7.8 Lab: Non-linear Modeling

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7.8 Lab: Non-linear Modeling

7.8.1 Polynomial Regression and Step Functions

```
library(ISLR)
attach(Wage)
```

First to have the model fitted:

```
fit=lm(wage~poly(age,4),data=Wage)
coef(summary(fit))
```

```
##                  Estimate Std. Error    t value     Pr(>|t|)
## (Intercept)    111.70361  0.7287409 153.283015 0.000000e+00
## poly(age, 4)1  447.06785 39.9147851  11.200558 1.484604e-28
## poly(age, 4)2 -478.31581 39.9147851 -11.983424 2.355831e-32
## poly(age, 4)3  125.52169 39.9147851   3.144742 1.678622e-03
## poly(age, 4)4  -77.91118 39.9147851  -1.951938 5.103865e-02
```

To obtain `age` and its timed terms, we can use `raw=TRUE` or write them down:

```
fit2=lm(wage~poly(age,4,raw=T),data=Wage)
coef(summary(fit2))
```

```
##                             Estimate   Std. Error   t value     Pr(>|t|)
## (Intercept)            -1.841542e+02 6.004038e+01 -3.067172 0.0021802539
## poly(age, 4, raw = T)1  2.124552e+01 5.886748e+00  3.609042 0.0003123618
## poly(age, 4, raw = T)2 -5.638593e-01 2.061083e-01 -2.735743 0.0062606446
## poly(age, 4, raw = T)3  6.810688e-03 3.065931e-03  2.221409 0.0263977518
## poly(age, 4, raw = T)4 -3.203830e-05 1.641359e-05 -1.951938 0.0510386498
```

```
fit2a=lm(wage~age+I(age^2)+I(age^3)+I(age^4),data=Wage)
coef(fit2a)
```

```
##   (Intercept)           age      I(age^2)      I(age^3)      I(age^4)
## -1.841542e+02  2.124552e+01 -5.638593e-01  6.810688e-03 -3.203830e-05
```

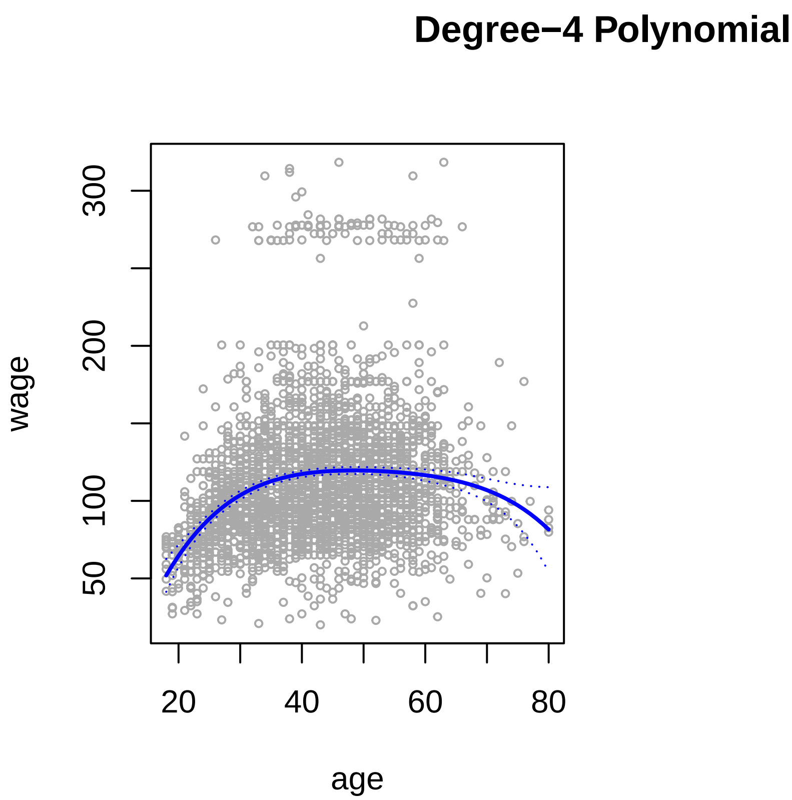
```
fit2b=lm(wage~cbind(age,age^2,age^3,age^4),data=Wage)
```

Then we can use the `predict()` function to see the predictions:

```
agelims=range(age)
age.grid=seq(from=agelims[1],to=agelims[2])
preds=predict(fit,newdata=list(age=age.grid),se=TRUE)
se.bands=cbind(preds$fit+2*preds$se.fit,preds$fit-2*preds$se.fit)
```

To plot the graphs:

```
par(mfrow=c(1,2),mar=c(4.5,4.5,1,1),oma=c(0,0,4,0))
plot(age,wage,xlim=agelims,cex=.5,col="darkgrey")
title("Degree-4 Polynomial",outer=T)
lines(age.grid,preds$fit,lwd=2,col="blue")
matlines(age.grid,se.bands,lwd=1,col="blue",lty=3)
```



To have a good model, we must know the degree of the polynomial to use, and one way to do this is by hypothesis testing with the anova table:

```
fit.1=lm(wage~age,data=Wage)
fit.2=lm(wage~poly(age,2),data=Wage)
fit.3=lm(wage~poly(age,3),data=Wage)
fit.4=lm(wage~poly(age,4),data=Wage)
fit.5=lm(wage~poly(age,5),data=Wage)
anova(fit.1,fit.2,fit.3,fit.4,fit.5)
```

```
## Analysis of Variance Table
##
## Model 1: wage ~ age
## Model 2: wage ~ poly(age, 2)
## Model 3: wage ~ poly(age, 3)
## Model 4: wage ~ poly(age, 4)
## Model 5: wage ~ poly(age, 5)
##   Res.Df     RSS Df Sum of Sq        F    Pr(>F)
## 1   2998 5022216
## 2   2997 4793430  1    228786 143.5931 < 2.2e-16 ***
## 3   2996 4777674  1     15756   9.8888  0.001679 **
## 4   2995 4771604  1      6070   3.8098  0.051046 .
## 5   2994 4770322  1      1283   0.8050  0.369682
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

By the p-values, we can know that the cubic or a quartic polynomial appear to provide a reasonable fit to the data, and Model 1, 2, 5 are not as good as they are.

```
coef(summary(fit.5))
```

```
##                  Estimate Std. Error     t value     Pr(>|t|)
## (Intercept)    111.70361  0.7287647 153.2780243 0.000000e+00
## poly(age, 5)1  447.06785 39.9160847  11.2001930 1.491111e-28
## poly(age, 5)2 -478.31581 39.9160847 -11.9830341 2.367734e-32
## poly(age, 5)3  125.52169 39.9160847   3.1446392 1.679213e-03
## poly(age, 5)4  -77.91118 39.9160847  -1.9518743 5.104623e-02
## poly(age, 5)5  -35.81289 39.9160847  -0.8972045 3.696820e-01
```

However, the ANOVA method works whether or not we used orthogonal polynomials; it also works when we have other terms in the model as well. For example, we can use `anova()` to compare these three models:

```
fit.1=lm(wage~education+age,data=Wage)
fit.2=lm(wage~education+poly(age,2),data=Wage)
fit.3=lm(wage~education+poly(age,3),data=Wage)
anova(fit.1,fit.2,fit.3)
```

