Math 217 – Midterm 2

Winter 2022

Solutions

Student ID Number: ______________________ Section: ________

Question	Points	Score
1	12	
2	16	
3	12	
4	12	
5	12	
6	12	
7	11	
8	13	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

(a) The *dimension* of the subspace V of $\mathbb{R}^n$

Solution: The *dimension* of the subspace V of $\mathbb{R}^n$ is the number of vectors in any basis of V.

(b) The function $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal transformation*

Solution: The function $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal transformation* if for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$.

Solution: The function $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal transformation* if the standard matrix of T is an orthogonal matrix.

Solution: The function $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal transformation* if T is linear and for all $\vec{x} \in \mathbb{R}^n$, we have $\|T(\vec{x})\| = \|\vec{x}\|$.

(c) The *orthogonal complement* $W^\perp$ of the subspace W inside the inner product space $(V, \langle \cdot, \cdot \rangle)$

Solution: The *orthogonal complement* of the subspace W in V is the set $W^\perp = \{v \in V \ : \ \langle v, w \rangle = 0 \text{ for all } w \in W.\}$.

(d) For $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation and $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ an ordered basis of $\mathbb{R}^n$, the *matrix* $[T]_\mathcal{B}$ *of* T *relative to* $\mathcal{B}$, also called the $\mathcal{B}$*-matrix of* T

Solution: The $\mathcal{B}$-matrix of T is the standard matrix of $L_\mathcal{B} \circ T \circ L_\mathcal{B}^{-1}$, where $L_\mathcal{B} : V \to \mathbb{R}^n$ is the $\mathcal{B}$-coordinate isomorphism.

Solution: The $\mathcal{B}$-matrix of T is $[T]_\mathcal{B} = \begin{bmatrix} | & & | \\ [T(\vec{b}_1)]_\mathcal{B} & \cdots & [T(\vec{b}_n)]_\mathcal{B} \\ | & & | \end{bmatrix}$.

Solution: The $\mathcal{B}$-matrix of T is the unique $n \times n$ matrix $[T]_\mathcal{B}$ such that $[T]_\mathcal{B}[v]_\mathcal{B} = [T(v)]_\mathcal{B}$ for all $v \in V$.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (4 points) For all vectors $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$, $\det \begin{bmatrix} \vec{x} - 2\vec{y} & \vec{x} & \vec{z} \end{bmatrix} = 2 \cdot \det \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}^\top$.

Solution: TRUE. Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $a, b, c, d \in \mathbb{R}$. Using multilinearity of the determinant and the fact that $\det A = \det A^\top$ for every $A \in \mathbb{R}^3$, we have

$$\begin{aligned} \det \begin{bmatrix} \vec{x} - 2\vec{y} & \vec{x} & \vec{z} \end{bmatrix} &= \det \begin{bmatrix} \vec{x} & \vec{x} & \vec{z} \end{bmatrix} - 2 \det \begin{bmatrix} \vec{y} & \vec{x} & \vec{z} \end{bmatrix} \\ &= 0 + 2 \det \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} = 2 \det \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}^\top. \end{aligned}$$

(b) (4 points) For every pair of ordered bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$, if there exists a nonzero vector $\vec{x} \in \mathbb{R}^2$ such that $[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$, then $\mathcal{B} = \mathcal{C}$.

Solution: FALSE. For a counterexample, let $\vec{x} = \vec{e}_1$, and let $\mathcal{B} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{C} = (\vec{e}_1, -\vec{e}_2)$. Then $[\vec{x}]_{\mathcal{B}} = \vec{e}_1 = [\vec{x}]_{\mathcal{C}}$, but $\mathcal{B} \neq \mathcal{C}$.

(c) (4 points) The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are similar to each other.

