MATH 217 - LINEAR ALGEBRA
HOMEWORK 8, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:

Section 5.2: 12, 26
Section 5.3: 36, 38
Section 5.4: 26, 32

Solution.

5.2.12

$$v_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 13 \end{bmatrix}$$

$$\Rightarrow \qquad u_1 = \frac{1}{\sqrt{2^2+3^2+6^2}} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}$$

$${v_2}^{\perp} = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 13 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 4 \\ 2 \\ 13 \end{bmatrix} \cdot \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right) \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{2^2+2^2+1^2}} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}.$$

5.2.26 Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 13 \end{bmatrix}$ be the columns of the given matrix M. The sequence of orthonormal vectors produced by the Gram-Schmidt process is:

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \quad \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \vec{v}_2 - (14)\vec{u}_1 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix},$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{3} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}.$$

Hence, the QR factorization of M is given by:

$$M = \begin{bmatrix} 2 & 4 \\ 3 & 4 \\ 0 & 2 \\ 6 & 13 \end{bmatrix} = \begin{bmatrix} 2/7 & 0 \\ 3/7 & -2/3 \\ 0 & 2/3 \\ 6/7 & 1/3 \end{bmatrix} \begin{bmatrix} 7 & 14 \\ 0 & 3 \end{bmatrix} = QR.$$

5.3.36: Let A be the given matrix. A is orthogonal if its columns form an orthonormal basis of $\mathbb{R}^3$, so let $\vec{u}_i$ be the ith column of A. Note that $\|\vec{u}_1\| = \|\vec{u}_2\| = 1$ and $\vec{u}_1 \cdot \vec{u}_2 = 0$. The third column of A, $\vec{u}_3$, must satisfy $\vec{u}_1 \cdot \vec{u}_3 = 0$ and $\vec{u}_2 \cdot \vec{u}_3 = 0$, i.e. $\vec{u}_3 \in (\text{span}(\vec{u}_1, \vec{u}_2))^\perp = \ker(B)$, where $B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. But $\ker(B) = \text{span}\left(\begin{bmatrix} 1 & 1 & -4 \end{bmatrix}^T\right)$. Hence $\vec{u}_3 = 1/(3\sqrt{2}) \begin{bmatrix} 1 & 1 & -4 \end{bmatrix}$, and the matrix is:

$$A = \begin{bmatrix} 2/3 & 1/\sqrt{2} & 1/(3\sqrt{2}) \\ 2/3 & -1/\sqrt{2} & 1/(3\sqrt{2}) \\ 1/3 & 0 & -4/(3\sqrt{2}) \end{bmatrix}.$$

5.3.38

a. One such example is $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Here $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

b. Note that $A^2 = AA = -A^T A$. Thus, with the columns of A given by $\vec{v}_i$, we have $A^2 = -\begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \vec{v}_1 \cdot \vec{v}_3 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \vec{v}_2 \cdot \vec{v}_3 \\ \vec{v}_3 \cdot \vec{v}_1 & \vec{v}_3 \cdot \vec{v}_2 & \vec{v}_3 \cdot \vec{v}_3 \end{bmatrix} = -\begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \vec{v}_1 \cdot \vec{v}_3 \\ \vec{v}_1 \cdot \vec{v}_2 & \vec{v}_2 \cdot \vec{v}_2 & \vec{v}_2 \cdot \vec{v}_3 \\ \vec{v}_1 \cdot \vec{v}_3 & \vec{v}_2 \cdot \vec{v}_3 & \vec{v}_3 \cdot \vec{v}_3 \end{bmatrix}$, which is clearly symmetric.