```
## Analysis of Variance Table
##
## Model 1: wage ~ education + age
## Model 2: wage ~ education + poly(age, 2)
## Model 3: wage ~ education + poly(age, 3)
##   Res.Df     RSS Df Sum of Sq        F Pr(>F)
## 1   2994 3867992
## 2   2993 3725395  1    142597 114.6969 <2e-16 ***
## 3   2992 3719809  1      5587   4.4936 0.0341 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Then we will try to predict if the individual earning is more than 250k per year.

```
fit=glm(I(wage>250)~poly(age,4),data=Wage,family=binomial)
preds=predict(fit,newdata=list(age=age.grid),se=T)
```

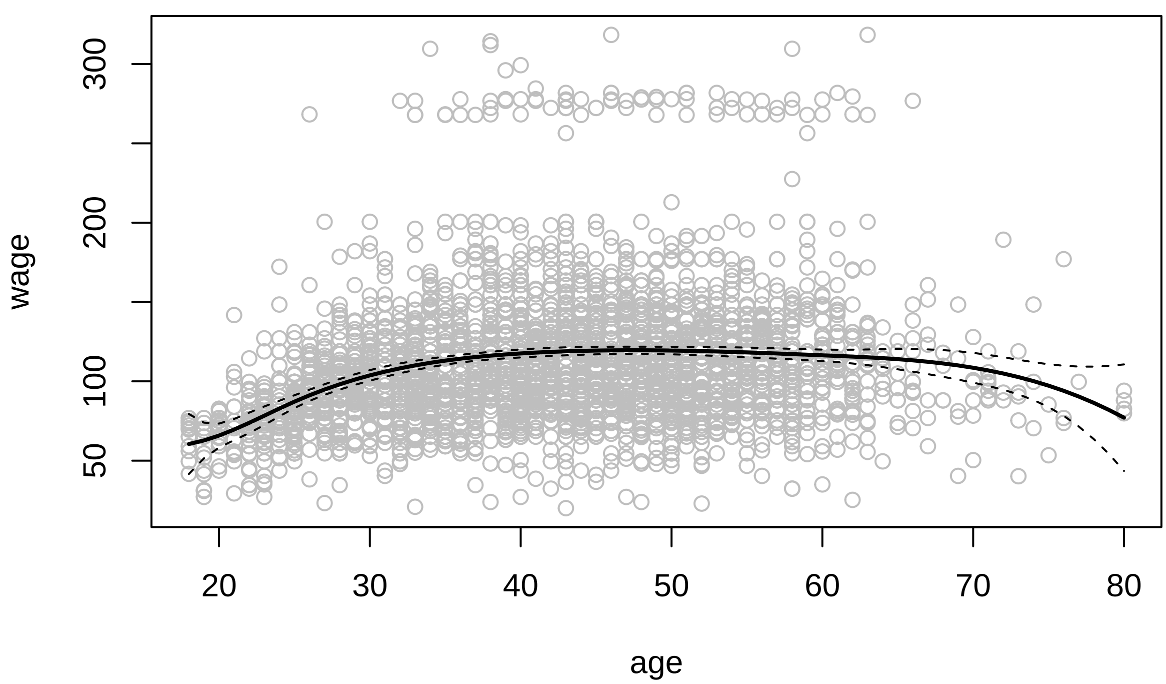
```
pfit=exp(preds$fit)/(1+exp(preds$fit))
se.bands.logit = cbind(preds$fit+2*preds$se.fit, preds$fit-2*preds$se.fit)
se.bands = exp(se.bands.logit)/(1+exp(se.bands.logit))
```

7.8.2 Splines

In order to fit regression splines, we need to use the `spline` library.

We will fit `wage` to `age` using a regression spline as below:

```
library(splines)
fit=lm(wage~bs(age,knots=c(25,40,60)),data=Wage)
pred=predict(fit,newdata=list(age=age.grid),se=T)
plot(age,wage,col="gray")
lines(age.grid,pred$fit,lwd=2)
lines(age.grid,pred$fit+2*pred$se,lty="dashed")
lines(age.grid,pred$fit-2*pred$se,lty="dashed")
```



Then we pre-specified knots at ages 25, 40, and 60, which would produce a spline with six basis functions.

```
dim(bs(age,knots=c(25,40,60)))
```

```
## [1] 3000    6
```

```
dim(bs(age,df=6))
```

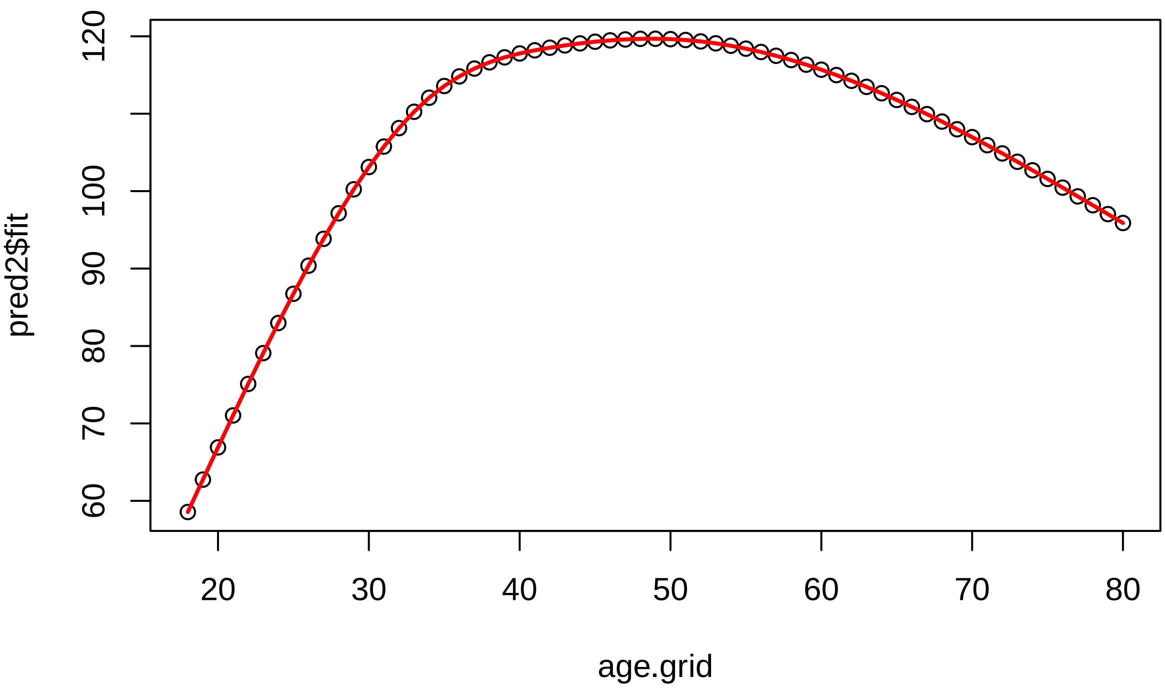
```
## [1] 3000    6
```

```r
attr(bs(age,df=6),"knots")
```

```
## [1] 33.75 42.00 51.00
```

In order to fit a natural spline, we would use the `ns()` function. Here we fit a natural spline with four degrees of freedom.

```r
fit2=lm(wage~ns(age,df=4),data=Wage)
pred2=predict(fit2,newdata=list(age=age.grid),se=T)
plot(age.grid, pred2$fit)#Not Sure
lines(age.grid, pred2$fit,col="red",lwd=2)
```



With the `bs()` function, we could instead specify the knots directly using the `knots` option.

