Solutions for Proposed Homework 7:

Part A:

4.3.23) $\left[T(1)\right]_{\mathfrak{U}} = \left[1\right]_{\mathfrak{U}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\left[T(t)\right]_{\mathfrak{U}} = \left[3\right]_{\mathfrak{U}} = \begin{bmatrix}3\\0\\0\end{bmatrix}$ and $\left[T(t^2)\right]_{\mathfrak{U}} = \left[9\right]_{\mathfrak{U}} = \begin{bmatrix}9\\0\\0\end{bmatrix}$, so we see that

$$[T]_{\mathfrak{U}} = \begin{bmatrix}1 & 3 & 9\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

T is not an isomorphism and $\{(t-3),(t-3)^2\}$ is a basis for $\ker(T)$, while $\{1\}$ is a basis for $\operatorname{im}(T)$, so T has rank 1.

4.3.24) $\left[T(1)\right]_{\mathfrak{U}} = \left[1\right]_{\mathfrak{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\left[T(t-3)\right]_{\mathfrak{U}} = \left[0\right]_{\mathfrak{B}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ and $\left[T((t-3)^2)\right]_{\mathfrak{U}} = \left[0\right]_{\mathfrak{B}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, so we see that

$$[T]_{\mathfrak{U}} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

T is not an isomorphism and $\{(t-3),(t-3)^2\}$ is a basis for $\ker(T)$, while $\{1\}$ is a basis for $\operatorname{im}(T)$, so T has rank 1.

4.3.46) a) Since $\left[1\right]_{\mathfrak{U}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$, $\left[t-3\right]_{\mathfrak{U}} = \begin{bmatrix}-3 & 1 & 0\end{bmatrix}$, and $\left[(t-3)^2\right]_{\mathfrak{U}} = \begin{bmatrix}9 & -6 & 1\end{bmatrix}$, we see that

$$S_{\mathfrak{B}\to\mathfrak{U}} = \begin{bmatrix}1 & -3 & 9\\0 & 1 & -6\\0 & 0 & 1\end{bmatrix}.$$

b)Notice that

$$\begin{bmatrix}1 & -3 & 9\\0 & 1 & -6\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 3 & 9\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -3 & 9\\0 & 1 & -6\\0 & 0 & 1\end{bmatrix}.$$

c) Similarly,

$$S_{\mathfrak{U}\to\mathfrak{B}} = \begin{bmatrix}1 & 3 & 9\\0 & 1 & 6\\0 & 0 & 1\end{bmatrix}.$$

4.3.54) Since $\left[T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)\right]_{\mathfrak{B}} = \left[\begin{bmatrix}1\\1\\-1\end{bmatrix}\right]_{\mathfrak{B}} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\left[T\left(\begin{bmatrix}5\\-4\\1\end{bmatrix}\right)\right]_{\mathfrak{B}} = \left[\begin{bmatrix}0\\0\\0\end{bmatrix}\right]_{\mathfrak{B}} = \begin{bmatrix}0\\0\end{bmatrix}$ (since $\begin{bmatrix}5\\-4\\1\end{bmatrix}$ is orthogonal to $\begin{bmatrix}1\\1\\-1\end{bmatrix}$), we see that

$$[T]_{\mathfrak{B}} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

5.1.12) The Triangle Inequality: If $\vec{v}, \vec{w} \in \mathbb{R}^n$, then

$$||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||.$$

Proof: Recall that, by definition,

$$||\vec{v} + \vec{w}||^2 = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} = ||\vec{v}||^2 + 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2.$$

The Cauchy-Schwartz inequality implies that $\vec{v} \cdot \vec{w} \leq ||\vec{v}||||\vec{w}||$, so we conclude that

$$||\vec{v} + \vec{w}||^2 \leq ||\vec{v}||^2 + 2||\vec{v}||||\vec{w}|| + ||\vec{w}||^2 = (||\vec{v}|| + ||\vec{w}||)^2.$$

Since, $||\vec{v} + \vec{w}|| \geq 0$ and $||\vec{v}|| + ||\vec{w}|| \geq 0$, this implies that

$$||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||.$$

5.1.28) Let $\mathfrak{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)$ and let V be the subspace of $\mathbb{R}$ spanned by $\mathfrak{B}$. Notice that each element of $\mathfrak{B}$ has length 2 and that distinct elements are orhogonal. Therefore if we scale each vector in $\mathfrak{B}$ by $\frac{1}{2}$ we obtain an orthonormal basis $\mathfrak{C} = \left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right)$ for V.

