Lecture 1, IEOR 4732

Options, Pricing, and CF

期貨, 對其定價, 以及特徵方程

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Contents

1 Overview

Basically there will be two parts:

Part I - pricing/valuation

- Transform Techniques
- Numerical solution of PDEs/PIDEs
- Simulation
- Deep Neural Networks (DNNs)

Part II- calibration & estimation

- Model Calibration
- Construction/Cooking
- Parameter Estimation

概述來講, 這本書分成兩個部分, 一個是定價和估值, 就是進行轉換, 對 PDE 等進行數值求解. 另一個就是校準, 完善模型.

1.1 Transform Techniques

Transform techniques are used for option pricing and data de-noising

- Fast Fourier transform (FFT)
- Fractional fast Fourier trânsform (FrFFT)
- COS method
- Saddle-point method

1.2 Numerical solution of PDEs/PIDEs

- In general, transform techniques cannot be utilized for path dependent options
- For Markov processes, there exisists a PDE/PIDE (FeynmanKac formula) that gives the value of the option
- Need to solve partial (-integro) differential equations for option pricing
- However, most cases they need to be solved numerically

1.3 Monte-Carlo Methods

Lots of applications:

- Sampling from various distributions
- Scenario analysis/stress testing
- VaR
- Options pricing

1.4 Model Calibration

- In the first part, we assume the model parameter set, Θ, is known/given.
- Now having (market) prices what is the optimal parameter set Θ^* ?
- The process of finding a parameter set that market prices and model prices match closely is called model calibration
- Usage: marking/extrapolation/risk management/scenario analysis/trading

1.5 Construction/Cooking

- Fitting: finding a function/polynomial or group of them to fit implied volatility surface
- Construction: construction of the yield curve or the zero-coupon bond currve from LIBOR rates, Swap rates, and Eurodollar futures

1.6 Parameter Estimation

- Snapshot of the market vs. Time-series-cross-section data
- times series of prices
- Smaller number of parameters better off we are (to avoid overfitting)
- Mis-pricing is an indicator being rich/cheap and a signal for trading
- Usage:nstrategy/trading

2 Vanilla Options

2.1 The Call Option

Definition The buyer has the right (not the obligation) to buy the underlying from the seller at maturity for a certain price (strike price).

The seller is obligated to sell underlying to the buyer upon buyer's call. The buyer pays a fee (premium) for this right.

2.2 The Put Option

Definition The buyer has the right (not the obligation) to sell the underlying at a specified price (strike price) at maturity to the seller.

The seller is obligated to purchase underlying from the buyer upon buyer's call. The buyer pays a fee (premium) for this right.

Q&A

Q: Who are buyers/sellers of option contracts?
A: Hedgers & Speculators
hedgers & speculators make the market & provide liquidity
Q: What determines the price of an option?
A: Supply & Dermand

2.3 A Speculator Case

Q: Assume you speculate that Apple Stock is going to go higher over time. Would you buy a call option or the stock?
A: A call option
Q: What is the benefit of buying the option?
A: The benefit of buying one call option vs. purchasing 100 shares of Apple is that the maximum loss is lower in the former

A Speculator Example

Scenario	Today price	Future Price	P & L
1	\$190	\$140	$-\$5,000$
2	\$190	\$170	$-\$2,800$
3	\$190	\$190	$-\$0$
4	\$190	$\$210^{20105}$	$-\$2,000$
5	\$190	\$240	$+\$5,000$

Table 1: Speculator bought 100 Apple stocks at \$190

Scenario	Today price	Future Price	Payoff	P&L
1	\$190	\$140	\$0	$-\$500$
2 ducal	\$190	\$170	\$0	$-\$500$
For	\$190	\$190	\$0	$-\$500$
4	\$190	\$210	\$1,000	$+\$500$
5	\$190	\$240	\$4,000	$+\$3,500$

Table 2: Table: Speculator bought a 3-month maturity call option at strike of \$200 and paid \$5

Observations

Q: What is trade-off?

A: By buying the option, speculator capped her downside risk but lowering her upside gain

Q: Do speculators have to hold on the contract until maturity?

A: No, they can sell the contract before maturity.

2.4 Hedgers

Q: Assume you own Apple Stock in your portfolio. How would you hedge your position?

