Worksheet 1: Solving Linear Equations (§1.1)

Problem 1. Consider the set A of points in $\mathbb{R}^3$ that satisfy the equation $2x - y + z = 0$, and the set B of points in $\mathbb{R}^3$ satisfying $x + 3y + 4z = 0$.

(a) What type of geometric objects are the sets A and B?

Solution: Planes.

(b) What type of geometric object is the set $A \cap B$?

Solution: A line.

(c) Find the set of all points in $\mathbb{R}^3$ that satisfy both of the linear constraints

$$\begin{aligned} 2x - y + z &= 0 \\ x + 3y + 4z &= 0 \end{aligned}$$

using the method of "elimination" (subtracting a multiple of one equation from another). What does this have to do with your answer to (b)?

Solution: $\{\langle x, x, -x\rangle : x \in \mathbb{R}\}$, or $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} t \;:\; t \in \mathbb{R} \right\}$. This solution set *is* the line from part (a).

(d) What does the "slope," or "direction" of a line in $\mathbb{R}^3$ mean? Describe your answer to (c) as a line with a given slope through a given point. Do *planes* in $\mathbb{R}^3$ have "slopes"? Describe A and B by giving their normal vectors and points that they pass through.

Solution: "Slope" means direction, and is given by a vector. The solution set in part (c) is the line with slope $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ through the origin, $\vec{0}$. The "slope" of a plane (in $\mathbb{R}^3$) is determined by two direction vectors, or equivalently one normal vector. Planes A and B both contain $\vec{0}$, and have normal vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, respectively.

Problem 2. Suppose we have a system of three (nontrivial[†]) linear equations in three unknowns:

$$\begin{aligned} ax + by + cz &= p \\ dx + ey + fz &= q \\ gx + hy + kz &= r, \end{aligned}$$

where the coefficients $a, b, c, d, e, f, g, h, k, p, q, r$ are real numbers.

(a) Discuss with your group why we can think of the solution space as an intersection of three planes in $\mathbb{R}^3$.[‡] What sorts of geometric objects can the intersection of three planes in $\mathbb{R}^3$ be?

(b) Find an explicit example (values of the constants) in which the solution space is a plane.

(c) Find an explicit example (values of the constants) in which the solution space is a line.

(d) Find an explicit example (values of the constants) in which the solution space is a point.

(e) Find an explicit example (values of the constants) so that the system is *inconsistent* (i.e., has no solutions).

(f) Are there values of the constants so that the solution space is a circle? A parabola? A union of two different lines? Place a bet on the shape of the solution space if you were to pick the constants at random.

Solution: (a) The intersection of three planes in $\mathbb{R}^3$ could be a plane, a line, a point, or the empty set. (f) The solution space could never be a circle, parabola, or union of two different lines. If you choose the constants randomly, the solution space is likely to be a single point.

[†]Here, *nontrivial* just means that that coefficients are not all zero.

[‡]A useful technique: if stuck on a problem, try a simpler version of it. In this case, you can start by thinking about the solution space for a system of *two* linear equations in *two* unknowns, which will be a subset of $\mathbb{R}^2$.

Problem 3. Solve each of the following systems of equations:

(a)

$$\begin{aligned} y &= 2w + 3z - 8 \\ x &= w + z - 4 \\ y &= 6w - 6x + 6z - 24 \\ w + z &= 3 \end{aligned}$$

Solution: The unique solution is $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$, so the solution set is a single point.

(b)

$$3w + 3x - 5z \;=\; 3w + 3x - 3y \;=\; 6w + 6x - 6y - 5z \;=\; w + x \;=\; 0$$

Solution: The augmented matrix row reduces to $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so the solution set is the line given by $w = -x$, $y = z = 0$.

(c)

$$\begin{aligned} -5x + 3y + 3z &= -5 \\ -7x + 4y + 4z &= -5 \\ -2x + y + z &= 5 \end{aligned}$$

Solution: The augmented matrix row reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so the system is inconsistent (i.e., the solution set is $\varnothing$). Alternatively, notice that subtracting the first equation from the second produces the equation

$$-2x + y + z = 0,$$

which is inconsistent with the third equation.

(d) For each of the linear systems in parts (a) – (c) above, describe the solution set geometrically.

Mathematical Language: Proofs and Logic

Problem 4. Consider the *Axiom of Parental Support*: If you get a "B" or better in this course, your parents will buy you a new car. Let us accept this as true (your experience notwithstanding), and take the following definitions:

Definition: An *"A" student* never gets a grade lower than "A−" in a given semester.

Definition: A *"B" student* gets at most one grade lower than a "B" in a given semester.

Definition: A *"C" student* gets no grade higher than "C" in a given semester.

Given these axioms and definitions, decide which of the following statements are THEOREMS.*
Justify each of your claims with either an argument or a counterexample.

(a) If I am an "A" student, I will get a new car from my parents at the end of the semester.
(b) If I am a "B" student, I will get a new car from my parents at the end of the semester.
(c) If I am a "C" student, I will not get a new car from my parents at the end of the semester.

Solution: (a) is a theorem, while (b) and (c) are not theorems. To prove (a), suppose you are an "A" student. Then by definition, you never get a grade lower than "A−" in a given semester. In particular, you will not get a grade lower than "A−" in *this* course. This means you will get a "B" or better in this course, which by the Axiom of Parental Support means that your parents will indeed buy you a new car.

For a counterexample to (b), I could be a "B" student who gets a "C" in this class and thus my parents refuse to buy me a car.

For a counterexample to (c), I might be a "C" student who gets a "C" in all my classes this semester but my parents decide to buy me a new car anyway. This does not contradict the Axiom of Parental Support because in math a statement of the form "If P then Q" is taken to be true whenever Q is true or P is false (or both! — remember that "or" is always used inclusively in math).

*A *theorem* in an axiomatic system is a statement that is logically implied by the axioms, so that it *must be true* provided that the axioms themselves are true. Another way to think of theorems is that they are the statements that can be proved using the axioms.