Worksheet 23: Eigenvalues and Eigenvectors (§§7.1, 7.2)

Let V be a vector space of dimension n, and let $T : V \to V$ be a linear transformation.

Definition: The scalar $\lambda \in \mathbb{R}$ is an *eigenvalue* of T if there is a non-zero vector $v \in V$ such that

$$T(v) = \lambda v;$$

such a vector v is called an *eigenvector* of T, with corresponding eigenvalue λ.

Note that, by definition, eigenvectors are required to be nonzero!

Problem 1. Show that for any scalar λ, the set

$$E_\lambda \;=\; \{v \in V : \; T(v) = \lambda v\}$$

is a subspace of V, called the *eigenspace* corresponding to λ. Then complete the statement: "λ is an eigenvalue of T if and only if E_λ is . . . "

Solution: Let $T : V \to V$ be a linear transformation, and let $\lambda \in \mathbb{R}$. Since $T(\vec{0}) = \vec{0} = \lambda\vec{0}$, we have $\vec{0} \in E_\lambda$. If $v \in E_\lambda$ and $c \in \mathbb{R}$, then $T(cv) = cT(v) = \lambda(cv)$, so $cv \in E_\lambda$. Finally, if $v, w \in E_\lambda$ then $T(v + w) = T(v) + T(w) = \lambda v + \lambda w = \lambda(v + w)$, so $v + w \in E_\lambda$. This shows that E_λ is a subspace of V. Furthermore, λ is an eigenvalue of T if and only if $E_\lambda \neq \{\vec{0}\}$.

Problem 2. For the following transformations, try to find an eigenvector using any methods you can think of, including basic geometry, if this is possible. What are the corresponding eigenvalues?

(a) $V = \mathbb{R}^2$, $T =$ reflection over the x-axis.

(b) $V = \mathbb{R}^2$, $T =$ reflection over the line $x = y$.

(c) $V = \mathbb{R}^2$, $T =$ rotation by 90°.

(d) $V = \mathbb{R}^2$, $T =$ left multiplication by $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

(e) $V = \mathbb{R}^2$, $T =$ left multiplication by $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

(f) $V = \mathcal{P}_3$ the space of polynomials of degree less than or equal 3 in the variable t,

$$T(f) = f'.$$

(g) $V = \mathbb{R}^{2\times 2}$, T is the zero transformation.

Solution:

(a) eigenvectors $\vec{e}_1$ and $\vec{e}_2$, with corresponding eigenvalues 1 and -1

(b) eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$, with corresponding eigenvalues 1 and -1

(c) no eigenvectors or eigenvalues

(d) eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$, with corresponding eigenvalues 2 and 3

(e) eigenvector $\vec{e}_1$, with corresponding eigenvalue 2

(f) any constant function is an eigenvector, with eigenvalue 0

(g) Any zero transformation has $\lambda = 0$ as its only eigenvalue, with the entire space $\mathbb{R}^{2\times 2}$ the corresponding eigenspace.

Problem 3. Suppose that λ is an eigenvalue of the linear transformation $T : V \to V$.

(a) Given $n \in \mathbb{N}$, is λ an eigenvalue of T^n? Can you find any eigenvalues of T^n?

(b) Supposing that T is invertible, is λ an eigenvalue of T^{-1}? Can you find any eigenvalues of T^{-1}?

Solution:

(a) If v is an eigenvector of T corresponding to λ, then $T^n(v) = \lambda^n v$, so v is an eigenvector of T^n with corresponding eigenvalue λ^n. Thus λ is not an eigenvalue of T^n unless there is some eigenvalue μ of T such that $\mu^n = \lambda$.

(b) If T is invertible and v is an eigenvector of T with corresponding eigenvalue λ, then $T^{-1}(\lambda v) = v$, so $T^{-1}(v) = \frac{1}{\lambda}v$, which shows that v is an eigenvector of T^{-1} with corresponding eigenvalue λ^{-1}. Thus λ is not an eigenvalue of T^{-1} unless λ^{-1} also happens to be an eigenvalue of T.

Problem 4. The *characteristic polynomial* of the linear transformation $T : V \to V$ is the polynomial f_T (which we write here in the variable λ) given by

$$f_T(\lambda) = \det(\lambda I - T),$$

where $I : V \to V$ is the identity transformation. Here we are thinking of $\lambda I - T$ as a new linear transformation from V to V, defined by

$$(\lambda I - T)(v) = \lambda v - T(v) \qquad \text{for all } v \in V.$$

Recall that in order to compute the determinant of a linear transformation $S : V \to V$, we just compute the determinant of the $\mathcal{B}$-matrix $[S]_{\mathcal{B}}$ of S where $\mathcal{B}$ is *any* basis of V.

Write down the characteristic polynomials for all the linear transformations in Problem 2.

Solution:

(a) $(\lambda - 1)(\lambda + 1)$

(b) $(\lambda - 1)(\lambda + 1)$

(c) $\lambda^2 + 1$

(d) $(\lambda - 2)(\lambda - 3)$

(e) $(\lambda - 2)^2$

(f) λ^4

(g) λ^4

Problem 5: A systematic way of finding eigenvalues. If v is an eigenvector of T with corresponding eigenvalue λ, then

$$(\lambda I - T)(v) = \vec{0},$$

so the linear transformation $\lambda I - T$ has a nontrivial kernel, hence is not invertible, which means

$$\det(\lambda I - T) = 0.$$

Conversely, reversing the argument shows that if $\det(\lambda I - T) = 0$, then λ is an eigenvalue of T. This shows that *the eigenvalues of T are just the roots of the characteristic polynomial of T*.

