Homework 1, IEOR 4732

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1 Question 1

The characteristic function of the log of stock price in Black-Scholes framework is given by:

$$\begin{aligned}\mathbb{E}\left(e^{iu\ln S_t}\right) &= \mathbb{E}\left(e^{ius_t}\right)\\ &= \exp\left(i\left(\ln S_0 + \left(r - q - \frac{\sigma^2}{2}\right)t\right)u - \frac{1}{2}\sigma^2u^2t\right)\\ &= \exp\left(i\left(s_0 + \left(r - q - \frac{\sigma^2}{2}\right)t\right)u - \frac{1}{2}\sigma^2u^2t\right)\end{aligned}$$

For the following parameters: Spot price, $S_0 = \$1900$; maturity, $T = 0.25$ year; volatility, $\sigma = 0.36$; risk-free interest rate, $r = 2.00\%$, continuous dividend rate, $q = 1.87\%$ and strike range of $K = 2000, 2100, 2200$ price European call options via the following transform techniques:

- (a) Fast Fourier transform (FFT): consider $\eta = \Delta\nu = 0.25, \alpha = 0.4, 1.0, 1.4, 3.0, N = 2^n$ for $n = 9, 11, 13, 15$, and $\beta = \ln K - \frac{\lambda N}{2}$
- (b) Fractional fast Fourier transform (FrFFT): consider $\eta = \Delta\nu = 0.25, \lambda = \Delta k = 0.1, \alpha = 0.4, 1.0, 1.4, 3.0, N = 2^n$ for $n = 6, 7, 8, 9$, and $\beta = \ln K - \frac{\lambda N}{2}$
- (c) Fourier-cosine (COS) method: consider values $[-1, 1], [-4, 4], [-8, 8], [-12, 12]$ for the interval $[a, b]$ and find the sensitivity of your results to the choice of $[a, b]$

Compare and conclude.
Answer

1.1 Part a

I modified the code given in the `sample code` folder, and get the result of Fast Fourier transform for different strike prices.

Table 1: Results for Different Values of K

	$K = 2000,\ \eta = 0.25$				$K = 2100,\ \eta = 0.25$				$K = 2200,\ \eta = 0.25$			
α	$N = 2^9$	2^{11}	2^{13}	2^{15}	$N = 2^9$	2^{11}	2^{13}	2^{15}	$N = 2^9$	2^{11}	2^{13}	2^{15}
0.4	95.3281	95.3281	95.3281	95.3281	64.9160	64.9160	64.9160	64.9160	43.0286	43.0286	43.0286	43.0286
1.0	95.2467	95.2467	95.2467	95.2467	64.8346	64.8346	64.8346	64.8346	42.9472	42.9472	42.9472	42.9472
1.4	95.2467	95.2467	95.2467	95.2467	64.8346	64.8346	64.8346	64.8346	42.9472	42.9472	42.9472	42.9472
3.0	95.2467	95.2467	95.2467	95.2467	64.8346	64.8346	64.8346	64.8346	42.9472	42.9472	42.9472	42.9472

1.2 Part b

The fractional FFT procedure computes a sum of the form

$$\sum_{\gamma=1}^{N} e^{-i2\pi\gamma(j-1)(m-1)}x(j)$$

for any value of γ I modified the sample codes given, which only contains methods for `VG`, `GBM`, and `Heston`.
I added the chunk like:

```
        elif model == 'BlackScholes':
        sigma = 0.36   # Volatility
        params.append(sigma)
```

and in `def generic CF`, define the model:

```
        elif (model == 'BlackScholes'):
        sigma = params[0]   # Volatility

        mu = np.log(S0) + (r - q - 0.5 * sigma**2) * T   # Drift
        a = sigma * np.sqrt(T)
        phi = np.exp(1j * mu * u - 0.5 * a**2 * u**2)
```

Which defines the Black-Scholes model, and we can get the result for different of strike price.

