Math 217 – Midterm 1

Winter 2020

Solutions

Student ID Number: ______________________ Section: ________

Question	Points	Score
1	12	
2	15	
3	12	
4	14	
5	12	
6	11	
7	12	
8	12	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   Note: in stating these definitions, please write out fully what you mean instead of using shorthand phrases such as "preserves" or "closed under."

   (a) The function $T : V \to W$ from the vector space V to the vector space W is a *linear transformation*

   Solution: The function $T : V \to W$ from the vector space V to the vector space W is a *linear transformation* if $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(c\vec{x}) = cT(\vec{x})$ for all $\vec{x}, \vec{y} \in V$ and $c \in \mathbb{R}$.

   (b) The *span* of the list of vectors $(\vec{v}_1, \dots, \vec{v}_n)$ in the vector space V

   Solution: The *span* of the list of vectors $(\vec{v}_1, \dots, \vec{v}_n)$ in the vector space V is the set
   $$\left\{ \sum_{i=1}^{n} c_i \vec{v}_i \ : \ c_1, \dots, c_n \in \mathbb{R} \right\}.$$

   (c) A *subspace* V of $\mathbb{R}^n$

   Solution: A *subspace* V of $\mathbb{R}^n$ is a subset $V \subseteq \mathbb{R}^n$ such that $\vec{0} \in V$ and for all $\vec{x}, \vec{y} \in V$ and $c \in \mathbb{R}$, we have $\vec{x} + \vec{y} \in V$ and $c\vec{x} \in V$.

   (d) The *rank* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

   Solution: The *rank* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the dimension of the image of T.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For all vectors $\vec{v} \in \mathbb{R}^n$, the set $\{\vec{v}\}$ is linearly independent.

Solution: FALSE. The set $\{\vec{0}\}$ is linearly dependent, since $1\vec{0} = \vec{0}$ is a nontrivial linear dependence relation on it.

(b) (3 points) For all $A \in \mathbb{R}^{n\times n}$, if $A^3 = I_n$ and $A^5 = I_n$, then $A = I_n$.

Solution: TRUE. Let $A \in \mathbb{R}^{n\times n}$ and suppose $A^3 = I_n = A^5$. Then

$$I_n \;=\; I_n^2 \;=\; (A^5)^2 \;=\; A^{10} \;=\; (A^3)^3 A \;=\; (I_n)^3 A \;=\; I_n A \;=\; A.$$

(c) (3 points) For every $m \times n$ matrix $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{m\times n}$, if $\ker(A) \neq \{\vec{0}\}$ then the columns of A form a linearly dependent list of vectors in $\mathbb{R}^m$.

Solution: TRUE. Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{m\times n}$, and suppose $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is a nonzero vector in $\ker(A)$. Then

$$\sum_{i=1}^{n} v_i \vec{a}_i \;=\; A\vec{v} \;=\; \vec{0}$$

is a nontrivial linear dependence relation on the columns of A, so $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent.

(Problem 2, Continued).

(d) (3 points) For every vector space V and subspace U of V, if $\mathcal{B}_U$ is a basis of U and $\mathcal{B}_V$ is a basis of V then $\mathcal{B}_U \subseteq \mathcal{B}_V$.

Solution: FALSE. For instance, let $V = \mathbb{R}^2$, let $U = \mathrm{Span}(\vec{e}_1 + \vec{e}_2)$, let $\mathcal{B}_V = \{\vec{e}_1, \vec{e}_2\}$, and let $\mathcal{B}_U = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Then $U \subseteq V$, $\mathcal{B}_U$ is a basis of U, and $\mathcal{B}_V$ is a basis of V, but $\mathcal{B}_U \not\subseteq \mathcal{B}_V$.

(e) (3 points) There is a 2×2 matrix $A \in \mathbb{R}^{2\times 2}$ such that the x-axis in $\mathbb{R}^2$ is both the kernel and image of A.

Solution: TRUE. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then the x-axis in $\mathbb{R}^2$ is both $\mathrm{im}(A)$ and $\ker(A)$.

3. Let $A = \begin{bmatrix} 2 & 0 & m \\ m & 1 & 8 \\ 0 & 1 & 0 \end{bmatrix}$, where m is a real number, and note that A can be transformed by a sequence of elementary row operations into the matrix $B = \begin{bmatrix} 1 & 0 & m/2 \\ 0 & 1 & 0 \\ 0 & 0 & 8 - \frac{m^2}{2} \end{bmatrix}$.

Note: the parts below are all independent of each other.

(a) (3 points) Without using determinants, find all values of m for which A is invertible.

Solution: A is invertible iff $\operatorname{rank}(A) = 3$ iff $8 - \frac{m^2}{2} \neq 0$ iff $m \neq \pm 4$.