Therefore the projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ to V is given by

$$\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}.$$

5.1.32) The 2×2 matrix $G = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 \end{bmatrix}$ is not invertible if and only if it has rank less than 2, which occurs if and only if its determinant

$$(\vec{v}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{v}_2) - (\vec{v}_1 \cdot \vec{v}_2)^2 = ||\vec{v}_1||^2||\vec{v}_2||^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 = 0.$$

However, by the Cauchy-Schwartz inequality, this occurs if and only if $\vec{v}_1$ and $\vec{v}_2$ are parallel. Therefore, G is invertible if and only if $\vec{v}_1$ and $\vec{v}_2$ are not parallel.

Part B:

Problem 1:

Let V and W be vector spaces and let $\mathcal{B} = (b_1, \ldots, b_n)$ be an ordered basis for V. Let $f : \mathcal{B} \to W$ is a function. If $\vec{v} \in V$ be arbitrary, then, since $\mathcal{B}$ is an ordered basis of V, there exist unique scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that $\vec{v} = \sum_{i=1}^n c_i \vec{b}_i$. We then define $T : V \to W$ by setting

$$T(\vec{v}) \; = \; \sum_{i=1}^n c_i f(\vec{b}_i).$$

In particular, T is an extension of the function f since $T(b_i) = f(b_i)$ for all i.
Suppose that $\vec{x}, \vec{y} \in V$ and that $\vec{x} = \sum_{i=1}^n c_i \vec{b}_i$ and $\vec{y} = \sum_{i=1}^n d_i \vec{b}_i$. Then $\vec{x} + \vec{y} = \sum_{i=1}^n (c_i + d_i)\vec{b}_i$, so

$$\begin{aligned} T(\vec{v} + \vec{w}) &= T\left(\sum_{i=1}^n c_i\vec{b}_i + \sum_{i=1}^n d_i\vec{b}_i\right) = T\left(\sum_{i=1}^n (c_i + d_i)\vec{b}_i\right) \\ &= \sum_{i=1}^n (c_i + d_i) f(\vec{b}_i) = \left(\sum_{i=1}^n c_i f(\vec{b}_i)\right) + \left(\sum_{i=1}^n d_i f(\vec{b}_i)\right) = T(\vec{x}) + T(\vec{y}) \end{aligned}$$

so T respects addition. If $k \in \mathbb{R}$, $\vec{x} \in V$ and $\vec{x} = \sum_{i=1}^n c_i \vec{b}_i$, then $k\vec{x} = \sum_{i=1}^n kc_i \vec{b}_i$, so

$$T(k\vec{x}) = T\left(k\sum_{i=1}^n c_i\vec{b}_i\right) = T\left(\sum_{i=1}^n kc_i\vec{b}_i\right) = \sum_{i=1}^n kc_i f(\vec{b}_i) = k\sum_{i=1}^n c_i f(\vec{b}_i) = kT(\vec{x}),$$

which implies that T respects scalar multiplication. Therefore, T is a linear transformation.
Finally, to see that T is unique, suppose the linear map $S : V \to W$ also has the property that S agrees with f on $\mathcal{B}$. Then for all $\vec{v} = \sum_{i=1}^n c_i\vec{b}_i \in V$ we have

$$S(\vec{v}) = S\left(\sum_{i=1}^n c_i\vec{b}_i\right) = \sum_{i=1}^n c_i S(\vec{b}_i) = \sum_{i=1}^n c_i f(\vec{b}_i) = \sum_{i=1}^n c_i T(\vec{b}_i) = T\left(\sum_{i=1}^n c_i\vec{b}_i\right) = T(\vec{v}).$$

Problem 2:
Let $T :\to W$ be a linear transformation and let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases for V.

(a) TRUE If $T : V \to W$ is a linear transformation and$\ker[T]_\mathcal{B} = \{[\vec{v}]_\mathcal{B} \,:\, \vec{v} \in \ker(T)\}$.

