# Worksheet 15: Orthogonal Projections and Orthonormal Bases (§5.1)



Recall the following definitions:

Two vectors $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ are said to be *orthogonal* if $\vec{v} \cdot \vec{w} = 0$.

The *length* of a vector $\vec{v}$ in $\mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

Given any set $S \subseteq \mathbb{R}^n$, the *orthogonal complement* $S^\perp$ of $S$ is the set

$$S^\perp = \{\vec{w} \in \mathbb{R}^n \ : \ \vec{w} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in S\}.$$

**Problem 1.**

(a) What is the orthogonal complement of the plane $2x - 3y + z = 0$ in $\mathbb{R}^3$?

(b) What is the orthogonal complement of the line $\text{Span}(\vec{e}_2)$ in $\mathbb{R}^3$?

**Solution:** (a) The line spanned by $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. (b) The $xz$-plane in $\mathbb{R}^3$.

**Problem 2.**

(a) Prove that for any $S \subseteq \mathbb{R}^n$, $S^\perp$ is a subspace of $\mathbb{R}^n$.

**Solution:** Let $S \subseteq \mathbb{R}^n$. Since $\vec{0} \cdot \vec{v} = 0$ for all $\vec{v} \in S$, $\vec{0} \in S^\perp$. If $\vec{u}, \vec{w} \in S^\perp$, then for all $\vec{v} \in S$ we have

$$(\vec{u} + \vec{w}) \cdot \vec{v} \ = \ \vec{u} \cdot \vec{v} + \vec{w} \cdot \vec{v} \ = \ 0 + 0 \ = \ 0,$$

so $\vec{u} + \vec{w} \in S^\perp$. Similarly, if $\vec{w} \in S^\perp$ and $c \in \mathbb{R}$ then $(c\vec{w}) \cdot \vec{v} = c(\vec{w} \cdot \vec{v}) = c0 = 0$ for all $\vec{v} \in S$, so $c\vec{w} \in S^\perp$. This shows that $S^\perp$ is a subspace of $\mathbb{R}^n$.

(b) Let $\vec{v} \in \mathbb{R}^n$, let $W$ be any subspace of $\mathbb{R}^n$, and suppose the subset $\{\vec{w}_1, \ldots, \vec{w}_r\} \subseteq W$ is a spanning set for $W$. Prove that $\vec{v} \in W^\perp$ if and only if $\vec{v} \cdot \vec{w}_i = 0$ for each $1 \leq i \leq r$.

**Solution:** If $\vec{v} \in W^\perp$, then of course $\vec{v} \cdot \vec{w}_i$ for each $i$ since each $\vec{w}_i$ belongs to $W$. Conversely, suppose $\vec{v} \cdot \vec{w}_i = 0$ for each $i$, and let $\vec{w} \in W$. Then since $\{\vec{w}_1, \ldots, \vec{w}_r\}$ spans $W$, we can choose scalars $c_1, \ldots, c_r$ such that $\vec{w} = c_1\vec{w}_1 + \cdots + c_r\vec{w}_r$. Then

$$\vec{v} \cdot \vec{w} \ = \ \vec{v} \cdot (c_1\vec{w}_1 + \cdots + c_r\vec{w}_r) \ = \ c_1(\vec{v} \cdot \vec{w}_1) + \cdots + c_r(\vec{v} \cdot \vec{w}_r) \ = \ 0 + \cdots + 0 \ = \ 0,$$

which shows $\vec{v} \in W^\perp$.

**Problem 3.** Let $(\vec{v}_1, \ldots, \vec{v}_r)$ be an *orthonormal* set of vectors in $\mathbb{R}^n$. This means[†] for each $i, j$,

$$\vec{v}_i \cdot \vec{v}_j \;=\; \delta_{ij} \;=\; \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

(Thus each $\vec{v}_i$ is a *unit vector*, i.e., has length one.) Show that $(\vec{v}_1, \ldots, \vec{v}_r)$ is linearly independent.

**Solution:** Let $c_1, \ldots, c_r \in \mathbb{R}$, suppose

$$c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = \vec{0},$$

and let $1 \leq i \leq r$ be arbitrary. Dotting both sides of the above equation by $\vec{v}_i$, we have

$$0 \;=\; \vec{v}_i \cdot \vec{0} \;=\; \vec{v}_i \cdot (c_1\vec{v}_1 + \cdots + c_r\vec{v}_r) \;=\; c_1(\vec{v}_i \cdot \vec{v}_1) + \cdots + c_r(\vec{v}_i \cdot \vec{v}_r) \;=\; c_i.$$

So each $c_i$ is zero, which shows that $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent.

**Problem 4.** Let $W$ be a subspace of $\mathbb{R}^n$ with orthonormal basis $(\vec{w}_1, \ldots, \vec{w}_r)$, and let $\vec{v} \in \mathbb{R}^n$. Prove that there exists a unique vector $\vec{w} \in W$ such that $\vec{v} - \vec{w} \in W^\perp$. The unique vector $\vec{w} \in W$ such that $\vec{v} - \vec{w} \in W^\perp$ is called the *(orthogonal) projection* of $\vec{v}$ onto $W$, written $\text{proj}_W(\vec{v})$.
[HINT: find a formula for $\vec{w}$ in terms of $\vec{w}_1, \ldots, \vec{w}_r$. Also, draw a picture!]