5.4.26: The least squares solutions of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

are the solutions to the (consistent) system

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which we can re-write as

$$\begin{bmatrix} 66 & 78 & 90 \\ 78 & 93 & 108 \\ 90 & 108 & 126 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and solve by row-reducing the augmented matrix

$$\begin{bmatrix} 66 & 78 & 90 & 1 \\ 78 & 93 & 108 & 2 \\ 90 & 108 & 126 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -7/6 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we obtain:

$$\left\{ \begin{bmatrix} -7/6 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t \ : \ t \in \mathbb{R} \right\}.$$

5.4.32: If we name our quadratic polynomial $y = ax^2 + bx + c$, we are looking to least squares solutions to

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Really, we need solutions to

$$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

which amounts to row-reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 98 & 36 & 14 & 0 \\ 36 & 14 & 6 & 0 \\ 14 & 6 & 4 & 27 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 27/4 \\ 0 & 1 & 0 & -567/20 \\ 0 & 0 & 1 & 513/20 \end{array}\right]$$

Thus, $y = \frac{27}{4}x^2 - \frac{567}{20}x + \frac{513}{20}$ or equivalently $y = 6.75x^2 - 28.35x + 25.65$ is the quadratic polynomial which best fits the given data points via least-squares.

Part B (25 points)

Problem 1. For each subspace V of $\mathbb{R}^n$, we write $\mathrm{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$ for the orthogonal projection onto V in $\mathbb{R}^n$.

(a) Find the standard matrix $[\mathrm{proj}_V]_{\mathcal{E}}$ of $\mathrm{proj}_V : \mathbb{R}^4 \to \mathbb{R}^4$ where

$$V \;=\; \mathrm{Span}\left(\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) \;\subseteq\; \mathbb{R}^4.$$

(b) Given $n \in \mathbb{N}$, find the standard matrix of $\mathrm{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$ where V is the one-dimensional subspace of $\mathbb{R}^n$ spanned by the vector $\vec{w} = \sum_{i=1}^{n} \vec{e}_i = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$.

(c) Prove that for every subspace V of $\mathbb{R}^n$ there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix of proj_V has the block form

$$[\mathrm{proj}_V]_{\mathcal{B}} \;=\; \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

for some integer $0 \le r \le n$, where each "0" is an appropriately sized zero matrix.

Solution.

(a) Notice that

$$V \;=\; \mathrm{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}\right),$$

and since these vectors give an orthonormal basis of V, Theorem 5.3.10 tells us the standard matrix for the projection is

$$[\mathrm{proj}_V]_{\mathcal{E}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

(b) We have that

$$\mathrm{proj}_V(\underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{\vec{v}}) = \frac{\vec{v}\cdot\vec{w}}{||\vec{w}||^2}\vec{w} = \frac{1}{n}\begin{bmatrix} v_1 + \ldots + v_n \\ \vdots \\ v_1 + \ldots + v_n \end{bmatrix},$$

and so

$$[\mathrm{proj}_V]_{\mathcal{E}} = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n}. \end{bmatrix},$$

that is the matrix with each term being $\frac{1}{n}$.

(c) Let $\{v_1, \ldots, v_k\}$ be an orthonormal basis of V, and let $\{v_{k+1}, \ldots, v_n\}$ be an orthonormal basis of $V^\perp$. Then $\mathcal{B} = \{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ such that

$$[\mathrm{proj}_V]_{\mathcal{B}} = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right].$$

Indeed, $\mathrm{proj}_V(v_i) = v_i$ for $1 \leq i \leq k$ and $\mathrm{proj}_V(v_i) = 0$ for $k < i \leq n$; therefore, $[\mathrm{proj}_V]_{\mathcal{B}} \cdot e_i = e_i$ for $1 \leq i \leq k$, and $[\mathrm{proj}_V]_{\mathcal{B}} \cdot e_i = 0$ for $k+1 \leq i \leq n$.

Problem 2. Recall that two subspaces V_1 and V_2 of $\mathbb{R}^n$ are *orthogonal*, denoted $V_1 \perp V_2$, if we have $\vec{v}\cdot\vec{w} = 0$ for all $\vec{v} \in V_1$ and $\vec{w} \in V_2$. Suppose that $V_1, \ldots, V_k$, with $k \geq 2$, are mutually orthogonal subspaces of $\mathbb{R}^n$ (that is, $V_i \perp V_j$ for all i and j with $i \neq j$). Show that $\dim V_1 + \cdots + \dim V_k \leq n$.

