MATH 217 - LINEAR ALGEBRA
HOMEWORK 10, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:
Section 6.1: 20
Section 6.2: 20, 42
Section 6.3: 14, 18
Section 7.1: 18, 42

Solution.
6.1.20) By performing row operations we see

$$\begin{bmatrix} 1 & k & 1 \\ 1 & k+1 & k+2 \\ 1 & k+2 & 2k+4 \end{bmatrix} \to \begin{bmatrix} 1 & k & 1 \\ 0 & 1 & k+1 \\ 0 & 2 & 2k+3 \end{bmatrix} \to \begin{bmatrix} 1 & k & 1 \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that the determinant is 1 so the matrix is invertible for all $k \in \mathbb{R}$
6.2.20) Let $\mathcal{U} = (E_{11}, E_{12}, E_{21}, E_{22})$ be the standard basis for $\mathbb{R}^{2\times 2}$. Then

$$[T]_{\mathfrak{U}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so T has determinant -1.
6.2.42) $\det(A^\top A) = \det(R^\top Q^\top Q R) = \det(R^\top R)$ since $Q^\top Q = I_m$. However, if R is upper triangular with diagonal entries $r_{11}, \dots, r_{mm} > 0$, then by a worksheet problem and the properties of the determinant, we see that

$$\det(A^\top A) = \det(R^\top R) = \det(R^\top)\det(R) = \det(R)^2 = r_{11}^2 \cdots r_{mm}^2 > 0.$$

6.3.14) We consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, so $A^\top A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{bmatrix}$. The 3-volume of the 3-parallepiped is $\sqrt{\det(A^\top A)} = \sqrt{6}$.
6.3.18) a) It is an ellipse with horizontal axes from $(-p, 0)$ to $(0, p)$ and vertical axis from $(0, -q)$ to $(0, q)$. Its area is $pq\pi = \det(A)\pi$.
b) If we rotate the ellipse to align the axes, the image is the same as in part a, with $p = a$ and $q = b$, since rotation preserves area we see that the area is $ab\pi$, so $\det(A) = ab$ as rotations have determinant equal to 1.
c) Its axes are $\pm\frac{1}{\sqrt{2}} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \pm \begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$ and $\pm\frac{1}{\sqrt{2}} T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \pm \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$.
7.1.18) Any non-zero vector in the plane V is taken to itself, hence is an eigenvector with eigenvalue 1. Any non-zero vector $\vec{v} \in V^\perp$ is taken to $-\vec{v}$, so is an eigenvector with eigenvalue

-1. One may then construct an eigenbasis for $\mathbb{R}^3$ by combining a basis for V with a non-zero vector in $V^\perp$. Therefore, the transformation is diagonalizable.

7.1.42) A has to be of the form $\begin{bmatrix} a & b & 0 \\ 0 & e & 0 \\ 0 & h & i \end{bmatrix}$, so a basis is given by

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

and V has dimension 5.

Part B (25 points)

Problem 1. In each part below, determine whether or not the given statement is true for all matrices A and B, and justify your answer with a proof or counterexample.

(a) $\det(AA^\top) = \det(A^\top A)$.
(b) If A is invertible, then $\det(A) = 1$.
(c) If A is an invertible $n \times n$ matrix and B, C and D are $n \times n$ matrices such that $D = CA^{-1}B$, then the 2×2 matrix $\begin{bmatrix} \det(A) & \det(B) \\ \det(C) & \det(D) \end{bmatrix}$ is not invertble.
(d) If A and B are $n \times n$ matrices and $\det(AB) = 0$, then A and B are both non-invertible.
(e) If A, B, C, and D are 2×2 matrices, then the determinant of the 4×4 block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $(\det A)(\det D) - (\det B)(\det C)$.