Table 2: Results for Different Values of K

	$K = 2000,\ \eta = 0.25$				$K = 2100,\ \eta = 0.25$				$K = 2200,\ \eta = 0.25$			
α	$N = 2^6$	2^7	2^8	2^9	$N = 2^6$	2^7	2^8	2^9	$N = 2^6$	2^7	2^8	2^9
0.4	95.3859	95.3295	95.3295	95.3295	64.9425	64.9174	64.9174	64.9174	43.0108	43.0298	43.0298	43.0298
1.0	95.3051	95.2477	95.2477	95.2477	64.8774	64.8357	64.8357	64.8357	42.9511	42.9482	42.9482	42.9482
1.4	95.3006	95.2475	95.2475	95.2475	64.8842	64.8355	64.8355	64.8355	42.9641	42.9480	42.9480	42.9480
3.0	95.2521	95.2467	95.2467	95.2467	64.8762	64.8349	64.8349	64.8349	42.9915	42.9476	42.9476	42.9476

The FrFFT analyzes a signal at non-integer frequency intervals, whereas the FFT simply calculates the standard frequency spectrum of a discrete signal with increased speed.

1.3 Part c

The Fourier cosine series expansion of a function $f(\theta)$ on $[0, \pi]$ is

$$\begin{aligned} f(\theta) &= \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k_k^s \theta\right) \\ &= \sum_{in}^{\infty} {}_{k=0}^{\infty} A_k \cos(k\theta) \end{aligned}$$

with the Fourier cosine coefficient

$$A_k = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(k\theta)d\theta$$

where $\bar{\sum}$ indicates the first term in the summation is weighted by one-half.

Then the option value at time t can be written as

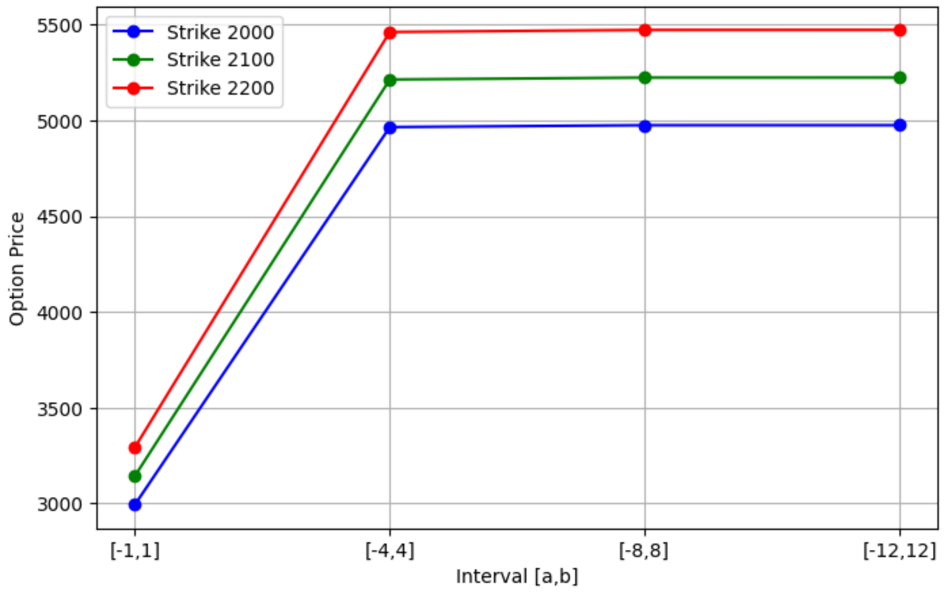
$$\begin{aligned} v(x,t) &= C\int_a^b v(y,T)f(y \mid x)^0 dy \\ &= C\int_a^{bb^{25}} v(y,T)\bar{\sum}_{k=0}^{\infty} A_k \cos\left(k\frac{y-a}{b-a}\pi\right) dy \\ &= C\sum_{k=0}^{\infty} A_k \left(\int_a^b v(y,T)\cos\left(k\frac{y-a}{b-a}\pi\right) dy\right) \end{aligned}$$

Here I tested different values for the interval $[a, b]$, and results are different.