[Hint: consider taking orthonormal bases of the subspaces V_i.]

Solution. For a fixed $1 \leq i \leq k$, let $\mathcal{B}_i$ be an orthonormal basis for V_i. We can always find such a basis by Gram-Schmidt. We claim that all the vectors in the union $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ are mutually orthogonal. Indeed, take any $v, w \in \mathcal{B}$, then either $v, w \in \mathcal{B}_i$ for some i, or $v \in B_i$ and $w \in B_j$ with $i \neq j$. In the first case, $v \cdot w = 0$ by construction. In the second case, $v \cdot w = 0$ because V_i and V_j are orthogonal.

So $\mathcal{B}$ is a list of $\dim(V_1) + \cdots + \dim(V_k)$ mutually orthogonal vectors in $\mathbb{R}^n$. Since the vectors in $\mathcal{B}$ are mutually orthogonal, they are also linearly independent by 5.1.3. But we know $\dim(\mathbb{R}^n)$ is equal to the size of a maximal set of linearly independent vectors, so it must be the case that $\dim(V_1) + \cdots + \dim(V_k) = |\mathcal{B}| \leq n$.

Solution. We prove this by induction on k. The statement P_k to be proved for all k is "if $V_1, \ldots, V_k$ are k mutually orthogonal subspaces of $\mathbb{R}^n$, then $\dim V_1 + \cdots + \dim V_k \leq n$." The base case is $k = 2$, which we proved in part (2).

Now we must prove that $P_k \Rightarrow P_{k+1}$. So assume the induction hypothesis, which is that P_k is true - this means that for *any* k mutually orthogonal subspaces, the sum of their dimensions is less than or equal to n. Then, to show that P_{k+1} is true, assume that $V_1, \ldots, V_k, V_{k+1}$ are mutually orthogonal subspaces of $\mathbb{R}^n$. Let $W = V_k + V_{k+1}$. Since $V_k \perp V_{k+1}$, part (1) shows that $\dim W = \dim V_k + \dim V_{k+1}$. Moreover, we claim W is orthogonal to $V_1, \ldots, V_{k-1}$; let $\vec{x} \in W$ and $\vec{y} \in V_i$ with $1 \leq i \leq k-1$. (To prove the claim, observe by definition there exist $\vec{x}_k \in V_k$ and $\vec{x}_{k+1} \in V_{k+1}$ with $\vec{x} = \vec{x}_k + \vec{x}_{k+1}$. Then $\vec{w} \cdot \vec{x} = \vec{w} \cdot \vec{x}_k + \vec{w} \cdot \vec{x}_{k+1}$; however, both of these terms are zero since V_i is orthogonal to V_k and to V_{k+1}). So $V_1, \ldots, V_{k-1}, W$ are mutually orthogonal. By the induction hypothesis P_k applied to the mutually orthogonal subspaces $V_1, \ldots, V_{k-1}, W$, we know that $\dim V_1 + \cdots + \dim V_{k-1} + \dim W \leq n$. Since $\dim W = \dim V_k + \dim V_{k+1}$, we have proven P_{k+1}. By induction, this completes the proof.

Solution. First we prove the following: if $(\vec{v}_1, \ldots, \vec{v}_k)$ is a list of nonzero vectors in $\mathbb{R}^n$ such that $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$, then $(\vec{v}_1, \ldots, \vec{v}_k)$ is linearly independent. Our proof is exactly like the proof that orthonormal lists are linearly independent. Let $c_1, \ldots, c_k \in \mathbb{R}$ and suppose $\sum_{i=1}^k c_i \vec{v}_i = \vec{0}$. Fix $1 \leq j \leq k$. Then

$$0 \;=\; \vec{v}_j \cdot \vec{0} \;=\; \vec{v}_j \cdot \sum_{i=1}^{k} c_i \vec{v}_i \;=\; c_j(\vec{v}_j \cdot \vec{v}_j).$$

