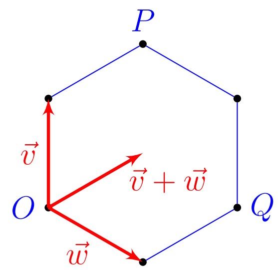
MATH 217 - LINEAR ALGEBRA
HOMEWORK 6, SOLUTIONS

Part A (10 points)

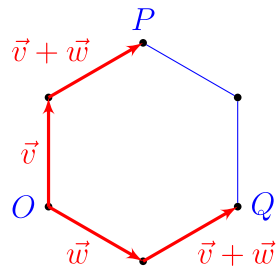
Solve the following problems from the book:
Section 3.4: 50, 66
Section 4.1: 58
Section 4.2: 26, 68

Solution.

3.4.50: It is helpful to note that the 'third direction' is $\vec{v}+\vec{w}$:



(a) We have $\overrightarrow{OP} = 2\vec{v}+\vec{w}$ and $\overrightarrow{OQ} = \vec{v}+2\vec{w}$:



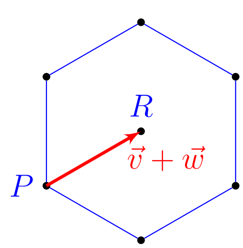
Therefore

$$\left[\overrightarrow{OP}\right]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \left[\overrightarrow{OQ}\right]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(b) Note that

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (3\vec{v}+2\vec{w}) - (2\vec{v}+\vec{w}) = \vec{v}+\vec{w},$$

so we have the following picture:



Thus R is the center of a tile.

(c) Note that every vertex looks locally either like Y or ⅄. Starting at a ⅄ vertex and adding either $2\vec{v}+\vec{w}$ or $\vec{v}+2\vec{w}$ takes us to another ⅄ vertex. Since O is a ⅄ vertex and $17\vec{v}+13\vec{w} = 7(2\vec{v}+\vec{w}) + 3(\vec{v}+2\vec{w})$, we see that S is a ⅄ vertex.

3.4.66 We have that

$$T\left(\begin{bmatrix} b \\ 1-a \end{bmatrix}\right) = \begin{bmatrix} b \\ 1-a \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} a-1 \\ b \end{bmatrix}\right) = \begin{bmatrix} 1-a \\ -b \end{bmatrix}.$$

Hence,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, T is the reflection over the line $\ell = \text{Span}\left(\begin{bmatrix} b \\ 1-a \end{bmatrix}\right)$.

4.1.58: First observe that V is a subspace of vector space of functions from $\mathbb{R}$ to $\mathbb{R}$:

- the zero function $\vec{0}$ is in V;
- if $f, g \in V$, then $(f+g)'' = f'' + g'' = -f + (-g) = -(f+g)$, so $f+g \in V$;
- if $f \in V$ and $c \in \mathbb{R}$, then $(cf)'' = cf'' = c(-f) = -(cf)$, so $cf \in V$.

(a) Let $g \in V$, so that $g'' = -g$. In order to show that $g^2 + (g')^2$ is constant, it suffices to show that its derivative is the zero function $\vec{0}$. We have

$$(g^2 + (g')^2)' = 2gg' + 2g'g'' = 2gg' + 2g'(-g) = \vec{0},$$

as desired.

(b) Suppose that $g \in V$ such that $g(0) = g'(0) = 0$. By part (a), the function $g^2 + (g')^2$ is constant, and by assumption, it equals 0 when $x = 0$. Therefore $g^2 + (g')^2 = \vec{0}$. Since the square of any function is everywhere nonnegative, we conclude that both g and g' are identically zero.

