# Math 217 – Final Exam

Fall 2021

**Time: 120 mins.**

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary (in which case you may still provide justification if you wish, since this may lead to partial credit).
3. As usual, in any vector space whose elements are functions (say, in the variable $x$), we may write "1" to refer to the constant function with constant value 1, "$x$" to refer to the identity function, "$x^2$" to refer to the squaring function, etc. If you have any questions about this notation, please ask.
4. In any problem involving an inner product space, terms such as "orthogonal" or "orthonormal" mean with respect to the given inner product; if the vector space in question is $\mathbb{R}^n$ and no alternative inner product has been specified, you may assume the inner product to be the usual dot product on $\mathbb{R}^n$.
5. On this exam, all vector spaces are assumed to be *real* vector spaces and all eigenvalues and eigenvector are assumed to be real *unless explicitly indicated otherwise.*
6. You may be expected to recall elementary facts from calculus and trigonometry such as: (i) how to integrate and differentiate polynomials and the sine, cosine, and exponential functions; (ii) how to simplify integrals of even or odd functions on an interval that is symmetric about the origin. Any other fact from calculus that you need on this exam will be provided to you in the problem statement.
7. Do not trivialize problems. You are free to quote results from the worksheets, the textbook, or homework as a step in proving something else, but if an entire problem is asking you to reprove a result from class, we expect you to reproduce a proof.

Student ID Number: ______________________ Section: __________

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 12 | 16 | 12 | 13 | 12 | 12 | 11 | 12 | 100 |
| Score: | | | | | | | | | |

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The *image* of the linear transformation $T : V \to W$ from the vector space $V$ to the vector space $W$

   (b) The vector $\vec{v}$ in the vector space $V$ is an *eigenvector* of the linear transformation $T : V \to V$

   (c) The *geometric multiplicity* $\mathrm{gemu}(\lambda)$ of the eigenvalue $\lambda$ of the linear transformation $T : V \to V$, where $V$ is a finite-dimensional vector space

   (d) The function $f : X \to Y$ is *injective*

2. State whether each statement is True or False and provide a short proof of your claim. For each part, indicate your answer by clearly writing "T" or "F" in the box on the left.

(a) (4 points) For all matrices $A \in \mathbb{R}^{n\times n}$, $\det(A) = 0$ if and only if $A$ is not diagonalizable.

(b) (4 points) For all matrices $A \in \mathbb{R}^{n\times m}$ and vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, if $AA^\top \vec{v} = \lambda_1 \vec{v}$ and $AA^\top \vec{w} = \lambda_2 \vec{w}$ where $\lambda_1 \neq \lambda_2$, then $\vec{v} \cdot \vec{w} = 0$.

(c) (4 points) For every symmetric matrix $A \in \mathbb{R}^{n\times n}$, the equation $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top A \vec{y}$ defines an inner product on $\mathbb{R}^n$.

(d) (4 points) For every $n \times n$ matrix $A$, $\det A = 0$ if and only if $\det(\mathrm{rref}(A)) = 0$.

3. Consider the $3 \times 3$ matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & a & 0 \\ 0 & b & 3 \end{bmatrix}$, where $a, b \in \mathbb{R}$. In parts (a) and (b) below, find *all* values of $a$ and $b$ in $\mathbb{R}$ for which the given condition holds, or else write "none" if there are no such values. Justify your answers.

   (a) (4 points) $A$ is invertible.

   (b) (3 points) There exists an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$.

   (c) (5 points) Assuming $a = 2$, find all $b \in \mathbb{R}$ for which $A$ is diagonalizable (over $\mathbb{R}$).

4. Let $\mathbb{R}^{2\times 2}$ be the vector space of $2 \times 2$ real matrices, and let $\mathcal{E}$ be the ordered basis of $\mathbb{R}^{2\times 2}$ given by

$$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

Let $M = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation defined by

$$T(A) = MA - A^\top$$

for every $A \in \mathbb{R}^{2\times 2}$. (You do *not* need to prove $T$ is linear, or that $\mathcal{E}$ is a basis of $\mathbb{R}^{2\times 2}$.)

(a) (3 points) Find the $\mathcal{E}$-matrix $[T]_\mathcal{E}$ of $T$.

(b) (3 points) Find the characteristic polynomial of $T$.

(Problem 4, Continued).

(c) (4 points) For each eigenvalue of $T$, find a basis of the corresponding eigenspace. Clearly indicate your eigenvalues and which basis goes with which eigenvalue.

(d) (3 points) Is $T$ diagonalizable? Briefly explain your answer.

5. (12 points) In each part, find the smallest positive integer $n$ such that there is a matrix $A \in \mathbb{R}^{n\times n}$ with the indicated property.

   *No justification is required for this problem. In particular, you do not have to provide an example matrix; just say what n is, and write your answer clearly in the box.*

   (a) $A$ has at least 5 distinct eigenvalues.

   $n = \boxed{\phantom{0}}$

   (b) $A$ has no real eigenvalues.

   $n = \boxed{\phantom{0}}$

   (c) $A$ has at least 3 distinct real eigenvalues and is not diagonalizable (over $\mathbb{R}$).

   $n = \boxed{\phantom{0}}$

   (d) The complex numbers $i$, $1+i$, $1-i$, $2+i$, and 7 are some of the complex eigenvalues of $A$.

   $n = \boxed{\phantom{0}}$

   (e) $A$ has at least one real eigenvalue and is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$.

   $n = \boxed{\phantom{0}}$

   (f) $A$ is not invertible and $\pm 1$ are eigenvalues of $A$ with

   $$\mathrm{gemu}(1) < \mathrm{gemu}(-1) < \mathrm{almu}(1).$$

   $n = \boxed{\phantom{0}}$

6. Let $V = C^\infty([-1,1])$ be the inner product space of smooth functions from $[-1,1]$ to $\mathbb{R}$ with inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$, and let $W$ be the subspace of $V$ given by $W = \text{Span}(1, t)$.

   (a) (4 points) Find an orthonormal basis $\mathcal{U}$ of the subspace $W$.

   (b) (4 points) Find the orthogonal projection of $t^2$ onto $W^\perp$; that is, find $\text{proj}_{W^\perp}(t^2)$.

   (c) (4 points) Let $g$ be the cosine function on $[-1,1]$, so $g(t) = \cos t$ for all $t \in [-1,1]$. Find the function $p$ in $W$ that is *closest* to the cosine function in $V$, in the sense that $\|p - g\| \leq \|q - g\|$ for all $q \in W$.

7. Let $U$ and $A$ be $n \times n$ matrices with real entries, and suppose $U$ is upper-triangular and $A$ is invertible with QR-factorization $A = QR$.

   (a) (3 points) Prove that all the complex eigenvalues of $U$ are real.

   (b) (3 points) Prove that if $U$ is orthogonally diagonalizable, then $U$ is already diagonal.

   (c) (5 points) Prove or disprove the following statement: $A$ is orthogonally diagonalizable if and only if $R$ is orthogonally diagonalizable.

8. Suppose $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ are finite-dimensional inner product spaces.

   (a) (7 points) Prove that if $\dim V \leq \dim W$, then there exists a linear transformation $T : V \to W$ such that $\langle x, y \rangle_V = \langle T(x), T(y) \rangle_W$ for all $x, y \in V$.

   (b) (5 points) Prove that if $\dim V > \dim W$, then there does *not* exist a linear transformation $T : V \to W$ such that $\langle x, y \rangle_V = \langle T(x), T(y) \rangle_W$ for all $x, y \in V$.

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