Math 217 – Midterm 1

Fall 2021

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary. (Even if a problem states that no justification is necessary, you may provide justification if you wish, since in rare cases this may lead to partial credit for an incorrect final answer.)
3. No calculators, notes, or other outside assistance is allowed.

Student ID Number: ______________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	13	
4	14	
5	14	
6	13	
7	11	
8	8	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The function $f : X \to Y$ is *surjective*

   (b) The *span* of the finite set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$

   (c) For vector spaces V and W, the function $T : V \to W$ is a *linear transformation*

   (d) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly dependent*

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(T)$ contains exactly two elements.

   (b) (3 points) For every matrix $A \in \mathbb{R}^{4\times 3}$ and $\vec{x}, \vec{y} \in \mathbb{R}^3$, if the columns of A are linearly independent and $A\vec{x} = A\vec{y}$ then $\vec{x} = \vec{y}$.

   (c) (3 points) For any linear maps $S : V \to U$ and $T : U \to W$ between vector spaces V, U, and W, if $T \circ S = 0$ then $S = 0$ or $T = 0$. (Here, we write 0 for the zero map between the appropriate spaces.)

(Problem 2, Continued).

(d) (3 points) There exists a linear transformation $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{2\times 2}$ whose kernel has dimension 3.

(e) (3 points) For every $n \in \mathbb{N}$ and subspace V of $\mathbb{R}^n$, there is $m \in \mathbb{N}$ and a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ such that $V = \mathrm{im}(T)$.

3. Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_5 \\ | & & | \end{bmatrix} \in \mathbb{R}^{4\times 5}$ be a 4×5 matrix with columns $\vec{a}_1, \ldots, \vec{a}_5$, let $\vec{b} \in \mathbb{R}^4$, and suppose the reduced row echelon form of the matrix $\begin{bmatrix} A \mid \vec{b} \end{bmatrix}$ is

$$\operatorname{rref}\begin{bmatrix} | & & | & | \\ \vec{a}_1 & \cdots & \vec{a}_5 & \vec{b} \\ | & & | & | \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (2 points) Find all values of c and d for which the linear system $A\vec{x} = \vec{b}$ is inconsistent. *(No justification required.)*

(b) (3 points) Write $\vec{a}_4$ as a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$, or else briefly explain why this is impossible.

(c) (3 points) Find A^{-1}, or else briefly explain why this is impossible.

(d) (5 points) Assuming that $c = 1$ and $d = 4$, find the solution set of the linear system $A\vec{x} = \vec{b}$. (Please write your answer parametrically, using proper set notation.)

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects any vector in $\mathbb{R}^2$ onto the line $y = -x$, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects any vector in $\mathbb{R}^2$ across the x-axis.

   No justification is required on any part of this problem.

   (a) (4 points) Find matrices A and B such that $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

   (b) (3 points) Find a matrix C such that $(S \circ T)(\vec{x}) = C\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

   (c) (4 points) Find a basis of $\mathrm{im}(T)$ and a basis of $\ker(T)$.

   (d) (3 points) For each of the matrices A, B, and C above, either find its inverse or write "DNE" if the matrix has no inverse.

   $A^{-1} =$ $B^{-1} =$ $C^{-1} =$

5. Let V be the subspace of $C^\infty(\mathbb{R})$ given by $V = \{a\cos x + b\sin x \ : \ a, b \in \mathbb{R}\}$. Let $T : V \to \mathbb{R}^2$ be the linear transformation given by

$$T(f) = \begin{bmatrix} f(0) \\ f''(0) \end{bmatrix} \text{ for each } f \in V.$$

$\Big($*Recall that* $\frac{d}{dx}(\cos x) = -\sin x$ *and* $\frac{d}{dx}(\sin x) = \cos x$, *and that* $\cos(0) = 1$ *and* $\sin(0) = 0.\Big)$

(a) (3 points) Find a basis of $\text{im}(T)$.

(b) (3 points) Find a basis of $\ker(T)$.

(c) (4 points) Is T injective? Is T surjective? Justify your answers.

(d) (4 points) Show that $\{\cos x, \sin x\}$ is a linearly independent subset of V.

6. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a *scaling* if there is $c \in \mathbb{R}$ for which $T(\vec{x}) = c\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) (4 points) Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is not a scaling but $T \circ T$ is the identity map on $\mathbb{R}^2$? Clearly state *yes* or *no* and justify your answer.

   (b) (4 points) Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is not a scaling but $T \circ T = T$? Clearly state *yes* or *no* and justify your answer.

   (c) (5 points) Prove that for all $n \in \mathbb{N}$ and matrices $A \in \mathbb{R}^{n\times n}$, if $A \neq I_n$ and $A^2 = A$ then A is not invertible.

7. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Recall that for all $X \subseteq V$ and $Y \subseteq W$, we define

$$T[X] = \{T(\vec{x}) \ : \ \vec{x} \in X\} \quad \text{and} \quad T^{-1}[Y] = \{\vec{x} \in V \ : \ T(\vec{x}) \in Y\}.$$

   (a) (2 points) Clearly state what $T[\ker(T)]$ and $T^{-1}[\operatorname{im}(T)]$ are. *(No justification necessary.)*

   (b) (5 points) Prove that for every subspace U of V, the set $T[U]$ is a subspace of W.

   (c) (4 points) Is it true that for every subspace $S \subseteq W$ we have $T[T^{-1}[S]] = S$? Clearly state your answer and prove your claim.

8. (8 points) Let V be a vector space, let X be a subset of V, and let $u, v \in V$. Prove that if $v \in \operatorname{Span}(X \cup \{u\})$ and $v \notin \operatorname{Span}(X)$, then $u \in \operatorname{Span}(X \cup \{v\})$.

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