Also, to fit a smoothing spline, we can use the `smooth.spline()` function.

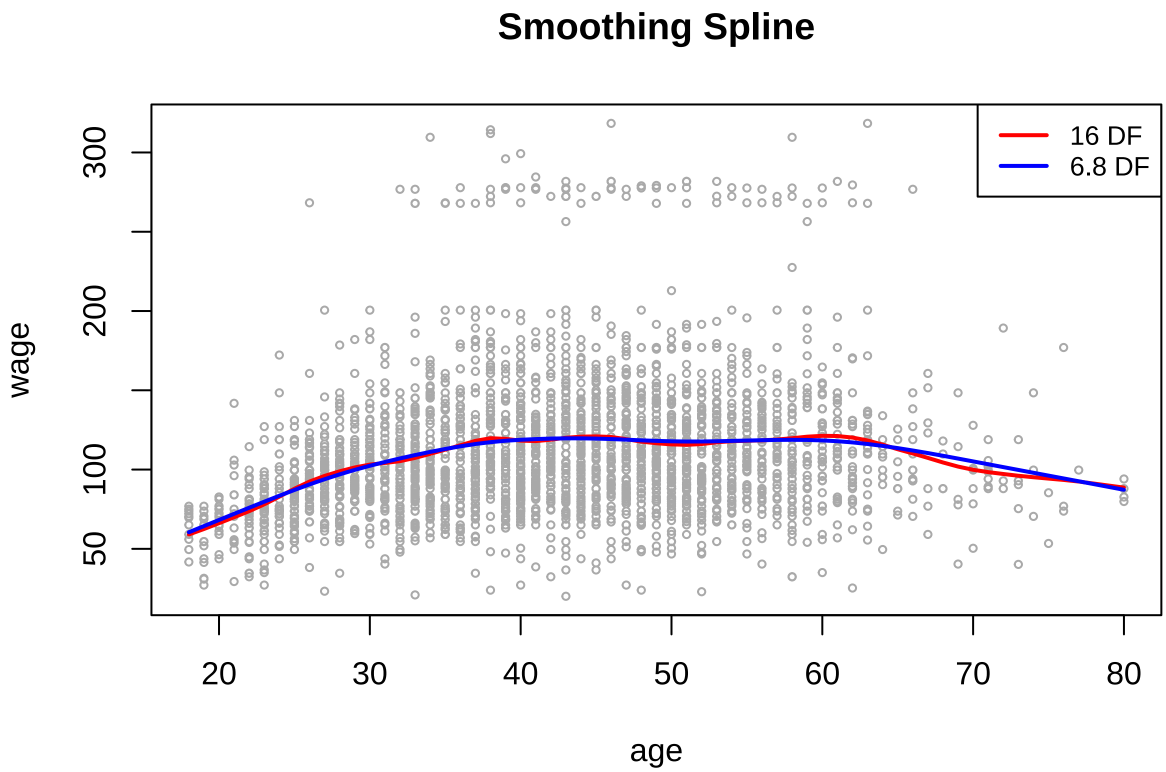
```r
plot(age, wage, xlim=agelims, cex=.5, col="darkgrey")
title (" Smoothing Spline ")
fit=smooth.spline(age,wage,df=16)
fit2=smooth.spline(age,wage,cv=TRUE)
```

```
## Warning in smooth.spline(age, wage, cv = TRUE): cross-validation with
## non-unique 'x' values seems doubtful
```

```r
fit2$df
```

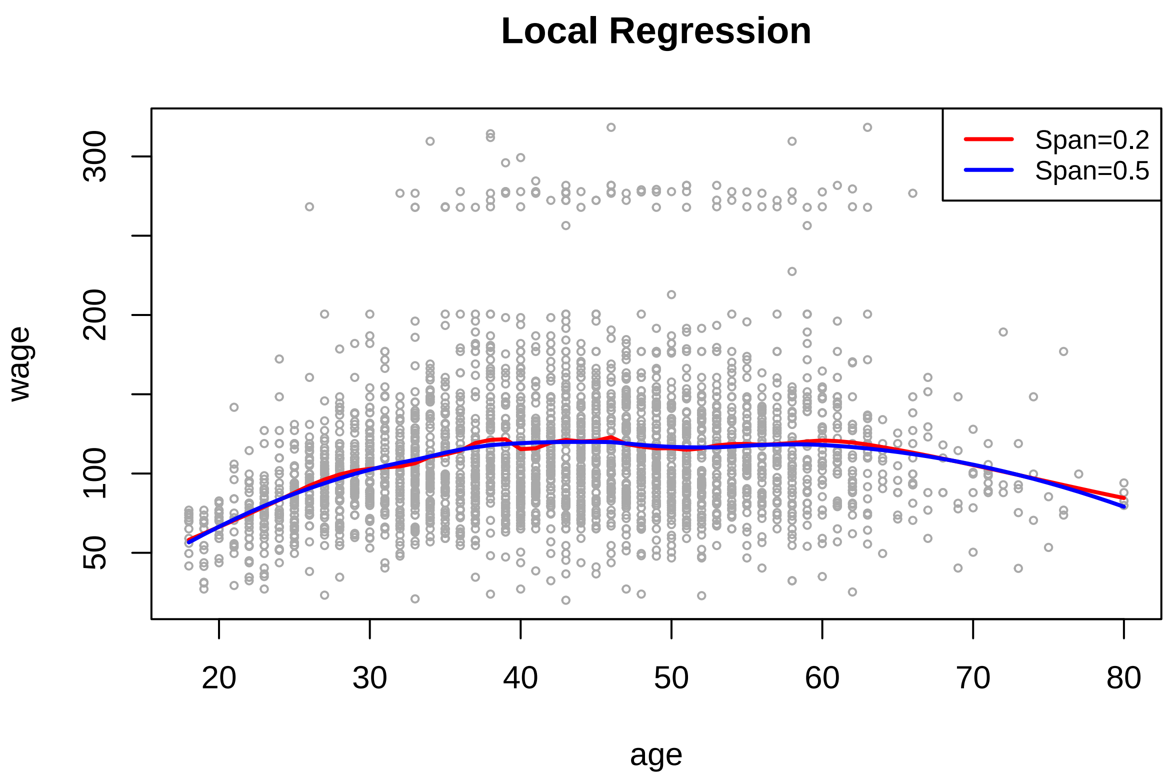
```
## [1] 6.794596
```

```
lines(fit,col="red",lwd=2)
lines(fit2,col="blue",lwd=2)
legend("topright",legend=c("16 DF","6.8 DF"), col=c("red","blue"),lty=1,lwd=2,cex=.8)
```



We can perform a local regression using the `loess()` function

```
plot(age,wage,xlim=agelims ,cex=.5,col="darkgrey")
title (" Local Regression ")
fit=loess(wage~age,span=.2,data=Wage)
fit2=loess(wage~age,span=.5,data=Wage)
lines(age.grid,predict(fit,data.frame(age=age.grid)), col="red",lwd=2)
lines(age.grid,predict(fit2,data.frame(age=age.grid)), col="blue",lwd=2)
legend("topright",legend=c("Span=0.2","Span=0.5"), col=c("red","blue"),lty=1,lwd=2,cex=.8)
```



Here we have performed local linear regression using spans of 0.2 and 0.5: that is, each neighborhood consists of 20% or 50% of the observations. The larger the span, the smoother the fit.