Proof: Suppose that $\vec{x} \in \ker[T]_\mathcal{B}$. Let $\vec{v} \in V$ be the unique vector in V such that $[\vec{v}]_\mathcal{B} = \vec{x}$. Then by definition of $[T]_\mathcal{B}$ and of $\ker[T]_\mathcal{B}$, respectively,

$$[T(\vec{v})]_\mathcal{B} = [T]_\mathcal{B}[\vec{v}]_\mathcal{B} = [T]_\mathcal{B}\vec{x} = \vec{0}.$$

Since $[T(\vec{v})]_\mathcal{B} = \vec{0}$, $T(\vec{v}) = \vec{0}_V$. Thus, $\vec{x} = [\vec{v}]_\mathcal{B}$ for some $\vec{v} \in \ker T$, which proves that $\ker[T]_\mathcal{B} \subseteq \{[\vec{v}]_\mathcal{B} \,:\, \vec{v} \in \ker T\}$.

Now suppose that $\vec{x} \in \{[\vec{v}]_\mathcal{B} \,:\, \vec{v} \in \ker T\}$. Then there exists a vector $\vec{v} \in V$ such that $\vec{x} = [\vec{v}]_\mathcal{B}$ and

$$[T]_\mathcal{B}\vec{x} = [T]_\mathcal{B}[\vec{v}]_\mathcal{B} = [T(\vec{v})]_\mathcal{B} = [\vec{0}_V]_\mathcal{B} = \vec{0},$$

hence $\vec{x} \in \ker[T]_\mathcal{B}$. Therefore, $\{[\vec{v}]_\mathcal{B} \,:\, \vec{v} \in \ker T\} \subseteq \ker[T]_\mathcal{B}$ and the proof is complete. □

(b) TRUE $\operatorname{im}[T]_\mathcal{B} = \{[\vec{v}]_\mathcal{B} \,:\, \vec{v} \in \operatorname{im}(T)\}$.

Proof: Suppose that $\vec{y} \in \operatorname{im}[T]_\mathcal{B}$. Then there exists a vector $\vec{x} \in \mathbb{R}^n$ such that $[T]_\mathcal{B}\vec{x} = \vec{y}$, by definition of the image of $[T]_\mathcal{B}$. Let $\vec{v}$ be the unique vector in V for which $[\vec{v}]_\mathcal{B} = \vec{x}$, and let $\vec{w} = T(\vec{v})$. Then $\vec{w} \in \operatorname{im} T$, and

$$\vec{y} = [T]_\mathcal{B}\vec{x} = [T]_\mathcal{B}[\vec{v}]_\mathcal{B} = [T(\vec{v})]_\mathcal{B} = [\vec{w}]_\mathcal{B}$$

implying that $\vec{y} \in \{[\vec{w}]_\mathcal{B} \,:\, \vec{w} \in \operatorname{im}(T)\}$. Thus, $\operatorname{im}[T]_\mathcal{B} \subseteq \{[\vec{w}]_\mathcal{B} \,:\, \vec{w} \in \operatorname{im}(T)\}$.

On the other hand, suppose that $\vec{y} \in \{[\vec{w}]_\mathcal{B} : \vec{w} \in \operatorname{im}(T)\}$. Then $\vec{y} = [\vec{w}]_\mathcal{B}$ for some $\vec{w} \in \operatorname{im} T$. By definition of the image of T, there exists a vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Let $\vec{x} = [\vec{v}]_\mathcal{B} \in \mathbb{R}^n$. Then

$$\vec{y} = [\vec{w}]_\mathcal{B} = [T(\vec{v})]_\mathcal{B} = [T]_\mathcal{B}[\vec{v}]_\mathcal{B} = [T]_\mathcal{B}\vec{x}$$

and we conclude $\vec{y} \in \operatorname{im}[T]_\mathcal{B}$. Therefore, $\{[\vec{w}]_\mathcal{B} \,:\, \vec{w} \in \operatorname{im}(T)\} \subseteq \operatorname{im} T_\mathcal{B}$ and our proof is complete. □

(c) FALSE $\ker[T]_{\mathcal{B}}$ need not be the same as $\ker[T]_{\mathcal{C}}$.

Counterexample: Consider the map $T : \mathcal{P}_1 \to \mathcal{P}_1$ given by $T(ax+b) = b$. If $\mathcal{B} = (x, 1)$, then $\mathrm{Ker}[T]_{\mathcal{B}} = \mathrm{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$, but if $\mathcal{C} = (1, x)$, then $\mathrm{Ker}[T]_{\mathcal{C}} = \mathrm{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

(d) FALSE If $\vec{v}, \vec{w} \in V$, $[\vec{v}]_{\mathcal{B}} \cdot [\vec{w}]_{\mathcal{B}}$ need not be the same as $[\vec{v}]_{\mathcal{C}} \cdot [\vec{w}]_{\mathcal{C}}$.

