Math 217 – Midterm 1

Winter 2019

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
3. No calculators, notes, or other outside assistance allowed.

Name: ______________________________ Section: ____________

Question	Points	Score
1	12	
2	15	
3	14	
4	11	
5	12	
6	14	
7	11	
8	11	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The function $f : X \to Y$ is *surjective*

   (b) Given vector spaces V and W, the *kernel* of the linear transformation $T : V \to W$

   (c) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   (d) The subset V of $\mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) For every 3×3 matrix A, if $A^2 = A$ then $A^3 = A$.

   (b) (3 points) For all vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{v}$ in $\mathbb{R}^3$, if the list $(\vec{x}, \vec{y}, \vec{z})$ is linearly independent, then the list $(\vec{x} + \vec{v}, \vec{y}, \vec{z})$ is also linearly independent.

   (c) (3 points) For every $n \in \mathbb{N}$, the set of non-invertible $n \times n$ matrices is a subspace of $\mathbb{R}^{n \times n}$.

(Problem 2, Continued).

(d) (3 points) For every $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$, if there exists a vector $\vec{b} \in \mathbb{R}^n$ such that the linear system $A\vec{x} = \vec{b}$ is inconsistent, then $\operatorname{rank}(A) < n$.

(e) (3 points) There exists a linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ such that $\operatorname{im}(T) = \ker(T)$, where $\mathcal{P}_2$ is the vector space of all polynomial functions in the variable x with real coefficients of degree at most 2.

3. Let $(\vec{u}, \vec{v}, \vec{w})$ be a basis of $\mathbb{R}^3$, and suppose that $\vec{z} = \vec{u} + \vec{v} + \vec{w}$. Let A be the 3×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{u} & \vec{v} & \vec{w} & \vec{z} \\ | & | & | & | \end{bmatrix},$$

and let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be the map with standard matrix A, so $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^4$. *(No justification is required on any part of this problem.)*

(a) (3 points) Find the reduced row echelon form of A.

(b) (2 points) Find $\operatorname{rank}(A)$ and $\operatorname{nullity}(A)$.

(c) (3 points) Find a basis of $\operatorname{im}(A)$.

(d) (3 points) Find a basis of $\ker(A)$.

(e) (3 points) Assuming that $\vec{z} = \vec{e}_1$, find the first column of the inverse of the 3×3 matrix $\begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^m$, $S : \mathbb{R}^n \to \mathbb{R}^d$, and $R : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations.

   (a) (2 points) State exact conditions on m, n, and d for which the composition $R \circ S \circ T$ is defined. *(No justification necessary).*

   (b) (2 points) Let A, B, and C be the standard matrices of R, S, and T, respectively. Assuming that $R \circ S \circ T$ is defined, find its standard matrix in terms of A, B, and C. *(No justification necessary).*

   (c) (3 points) Again assume that $R \circ S \circ T$ is defined, and suppose $T(\vec{e}_1) = \vec{v}_1$, $T(\vec{e}_2) = \vec{v}_2$, $S(\vec{v}_1) = \vec{w}_1$, $S(\vec{v}_2) = \vec{w}_2$, $R(\vec{w}_1) = \vec{e}_1 + 2\vec{e}_2$, and $R(\vec{w}_2) = 2\vec{e}_1 + 5\vec{e}_2$. Find the standard matrix of $R \circ S \circ T$.

   (d) (4 points) Assume $m = n = d = 2$, and suppose R is reflection over the x-axis, S is projection onto the y-axis, and T is reflection over the line $y = -x$. Find the standard matrix of $R \circ S \circ T$.

5. Consider the system of linear equations $A\vec{x} = \vec{b}$, where $A \in \mathbb{R}^{3\times 4}$ and $\vec{b} \in \mathbb{R}^3$. Throughout this problem, suppose the reduced row-echelon form of the augmented matrix $[\,A \mid \vec{b}\,]$ is

$$\left[\begin{array}{cccc|c} p & q & 3 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & r & s \end{array}\right], \qquad \text{where } p, q, r, s \in \mathbb{R}.$$

(No justification is required on any part of this problem.)

(a) (3 points) Find all values of p and q that are consistent with the given information, or else write *none* if there are no such values.

(b) (3 points) Find all values of r and s that are consistent with both the given information and the assumption that the linear system $A\vec{x} = \vec{b}$ has no solutions, or else write *none* if there are no such values.

(c) (3 points) Find all values of r and s that are consistent with both the given information and the assumption that the linear system $A\vec{x} = \vec{b}$ has a unique solution, or else write *none* if there are no such values.

(d) (3 points) Find all values of r and s that are consistent with both the given information and the assumption that $\operatorname{rank}(A) = 2$, or else write *none* if there are no such values.

6. Given $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the map given by $T_\alpha(A) = SA$, where $S = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

   (a) (4 points) Show that for all $\alpha \in \mathbb{R}$, T_α is a linear transformation.

   (b) (4 points) Show that if S is an invertible matrix, then T_α is an isomorphism.

(Problem 6, Continued). *Recall the instructions for Problem 6:*

Given $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the map given by $T_\alpha(A) = SA$, where $S = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

(c) (3 points) Letting $\alpha = 1$, find a basis of $\ker(T_1)$. *You should indicate how you obtain your answer, but you do not need to prove that your answer is actually a basis of* $\ker(T_1)$.

(d) (3 points) Again letting $\alpha = 1$, find a basis of $\operatorname{im}(T_1)$. *You should indicate how you obtain your answer, but you do not need to prove that your answer is actually a basis of* $\operatorname{im}(T_1)$.

7. Let V and W be vector spaces, and suppose that $T : V \to W$ is a linear transformation.

   (a) (2 points) Prove that $T(\vec{0}_V) = \vec{0}_W$.

   (b) (4 points) Prove that if T is injective, then $\ker(T) = \{\vec{0}_V\}$.

   (c) (5 points) Prove that if $\ker(T) = \{\vec{0}_V\}$, then T is injective.

8. Let $n \geq 2$, let A and B be $n \times n$ matrices, and write O for the $n \times n$ zero matrix.

   (a) (6 points) Prove that if $AB = O$, then $\operatorname{rank}(A) + \operatorname{rank}(B) \leq n$.

   (b) (2 points) State the converse of the statement you were asked to prove in part (a). *(No justification required).*

   (c) (3 points) Prove the converse of the statement in part (a) if it is true for all $n \geq 2$, or else give a counterexample to show that it can fail for some $n \geq 2$.

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