, which implies that $L_{\mathcal{B}}[\mathcal{C}]$ has exactly n elements in it. Since $L_{\mathcal{B}}$ is bijective, it follows that $\mathcal{C}$ has n elements as well.

[†] Hint: recall from Worksheet 10 that any minimal spanning set of V is a basis of V.

[‡] Hint: use Problems 3, 4(b), and the fact that every basis of $\mathbb{R}^n$ has exactly n vectors in it.

(e) Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{C}$, respectively. Let $\dim V = n$ and $\dim W = m$. Then $L_{\mathcal{B}}$ is an isomorphism of V with $\mathbb{R}^n$ and $L_{\mathcal{C}}$ is an isomorphism of W with $\mathbb{R}^m$. Since $\mathbb{R}^n \cong \mathbb{R}^m$ if and only if $m = n$, it follows that $V \cong W$ if and only if $\dim V = \dim W$.