Worksheet 10: Bases §§3.3

On previous worksheets we have defined *span* and *linear independence / dependence* of finite lists of vectors, but now we give the general definitions which apply equally well to infinite sets.

Definition: Let V be a vector space and let X be a (possibly infinite) subset of V.

- The *span* of X, written $\mathrm{Span}(X)$, is the set of all vectors in V that can be expressed as a linear combination of a finite list of vectors in X. That is,

$$\mathrm{Span}(X) = \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \ : \ n \in \mathbb{N} \text{ and for each } 1 \le i \le n,\ \vec{v}_i \in X \text{ and } c_i \in \mathbb{R}\}.$$

- X is *linearly independent* if for every finite list of distinct vectors $\vec{v}_1, \ldots, \vec{v}_n$ in X and for all scalars $c_1, \ldots, c_n \in \mathbb{R}$, if $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ then $c_i = 0$ for each $1 \le i \le n$.
- X is *linearly dependent* if there exists a finite (but nonempty) list of distinct vectors $\vec{v}_1, \ldots, \vec{v}_n$ in X and scalars $c_1, \ldots, c_n \in \mathbb{R}$ that are not all zero such that $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$.
- A *basis* of V is a linearly independent subset $\mathcal{B}$ of V such that $\mathrm{Span}(\mathcal{B}) = V$.

Note that the subset $X \subseteq V$ is linearly dependent if and only if it is *not* linearly independent.

Problem 1: A lemma on linear independence.

Let V be a vector space, let X be a linearly independent subset of V, and let $v \in V \setminus X$.

(a) Prove that $X \cup \{v\}$ is linearly independent or $v \in \mathrm{Span}(X)$.

Solution: Since $\vec{0} \in \mathrm{Span}(S)$ for any $S \subseteq V$, the claim is true if $\vec{v} = \vec{0}$, so assume $\vec{v} \neq \vec{0}$ and suppose $X \cup \{v\}$ is linearly dependent. This means there is a finite nonempty list of distinct vectors $x_1, \ldots, x_n$ in X along with scalars $c_0, \ldots, c_n \in \mathbb{R}$, not all zero, such that

$$c_0v + c_1x_1 + \cdots + c_nx_n \ = \ \vec{0}. \qquad (*)$$

Since X is linearly independent, we must have $c_0 \neq 0$, since otherwise $c_1x_1 + \cdots + c_nx_n = \vec{0}$ would be a nontrivial linear relation on X. Since $c_0 \neq 0$, we can rewrite $(*)$ as

$$v \ = \ (-c_0^{-1}c_1)x_1 + \cdots + (-c_0^{-1}c_n)x_n,$$

which shows that $v \in \mathrm{Span}(X)$.

(b) Prove that *only one* of the two possibilities given above can hold.

Solution: Suppose $v \in \mathrm{Span}(X)$, say $v = c_1x_1 + \cdots c_nx_n$ where $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{R}$. Then

$$c_1x_1 + \cdots + c_nx_k - v \ = \ \vec{0}$$

is a nontrivial linear relation on $X \cup \{v\}$, which means that $X \cup \{v\}$ is linearly dependent.

Problem 2: A lemma on span

Again let V be a vector space, and let X and Y be subsets of V.

(a) Prove that $X \subseteq \text{Span}(X)$.

(b) Prove that if $X \subseteq Y$, then $\text{Span}(X) \subseteq \text{Span}(Y)$.

(c) Prove that $\text{Span}\big(\text{Span}(X)\big) = \text{Span}(X)$.

Solution:

(a) Let $x \in X$. Then $x = 1x$, which shows $x \in \text{Span}(X)$.

(b) Suppose $X \subseteq Y$, and let $z \in \text{Span}(X)$. Fix vectors $x_1, \ldots, x_n \in X$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that $z = \sum_{i=1}^n c_i x_i$. Then since each $x_i \in Y$, we see that $z \in \text{Span}(Y)$ as well.