Solution: TRUE. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose standard matrix is B, and let $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B} = (\vec{b}_1, \vec{b}_2) = (\vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2)$. Then $T(\vec{b}_1) = \vec{b}_1$ and $T(\vec{b}_2) = -\vec{b}_2$, so $[T]_{\mathcal{B}} = A$. Thus

$$A = [T]_{\mathcal{B}} = S_{\mathcal{E}\to\mathcal{B}} [T]_{\mathcal{E}} S_{\mathcal{E}\to\mathcal{B}}^{-1} = S_{\mathcal{E}\to\mathcal{B}} B S_{\mathcal{E}\to\mathcal{B}}^{-1},$$

showing that A and B are similar.

Solution: TRUE. Note that

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1},$$

and more generally $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} A \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ if and only if $a = c$ and $b = -d$.

(d) (4 points) For every $m \times n$ matrix A, we have $\operatorname{im}(AA^\top) = \operatorname{im} A$.

Solution: TRUE. Let $A \in \mathbb{R}^{m\times n}$. Using the worksheet (or textbook) identites $\ker B^\top = (\operatorname{im} B)^\perp$ and $\ker(B^\top B) = \ker(B)$ for every matrix B, we have

$$(\operatorname{im} AA^\top)^\perp = \ker(AA^\top)^\top = \ker\left((A^\top)^\top A^\top\right) = \ker AA^\top = \ker A^\top = (\operatorname{im} A)^\perp.$$

Taking orthogonal complements gives us $\operatorname{im} AA^\top = \operatorname{im} A$, as desired.

Solution: TRUE. Let $A \in \mathbb{R}^{m \times n}$. Given $\vec{x} \in \operatorname{im} AA^\top$, we can fix $\vec{y} \in \mathbb{R}^m$ such that $\vec{x} = (AA^\top)\vec{y} = A(A^\top \vec{y})$, which shows $\vec{x} \in \operatorname{im} A$. Thus in order to show $\operatorname{im} AA^\top = \operatorname{im} A$, it will suffice to show $\dim \operatorname{im} AA^\top = \dim \operatorname{im} A$. For this we use Rank-Nullity along with the identites $\ker B^\top B = \ker B$ and $\ker B^\top = (\operatorname{im} B)^\perp$ which hold for every matrix B. Indeed, we have

$$\dim \operatorname{im} AA^\top = m - \dim \ker AA^\top = m - \dim \ker A^\top = m - \dim(\operatorname{im} A)^\perp = \dim \operatorname{im} A,$$

completing the proof.

Solution: TRUE. If $\vec{y} \in \operatorname{im}(AA^\top)$, then we can fix $\vec{x} \in \mathbb{R}^n$ such that $\vec{y} = (AA^\top)\vec{x} = A(A^\top \vec{x})$, which shows $\vec{y} \in \operatorname{im} A$. Conversely, let $\vec{y} \in \operatorname{im} A$ and fix $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{y}$. Then $\operatorname{proj}_{\operatorname{im} A^\top}(\vec{x}) \in \operatorname{im} A^\top$, so we can fix $\vec{z} \in \mathbb{R}^m$ such that $A^\top \vec{z} = \operatorname{proj}_{\operatorname{im} A^\top}(\vec{x})$. But then by definition of orthogonal projection and the identity $(\operatorname{im} A^\top)^\perp = \ker A$ from the worksheets (or text), we have

$$\vec{x} - A^\top \vec{z} \;\in\; (\operatorname{im} A^\top)^\perp \;=\; \ker(A),$$

so $A(\vec{x} - A^\top \vec{z}) = \vec{0}$, or equivalently $A\vec{x} = AA^\top \vec{z}$. This shows $\vec{y} \in \operatorname{im}(AA^\top)$. Thus $\operatorname{im} A \subseteq \operatorname{im}(AA^\top)$ and $\operatorname{im} A \supseteq \operatorname{im}(AA^\top)$, so $\operatorname{im} A = \operatorname{im}(AA^\top)$ as desired.

3. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ a \\ a \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ b \\ c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$, and let $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ and $V = \operatorname{Span}(\mathcal{B})$.

