Math 217 – Midterm 2

Fall 2019

Time: 120 mins.

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space, if you do this.*** If you need additional blank paper, ask an instructor. You may not use any paper not provided with this exam.
2. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
3. No calculators, notes, or other outside assistance allowed.
4. On this exam, *unless stated otherwise*, terms such as "orthogonal" and "orthonormal" as applied to vectors in $\mathbb{R}^n$ mean with respect to the usual dot product on $\mathbb{R}^n$.

Student ID Number: ______________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	12	
4	17	
5	11	
6	13	
7	10	
8	10	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The $n \times n$ matrix A is *orthogonal*

   (b) The vector $\vec{x}^*$ is a *least-squares solution* of the linear system $A\vec{x} = \vec{b}$

   (c) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the inner product space V is *orthonormal* relative to the inner product $\langle \cdot, \cdot \rangle$ on V

   (d) The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) There exists an orthonormal list of 3 vectors in $\mathbb{R}^3$ that does not span $\mathbb{R}^3$.

   (b) (3 points) For every inner product space V (with inner product $\langle \cdot, \cdot \rangle$) and for all vectors $\vec{x}, \vec{y} \in V$, if $\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2$ then $\vec{x}$ and $\vec{y}$ are orthogonal.

   (c) (3 points) For all vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ and matrices $A, B \in \mathbb{R}^{n \times n}$, if $A^\top B = I_n$ then $A\vec{x} \cdot B\vec{y} = \vec{x} \cdot \vec{y}$.

(Problem 2, Continued).

(d) (3 points) For every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and for every pair of ordered bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^n$, if $[T]_{\mathcal{B}} = [T]_{\mathcal{C}}$ then $\mathcal{B} = \mathcal{C}$.

(e) (3 points) For every symmetric matrix A, $\ker(A^2) = \mathrm{im}(A)^{\perp}$.

3. Let V be the subspace of $\mathbb{R}^3$ consisting of solutions of the linear equation $x+2y-z=0$, and let $\mathcal{B}$ be the orthonormal basis of V given by

$$\mathcal{B} \;=\; (\vec{b}_1, \vec{b}_2) \;=\; \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

(a) (3 points) Find the $\mathcal{B}$-coordinate vector of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in V. *(No justification required).*

(b) (3 points) Find a vector $\vec{v} \in V$ such that $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *(No justification required).*

(c) (3 points) Let $L_{\mathcal{B}} : V \to \mathbb{R}^2$ be the coordinate isomorphism defined by $L_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$ for each $\vec{v} \in V$, and define the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = L_{\mathcal{B}}^{-1}(\vec{x})$ for each $\vec{x} \in \mathbb{R}^2$. Find the standard matrix of T.

(d) (3 points) Find a vector $\vec{b}_3 \in \mathbb{R}^3$ such that the matrix $B = \begin{bmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{bmatrix}$ is orthogonal, or else explain why this is impossible.

4. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable x.

   (a) (6 points) In each of (i) – (iii) below, determine whether the given rule defines an inner product on $\mathcal{P}_2$. Write YES or NO. *(No justification necessary.)*

   (i) for all $p, q \in \mathcal{P}_2$, $\langle p, q\rangle = p'q' \in \mathcal{P}_2$ ____________________

   (ii) for all $p, q \in \mathcal{P}_2$, $\langle p, q\rangle = p(1)q(1)$ ____________________

   (iii) for all $p, q \in \mathcal{P}_2$, $\langle p, q\rangle = \int_0^1 \big[p(x)q(x)\big]^2 dx$ ____________________

   For the rest of this problem, let p, q, and r be the polynomials in $\mathcal{P}_2$ defined for all $x \in \mathbb{R}$ by the rules $p(x) = 1$, $q(x) = x$, and $r(x) = x^2$. Also let $\langle\cdot,\cdot\rangle$ be the inner product on $\mathcal{P}_2$ given by

$$\langle f, g\rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

   for all $f, g \in \mathcal{P}_2$. (You do not have to prove that this is an inner product.)

   (b) (3 points) Fill in the missing values in the following table of inner products, where, for instance, the inner product $\langle q, p\rangle = \frac{1}{2}$ is one of those given.

	p	q	r
p	1	$\mathbf{\frac{1}{2}}$	$\frac{1}{3}$
q	$\frac{1}{2}$	$\frac{1}{3}$	$\mathbf{\frac{1}{4}}$
r	$\mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{4}}$	$\mathbf{\frac{1}{5}}$

   (c) (2 points) Find an orthonormal basis of Span(q).

   (d) (3 points) Find an orthonormal basis of Span(p, q).

   (e) (3 points) Find the vector in Span(q) that is closest to r.

5. Let $a, b, c \in \mathbb{R}$, and suppose that

$$\mathcal{B} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ b \end{bmatrix} \right)$$

is an ordered basis of $\mathbb{R}^3$.

(a) (3 points) Find the change-of-coordinates matrix $S_{\mathcal{B}\to\mathcal{E}}$, where $\mathcal{E} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is the standard basis of $\mathbb{R}^3$. (Your answer may depend on a, b, and c.)

(b) (4 points) Suppose $c = 0$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection over the xy-plane. Find the $\mathcal{B}$-matrix $[T]_\mathcal{B}$ of T. (Your answer may depend on a and b.)

(c) (4 points) Suppose $a = 1$, and let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be orthogonal projection onto the plane in $\mathbb{R}^3$ defined by the equation $x + py + qz = 0$. Find values of b, c, p, q such that $[P]_\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Circle your answers.

6. Let $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \in \mathbb{R}^{4\times 3}$ and $B = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \in \mathbb{R}^{4\times 2}$, where $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a linearly independent list of vectors in $\mathbb{R}^4$. Suppose A has QR-factorization

$$A = \underbrace{\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}}_{R} \in \mathbb{R}^{4\times 3}.$$

(a) (3 points) Compute $A^\top A$.

(b) (3 points) Compute $\vec{v}_1 \cdot (\vec{v}_2 + 2\vec{v}_3)$.

(c) (3 points) Find a vector $\vec{x} \in \mathbb{R}^2$ such that $B\vec{x} = \vec{u}_2$.

(d) (4 points) Find a least-squares solution $\vec{x}^*$ of the linear system $B\vec{x} = \vec{v}_3$.

7. Let $n \in \mathbb{N}$, and let $\mathcal{B} = (\vec{u}_1, \ldots, \vec{u}_n)$ be an orthonormal basis of $\mathbb{R}^n$.

   (a) (4 points) Prove that for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, $[\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}} = \vec{x} \cdot \vec{y}$.

   (b) (6 points) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, and let k be an integer such that $1 \leq k < n$. Let $V = \text{Span}(\vec{u}_1, \ldots, \vec{u}_k)$, and let $W = \text{Span}(\vec{u}_{k+1}, \ldots, \vec{u}_n)$. Prove that if $\ker(A) = V$ then $\text{im}(A) = W$.

8. Let $n \in \mathbb{N}$, let V be an n-dimensional inner product space with inner product $\langle \cdot\,, \cdot \rangle$, let $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ be an ordered basis of V, and let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ be a list of vectors in V. Let S be the $n \times n$ matrix whose jth column is $[\vec{b}_j]_{\mathcal{U}}$.

   (a) (5 points) Prove that if S is invertible, then $\mathcal{B}$ is a basis of V.

   (b) (5 points) Prove that if $\mathcal{U}$ is an orthonormal basis of V and S is orthogonal, then $\mathcal{B}$ is an orthonormal basis of V.

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