# Worksheet 24: Diagonalizability (§§7.1, 7.2)



**Problem 1.** Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, and let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the associated linear transformation.

(a) Find the characteristic polynomial of $A$.

(b) Find the eigenvalues of $A$.

(c) For each eigenvalue $\lambda$ of $A$, find a basis of the (nontrivial) subspace

$$E_\lambda \;=\; \{\vec{v} \in \mathbb{R}^2 \;:\; A\vec{v} = \lambda\vec{v}\}.$$

(d) Does $\mathbb{R}^2$ have a basis $\mathcal{B}$ consisting of eigenvectors of $A$? If so, find the matrix of $T_A$ relative to such a basis, $\mathcal{B}$.

(e) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

**Solution:**

(a) $(t-4)(t-1)$

(b) 4 and 1

(c) $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is a basis of $E_4$ and $\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ is a basis of $E_1$

(d) Yes — $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$, and $[T_A]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

(e) $P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

**Definition**: A linear transformation $T : V \to V$ of the finite-dimensional vector space $V$ is said to be *diagonalizable* if there is a basis $\mathcal{B}$ of $V$ such that $[T]_{\mathcal{B}}$ is diagonal. A square matrix $A$ is said to be *diagonalizable* if the linear transformation $T_A$ is diagonalizable, or equivalently if $A$ is similar to a diagonal matrix.

**Problem 2.** Why are the two characterizations of diagonalizability of $A$ just given equivalent to each other?

**Solution:** $T_A$ is diagonalizable *iff* there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that $[T]_{\mathcal{B}}$ is diagonal *iff* there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that $S_{\mathcal{B}\to\mathcal{E}}^{-1} A S_{\mathcal{B}\to\mathcal{E}}$ is diagonal *iff* $A$ is similar to a diagonal matrix.

**Problem 3.** Let $T : V \to V$ be a linear transformation of the finite-dimensional vector space $V$.

(a) Prove that if $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis of $V$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad \text{if and only if} \qquad T(v_i) = \lambda_i v_i \text{ for all } 1 \le i \le n.$$

(b) Prove that $T$ is diagonalizable if and only if there is a basis $\mathcal{B}$ of $V$ consisting of eigenvectors of $T$. (Such a basis of $V$ is called an *eigenbasis* for $T$).

**Solution:**

(a) Let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis of $V$, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then for each $1 \le i \le n$,

$$[T]_{\mathcal{B}}\vec{e}_i = [T]_{\mathcal{B}}[v_i]_{\mathcal{B}} = [T(v_i)]_{\mathcal{B}} \qquad \text{and} \qquad \lambda_i\vec{e}_i = \lambda_i[v_i]_{\mathcal{B}} = [\lambda_i v_i]_{\mathcal{B}},$$

which shows that the $i$th column of $[T]_{\mathcal{B}}$ is $\lambda_i\vec{e}_i$ if and only if $T(v_i) = \lambda_i v_i$.

(b) Using part (a) and the definitions, we have that

$$\begin{aligned} T \text{ is diagonalizable} \quad &\iff \quad \text{there is a basis } \mathcal{B} \text{ of } V \text{ and scalars } \lambda_1, \ldots, \lambda_n \in \mathbb{R}^n \\ &\qquad\quad \text{such that } [T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \\ &\iff \quad \text{there is a basis } \mathcal{B} = (v_1, \ldots, v_n) \text{ of } V \text{ and scalars } \lambda_1, \ldots, \lambda_n \\ &\qquad\quad \text{such that } T(v_i) = \lambda_i v_i \text{ for all } 1 \le i \le n \\ &\iff \quad \text{there is a basis } \mathcal{B} \text{ of } V \text{ consisting of eigenvectors of } T. \end{aligned}$$

**Problem 4.** Let $A$ be an $n \times n$ matrix.

(a) Prove that if the $n \times n$ matrix $P = [\vec{v}_1 \ \cdots \ \vec{v}_n]$ is invertible and if $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad \text{if and only if} \qquad A\vec{v}_i = \lambda_i\vec{v}_i \text{ for all } 1 \le i \le n.$$

(b) Prove that $A$ is diagonalizable if and only if there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of $A$. (Again, such a basis of $\mathbb{R}^n$ is called an *eigenbasis* for $A$).

**Solution:**

(a) Let $A$ and $P$ be $n \times n$ matrices, suppose $P = [\vec{v}_1 \ \cdots \ \vec{v}_n]$ is invertible, let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and write $D$ for the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Then for each $1 \leq i \leq n$ we have

$$AP\vec{e}_i = A\vec{v}_i \qquad \text{and} \qquad PD\vec{e}_i = \lambda_i \vec{v}_i,$$

which shows that $AP = PD$ if and only if $A\vec{v}_i = \lambda_i \vec{v}_i$ for each $1 \leq i \leq n$.