7.8.3 GAMs

Now we will fit a GAM, using `lm()` function.

```
gam1=lm(wage~ns(year,4)+ns(age,5)+education,data=Wage)
```

We now fit the model using smoothing splines rather than natural splines

```
library(gam)
```

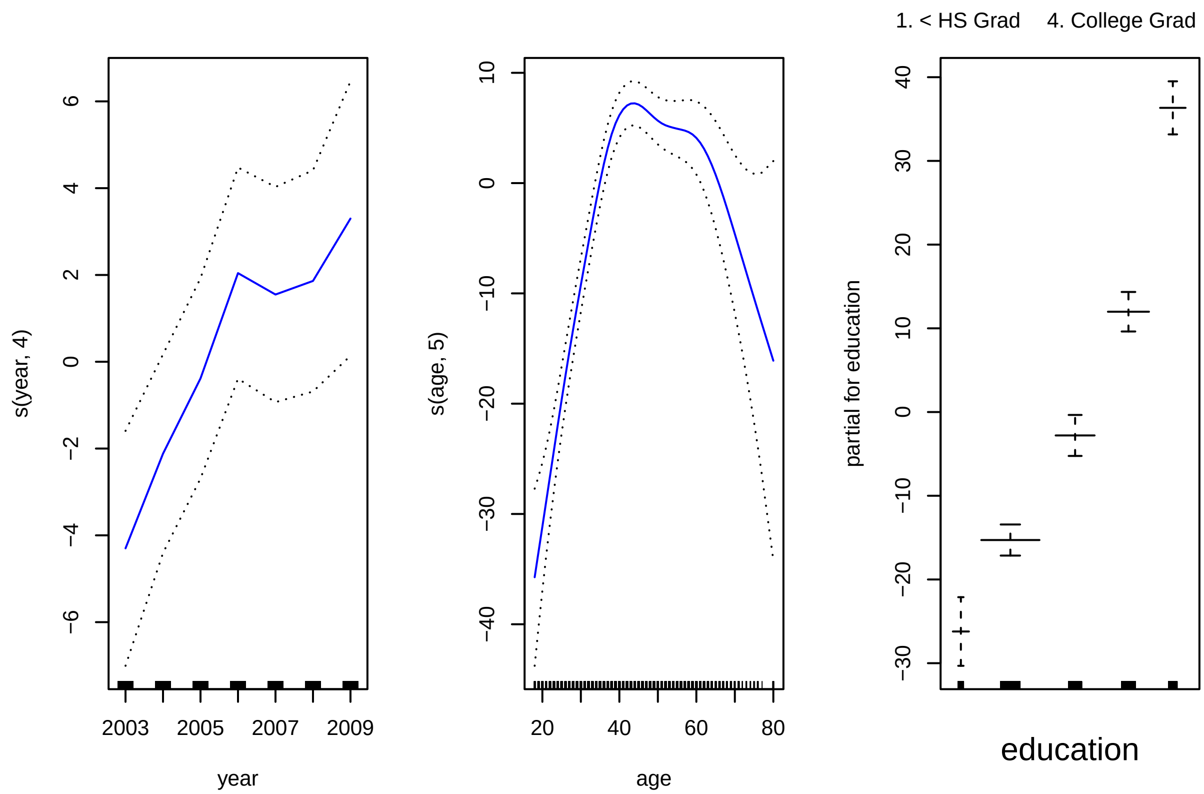
```
## Loading required package: foreach
```

```
## Loaded gam 1.22-2
```

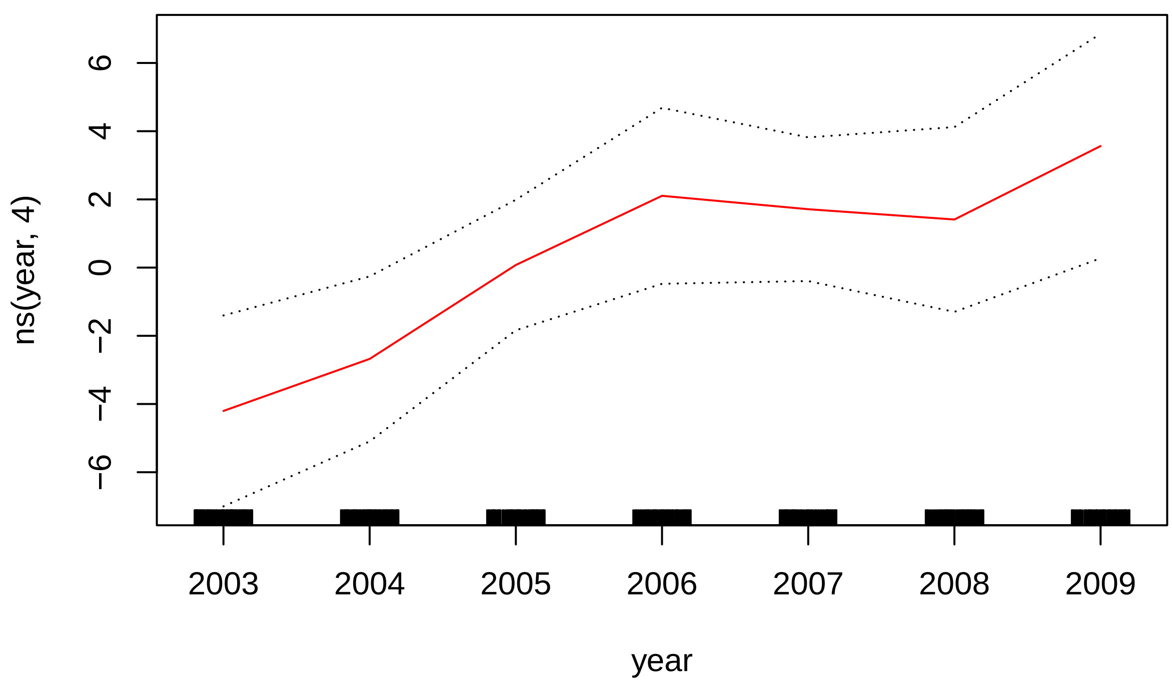
```
gam.m3=gam(wage~s(year,4)+s(age,5)+education,data=Wage)
```