Fourier-Cosine Method				
K	$[-1, 1]$	$[-4, 4]$	$[-8, 8]$	$[-12, 12]$
2000	2994.5770	4965.5228	4975.0742	4975.2406
2100	3144.3059	5213.7990	5223.8279	5224.0026
2200	3294.0347	5462.0751	5472.5816	5472.7646

And the result can be plotted below:

Figure 1: Sensitivity of Option Prices to Interval [a,b]



I later looked up in Fang and Oosterlee's paper, they proposed some rule of thumb based on cumulants to get an idea how to choose $[a, b]$. In particular, they suggested (In equation (49)):

$$[a,b] = \begin{cases} \left[c_1 \pm 12\sqrt{c_2}\right] & , n_c = 2 \\ \left[c_1 \pm 10\sqrt{c_2 + \sqrt{c_4}}\right] & , n_c = 4 \\ \left[c_1 \pm 10\sqrt{c_2 + \sqrt{c_4 + \sqrt{c_6}}}\right] & , n_c = 6 \end{cases}$$

where c_1, c_2, c_4, c_6 are the first, second, forth and sixth cumulants of $\ln(S_t/K)$. The cumulants, c_n, are defined by the cumulant-generating function $g(t)$:

$$g(t) = \log\left(E\left(e^{t \cdot X}\right)\right)$$

for some random variable X. The cumulants are given by the derivatives, at zero, of $g(t)$. Back to our model, when L is chosen as 8, which corresponds to $[-4, 4]$, the premiums are more accurate and consistent.

2 Appendix

2.1 Part a

```
import numpy as np
import math
import time
import cmath

# Fixed Parameters
S0 = 1900  # Initial stock price
K = 2000    # Strike price
k = math.log(K)  # Log of strike price
r = 0.02  # Risk-free rate
q = 0.0187  # Dividend yield
T = 0.25     # Time to maturity

# Parameters for FFT
n = 15     # n determines the size of FFT: N = 2^n
N = 2**n
eta = 0.25 # Step-size
alpha = 1.0 # Damping factor

# Black-Scholes model parameters
sigma = 0.36  # Volatility

# Characteristic function for Black-Scholes
def generic_CF(u, S0, r, q, T, sigma):
mu = np.log(S0) + (r - q - 0.5 * sigma**2) * T
a = sigma * np.sqrt(T)
phi = np.exp(1j * u * mu - 0.5 * a**2 * u**2)
return phi

# FFT implementation for Black-Scholes model
def genericFFT(S0, K, r, q, T, alpha, eta, n, sigma):
N = 2**n
lda = (2 * np.pi / N) / eta  # Step-size in log strike space
beta = np.log(K)

# Initialize arrays
km = np.zeros(N)
xX = np.zeros(N, dtype=complex)

# Discount factor
df = np.exp(-r * T)

# Define the frequency range
nuJ = np.arange(N) * eta
psi_nuJ = generic_CF(nuJ - (alpha + 1) * 1j, S0, r, q, T, sigma) / ((alpha + 1j *
    nuJ) * (alpha + 1 + 1j * nuJ))

# Compute the xX vector (values for the FFT)
for j in range(N):
km[j] = beta + j * lda
wJ = eta if j > 0 else eta / 2  # Weighting for j = 0
xX[j] = np.exp(-1j * beta * nuJ[j]) * df * psi_nuJ[j] * wJ

# Apply FFT to the xX vector
yY = np.fft.fft(xX)

# Compute the option prices
cT_km = np.zeros(N)
```

```
        for i in range(N):
        multiplier = np.exp(-alpha * km[i]) / np.pi
        cT_km[i] = multiplier * np.real(yY[i])

        return km, cT_km

        # Function to calculate option price using FFT
        def calculate_option_price(S0, K, r, q, T, alpha, eta, n, sigma):
        start_time = time.time()

        km, cT_km = genericFFT(S0, K, r, q, T, alpha, eta, n, sigma)
        cT_k = np.interp(k, km, cT_km)  # Interpolating the option price for the given
            strike

        elapsed_time = time.time() - start_time
        print(f"Option␣via␣FFT:␣for␣strike␣{np.exp(k):.4f}␣the␣option␣premium␣is␣{cT_k:.4f
            }")
        print(f"FFT␣execution␣time␣was␣{elapsed_time:.7f}␣seconds")

        return cT_k

        # Example usage
        option_price = calculate_option_price(S0, K, r, q, T, alpha, eta, n, sigma)
```

