Worksheet 25: Eigenvalues and Diagonalizability (§§7.1–7.3)

Problem 1. Let $\mathcal{P}_2$ be the vector space of polynomial functions of degree at most 2, and let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by $T(f)(x) = f(x) + (1-x)f'(x) + 2x^2 f''(x)$.

(a) Find the characteristic polynomial of T.

(b) Determine whether or not T is diagonalizable, and if it is, find a basis $\mathcal{B}$ of $\mathcal{P}_2$ in which T is diagonal.

Solution:

(a) In the basis $\mathcal{E} = (1, x, x^2)$, we have $[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, so the characteristic polynomial of T is $\det\big(tI - [T]_{\mathcal{E}}\big) = t(t-1)(t-3)$.

(b) The eigenvalues of T are the roots of its characteristic polynomial: 0, 1, and 3. Since T has three distinct eigenvalues, it is diagonalizable with $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ for an appropriate eigenbasis $\mathcal{B}$. By finding vectors in the eigenspaces $\ker(\lambda I - T)$ for each eigenvalue λ of T, we obtain the eigenbasis $\mathcal{B} = (1, 1-x, 1+2x+3x^2)$.

Problem 2. Suppose $\lambda_1, \ldots, \lambda_r$ are *distinct* eigenvalues of the $n \times n$ matrix A, and for each $1 \le k \le r$ let $\mathcal{B}_k$ be a basis of the subspace

$$E_{\lambda_k} = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda_k \vec{v}\}.$$

Prove that the set $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$ is linearly independent.

Solution: Write $\mathcal{B}_k = \{\vec{v}_{k1}, \ldots, \vec{v}_{km_k}\}$ for each $1 \le k \le r$. Suppose

$$\sum_{k=1}^{r} \sum_{j=1}^{n_k} c_{kj} \vec{v}_{kj} = \vec{0},$$

and for each $1 \le k \le r$ write $\vec{w}_k = \displaystyle\sum_{j=1}^{m_k} c_{kj} \vec{v}_{kj}$, so each $\vec{w}_k \in V_{\lambda_k}$ and $\vec{w}_1 + \cdots + \vec{w}_r = \vec{0}$. Then each $\vec{w}_k$ must be $\vec{0}$ by Problem 8 of Worksheet 23, which implies that $c_{kj} = 0$ for all k, j since each $\mathcal{B}_k$ is a basis. Hence $\mathcal{B}$ is linearly independent.

Problem 3. Let A be an $n \times n$ matrix, and let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ be the set of eigenvalues of A (also called the *spectrum* of A). For each $1 \le k \le r$, let g_k be the geometric multiplicity of λ_k as an eigenvalue of A. Prove that A is diagonalizable if and only if $g_1 + \cdots + g_r = n$.

Solution: Suppose A is diagonalizable. Then A admits an eigenbasis of $\mathbb{R}^n$, say $\mathcal{B}$. For each $1 \le k \le r$, let $\mathcal{B}_k = \{\vec{v} \in \mathcal{B} : A\vec{v} = \lambda_k \vec{v}\}$, so that $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for each $i \neq j$. Note that $\mathcal{B}_k$ is a linearly independent subset of E_{λ_k}, so $|\mathcal{B}_k| \le g_k$. But $\sum_{k=1}^r |\mathcal{B}_k| = |\mathcal{B}| = n$, so $\sum_{k=1}^r g_k \ge n$. Since we also know $\sum_{k=1}^r g_k \le n$ by Problem 2, we conclude that $\sum_{k=1}^r g_k = n$.

Conversely, for each $1 \le k \le r$ let $\mathcal{B}_k$ be a basis of E_{λ_k}, so that $g_k = \dim(E_{\lambda_k})$. By Problem 2, $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$ is linearly independent. So if $g_1 + \cdots + g_r = n$, then $\mathcal{B}$ is a linearly independent subset of $\mathbb{R}^n$ containing n vectors, which means that $\mathbb{R}^n$ has a basis consisting of eigenvectors of A and thus A is diagonalizable.

Problem 4.

(a) Prove that an $n \times n$ matrix with n distinct eigenvalues must be diagonalizable.

(b) Prove that if A is a diagonalizable $n \times n$ matrix with only one eigenvalue, then A is already a diagonal matrix.

Solution:

(a) Let A be an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Since each of these eigenvalues has geometric multiplicity at least 1, the sum of their geometric multiplicities is at least n, so $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. It follows that A is diagonalizable.

(b) Suppose A is a diagonalizable $n \times n$ matrix with only one eigenvalue, λ. Let P be an invertible matrix such that $A = P\lambda I_n P^{-1}$. Then using the fact that scalar matrices commute with all other matrices (of the same square size), we have

$$A = P\lambda I_n P^{-1} = \lambda I_n P P^{-1} = \lambda I_n,$$

showing that A is diagonal.

Another way to think about this is that for A to be diagonalizable, λ must have geometric multiplicity n. Thus $\dim \ker(\lambda I_n - A) = n$, which implies $\lambda I_n - A = 0$.