(c) From part (a) we have $X \subseteq \text{Span}(X)$, which by part (b) implies $\text{Span}(X) \subseteq \text{Span}\big(\text{Span}(X)\big)$. For the reverse inclusion, let $z \in \text{Span}\big(\text{Span}(X)\big)$. Fix vectors $y_1, \ldots, y_n \in \text{Span}(X)$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that $z = \sum_{i=1}^n c_i y_i$. Using the fact that each $y_i \in \text{Span}(X)$, for each $1 \leq i \leq n$ fix vectors $x_{i1}, \ldots, x_{ik(i)}$ such that

$$y_i \;=\; \sum_{j=1}^{k(i)} c_{ij} x_{ij}.$$

Then

$$z \;=\; \sum_{i=1}^{n} c_i y_i \;=\; \sum_{i=1}^{n} \sum_{j=1}^{k(i)} c_{ij} x_{ij},$$

which shows $z \in \text{Span}(X)$.

Problem 3: Other ways to think of bases

Once again, let V be a vector space, and let $\mathcal{B}$ be a subset of V. By definition, $\mathcal{B}$ will be a basis of V if and only if $\mathcal{B}$ is a linearly independent spanning set for V. Prove that bases can alternatively be characterized in the following two ways:

(a) $\mathcal{B}$ is a basis of V if and only if $\mathcal{B}$ is linearly independent and no set of vectors properly containing $\mathcal{B}$ is linearly independent. (*Hint*: use the lemmas.)

Solution: For the forward direction, suppose $\mathcal{B}$ is a basis of V, so in particular $\mathcal{B}$ is linearly independent. If $\vec{v} \in V \setminus \mathcal{B}$, then since $\mathcal{B}$ spans V we have $\vec{v} \in \text{Span}(\mathcal{B})$, which implies that $\mathcal{B} \cup \{\vec{v}\}$ is not linearly independent by Problem 1b.

For the converse, suppose $\mathcal{B}$ is a maximal linearly independent set in V. Then $\mathcal{B} \subseteq \text{Span}(\mathcal{B})$ by 2a, and for each $\vec{v} \in V \setminus \mathcal{B}$ we know $\vec{v}$ belongs to $\text{Span}(\mathcal{B})$ by 1a. This shows that $\mathcal{B}$ spans V and hence is a basis of V.

(b) $\mathcal{B}$ is a basis of V if and only if $\mathcal{B}$ spans V and no proper subset of $\mathcal{B}$ spans V.

Solution: For the forward direction, suppose $\mathcal{B}$ is a basis of V. Then $\mathcal{B}$ spans V, by definition of basis. Now let S be a proper subset of $\mathcal{B}$, say $\vec{b} \in \mathcal{B} \setminus S$. Then $S \cup \{\vec{b}\}$ is linearly independent since it is a subset of the linearly independent set $\mathcal{B}$, so $\vec{b} \notin \text{Span}(S)$ by Problem 1b, which shows S does not span V.

For the converse, let $\mathcal{B}$ be a minimal spanning set for V, and suppose for contradiction that $\mathcal{B}$ is linearly dependent. Then we can we can find $\vec{b} \in \mathcal{B}$ that is a linear combination of the vectors in $\mathcal{B} \setminus \{\vec{b}\}$, which implies that $\mathcal{B} \setminus \{\vec{b}\}$ spans V by Problem 2, a contradiction. Thus $\mathcal{B}$ is linearly independent, but it also spans V, so it is a basis of V.

Problem 4: Yet another way to think about bases

Let V be a subspace of $\mathbb{R}^m$, and let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ be an ordered list of vectors in V. Prove that if $\mathcal{B}$ is an ordered basis of V, then for every vector $\vec{v} \in V$ there is a unique list of scalars $(c_1, \ldots, c_n)$ such that $\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$.