(b) (3 points) Find a value of m for which $\ker(A) \neq \{\vec{0}\}$, and for the value of m that you choose, find $\ker(A)$.

Solution: If $m = 4$, then $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $\ker(A) = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} t \ : \ t \in \mathbb{R} \right\}$.

(c) (3 points) Find all possible values of a, b, and m such that $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$.

(No justification is necessary for this part.)

Solution: $a = 1$, $b = 0$, $m = -4$.

(d) (3 points) Assuming that $m = 0$, find A^{-1}.

Solution: Assuming $m = 0$ and using $\operatorname{rref}[A \mid I_3] = [I_3 \mid A^{-1}]$, we see that $A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1/8 & -1/8 \end{bmatrix}$.

4. Throughout this problem consider the following linear system, where $a, b, c \in \mathbb{R}$:

$$\begin{cases} ax + y + 2z = 5 \\ 3x + by + z = 1 \\ x + y + z = c \end{cases}$$

(*No justification is necessary on parts (a) – (c) of this problem.*)

(a) (4 points) Assuming that $a = 0$ and $b = 2$, find all values of c for which the linear system is inconsistent.

Solution: We row reduce $\begin{bmatrix} 0 & 1 & 2 & 5 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 1 & c \end{bmatrix}$ to obtain $\begin{bmatrix} 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 6-3c \\ 1 & 1 & 1 & c \end{bmatrix}$, which shows that the linear system is inconsistent if and only if $c \neq 2$.

(b) (4 points) Assuming that $b = 0$ and $c = 4$, find all values of a for which the linear system has infinitely many solutions.

Solution: Consider the augmented matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4] = \begin{bmatrix} a & 1 & 2 & 5 \\ 3 & 0 & 1 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$. Since $\vec{v}_4 = 3\vec{v}_2 + \vec{v}_3$, the system is consistent no mater what a is. Since $(\vec{v}_2, \vec{v}_3)$ is linearly independent by inspection, it follows that the system has infinitely many solutions iff $\vec{v}_1 \in \mathrm{Span}(\vec{v}_2, \vec{v}_3)$, which happens (by inspection) iff $a = 4$.

(c) (3 points) Assuming $a = 1$, $b = 2$, and $c = 3$, find all solutions of the linear system.

Solution: rref $\begin{bmatrix} 1 & 1 & 2 & 5 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, so the unique solution is $\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$.

(d) (3 points) Are there any values of a, b, and c such that the solution set of the linear system is a subspace of $\mathbb{R}^3$? Justify your answer.

Solution: No, there are no such values, because every subspace of $\mathbb{R}^3$ contains $\vec{0}$, but $\vec{0}$ cannot be a solution of the system since $\begin{bmatrix} 5 \\ 1 \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ no matter what a, b, c are.

5. Let $T : \mathbb{R}^3 \to \mathcal{P}_2$ and $S : \mathcal{P}_2 \to \mathbb{R}^3$ be the linear transformations defined by the rules

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)(x) \;=\; ax^2 + (b-c)x \qquad \text{and} \qquad S(p) \;=\; \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

(a) (4 points) Show that T is a linear transformation.

Solution: Let $k \in \mathbb{R}$ and let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \in \mathbb{R}^3$. Then

$$T\left(k\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T\left(\begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}\right) = kax^2 + (kb - kc)x = k(ax^2 + (b-c)x) = kT\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$$

and

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) \;=\; T\left(\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}\right) \;=\; (a+d)x^2 + \big((b+e) - (c+f)\big)x$$

$$=\; ax^2 + (b-c)x + dx^2 + (e-f)x \;=\; T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) + T\left(\begin{bmatrix} d \\ e \\ f \end{bmatrix}\right).$$

(b) (4 points) Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$. *(No justification needed.)*

Solution: $\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$ is a basis of $\ker(T)$ and (x, x^2) is a basis of $\operatorname{im}(T)$.

(c) (4 points) Find the standard matrix of $S \circ T$. *(No justification needed.)*

Solution: The standard matrix of $S \circ T$ is

$$\begin{bmatrix} | & | & | \\ S(T(\vec{e}_1)) & S(T(\vec{e}_2)) & S(T(\vec{e}_3)) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 4 & 2 & -2 \end{bmatrix}.$$

6. In each part below, either provide an explicit example of an object with the given properties if possible, or else state that no object with the given properties exists. Justify your answers.

   (a) (3 points) A system of three linear equations in two variables whose solution set S is a one-dimensional subspace of $\mathbb{R}^2$.

   Solution: $\begin{cases} x+y &= 0 \\ x+y &= 0 \\ x+y &= 0 \end{cases}$

   (b) (4 points) A pair of matrices $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 2}$ such that $\operatorname{im}(AB) = \mathbb{R}^2$ and $\operatorname{im}(BA) = \mathbb{R}^3$.

