Math 217 – Final Exam

Fall 2019

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
3. No calculators, notes, or other outside assistance allowed.
4. On this exam, *unless stated otherwise*, terms such as "orthogonal" and "orthonormal" as applied to vectors in $\mathbb{R}^n$ mean with respect to the usual dot product on $\mathbb{R}^n$.
5. On this exam, *unless stated otherwise*, "eigenvalue" means *real* eigenvalue, and terms such as "similar" and "diagonalizable" mean over $\mathbb{R}$.

Student ID Number: ______________________ Section: __________

Question	Points	Score
1	12	
2	16	
3	10	
4	16	
5	13	
6	12	
7	10	
8	11	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The *dimension* of the subspace V of $\mathbb{R}^n$

   (b) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   (c) The *rank* of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

   (d) For a subset X of the vector space V, the *span* of X in V

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) For all $n \in \mathbb{N}$, the set W of all orthogonal $n \times n$ matrices is a subspace of the vector space $\mathbb{R}^{n\times n}$.

   (b) (3 points) For all integers $0 \leq k \leq n$, if the $n \times n$ matrix A has k distinct eigenvalues, then rank $A \geq k$.

   (c) (3 points) If $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is a linear transformation whose image is contained in its kernel, then $\operatorname{rank}(T) \leq 4$.

(Problem 2, Continued).

(d) (3 points) For every matrix $A \in \mathbb{R}^{m\times n}$, if $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, then the columns of A are linearly independent.

(e) (4 points) For every matrix $A \in \mathbb{R}^{n\times n}$, if $A^2 = A$ then A is diagonalizable over $\mathbb{R}$.

3. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable x. Let $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by

$$T(p)(x) = p'(x) + p''(x) \quad \text{for all } x \in \mathbb{R}.$$

(a) (6 points) *(No justification is necessary for this part of the problem.)*

(i) Find a basis of $\operatorname{im}(T)$.

(ii) Find a basis of $\ker(T)$.

(iii) Compute $\det(T)$.

(b) (4 points) Find a polynomial p that is an eigenvector of T, and find the associated eigenvalue along with the geometric multiplicity of this eigenvalue. Justify your answer.

4. Consider the 3×3 matrix $A = \begin{bmatrix} a & 0 & 1 \\ 0 & b & 0 \\ -1 & 0 & 0 \end{bmatrix}$, where $a, b \in \mathbb{R}$. In parts (a) – (d) below, find *all* values of a and b for which the given condition holds, or else write "none" if there are no such values. *(No justification is needed for any part of this problem.)*

   (a) (2 points) A is invertible.

   (b) (2 points) A is orthogonal.

   (c) (2 points) A is orthogonally diagonalizable.

   (d) (4 points) A has one eigenvalue with algebraic multiplicity 3.

   For parts (e) and (f) below, fix $b = 1$, and find all values of a for which the given condition holds or else write "none" if there are no such values.

   (e) (3 points) A is diagonalizable over $\mathbb{R}$.

   (f) (3 points) A is diagonalizable over $\mathbb{C}$.

5. Let V be a k-dimensional subspace of $\mathbb{R}^n$, where $0 < k < n$. Let $\operatorname{refl}_V : \mathbb{R}^n \to \mathbb{R}^n$ be reflection through the subspace V, so $\operatorname{refl}_V(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$ and $\operatorname{refl}_V(\vec{w}) = -\vec{w}$ for all $\vec{w} \in V^{\perp}$. Let A be the standard matrix of refl_V.

   (a) (3 points) Find $\det(A)$ in terms of n and k. *(No justification needed.)*

   (b) (4 points) Is A symmetric? Answer yes or no, and briefly justify your answer.

   (c) (4 points) Assuming $n \geq 3$, find the area of the parallelogram P in $\mathbb{R}^n$ determined by the vectors $\vec{v}_1 = \vec{e}_1 + \vec{e}_2$ and $\vec{v}_2 = \vec{e}_1 + \vec{e}_3$ in $\mathbb{R}^n$.

   (d) (2 points) With P as in part (c), find the area of $\operatorname{refl}_V[P]$. You may give your answer in terms of the area of P, if you wish. *(No justification needed.)*

6. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Let $\mathcal{B} = (\vec{x}, \vec{y}, \vec{z})$ be a basis of $\mathbb{R}^3$, and assume that

$$T(\vec{x}) = \vec{y}, \qquad T(\vec{y}) = \vec{z}, \qquad T(\vec{z}) = \vec{x}.$$

Let A be the standard matrix of T, so that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

(a) (4 points) Compute $\det(A)$. Justify your answer.

(b) (4 points) Find an eigenvector of T and the corresponding eigenvalue. Justify your answer.

(c) (4 points) Determine whether T is diagonalizable, and justify your answer.

7. Fix an $n \times n$ matrix M, and let $T\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the linear transformation defined by $T(A) = MA$ for all $A \in \mathbb{R}^{n\times n}$.

   (a) (4 points) Prove that if $\vec{v}$ is an eigenvector of M with eigenvalue λ, then the matrix $A = \begin{bmatrix} | & & | \\ \vec{v} & \cdots & \vec{v} \\ | & & | \end{bmatrix}$ with all columns equal to $\vec{v}$ is an eigenvector of T.

   (b) (6 points) Prove that if M has n distinct real eigenvalues, then T is diagonalizable.

8. Let V be an inner product space of dimension n with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ be an orthonormal basis of V with respect to this inner product. Let $T\colon V \to V$ be a linear transformation and assume for all $\vec{x}, \vec{y} \in V$ that

$$\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T(\vec{y}) \rangle.$$

(a) (5 points) Prove that $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix of T, is a symmetric matrix.

(b) (6 points) Prove that there exists an orthonormal basis $\mathcal{U}$ of V which is an eigenbasis for the linear transformation T.

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