Solution: Suppose that $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ is an ordered basis of V, and let $\vec{v} \in V$. Since $\mathcal{B}$ spans V, there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that $\vec{v} = c_1\vec{b}_1 + \cdots + c_m\vec{b}_n$. To show that these coefficients are unique, suppose that also $\vec{v} = d_1\vec{b}_1 + \cdots + d_n\vec{b}_n$. Then

$$\begin{aligned} \vec{0} &= \vec{v} - \vec{v} \\ &= \left(\sum_{i=1}^{n} c_i\vec{b}_i\right) - \left(\sum_{i=1}^{n} d_i\vec{b}_i\right) \\ &= \sum_{i=1}^{n} (c_i - d_i)\vec{b}_i. \end{aligned}$$

Since $\mathcal{B}$ is linearly independent, we must have $c_i - d_i = 0$ for each i; that is, we have $c_i = d_i$ for each i, so the coefficients are indeed unique.

Remark: the converse of Problem 4 is also true; that is, if every vector in V can be expressed in a unique way as a linear combination of vectors in $\mathcal{B}$, then $\mathcal{B}$ is a basis of V. In fact, Problem 4 and its converse (properly formulated) are true even if $\mathcal{B}$ is infinite.

Remark: If a set X has some property, say property P, and no set strictly containing X has property P, then X is said to be "maximal" with respect to P. Similarly, if X has property P and no proper subset of X has property P, then X is said to be "minimal" with respect to P.

Problems 3 and 4 and the two remarks above can be summarized as follows:

Theorem. *For any set $\mathcal{B}$ in the subspace V of $\mathbb{R}^m$, the following are equivalent:*

(i) $\mathcal{B}$ is a basis of V;

(ii) $\mathcal{B}$ is a maximal linearly independent subset of V;

(iii) $\mathcal{B}$ *is a minimal spanning set for* V*;*

(iv) every vector in V *can be expressed in a unique way as a linear combination of vectors in* $\mathcal{B}$*.*

Problem 5: Bases of $\mathbb{R}^n$

Let $\vec{v}_1, \dots, \vec{v}_m$ be a list of vectors in $\mathbb{R}^n$.

(a) Prove that if $m > n$, then $(\vec{v}_1, \dots, \vec{v}_m)$ is not linearly independent.

(b) Prove that if $m < n$, then $(\vec{v}_1, \dots, \vec{v}_m)$ does not span $\mathbb{R}^n$.

[Hint for (a) and (b): write the vectors as columns in an $n \times m$ matrix A, and consider rref(A).]

(c) What does this imply about the number of vectors in any basis of $\mathbb{R}^n$?

Solution:

(a) Let $A = [\vec{v}_1 \ \cdots \ \vec{v}_m] \in \mathbb{R}^{n\times m}$. If $m > n$, then at least one column of rref(A) is not a pivot column, which means the corresponding column in A is redundant in the list $(\vec{v}_1, \dots, \vec{v}_m)$, which makes this list linearly dependent.

(b) Again let $A = [\vec{v}_1 \ \cdots \ \vec{v}_m] \in \mathbb{R}^{n\times m}$. If $m < n$, then at least one row of rref(A) does not have a leading 1 in it, which means we can find a vector $\vec{b} \in \mathbb{R}^n$ that does not belong to $\mathrm{im}(T_A) = \mathrm{Span}(\vec{v}_1, \dots, \vec{v}_m)$.

(c) Any basis of $\mathbb{R}^n$ has exactly n vectors in it.

Problem 6.

In each part below, underline the correct choice from the given options to make a true statement:

(a) Any set of vectors containing $\vec{0}$ is (<u>linearly dependent</u> / linearly independent)

(b) Any (subset / <u>superset</u>) of a linearly dependent set of vectors is linearly dependent.

(c) Any (<u>subset</u> / superset) of a linearly independent set of vectors is linearly independent.

Problem 7.

Let W be a subspace of the vector space V. Determine whether the following are *true* or *false*:

(a) Every basis of V contains a basis of W. FALSE!

(b) Every basis of W is contained in a basis of V. TRUE!