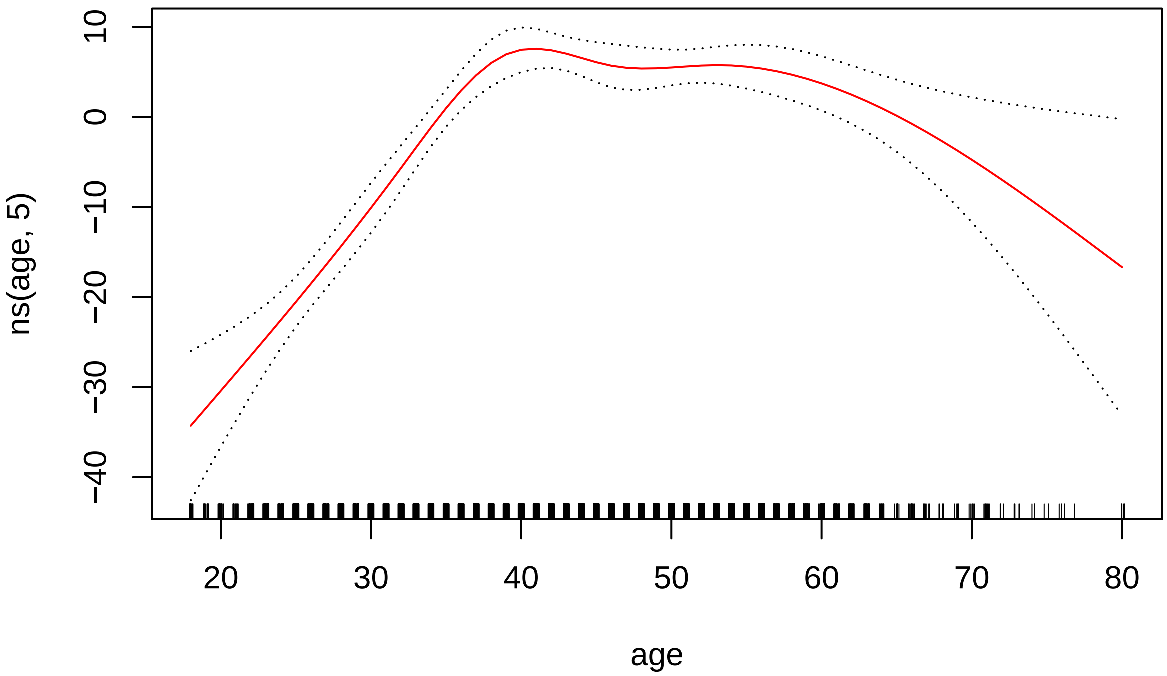
In order to produce the figure, we would use the `plot()` function:

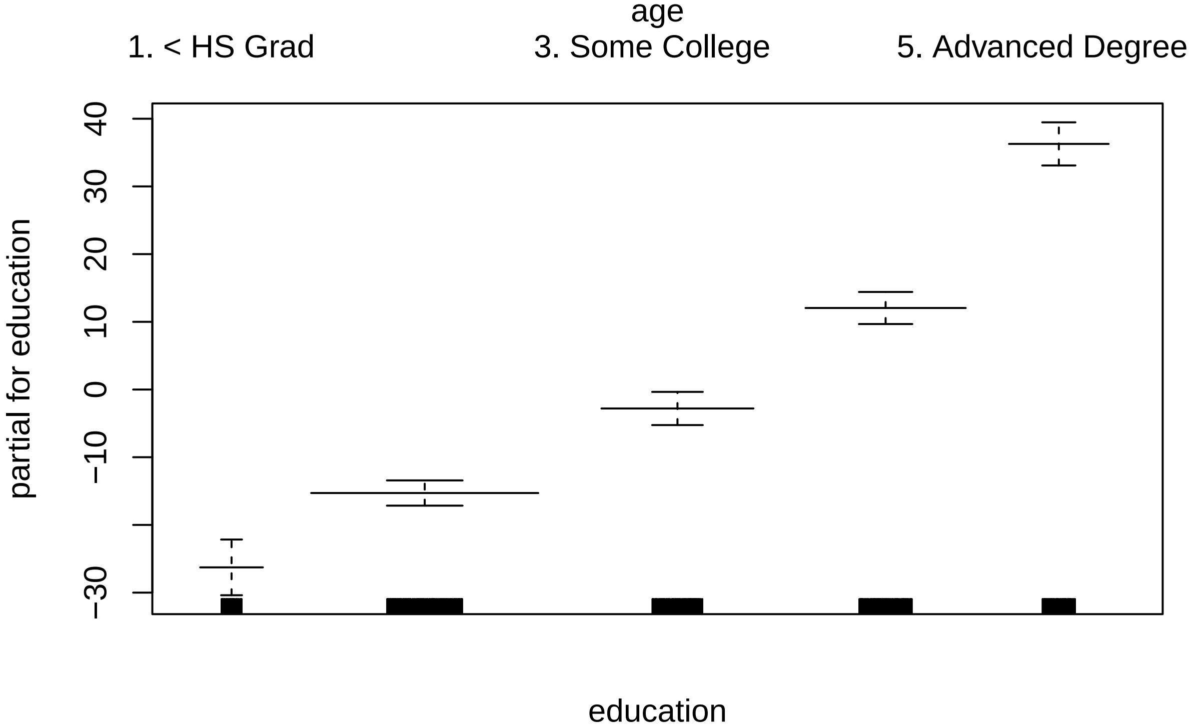
```
par(mfrow=c(1,3))
plot(gam.m3, se=TRUE,col="blue")
```




```
plot.Gam(gam1, se=T, col="red")
```








```
gam.m1=gam(wage~s(age,5)+education, data=Wage)
gam.m2=gam(wage~year+s(age,5)+education, data=Wage)
anova(gam.m1,gam.m2,gam.m3,test="F")
```

```
## Analysis of Deviance Table
##
## Model 1: wage ~ s(age, 5) + education
## Model 2: wage ~ year + s(age, 5) + education
## Model 3: wage ~ s(year, 4) + s(age, 5) + education
##   Resid. Df Resid. Dev Df Deviance       F    Pr(>F)
```

```
## 1      2990    3711731
## 2      2989    3693842  1  17889.2 14.4771 0.0001447 ***
## 3      2986    3689770  3   4071.1  1.0982 0.3485661
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We find that there is compelling evidence that a GAM with a linear function of `year` is better than a GAM that doesn't include `year` at all.

However, there is no evidence that a non-linear function of year is needed (p-value = 0.349). In other words, based on the results of this ANOVA, M_2 is preferred.

Then we can get a summary of the gam fit