Solution. a) FALSE: let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2$, then

$$\det(A^\top A) = \det(1) = 1 \neq 0 = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \det(AA^\top).$$

b) FALSE: Notice that $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible (it has rank equal 2), but $\det(A) = 2$.

c) TRUE: $\det(D) = \det(C)\det(A^{-1})\det(B)$ and since $\det(A^{-1}) = \frac{1}{\det(A)}$ is non-zero, we conclude that $\det(D)\det(A) = \det(C)\det(B)$. Therefore, $\begin{bmatrix} \det(A) & \det(B) \\ \det(C) & \det(D) \end{bmatrix}$ has determinant zero, so is not invertible.

d) FALSE: If $A = I_2$ and B is the zero matrix, then AB is the zero matrix, so $\det(AB) = 0$. However, A is invertible.

e) FALSE: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $\det A = \det B = \det C = \det D = 0$, so $(\det A)(\det D) - (\det B)(\det C) = 0$, but the 4×4 block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ has determinant equal to -1.

Problem 2. Let V be a vector space. A function $F\colon V \times V \to \mathbb{R}$ is called *bilinear* if all of the following hold:

- for all $v, u, w \in V$, $F(v+u, w) = F(v, w) + F(u, w)$ and $F(v, u+w) = F(v, u) + F(v, w)$;
- for all $v, w \in V$ and $k \in \mathbb{R}$, $F(kv, w) = kF(v, w)$ and $F(v, kw) = kF(v, w)$.

If $F\colon V \times V \to \mathbb{R}$ is a bilinear function, we say that F is *alternating* if $F(v, v) = 0$ for all $v \in V$.

(a) Prove that for every bilinear function $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, there is a unique 2×2 matrix A such that $F(\vec{x}, \vec{y}) = \vec{x}^\top A\vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$.
(b) Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a bilinear function, and let $A \in \mathbb{R}^{2\times 2}$ be the matrix given in part (a), that is, the unique matrix such that $F(\vec{x}, \vec{y}) = \vec{x}^\top A\vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$. Show that F is alternating if and only if A is *skew-symmetric*, meaning that $A^\top = -A$.
(c) Prove that if $F\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is bilinear, alternating, and satisfies $F(\vec{e}_1, \vec{e}_2) = 1$, then F is the determinant map on $\mathbb{R}^{2\times 2}$ in the sense that $F(\vec{x}, \vec{y}) = \det \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$.

Remark: An analogue of Problem 2(c) holds for larger matrices as well, and gives the following alternative definition of determinant: *for each* $n \geq 1$, $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ *is the unique alternating multilinear map that assigns a value of* 1 *to* I_n.

Solution.

(a) Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be bilinear. Let $A = \begin{bmatrix} F(\vec{e}_1, \vec{e}_1) & F(\vec{e}_1, \vec{e}_2) \\ F(\vec{e}_2, \vec{e}_1) & F(\vec{e}_2, \vec{e}_2) \end{bmatrix}$. Then for all $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, we have

$$\begin{aligned} F(\vec{x}_1, \vec{x}_2) &= F(x_1\vec{e}_1 + x_2\vec{e}_2, y_1\vec{e}_1 + y_2\vec{e}_2) \\ &= x_1y_1F(\vec{e}_1, \vec{e}_1) + x_1y_2F(\vec{e}_1, \vec{e}_2) + x_2y_1F(\vec{e}_2, \vec{e}_1) + x_2y_2F(\vec{e}_2, \vec{e}_2) \\ &= \vec{x}^\top A\vec{y}. \end{aligned}$$

For uniqueness, if $B \in \mathbb{R}^{2\times 2}$ is any matrix satisfying $F(\vec{x}, \vec{y}) = \vec{x}^\top B\vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$, then for each $1 \leq i, j \leq 2$, the (i, j)-entry of B is $\vec{e}_i^\top B\vec{e}_j = F(\vec{e}_i, \vec{e}_j)$, so $B = A$.

