MATH 217 - LINEAR ALGEBRA
HOMEWORK 3, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:
Section 2.2: 20, 22, 38;
Section 2.3: 18, 34;
Section 2.4: 12, 30.

Solution.

2.2.20: Denote by T this transformation and

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

then

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad T(\vec{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \qquad T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By Theorem 2.1.2, the matrix of T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.2.22: Let T be the rotation about the y-axis. Then $T(e_2) = e_2$ and $T(e_1)$ and $T(e_3)$ lies in the xz-plane, i.e., second coordinate of $T(e_1)$ and $T(e_3)$ is zero. In the xz-plane T behaves like a rotation in $\mathbb{R}^2$. We conclude that

$$T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

2.2.38 a. Orthogonal projection: $\det \begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix} = 0$.

b. Reflection: $\det \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = -1$.

c. Rotation: $\det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = 1$.

d. Shear: $\det \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = 1$.

2.3.18 If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{cases} 2a - 3b = 2a + 3c \\ 3a + 2b = 2b + 3d \\ 2c - 3d = -3a + 2c \\ 3c + 2d = -3b + 2d \end{cases} \Rightarrow \begin{cases} a = s \\ b = t \\ c = -t \\ d = s, \end{cases}$$

hence the set of all matrices which commute with A is equal to $\left\{ \begin{bmatrix} s & t \\ -t & s \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$.

2.3.34 $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

By induction, $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, $A^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}$.

2.4.12 We have

$$\operatorname{rref}\left[\begin{array}{cccc|cccc} 2 & 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 \end{array}\right],$$

so the matrix is invertible with the inverse $\begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

2.4.30 Putting this into rref we have

$$\begin{pmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & -c & -bc \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}$$

and the matrix is never invertible.

Part B (25 points)

Problem 1. Suppose A is an $m \times n$ matrix, B is an $n \times m$ matrix, and $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$.

(a) Prove that if $AB = I_m$, then T is surjective.
(b) Prove that if $BA = I_n$, then T is injective.

Solution.

(a) Suppose $AB = I_m$, and let $\vec{y} \in \mathbb{R}^m$. Then $\vec{y} = (AB)\vec{y} = A(B\vec{y}) = T(B\vec{y}) \in \operatorname{im}(T)$, which shows that T is surjective.

(b) Suppose $BA = I_n$. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and suppose $T(\vec{x}) = T(\vec{y})$, so $A\vec{x} = A\vec{y}$. Then

$$\vec{x} = (BA)\vec{x} = B(A\vec{x}) = B(A\vec{y}) = (BA)\vec{y} = \vec{y}.$$

So we have shown that $\vec{x} = \vec{y}$ whenever $T(\vec{x}) = T(\vec{y})$, which proves that T is injective.

Notation: For any line L in $\mathbb{R}^2$ that passes through the origin, we write proj_L for the *orthogonal projection onto* L and ref_L for *reflection over* L, as in Section 2.2 of the textbook.

Problem 2. Let L_1 be a line in $\mathbb{R}^2$ passing through the origin, let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a unit vector parallel to L_1, and let L_2 be a line passing through the origin that is perpendicular to L_1.

(a) Find the standard matrix of $\operatorname{proj}_{L_1} \circ \operatorname{proj}_{L_2}$.

(b) Prove that for all $\vec{x} \in \mathbb{R}^2$, $\operatorname{proj}_{L_1}(\vec{x}) + \operatorname{proj}_{L_2}(\vec{x}) = \vec{x}$.

(c) Prove that $\operatorname{ref}_{L_1} \circ \operatorname{ref}_{L_2}$ is a rotation, and find the angle of rotation.

Solution. First, note that a unit vector parallel to L_2 is $\vec{u} = \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix}$.