```
summary(gam.m3)
```

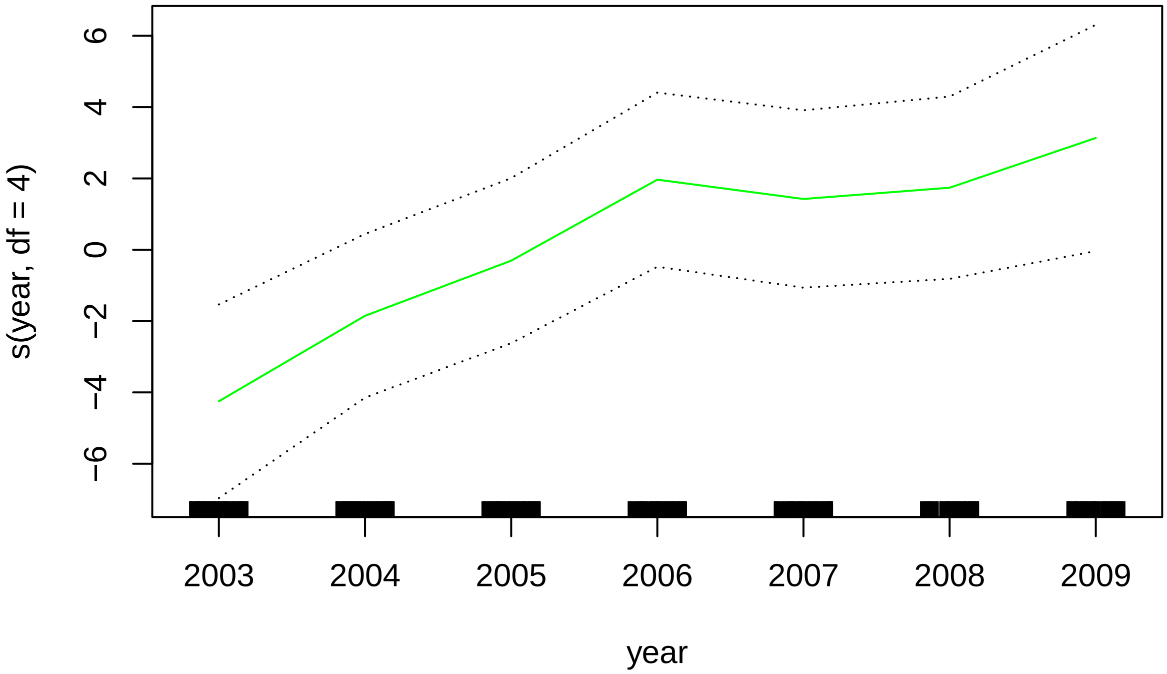
```
##
## Call: gam(formula = wage ~ s(year, 4) + s(age, 5) + education, data = Wage)
## Deviance Residuals:
##     Min      1Q  Median      3Q     Max
## -119.43  -19.70   -3.33   14.17  213.48
##
## (Dispersion Parameter for gaussian family taken to be 1235.69)
##
##     Null Deviance: 5222086 on 2999 degrees of freedom
## Residual Deviance: 3689770 on 2986 degrees of freedom
## AIC: 29887.75
##
## Number of Local Scoring Iterations: NA
##
## Anova for Parametric Effects
##              Df  Sum Sq Mean Sq F value    Pr(>F)
## s(year, 4)    1   27162   27162  21.981 2.877e-06 ***
## s(age, 5)     1  195338  195338 158.081 < 2.2e-16 ***
## education     4 1069726  267432 216.423 < 2.2e-16 ***
## Residuals  2986 3689770    1236
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Anova for Nonparametric Effects
##             Npar Df Npar F  Pr(F)
## (Intercept)
## s(year, 4)        3  1.086 0.3537
## s(age, 5)         4 32.380 <2e-16 ***
## education
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

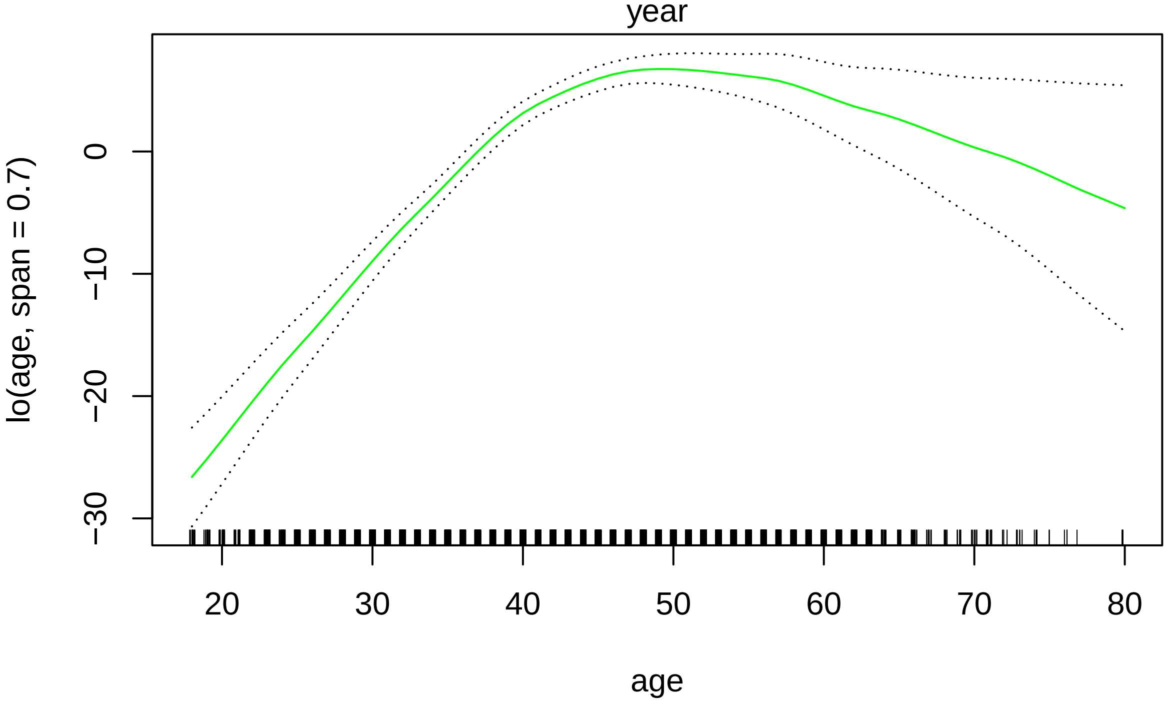
We can also use the `predict()` function for the class `gam`. Here we can make predictions on the training set

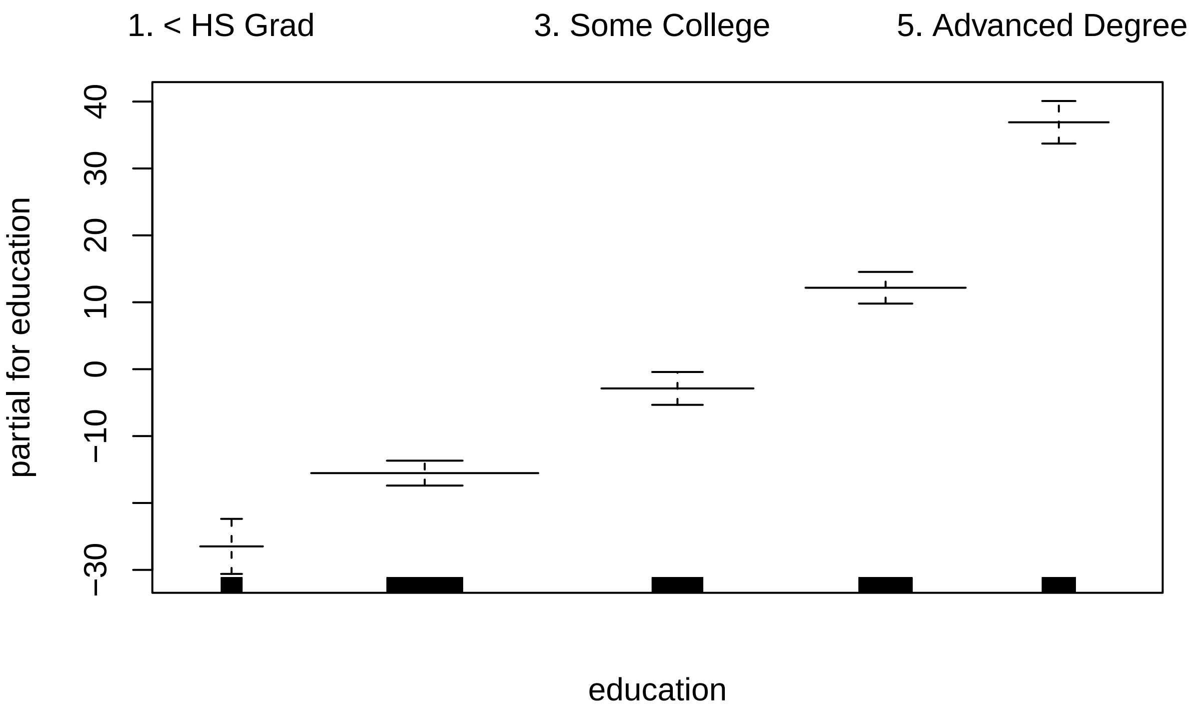
```
preds=predict(gam.m2, newdata=Wage)
```