(c) Let $f \in V$. Observe that $\sin(x)$ and $\cos(x)$ are both in V, since

$$(\sin(x))'' = -\sin(x), \qquad (\cos(x))'' = -\cos(x).$$

Since V is closed under addition and scalar multiplication, get that $g \in V$, where by definition $g(x) := f(x) - f(0)\cos(x) - f'(0)\sin(x)$ for all $x \in \mathbb{R}$. We have

$$g(0) = f(0) - f(0)\cos(0) - f'(0)\sin(0) = f(0) - f(0) - 0 = 0,$$

and since $g'(x) = f'(x) + f(0)\sin(x) - f'(0)\cos(x)$, we have

$$g'(0) = f'(0) + f(0)\sin(0) - f'(0)\cos(0) = f'(0) + 0 - f'(0) = 0.$$

Thus by part (b), we get $g = \vec{0}$, i.e. $f(x) = f(0)\cos(x) + f'(0)\sin(x)$ for all $x \in \mathbb{R}$.

4.2.26: We prove that this function is an isomorphism. Let's start by proving that it is linear. For every $a, b, c, a', b', c', k \in \mathbb{R}$

$$\begin{aligned} T(a + bt + ct^2 + a' + b't + c't^2) &= T((a+a') + (b+b')t + (c+c')t^2) \\ &= (a+a') - (b+b')t + (c+c')t^2 = a - bt + ct^2 + a' - b't + c't^2 \\ &= T(a + bt + ct^2) + T(a' + b't + c't^2) \end{aligned}$$

and

$$T(k(a+bt+ct^2)) = T(ka+kbt+kct^2) = ka-kbt+kct^2 = k(a-bt+ct^2) = kT(a-bt+ct^2).$$

Note that it will suffice to prove that T is surjective. For every $a+bt+ct^2 \in P_2$,

$$a+bt+ct^2 = T(a-bt+ct^2)$$

therefore T is surjective.

4.2.68: We claim that T is an isomorphism if and only if $k \neq 1, 5$. To prove this, we first observe that since T is a linear transformation from a finite-dimensional vector space to itself, it is injective if and only if it is surjective (this follows from the rank-nullity theorem). Therefore, T is an isomorphism if and only if it is injective, if and only if $\ker(T) = \{\vec{0}\}$. Thus we must show that $\ker(T) = \{\vec{0}\}$ if and only if $k \neq 1, 5$.

Note that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 3a & -b \\ (5-k)c & (1-k)d \end{bmatrix} \quad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

We see that if $k \neq 1, 5$, then $\ker(T) = \{\vec{0}\}$. Conversely, if $k = 1$, then $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is in $\ker(T)$, while if $k = 5$, then $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is in $\ker(T)$.

Part B (25 points)

Problem 1. Let V be a vector space, and let $(\vec{v}_1, \dots, \vec{v}_n)$ be a list of vectors in V. Define the function $T\colon \mathbb{R}^n \to V$ by

$$T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \quad \text{for all } \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

(a) Prove that T is a linear transformation.
(b) Prove that if $(\vec{v}_1, \dots, \vec{v}_n)$ is an ordered basis of V, then T is an isomorphism.
(c) Prove that if $(\vec{v}_1, \dots, \vec{v}_n)$ does not span V, then T is not surjective.
(d) Prove that if $(\vec{v}_1, \dots, \vec{v}_n)$ is not linearly independent, then T is not injective.
(e) Prove that if T is an isomorphism, then $(\vec{v}_1, \dots, \vec{v}_n)$ is an ordered basis of V.

Solution.

(a) For every $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \vec{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in \mathbb{R}^n$ and $k \in \mathbb{R}$, we have

$$T(\vec{c}+\vec{d}) = (c_1+d_1)\vec{v}_1 + \cdots + (c_n+d_n)\vec{v}_n = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n + d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = T(\vec{c}) + T(\vec{d}).$$

and

$$T(k\vec{c}) = (kc_1)\vec{v}_1 + \cdots + (kc_n)\vec{v}_n = k(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = kT(\vec{c}).$$

(b) Since $(\vec{v}_1, \dots, \vec{v}_n)$ is a basis of V, we know $\dim V = n$. Thus, it suffices to show that T is injective. Suppose $\vec{c} \in \mathbb{R}^n$ is in $\ker T$. Then, $T(\vec{c}) = \vec{0}$, i.e. $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$, is a linear relation among the linearly independent vectors $(\vec{v}_1, \dots, \vec{v}_n)$. By definition, $\vec{c} = \vec{0} \in \mathbb{R}^n$ and we are done as $\ker T = \{\vec{0}\}$.