A: By buying a put option

Q: What is the benefit of buying the put option

A: She protects the value of her portfolio against big drops in the value of the stock

A Hedger Example

Scenario	Today price	Future Price	Future Portfolio Value
1	\$190	\$130	\$13,000
2	\$190	\$160	\$16,000
3	\$190	\$190	\$19,000
4	\$190	\$220	\$22,000
5	\$190	\$250	\$25,000

Table 3: Table: Hedger did not buy a put option, her today's portfolio value is \$19,000

Scenario	Today price [5]	Future Price	Payoff	Future Portfolio Value
1	\$190	\$130	\$5,000	$\$13,000 + \$5,000 - \$800 = \$17,200$
2	\$190	\$160	\$2,000	$\$16,000 + \$2,000 - \$800 = \$17,200$
3	\$190	\$190	\$0	$\$19,000 - \$800 = \$18,200$
4	\$190	\$190	\$220	\$0
5	\$190	\$250	\$0	$\$25,000 - \$800 = \$21,200$

Table 4: Table: Speculator bought a 6-month maturity put option at strike of \$180 and paid \$8

Observations

- Q: What was the effect of buying the put option?
- A: By buying the put option, she hedged her position as the portfolio value would not go below certain level not matter how low the Apple stock goes.
- Q: How to value an option?

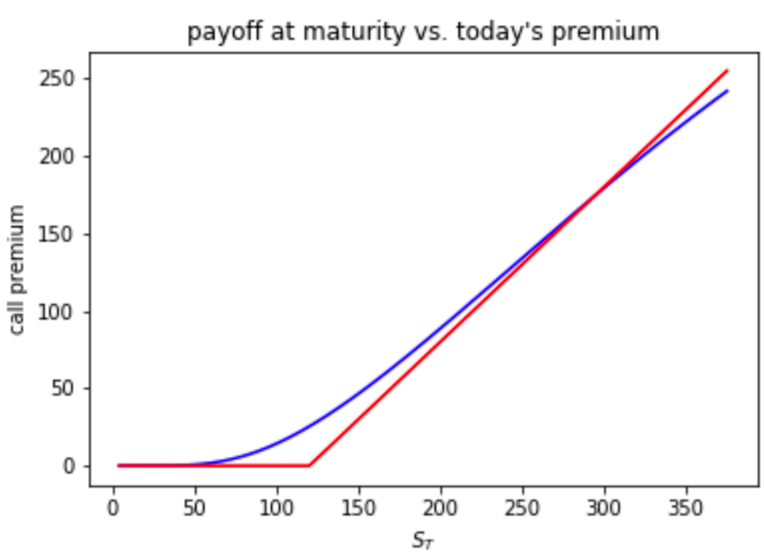
As for terms of the option contract:

- S_0 : today's price (known)
- K: strike price (pre-specified)
- T : maturity (pre-specified)
- $V_0(K,T)$: today's premium for K and T
- $S_{T:}$: Dpfice at time T (unknown)

2.5 Call

Call option:

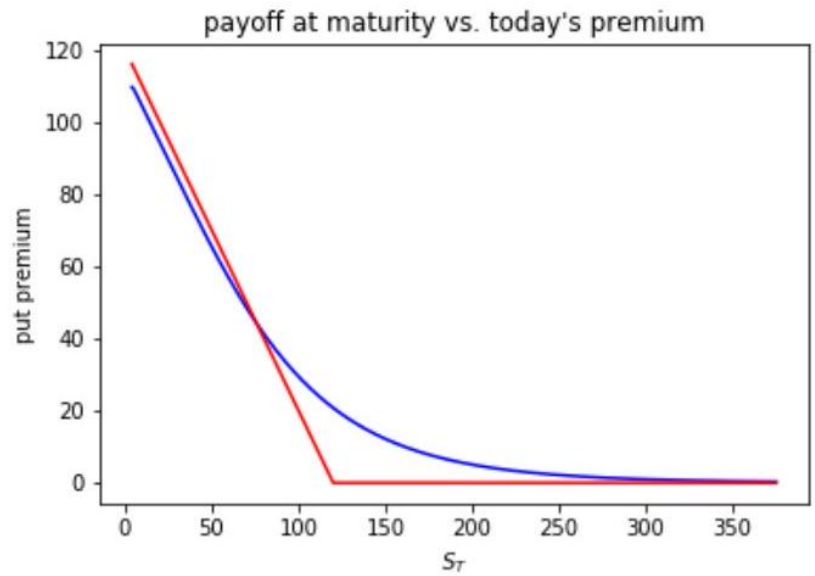
- Today's premium: $C_0(K,T)$
- Payoff at maturity: $(S_T - K)^+$



