Worksheet 9: Linear Independence (§§3.2, 3.3)

Warm-Up Problem: Thinking about subspaces.

(a) Suppose that W is a subspace of the vector space V. Is W also a vector space?

(b) Describe all subspaces of $\mathbb{R}^3$. How many are there? Can you classify them into different "types?" If so, how many types are there?

(c) Repeat the previous question for $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^4$.

Solution:

(a) Yes!

(b) There are four different types: $\{\vec{0}\}$; lines through $\vec{0}$; planes through $\vec{0}$; and $\mathbb{R}^3$ itself.

(c) $\mathbb{R}$ has only two subspaces, namely $\{\vec{0}\}$ and $\mathbb{R}$ itself. In addition to the trivial subspaces $\{\vec{0}\}$ and $\mathbb{R}^2$ itself, every line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. In general, $\mathbb{R}^n$ has $n+1$ different types of subspaces, namely those of "dimension" k for $0 \le k \le n$ an integer.

Problem 1: Building up subspaces.

In this problem we will "build up" subspaces by adding one vector at a time to a growing list and taking the span at each stage.

(a) Since it can be a difficult definition to remember, remind yourself again of the definition of *span*. Do not look it up — try to recall from memory. Just complete this sentence: "If V is a vector space and $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a finite set of vectors in V, then $\operatorname{Span}(S)$ is ..."

Solution: The span of the set $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is

$$\operatorname{Span}(S) \;=\; \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \;:\; c_1, \dots, c_n \in \mathbb{R}\},$$

which is just the set of all vectors in V that can be expressed as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

(b) Let's build up subspaces in $\mathbb{R}^3$. Below is an ordered list of six vectors in $\mathbb{R}^3$. Describe, and try to visualize, the six subspaces $\operatorname{Span}(\vec{u}_1)$, $\operatorname{Span}(\vec{u}_1, \vec{u}_2)$, ..., $\operatorname{Span}(\vec{u}_1, \dots, \vec{u}_6)$.

$$\vec{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix}-4\\0\\-4\end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix}-3\\-1\\-3\end{bmatrix}, \quad \vec{u}_5 = \begin{bmatrix}0\\2\\0\end{bmatrix}, \quad \vec{u}_6 = \begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

Solution: $\operatorname{Span}(\vec{u}_1)$ is the line in $\mathbb{R}^3$, call it ℓ, that is generated by $\vec{u}_1$, i.e., it is the line in $\mathbb{R}^3$ that goes through $\vec{u}_1$ and $\vec{0}$. $\operatorname{Span}(\vec{u}_1, \vec{u}_2)$ and $\operatorname{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are both still just this line ℓ, because $\vec{u}_2$ and $\vec{u}_3$ already lie on ℓ, so tacking on $\vec{u}_2$ or $\vec{u}_3$ adds nothing new. In contrast, $\vec{u}_4$ does not lie on ℓ, so including $\vec{u}_4$ in the list will increase the span. In fact, $\operatorname{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ is the plane $\mathcal{P}$ in $\mathbb{R}^3$ that contains $\vec{0}$, $\vec{u}_1$, and $\vec{u}_4$, so it also contains the entire line ℓ. A little algebra shows that $\vec{u}_5$ is also contained in the plane $\mathcal{P}$, so

$$\operatorname{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5) \;=\; \operatorname{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4) \;=\; \mathcal{P},$$

that is, tacking on $\vec{u}_5$ adds nothing new. Finally, since $\vec{u}_6$ does *not* belong to $\mathcal{P}$, adding it to the list *does* generate something new, and we have

$$\operatorname{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5, \vec{u}_6) \;=\; \operatorname{Span}(\vec{u}_1, \vec{u}_4, \vec{u}_6) \;=\; \mathbb{R}^3.$$

(c) Which vectors in the list $\vec{u}_1, \ldots, \vec{u}_6$ were "redundant?" Explain what is meant by this.

Solution: The vectors $\vec{u}_2$, $\vec{u}_3$, and $\vec{u}_5$ were redundant, since each of these was already contained in the span of the previous vectors in the list, so each one added "nothing new" when it was tossed in. In general, a vector v_k in a list $(v_1, \ldots, v_n)$ is *redundant* if v_k can be written as a linear combination of $v_1, \ldots, v_{k-1}$.