We can also use local regression fits as building blocks in a GAM, using the `lo()` function

```r
gam.lo=gam(wage~s(year,df=4)+lo(age,span=0.7)+education, data=Wage)
plot.Gam(gam.lo, se=TRUE, col="green")
```







We can also use the `lo()` to create interactions before calling the `gam()` function.

```
gam.lo.i=gam(wage~lo(year,age,span=0.5)+education,data=Wage)
```

```
## Warning in lo.wam(x, z, wz, fit$smooth, which, fit$smooth.frame, bf.maxit, :
## liv too small.  (Discovered by lowesd)

## Warning in lo.wam(x, z, wz, fit$smooth, which, fit$smooth.frame, bf.maxit, : lv
## too small.  (Discovered by lowesd)

## Warning in lo.wam(x, z, wz, fit$smooth, which, fit$smooth.frame, bf.maxit, :
## liv too small.  (Discovered by lowesd)

## Warning in lo.wam(x, z, wz, fit$smooth, which, fit$smooth.frame, bf.maxit, : lv
## too small.  (Discovered by lowesd)
```

To plot the resulting surface, we can use the `akima` package.

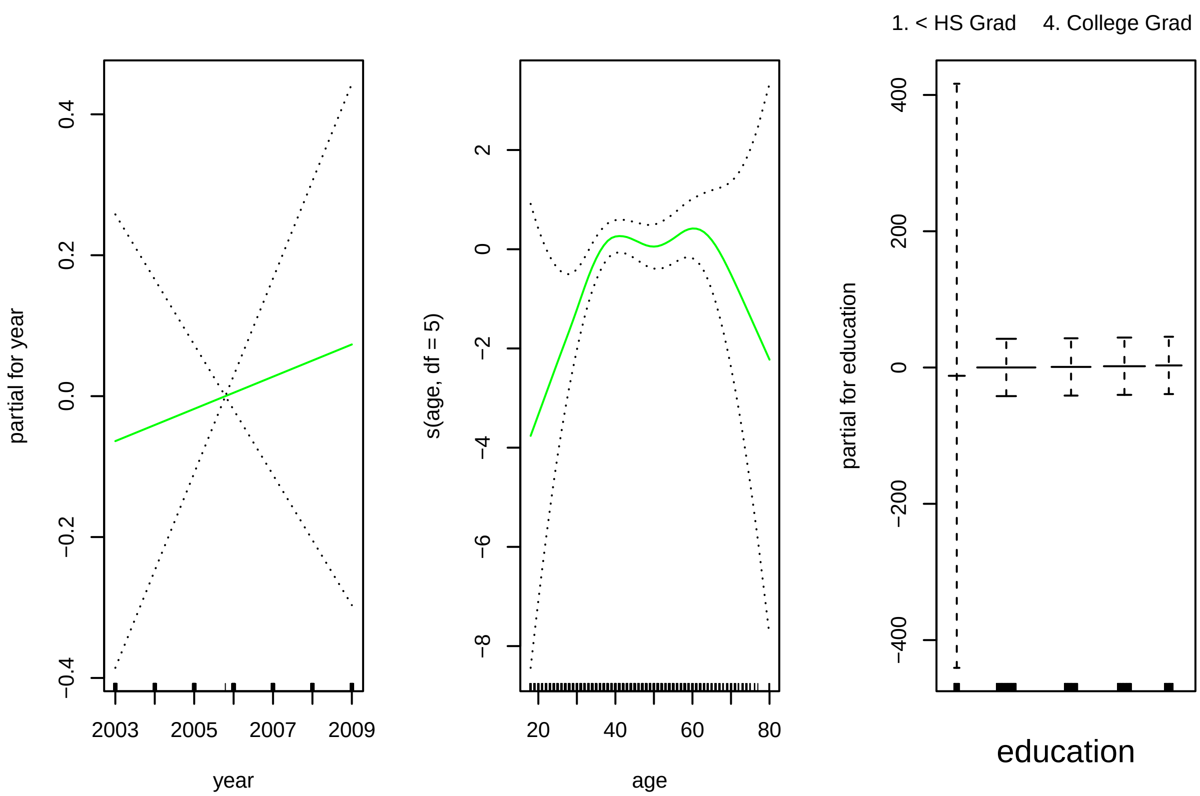
```
library(akima)
```

```
## The legacy packages maptools, rgdal, and rgeos, underpinning the sp package,
## which was just loaded, will retire in October 2023.
## Please refer to R-spatial evolution reports for details, especially
## https://r-spatial.org/r/2023/05/15/evolution4.html.
## It may be desirable to make the sf package available;
## package maintainers should consider adding sf to Suggests:.
## The sp package is now running under evolution status 2
##      (status 2 uses the sf package in place of rgdal)
```

```r
#plot(gam.lo.i)
```

In order to fit a logistic regression GAM, we again use the `I()` function in constructing the binary response variable, and set `family=binomial`

```r
gam.lr=gam(I(wage>250)~year+s(age,df=5)+education, family=binomial,data=Wage)
par(mfrow=c(1,3))
plot(gam.lr,se=T,col="green")
```



It is easy to see that there are no high earners in the `<HS` category:

