Worksheet 8: Subspaces (§§3.1, 3.2)

Key definitions:

- For any $m \times n$ matrix A, let T_A be the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ induced by A, i.e., given by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.
- The *image* of a function $f : X \to Y$ is the set $\mathrm{im}(f) = \{f(x) \; : \; x \in X\}$.
- The *span* of a finite set of vectors $\{v_1, \ldots, v_n\}$ in a vector space V is the set of all linear combinations of $v_1, \ldots, v_n$. That is,†

$$\mathrm{Span}\big(\{v_1, \ldots, v_n\}\big) \; = \; \{c_1 v_1 + \cdots + c_n v_n \; : \; c_1, \ldots, c_n \in \mathbb{R}\} \; = \; \left\{ \sum_{i=1}^{n} c_i \vec{v}_i \; : \; c_i \in \mathbb{R} \text{ for each } i \right\}.$$

- The *kernel* of a linear transformation $T : V \to W$ from the vector space V to the vector space W is the set

$$\ker(T) \; = \; \{v \in V \; : \; T(v) = \vec{0}\}.$$

- If V is a vector space and $S \subseteq V$, then S is *closed under vector addition* if $u + v \in S$ whenever $u \in S$ and $v \in S$, and *closed under scalar multiplication* if $cv \in S$ whenever $v \in S$ and $c \in \mathbb{R}$.
- If V is a vector space and $S \subseteq V$, then S is called a *subspace* of V if $\vec{0} \in S$ and S is closed under vector addition and scalar multiplication.

Problem 1: Span.

(a) What sort of geometric object is the span of a nonzero vector in $\mathbb{R}^3$? (a line)

(b) What sort of geometric object could the span of a pair of nonzero vectors in $\mathbb{R}^3$ be? Explain. (A line or a plane, depending on whether the two vectors are scalar multiples of each other.)

(c) Describe the following subsets of $\mathbb{R}^3$:

$$\mathrm{Span}(\vec{0}), \quad \mathrm{Span}(\vec{e}_1), \quad \mathrm{Span}(\vec{e}_1, \vec{e}_2), \quad \mathrm{Span}\big(\vec{e}_1, \vec{e}_2, \vec{e}_1 + 3\vec{e}_2\big), \quad \mathrm{Span}\big(\vec{e}_1, \vec{e}_1 + \vec{e}_2, \vec{e}_1 + \vec{e}_2 + \vec{e}_3\big).$$

$\big(\{\vec{0}\} \; \subseteq \;$ the x-axis $\; \subseteq \;$ the xy-plane $\; \subseteq \;$ the xy-plane $\; \subseteq \; \mathbb{R}^3.\big)$

(d) Prove that for any finite set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$, there is a linear transformation T such that $\mathrm{Span}(S) = \mathrm{im}(T)$.

Solution: Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a finite set of vectors in $\mathbb{R}^m$. Let $A = [\vec{v}_1 \cdots \vec{v}_n]$ be the $m \times n$ matrix whose jth column is $\vec{v}_j$, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by A. Then

$$\mathrm{im}(T_A) \; = \; \{T_A(\vec{x}) \; : \; \vec{x} \in \mathbb{R}^n\} \; = \; \{A\vec{x} \; : \; \vec{x} \in \mathbb{R}^n\} \; = \; \left\{ \sum_{i=1}^{n} x_i \vec{v}_i \; : \; \vec{x} \in \mathbb{R}^n \right\} \; = \; \mathrm{Span}(S).$$

† While technically correct, $\mathrm{Span}\big(\{v_1, \ldots, v_n\}\big)$ looks a bit silly, so we will often abbreviate it $\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_n)$.

Problem 2: Kernel.

(a) Describe geometrically the kernels of the following linear operators on $\mathbb{R}^3$:

(i) the orthogonal projection onto the yz-plane;

(ii) the orthogonal projection onto the z-axis;

(iii) the reflection about the plane $y = z$;

(iv) The mapping T_A, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{bmatrix}.$$

Solution:

(i) the x-axis

(ii) the xy-plane

(iii) the origin

(iv) the line through $(0, 0, 0)$ and $(1, 1, 1)$

(b) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Prove that T is injective if and only if $\ker(T) = \{\vec{0}\}$.