(a) We know that the standard matrix for $\operatorname{proj}_{L_1}$ is given by $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$ and the standard matrix for $\operatorname{proj}_{L_2}$ is given by $\begin{bmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_1^2 \end{bmatrix}$. Then the matrix for $\operatorname{proj}_{L_1} \circ \operatorname{proj}_{L_2}$ is the product of these two matrices:

$$\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \begin{bmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_1^2 \end{bmatrix} = \begin{bmatrix} u_1^2u_2^2 - (u_1u_2)^2 & -u_1^3u_2 + u_1^3u_2 \\ u_1u_2^3 - u_1u_2^3 & -(u_1u_2)^2 + u_1^2u_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Geometrically, this makes sense! The first projection will send every vector onto the line L_2; since L_2 is orthogonal to L_1, the second projection will send every vector on L_2 to $\vec{0}$. Thus the composition of these two projections will send every vector to $\vec{0}$.

(b) This is equivalent to proving that the sum of the matrices for $\operatorname{proj}_{L_1}$ and $\operatorname{proj}_{L_2}$ is I_2. We can find this:

$$\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} + \begin{bmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_1^2 \end{bmatrix} = \begin{bmatrix} u_1^2 + u_2^2 & u_1u_2 - u_1u_2 \\ u_1u_2 - u_1u_2 & u_2^2 + u_1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where the last equality is true because $\vec{u}$ is a unit vector, and therefore $u_1^2 + u_2^2 = 1$.

Note that this also makes sense geometrically, as we know that $\operatorname{proj}_{L_2}(\vec{x})$ is what Bretscher calls $\vec{x}^{\perp}$ for $\operatorname{proj}_{L_1}$.

(c) We know that the matrix for ref_{L_1} is $\begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}$ and the matrix for ref_{L_2} is $\begin{bmatrix} 2u_2^2-1 & -2u_1u_2 \\ -2u_1u_2 & 2u_1^2-1 \end{bmatrix}$. Then the matrix for the composition of the reflections is given by the product

$$\begin{aligned}
&\begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}\begin{bmatrix} 2u_2^2-1 & -2u_1u_2 \\ -2u_1u_2 & 2u_1^2-1 \end{bmatrix} \\
&= \begin{bmatrix} (2u_1^2-1)(2u_2^2-1)-4u_1^2u_2^2 & -4u_1^3u_2+2u_1u_2+4u_1^3u_2-2u_1u_2 \\ 4u_1u_2^3-2u_1u_2-4u_2^3u_1+2u_1u_2 & -4u_1^2u_2^2+(2u_2^2-1)(2u_2^2-1) \end{bmatrix} \\
&= \begin{bmatrix} 4u_1^2u_2^2-2u_1^2-2u_2^2+1-4u_1^2u_2^2 & 0 \\ 0 & -4u_1^2u_2^2+4u_1^2u_2^2-2u_1^2-2u_2^2+1 \end{bmatrix} \\
&= \begin{bmatrix} -2(u_1^2+u_2^2)+1 & 0 \\ 0 & -2(u_1^2+u_2^2)+1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
\end{aligned}$$

In order to show that this is the matrix for a rotation, we need to find an angle θ such that $\cos(\theta) = -1$ and $\sin(\theta) = 0$. This is true when $\theta = \pi$. So this transformation is a rotation by π radians.

In fact, a more general result is true: composing the reflections across any two lines through the origin will always be a rotation about the origin by an angle equal to twice the angle between the two lines. You can convince yourself that this makes sense geometrically by drawing pictures. A proof using matrices is best done using the angle definition for a rotation matrix and applying the appropriate trigonometric identities.

Definition: For any square matrix A, we define the *trace* of A to be the sum of the diagonal entries of A, written $\mathrm{tr}(A)$. That is, for any $n \times n$ matrix A whose (i,j)-entry is a_{ij} for each $1 \le i, j \le n$, we define

$$\mathrm{tr}(A) \;=\; \sum_{i=1}^{n} a_{ii}.$$

Problem 3.

(a) Let $m, n \in \mathbb{N}$. Prove that for all $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$, we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. [Hint: you may use 5(e) below.]