(a) (4 points) Find all values of a, b, c for which $\mathcal{B}$ is an orthonormal basis of V. *(No justification needed.)*

Solution: $a = \pm\frac{1}{\sqrt{2}}$ and $b = c = 0$.

For the remainder of this problem, let $a = b = c = 1$ and $\mathcal{C} = (\vec{v}_1,\ \vec{v}_2,\ \vec{e}_2 - \vec{e}_3)$, so that $\mathcal{C}$ is a basis of $\mathbb{R}^3$ (you do not have to prove this).

(b) (4 points) Find the $\mathcal{C}$-matrix $[\operatorname{proj}_V]_{\mathcal{C}}$ of orthogonal projection onto V.

Solution: Since $\vec{v}_1, \vec{v}_2 \in V$, we have $\operatorname{proj}_V(\vec{v}_1) = \vec{v}_1$ and $\operatorname{proj}_V(\vec{v}_2) = \vec{v}_2$, and since $\vec{e}_2 - \vec{e}_3 \in V^\perp$ we have $\operatorname{proj}_V(\vec{e}_2 - \vec{e}_3) = \vec{0}$. Thus

$$[\operatorname{proj}_V]_{\mathcal{C}} = \begin{bmatrix} | & | & | \\ [\operatorname{proj}_V(\vec{v}_1)]_{\mathcal{C}} & [\operatorname{proj}_V(\vec{v}_2)]_{\mathcal{C}} & [\operatorname{proj}_V(\vec{e}_2 - \vec{e}_3)]_{\mathcal{C}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) (4 points) Apply the Gram-Schmidt process to $\mathcal{B}$ in order to obtain an orthonormal basis $\mathcal{U} = (\vec{u}_1, \vec{u}_2)$ of V.

Solution: We have $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{u}_2$ is the normalization of

$$\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{2}} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since this is already a unit vector, we conclude that $\mathcal{U} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$.

4. Consider the space $\mathcal{P}_2$ of polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree at most 2. Let $\mathcal{E}$ be the ordered basis $\mathcal{E} = (1, x, x^2)$ of $\mathcal{P}_2$, and let $\mathcal{B}$ be another basis of $\mathcal{P}_2$ with

$$S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) (4 points) Find $[6 + 8x + 12x^2]_{\mathcal{B}}$.

Solution: Noting that $[6 + 8x + 12x^2]_{\mathcal{E}} = \begin{bmatrix} 6 & 8 & 12 \end{bmatrix}^\top$ and $S_{\mathcal{B}\to\mathcal{E}}[p]_{\mathcal{B}} = [p]_{\mathcal{E}}$ for all $p \in \mathcal{P}_2$, we see that we need to solve the linear system $S_{\mathcal{B}\to\mathcal{E}}\vec{x} = \begin{bmatrix} 6 & 8 & 12 \end{bmatrix}^\top$.

$$\operatorname{rref} \begin{bmatrix} 2 & 0 & -1 & 6 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 3 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix},$$

so we conclude that $[6 + 8x + 12x^2]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$.

(b) (4 points) Find a nonzero polynomial $p \in \mathcal{P}_2$ such that $[p]_{\mathcal{E}} = [p]_{\mathcal{B}}$.

Solution: Write $\mathcal{B} = (p_1, p_2, p_3)$. Note that

$$[x]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = S_{\mathcal{B}\to\mathcal{E}}\vec{e}_2 = [p_2]_{\mathcal{E}},$$

so p_2 is given by $p_2(x) = x$, and therefore $[x]_{\mathcal{B}} = \vec{e}_2 = [x]_{\mathcal{E}}$.

(c) (4 points) Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by $T(p)(x) = p(x) + 2p'(x) + p(0)$. Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T.

