8.2 The Posterior via Exact Formal Analysis

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Suppose we want to purpose a solution to Bayes' rule using formal analysis? Since we know that the beta distribution is conjugate to Bernoulli likelihood function for single proportions, this suggests that a product of beta distributions would be conjugate to a product of Bernoulli functions.

Then we use the same logic as before; we assume a beta$(\theta_1|a_1, b_1)$ prior on θ_1, and an independent beta$(\theta_2|a_2, b_2)$ on θ_2.

$$\begin{aligned} p(\theta_1, \theta_2|D) &= p(D|\theta_1, \theta_2)p(\theta_1, \theta_2)/p(D) \\ &= \frac{\theta_1^{(z_1+a_1-1)}(1-\theta_1)^{(N_1-z_1+b_1-1)}\theta_2^{(z_2+a_2-1)}(1-\theta_2)^{(N_2-z_2+b_2-1)}}{p(D)B(a_1, b_1)B(a_2, b_2)} \end{aligned}$$

We know that the left side of equation above must be probability density function, and we see that the numerator of the right side has the form of product of beta distribution. Therefore the denominator of the equation above must be the corresponding normalizer for the product of beta distributions:

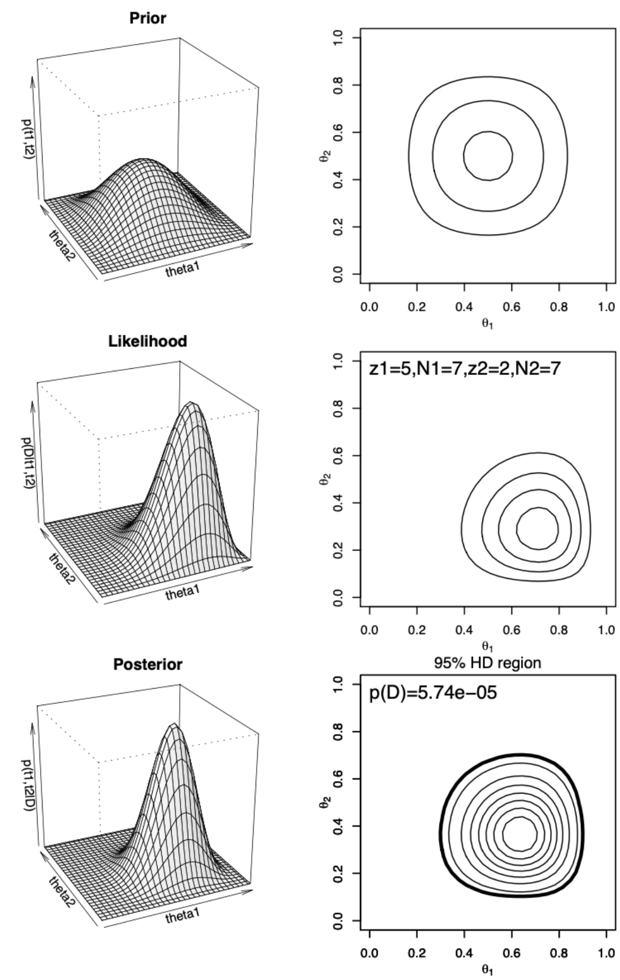
$$p(D)B(a_1, b_1)B(a_2, b_2) = B(z_1+a_1, N_1-z_1+b_1)\, B(z_2+a_2, N_2-z_2+b_2)$$

By rearranging terms we can get

$$p(D) = \frac{B(z_1+a_1, N_1-z_1+b_1)\, B(z_2+a_2, N_2-z_2+b_2)}{B(a_1, b_1)B(a_2, b_2)}$$

This is exactly analogous to the result we found before.

Recapitulation: When the prior is a product of independent beta distributions, the posterior is also a product of independent beta distributions.



This plot shows graphs for updating a product of beta distribution. Left panels show perspective surface plots; right panels show contour plots of the same distribution. The posterior contour plot (lower right) includes the value of $p(D)$, and shows the 95% highest density region as a darker contour line.

The main point of this section was to graphically display the meaning of a prior, likelihood, and posterior on a two-parameter space. An example was shown above.