(c) By definition, the image of T is the space of all possible linear combinations of $(\vec{v}_1, \dots, \vec{v}_n)$. It follows that, if $\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n) \neq V$, then $\mathrm{im}\, T = \mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n) \neq V$ and T is not surjective.

(d) If $(\vec{v}_1, \dots, \vec{v}_n)$ is linearly dependent, then there exist $c_1, \dots, c_n \in \mathbb{R}$ not all zero such that $c_1\vec{v}_1 + \ldots + c_n v_n = \vec{0}$. In other words, the nonzero vector $\begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix}^\top$ is in the kernel of T. Thus, T is not injective as $\ker T \neq \{\vec{0}\}$.

(e) If T is an isomorphism, then T is both injective and surjective. By taking the contrapositive of parts (c) and (d), we have that $(\vec{v}_1, \dots, \vec{v}_n)$ is linearly independent and spans V. Thus, $(\vec{v}_1, \dots, \vec{v}_n)$ is a basis of V.

Problem 2. Let $\mathbb{C}$ be the vector space of complex numbers. Consider the lists $\mathcal{B} = (1, i)$ and $\mathcal{C} = (1+i, 1-i)$ of vectors in $\mathbb{C}$.

(a) Show that $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{C}$.

(b) In each part below, determine whether or not the given function is a linear transformation, with justification.
   - (i) $M \colon \mathbb{C} \to \mathbb{C}$, defined by $M(z) = |z|$.
   - (ii) $\sigma \colon \mathbb{C} \to \mathbb{C}$, defined by $\sigma(z) = \bar{z}$.
   - (iii) $T \colon \mathbb{C} \to \mathbb{C}$, defined by $T(z) = 2i\bar{z}$.
   - (iv) $\mathrm{Re} \colon \mathbb{C} \to \mathbb{C}$ defined by $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$.

Solution.

(a) Recall that $\mathbb{C} = \{a + ib \mid \text{ for some } a, b \in \mathbb{R}\}$. It follows immediately that $\mathbb{C} = \mathrm{Span}(\mathcal{B})$. Moreover, $a + ib = 0$ if and only if $a = b = 0$ therefore $\mathcal{B}$ is also linearly independent. Thus, $\mathcal{B}$ is a basis for the two dimensional vector space $\mathbb{C}$. In particular, it suffices to show that $\mathrm{Span}(\mathcal{C}) = \mathbb{C}$ since $\mathcal{C}$ consists of two vectors. This follows at once if we observe that $1 = \frac{1}{2}(1+i) + \frac{1}{2}(1-i)$ and $i = \frac{1}{2}(1+i) - \frac{1}{2}(1-i)$.

(b) (i) M is not a linear transformation as $M(-i) = |-i| = 1 \neq -1 = -1 \cdot |i| = -1 \cdot M(i)$.

(ii) σ is a linear transformation. Indeed we can check that for all $a, b, c, d, k \in \mathbb{R}$ we have

$$\begin{aligned} M((a+ib) + (c+id)) &= M(a + c + i(b+d)) = a + c - i(b+d) \\ &= a - ib + c - id = M(a+ib) + M(c+id). \end{aligned}$$

and

$$M(k(a+ib)) = M(ka + ikb) = ka - ikb = k(a - ib) = kM(a+ib).$$

Note that we just proved that for all $z, w \in \mathbb{C}$ and $k \in \mathbb{R}$

$$\overline{z+w} = \overline{z} + \overline{w}, \text{ and } \overline{kz} = \overline{k}\overline{z} = k\overline{z}.$$

We can, and will, use in these equalities in the rest of the problem!

