Worksheet 19: Inner Product Spaces (§5.5)

Let V be a vector space. "Inner products" on V are abstractions of the dot product on $\mathbb{R}^n$ in much the same way that V itself is an abstraction of $\mathbb{R}^n$. An *inner product* on V is a function that assigns a real number $\langle v, w\rangle \in \mathbb{R}$ to pairs of vectors $v, w \in V$ and has all the important algebraic properties of the dot product. Specifically, inner products are:

- *symmetric*
- "linear in both arguments," or *bilinear*
- *positive definite*

Problem 1. Make sure you understand each of the properties listed above. Write down what they mean using equations!

Solution: *Symmetric* means that $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$. *Linear in the first argument* means that $\langle au+bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$ for all $a, b \in \mathbb{R}$ and $u, v, w \in V$. *Linear in the second argument* means the analogous thing for the second argument, and *linear in both arguments* means exactly what it sounds like. Finally, *positive definite* means that $\langle v, v\rangle > 0$ for all nonzero $v \in V$.

Problem 2. Show that the rule

$$\langle \vec{v}, \vec{w}\rangle \;=\; \vec{v}^{\top} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \vec{w}$$

is an inner product on $\mathbb{R}^2$ which is different from the dot product. Can you find an explicit formula for it in terms of the components of $\vec{v}$ and $\vec{w}$? Can you come up with other examples?

Solution: By multiplying out the product $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, we see that

$$\langle \vec{v}, \vec{w}\rangle \;=\; v_1 w_1 + v_1 w_2 + v_2 w_1 + 2 v_2 w_2.$$

Symmetry follows easily from the above formula, linearity in both arguments follows from the rules of matrix multiplication, and positive definiteness follows from the fact that the square of a nonzero number is always positive. This inner product is different from the dot product because, for instance, $\langle \vec{e}_2, \vec{e}_2\rangle = 2 \neq 1 = \vec{e}_2 \cdot \vec{e}_2$. For another example, we could define

$$\langle \vec{v}, \vec{w}\rangle \;=\; \vec{v}^{\top} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \vec{w}.$$

Problem 3. Which of the following are inner product spaces? You do not need to write a proof, but you should briefly explain why or why not.

(a) $V =$ the space of continuous functions from $[-\pi, \pi]$ to $\mathbb{R}$,

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt.$$

(b) $V =$ the space of polynomials in the variable t of degree at most 2,

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

(c) $V =$ the space of infinite sequences $a_1, a_2, \ldots$,

$$\langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} a_n b_n$$

(d) $V = \mathbb{R}^{n \times n} =$ the space of all $n \times n$ matrices,

$$\langle A, B \rangle = \operatorname{tr}(A + B).$$

(e) $V = \mathbb{R}^{m \times n} =$ the space of all $m \times n$ matrices,

$$\langle A, B \rangle = \operatorname{tr}(A^\top B).$$

Solution: (a), (b), and (e) are inner products. (c) is not because $\langle f, g \rangle$ may not be a real number (it may be ∞). (d) is not because it is not positive definite.

Problem 4. Show that if $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space, then V has an orthonormal basis.[†]

Solution: Since V is in particular a finite dimensional vector space, we know that V has a (finite) basis, say $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$. But now if we apply the Gram-Schmidt process to $\mathcal{B}$, we will obtain an orthonormal set $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ of n vectors in V. Since orthonormal sets of vectors are linearly independent, $\mathcal{U}$ must be a basis of V.

Problem 5. Find an orthonormal basis for the examples in Problem 2 and Problem 3(b) above.

[†]Let's explain this strange notation "$(V, \langle \cdot, \cdot \rangle)$." Remember that an inner product space is a vector space V on which we have defined a function $f : V \times V \to \mathbb{R}$ called an *inner product* that is bilinear, symmetric, and positive-definite. But we don't usually call inner products "f"; rather, we use the notation $\langle \vec{x}, \vec{y} \rangle$ for the inner product of $\vec{x}$ and $\vec{y}$. So $\langle \cdot, \cdot \rangle$ is just a name for our inner product, where the "dots" are placeholders indicating that the (two) arguments of the inner product go there. So, taken as a whole, the notation "$(V, \langle \cdot, \cdot \rangle)$" refers to our inner product space and indicates that its underlying vector space is V and that $\langle \vec{x}, \vec{y} \rangle$ is the notation we will be using for the inner product of $\vec{x}, \vec{y} \in V$. Phew!

Solution: Relative to the inner product defined in Problem 2, $\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right)$ is an orthonormal basis of $\mathbb{R}^2$. Relative to the inner product defined on V in Problem 3b,

$$\left(1, \sqrt{12}(t-\tfrac{1}{2}), \sqrt{180}(t^2-t+\tfrac{1}{6})\right)$$

is an orthonormal basis of V.

Problem 6. Let $(V, \langle -,-\rangle)$ be the inner product space from Problem 3(a) above. Find the orthogonal projection of the function $y = e^t$ onto the subspace T_1 spanned by 1, $\sin t$, and $\cos t$. HINT: Recall that $\displaystyle\int e^t \sin t\, dt = \tfrac{1}{2}e^t(\sin t - \cos t) + C$ and $\displaystyle\int e^t \cos t\, dt = \tfrac{1}{2}e^t(\sin t + \cos t) + C$.

