# Math 217 – Midterm 1

## Winter 2020

**Time: 120 mins.**

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
3. No calculators, notes, or other outside assistance allowed.

Student ID Number: ______________________ Section: __________

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 15 | |
| 3 | 12 | |
| 4 | 14 | |
| 5 | 12 | |
| 6 | 11 | |
| 7 | 12 | |
| 8 | 12 | |
| Total: | 100 | |

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   *Note: in stating these definitions, please write out fully what you mean instead of using shorthand phrases such as "preserves" or "closed under."*

   (a) The function $T : V \to W$ from the vector space $V$ to the vector space $W$ is a *linear transformation*

   (b) The *span* of the list of vectors $(\vec{v}_1, \dots, \vec{v}_n)$ in the vector space $V$

   (c) A *subspace* $V$ of $\mathbb{R}^n$

   (d) The *rank* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For all vectors $\vec{v} \in \mathbb{R}^n$, the set $\{\vec{v}\}$ is linearly independent.

(b) (3 points) For all $A \in \mathbb{R}^{n\times n}$, if $A^3 = I_n$ and $A^5 = I_n$, then $A = I_n$.

(c) (3 points) For every $m \times n$ matrix $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{m\times n}$, if $\ker(A) \neq \{\vec{0}\}$ then the columns of $A$ form a linearly dependent list of vectors in $\mathbb{R}^m$.

(Problem 2, Continued).

(d) (3 points) For every vector space $V$ and subspace $U$ of $V$, if $\mathcal{B}_U$ is a basis of $U$ and $\mathcal{B}_V$ is a basis of $V$ then $\mathcal{B}_U \subseteq \mathcal{B}_V$.

(e) (3 points) There is a $2 \times 2$ matrix $A \in \mathbb{R}^{2\times 2}$ such that the $x$-axis in $\mathbb{R}^2$ is both the kernel and image of $A$.

3. Let $A = \begin{bmatrix} 2 & 0 & m \\ m & 1 & 8 \\ 0 & 1 & 0 \end{bmatrix}$, where $m$ is a real number, and note that $A$ can be transformed by a sequence of elementary row operations into the matrix $B = \begin{bmatrix} 1 & 0 & m/2 \\ 0 & 1 & 0 \\ 0 & 0 & 8 - \frac{m^2}{2} \end{bmatrix}$.

*Note: the parts below are all independent of each other.*

(a) (3 points) Without using determinants, find all values of $m$ for which $A$ is invertible.

(b) (3 points) Find a value of $m$ for which $\ker(A) \neq \{\vec{0}\}$, and for the value of $m$ that you choose, find $\ker(A)$.

(c) (3 points) Find all possible values of $a$, $b$, and $m$ such that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$.

*(No justification is necessary for this part.)*

(d) (3 points) Assuming that $m = 0$, find $A^{-1}$.

4. Throughout this problem consider the following linear system, where $a, b, c \in \mathbb{R}$:

$$\begin{cases} ax & + & y & + & 2z & = & 5 \\ 3x & + & by & + & z & = & 1 \\ x & + & y & + & z & = & c \end{cases}$$

(*No justification is necessary on parts (a) – (c) of this problem.*)

(a) (4 points) Assuming that $a = 0$ and $b = 2$, find all values of $c$ for which the linear system is inconsistent.

(b) (4 points) Assuming that $b = 0$ and $c = 4$, find all values of $a$ for which the linear system has infinitely many solutions.

(c) (3 points) Assuming $a = 1$, $b = 2$, and $c = 3$, find all solutions of the linear system.

(d) (3 points) Are there any values of $a$, $b$, and $c$ such that the solution set of the linear system is a subspace of $\mathbb{R}^3$? Justify your answer.

5. Let $T : \mathbb{R}^3 \to \mathcal{P}_2$ and $S : \mathcal{P}_2 \to \mathbb{R}^3$ be the linear transformations defined by the rules

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)(x) \;=\; ax^2 + (b-c)x \qquad \text{and} \qquad S(p) \;=\; \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

(a) (4 points) Show that $T$ is a linear transformation.

(b) (4 points) Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$. *(No justification needed.)*

(c) (4 points) Find the standard matrix of $S \circ T$. *(No justification needed.)*

6. In each part below, either provide an explicit example of an object with the given properties if possible, or else state that no object with the given properties exists. Justify your answers.

   (a) (3 points) A system of three linear equations in two variables whose solution set $S$ is a one-dimensional subspace of $\mathbb{R}^2$.

   (b) (4 points) A pair of matrices $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 2}$ such that $\operatorname{im}(AB) = \mathbb{R}^2$ and $\operatorname{im}(BA) = \mathbb{R}^3$.

   (c) (4 points) A surjective linear transformation $T : \mathcal{P}_4 \to \mathbb{R}^{2\times 2}$ from the space of all polynomials of degree at most 4 to the space of all $2 \times 2$ real matrices such that $\ker(T) = \{p \in \mathcal{P}_4 \ : \ p(0) = 0\}$.

7. Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in the vector space $V$, let $T : V \to V$ be a linear transformation, and suppose that the list $\mathcal{B} = (T(\vec{v}_1), \ldots, T(\vec{v}_n))$ is an ordered basis of $V$.

   (a) (6 points) Prove that the list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent.

   (b) (6 points) Prove that $T$ is invertible.

8. Let $n \in \mathbb{N}$, and let $A \in \mathbb{R}^{2n\times 2n}$ be a $2n \times 2n$ matrix with the property that $A^2 = 0$.

   (a) (6 points) Prove that $\operatorname{rank}(A) \leq n$.

   (b) (6 points) Prove that if $\operatorname{rank}(A) = n$, then $\operatorname{im}(A) = \ker(A)$.

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