Solution:

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | \\ [T(1)]_{\mathcal{E}} & [T(x)]_{\mathcal{E}} & [T(x^2)]_{\mathcal{E}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Let $\vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^4$ with $V = \operatorname{Span}(\vec{v}, \vec{w})$ and $\vec{x} \notin V$. Assume that

$$\vec{v} \cdot \vec{v} = 4, \qquad \vec{v} \cdot \vec{w} = 0, \qquad \vec{w} \cdot \vec{w} = 16, \qquad \vec{x} \cdot \vec{v} = 1, \qquad \vec{x} \cdot \vec{w} = -2, \qquad \vec{x} \cdot \vec{x} = 4.$$

(a) (4 points) Find $\operatorname{proj}_V(\vec{x})$ in terms of $\vec{v}$ and $\vec{w}$.

Solution: Since $\vec{v} \cdot \vec{w} = 0$, $\vec{v}$ and $\vec{w}$ are orthogonal, so

$$\operatorname{proj}_V(\vec{x}) \;=\; \left(\frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)\vec{v} + \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w} \;=\; \frac{1}{4}\vec{v} + \frac{-2}{16}\vec{w} \;=\; \frac{1}{4}\vec{v} - \frac{1}{8}\vec{w}.$$

(b) (4 points) Find a nonzero vector in $V^\perp \cap \operatorname{Span}(\vec{v}, \vec{w}, \vec{x})$.

Solution: Projecting $\vec{x}$ orthogonally onto $V^\perp$, we obtain

$$\operatorname{proj}_{V^\perp}(\vec{x}) \;=\; \vec{x} - \operatorname{proj}_V(\vec{x}) \;=\; \vec{x} - \frac{1}{4}\vec{v} + \frac{1}{8}\vec{w} \;\in\; V^\perp \cap \operatorname{Span}(\vec{v}, \vec{w}, \vec{x}).$$

(c) (4 points) Suppose $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ is the 4×2 matrix with columns $\vec{v}$ and $\vec{w}$. Letting $A = QR$ be the QR-factorization of A, find the matrix R.

Solution: Since $\vec{v}$ and $\vec{w}$ are already orthogonal, the columns of Q are just

$$Q \;=\; \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \;=\; \begin{bmatrix} \frac{\vec{v}}{\|\vec{v}\|} & \frac{\vec{w}}{\|\vec{w}\|} \end{bmatrix} \;=\; \begin{bmatrix} \frac{1}{2}\vec{v} & \frac{1}{4}\vec{w} \end{bmatrix},$$

so we have

$$R \;=\; \begin{bmatrix} \|\vec{v}\| & \vec{w} \cdot \frac{\vec{v}}{\|\vec{v}\|} \\ 0 & \|\vec{w}\| \end{bmatrix} \;=\; \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

6. Suppose you are given sample data points

$$(a_0, b_0), \quad (a_1, b_1), \quad (a_2, b_2), \quad (a_3, b_3)$$

in the plane, $\mathbb{R}^2$.

(a) (4 points) Find a matrix A and vector $\vec{b}$, in terms of the data points above, such that if $\vec{x}^* = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ is a least-squares solution of the linear system $A\vec{x} = \vec{b}$, then the function $f(t) = c_0 + c_1 t$ "best fits" these data points. *(No justification necessary.)*

Solution: The given data points lead to the constraints

$$\begin{aligned} c_0 + c_1 a_0 &= b_0 \\ c_0 + c_1 a_1 &= b_1 \\ c_0 + c_1 a_2 &= b_2 \\ c_0 + c_1 a_3 &= b_3, \end{aligned}$$

so we can let $A = \begin{bmatrix} 1 & a_0 \\ 1 & a_1 \\ 1 & a_2 \\ 1 & a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(b) (3 points) State a condition on the sample points that characterizes when the linear system you found in part (a) has a unique least-squares solution. *(No justification necessary.)*

Solution: There is a unique least-squares solution if and only if the columns of A are linearly independent, which happens if and only if the numbers a_i are not all the same.

