MATH 217 - LINEAR ALGEBRA
HOMEWORK 1, SOLUTIONS

Part A (10 points)

(4 pts) Complete the Math 217 Learning and Student Experience Survey (also linked on Canvas).
(6 pts) Solve the following problems from the book:
Section 1.1: 18, 36
Section 1.2: 8, 26, 36, 48

Solution.

1.1.18: We label the equations in the natural order, I, II, and III from top to bottom. Subtracting the first equation from both the second and third yields the linear system

$$\left|\begin{array}{rrrl} x & +2y & +3z = & a \\ & y & +5z = & b-a \\ & & -z = & c-a \end{array}\right|.$$

Adding 3 times equation III to equation I and 5 times equation III to equation II then yields the linear system

$$\left|\begin{array}{rrrl} x & +2y & = & 3c-2a \\ & y & = & b+5c-6a \\ & & -z = & c-a \end{array}\right|.$$

Finally, subtracting twice equation II from equation I and multiplying equation III by -1 gives us

$$\left|\begin{array}{rrrl} x & & = & 10a-2b-7c \\ & y & = & -6a+b+5c \\ & & z = & a-c \end{array}\right|.$$

1.1.36: Let $f(t) = a + bt + ct^2$ be a polynomial of degree at most 2. The graph of f passing through $(1,1)$ gives the constraint $a+b+c=1$, and the graph of f passing through $(3,3)$ gives the constraint $a+3b+9c=3$. Subtracting the second of these two equations from the first gives us $2b+8c=2$, which implies $b+4c=1$. However, if $f'(2)=3$ then we have $b+4c=3$, contradicting the fact that $b+4c=1$. Therefore there are no polynomials f of degree at most 2 whose graphs pass through $(1,1)$ and $(3,3)$ and satisfy $f'(2)=3$.

1.2.8: A sequence of steps to transform the augmented matrix of the system to its reduced row-echelon form is:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right].$$

Solving for the leading variables in terms of the free variables gives us

$$\begin{aligned} x_2 &= x_5 \\ x_4\ 7 &= -2x_5 \end{aligned}$$

with x_1, x_3, and x_5 free. Therefore the solution set of the linear system can be expressed parametrically as

$$\left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \ : \ x_1, x_3, x_5 \in \mathbb{R} \right\}.$$

1.2.26: Yes. If matrix A is transformed into matrix B by swapping two rows, then this same elementary row operation transforms B back into A. If A is transformed into B by scaling row i of A by the nonzero scalar k, then B is transformed back into A by scaling row i of B by the nonzero scalar $1/k$. Finally, if A is transformed into B by adding k times row i of A to row j of A, then B is transformed back into A by subtracting k times row i of B to row j of B.

1.2.36: The vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ is perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ if and only if $x + 3y - z = 0$. Therefore the set of all vectors in $\mathbb{R}^3$ perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ is the solution set of this linear equation, which is the set $\left\{ y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \ : \ y, z \in \mathbb{R} \right\}$.

1.2.48: The augmented matrix of this linear system is $\begin{bmatrix} 0 & 1 & 2k & 0 \\ 1 & 2 & 6 & 2 \\ k & 0 & 2 & 1 \end{bmatrix}$, which can be row-reduced to the matrix

$$\begin{bmatrix} 1 & 0 & 6-4k & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 2(k-1)(2k-1) & 1-2k \end{bmatrix}.$$

From this we see that:

(a) The system has a unique solution if and only if $k \neq 1$ and $k \neq \frac{1}{2}$.

(b) The system is inconsistent if and only if $k = 1$, since in this case the last column of the augmented matrix is a pivot column.

(c) The system has infinitely many solutions if and only if $k = \frac{1}{2}$, since in this case the last column of the augmented matrix is not a pivot column (so the system is consistent), but the third column is also not a pivot column and therefore corresponds to a free variable.

Part B (25 points)

The following covers material that is in the “Joy of Sets” and “Mathematical Hygiene” handouts, available on Canvas. Some of the material in these handouts, but not all of it, is recapped here.

1. Logical Connectives.

Every mathematical statement is either true or false. Starting from given mathematical statements, we can use logical operations to form new mathematical statements which are again either true or false. Let P and Q be two statements. Here are four basic logical constructions:

- The statement "P *and* Q" is true exactly when both P and Q are true statements.
- The statement "P *or* Q" is true exactly when at least one (possibly both!) of P or Q is true.
- The statement "*if* P *then* Q" is true exactly when Q is true or P is false. The shorthand notation for "*if* P *then* Q" is $P \implies Q$, read "P implies Q."
- The statement "P *if and only if* Q" is true exactly when both $P \implies Q$ and $Q \implies P$ are true statements. The shorthand notation for "P *if and only if* Q" is $P \iff Q$.