(iii) T is a linear transformation. Indeed we can check that for all $z, w \in \mathbb{C}$ and $k \in \mathbb{R}$ we have

$$T(z+w) = 2i(\overline{z+w}) = 2i(\overline{z}+\overline{w}) = T(z)+T(w).$$

and

$$T(kz) = 2i\overline{kz} = 2i\overline{k}\overline{z} = k(2i\overline{z}) = kT(z).$$

(iv) Re is a linear transformation. Indeed we can check that for all $z, w \in \mathbb{C}$ and $k \in \mathbb{R}$ we have

$$\mathrm{Re}(z+w) = \frac{z+w+\overline{z+w}}{2} = \frac{z+w+\overline{z}+\overline{w}}{2} = \frac{z+\overline{z}}{2} + \frac{w+\overline{w}}{2} = \mathrm{Re}(z)+\mathrm{Re}(w).$$

and

$$\mathrm{Re}(kz) = \frac{kz+\overline{kz}}{2} = \frac{kz+\overline{k}\overline{z}}{2} = \frac{k(z+\overline{z})}{2} = k\mathrm{Re}(z).$$

Problem 3. Let $C^\infty(\mathbb{R})$ be the vector space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$. In other words, every vector $f \in C^\infty(\mathbb{R})$ is a function $f\colon \mathbb{R} \to \mathbb{R}$ that is differentiable k-times for all $k \in \mathbb{N}$. Let $f_1, \dots, f_6$ be the six functions in $C^\infty(\mathbb{R})$ defined by

$$f_1(x) = 1, \quad f_2(x) = \sin(2x), \quad f_3(x) = \cos(2x),$$

$$f_4(x) = \sin^2(x), \quad f_5(x) = \cos^2(x), \quad f_6(x) = \sin x \cos x.$$

Let $V = \mathrm{Span}(f_1, f_2, f_3, f_4, f_5, f_6)$, and let $\mathcal{B} = (f_1, f_2, f_4) = (1, \sin 2x, \sin^2 x)$.

(a) Prove that $\mathcal{B}$ is an ordered basis of V. [*Hint*: remember (or look up) some trig identities.]
(b) For each $i \in \{1, \dots, 6\}$, find $[f_i]_\mathcal{B}$.
(c) Let $\mathcal{C} = (f_4, f_5, f_6)$, so that $\mathcal{C}$ is an ordered basis of V (you do not have to prove this). Find a matrix S such that $S[f]_\mathcal{B} = [f]_\mathcal{C}$ for all $f \in V$.

Solution.

(a) First we show that $\mathcal{B}$ is linearly independent. Let $a, b, c \in \mathbb{R}$, let $\vec{0}_V$ be the zero vector in V, and suppose $af_1 + bf_2 + cf_4 = \vec{0}_V$. Since $\vec{0}_V$ is the constant zero function, this means that

$$a + b\sin 2x + c\sin^2 x \ = \ 0 \quad \text{for all } x \in \mathbb{R}.$$

In particular, setting $x = 0$ gives us

$$0 \ = \ a + b\sin(2\cdot 0) + c\sin^2(0) \ = \ a+0+0 \ = \ a.$$

Then, using $a = 0$ and setting $x = \frac{\pi}{2}$, we get

$$0 \ = \ b\sin(\pi) + c\sin^2\left(\tfrac{\pi}{2}\right) \ = \ 0 + c \ = \ c.$$

Finally, now that we know $a = c = 0$, setting $x = \frac{\pi}{4}$ gives us

$$0 \ = \ b\sin(2\cdot\frac{\pi}{4}) \ = \ b\sin(\frac{\pi}{2}) \ = \ b.$$

Thus $a = b = c = 0$, showing that $\mathcal{B}$ is indeed linearly independent.