2.6 Put

Put:

- Today's premium: $P_0(K, T)$
- Payoff at maturity: $(K - S_T)^+$



3 Simple Pricing

What do we need for option pricing?

- Payoff function

$$\begin{aligned} \text{call: } & (S_T - K)^+ \\ \text{put: } & (K - S_T)^+ \end{aligned}$$

- (Conditional) probability distribution function of stock price at time T, i.e. $S_{\bar{c}}$, given today spot S_0 namely $f(S_T \mid S_0)$

3.1 How to Price an Option?

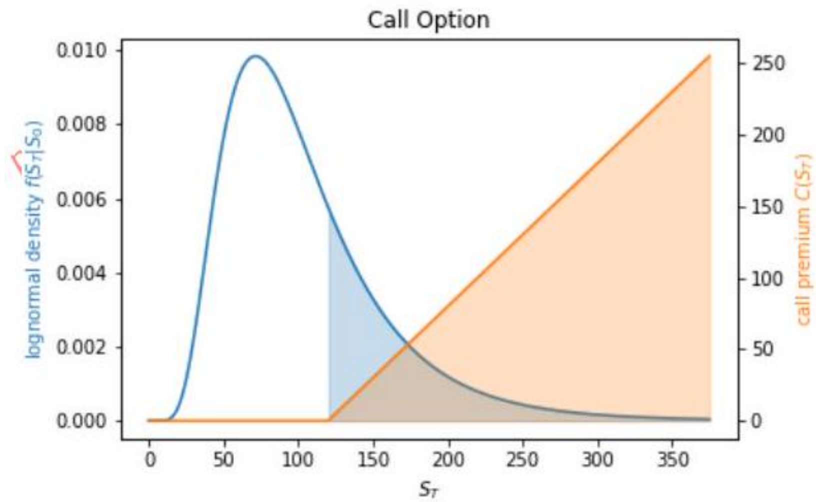
Integrate the payoff against probability distribution $f(S_T \mid S_0)$ i.e. expectation.
Q: What is missing?
A: Discounting, so far we have described the option value at expiration. Wes need to discount to get its today's value.

3.2 Price the Options Mathematically

For a Call Option

The European call option price $C_0(K, T)$ can be expressed as:

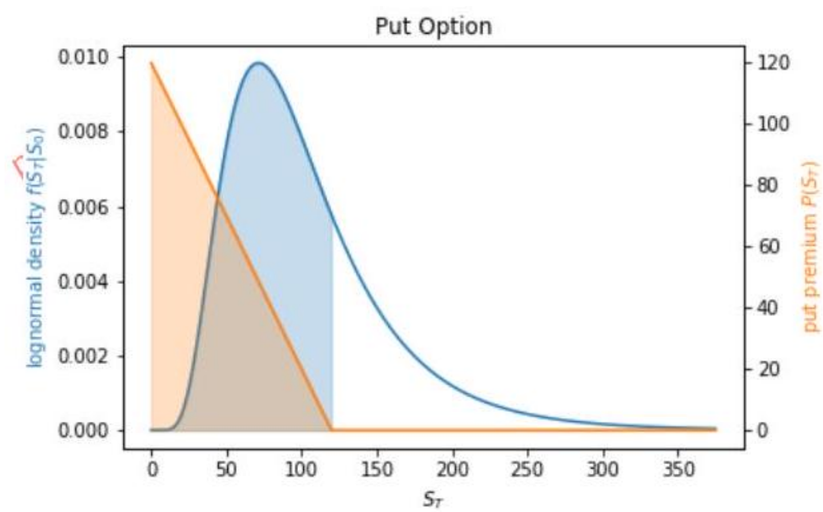
$$
\begin{aligned}
C_0(K,T) &= e^{-rT}\mathbb{E}_0\left[(S_T-K)^+\right] \\
&= e^{-rT}\int_0^{\infty} (S_T-K)^+ f\left(S_T \mid S_0^2\right) dS_T \\
&= e^{-rT}\int_K^{\infty} \left(S_{T\in T}0^2 K\right) f\left(S_T \mid S_0\right) dS_T
\end{aligned}
$$