   Solution: This is impossible. If $A \in \mathbb{R}^{2\times 3}$, then $\operatorname{im}(A) \subseteq \mathbb{R}^2$ and therefore $\operatorname{rank}(A) \leq 2$, so $\dim(\ker(A)) \geq 1$ by Rank-Nullity. But $\ker(A) \subseteq \ker(BA)$, so $\dim(\ker(BA)) \geq 1$ and thus $\operatorname{rank}(BA) \leq 2$ again by Rank-Nullity, which implies $\operatorname{im}(BA) \neq \mathbb{R}^3$.

   (c) (4 points) A surjective linear transformation $T : \mathcal{P}_4 \to \mathbb{R}^{2\times 2}$ from the space of all polynomials of degree at most 4 to the space of all 2×2 real matrices such that $\ker(T) = \{p \in \mathcal{P}_4 \,:\, p(0) = 0\}$.

   Solution: This is also impossible. The set $\{x, x^2, x^3, x^4\}$ is a basis of $\{p \in \mathcal{P}_4 : p(0) = 0\}$, so if $\ker(T)$ is this set, then $\dim(\ker(T)) = 4$. Using $\dim(\mathcal{P}_4) = 5$ and Rank-Nullity, this implies $\operatorname{rank}(T) = 1$. So $\operatorname{im}(T)$ is a one-dimensional subspace of $\mathbb{R}^{2\times 2}$, which means $\operatorname{im}(T)$ cannot be all of $\mathbb{R}^{2\times 2}$ since $\dim(\mathbb{R}^{2\times 2}) = 4$, showing that T is not surjective.

7. Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in the vector space V, let $T : V \to V$ be a linear transformation, and suppose that the list $\mathcal{B} = (T(\vec{v}_1), \ldots, T(\vec{v}_n))$ is an ordered basis of V.

   (a) (6 points) Prove that the list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent.

   Solution: Let $c_1, \ldots, c_n \in \mathbb{R}$, and suppose $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$. Then

   $$\vec{0} \;=\; T(\vec{0}) \;=\; T\left(\sum_{i=1}^n c_i\vec{v}_i\right) \;=\; \sum_{i=1}^n c_iT(\vec{v}_i).$$

   Since $\mathcal{B}$ is a basis of V, $\mathcal{B}$ is linearly independent, which implies that $c_i = 0$ for each $1 \le i \le n$. This shows that $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent as well.

   (b) (6 points) Prove that T is invertible.

   Solution: Let $\vec{y} \in V$. Since $\mathcal{B}$ spans V, we may fix $c_1, \ldots, c_n \in \mathbb{R}$ such that $\vec{y} = \sum_{i=1}^n c_iT(\vec{v}_i) = T\left(\sum_{i=1}^n c_i\vec{v}_i\right)$. This shows that T is surjective, so $\mathrm{rank}(T) = n$. By the Rank-Nullity Theorem, it follows that $\dim(\ker(T)) = 0$. Thus T is both surjective and injective, hence bijective, from which it follows that T is invertible.

8. Let $n \in \mathbb{N}$, and let $A \in \mathbb{R}^{2n\times 2n}$ be a $2n \times 2n$ matrix with the property that $A^2 = 0$.

(a) (6 points) Prove that $\operatorname{rank}(A) \leq n$.

Solution: Let $A \in \mathbb{R}^{2n\times 2n}$ and suppose $A^2 = 0$. Then for all $\vec{y} = A\vec{x} \in \operatorname{im}(A)$, we have $A\vec{y} = A(A\vec{x}) = A^2\vec{x} = 0\vec{x} = 0$, which shows $\operatorname{im}(A) \subseteq \ker(A)$ and thus $\operatorname{rank}(A) \leq \operatorname{nullity}(A)$. By the Rank-Nullity Theorem we know $\operatorname{nullity}(A) + \operatorname{rank}(A) = 2n$. Thus we have

$$\operatorname{rank}(A) \ \leq \ \operatorname{nullity}(A) \ = \ 2n - \operatorname{rank}(A),$$

which implies $2\operatorname{rank}(A) \leq 2n$, so $\operatorname{rank}(A) \leq n$ as desired.

(b) (6 points) Prove that if $\operatorname{rank}(A) = n$, then $\operatorname{im}(A) = \ker(A)$.

Solution: Suppose $\operatorname{rank}(A) = n$. Then $\operatorname{nullity}(A) = 2n - \operatorname{rank}(A) = n$, so $\dim(\operatorname{im}(A)) = n = \dim(\ker(A))$. Since $\operatorname{im}(A) \subseteq \ker(A)$ as shown in (a), it follows that $\operatorname{im}(A) = \ker(A)$.