(b) Let $n \in \mathbb{N}$. Prove by induction on k that for all $k \in \mathbb{N}$ and for all matrices $A_1, \dots A_k \in \mathbb{R}^{n\times n}$, $\mathrm{tr}(A_1 + \cdots + A_k) = \mathrm{tr}(A_k + \cdots + A_1)$. [Note that part (b) is independent of part (a), since part (a) involves multiplying matrices while part (b) involves adding them.]

Solution.

(a) We have

$$\begin{align}
\operatorname{tr}(AB) &= \sum_{i=1}^{n}(AB)_{ii} \tag{0.1}\\
&= \sum_{i=1}^{n}\left(\sum_{k=1}^{m} a_{ik}b_{ki}\right) \tag{0.2}\\
&= \sum_{k=1}^{m}\sum_{i=1}^{n} b_{ki}a_{ik} \tag{0.3}\\
&= \sum_{k=1}^{m}(BA)_{kk} \tag{0.4}\\
&= \operatorname{tr}(BA). \tag{0.5}
\end{align}$$

Line (0.1) was the definition of the trace of AB, line (0.2) was using the definition of matrix multiplication, line (0.3) was switching the order of summation, line (0.4) was rewriting the inside sum as matrix multiplication, and line (0.5) was recognizing this as the trace of BA.

(b) To help us prove this by induction, we will first prove the following Lemma, which you may or may not have proved in class:

Lemma. For any $A, B \in \mathbb{R}^{n\times n}$, $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$. Proof: $\operatorname{tr}(A + B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{tr}(A) + \operatorname{tr}(B)$, where the two summations are equal due to associativity and commutativity of addition of real numbers.

Now we use induction on $k \in \mathbb{N}$ to prove that for all matrices $A_1, \dots A_k \in \mathbb{R}^{n\times n}$,

$$\operatorname{tr}(A_1 + \cdots + A_k) = \operatorname{tr}(A_k + \cdots + A_1).$$

Base case. For $k = 1$, we have $\operatorname{tr}(A) = \operatorname{tr}(A)$.

Inductive step. Let $j \in \mathbb{N}$ be fixed, and assume for inductive hypothesis that for all $n \times n$ matrices $A_1, \dots, A_j$ we have $\operatorname{tr}(A_1 + \cdots + A_j) = \operatorname{tr}(A_j + \cdots + A_1)$. Let $A_1, \dots, A_{j+1}$ be arbitrary $n \times n$ matrices, and consider $\operatorname{tr}(A_1 + \cdots + A_j + A_{j+1})$. We have

$$\begin{align*}
\operatorname{tr}(A_1 + \cdots + A_j + A_{j+1}) &= \operatorname{tr}(A_1 + \cdots + A_j) + \operatorname{tr}(A_{j+1}) && \text{(by the lemma)}\\
&= \operatorname{tr}(A_j + \cdots + A_1) + \operatorname{tr}(A_{j+1}) && \text{(by the inductive hypothesis)}\\
&= \operatorname{tr}(A_{j+1}) + \operatorname{tr}(A_j + \cdots + A_1) && \text{(commutativity of addition)}\\
&= \operatorname{tr}(A_{j+1} + A_j + \cdots + A_1) && \text{(by the lemma)}
\end{align*}$$

Problem 4. In each part below, determine whether the given statement is true or false and justify your claim with a proof or a counterexample.

(a) If A and B are invertible $n \times n$ matrices, then $A + B$ is invertible.
(b) If A and B are invertible $n \times n$ matrices, then ABA^{-1} is invertible.
(c) If A is a diagonal[1] matrix, then A is invertible.
(d) If A and B are $n \times n$ matrices such that A and AB are invertible, then B is invertible.

[1]Recall that a *diagonal* matrix is a square matrix whose non-diagonal entries are all zero; that is, the $n \times n$ matrix A whose (i, j)-entry is a_{ij} for each $1 \le i, j \le n$ is *diagonal* if $a_{ij} = 0$ whenever $i \ne j$.