Solution: By computation we see that $\left(\frac{1}{\sqrt{2}}, \sin t, \cos t\right)$ is an orthonormal basis of T_1. Then

$$\begin{aligned}
\langle e^t, \tfrac{1}{\sqrt{2}}\rangle &= \frac{e^\pi - e^{-\pi}}{\sqrt{2}\pi}\\
\langle e^t, \sin t\rangle &= \frac{e^\pi - e^{-\pi}}{2\pi}\\
\langle e^t, \sin t\rangle &= -\frac{e^\pi - e^{-\pi}}{2\pi},
\end{aligned}$$

so the projection of e^t onto T_1 is

$$\frac{e^\pi - e^{-\pi}}{2\pi} + \frac{e^\pi - e^{-\pi}}{2\pi}\sin t - \frac{e^\pi - e^{-\pi}}{2\pi}\cos t.$$

Problem 7. Prove that if $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ is an orthonormal basis of the inner product space V, then

$$\langle \vec{x}, \vec{y}\rangle = [\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}}$$

for all $\vec{x}, \vec{y} \in V$. Is this still true if $\mathcal{U}$ is any basis of V, not necessarily orthonormal?

Solution: Let $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ be an orthonormal basis of V, let $\vec{x}, \vec{y} \in V$, and write $\vec{x} = \sum_{i=1}^n a_i\vec{u}_i$ and $\vec{y} = \sum_{i=1}^n b_i\vec{u}_i$. Then, using the fact that $\mathcal{U}$ is orthonormal, we get

$$\langle \vec{x}, \vec{y}\rangle = \left\langle \sum_{i=1}^n a_i\vec{u}_i, \sum_{i=1}^n b_i\vec{u}_i \right\rangle = \sum_{i=1}^n\sum_{j=1}^n a_ib_j\langle \vec{u}_i, \vec{u}_j\rangle = \sum_{i=1}^n a_ib_i = [\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}}.$$

If $\mathcal{U}$ is not orthonormal, then this could fail. For instance, letting V be the inner product space in Problem 2 with (non-orthonormal) basis $\mathcal{U} = \mathcal{E}$, we see that $\langle \vec{e}_1, \vec{e}_2\rangle = 1 \neq 0 = [\vec{e}_1]_{\mathcal{E}} \cdot [\vec{e}_2]_{\mathcal{E}}$.

Problem 8. Let $\langle -, - \rangle$ be an inner product on $\mathbb{R}^n$. Show that there exists an $n \times n$ matrix A such that

$$\langle \vec{x}, \vec{y} \rangle \;=\; \vec{x}^\top A\, \vec{y}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Show moreover that A must be symmetric. (Hint: can you recover A from the inner products $\langle \vec{e}_i, \vec{e}_j \rangle$?)

Solution: Given an $n \times n$ matrix A and $1 \le i, j \le n$, the (i,j)-entry of A is just $\vec{e}_i^\top A \vec{e}_j$. Thus we let A be the $n \times n$ matrix whose (i,j)-entry is $a_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$. Then for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have

$$\langle \vec{x}, \vec{y} \rangle \;=\; \left\langle \sum_{i=1}^{n} x_i \vec{e}_i,\; \sum_{j=1}^{n} y_j \vec{e}_j \right\rangle \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \langle \vec{e}_i, \vec{e}_j \rangle \;=\; \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} y_j \right) \;=\; \vec{x}^T A \vec{y}.$$

Note that A is symmetric, since for all $1 \le i, j \le n$, $A(i,j) = \langle \vec{e}_i, \vec{e}_j \rangle = \langle \vec{e}_j, \vec{e}_i \rangle = A(j,i)$.

Problem 9. Describe/classify *all* inner products on $\mathbb{R}^2$.

Solution: For any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the function $\beta_A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\beta_A(\vec{x}, \vec{y}) \;=\; \vec{x}^\top A \vec{y}$$

will be linear in both arguments by the properties of matrix multiplication. Furthermore, β_A will be symmetric precisely when A is symmetric (i.e., when $b = c$). What is a bit harder to see is that β_A will be positive definite if and only if both $a > 0$ and $\det(A) > 0$. Thus by Problem (7), the inner products on $\mathbb{R}^2$ are just those functions of the form β_A where A is a symmetric matrix whose determinant and upper-left entry are both positive.

For the claim about positive definiteness, first note that a simple example such as $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ shows that the conditions $a > 0$ and $\det A > 0$ are necessary. To see that these conditions are sufficient, assume them, and let $(x, y) \neq \vec{0}$; we must show

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; ax^2 + 2bxy + dy^2 \;>\; 0. \qquad (*)$$

Since $a > 0$, $(*)$ holds if $y = 0$, so we may assume $y \neq 0$. Then by linearity, we may further assume after scaling by a positive number that $y = \pm 1$. But then, using the discriminant, $(*)$ reduces to the condition $(2b)^2 - 4ad = 4(b^2 - ad) < 0$, which holds whenever $\det(A) > 0$.