7.7 Generalized Additive Models

Zongyi Liu

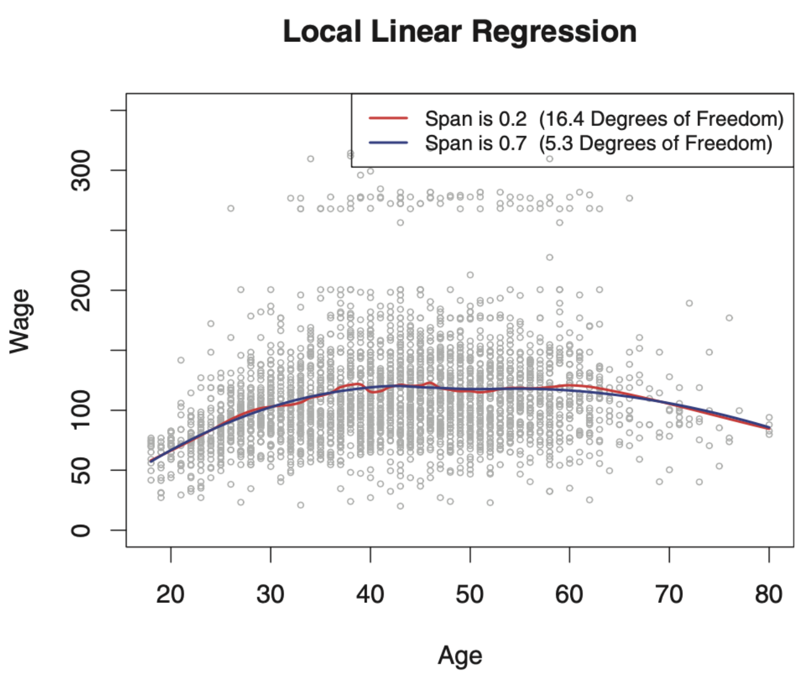
2023-06-06

7.7 Generalized Additive Models

In previous sections we learned a number of approaches for flexibly predicting a response Y on the basis of a single predictor X, which can be seen as extensions of simple linear regressions. Here we explore the problem of flexibly predicting Y on the basis of several predictors, $X_1, ..., X_p$.

In this chapter we will learn about Generalized Additive Models (GAM), which provide a general framework for extending a standard linear model by allowing non-linear functions of each of the variables, while maintaining additivity.

Here we have the plot of local linear regression:



7.7.1 GAMs for Regression Problems

A natural way to extend the multiple linear regression model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i$$

in order to allow for non-linear relationships between each feature and the response is to replace each linear component $\beta_j x_i j$ with a (smooth) non-linear function $f_j(x_i j)$, we can then write the model as

$$
\begin{aligned}
y_i &= \beta_0 + \sum_{j=1}^{p} f_j(x_{ij}) + \epsilon_i \\
&= \beta_0 + f_1(x_{i1}) + f_2(x_{i2}) + \cdots + f_p(x_{ip}) + \epsilon_i.
\end{aligned}
$$

This is an example of a GAM, which is called an additive model because we calculate a separate f_j for each X_j, and then add together all of their contributions.

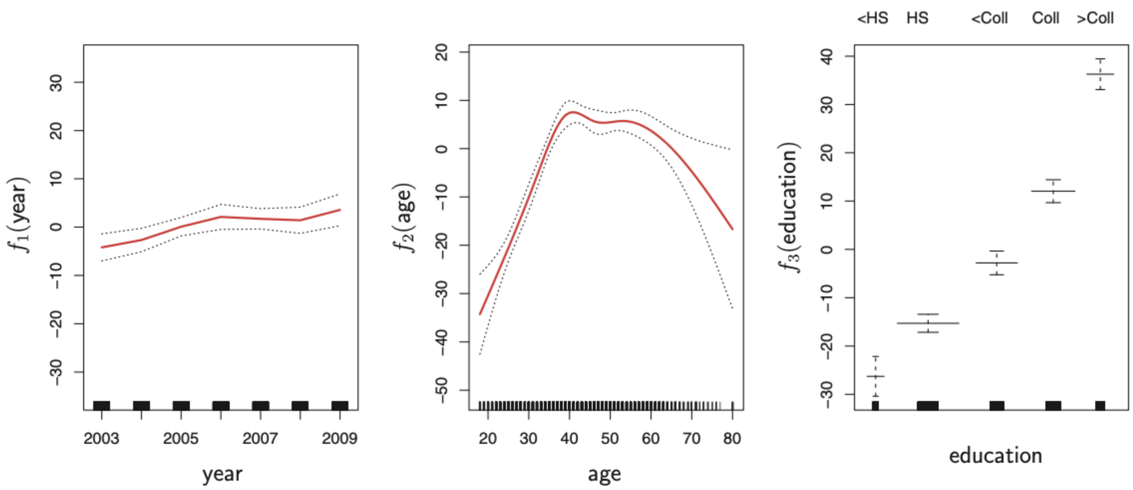


FIGURE 7.11. *For the* `Wage` *data, plots of the relationship between each feature and the response,* `wage`*, in the fitted model (7.16). Each plot displays the fitted function and pointwise standard errors. The first two functions are natural splines in* `year` *and* `age`*, with four and five degrees of freedom, respectively. The third function is a step function, fit to the qualitative variable* `education`*.*