For a Put Option

The European put option price $P_0(K,T)$ can be expressed as:

$$
\begin{aligned}
P_0(K,T) &= e^{-rT}\mathbb{E}_0\left[(K-S_T)^+\right] \\
&= e^{-rT}\int_0^{\infty} (K-S_T)^+ f\left(S_T Y S_0^{2ce^2}\right) dS_T \\
&= e^{-rT}\int_0^{K} \left(K_1+S_T\right) f\left(S_T \mid S_0\right) dS_T
\end{aligned}
$$



3.3 Evaluation of the Integral

How to do it? $\rightarrow$ First make it a definite integral (call options). Then, set N equal sub-intervals of length η and evaluate each sub-integral and add them up.

$$\begin{aligned}C_0(K,T) &= e^{-rT}\int_K^{\infty}\left(S_T-K\right)f\left(S_T\right)dS_T\\&\approx e^{-rT}\int_K^{B}(S-K)f(S)dS\end{aligned}$$

- Call options: choose η and N independently and set grid points to be:

$$s_j = K+(j-1)\eta \text{ for } j=1,\ldots,N+1$$

where the upper bound is $B=s_{N+1}=K+N\eta$

- Put options: upper bound is known (i.e. K), choose N and set $\eta=\frac{K}{N}$:

$$s_j=(j-1)\eta \text{ for } j=1,\ldots,N+1$$

$$C_0(K,T)\approx e^{-rT}\sum_{j=1}^{N}\int_{s_j}^{s_{j+1}}(S-K)f(S)dS^2$$

Using Trapezoidal Rule we can approximate each sub-integral:

$$\int_{s_j}^{s_{j+1}}(S-K)f(S)dS\approx\frac{\eta}{2}\left(\left(s_j-K\right)f\left(s_j\right)+\left(s_{j+1}-K\right)f\left(s_{j+1}\right)\right)$$

Substituting to obtain:

$$\begin{aligned}C_0\left(K_sT\right)^2\Big) &\approx e^{-rT}\sum_{j=1}^{N}\int_{s_j}^{s_{j+1}}(S-K)f(S)dS\\&\approx e^{-rT}\sum_{j=1}^{N}\frac{\eta}{2}\left(\left(s_j-K\right)f\left(s_j\right)+\left(s_{j+1}-K\right)f\left(s_{j+1}\right)\right)\end{aligned}$$

3.4 Approximated Call/Put Option

$$\begin{aligned}C_0(K,T)&\approx e^{-rT}\sum_{j=1}^{N}\left(s_j-K\right)f\left(s_j\mid S_0\right)w_j\\P_0(K,T)&\approx e^{-\kappa r}\sum^{N}\left(K-s_j\right)_{j=1}f\left(s_j\mid S_0\right)w_j\end{aligned}$$

where

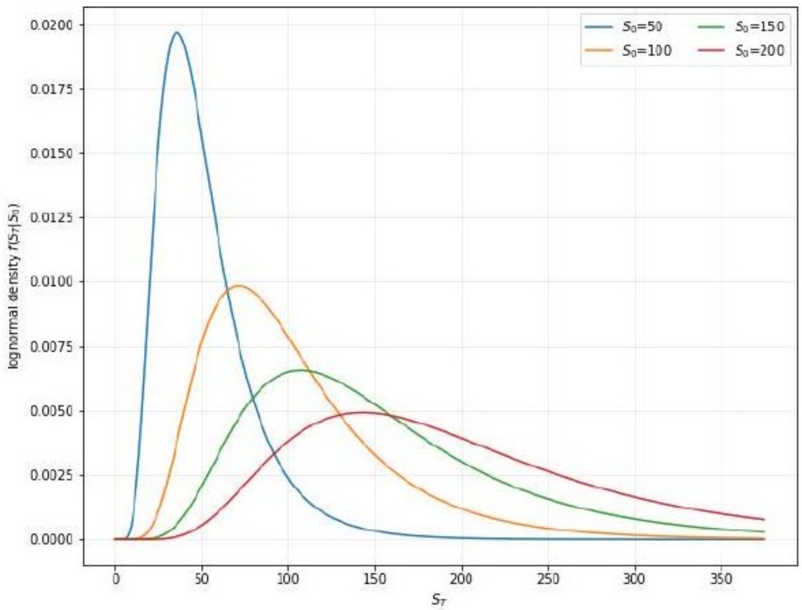
$$w_j=\begin{cases}\frac{1}{2}\eta & j=1\\ \eta & \text{otherwise}\end{cases}$$

