8.1 Prior, Likelihood and Posterior for Two Proportions

Zongyi Liu

2023-06-17

This chapter addresses the question of how to make inferences regarding two independent binomial proportions.

8.1 Prior, Likelihood and Posterior for Two Proportions

We are considering situations in which there are two underlying proportions, namely θ_1 and θ_2 for the two groups. We are trying to determine what we should believe about these proportions after we have observed some data from the two groups.

Our prior beliefs are about combinations of paramter values, which can be expressed as $p(\theta_1, \theta_2)$, and they must follow the rule $\int \int d\theta_1 \theta_2 p(\theta_1, \theta_2) = 1$. In some of applications in this chapter, we would assume that our beliefs about θ_2 is uninfluenced by our belief about θ_1; this kind of independent relationship can also be expressed as $p(\theta_1, \theta_2) = p(\theta_1)p(\theta_2)$ for every value of θ_1 and θ_2.

Along with the prior beliefs, we have some observed data. We assume that the flips within a group are independent of each other, and that the flips across groups are independent of each other.

Independence of the data from the two coins means that $p(y_1|\theta_1, \theta_2) = p(y_1|\theta_1)$. We can then denote the probability of D just as follows:

$$
\begin{aligned}
p(D|\theta_1, \theta_2) &= \prod_{y_{1i} \in D_1} p(y_{1i}|\theta_1, \theta_2) \prod_{y_j \in D_2} p(y_j|\theta_1, \theta_2) \\
&= \prod_{y_{1i} \in D_1} \theta_1^{y_{1i}} (1 - \theta_1)^{(1 - y_{1i})} \prod_{y_{2j} \in D_2} \theta_2^{y_{2j}} (1 - \theta_2)^{(1 - y_{2j})} \\
&= \theta_1^{z_1} (1 - \theta_1)^{(N_1 - z_1)} \, \theta_2^{z_2} (1 - \theta_2)^{(N_2 - z_2)}
\end{aligned}
$$

The posterior distribution of our beliefs about the underlying proportions is derived in the usual way by applying Bayes' rule, but now the functions involve two parameters:

$$
\begin{aligned}
p(\theta_1, \theta_2|D) &= p(D|\theta_1, \theta_2) p(\theta_1, \theta_2) / p(D) \\
&= p(D|\theta_1, \theta_2) p(\theta_1, \theta_2) \Big/ \iint d\theta_1 d\theta_2 p(D|\theta_1, \theta_2) p(\theta_1, \theta_2)
\end{aligned}
$$