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Problem 5. For each of (a) – (f) below, either prove the stated conclusion or show that it is false by giving a counterexample:

For all $n \times n$ matrices A and B, if A and B are similar then they have the same ...

(a) characteristic polynomial

(b) eigenvalues

(c) eigenvectors

(d) determinant

(e) trace

(f) reduced row echelon form

Solution: (a), (b), (d), and (e) are true. For (d), we have

$$\det(PAP^{-1}) \;=\; (\det P)(\det A)(\det P^{-1}) \;=\; (\det P)(\det A)(\det P)^{-1} \;=\; \det A.$$

For (e), using the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we have

$$\operatorname{tr}(PAP^{-1}) \;=\; \operatorname{tr}(AP^{-1}P) \;=\; \operatorname{tr}(A).$$

For (a), the characteristic polynomial of PAP^{-1} is

$$\det(xI - PAP^{-1}) \;=\; \det(PxIP^{-1} - PAP^{-1}) \;=\; \det(P(xI - A)P^{-1}) \;=\; \det(xI - A),$$

which is the characteristic polynomial of A. Finally, (b) follows from (a) since the eigenvalues of a matrix are just the roots of its characteristic polynomial.

For counterexamples, it is easy to see geometrically that two reflections over different lines in $\mathbb{R}^2$ will be similar but will have different eigenvectors. And the projection matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are similar but have different rref. (Intuitively, rref has nothing to do with similarity).

Problem 6. For each of (a) – (f) below, find two $n \times n$ matrices that are *not* similar to each other, but nevertheless have the same ...

(a) characteristic polynomial

(b) eigenvalues

(c) eigenvectors

(d) determinant

(e) trace

(f) reduced row echelon form

Then show that two *diagonalizable* $n \times n$ matrices with the same characteristic polynomial *are* similar to each other.

Solution: The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

are not similar to each other, since the first is diagonalizable and the second is not. But they have the same characteristic polynomial, eigenvalues, trace, determinant, and rref. For (c), I_2 and $2I_2$ have the same eigenvectors but different eigenvalues, so they are not similar.

If A and B are two diagonalizable $n \times n$ matrices with the same characteristic polynomial, and hence the same eigenvalues, then A and B are both similar to the same $n \times n$ diagonal matrix D whose diagonal entries are the common eigenvalues of A and B, and therefore A and B are similar to each other.

Problem 7. In each part below, either give an example of a square matrix A with the stated properties, or else explain why this is impossible:

(a) invertible but not diagonalizable

(b) diagonalizable but not invertible

(c) neither diagonalizable nor invertible

(d) both diagonalizable and invertible

Solution:

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Problem 8. Show that for any 2×2 matrix A, the characteristic polynomial of A is

$$x^2 - \operatorname{tr}(A)x + \det(A).$$

Solution:

$$\det\left(xI_2 - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \det\begin{bmatrix} x-a & -b \\ -c & x-d \end{bmatrix} = (x-a)(x-d) - bc = x^2 - (a+d)x + ad - bc.$$

Definition: If A is an $n \times n$ matrix and $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$ is any polynomial, define $p(A)$ to be the matrix

$$p(A) = a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_k A^k.$$

Problem 9. Let A be any 2×2 matrix and suppose $p(t)$ is the characteristic polynomial of A. Find $p(A)$ in terms of the entries of A.

Solution:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)I_2 \;=\; \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem 10. Let A be an $n \times n$ matrix, and let $p(t)$ be the characteristic polynomial of A. Prove that if A is diagonalizable, then $p(A)$ is the $n \times n$ zero matrix.†

[Hint: compute $p(A)\vec{v}$ for $\vec{v}$ an eigenvector of A.]

Solution: Let A be an $n \times n$ matrix, and suppose $\vec{v} \in \mathbb{R}^n$ is an eigenvector of A with corresponding eigenvalue λ. Since the eigenvalues of A are just the roots of the characteristic polynomial p of A, we can factor p as $p(t) = q(t)(t-\lambda)$ for some polynomial q. Then

$$p(A)\vec{v} \;=\; q(A)(A-\lambda I_n)\vec{v} \;=\; q(A)(A\vec{v}-\lambda\vec{v}) \;=\; q(A)\vec{0} \;=\; \vec{0}.$$

Thus $p(A)\vec{v} = \vec{0}$ for every eigenvector $\vec{v}$ of A. Assuming now that A is diagonalizable, we can choose a basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$ consisting of eigenvectors of A. Then for any vector $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \mathbb{R}^n$, we have

$$p(A)\vec{v} \;=\; p(A)(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \;=\; c_1 p(A)\vec{v}_1 + \cdots + c_n p(A)\vec{v}_n \;=\; \vec{0} + \cdots + \vec{0} \;=\; \vec{0}.$$

Thus $p(A)$ is the $n \times n$ zero matrix.

† Actually, this remarkable fact holds for *any* square matrix, but it is easier to prove for diagonalizable matrices.