(b) Again let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be bilinear, and let $A = [a_{ij}] \in \mathbb{R}^{2\times 2}$ represent F as in part (a), so $a_{ij} = F(\vec{e}_i, \vec{e}_j)$ for each $1 \leq i, j \leq 2$. If F is alternating, then $F(\vec{e}_1, \vec{e}_1) = 0 = F(\vec{e}_2, \vec{e}_2)$, so the diagonal entries of A are zero, and

$$0 \;=\; F(\vec{e}_1 + \vec{e}_2, \vec{e}_1 + \vec{e}_2) \;=\; \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;=\; a_{12} + a_{21},$$

so $a_{12} = -a_{21}$. Thus A is indeed skew-symmetric.

Conversely, if A is skew-symmetric, then for all $\vec{v} \in \mathbb{R}^2$ we have

$$\begin{aligned} 2F(\vec{v}, \vec{v}) &= \vec{v}^\top A\vec{v} + \vec{v}^\top A\vec{v} \;=\; \vec{v}^\top A\vec{v} + \vec{v}^\top(-A^\top)\vec{v} \\ &= \vec{v}^\top A\vec{v} - \vec{v}^\top A^\top\vec{v} \;=\; \vec{v} \cdot A\vec{v} - A\vec{v} \cdot \vec{v} \;=\; 0, \end{aligned}$$

so $F(\vec{v}, \vec{v}) = 0$ as desired.

(c) Suppose $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is bilinear, alternating, and satisfies $F(\vec{e}_1, \vec{e}_2) = 1$. By parts (a) and (b), we have that for all $\vec{x}, \vec{y} \in \mathbb{R}^{2\times 2}$,

$$F(\vec{x}, \vec{y}) \;=\; \vec{x}^\top \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{y}.$$

Thus for all $a, b, c, d \in \mathbb{R}$ we have

$$F\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) = \begin{bmatrix} a & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = ad - bc.$$

This shows that F is indeed the determinant map on $\mathbb{R}^{2\times 2}$.

Problem 3. Let $M \in \mathbb{R}^{2\times 2}$ be a 2×2 matrix of rank 1. Prove the following:

(a) Every non-zero vector in the image of M is an eigenvector of M.
(b) If $\operatorname{im} M \neq \ker M$, then M is diagonalizable.
(c) If M is diagonalizable, then $\operatorname{im} M \neq \ker M$. *(Hint: What are the possible eigenvalues of M if $\operatorname{im} M = \ker M$?)*
(d) If 1 is an eigenvalue of M, then $M^2 = M$.
(e) If $M^2 = M$, then 1 is an eigenvalue of M.

Solution. a) Since M has rank one, $\operatorname{im} M$ is one-dimensional. If $\vec{v} \in \operatorname{im} M$, then $M\vec{v}$ also lies in $\operatorname{im} M$. Since $\operatorname{im} M$ is one-dimensional, if $\vec{v}$ is non-zero, then $M\vec{v} = c\vec{v}$ for some $c \in \mathbb{R}$. Therefore, $\vec{v}$ is an eigenvector of M.

b) Suppose that $\operatorname{im} M \neq \ker M$. If $\vec{v}$ is a non-zero eigenvector in $\operatorname{im} M$, then $M\vec{v} \neq \vec{0}$, so its eigenvalue is non-zero. On the other hand, $\dim(\ker M) = 1$ (by the rank-nullity theorem), so if $\vec{w} \in \ker M$ is non-zero, then $\vec{w}$ is an eigenvector with eigenvalue 0. Since $\operatorname{im} M \neq \ker M$, $\vec{v}$ and $\vec{w}$ are linearly independent, so M is diagonalizable (since $(\vec{v}, \vec{w})$ will form an eigenbasis for M).

c) Suppose that M is diagonalizable and $\operatorname{im} M = \ker M$. Then, if $\vec{v}$ is an eigenvector of M, then $M\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$. But then, $M(\lambda\vec{v}) = \vec{0}$, since $\lambda\vec{v} \in \operatorname{im} M = \ker M$, so

$$\vec{0} = M(\lambda\vec{v}) = \lambda M\vec{v} = \lambda^2\vec{v}$$

which implies that $\lambda = 0$ (since $\vec{v}$ is non-zero). Therefore, all eigenvalues are 0, and M would be similar to the zero matrix and hence equal to the zero matrix. However, this contradicts our assumption that M has rank 1. Therefore, if M is diagonalizable, then $\operatorname{im}(M) \neq \ker(M)$.