Problem 1. Decide whether the given statements are true or false. *Briefly* justify your answers.[1]

(a) Every square is a rectangle or $0! = 1$.
(b) If every isosceles triangle is an equilateral triangle, then the absolute value function is differentiable at zero.
(c) $\dfrac{d}{dx}\displaystyle\int_0^x e^{t^2}\,dt = e^{x^2}$ if and only if $(\pi+e)^2 = \pi^2 + e^2$.
(d) If every polynomial function is differentiable, then $\cos\frac{\pi}{6} = \frac{1}{2}$ or the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges.
(e) If the infinite series $\displaystyle\sum_{n=1}^\infty 2^{-n}$ converges and cubes have eight vertices, then 60 is a prime number.

Solution.

(a) TRUE, by the meaning of *or*, since "Every square is a rectangle" is true (or alternatively, since "$0! = 1$" is true).
(b) TRUE, by the meaning of *if…then*, since the premise "every isosceles triangle is an equilateral triangle" of the implication is false.
(c) FALSE, by the meaning of *if and only if*, since "$\dfrac{d}{dx}\displaystyle\int_0^x e^{t^2}\,dt = e^{x^2}$" is true but "$(\pi+e)^2 = \pi^2 + e^2$" is false.
(d) TRUE, by the meaning of *if…then* and *or*. Observe that "$\cos\frac{\pi}{6} = \frac{1}{2}$ or the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges" is true since "the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges" is true. Therefore the implication "If every polynomial function is differentiable, then $\cos\frac{\pi}{6} = \frac{1}{2}$ or the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges" is true since it is an implication with a true conclusion.
(e) FALSE, by the meaning of *if…then* and *and*. Note that "the infinite series $\displaystyle\sum_{n=1}^\infty 2^{-n}$ convergesand cubes have eight vertices" is true since both "the infinite series $\displaystyle\sum_{n=1}^\infty 2^{-n}$

[1] Don't work too hard here. For instance, sufficient justification for claiming that "If 13 is prime then $\sqrt{2}$ is rational" is false could be: "FALSE by the meaning of 'if…then', since '13 is prime' is true, but '$\sqrt{2}$ is rational' is false." (In particular, you would *not* need to prove that 13 is prime or that $\sqrt{2}$ is irrational!)

converges" and "cubes have eight vertices" are true, which means the implication "If the infinite series $\sum_{n=1}^{\infty} 2^{-n}$ converges and cubes have eight vertices, then 60 is a prime number" is false since it is an implication with a true premise but a false conclusion.

2. Quantifiers.

Starting from a mathematical statement or predicate which involves a variable, we can form a new one by quantifying the given variable.

- The quantifier "for all" indicates that something is true about every element in a given set and is abbreviated $\forall$. It is often appropriate to read "for all" as "for every" or "for each". For example, the truth value of $x^2 > 0$ depends on the value of x. So the quantified statement "$\forall x \in \mathbb{R},\ x^2 > 0$" is false, since it fails for $x = 0$.
- The quantifier "there exists" indicates that something is true for at least one element in a given set and is abbreviated $\exists$. It is often read as "for some", where "some" is not necessarily plural. For example, the truth value of $x^2 = 0$ depends on the value of x. So the quantified statement "$\exists x \in \mathbb{R}$ such that $x^2 = 0$" is true, since it holds for $x = 0$.
- The abbreviation "s.t." stands for "such that". (Yes, mathematicians can be lazy!)
- Since "for all" and "for some" are different quantifiers, it is very important that you never just write "for", since this would be ambiguous! (For instance, how should we interpret "$x^2 > 0$ for $x \in \mathbb{R}$". It's true if we mean "for some" but false if we mean "for all.") *Usually, but not always,* "for" by itself means "for all"; but it's always best to help out your reader (ahem, *grader*) by making your quantifiers explicit!

Problem 2.

(a) Let $P(x)$ be a statement whose truth value depends on x. An *example* is a value of x that makes $P(x)$ true, and a *counterexample* is a value of x that makes $P(x)$ false. Fill in the blank spaces with "is true", "is false", or "nothing" as appropriate:

	"$\forall x, P(x)$"	"$\exists x$ s.t. $P(x)$"
An example proves		
A counterexample proves		

(b) Write the negation of each of the following statements, using quantifiers. Your answers should involve the statement "not $P(x)$".
  (i) $\forall x, P(x)$.
  (ii) $\exists x$ s.t. $P(x)$.