Solution.

(a) False. For example, if $A = I_n$ and $B = -I_n$, then A and B are invertible but $A + B = 0$ so $A + B$ is not invertible.

(b) True. Let A and B be invertible $n \times n$ matrices, and consider the matrix $AB^{-1}A^{-1}$. Since $AB^{-1}A^{-1}ABA^{-1} = I_n = ABA^{-1}AB^{-1}A^{-1}$, the matrix $AB^{-1}A^{-1}$ is the inverse of ABA^{-1}.

(c) False. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. This matrix is diagonal, since all of the non-diagonal entries are zero, but it is not invertible, because it has rank one.

(d) True. Let A and B be $n \times n$ matrices such that both A and AB are invertible. Let A^{-1} be the inverse of A and let C be the inverse of AB, so we have $AA^{-1} = I_n = A^{-1}A$ and $(AB)C = I_n = C(AB)$. We claim that B is invertible with inverse $B^{-1} = CA$. To prove this, we must show that $(CA)B = I_n$ and $B(CA) = I_n$. The first of these follows from associativity of matrix multiplication and the fact that $C(AB) = I_n$. In fact, this alone is sufficient to complete the proof, since 2.4.8 in the text now implies that $B(CA) = I_n$.

However, even without invoking 2.4.8, we can prove $B(CA) = I_n$ directly as follows. First, we know that $(AB)C = I_n$, which implies $A(BC) = I_n$. Multiplying each side by A^{-1} on the left gives us $BC = A^{-1}$, and then multiplying each side by A on the right gives us $(BC)A = I_n$. Using associativity one last time gives us $B(CA) = I_n$, as desired.

Problem 5 (Recreational Problem). [2] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, and write a_{ij} and b_{ij} for the (i, j)-entries of A and B, respectively.

(a) Prove that for all $1 \le j \le n$, $A\vec{e}_j$ is the jth column of A.
(b) Prove that[3] for all $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v}^\top\vec{w} = \vec{v} \cdot \vec{w}$.
(c) Prove that for all $1 \le i \le m$, $\vec{e}_i^\top A$ is the ith row of A.
(d) Prove that for all $1 \le i \le m$ and $1 \le j \le p$, $\vec{e}_i^\top A\vec{e}_j$ is the (i, j)-entry of A.
(e) Prove that for all $1 \le i \le m$ and $1 \le j \le p$, the (i, j)-entry of AB is $\sum_{k=1}^n a_{ik}b_{kj}$.
(f) Prove that $(AB)^\top = B^\top A^\top$.
(g) Prove that for all $1 \le i \le m$, the ith row of AB is $\alpha_i B$ where α_i is the ith row of A.
(h) Prove that $AB = \sum_{k=1}^n (A\vec{e}_k)(\vec{e}_k^\top B)$.

Solution.

(a) This follows immediately from our definition of *matrix-vector products*, where we defined $A\vec{x}$ to be the linear combination of the columns of A using the entries in $\vec{x}$ as weights.

(For more details, if A has columns $\vec{a}_1, \ldots, \vec{a}_n$, then for each $1 \le j \le n$ we have

$$A\vec{e}_j = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \vec{e}_j = \sum_{i=1}^n \delta_{ij}\vec{a}_j = \vec{a}_j,$$

[2] "Recreational Problems" may come up from time to time and exist for your amusement and edification, but they are *optional* and will not be graded. Typically they can be safely ignored unless you are looking for an extra challenge; however, this particular one has some useful facts about matrix multiplication that you can (and will likely need) to use often in the future, so we recommend that you read and make sure you understand each part.