Remark: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is an example of a rank one matrix which is not diagonalizable and one can check that $\operatorname{im}(M) = \ker(M) = \operatorname{span}(\vec{e}_1)$.

d) If 1 is an eigenvalue of M, then M has two distinct eigenvalues, since as we observed above 0 is an eigenvalue of M. Letting $\vec{v}$ and $\vec{w}$ be eigenvectors of M with corresponding eigenvalues 0 and 1, respectively, we have that $(\vec{v}, \vec{w})$ is linearly independent since the (one-dimensional) eigenspaces E_0 and E_1 are distinct. Therefore, M is diagonalizable and there exists an invertible matrix S so that $M = SDS^{-1}$ where D is a diagonal matrix with diagonal entries 0 and 1. So, $M^2 = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1} = SDS^{-1} = M$, since $D^2 = D$.

e) Suppose that $M^2 = M$. Let $\vec{w}$ be a non-zero vector in $\operatorname{im} M$. Then $\vec{w} = M\vec{v}$ for some non-zero vector $\vec{v}$. So, $M\vec{w} = M^2\vec{v} = M\vec{v} = \vec{w}$. Thus, 1 is an eigenvector of M.

Problem 4. In each part below, determine whether or not the given statement is true for all $n \times n$ matrices A and B, and justify your answer with a proof or counterexample.

(a) If A is diagonalizable, then $A^\top$ is diagonalizable.
(b) If A and B are diagonalizable, then $A + B$ is diagonalizable.
(c) If A and B are diagonalizable, then AB is diagonalizable.
(d) If A^2 is diagonalizable, then A is diagonalizable.
(e) If A is diagonalizable and invertible, then A^{-1} is diagonalizable.

Solution. a) TRUE: If A is diagonalizable, then there exists an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$. But then $A^\top = (S^{-1})^\top D^\top S^\top$. However, $(S^{-1})^\top = (S^\top)^{-1}$ and $D^\top = D$, so if $T = (S^\top)^{-1}$, then $A^\top = TDT^{-1}$, so $A^\top$ is diagonalizable.

b) FALSE: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, then A is diagonal (hence diagonalizable) and B is diagonalizable, since it has rank 1 and $\operatorname{im} B \neq \ker B$. However, $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable since its only eigenvalue is 1, so if it were diagonalizable it would have to be I_2. (Explicitly, if A were diagonalizable, then there would exist S so that $A = SI_2S^{-1} = I_2$.)

c) FALSE: If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A is diagonal (hence diagonalizable) and B is diagonalizable, since it is a reflection about the line $y = x$ (hence it's similar to A). However, $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a rotation by angle $3\pi/2$, which is not diagonalizable (no non-zero vector gets sent to a multiple of itself).

d) FALSE: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal (hence diagonalizable), but A is not diagonalizable, since its only eigenvalue is 0 and hence if it were diagonalizable it would have to be the zero matrix. (Explicitly, if A were diagonalizable, then there would exist S so that $A = S0S^{-1} = 0$.)

e) TRUE: If A is diagonalizable and invertible, then there exists an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$. But then D would have to be invertible, since $\det(D) = \det(A) \neq 0$. If D is diagonal and invertible, then its inverse D is also diagonal (with each diagonal entry of D^{-1} equal to the reciprocal of the corresponding diagonal entry of D.) So $A^{-1} = SD^{-1}S^{-1}$ and A^{-1} is also diagonalizable.