Computational cost-$O(N)$

4 Example: Lognormal Distribution

4.1 Settings

$$f\left(S_T \mid S_0\right)=\frac{e^{-\frac{1}{2}\left(\frac{\ln S_T-\ln S_0-\left(r-q-\frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}\right)^2}}{\sigma S_T \sqrt{2 \pi T}}$$



- spot S_0 : \$100
- risk-free rate: 5%
- dividend rate: 1%
- maturity T : one year
- volatility 0.30%
- strike K : \$140

	η		
N	0.25	0.5	150
2^8	2.2247	2.8446	2.9130
2^9	2.8450	2.9133	2,9136
2^{10}	2.9134	2.9139	2.9136
2^{11}	2.9140	2.9139	2.9136
2^{12}	2.9140	2.9139	2.9136
2^{13}	2.9140	2.9139	2.9136

Table 5: Call prices for strike $K=\$140$ (out-the-money) for various values of $\eta \& N$

N	η	**Put Premium**
2^8	0.546	37.0810
2^9	0.273	37.0811
2^{10}	0.136	27.0811
2^{11}	0.068	37.0811
2^{12}	0.034	37.0811
2^{13}	0.017	37.0811

Table 6: Tables Put prices for strike $K = \$140$ (in-the-money) for various values N

4.2 Assessment of this Approach

- How good is this approximation? $\rightarrow$ For appropriate choices of N and η, the approximation is pretty good.
- How feasible is this approach?$\rightarrow$ If conditional probabilityodensity function, $f(S_T \mid S0)$, is known and available in closed-form, any numerical integration can be utilized to approximate the value of the option. However, in general $f(S_T \mid S_0)$ might not be available in an integrated form.
- Is there an alternative or a better approach? $\rightarrow$ Yes.

5 Characteristic Function

5.1 Fourier and Inverse Fourier Transform

Definition

For function $f(x)$, its Fourier transform is:

$$\Phi(\nu) = \int_{-\infty}^{\infty} e^{i\nu x} f(x) dx$$

Definition

Having $\Phi(\nu)$, the function $f(x)$ can be recovered via inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\nu x} \Phi(\nu) d\nu$$

5.2 Characteristic Function

Definition

If $f(x)$ is PDF of a r.v. x, its Fourier transform is called the characteristic function

$$\begin{aligned} \Phi(\nu) &= \int_{-\infty}^{\infty} e^{i\nu x} f(x) dx \\ &= \mathbb{E}\left(e^{i\nu x}\right) \end{aligned}$$

$f(x)$ can be recovered from its CF via IFT.

5.3 Example: Black-Merton-Scholes

S_t follows the B-M-S SDE:

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t$$

CF of the log of the stock price:

$$\Phi(\nu) = \exp\left(i\left(\ln S_0 + \left(r - q - \frac{\sigma^2}{2}\right)T\right)\nu - \frac{\sigma^2\nu^2}{2}T\right)$$

6 Continuation of Pricing

6.1 Set-Up

- S_T : T-time price
- $f\left(S_T \mid S_0\right)$: density of S_T
- $q\left(s_T \mid s_0\right)$: density of $s_T = \ln\left(S_T\right)$
- $k = \ln(K)$: log of the strike price
- $\Phi(\nu)$: CF of log of stock price process
- $\mathcal{C}_T(k)$: price of a T-maturity call with $K = e^k$