2.2 Part b

```

        import warnings
        warnings.filterwarnings("ignore")

        import numpy as np
        import cmath
        import math
        import time

        # Fixed Parameters
        S0 = 1900
        K = 2100
        k = math.log(K)
        r = 0.02
        q = 0.0187

        # Parameters for FFT and FrFFT

        n_FFT = 7
        N_FFT = 2**n_FFT

        n_FrFFT = 7
        N_FrFFT = 2**n_FrFFT

        N = 2000

        #step-size
        eta = 0.25
        # damping factor
        alpha = 3

        # step-size in log strike space
        lda_FFT = (2*math.pi/N_FFT)/eta # lda is fixed under FFT
        lda_FrFFT = 0.001 # lda is an adjustable parameter under FrFFT,


#Choice of beta
beta = np.log(S0)-N*lda_FFT/2
#beta = np.log(S0)-N*lda_FrFFT/2
#beta = np.log(K)

#model-specific Parameters
model = 'BlackScholes'

params = []
if (model == 'GBM'):

sig = 0.30
params.append(sig);

elif model == 'BlackScholes':
sigma = 0.36  # Volatility
params.append(sigma)

elif (model == 'VG'):

sig = 0.3
nu = 0.5
theta = -0.4
#
params.append(sig);
params.append(nu);
params.append(theta);



elif (model == 'Heston'):

kappa = 2.0
theta = 0.05
sig = 0.30
rho = -0.70
v0 = 0.04
#
params.append(kappa)
params.append(theta)
params.append(sig)
params.append(rho)
params.append(v0)

def generic_CF(u, params, S0, r, q, T, model):

if (model == 'GBM'):

sig = params[0]
mu = np.log(S0) + (r-q-sig**2/2)*T
a = sig*np.sqrt(T)
phi = np.exp(1j*mu*u-(a*u)**2/2)

elif (model == 'BlackScholes'):
sigma = params[0]  # Volatility

mu = np.log(S0) + (r - q - 0.5 * sigma**2) * T  # Drift
a = sigma * np.sqrt(T)  # Standard deviation over the maturity period
phi = np.exp(1j * mu * u - 0.5 * a**2 * u**2)  # Characteristic function for Black
    -Scholes