Since $\vec{v}_j \neq \vec{0}$, we know $\vec{v}_j \cdot \vec{v}_j \neq 0$, so dividing each side by $\vec{v}_j \cdot \vec{v}_j$ gives us $c_j = 0$. Since j was arbitrary, we conclude that $c_j = 0$ for each $1 \leq j \leq k$, showing that $(\vec{v}_1, \ldots, \vec{v}_k)$ is linearly independent as claimed.

Now, let $V_1, \ldots, V_k$ be mutually orthogonal subspaces of $\mathbb{R}^n$, and for each $1 \leq i \leq k$ let $\mathcal{B}_i = (\vec{b}_{i,1}, \ldots, \vec{b}_{i,r_i})$ be a basis of V_i, so $\dim(V_i) = r_i$. For each $1 \leq i \leq k$ and $1 \leq j \leq r_i$, let $c_{i,j} \in \mathbb{R}$ be arbitrary, and suppose $\sum_{i=1}^k \sum_{j=1}^{r_i} c_{i,j} \vec{b}_{i,j} = \vec{0}$. For each $1 \leq i \leq k$, let $\vec{v}_i = \sum_{j=1}^{r_i} c_{i,j} \vec{b}_{i,j}$. Then $\sum_{i=1}^k \vec{v}_i = \vec{0}$, and furthermore $\vec{v}_i \cdot \vec{v}_{i'} = 0$ whenever $i \neq i'$. Let I be the set of indices $i \in \{1, \ldots, k\}$ such that $\vec{v}_i \neq \vec{0}$, so $\{\vec{v}_i : i \in I\}$ is linearly independent by our claim above. If I is nonempty, then the equation $\sum_{i \in I} \vec{v}_i = \vec{0}$ is a nontrivial linear relation on a linearly independent set, which is impossible, so we have that $\vec{v}_i = \vec{0}$ for each $1 \leq i \leq k$. But then $\sum_{j=1}^{r_i} c_{i,j} \vec{b}_{i,j} = \vec{0}$ for each i, which by linear independence of each $\mathcal{B}_i$ shows that $c_{i,j} = 0$ for every $1 \leq i \leq k$ and $1 \leq j \leq r_i$. It follows that $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is linearly independent, which means $\dim V_1 + \cdots + \dim V_k = r_1 + \cdots + r_k = |\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k| \leq n$, as desired.

Definition: An invertible matrix A is called *orthogonal* if $A^{-1} = A^\top$.

Problem 3. Determine whether the following statements are True or False, and provide a short proof (or a counter-example) of your claim.

(a) If A and B are orthogonal $n \times n$ matrices, then AB is also orthogonal.
(b) If A^2 is an orthogonal matrix, then A is orthogonal.
(c) The set $S = \{A \in \mathbb{R}^{2\times 2} : A \text{ is orthogonal}\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

(d) If A is an orthogonal matrix and A^2 is the identity matrix, then A is symmetric.

Solution.

(a) TRUE. Suppose A and B are orthogonal $n \times n$ matrices, so A and B are invertible and $A^{-1} = A^\top$ and $B^{-1} = B^\top$. Then AB is also invertible, since products of invertible matrices are invertible, and we have $(AB)^{-1} = B^{-1}A^{-1} = B^\top A^\top = (AB)^\top$. This shows that AB is also an orthogonal matrix.

(b) FALSE: A^2 could be orthogonal without A being orthogonal. To see this, let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}$. Then A is not orthogonal since its columns are not unit vectors, but $A^2 = I_2$, which is orthogonal.

(c) FALSE. The set $S = \{A \in \mathbb{R}^{2\times 2} : A \text{ is orthogonal}\}$ is not a subspace of $\mathbb{R}^{2\times 2}$, since the 2×2 zero matrix is not orthogonal, so S does not contain the zero vector in $\mathbb{R}^{2\times 2}$.