6.2 Formulation

The European call option price $C_T(k)$ can be expressed as:

$$\begin{aligned} e^{-rT}\mathbb{E}_0\left[\left(S_T - K\right)^+\right] &= e^{-rT}\int_K^{\infty}\left(S_T - K\right) f\left(S_T \mid S_0\right) dS_T \\ &= C_T(k) \end{aligned}$$

6.3 $f\left(S_T \mid S_0\right)$ and $q\left(s_T \mid s_0\right)$

資產價格的 PDF:

$$f(S_T \mid S_0) = \frac{1}{\sigma S_T\sqrt{2\pi T}}\exp\left\{-\frac{1}{2}\left(\frac{\ln S_T - \ln S_0 - (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)^2\right\}$$

對數資產價格的 PDF:

$$q(s_T \mid s_0) = \frac{1}{\sigma\sqrt{2\pi T}}\exp\left\{-\frac{1}{2}\left(\frac{s_T - s_0 - (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)^2\right\}$$

6.4 Modified Call and its Fourier Transform

Define $c_T(k) = e^{\alpha k} C_T(k)$. Let $\Psi_T(\nu)$ be its Fourier transform

$$\begin{aligned}
\Psi_T(\nu) &= \int_{-\infty}^{\infty} e^{i\nu k} c_T(k) dk \\
&= \int_{-\infty}^{\infty} e^{i\nu k} e^{\alpha k} C_T(k) dk
\end{aligned}$$

Can use the inverses Fourier transform to get

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-i\nu k} \Psi_T(\nu) d\nu$$

6.5 Evaluation by Switching Order of Integration

通過調整積分的順序, 展現了傅立葉轉換下的 Modified Call Price 的推導:

$$\begin{aligned}
\Psi_T(\nu) &= \int_{-\infty}^{\infty} e^{i\nu k} c_T(k)\, dk \\
&= \int_{-\infty}^{\infty} e^{i\nu k} e^{\alpha k} \left(e^{-rT} \int_{k}^{\infty} (e^s - e^k) q(s)\, ds \right) dk \\
&= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{s} e^{(\alpha + i\nu)k} (e^s - e^k) q(s)\, dk\, ds \\
&= e^{-rT} \int_{-\infty}^{\infty} q(s) \left(\int_{-\infty}^{s} e^{(\alpha + i\nu)k} (e^s - e^k)\, dk \right) ds
\end{aligned}$$

6.6 Evaluation of Inner Integral

$$\begin{aligned}
\int_{-\infty}^{s} e^{(\alpha + i\nu)k} (e^s - e^k)\, dk &= e^s \left. \frac{e^{(\alpha + i\nu)k}}{\alpha + i\nu} \right|_{-\infty}^{s} - \left. \frac{e^{(\alpha + i\nu + 1)k}}{\alpha + i\nu + 1} \right|_{-\infty}^{s} \\
&= e^s \cdot \frac{e^{(\alpha + i\nu)s}}{\alpha + i\nu} - \frac{e^{(\alpha + i\nu + 1)s}}{\alpha + i\nu + 1} \\
&= \frac{e^{(\alpha + i\nu + 1)s}}{(\alpha + i\nu)(\alpha + i\nu + 1)}
\end{aligned}$$

6.7 $\Psi_T(\nu)$ with Respect to $\Phi(\nu)$

$$\begin{aligned}
\Psi_T(\nu) &= e^{-rT} \int_{-\infty}^{\infty} q(s) \cdot \frac{e^{(\alpha + i\nu + 1)s}}{(\alpha + i\nu)(\alpha + i\nu + 1)}\, ds \\
&= \frac{e^{-rT}}{(\alpha + i\nu)(\alpha + i\nu + 1)} \int_{-\infty}^{\infty} e^{(\alpha + i\nu + 1)s} q(s)\, ds \\
&= \frac{e^{-rT}}{(\alpha + i\nu)(\alpha + i\nu + 1)} \cdot \Phi(\nu - (\alpha + 1)i)
\end{aligned}$$

6.8 Call Option via Inverse Fourier Transform

Recall

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-i\nu k} \boxed{\Psi_T(\nu)} \, d\nu$$

where

$$\Psi_T(\nu) = \frac{e^{-rT}}{(\alpha + i\nu)(\alpha + i\nu + 1)} \cdot \Phi(\nu - (\alpha + 1)i)$$

6.9 Integral Evaluation

$$\begin{aligned} C_T(k) &= \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-i\nu k} \Psi_T(\nu) d\nu \\ &\approx \frac{e^{-\alpha k}}{\pi} \int_0^{B} e^{-i\nu k} \Psi_T(\nu) d\nu \end{aligned}$$

Set N equal sub-intervals of length η where

$$\eta = \frac{B}{N}$$

$\nu_j = (j-1)^2$ for $j = 1, \ldots, N+1$.