(d) Find the reduced row echelon form of the 3×6 matrix $A = [\vec{u}_1 \;\cdots\; \vec{u}_6]$.

Solution: $\begin{bmatrix} 1 & 0 & -4 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(e) Identify the columns in A that correspond to free variables, and those that correspond to leading variables. (The columns corresponding to leading variables are sometimes called "pivot columns.") What can you say about these two types of columns in the matrix A, as compared with your list of subspaces from part (b)?

Solution: The second, third, and fifth columns of A correspond to free variables. These are exactly the "redundant" columns from part (c). The remaining columns correspond to leading variables, so columns one, four, and six are pivot columns. The pivot columns are precisely the ones that "generate something new" when we use the column vectors in A to build up subspaces of $\mathbb{R}^3$.

(f) For each of the "redundant" vectors in the list $\vec{u}_1, \ldots, \vec{u}_6$, find the weights that are used to write this redundant vector as a linear combination of the *non*-redundant vectors that precede it in the list. [Hint: look at $\operatorname{rref}(A)$]

Solution: $\vec{u}_2 = 0\vec{u}_1$, $\vec{u}_3 = -4\vec{u}_1$, and $\vec{u}_5 = -6\vec{u}_1 - 2\vec{u}_4$.

Definition: A *linear relation* on the ordered list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ is an equation of the form

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \;=\; \vec{0}. \qquad (*)$$

Such a relation is *trivial* if each scalar $c_1, \ldots, c_n$ is 0, and *nontrivial* otherwise, i.e., if at least one c_i is nonzero. The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ is *linearly independent* if every linear relation of the form $(*)$ satisfied by $\vec{v}_1, \ldots, \vec{v}_n$ is trivial. Otherwise $(\vec{v}_1, \ldots, \vec{v}_n)$ is *linearly dependent.*

More explicitly, the list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ is *linearly independent* if for all $c_1, \ldots, c_n \in \mathbb{R}$, $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ implies that $c_i = 0$ for each $1 \leq i \leq n$; on the other hand, this list is *linearly dependent* if there exist scalars $c_1, \ldots, c_n$, not all zero, such that $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$.

Problem 2: Linear independence.

(a) Is the list $(\vec{u}_1, \ldots, \vec{u}_6)$ from Problem 1 linearly dependent or linearly independent? (Notice that *every* list of vectors is one or the other and not both!)

Solution: The list $(\vec{u}_1, \ldots, \vec{u}_6)$ from Problem 1 is linearly dependent, since for instance

$$0\vec{u}_1 + 1\vec{u}_2 + 0\vec{u}_3 + 0\vec{u}_4 + 0\vec{u}_5 + 0\vec{u}_6 \;=\; \vec{0}.$$

This calculation shows why any list containing $\vec{0}$ will be linearly dependent.

(b) What happens if you take the list $(\vec{u}_1, \ldots, \vec{u}_6)$ from Problem 1 and eliminate all the "redundant" vectors. Which vectors are left? Is the resulting list linearly dependent or linearly independent?

Solution: If you eliminate the redundant vectors from $(\vec{u}_1, \ldots, \vec{u}_6)$, then you are left with $(\vec{u}_1, \vec{u}_4, \vec{u}_6)$, which is a linearly independent list.

(c) Take a moment to convince yourself that a list of vectors is linearly independent if and only if it contains no redundant vectors. In other words, convince yourself that our definition of *linearly independent* is the same as the book's. Can you sketch a proof of this?

Solution: If $(v_1, \ldots, v_n)$ is linearly dependent, we can find $c_1, \ldots, c_n \in \mathbb{R}$, not all zero, such that $\sum c_i v_i = 0$; then if k is largest such that $c_k \neq 0$, we have $v_k = -c_k^{-1}(c_1v_1 + \cdots + c_{k-1}v_{k-1})$, so that v_k is redundant. Conversely, if v_k in the list $(v_1, \ldots, v_n)$ is redundant, say $v_k = c_1v_1 + \cdots + c_{k-1}v_{k-1}$, then

$$c_1v_1 + \cdots + c_{k-1}v_{k-1} - v_k \;=\; \vec{0}$$

is a nontrivial linear relation showing that $(v_1, \ldots, v_n)$ is linearly dependent.