(c) (5 points) Suppose that the sample data points are $(0, 1)$, $(1, 3)$, $(2, 4)$, and $(3, 4)$. Use least-squares to fit a line $f(t) = c_0 + c_1 t$ to these data.

Solution: We must find the unique least-squares solution of the linear system $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$. The least-squares solution of this system is the solution of the associated normal equation $A^\top A\vec{x} = A^\top \vec{b}$. Since

$$A^\top A = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \quad \text{and} \quad A^\top \vec{b} = \begin{bmatrix} 12 \\ 23 \end{bmatrix},$$

we get a solution of $\vec{x}^* = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. That is, the line $f(t) = \frac{3}{2} + t$ best fits these data points.

7. Let V be a finite-dimensional vector space with ordered basis $\mathcal{B} = (b_1, \ldots, b_n)$. Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be the map defined by

$$\langle x, y \rangle \;=\; [x]_{\mathcal{B}} \cdot [y]_{\mathcal{B}} \qquad \text{for all } x, y \in V.$$

(a) (7 points) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V.

Solution: For all $x, y, z \in V$ and $c \in \mathbb{R}$, we have

$$\langle x+y, z \rangle \;=\; [x+y]_{\mathcal{B}} \cdot [z]_{\mathcal{B}} \;=\; \big([x]_{\mathcal{B}} + [y]_{\mathcal{B}}\big) \cdot [z]_{\mathcal{B}} \;=\; [x]_{\mathcal{B}} \cdot [z]_{\mathcal{B}} + [y]_{\mathcal{B}} \cdot [z]_{\mathcal{B}} \;=\; \langle x, z \rangle + \langle y, z \rangle$$

and

$$\langle cx, y \rangle \;=\; [cx]_{\mathcal{B}} \cdot [y]_{\mathcal{B}} \;=\; \big(c[x]_{\mathcal{B}}\big) \cdot [y]_{\mathcal{B}} \;=\; c\big([x]_{\mathcal{B}} \cdot [y]_{\mathcal{B}}\big) \;=\; c\langle x, y \rangle$$

since $L_{\mathcal{B}}$ is linear and the dot product is linear in the first component. Thus $\langle \cdot, \cdot \rangle$ is linear in the first component. Furthermore,

$$\langle x, y \rangle \;=\; [x]_{\mathcal{B}} \cdot [y]_{\mathcal{B}} \;=\; [y]_{\mathcal{B}} \cdot [x]_{\mathcal{B}} \;=\; \langle y, x \rangle$$

since the dot product is symmetric, so $\langle \cdot, \cdot \rangle$ is symmetric, and thus also linear in the second component. Finally, since $L_{\mathcal{B}}$ is an isomorphism and the dot product is positive-definite, for all nonzero $x \in V$ the coordinate vector $[x]_{\mathcal{B}}$ is also nonzero and we have

$$\langle x, x \rangle \;=\; [x]_{\mathcal{B}} \cdot [x]_{\mathcal{B}} \;>\; 0.$$

Thus $\langle \cdot, \cdot \rangle$ is bilinear, symmetric, and positive-definite, so it is an inner product.

(b) (4 points) Prove that $\mathcal{B}$ is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$.

Solution: Using the fact that $\mathcal{E} = (\vec{e}_1, \ldots, \vec{e}_n)$ is orthonormal with respect to the dot product on $\mathbb{R}^n$, we have for all $1 \le i, j \le n$ that

$$\langle b_i, b_j \rangle \;=\; [b_i]_{\mathcal{B}} \cdot [b_j]_{\mathcal{B}} \;=\; \vec{e}_i \cdot \vec{e}_j \;=\; \delta_{ij}.$$

This shows that $\mathcal{B}$ is orthonormal with respect to $\langle \cdot, \cdot \rangle$.