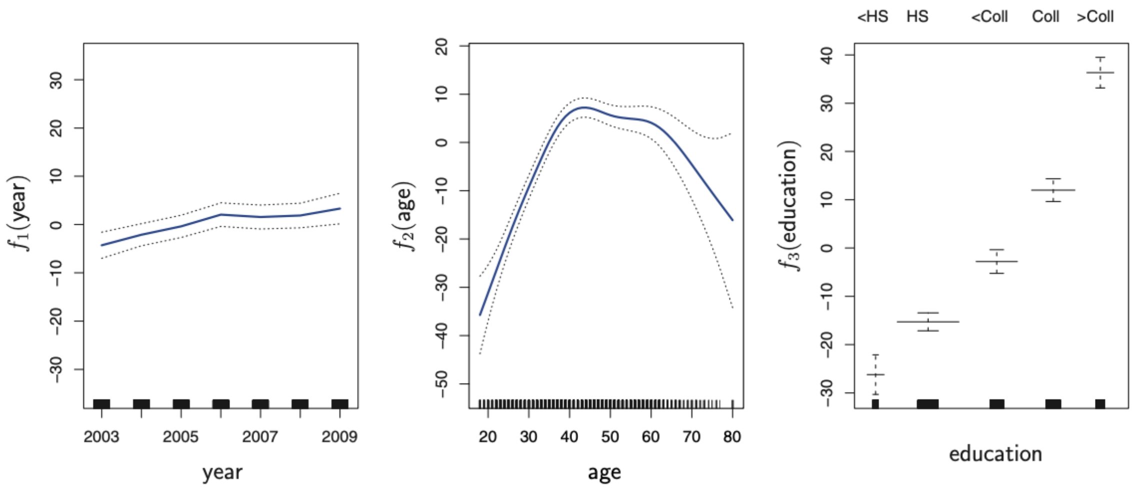


FIGURE 7.12. *Details are as in Figure 7.11, but now f_1 and f_2 are smoothing splines with four and five degrees of freedom, respectively.*

Pros and Cons of GAMs

- GAMs allow us to fit a non-linear f_j to each X_j, so that we can automatically model non-linear relationships that standard linear regression will miss. This means that we do not need to manually try out many different transformations on each variable individually.
- The non-linear fits can potentially make more accurate predictions for the response Y.
- Because the model is additive, we can still examine the effect of each X_j on Y individually while holding all of the other variables fixed. Hence if we are interested in inference, GAMs provide a useful representation.
- The smoothness of the function f_j for the variable X_j can be summarized via degrees of freedom.
- (Con) The main limitation of GAMs is that the model is restricted to be additive. With many variables, important interactions can be missed.

7.7.2 GAMs for Classification Problems

GAMs can also be used in situations where Y is qualitative. For simplicity, here we will assume Y takes on values zero or one, and let $p(X) = Pr(Y = 1|X)$ be the conditional probability that the response equals one. Recall the logistic regression model

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p.$$

We can allow for non-linear relationships to use the model

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + f_1(X_1) + f_2(X_2) + \cdots + f_p(X_p).$$

We fit a GAM to the `Wage` data in order to predict the probability that an individual's income exceeds \$250,000 per year. The GAM that we fit takes the form

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 \times \texttt{year} + f_2(\texttt{age}) + f_3(\texttt{education}),$$

where

$$p(X) = \Pr(\texttt{wage} > 250|\texttt{year}, \texttt{age}, \texttt{education}).$$

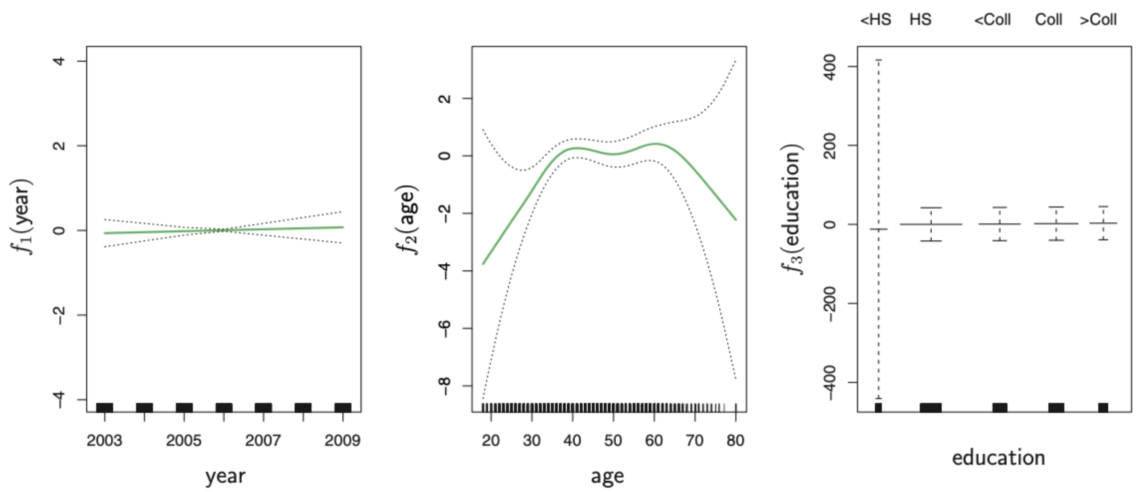


FIGURE 7.13. *For the* `Wage` *data, the logistic regression GAM given in (7.19) is fit to the binary response* `I(wage>250)`*. Each plot displays the fitted function and pointwise standard errors. The first function is linear in* `year`*, the second function a smoothing spline with five degrees of freedom in* `age`*, and the third a step function for* `education`*. There are very wide standard errors for the first level* `<HS` *of* `education`*.*