[3] Note that technically $\vec{v}^\top\vec{w}$ is a 1×1 matrix and $\vec{v} \cdot \vec{w}$ is a scalar, so this equation is only true if you are willing to treat 1×1 matrices and scalars as the same thing. Hopefully this does not bother you too much, but if it does, don't worry: we will be able to explain the relationship between $\mathbb{R}$ and $\mathbb{R}^{1\times1}$ more precisely in a few weeks!

where δ_{ij} is defined to be 1 if $i = j$ and 0 otherwise. This function δ_{ij} is called the "Kronicker delta" and we will meet it again in Ch 5.)

(b) Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, where as usual we view $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ as $n \times 1$ column vectors.

Then by the same definition of *matrix-vector product* that we used in part (a), we have

$$\vec{v}^\top \vec{w} \;=\; \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \;=\; \sum_{i=1}^{n} w_i v_i \;=\; \vec{v} \cdot \vec{w}.$$

(c) Write $\vec{a}_1, \ldots, \vec{a}_n$ for the columns of A. Then by our definition of matrix multiplication and part (b), for each $1 \le i \le m$ we have

$$\vec{e}_i^\top A \;=\; \begin{bmatrix} \vec{e}_i^\top \vec{a}_1 & \cdots & \vec{e}_i^\top \vec{a}_n \end{bmatrix} \;=\; \begin{bmatrix} \vec{e}_i \cdot \vec{a}_1 & \cdots & \vec{e}_i \cdot \vec{a}_n \end{bmatrix}.$$

Since $\vec{e}_i \cdot \vec{a}_i$ is the ith entry in $\vec{a}_j$, which is the (i, j)-entry of A, we see that $\vec{e}_i^\top A$ is indeed the ith row of A (as a row vector).

(d) By parts (a) and (c) and the associativity of matrix multiplication, $\vec{e}_i^\top A \vec{e}_j$ is the ith entry of the jth column of A, or equivalently the jth entry of the ith row of A; either way, it is the (i, j)-entry of A.

(e) Let $1 \le i \le m$ and $1 \le j \le p$, and write $\vec{b}_j$ for the jth column of B. By our definition of matrix multiplication, the jth column of AB is $A\vec{b}_j$. So the (i, j)-entry of AB, which is just the ith entry of $A\vec{b}_j$, will be $\vec{e}_i^\top A\vec{b}_j$ by part (c). But by parts (b) and (c), $\vec{e}_i^\top A\vec{b}_j$ is the the dot product of the ith row of A with the jth column of B; that is, it is $\sum_{k=1}^n a_{ik} b_{kj}$.

(f) First note that $(AB)^\top$ and $B^\top A^\top$ are both $m \times p$ matrices. Next, let $1 \le i \le m$ and $1 \le j \le n$. Then the (i, j)-entry of $(AB)^\top$ is the (j, i)-entry of AB, which is $\sum_{k=1}^n a_{jk} b_{ki}$ by part (e), and, again by part (e) (and the definition of transpose), the (i, j)-entry of $B^\top A^\top$ is $\sum_{k=1}^n b_{ki} a_{jk}$. Since these sums are equal, we see that $(AB)^\top$ and $B^\top A^\top$ have the same (i, j)-entry for all $1 \le i \le m$ and $1 \le j \le n$, which means they are the same matrix.

(g) Using part (c), the ith row of AB is $\vec{e}_i^\top (AB) = (\vec{e}_i^\top A)B = \alpha_i B$, where α_i is the ith row of A (as a row vector).

(h) For all $1 \le i \le m$ and $1 \le j \le n$, the (i, j)-entry of $\sum_{k=1}^n (A\vec{e}_k)(\vec{e}_k^\top B)$ is

$$\vec{e}_i^\top \left(\sum_{k=1}^{n} (A\vec{e}_k)(\vec{e}_k^\top B) \right) \vec{e}_j \;=\; \sum_{k=1}^{n} \vec{e}_i^\top A \vec{e}_k \vec{e}_k^\top B \vec{e}_j \;=\; \sum_{k=1}^{n} a_{ik} b_{kj}$$

by part (d), so the claim follows from part (e).