Solution: For the forward direction, suppose T is injective. Since T is linear, we know that $T(\vec{0}) = \vec{0}$, so $\vec{0} \in \ker(T)$. If $\vec{v}$ is any other vector in $\ker(T)$ then $T(\vec{v}) = \vec{0}$ which forces $\vec{v} = \vec{0}$ since T is injective, so we conclude that $\ker(T) = \{\vec{0}\}$ as desired. For the converse, suppose that $\ker(T) = \{\vec{0}\}$, let $\vec{x}, \vec{y} \in V$, and suppose that $T(\vec{x}) = T(\vec{y})$. Then by linearity of T,

$$\vec{0} \;=\; T(\vec{x}) - T(\vec{y}) \;=\; T(\vec{x} - \vec{y}),$$

so we have $\vec{x} - \vec{y} \in \ker(T)$. By assumption this gives $\vec{x} - \vec{y} = \vec{0}$, so $\vec{x} = \vec{y}$ as desired.

Problem 3: Image and Kernel

(a) Let S be the solution set in $\mathbb{R}^4$ of the linear system $\begin{cases} x_1 + 2x_3 + 3x_4 & = & 0 \\ x_3 + 2x_4 & = & 0 \end{cases}$

(i) Find a linear transformation T such that $\operatorname{im}(T) = S$.

(ii) Find a linear transformation T such that $\ker(T) = S$.

Solution: For (i), let T be the linear transformation induced by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$ whose columns span the solution set of the linear system. For (ii), let T be the linear transformation induced by the coefficient matrix $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ of the linear system.

(b) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Prove that $\ker(T)$ is a subspace of V and that $\operatorname{im}(T)$ is a subspace of W.

Solution: Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Since T is linear we know that $T(\vec{0}_V) = \vec{0}_W$, so $\vec{0}_V \in \ker(T)$ and $\vec{0}_W \in \operatorname{im}(T)$. For any $\vec{x}, \vec{y} \in \ker(T)$ and $c \in \mathbb{R}$ we have

$$T(\vec{x} + \vec{y}) \;=\; T(\vec{x}) + T(\vec{y}) \;=\; \vec{0} + \vec{0} \;=\; \vec{0},$$

so $\vec{x} + \vec{y} \in \ker(T)$, and

$$T(c\vec{x}) \;=\; cT(\vec{x}) \;=\; c\vec{0} \;=\; \vec{0},$$

so $c\vec{x} \in \ker(T)$. Thus $\ker(T)$ contains $\vec{0}_V$ and is closed under vector addition and scalar multiplication, and is therefore a subspace of V. For $\operatorname{im}(T)$, let $\vec{x}, \vec{y} \in \operatorname{im}(T)$ and $c \in \mathbb{R}$. Fix $\vec{u}, \vec{v} \in V$ such that $T(\vec{u}) = \vec{x}$ and $T(\vec{v}) = \vec{y}$. Then

$$\vec{x} + \vec{y} \;=\; T(\vec{u}) + T(\vec{v}) \;=\; T(\vec{u} + \vec{v}) \;\in\; \operatorname{im}(T)$$

and

$$c\vec{x} \;=\; cT(\vec{u}) \;=\; T(c\vec{u}) \;\in\; \operatorname{im}(T),$$

so $\operatorname{im}(T)$ contains $\vec{0}_W$ and is closed under vector addition and scalar multiplication, and is therefore a subspace of W.

Problem 4: Subspaces

For each part below, determine whether or not the given sub*set* is a sub*space* of the vector space:

(a) the solution set in $\mathbb{R}^n$ of a homogeneous* linear system in n variables YES

(b) the solution set in $\mathbb{R}^n$ of a *non*-homogeneous linear system in n variables NO

(c) the set of all points (x, y) in $\mathbb{R}^2$ such that $x^2 - y^2 = 0$ NO

(d) the set of all points (x, y) in $\mathbb{R}^2$ such that $x^3 - y^3 = 0$ YES

(e) the set of even polynomials‡ in the space $\mathcal{P}$ of all polynomials YES

*Recall that the linear system $A\vec{x} = \vec{b}$ is *homogeneous* if $\vec{b} = \vec{0}$.

‡Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(f) the set of odd polynomials in the space $\mathcal{P}$ of all polynomials YES

(g) the set of functions in $C^\infty([0,1])$ that are proportional to their own derivative NO

(h) the set of invertible $n \times n$ matrices in $\mathbb{R}^{n\times n}$ NO

(i) the set of symmetric $n \times n$ matrices in $\mathbb{R}^{n\times n}$ (recall that a matrix A is *symmetric* if $A = A^\top$, where $A^\top$ is the *transpose* of A) YES