8. Letting $\mathcal{E}$ be the ordered basis

$$\mathcal{E} \;=\; (E_{11}, E_{12}, E_{21}, E_{22}) \;=\; \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of $\mathbb{R}^{2\times 2}$, define an inner product on $\mathbb{R}^{2\times 2}$ as in Problem 7 by $\langle A, B\rangle = [A]_{\mathcal{E}} \cdot [B]_{\mathcal{E}}$.

(a) (4 points) Show that for all $A, B \in \mathbb{R}^{2\times 2}$, $\langle A, B\rangle = \operatorname{tr}(A^\top B)$.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Then

$$\langle A, B\rangle \;=\; [A]_{\mathcal{E}} \cdot [B]_{\mathcal{E}} \;=\; \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \;=\; ae + bf + cg + dh,$$

and

$$\operatorname{tr}(A^\top B) \;=\; \operatorname{tr}\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \;=\; \operatorname{tr}\left(\begin{bmatrix} ae + cg & af + ch \\ be + dg & bf + dh \end{bmatrix} \right) \;=\; ae{+}cg{+}bf{+}dh.$$

(b) (5 points) Prove that for every basis $\mathcal{U} = (U_1, U_2, U_3, U_4)$ of $\mathbb{R}^{2\times 2}$ that is orthonormal with respect to $\langle \cdot, \cdot \rangle$, the change-of-coordinates matrix $S_{\mathcal{U}\to\mathcal{E}}$ is orthogonal.

Solution: Let $\mathcal{U} = (U_1, U_2, U_3, U_4)$ be a basis of $\mathbb{R}^{2\times 2}$ that is orthonorml with respect to $\langle \cdot, \cdot \rangle$. Then for all $1 \le i, j \le 4$, $\delta_{ij} = \langle U_i, U_j \rangle = [U_i]_{\mathcal{E}} \cdot [U_j]_{\mathcal{E}}$. Thus since

$$S_{\mathcal{U}\to\mathcal{E}} \;=\; \begin{bmatrix} [U_1]_{\mathcal{E}} & [U_2]_{\mathcal{E}} & [U_3]_{\mathcal{E}} & [U_4]_{\mathcal{E}} \end{bmatrix},$$

we see that the columns of $S_{\mathcal{U}\to\mathcal{E}}$ are orthonormal, so $S_{\mathcal{U}\to\mathcal{E}}$ is orthogonal.

(c) (4 points) Let $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation such that $[L]_{\mathcal{E}}$ is symmetric. Prove that for every ordered basis $\mathcal{U}$ of $\mathbb{R}^{2\times 2}$ that is orthonormal with respect to $\langle \cdot, \cdot \rangle$, the $\mathcal{U}$-matrix of L is also symmetric.

Solution: Let L be as stated, and let $\mathcal{U}$ be a basis of $\mathbb{R}^{2\times 2}$ that is orthonormal with respect to $\langle \cdot, \cdot \rangle$, so $S_{\mathcal{U}\to\mathcal{E}}$ is orthogonal by part (b). Then $[L]_{\mathcal{U}} = S_{\mathcal{E}\to\mathcal{U}}[L]_{\mathcal{E}}S_{\mathcal{U}\to\mathcal{E}}$, so (using the fact that inverses of orthogonal matrices are orthogonal and that $S_{\mathcal{E}\to\mathcal{U}}$ and $S_{\mathcal{U}\to\mathcal{E}}$ are inverses of each other),

$$[L]_{\mathcal{U}}^\top \;=\; \left(S_{\mathcal{E}\to\mathcal{U}}[L]_{\mathcal{E}}S_{\mathcal{U}\to\mathcal{E}}\right)^\top \;=\; S_{\mathcal{U}\to\mathcal{E}}^\top [L]_{\mathcal{E}}^\top S_{\mathcal{E}\to\mathcal{U}}^\top \;=\; S_{\mathcal{E}\to\mathcal{U}}[L]_{\mathcal{E}}S_{\mathcal{U}\to\mathcal{E}} \;=\; [L]_{\mathcal{U}}.$$