4.2 Applied to Models and Data

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One of the key application that makes Bayes' rule so useful is when the row and column variables are data values and model parameter values, respectively

Generally, a model specifies

$$p(data\ values|parameters\ values\ and\ model\ structure)$$

Given the data:

$$p(parameters\ values\ and\ model\ structure|data\ values)$$

When we have observed some data, we use Bayes' rule to determine our beliefs across competing parameter values in a model, and to determine our beliefs across competing models.

After applying Bayes' rule to data and model parameter, we can get a new table:

Data	**Model Parameter**			**Marginal**
		θ value		
		$\vdots$		
D value	$\cdots$	$p(D,\theta)$ $= p(D\|\theta)p(\theta)$ $= p(\theta\|D)p(D)$	$\cdots$	$p(D)$
		$\vdots$		
Marginal:		$p(\theta)$		

The prior probability of the parameter values is the marginal distribution, $p(\theta)$, which appears in the lower margin of the table.

Then we can introduce some notations:

$$\underbrace{p(\theta|D)}_{\text{posterior}} = \underbrace{p(D|\theta)}_{\text{likelihood}}\ \underbrace{p(\theta)}_{\text{prior}} \Big/ \underbrace{p(D)}_{\text{evidence}}$$

where the evidence is

$$p(D) = \int d\theta\, p(D|\theta)p(\theta).$$

We can also add the conditional term into the equation:

$$\underbrace{p(\theta|D,M)}_{\text{posterior}} = \underbrace{p(D|\theta,M)}_{\text{likelihood}}\,\underbrace{p(\theta|M)}_{\text{prior}} \,/\, \underbrace{p(D|M)}_{\text{evidence}}$$

where the evidence is

$$p(D|M) = \int d\theta\, p(D|\theta,M)p(\theta|M).$$

By Bayes' rule and take the ratio of those equations, we can get:

$$\frac{p(M1|D)}{p(M2|D)} = \underbrace{\frac{p(D|M1)}{p(D|M2)}}_{\text{Bayes factor}}\,\frac{p(M1)}{p(M2)}.$$

This equation says that the posterior beliefs is the ration of the evidences times the ratio of the prior beliefs.

The ratio of the evidences is called the **Bayes factor**.

Also, the quantity $p(D|M)$, which is called the "evidence" in this book, is sometimes instead called the "marginal likelihood" or "prior predictive" by other authors.

$p(M1|D) = p(D|M1)p(M1)/p(D)$, is the "likelihood of the model M1 for the data D".

4.2.1 Data Order In-variance

First we will unpack $p(\theta|D', D)$ and have

$$p(\theta|D',D) = \frac{p(D'|\theta,D)\,p(\theta|D)}{\int d\theta\, p(D'|\theta,D)\,p(\theta|D)}$$

The equation then can be rewritten as

$$p(\theta|D',D) = \frac{p(D'|\theta)\,p(\theta|D)}{\int d\theta\, p(D'|\theta)\,p(\theta|D)}$$

Then we use Bayes' rule again, and we can convert the equation into

$$p(\theta|D',D) = \frac{p(D'|\theta)\,p(D|\theta)\,p(\theta)/p(D)}{\int d\theta\, p(D'|\theta)\,p(D|\theta)\,p(\theta)/p(D)}$$

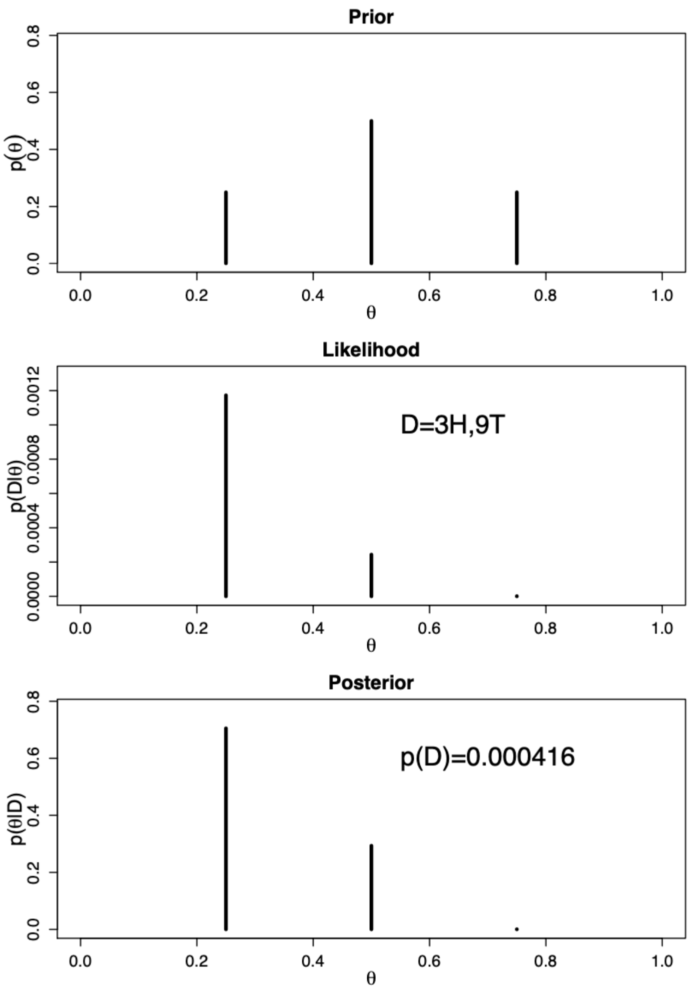
4.2.2 An Example with Coin Flipping

This section we will consider an example when flipping the coin.

First, we specify our prior beliefs. We denote the bias as $\theta = p(H)$, the probability of the coin coming up heads. To keep the example straightforward, suppose that we believe there are only three possible values for the coin's bias: Either the coin is fair, with $\theta = .50$, or the coin is biased with $\theta = .25$ or $\theta = .75$. We believe that the coin is probably fair, but there's some smaller chance it could be biased high or low.

Next, we flip the coin to get some data D and determine the likelihood, $p(D|\theta)$. Suppose we flip the coin 12 times, and it comes up heads 3 times. According to our model of the coin, the probability of coming up heads is θ, and the probability of coming up tails is $1-\theta$.

We can show the results in the plot below



4.2.2.1 $p(D|\theta)$ is not θ

In the examples involving coin flips, it is easy to lose sight of the important fact that $p(D|\theta)$ is different from θ, even though they both are values between 0 and 1 for our current examples. The likelihood $p(D|\theta)$ is a mathematical function of θ. The value of the likelihood function is always a probability. Adding to the confusability is the fact that, in our examples so far, the function that maps θ to $p(D=H|\theta)$ has been the identity function:

$$p(D=H|\theta)=\theta$$