Counterexample: Consider the ordered bases $\mathcal{B} = (1)$ and $\mathcal{C} = (2)$ for $\mathbb{R}$. Then if $\vec{v} = \vec{w} = 2$, then

$$[\vec{v}]_{\mathcal{B}} \cdot [\vec{w}]_{\mathcal{B}} = 2 \cdot 2 = 4 \neq 1 = 1 \cdot 1 = [\vec{v}]_{\mathcal{C}} \cdot [\vec{w}]_{\mathcal{C}}.$$

Problem 3:

a) Suppose that $A, B \in \mathbb{R}^{2\times 2}$ and $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $B = \begin{bmatrix}e & f\\ g & h\end{bmatrix}$. Then $A+B = \begin{bmatrix}a+e & b+f\\ c+g & d+h\end{bmatrix}$ and

$$T(A+B) = \begin{bmatrix}a+e & 0 & b+f\\ 0 & 0 & 0\\ c+g & 0 & d+h\end{bmatrix} = \begin{bmatrix}a & 0 & b\\ 0 & 0 & 0\\ c & 0 & d\end{bmatrix} + \begin{bmatrix}e & 0 & f\\ 0 & 0 & 0\\ g & 0 & h\end{bmatrix} = T(A) + T(B)$$

so T respects addition. Similarly, if $k \in \mathbb{R}$, $A \in \mathbb{R}^{2\times 2}$ and $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then $kA = \begin{bmatrix}ka & kbf\\ kc & kd\end{bmatrix}$ and

$$T(kA) = \begin{bmatrix}ka & 0 & kb\\ 0 & 0 & 0\\ kc & 0 & kd\end{bmatrix} = k\begin{bmatrix}a & 0 & b\\ 0 & 0 & 0\\ c & 0 & d\end{bmatrix} = kT(A)$$

so T respects scalar multiplication. Since T respects addition and scalar multiplication, it is a linear transformation.

b) $\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$ is a basis for $\mathrm{im}(T)$. So,

$$\left\{\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}\right\} = \{\vec{e}_1, \vec{e}_3, \vec{e}_7, \vec{e}_9\}$$

is a basis for $[\mathrm{im}(T)]_{\mathfrak{B}}$.

Since T is injective, $\ker T = \{\vec{0}\}$, and, by convention, its basis is empty. Similarly, the basis for $[\ker T]_{\mathfrak{B}}$ is empty.

c) Since $\left[T\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)\right]_{\mathfrak{B}_3} = \left[\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right]_{\mathfrak{B}_3} = \vec{e}_1$, $\left[T\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)\right]_{\mathfrak{B}_3} = \left[\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right]_{\mathfrak{B}_3} = \vec{e}_3$, $\left[T\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right)\right]_{\mathfrak{B}_3} = \left[\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}\right]_{\mathfrak{B}_3} = \vec{e}_7$, and $\left[T\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\right]_{\mathfrak{B}_3} = \left[\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right]_{\mathfrak{B}_3} = \vec{e}_9$, we see that if

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then $[T(A)]_{\mathcal{B}_3} = C[A]_{\mathcal{B}_2}$ for all $A \in \mathbb{R}^{2\times 2}$.

Problem 4:

(a) First notice that $\vec{0} \in V$, since V is a subspace of $\mathbb{R}^n$ and that $\vec{0} \in V^\perp$, so $\vec{0} \in V^\perp$. On the other hand, if $\vec{v} \in V \cap V^\perp$, then $\vec{v} \cdot \vec{v} = 0$, so $\vec{v} = \vec{0}$. Therefore, $V \cap V^\perp = \{\vec{0}\}$.

(b) First we show that $V \subseteq (V^\perp)^\perp$: Let $\vec{v} \in V$. Then, $\vec{v}$ is perpendicular to every vector in $V^\perp$, so $\vec{v} \in (V^\perp)^\perp$. Now, we notice that by Theorem 5.1.8, part c of the book,

$$\dim(V^\perp)^\perp = n - \dim(V^\perp) = n - (n - \dim V) = \dim V.$$

Hence, since $V \subseteq (V^\perp)^\perp$ and they are subspaces of the same dimension, they must be equal.