Solution.

(a)

	"$\forall x, P(x)$"	"$\exists\ x$ s.t. $P(x)$"
An example proves	nothing	is true
A counterexample proves	is false	nothing

(b) (1) $\exists x$ s.t. (not $P(x)$).
  (2) $\forall x$, (not $P(x)$).

In (c) – (h), determine whether the given statement is true or false, and briefly justify your answer (as you did for Problem 1).

(c) Some integers greater than 10 are prime and even.
(d) Some integers greater than 10 are prime and some integers greater than 10 are even.

(e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x^y = x$.
(f) For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $x^y = x$.
(g) Some triangles have three vertices.
(h) For every positive real number a, there exists a unique[2] real number x such that $x^2 = a$.

Solution.

(c) FALSE, since for every integer $n > 10$, if n is even then n is divisible by 2 and $2 < n$, so n is not prime.
(d) TRUE, since for instance 11 is an integer greater than 10 that is prime, and 12 is an integer greater than 10 that is even.
(e) TRUE. Let $x = 1$. Then for all $y \in \mathbb{R}$, $x^y = 1^y = 1$. Thus there exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x^y = x$.
(f) TRUE. Given any $x \in \mathbb{R}$, we have $x^1 = x$, so it is true that for all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $x^y = x$.
(g) TRUE; in fact, all triangles have three vertices.
(h) FALSE. For any given positive real number a, there are in fact two distinct real numbers x such that $x^2 = a$ (namely $\sqrt{a}$ and $-\sqrt{a}$), so there does not exist a *unique* such number.

3. Negation.

The *negation* of a statement P, denoted "*not* P," is a statement that is true whenever P is false and false whenever P is true. There may be many different ways to formulate the negation of P, but all of them will be logically equivalent. Note that the negation of the if-then statement "$P \implies Q$" is "P and not Q," as $P \implies Q$ is false if and only if P is true and Q is false.

Problem 3. Formulate the negation of each of the statements below in a meaningful way (some of these statements have been recycled from Problems 1 and 2). Note: just writing "It is not the case that ..." before each statement will not receive credit, as that does not help the reader understand the meaning of the negation. (No justification is needed – you may just write the negation).

(a) Every square is a rectangle or $0! = 1$.
(b) $\frac{d}{dx}\int_0^x e^{t^2}\,dt = e^{x^2}$ if and only if $(\pi + e)^2 = \pi^2 + e^2$.
(c) If every polynomial function is differentiable, then $\cos\frac{\pi}{6} = \frac{1}{2}$ or the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges.
(d) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x^y = x$.
(e) Some triangles have three vertices.
(f) Every irrational number is real, but[3] not every real number is irrational.

Solution.

(a) Some square is not a rectangle and $0! \neq 1$.
(b) Either $\frac{d}{dx}\int_0^x d^{t^2}dt = e^{x^2}$ and $(\pi+e)^2 \neq \pi^2+e^2$ or else $\frac{d}{dx}\int_0^x d^{t^2}dt \neq e^{x^2}$ and $(\pi+e)^2 = \pi^2 + e^2$.

[2]The statement "there exists unique $x \in X$ such that $P(x)$" means that there is one and only one element in the set X having property P. For those who like fancy symbolisms, this is sometimes abbreviated "$\exists! x \in X$ s.t. $P(x)$."
[3]In logic, the word 'but' can always be interpreted to mean 'and'.

(c) Every polynomial function is differentiable and $\cos\frac{\pi}{6} \neq \frac{1}{2}$ and the improper integral $\int_1^\infty \frac{1}{x^2}dx$ diverges.
(d) For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $x^y \neq x$.
(e) No triangles have three vertices.
(f) Some irrational number is not real or every real number is irrational.

4. Converse and Contrapositive.

There are two additional logical statements that can be formed from a given "if-then" statement:

- The *converse* of the statement $P \implies Q$ is the statement $Q \implies P$. The converse may be true or false, independent of the truth value of the original "if-then" statement. To see this, compare the truth tables for both statements:

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

  The last two columns do not coincide.
- The *contrapositive* of the statement $P \implies Q$ is the statement not $Q \implies$ not P. The original "if-then" statement and its contrapositive have the *same* truth value. To see this, compare the truth tables for both statements:

P	Q	$P \implies Q$	not Q	not P	not $Q \implies$ not P
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

  The columns corresponding to $P \implies Q$ and not $Q \implies$ not P coincide.