Using this method, list *all* eigenvalues of the transformations in Problem 2, and thus verify that you didn't miss any the first time around. (Keep these eigenvalues handy for Problem 6 below).

Solution:

(a) $\lambda = \pm 1$

(b) $\lambda = \pm 1$

(c) none

(d) $\lambda = 2, 3$

(e) $\lambda = 2$

(f) $\lambda = 0$

(g) $\lambda = 0$

Definition: Let λ be an eigenvalue of T. The *algebraic multiplicity* "almu(λ)" of λ is the number of times that λ occurs as a root of the characteristic polynomial f_T of T; that is, if we write f_T in the variable x, then $\text{almu}(\lambda)$ is the largest power of $(x-\lambda)$ that is a factor of $f_T(x)$. The *geometric multiplicity* "$\text{gemu}(\lambda)$" of λ is the dimension of the corresponding eigenspace E_λ.

Problem 6. Compute the algebraic and geometric multiplicities of all the eigenvalues of the transformations in Problem 2. Do you find experimentally any relation between the algebraic and geometric multiplicities?

Solution: Every eigenvalue in Problem 2 has geometric multiplicity 1, except the eigenvalue $\lambda = 0$ in part (g), which has geometric multiplicity 4. The eigenvalue $\lambda = 2$ in part (e) has algebraic multiplicity 2, the eigenvalue $\lambda = 0$ in parts (f) and (g) have algebraic multiplicity 4, and all the other eigenvalues in Problem 2 have algebraic multiplicity 1. For each eigenvalue, the geometric multiplicity is always less than or equal to the algebraic multiplicity.

Problem 7. Suppose v_1 and v_2 are eigenvectors of T that correspond to *distinct* eigenvalues λ_1 and λ_2, respectively. Show that v_1 and v_2 are linearly independent. What is $E_{\lambda_1} \cap E_{\lambda_2}$?

Solution: Let $\vec{v}_1$ and $\vec{v}_2$ be eigenvectors of T corresponding to distinct eigenvalues λ_1 and λ_2, and suppose $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Applying T to this equation gives

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0},$$

and scaling by λ_1 gives

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}.$$

Subtracting the second of these equations from the first, we obtain

$$c_2(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0}.$$

Since $\lambda_1 \neq \lambda_2$ and $\vec{v}_2 \neq \vec{0}$, we see that $c_2 = 0$, and a similar argument shows $c_1 = 0$. Therefore $(\vec{v}_1, \vec{v}_2)$ is linearly independent.

Furthermore, it follows immediately from the definition of eigenspace that $E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$ whenever $\lambda_1 \neq \lambda_2$, since if $\vec{v} \in E_{\lambda_1} \cap E_{\lambda_2}$ where $\lambda_1 \neq \lambda_2$, then $\lambda_1\vec{v} = T(\vec{v}) = \lambda_2\vec{v}$, which implies $\vec{v} = \vec{0}$. (This gives another way of showing that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent).

Problem 8. Extend the statement and proof of the previous problem to r eigenvectors.

Solution: We show that if $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, respectively, then $(\vec{v}_1, \ldots, \vec{v}_r)$ is linearly independent.

We argue by induction on r. The base case $r = 1$ follows immediately from the fact that eigenvectors are nonzero. For the inductive step, let $r > 1$, suppose the claim holds for any set of $r-1$ eigenvectors with pairwise distinct eigenvalues, and let $\vec{v}_1, \ldots, \vec{v}_r$ be eigenvectors of T with corresponding pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, respectively. Suppose $\sum_{i=1}^r c_i\vec{v}_i = \vec{0}$. Separately applying T to this equation and multiplying it by λ_r, as in the proof of (6), we obtain

$$\sum_{i=1}^r c_i\lambda_i\vec{v}_i \;=\; \vec{0} \;=\; \sum_{i=1}^r c_i\lambda_r\vec{v}_i,$$

so we have $\sum_{i=1}^{r-1} c_i(\lambda_i - \lambda_r)\vec{v}_i = \vec{0}$ after subtracting. Using the fact that $\lambda_i \neq \lambda_r$ for each $i \neq r$, and applying the inductive hypothesis, we see that $c_i = 0$ for each $1 \leq i < r$. Therefore $c_r\vec{v}_r = \vec{0}$ (from the original equation), which implies $c_r = 0$ since $\vec{v}_r \neq \vec{0}$. Thus each c_i is zero, showing that $(\vec{v}_1, \ldots, \vec{v}_r)$ is linearly independent as claimed, completing the induction.

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Everything on this worksheet has been defined in terms of a linear transformation $T : V \to V$. If A is an $n \times n$ matrix, then the map $T_A(\vec{x}) = A\vec{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus we may redefine all the notions introduced in this worksheet in the context of matrices: the eigenvalues, eigenvectors, and characteristic polynomial of a square matrix A are just the eigenvalues, eigenvectors, and characteristic polynomial of the transformation T_A.