(d) Consider the following statement:

> "For any finite list of vectors $S = (\vec{v}_1, \ldots, \vec{v}_n)$ in $\mathbb{R}^m$, there is a linearly independent sublist $\mathcal{B}$ of S such that $\mathrm{Span}(\mathcal{B}) = \mathrm{Span}(S)$."

Is this statement true? If so, can you explain how to find such a $\mathcal{B}$ given S? [Hint: If you are stuck, think about the particular case $S = (\vec{u}_1, \ldots, \vec{u}_6)$ from Problem 1.]

Solution: Yes, this statement is true. Given the vectors $S = (\vec{v}_1, \ldots, \vec{v}_n)$ in $\mathbb{R}^m$, write these vectors as columns in a matrix, and then row reduce to find the pivot columns. The sublist B consisting of the pivot columns will have the same span as S.

(e) Why do you think the subset of S in the previous question was denoted by the letter $\mathcal{B}$?

Solution: It's a *b*asis of $\mathrm{Span}(S)$!

Definition: Let $m \geq 1$, and suppose V is a subspace of $\mathbb{R}^m$. A subset $S \subseteq V$ is said to *span* V, or is called a *spanning set* for V, if $\mathrm{Span}(S) = V$. An *ordered basis*† of V is a linearly independent list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in V that spans V.

Problem 3: Bases in $\mathbb{R}^n$

(a) Let $S = (\vec{u}_1, \ldots, \vec{u}_6)$ be the list of vectors in $\mathbb{R}^3$ from Problem 1. Find *all possible* lists of vectors from S that form an ordered basis of $\mathbb{R}^3$. Don't write out all of them all, but rather just write out a few and count how many there are in total.

Solution: There are 30 in total. They are given by the five sets $\{1, 4, 6\}$, $\{3, 4, 6\}$, $\{1, 5, 6\}$, $\{3, 5, 6\}$, and $\{4, 5, 6\}$, with 6 different possible orderings for each.

(b) How many vectors are there in an ordered basis of $\mathbb{R}^n$? Exactly n.

(c) What is the *standard (ordered) basis* of $\mathbb{R}^n$? (We will usually name standard bases $\mathcal{E}$). The standard basis of $\mathbb{R}^n$ is $\mathcal{E} = (\vec{e}_1, \ldots, \vec{e}_n)$, where for each $1 \leq k \leq n$, $\vec{e}_k$ is the vector in $\mathbb{R}^n$ with a 1 in the kth position and 0 everywhere else.

Problem 4: Finding ordered bases of subspaces

Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 3 & 2 & 5 \\ -2 & -5 & -2 & -11 \\ 3 & 1 & 0 & 3 \\ 4 & 4 & 2 & 8 \end{bmatrix}.$$

†Sometimes we will want bases to be ordered, and sometimes not. On the next worksheet, we will extend the notions of *linearly independent* and *linearly dependent* to make sense for arbitrary sets of vectors, rather than just for finite lists of vectors, and then we will define the notion of a *basis* (as opposed to an *ordered basis*) of a vector space.

(a) Find an ordered basis of $\mathrm{im}(T_A)$.[‡]

(b) Find an ordered basis of $\ker(T_A)$.

Solution: $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the first three columns form a basis of $\mathrm{im}(A)$, and the single vector $\begin{bmatrix} 0 \\ -3 \\ 2 \\ 1 \end{bmatrix}$ forms a basis of $\ker(A)$.

[‡]In our handout on functions, and also in class, we have defined the *image* and *kernel* of a function. Of course, matrices are not functions, but any $m \times n$ matrix A naturally induces the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$, so it makes sense to consider $\mathrm{im}(T_A)$ and $\ker(T_A)$. In order to simplify the notation, we will often just cheat and write $\mathrm{im}(A)$ and $\ker(A)$ instead, when what we really mean is $\mathrm{im}(T_A)$ and $\ker(T_A)$. Hopefully no one will be too offended by this!