Problem 4. Write both the converse and the contrapositive of the following "if-then" statements.
(a) If it sounds too good to be true, then it is too good to be true.
(b) If there exists $k \in \mathbb{N}$ such that $A^k = I_n$, then A is invertible.
(c) If z is a zero of the Riemann zeta function, then z is a negative even integer or z is a complex number with real part $\frac{1}{2}$.

Solution.
(a) Converse: If it is too good to be true, then it sounds too good to be true. Contrapostive: If it is not too good to be true, then it does not sound too good to be true.
(b) Converse: If A is invertible, then there exists $k \in \mathbb{N}$ such that $A^k = I_n$. Contrapostive: If A is not invertible, then for all $k \in \mathbb{N}$, $A^k \neq I_n$. (It is ok here to say: "If A is not invertible, then there does not exist $k \in \mathbb{N}$ such that $A^k = I_n$.")
(c) Converse: If z is a negative even integer or z is a complex number with real part $\frac{1}{2}$, then z is a zero of the Riemann zeta function. Contrapostive: If z is not a negative even integer and z is not a complex number with real part $\frac{1}{2}$, then z is not a zero of the Riemann zeta function.

5. Sets.

A *set* is a container with no distinguishing feature other than its contents. The objects contained in a set are called the *elements* of the set. We write $a \in S$ to signify that the object a is an element of the set S. The number of elements in a set S is called the *cardinality* of the set, denoted $|S|$.

Since a set has no distinguishing feature other than its contents, there is a unique set containing no elements which is called the *empty set* and is denoted $\emptyset$. Some other very common sets are the set $\mathbb{N}$ of all natural numbers, the set $\mathbb{Z}$ of all integers, the set $\mathbb{Q}$ of all rational numbers, the set $\mathbb{R}$ of all real numbers, and the set $\mathbb{C}$ of all complex numbers.

There are two important ways to specify a set.

- *Enumeration.* One can list the contents of the set, in which case the set is denoted by enclosing the list in curly braces. For example, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.[4]
- *Comprehension.* One can describe the contents of the set by a property of its elements. If $P(a)$ is a property of the object a, then the set of all objects a such that $P(a)$ is true is denoted by $\{a \mid P(a)\}$, or equivalently $\{a : P(a)\}$. For example,

$$\mathbb{Q} \;=\; \{x \mid x = \tfrac{a}{b} \text{ for some } a, b \in \mathbb{Z} \text{ with } b \neq 0\}.$$

  Comprehension can also be used together with functions. For instance, $\{n^2 : n \in \mathbb{N}\}$ is the set of all perfect squares, and $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is the set of all reciprocals of natural numbers.

Let X and S be sets. We say that S is a *subset* of X if $a \in S \implies a \in X$ holds for all objects a. We write $S \subseteq X$ to signify that S is a subset of X. This means that S is a set each of whose elements also belongs to X. The subset of X consisting of all elements a of X such that property $P(a)$ holds true is denoted $\{a \in X \mid P(a)\}$ or $\{a \in X : P(a)\}$.

Problem 5. *You do not need to justify your answers for any part of this problem.*

(a) Give common English descriptions of the following sets:
  (i) $\{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}$
  (ii) $\{n \in \mathbb{Z} : \text{there is no } k \in \mathbb{Z} \text{ such that } n = 2k + 1\}$

(b) Use set comprehension notation to give a description of each of the following sets:
  (i) The closed unit ball[5] (including boundary) in $\mathbb{R}^3$.
  (ii) The set of all perfect cubes in $\mathbb{Z}$.

(c) Given $c \in \mathbb{Z}$ and sets $A, B \subseteq \mathbb{Z}$, we define $cA := \{ca \mid a \in A\}$, $A + c = \{a + c \mid a \in A\}$, and $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$. Using enumeration, write out some (i.e., at least the first twelve) elements of the set $5\mathbb{N} + 9\mathbb{N} + 4$.

(d) Determine whether the following statements are true or false:

(i) $\pi \in \mathbb{R}$	(iii) $\{\pi\} \in \mathbb{R}$	(v) $\emptyset \in \mathbb{R}$	(vii) $\emptyset \in \emptyset$
(ii) $\pi \subseteq \mathbb{R}$	(iv) $\{\pi\} \subseteq \mathbb{R}$	(vi) $\emptyset \subseteq \mathbb{R}$	(viii) $\emptyset \subseteq \emptyset$

Solution.