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} \int_{\nu_j}^{\nu_{j+1}} e^{-i\nu k} \Psi_T(\nu) d\nu$$

Using Trapezoidal Rule we can approximate each sub-integral:

$$\int_{\nu_j}^{\nu_{j+1}} e^{-i\nu k} \Psi_T(\nu) d\nu \approx \frac{\eta}{2} \left(e^{-i\nu_j k} \Psi_T\left(\nu_j\right) + e^{-i\nu_{j+1} k} \Psi_T\left(\nu_{j+1}\right) \right)$$

Substituting to obtain:

$$\begin{aligned} C_T(k) &\approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} \int_{\nu_j}^{\nu_{j+1}} e^{-i\nu k} \Psi_T(\nu) \, d\nu \\ &\approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} \frac{\eta}{2} \left(e^{-i\nu_j k} \Psi_T(\nu_j) + e^{-i\nu_{j+1} k} \Psi_T(\nu_{j+1}) \right) \end{aligned}$$

6.10 Call Price for a Specific Strike

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\nu_j k} \psi_T\left(\nu_j\right) w_j$$

where

$$w_j = \begin{cases} \frac{1}{2}\eta & j = 1 \\ \eta & \text{otherwise} \end{cases}$$

Computational cost-$O(N)$

6.10.1 Comparison

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\nu_j k} \Psi_T(\nu_j) w_j$$

versus

$$C_0(K,T) \approx e^{-rT} \sum_{j=1}^{N} (s_j - K) f(s_j|S_0) w_j$$

where

$$w_j = \begin{cases} \frac{1}{2}\eta & j=1 \\ \eta & \text{otherwise} \end{cases}$$

6.11 How about Calls with Various Strikes

- If we are interested in just one specific strike, no extra step is needed.
- How about calculating option premium at that maturity for various different strikes; assuming m strikes, repeat the procedure m times, computational cost $O(m \times N)$.
- It turns out that we can use FFT to get premiums at N different strikes with cost of $O(N \ln N)$ as opposed to $O(N^2)$.

6.12 Calls with Various Strikes

For $m = 1, \ldots, N$

$$\begin{aligned} C_T(k_m) &\approx \frac{e^{-\alpha k_m}}{\pi} \sum_{j=1}^{N} e^{-i\nu_j k_m} \Psi_T(\nu_j) w_j \\ &= \frac{e^{-\alpha k_m}}{\pi} \sum_{j=1}^{N} e^{-i(j-1)\eta(m-1)\Delta k} e^{-i\beta\nu_j} \Psi_T(\nu_j) w_j \\ &= \frac{e^{-\alpha k_m}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(m-1)} \boxed{e^{-i\beta\nu_j} \Psi_T(\nu_j) w_j} \\ C_T(k_m) &\approx \frac{e^{-\alpha k_m}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(m-1)} \boxed{x(j)} \end{aligned}$$

6.13 Fast Fourier Transform

FFT provides an efficient algorithm for calculating the following

$$\omega(m) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(m-1)} x(j)$$

for $m = 1, \ldots, N$

Q: What is its efficiency?