*Alternate proof:* Let $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$, so $P = S_{\mathcal{B}\to\mathcal{E}}$, and let $D$ be the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then, using 3(a), we have

$$\begin{aligned} P^{-1}AP = D \quad &\iff \quad S_{\mathcal{E}\to\mathcal{B}}[T_A]_{\mathcal{E}} S_{\mathcal{B}\to\mathcal{E}} = D \\ &\iff \quad [T_A]_{\mathcal{B}} = D \\ &\iff \quad T_A(\vec{v}_i) = \lambda_i \vec{v}_i \text{ for each } i \quad \iff \quad A\vec{v}_i = \lambda_i \vec{v}_i \text{ for each } i. \end{aligned}$$

(b) By part (a), for any $n \times n$ matrix $A$ and invertible $n \times n$ matrix $P$, we have that $P^{-1}AP$ is diagonal (and thus $A$ is diagonalizable) if and only if the columns of $P$ are eigenvectors of $A$. Since the columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$, we see that $A$ is diagonalizable if and only if there is an eigenbasis of $\mathbb{R}^n$ for $A$.

*Alternate solution:* Using 3(b), we have that $A$ is diagonalizable *iff* $T_A$ is diagonalizable *iff* there exists an eigenbasis for $T_A$ *iff* there exists an eigenbasis for $A$.

To *diagonalize* a linear transformation $T : V \to V$ means to find a basis $\mathcal{B}$ of $V$ such that $[T]_{\mathcal{B}}$ is diagonal. Similarly, to *diagonalize* a square matrix $A$ means to factor $A$ as $A = PDP^{-1}$ where $D$ is diagonal.

**Problem 5.** Determine whether the following matrices or linear transformations are diagonalizable. Either diagonalize the given matrix or transformation, if possible, or else explain why this is impossible. (Pay close attention to the algebraic and geometric multiplicities of the eigenvalues you find!)

(a) $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection over the line spanned by $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a counterclockwise $90°$ rotation about the origin.

(e) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the line spanned by the nonzero vector $\vec{v} \in \mathbb{R}^2$.

(f) $T : \mathcal{P}_3 \to \mathcal{P}_3$ is the differentiation operator on the space $\mathcal{P}_3$ of polynomials of degree $\leq 3$.

(g) $A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(h) $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

**Solution:**

(a) $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}^{-1}$

(b) $A$ is not diagonalizable, since $A$ has only one eigenvalue ($\lambda = 1$), and the corresponding eigenspace is only 1-dimensional.

(c) In the basis $\mathcal{B} = \left( \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right)$ of $\mathbb{R}^2$, we have $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) $T$ is not diagonalizable, since it has no (real) eigenvalues or eigenvectors.

(e) If $\vec{w}$ is any nonzero vector in $\text{Span}(\vec{v})^{\perp}$, then $\mathcal{B} = (\vec{v}, \vec{w})$ is a basis of $\mathbb{R}^2$ and $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(f) $T$ is not diagonalizable, since $T$ has only one eigenvalue ($\lambda = 0$), and the corresponding eigenspace is only 1-dimensional.

(g) $A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1}.$

(h) $A$ is not diagonalizable. The eigenvalues of $A$ are $\lambda = 3$ and $\lambda = 2$, each with an algebraic multiplicity of 2. The geometric multiplicity of 2 is 2, but the geometric multiplicity of 3 is only 1, meaning that there are not enough eigenvectors corresponding to $\lambda = 3$ to span the eigenspace $E_3$.

**Problem 6.** Can you make a conjecture relating the algebraic and geometric multiplicities of the eigenvalues of a square matrix $A$ to the question of whether $A$ is diagonalizable?

**Solution:** The $n \times n$ matrix $A$ is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues of $A$ is $n$. Equivalently, the $n \times n$ matrix $A$ is diagonalizable if and only if the sum of the algebraic multiplicities of the eigenvalues of $A$ is $n$ *and additionally* for each eigenvalue $\lambda$ of $A$, the algebraic and geometric multiplicities of $\lambda$ are equal.

**Problem 7.** Let $n \geq 1$ and let $V$ be any subspace of $\mathbb{R}^n$. Prove that both the orthogonal projection of $\mathbb{R}^n$ onto $V$ and the reflection of $\mathbb{R}^n$ through $V$ are diagonalizable.

**Solution:** Let $n \geq 1$ and let $V$ be a subspace of $\mathbb{R}^n$ of dimension $k$. Let $(v_1, \ldots, v_k, v_{k+1}, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ such that $(v_1, \ldots, v_k)$ is a basis of $V$ and $(v_{k+1}, \ldots, v_n)$ is a basis of $V^\perp$. Then, letting $T$ be the orthogonal projection of $\mathbb{R}^n$ onto $V$ and letting $S$ be the reflection of $\mathbb{R}^n$ through $V$, the $\mathcal{B}$-matrices of $T$ and $S$ have the block matrix form

$$[T]_\mathcal{B} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad [S]_\mathcal{B} = \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix},$$

showing that each is diagonalizable.