**Solution:** Let $W$ be a subspace of $\mathbb{R}^n$, let $\vec{v} \in \mathbb{R}^n$, and let $\vec{w} = \sum_{i=1}^r c_i\vec{w}_i \in W$. Then for each $1 \leq j \leq r$, we have $\vec{w}_j \cdot (\vec{v} - \vec{w}) = 0$ if and only if

$$\vec{w}_j \cdot \vec{v} \;=\; \vec{w}_j \cdot \sum_{i=1}^r c_i\vec{w}_i \;=\; c_j.$$

Therefore $\vec{v} - \vec{w} \in W^\perp$ if and only if $c_j = \vec{w}_j \cdot \vec{v}$ for each $1 \leq j \leq r$. This shows that there is indeed a unique vector $\vec{w} \in W$ such that $\vec{v} - \vec{w} \in W^\perp$, namely $\vec{w} = \sum_{i=1}^r (\vec{v} \cdot \vec{w}_i)\vec{w}_i$.

**Problem 5.** Let $W$ be a subspace of $\mathbb{R}^n$ with orthonormal basis $(\vec{w}_1, \ldots, \vec{w}_r)$, and let $\text{proj}_W$ be the orthogonal projection onto $W$, viewed as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Is $\text{proj}_W$ linear? If so, what are its kernel and image? Can you find its standard matrix?

**Solution:** Yes, $\text{proj}_W$ is linear since the dot product is linear in each argument. The image of $\text{proj}_W$ is $W$, and its kernel is $W^\perp$. To find the standard matrix of $\text{proj}_W$, let $A = [\vec{w}_1 \;\cdots\; \vec{w}_r]$ and note that for each $1 \leq i \leq n$,

$$\text{proj}_W(\vec{e}_i) \;=\; \sum_{j=1}^r (\vec{e}_i \cdot \vec{w}_j)\vec{w}_j \;=\; A \begin{bmatrix} \vec{e}_i \cdot \vec{w}_1 \\ \vdots \\ \vec{e}_i \cdot \vec{w}_r \end{bmatrix},$$

[†] The symbol $\delta_{ij}$ is called the *Kronecker delta*, and is defined to be equal to 1 if $i = j$ and 0 if $i \neq j$.

so that

$$[\mathrm{proj}_W]_{\mathcal{E}} \;=\; A\begin{bmatrix} \vec{e}_1\cdot\vec{w}_1 & \cdots & \vec{e}_n\cdot\vec{w}_1 \\ \vdots & & \vdots \\ \vec{e}_1\cdot\vec{w}_r & \cdots & \vec{e}_n\cdot\vec{w}_r \end{bmatrix} \;=\; AA^T.$$

Alternatively, writing matrix multiplication using outer products, for any $\vec{v}\in\mathbb{R}^n$ we have

$$\mathrm{proj}_W(\vec{v}) \;=\; \sum_{i=1}^{r}(\vec{v}\cdot\vec{w}_i)\vec{w}_i \;=\; \sum_{i=1}^{r}\vec{w}_i\vec{w}_i^T\vec{v} \;=\; \left(\sum_{i=1}^{r}\vec{w}_i\vec{w}_i^T\right)\vec{v} \;=\; AA^T\vec{v}.$$

**Problem 6.** Show by induction on dimension that every subspace of $\mathbb{R}^n$ has an orthonormal basis. [HINT: for the inductive step, if $V$ is a $(k+1)$-dimensional subspace of $\mathbb{R}^n$, let $\vec{v}$ be some fixed nonzero vector in $V$ and consider the kernel of the linear transformation $T: V\to\mathbb{R}$ defined by $T(\vec{x}) = \vec{v}\cdot\vec{x}$.]

**Solution:** The induction base is trivial (if $V$ is a 1-dimensional subspace of $\mathbb{R}^n$, just normalize any non-zero vector in $V$). For the inductive step, let $1\le k<n$, suppose every subspace of $\mathbb{R}^n$ of dimension $k$ has an orthonormal basis, and let $V$ be a subspace of $\mathbb{R}^n$ of dimension $k+1$. Let $\vec{v}$ be some fixed nonzero vector in $V$, and define the linear transformation $T: V\to\mathbb{R}$ by $T(\vec{x}) = \vec{v}\cdot\vec{x}$. Then $\dim(\mathrm{im}(T)) = 1$, so by Rank-Nullity $\ker(T)$ has dimension $k$. Using the inductive hypothesis, let $(\vec{u}_1,\ldots,\vec{u}_k)$ be an orthonormal basis of $\ker(T)$. Then

$$\left(\vec{u}_1,\ldots,\vec{u}_k,\frac{\vec{v}}{\|\vec{v}\|}\right)$$

is an orthonormal basis of $V$, completing the induction.