```

```
        elif(model == 'Heston'):

        kappa  = params[0]
        theta  = params[1]
        sigma  = params[2]
        rho    = params[3]
        v0     = params[4]

        tmp = (kappa-1j*rho*sigma*u)
        g = np.sqrt((sigma**2)*(u**2+1j*u)+tmp**2)

        pow1 = 2*kappa*theta/(sigma**2)

        numer1 = (kappa*theta*T*tmp)/(sigma**2) + 1j*u*T*r + 1j*u*math.log(S0)
        log_denum1 = pow1 * np.log(np.cosh(g*T/2)+(tmp/g)*np.sinh(g*T/2))
        tmp2 = ((u*u+1j*u)*v0)/(g/np.tanh(g*T/2)+tmp)
        log_phi = numer1 - log_denum1 - tmp2
        phi = np.exp(log_phi)

        #g = np.sqrt((kappa-1j*rho*sigma*u)**2+(u*u+1j*u)*sigma*sigma)
        #beta = kappa-rho*sigma*1j*u
        #tmp = g*T/2

        #temp1 = 1j*(np.log(S0)+(r-q)*T)*u + kappa*theta*T*beta/(sigma*sigma)
        #temp2 = -(u*u+1j*u)*v0/(g/np.tanh(tmp)+beta)
        #temp3 = (2*kappa*theta/(sigma*sigma))*np.log(np.cosh(tmp)+(beta/g)*np.sinh(tmp))

        #phi = np.exp(temp1+temp2-temp3);


        elif (model == 'VG'):

        sigma  = params[0];
        nu     = params[1];
        theta  = params[2];

        if (nu == 0):
        mu = np.log(S0) + (r-q - theta -0.5*sigma**2)*T
        phi  = np.exp(1j*u*mu) * np.exp((1j*theta*u-0.5*sigma**2*u**2)*T)
        else:
        mu  = np.log(S0) + (r-q + np.log(1-theta*nu-0.5*sigma**2*nu)/nu)*T
        phi = np.exp(1j*u*mu)*((1-1j*nu*theta*u+0.5*nu*sigma**2*u**2)**(-T/nu))

        return phi
        def evaluateIntegral(params, S0, K, r, q, T, alpha, eta, N, model):

        # Just one strike at a time
        # no need for Fast Fourier Transform

        # discount factor
        df = math.exp(-r*T)

        sum1 = 0
        for j in range(N):
        nuJ = j*eta
        psi_nuJ = df*generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j
            *nuJ)*(alpha+1+1j*nuJ))
        if j == 0:
        wJ = (eta/2)
        else:
        wJ = eta
        sum1 += np.exp(-1j*nuJ*k)*psi_nuJ*wJ

```

```
cT_k = (np.exp(-alpha*k)/math.pi)*sum1

return np.real(cT_k)

def genericFFT(params, S0, K, r, q, T, alpha, eta, n, model):

N = 2**n

# step-size in log strike space
lda = (2*np.pi/N)/eta

#Choice of beta
#beta = np.log(S0)-N*lda/2
#beta = np.log(K)

# forming vector x and strikes km for m=1,...,N
km = np.zeros((N))
xX = np.zeros((N))

# discount factor
df = math.exp(-r*T)

nuJ = np.arange(N)*eta
psi_nuJ = generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*
    nuJ)*(alpha+1+1j*nuJ))

for j in range(N):
km[j] = beta+j*lda
if j == 0:
wJ = (eta/2)
else:
wJ = eta

xX[j] = np.exp(-1j*beta*nuJ[j])*df*psi_nuJ[j]*wJ

yY = np.fft.fft(xX)
cT_km = np.zeros((N))
for i in range(N):
multiplier = np.exp(-alpha*km[i])/math.pi
cT_km[i] = multiplier*np.real(yY[i])

return km, cT_km

def genericFrFFT(params, S0, K, r, q, T, alpha, eta, n, lda, model):

N = 2**n
gamma = eta*lda/(2*math.pi)

#Choice of beta
#beta = np.log(S0)-N*lda/2
beta = np.log(K)

# initialize x, y, z, and cT_km
km = np.zeros((N))
x = np.zeros((N))
y = np.zeros((2*N), dtype=np.complex128)
z = np.zeros((2*N), dtype=np.complex128)
cT_km = np.zeros((N))

# discount factor
df = math.exp(-r*T)

# compute x
```

```python
nuJ = np.arange(N)*eta
psi_nuJ = generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*
    nuJ)*(alpha+1+1j*nuJ))

for j in range(N):
km[j] = beta+j*lda
if j == 0:
wJ = (eta/2)
else:
wJ = eta
x[j] = np.exp(-1j*beta*nuJ[j])*df*psi_nuJ[j]*wJ

# set up y
for i in range(N):
y[i] = np.exp(-1j*math.pi*gamma*i**2)*x[i]
y[N:] = 0

# set up z
for i in range(N):
z[i] = np.exp(1j*math.pi*gamma*i**2)
z[N:] = z[:N][::-1]

# compute xi_hat
xi_hat = np.fft.ifft(np.fft.fft(y) * np.fft.fft(z))

# compute call prices
for i in range(N):
cT_km[i] = np.exp(-alpha*(beta + i*lda))/math.pi * (np.exp(-1j*math.pi*gamma*i**2)
    *xi_hat[i]).real

return km, cT_km

print('␣')
print('===================')
print('Model␣is␣%s' % model)
print('-------------------')

T = 0.25

# FFT
print('␣')
start_time = time.time()
km, cT_km = genericFFT(params, S0, K, r, q, T, alpha, eta, n_FFT, model)
#cT_k = cT_km[0]
cT_k = np.interp(k, km, cT_km)

elapsed_time = time.time() - start_time

#cT_k = np.interp(np.log(k), km, cT_km)
print("Option␣via␣FFT:␣for␣strike␣%s␣the␣option␣premium␣is␣%6.4f" % (np.exp(k),
    cT_k))
#print("Option via FFT: for strike %s the option premium is %6.4f" % (np.exp(k),
    cT_km[0]))
print('FFT␣execution␣time␣was␣%0.7f' % elapsed_time)

# FrFFT
print('␣')
start_time = time.time()
km, cT_km = genericFrFFT(params, S0, K, r, q, T, alpha, eta, n_FrFFT, lda_FrFFT,
    model)
#cT_k = cT_km[0]
cT_k = np.interp(k, km, cT_km)

