Math 217 – Midterm 2

Winter 2019

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
3. No calculators, notes, or other outside assistance allowed.
4. On this exam, *unless stated otherwise*, terms such as "orthogonal" and "orthonormal" as applied to vectors in $\mathbb{R}^n$ mean with respect to the usual dot product on $\mathbb{R}^n$.

Name: ______________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	12	
4	12	
5	16	
6	12	
7	10	
8	11	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B

   (b) The *orthogonal complement* of the subspace W of the inner product space V, with inner product $\langle \cdot, \cdot \rangle$

   (c) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal* transformation

   (d) The *norm* (or *magnitude*, or *length*) of the vector $\vec{v}$ in the inner product space $(V, \langle \cdot, \cdot \rangle)$

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For every orthogonal matrix A, we have $|\det(A)| = 1$.

(b) (3 points) For every square matrix A, if $\ker(A) = \mathrm{im}(A)^{\perp}$, then A is symmetric.

(c) (3 points) For all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$, if both S and $S \circ T$ are orthogonal transformations, then T is an orthogonal transformation.

(Problem 2, Continued).

(d) (3 points) The map $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\langle \vec{x}, \vec{y} \rangle = \det \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}$$

is an inner product on $\mathbb{R}^2$.

(e) (3 points) For all finite-dimensional vector spaces V, ordered bases $\mathcal{B}$ and $\mathcal{C}$ of V, and linear transformations $T : V \to V$,

$$\det[T]_{\mathcal{B}} = \det[T]_{\mathcal{C}}.$$

3. Let $\mathcal{P}_3$ be the vector space of polynomial functions of degree at most 3 in the variable x, and consider the ordered bases $\mathcal{E} = (1, x, x^2, x^3)$ and $\mathcal{B} = (1, x+1, 2x^2, x^3+x)$ of $\mathcal{P}_3$.

   (All solutions in this problem should be matrices or vectors having numerical entries. No justification is required on this problem, but including it might give you partial credit for incorrect final answers.)

   (a) (4 points) Let $f \in \mathcal{P}_3$ be the polynomial given by $f(x) = 2 + 3x + 4x^2 + 5x^3$. Find the coordinate vectors $[f]_{\mathcal{E}}$ and $[f]_{\mathcal{B}}$.

   (b) (4 points) Find the change-of-coordinates matrices $S_{\mathcal{B}\to\mathcal{E}}$ and $S_{\mathcal{E}\to\mathcal{B}}$.

   (c) (4 points) Find the $\mathcal{B}$-matrix $[D]_{\mathcal{B}}$ of D, where $D : \mathcal{P}_3 \to \mathcal{P}_3$ is the differentiation map defined by $D(f) = f'$ for all $f \in \mathcal{P}_3$.

4. Let $\mathcal{P}_1$ be the vector space of polynomials of degree at most 1 in the variable x. Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}_1$ by

$$\langle p, q \rangle \;=\; \int_0^1 p(x)q(x)\,dx.$$

Let $u_1(x) = 1$ and $u_2(x) = \sqrt{3}(2x-1)$ be polynomials in $\mathcal{P}_1$, and let $\mathcal{B} = (u_1, u_2)$.

(a) (5 points) Prove that $\mathcal{B}$ is an orthonormal basis of $\mathcal{P}_1$.

(b) (3 points) Find, in terms of a and b, the polynomial $p \in \mathcal{P}_1$ whose $\mathcal{B}$-coordinate vector is $\begin{bmatrix} a \\ b \end{bmatrix}$.

(Problem 4, Continued).

As above, let $\mathcal{P}_1$ be the inner product space of polynomials of degree at most 1 with inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p, q \rangle \;=\; \int_0^1 p(x)q(x)\,dx.$$

Also let $\mathcal{B} = (u_1, u_2)$, where $u_1(x) = 1$ and $u_2(x) = \sqrt{3}(2x-1)$.