(d) TRUE. Let A be an orthogonal $n \times n$ matrix such that $A^2 = I_n$. Then

$$A^\top \;=\; A^\top(A^2) \;=\; A^{-1}AA \;=\; I_nA \;=\; A,$$

so A is symmetric.

Problem 4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation.

(a) Show that T preserves angles. That is, prove that for all nonzero vectors $v, w \in \mathbb{R}^n$, if the angle between v and w is θ, then the angle between $T(v)$ and $T(w)$ is also θ.

(b) Conversely, if $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation that preserves angles, is T necessarily orthogonal? Prove your claim.

Solution.

(a) Since T is orthogonal, we have $\|T(v)\| = \|v\|$, $\|T(w)\| = \|w\|$ as well as $T(v) \cdot T(w) = v \cdot w$. We may then directly verify that

$$\cos(\theta) = \frac{v \cdot w}{\|v\|\|w\|} = \frac{T(v) \cdot T(w)}{\|T(v)\|\|T(w)\|} = \cos(\theta')\ ,$$

where θ' denotes the angle between $T(v)$ and $T(w)$, and therefore we have $\theta = \theta'$.

(b) Consider the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = 2x$. Then T is not orthogonal, since $\|T(x)\| = \|2x\| = 2\|x\|$ for all $x \in \mathbb{R}^n$. But T preservers angles, since

$$\begin{aligned} \cos(\theta') &= \frac{T(v) \cdot T(w)}{\|T(v)\|\|T(w)\|} \\ &= \frac{4(v \cdot w)}{4\|v\|\|w\|} \\ &= \frac{v \cdot w}{\|v\|\|w\|} \\ &= \cos(\theta)\ . \end{aligned}$$

and therefore $\theta = \theta'$.

Problem 5. Let A be an $m \times n$ matrix with linearly independent columns, and let $\vec{b} \in \mathbb{R}^m$. Suppose $A = QR$ is the QR-factorization of A. Show that $R^{-1}Q^\top\vec{b}$ is the unique least-squares solution of the linear system $A\vec{x} = \vec{b}$.

Solution. Let A be an $m \times n$ matrix with linearly independent columns. By 3.2.9 we know $\ker(A) = \{\vec{0}\}$, so by Theorem 5.4.6 we know that the unique least-squares solution of the linear system $A\vec{x} = \vec{b}$ is $\vec{x}^* = (A^\top A)^{-1}A^\top$. But since Q is an $m \times n$ matrix with orthonormal columns, we know $Q^\top Q = I_n$, so writing $A = QR$ we have

$$\begin{aligned} \vec{x}^* &= (A^\top A)^{-1}A^\top\vec{b} = ((QR)^\top QR)^{-1}(QR)^\top\vec{b} = (R^\top Q^\top QR)^{-1}R^\top Q^\top\vec{b} \\ &= (R^\top R)^{-1}R^\top Q^\top\vec{b} = R^{-1}(R^\top)^{-1}R^\top Q^\top\vec{b} = R^{-1}Q^\top\vec{b}. \end{aligned}$$

Solution. By 3.2.9 we know $\ker(A) = \{\vec{0}\}$, so by Theorem 5.4.6 there is a unique least squares solution. By Theorem 5.4.3, $\vec{x}^*$ is a least squares solution of the system $A\vec{x} = \vec{b}$ if and only if $A\vec{x}^* = \mathrm{proj}_V\vec{b}$ where $V = \mathrm{im}(A)$. But

$$AR^{-1}Q^\top\vec{b} = QRR^{-1}Q^\top\vec{b} = QQ^\top b.$$

By 5.3.10, $QQ^\top$ is the standard matrix for the orthogonal projection onto $\mathrm{im}\,A$, so $QQ^\top b = \mathrm{proj}_V\vec{b}$ and we are done by the previous paragraph.