```

```python
        elapsed_time = time.time() - start_time

        #cT_k = np.interp(np.log(), km, cT_km)
        print("Option via FrFFT: for strike %s the option premium is %6.4f" % (np.exp(k),
            cT_k))
        #print("Option via FFT: for strike %s the option premium is %6.4f" % (np.exp(k),
            cT_km[0]))
        print('FrFFT execution time was %0.7f' % elapsed_time)


        # Integral
        print(' ')
        start_time = time.time()
        cT_k = evaluateIntegral(params, S0, K, r, q, T, alpha, eta, N, model)
        elapsed_time = time.time() - start_time
        print("Option via Integration: for strike %s the option premium is %6.4f" % (np.
            exp(k), cT_k))
        print('Evaluation of integral time was %0.7f' % elapsed_time)
```

2.3 Part c

```python


                import numpy as np
                def char_func(u, S0, r, q, sigma, T):
                """ Black-Scholes characteristic function """
                return np.exp(1j * u * (np.log(S0) + (r - q - 0.5 * sigma**2) * T) - 0.5 *
                     sigma**2 * u**2 * T)

                def cos_method_call(S0, K, T, r, q, sigma, N, a, b):
                """ Fourier-Cosine method for European call option pricing """
                x0 = np.log(S0 / K)

                # Compute coefficients
                k = np.arange(N)
                omega_k = k * np.pi / (b - a)

                # Characteristic function evaluations
                phi_k = char_func(omega_k, S0, r, q, sigma, T) * np.exp(-1j * omega_k * a)

                # Payoff function coefficients
                chi_k = (np.sin(omega_k * (b - a)) - np.sin(omega_k * (-a))) / omega_k
                chi_k[0] = b - a  # Handle k=0 separately

                # COS method summation
                V = np.real(phi_k * chi_k * 2 / (b - a))

                # Discounted expectation
                call_price = np.exp(-r * T) * np.sum(V)
                return call_price * K

                # Parameters
                S0 = 1900  # Spot price
                T = 0.25  # Maturity in years
                sigma = 0.36  # Volatility
                r = 0.02  # Risk-free rate
                q = 0.0187  # Continuous dividend rate
                strike_prices = [2000, 2100, 2200]  # Strike prices
                N = 100  # Number of terms in Fourier series
                intervals = [(-1, 1), (-4, 4), (-8, 8), (-12, 12)]  # Intervals [a, b]

```

```python
# Compute option prices for different intervals
results = {}
for a, b in intervals:
results[(a, b)] = {K: cos_method_call(S0, K, T, r, q, sigma, N, a, b) for
    K in strike_prices}

# Print results
for (a, b), prices in results.items():
print(f"Interval [{a}, {b}]:")
for K, price in prices.items():
print(f"  Strike {K}: {price:.4f}")
```