(c) Let's prove the forward direction. Assume $V \subseteq W$, and let $\vec{x} \in W^\perp$. Then, $\vec{x} \cdot \vec{w} = 0$ for all $\vec{w} \in W$, and since $V \subseteq W$, that implies that $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v} \in V$. Hence, $\vec{x} \in V^\perp$.

Let's prove the backwards direction. Assume that $W^\perp \subseteq V^\perp$. By the forward direction, which we just proved, this implies that $(V^\perp)^\perp \subseteq (W^\perp)^\perp$, and the result follows from part (b).

(d) Let $\vec{x} \in V^\perp \cap W^\perp$, we want to show that $\vec{x} \in (V+W)^\perp$, that is, that for all $\vec{y} \in V + W$, we have that $\vec{x} \cdot \vec{y} = 0$.

Let $\vec{y} \in V + W$. There exists $\vec{v} \in V$ and $\vec{w} \in W$ such that $\vec{y} = \vec{v} + \vec{w}$. Since $\vec{x} \in V^\perp$, we have that $\vec{x} \cdot \vec{v} = 0$, and since $\vec{x} \in W^\perp$, we also have that $\vec{x} \cdot \vec{w} = 0$. Hence,

$$\vec{x} \cdot \vec{y} = \vec{x} \cdot \vec{v} + \vec{x} \cdot \vec{w} = 0,$$

which shows $(V+W)^\perp \subseteq V^\perp \cap W^\perp$.

Conversely, we note that $V \subseteq V + W$ and $W \subseteq V + W$. By part (c), this implies that $(V+W)^\perp \subseteq V^\perp$ and $(V+W)^\perp \subseteq W^\perp$. Hence, $(V+W)^\perp \subseteq V^\perp \cap W^\perp$.

Therefore, $(V+W)^\perp = V^\perp \cap W^\perp$.

Problem 5:

(a) TRUE For every subspace V of $\mathbb{R}^n$, the orthogonal projection $\text{proj}_V : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a projection.

Proof: By Theorem 5.1.4 in the book, we know that proj_V is a linear transformation. Recall that $\text{proj}_V(\vec{x}) = \vec{x}^{||}$, where $\vec{x}^{||}$ is the only vector in V such that $\vec{x} - \vec{x}^{||}$ is in $V^\perp$. Since $\vec{x}^{||} - \vec{x}^{||} = \vec{0} \in V^\perp$, we have that

$$\text{proj}_V(\text{proj}_V(\vec{x})) = \text{proj}_V(\vec{x}^{||}) = \vec{x}^{||} = \text{proj}_V(\vec{x})$$

for all $\vec{x} \in \mathbb{R}^n$. □

(b) TRUE For every projection $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and basis $\mathcal{B}$ of $\mathbb{R}^n$, the $\mathcal{B}$-matrix $[T]_\mathcal{B}$ of T is a *projection matrix*, meaning that $([T]_\mathcal{B})^2 = [T]_\mathcal{B}$.

Proof: Let A be the $n \times n$ matrix such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Since T is a projection, we have that $A^2 = A$. Let S be the change of basis matrix from $\mathfrak{B}$ to the standard basis $\mathcal{E}$. Then $[T]_\mathfrak{B} = S^{-1}AS$, so

$$([T]_\mathfrak{B})^2 = (S^{-1}AS)^2 = S^{-1}ASS^{-1}AS = S^{-1}AS = [T]_\mathfrak{B}.$$

□

(c) FALSE If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a projection and $V = \text{im}\, T$, T need not agree with proj_V.

Counterexample: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

We have that $A^2 = A$, so T is a projection. We have that $\text{im}(T)$ is the x-axis. However, the projection onto the x-axis has standard matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

so T is not the orthogonal projection onto $\text{im}(T)$.

Alternatively, we can see that $\ker(T) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$, but $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not perpendicular to the x-axis, so T is not the orthogonal projection onto $\text{im}(T)$.

(d) TRUE. For every vector space W and projection $T : W \longrightarrow W$, if T is surjective then T is the identity map on W.

Proof: Let $\vec{x} \in W$. Since T is surjective, there exists $\vec{y} \in W$ such that $T(\vec{y}) = \vec{x}$. Applying T to both sides of the equation, and using that $T(T(\vec{y})) = T(\vec{y}) = \vec{x}$, we get that $\vec{x} = T(\vec{x})$. Hence, T is the identity transformation.