Now we show that $\mathcal{B}$ spans V. It will suffice to show $f_3, f_5, f_6 \in \mathrm{Span}(\mathcal{B})$. But this follows from trig identities:

$$\begin{aligned} f_3 &= \cos(2x) &= 1 - 2\sin^2 x &= 1f_1 - 2f_4 \\ f_5 &= \cos^2 x &= 1 - \sin^2 x &= 1f_1 - f_4 \\ f_6 &= \sin x \cos x &= \tfrac{1}{2}\sin(2x) &= \tfrac{1}{2}f_2. \end{aligned}$$

(b) Using our calculations in part (a), we see that

$$[f_1]_\mathcal{B} = \begin{bmatrix}1\\0\\0\end{bmatrix},\ [f_2]_\mathcal{B} = \begin{bmatrix}0\\1\\0\end{bmatrix},\ [f_3]_\mathcal{B} = \begin{bmatrix}1\\0\\-2\end{bmatrix},\ [f_4]_\mathcal{B} = \begin{bmatrix}0\\0\\1\end{bmatrix},\ [f_5]_\mathcal{B} = \begin{bmatrix}1\\0\\-1\end{bmatrix},\ [f_6]_\mathcal{B} = \begin{bmatrix}0\\1/2\\0\end{bmatrix}.$$

(c) Using the formula for the change-of-coordinates matrix, we see that

$$S = S_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} | & | & | \\ [f_1]_\mathcal{C} & [f_2]_\mathcal{C} & [f_4]_\mathcal{C} \\ | & | & | \end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 2 & 0\end{bmatrix}.$$

Problem 4. Let $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ be an ordered basis of $\mathbb{R}^n$, let $\mathcal{C} = (\vec{w}_1, \dots, \vec{w}_m)$ be an ordered basis of $\mathbb{R}^m$, and let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) Prove that there exists a unique $m \times n$ matrix M such that $[T(\vec{x})]_\mathcal{C} = M[\vec{x}]_\mathcal{B}$ for all $\vec{x} \in \mathbb{R}^n$.

(b) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T : \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \mapsto \begin{bmatrix}x_1 + 5x_2\\ x_2 + 5x_3\end{bmatrix}.$$

(i) Find the standard matrix of T, i.e., the matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

(ii) Find a matrix M such that $[T(\vec{x})]_\mathcal{C} = M[\vec{x}]_\mathcal{B}$ for $\vec{x} \in \mathbb{R}^3$, where

$$\mathcal{B} = \left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}25\\-5\\1\end{bmatrix} \right) \text{ and } \mathcal{C} = \left(\begin{bmatrix}5\\1\end{bmatrix}, \begin{bmatrix}1\\5\end{bmatrix} \right).$$

Solution.

(a) Compare to Theorem 3.4.3 in the book. Let $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \mathbb{R}^n$. We have

$$\begin{aligned} [T(\vec{x})]_\mathcal{C} &= [T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)]_\mathcal{C} \\ &= [c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n)]_\mathcal{C} \\ &= c_1[T(\vec{v}_1)]_\mathcal{C} + \cdots + c_n[T(\vec{v}_n)]_\mathcal{C} \\ &= \big[[T(\vec{v}_1)]_\mathcal{C} \ \cdots [T(\vec{v}_n)]_\mathcal{C}\big] \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} \\ &= \big[[T(\vec{v}_1)]_\mathcal{C} \ \cdots [T(\vec{v}_n)]_\mathcal{C}\big]\, [\vec{x}]_\mathcal{B} \end{aligned}$$

Therefore the unique matrix is $M = \big[[T(\vec{v}_1)]_\mathcal{C} \ \cdots [T(\vec{v}_n)]_\mathcal{C}\big]$.

(b) (i) From class, we know that the standard matrix is

$$A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}.$$

(ii) Let $(\vec{b}_1, \vec{b}_2, \vec{b}_3) = \mathcal{B}$. From part (a), we know that

$$M = \begin{bmatrix} | & | & | \\ [T(\vec{b}_1)]_{\mathcal{C}} & [T(\vec{b}_2)]_{\mathcal{C}} & [T(\vec{b}_3)]_{\mathcal{C}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} \left[\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right]_{\mathcal{C}} & \left[\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right]_{\mathcal{C}} & \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$