(a) (i) The unit circle in $\mathbb{R}^2$.
  (ii) The set of even integers.

(b) (i) $\{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 \leq 1\}$ (ii) $\{n^3 : n \in \mathbb{Z}\}$

(c) $5\mathbb{N} + 9\mathbb{N} + 4 = \{18, 23, 27, 28, 32, 33, 36, 37, 38, 41, 42, 43, \ldots\}$.

[4]Typically the use of ellipses in math implies that there is some pattern the reader should be able to discern and continue from the elements that are given, so be sure to list enough elements for the pattern to recognizable!

[5]In math, *sphere* usually means just the surface and not the solid interior, while *open ball* means the solid interior without the surface, and *closed ball* means the solid interior together with the surface.

(d) (i) T (ii) F (iii) F (iv) T (v) F (vi) T (vii) F (viii) T

6. Set Operations.

Starting from given sets, we can use set operations to form new sets.

- Given sets X and Y, the *intersection* of X and Y is defined as
$$X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}.$$
- Given sets X and Y, the *union* of X and Y is defined as
$$X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}.$$
- Given sets X and Y, the *difference* of X and Y, denoted $X \setminus Y$ or $X - Y$, is the set
$$\{x \in X \mid x \notin Y\}.$$
- Given a set Y inside some larger set X, the *complement* of Y with respect to X, denoted Y^C, is $X \setminus Y$. (The larger set X, sometimes referred to as the *universe*, is often suppressed in the notation).

Problem 6. *You do not need to justify your answers for any part of this problem.*

(a) Given $c \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, let cA and $A + c$ be defined as in Problem 5(c).
   (i) For positive integers n and m such that $n \neq m$, use set comprehension notation to describe the sets $n\mathbb{Z} \cup m\mathbb{Z}$ and $n\mathbb{Z} \setminus m\mathbb{Z}$.
   (ii) Again letting n and m be distinct positive integers, use set comprehension notation to describe $n\mathbb{Z} \cap m\mathbb{Z}$, and find an integer p such that $6\mathbb{Z} \cap 9\mathbb{Z} = p\mathbb{Z}$.

(b) Determine whether each of the following statements is true or false:
   (i) $\cup$ is *commutative*, meaning that $A \cup B = B \cup A$ for all sets A and B.
   (ii) $\cap$ is commutative.
   (iii) $\setminus$ is commutative.
   (iv) $\cup$ is *associative*, meaning that $A \cup (B \cup C) = (A \cup B) \cup C$ for all sets A, B, and C.
   (v) $\cap$ is associative.
   (vi) $\setminus$ is associative.
   (vii) $\cup$ distributes over $\cap$, meaning that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B, and C.
   (viii) $\cap$ distributes over $\cup$, meaning that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

(c) Complete the following statements to make them true.
   (i) For all sets A and B, $A \subseteq B$ if and only if $A \cup B =$ ___ .
   (ii) For all sets A and B, $A \subseteq B$ if and only if $A \cap B =$ ___ .
   (iii) For all sets A and B, $A \subseteq B$ if and only if $A \setminus B =$ ___ .
   (iv) For every set X and subset $A \subseteq X$, $A \cup (X \setminus A) =$ ___ .
   (v) For every set X and subset $A \subseteq X$, $A \cap (X \setminus A) =$ ___ .

Solution.

(a) (i) $n\mathbb{Z} \cup m\mathbb{Z} = \{\ell \in \mathbb{Z} \,:\, \text{there is } k \in \mathbb{Z} \text{ s.t. } \ell = nk \text{ or } \ell = mk\}$.
$n\mathbb{Z} \setminus m\mathbb{Z} = \{\ell \in \mathbb{Z} \,:\, \text{there is } k \in \mathbb{Z} \text{ s.t. } \ell = nk \text{ and for all } j \in \mathbb{Z},\ \ell \neq mj\}$.

(ii) $n\mathbb{Z} \cap m\mathbb{Z} = \{\ell \in \mathbb{Z} : \text{there is } k \in \mathbb{Z} \text{ s.t. } \ell = nk \text{ and there is } j \in \mathbb{Z} \text{ s.t. } \ell = mj\}$.
$6\mathbb{Z} \cap 9\mathbb{Z} = 18\mathbb{Z}$, so we can let $p = 18$.

(b) (i) T (ii) T (iii) F (iv) T (v) T (vi) F (vii) T (viii) T

(c) (i) B (ii) A (iii) $\emptyset$ (iv) X (v) $\emptyset$