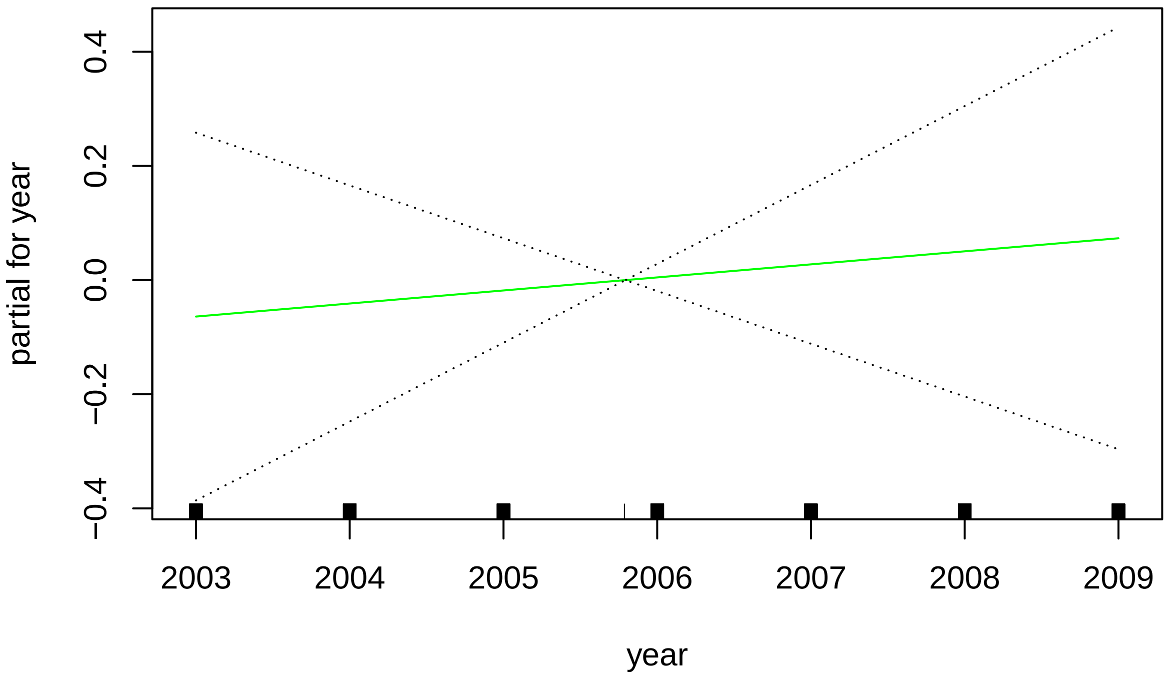
```r
table(education, I(wage >250))
```

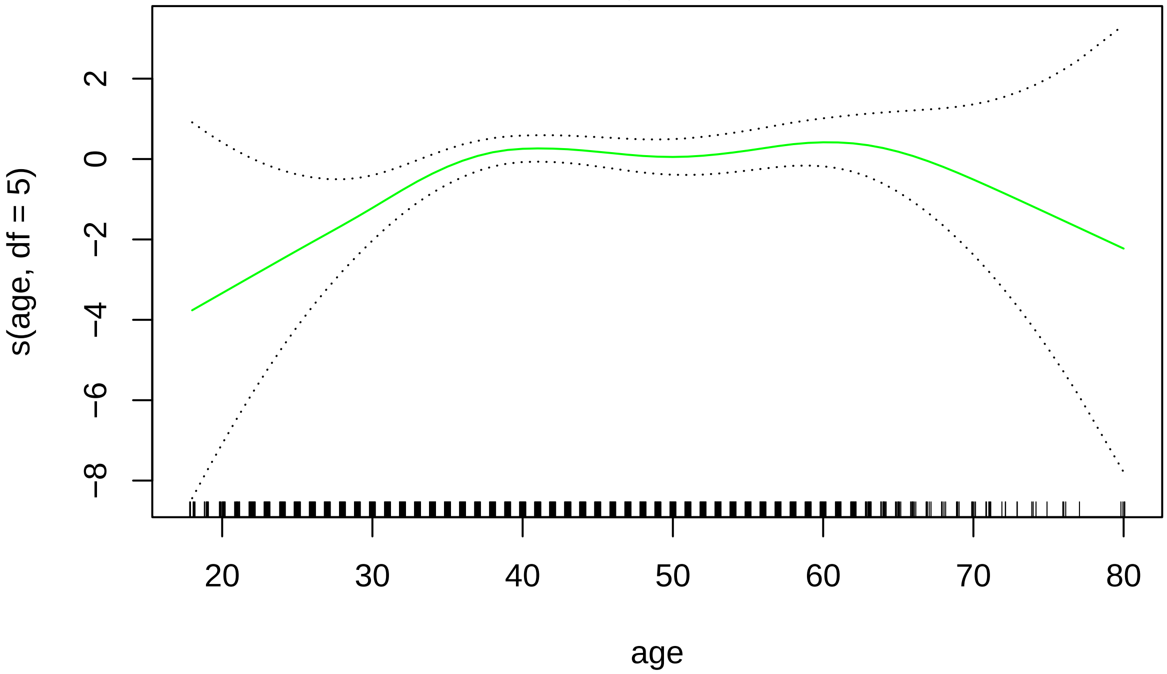
```
##
## education            FALSE TRUE
##   1. < HS Grad         268    0
##   2. HS Grad           966    5
##   3. Some College      643    7
##   4. College Grad      663   22
##   5. Advanced Degree   381   45
```

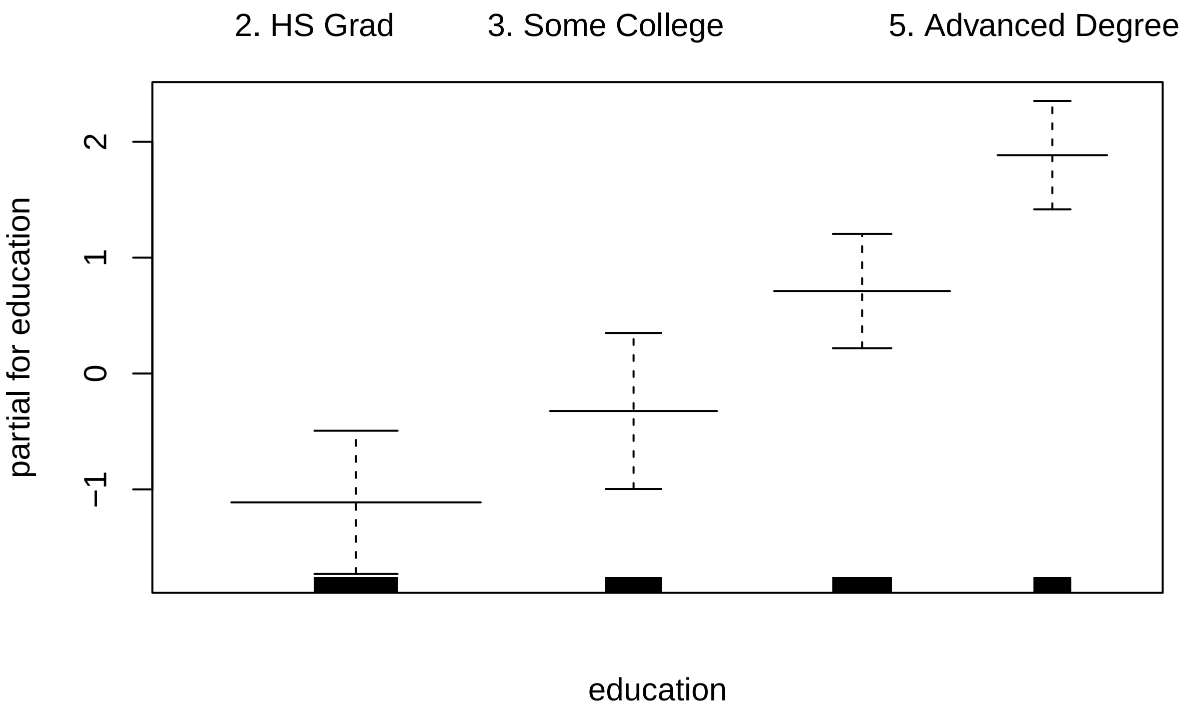
So we would fit a logistic regression GAM using all but this category, which provides more sensible results.

```r
gam.lr.s=gam(I(wage>250)~year+s(age,df=5)+education,family= binomial,data=Wage,subset=(education!="1. <

plot(gam.lr.s,se=T,col="green")
```


partial for year
0.4
0.2
0.0
−0.2
−0.4
2003
2004
2005
2006
2007
2008
2009
year


s(age, df = 5)
2
0
−2
−4
−6
−8
20
30
40
50
60
70
80
age


2. HS Grad
3. Some College
5. Advanced Degree
partial for education
2
1
0
−1
education