# Worksheet 5: Vector Spaces and Linear Transformations



**Problem 1: Vector Spaces**. On Worksheet 3 we saw that matrices of the same size can be added together and scaled by real numbers, and that these operations have certain nice properties like commutativity and associativity. Algebraic structures like this are called *vector spaces*. The precise definition is given below; as you read through the eight statements below, you should discuss them briefly with your group members. Can you think of names for the familiar properties any of them express?

**Definition.** Let $V$ be a set. Suppose an *addition operation* $+$ is defined on $V$, so that for every pair of elements $v, w$ of $V$ there is associated another element $v + w$ of $V$. Suppose also that a *scalar multiplication* by real numbers is defined on $V$, so that for every $c \in \mathbb{R}$ and $v \in V$ there is associated an element $cv$ in $V$. Then $V$ is called a *vector space* if all the following are true:

VS-1: for all $u, v, w \in V$, $(u + v) + w = u + (v + w)$;

VS-2: for all $u, v \in V$, $u + v = v + u$;

VS-3: there is an element $\vec{0} \in V$ such that $\vec{0} + v = v$ for all $v \in V$;

VS-4: for all $v \in V$ there is a unique element $-v \in V$ such that $v + (-v) = \vec{0}$;

VS-5: for all $a \in \mathbb{R}$ and $v, w \in V$, $a(v + w) = av + aw$;

VS-6: for all $a, b \in \mathbb{R}$ and for all $v \in V$, $(a + b)v = av + bv$;

VS-7: for all $a, b \in \mathbb{R}$ and for all $v \in V$, $a(bv) = (ab)v$;

VS-8: for all $v \in V$, $1v = v$.

$* * *$ An element of a vector space is called a *vector*. $* * *$

Most of these properties (called "axioms" for vector spaces) should look familiar. Of course, we already know some excellent examples of vector spaces: *for every positive integer $n$, $\mathbb{R}^n$ is a vector space*. But there are many more examples! For instance, on Worksheet 3 we basically showed that for any positive integers $m, n$, the collection of $m \times n$ matrices forms a vector space. You should mentally convince yourself of this now, and the same goes for some other important examples:

(a) Convince yourself that the set $\mathbb{R}^{m\times n}$ of $m \times n$ matrices does indeed form a vector space, by quickly going through Axioms 1–8 with $\mathbb{R}^{m\times n}$ in mind. What is the "zero vector" here?

(b) For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the set of all *polynomial functions* of degree at most $n$. So, for instance,

$$x^2 + 1 \in \mathcal{P}_2 \quad \text{and} \quad 3x^4 - \pi x^2 + x - 1 \in \mathcal{P}_5 \setminus \mathcal{P}_3.$$

Verify that for each $n$, $\mathcal{P}_n$ is a vector space. What are the vector addition and scalar multiplication operations? What is the zero vector?

(c) Does the set $\mathcal{P}$ of *all* polynomial functions (of any degree) form a vector space?

(d) Let $C^\infty([0, 1])$ be the set of all *smooth functions* from $[0, 1]$ to $\mathbb{R}$, where "smooth" means $n$-times differentiable for all $n \in \mathbb{N}$. Verify that $C^\infty([0, 1])$ is a vector space.

While there are many different examples of vectors spaces, our original examples $\mathbb{R}^n$ will end up playing a special role. For reasons that will become clearer later on, let's agree to call the special vector spaces $\mathbb{R}^n$ *coordinate vector spaces*, or more simply *coordinate spaces*. Of course, $\mathbb{R}^n$ still **is** a vector space, but you should remember that there are many other examples of vector spaces and just because something is called a "vector" does not necessarily mean that it belongs to some $\mathbb{R}^n$.

**Problem 2: Linear Transformations**. Now that we have defined the general concept of a *vector space*, we can also define the general concept of a *linear transformation*.

**Definition.** Suppose that $V$ and $W$ are vector spaces. A function $T : V \to W$ is called a *linear transformation*, or is said to be *linear*, if for all $u, v \in V$ and for all $c \in \mathbb{R}$,

$$\begin{aligned} T(u+v) &= T(u) + T(v) \\ \text{and} \quad T(cv) &= cT(v). \end{aligned}$$

If $T : V \to W$ is linear and if $V = W$, then we call $T$ a *linear operator* on $V$.

For each of the functions between vector spaces given below, determine whether or not the function is linear.

(a) $F : \mathcal{P}_2 \to \mathcal{P}_2$ defined by $F(p)(x) = x + p(x)$ for all $x \in \mathbb{R}$. No.

(b) $F : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $F(p)(x) = xp(x)$ for all $x \in \mathbb{R}$. Yes.

(c) $F : \mathcal{P} \to \mathcal{P}$ defined by $F(p) = p^2$. No.

(d) $F : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $F(g) = \frac{d}{dx}(g)$. Yes.

(e) $F : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $F(g)(x) = \displaystyle\int_0^x g(t)\,dt$. Yes.

(f) $F : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $F(g)(x) = g(1-x)$ for all $x \in \mathbb{R}$. Yes.

(g) $F : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $F(g)(x) = e^{g(x)}$ for all $x \in \mathbb{R}$. No.

(h) $\det : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. No.

(i) $\mathrm{tr} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $\mathrm{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$. Yes.

**Problem 3: Compositions**. If $f : X \to Y$ and $g : Y \to Z$ are functions (so that the domain of $g$ contains the image of $f$), then the *composition* of $f$ with $g$ is the function

$$g \circ f : X \to Z$$

defined[†] by $(g \circ f)(x) = g(f(x))$ for all $x \in \mathbb{R}$.

Prove that if $U$, $V$, and $W$ are vector spaces and if $T : U \to V$ and $S : V \to W$ are linear transformations, then the composite function $S \circ T$ is also a linear transformation (from $U$ to $W$).

[†]Notice that we write composition of functions "backwards": $g \circ f$ means first apply $f$, then apply $g$. Sometimes, if it is obvious that $f$ and $g$ are functions and that we want to compose them, we can just write "$gf$" instead of $g \circ f$.

**Solution:** Let $U$, $V$, $W$ be vector spaces, and suppose that $T : U \to V$ and $S : V \to W$ are linear. Let $\vec{x}, \vec{y} \in U$ and $c \in \mathbb{R}$ be arbitrary. Then, using linearity of $T$ and $S$ separately, we have

$$(S \circ T)(\vec{x}+\vec{y}) \;=\; S\big(T(\vec{x}+\vec{y})\big) \;=\; S\big(T(\vec{x})+T(\vec{y})\big) \;=\; S(T(\vec{x}))+S(T(\vec{y})) \;=\; (S\circ T)(\vec{x})+(S\circ T)(\vec{y})$$

and

$$(S \circ T)(c\vec{x}) \;=\; S\big(T(c\vec{x})\big) \;=\; S\big(cT(\vec{x})\big) \;=\; cS\big(T(\vec{x})\big) \;=\; c(S \circ T)(\vec{x}).$$

This shows that $S \circ T$ is linear.