(c) (4 points) Define the linear transformation $T : \mathcal{P}_1 \to \mathbb{R}$ by $T(ax+b) = a+b$. Find a polynomial $w(x) = cx + d$ in $\mathcal{P}_1$ such that $T(p) = \langle w, p \rangle$ for all $p \in \mathcal{P}_1$.

5. Let $A \in \mathbb{R}^{5\times 3}$ be a 5×3 matrix, and suppose that $A^\top A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix}$.

(a) (3 points) Show that $\operatorname{rank}(A) = 3$.

(Problem 5, Continued).

As above, let A be a 5×3 matrix such that $A^\top A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix}$.

Also, for the rest of this problem, suppose that $A = QR$ is the QR-factorization of A.

(b) (2 points) Circle the correct statement below. *(No justification necessary.)*

- Q is orthogonal. • Q is not orthogonal.
- There is not enough information given to determine whether Q is orthogonal.

(c) (3 points) Circle all matrices below that are equal to the standard matrix P of orthogonal projection onto $\operatorname{im} A$ in $\mathbb{R}^5$. If none of these are equal to P, circle *none*. *(No justification necessary.)*

$$A^\top A \qquad Q^\top Q \qquad R^\top R \qquad AA^\top \qquad QQ^\top \qquad RR^\top \qquad \textit{none}$$

(d) (4 points) Find the following determinants. *(No justification necessary.)*

- $\det Q^\top Q =$
- $\det R^\top R =$
- $\det A^\top A =$
- $\det AA^\top =$

(e) (4 points) Find R.

6. Let A be the 3×3 matrix $A = \begin{bmatrix} -1 & 0 & r \\ p & -1 & 1 \\ 1 & q & -2 \end{bmatrix}$, where $p, q, r \in \mathbb{R}$. In parts (a) – (c) below, there exist values of p, q, and r that satisfy the given conditions. You may assume this fact without proof. In each of (a) – (c), find values of p, q, and r that satisfy the given conditions. *No justification is needed in parts (a) – (c).*

   (a) (3 points) Both $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$ are least-squares solutions of $A\vec{x} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$.

   (b) (3 points) The linear systems $A\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ have exactly the same least-squares solutions.

   (c) (3 points) For all $\vec{x}, \vec{y} \in \mathbb{R}^3$, we have $\vec{x} \cdot A\vec{y} = A\vec{x} \cdot \vec{y}$.

   (d) (3 points) Do there exist values of p, q, and r for which the function $\langle \vec{v}, \vec{w} \rangle = \vec{v}^\top A\vec{w}$ defines an inner product on $\mathbb{R}^3$? Briefly justify your answer.

7. Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_k \\ | & & | \end{bmatrix}$ be an $n \times k$ matrix with orthonormal columns.

(a) (4 points) Prove directly from the definitions of *orthonormal* and *linearly independent* that the list $(\vec{a}_1, \ldots, \vec{a}_k)$ is linearly independent.

(b) (3 points) Prove that if $n = k$, then A has orthonormal rows. (You may use properties of matrices that were proved in class or stated in the textbook.)

(c) (3 points) Prove that if A has orthonormal rows, then $n = k$. (You may use properties of matrices that were proved in class or stated in the textbook.)

8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A, so that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) (4 points) Prove that $\ker(A^\top) = (\operatorname{im} A)^\perp$.

   (b) (4 points) Assume that $A^2 = A$ and $A^\top = A$.

      (i) Prove that $A\vec{y} = \vec{y}$ for all $\vec{y} \in \operatorname{im}(A)$.

      (ii) Prove that $A\vec{x} = \vec{0}$ for all $\vec{x} \in (\operatorname{im} A)^\perp$.

   (c) (3 points) Give an example of a rank 1 matrix $A \in \mathbb{R}^{2\times 2}$ such that $A^2 = A$ and $A^\top = A$. *(No justification required.)*

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