A: Computational cost $-O(N \ln N)$ as opposed to $O(N^2)$

Q: How can we use FFT?
Aso'By setting $\lambda\eta = \frac{2\pi}{N}$, we can use FFT to calculate the sum

6.14 Implementation

Having $\Phi(\nu)$, choose $\eta, N = 2^n$, and β, calculate $\lambda = \frac{2\pi}{N\eta}$, $\nu_j = (j-1)\eta$, and set $\alpha > 0$. Form vector $\mathbf{x}$.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \frac{\eta}{2} \cdot \frac{e^{-rT}}{(\alpha+i\nu_1)(\alpha+i\nu_1+1)} e^{-i\beta\nu_1}\, \Phi\left(\nu_1 - (\alpha+1)i\right) \\ \frac{\eta}{(\alpha+i\nu_2)(\alpha+i\nu_2+1)} \cdot e^{-rT}\, e^{-i\beta\nu_2}\, \Phi\left(\nu_2 - (\alpha+1)i\right) \\ \vdots \\ \frac{\eta}{(\alpha+i\nu_N)(\alpha+i\nu_N+1)} \cdot e^{-rT}\, e^{-i\beta\nu_N}\, \Phi\left(\nu_N - (\alpha+1)i\right) \end{pmatrix}$$

$$\mathbf{y} = \text{fft}(\mathbf{x})$$

Call prices at strike k_m for $m = 1, \ldots, N$

$$\begin{pmatrix} C_T(k_1) \\ C_T(k_2) \\ \vdots \\ C_T(k_N) \end{pmatrix} = \begin{pmatrix} \frac{e^{-\alpha k_1}}{\pi} \operatorname{Re}(y_1) \\ \frac{e^{-\alpha k_2}}{\pi} \operatorname{Re}(y_2) \\ \vdots \\ \frac{e^{-\alpha k_N}}{\pi} \operatorname{Re}(y_N) \end{pmatrix}$$

where $k_m = \beta + (m-1)\lambda$

6.15 Choice of β

$$k_m = \beta + (m-1)\lambda \text{ for } m = 1, \ldots, N$$

Two common choices:

- middle of the range corresponds to at-the-money: set$\beta = \ln(S_0) - \frac{N}{2}\lambda$
- the first call corresponds to a specific strike K : set $\beta = \ln(K)$

7 Examples

7.1 Parameters

- Spot S_0 : \$100
- Strike K : \$80
- risk-free rate: 5%
- dividend rate: 1%
- maturity T : one year

7.2 Example 1: Black-Merton-Scholes

S_t follows the B-M-S SDE:

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t$$

CF of the log of the stock price:

$$\Phi(\nu) = e^{i\left(\ln S_0 + \left(r-q-\frac{\sigma^2}{2}\right)T\right)\nu - \frac{\sigma^2\nu^2}{2}T}$$

In this example, we assume $\Theta = \{\sigma\} = \{0.3\}$

	$\eta = 0.10$		$\eta = 0.25$	
α	$N = 2^6$	2^{10}	2^6	2^{10}
0.01	138.7372	138.8336	372.1118	372.1118
0.5	25.4710	25.6146	25.6150	25.6150
1.0	25.4432	25.6146	25.6146	25.6146
1.5	25.4497	25.6146	25.6146	25.6146
2.0	25.5024	25.6146	25.6146	25.6146
5.0	26.4853	25.6146	25.6146	25.6146
10.0	9.0641	25.6146	25.6141	25.6146

Table 7: B-M-S premiums for various values of α, N, and η

7.3 Example 2: Heston Stochastic Volatility Model

S_t follows the following SDE:

$$dS_t = (r-q)S_t dt + \sqrt{v_t} S_t dW_t^{(1)}$$
$$dv_t = \kappa\left(\theta - v_t\right) dt + \lambda\sqrt{v_v} dW_t^{(2)}$$

CF of the log of the stock price:

$$\Phi(u) = \frac{\exp\left\{iu\ln S_0 + iu(r-q)T + \frac{\kappa\theta T(\kappa - i\rho\sigma u)}{\sigma^2}\right\}}{\left(\cosh\frac{\gamma T}{2} + \frac{\kappa - i\rho\sigma u}{\gamma}\sinh\frac{\gamma T}{2}\right)^{\frac{2\kappa\theta}{\sigma^2}}}$$
$$\times \exp\left\{\frac{-\left(u^2+iu\right)v_0}{\gamma\coth\frac{\gamma T}{2} + \kappa - i\rho\sigma u}\right\}$$

where

$$\gamma = \sqrt{\sigma^2\left(u^2+iu\right) + \left(\kappa - i\rho\sigma u\right)^2}$$

In this example, we assume

$$\Theta = \{\{k, \theta, \lambda, \rho, v_0\} = \{2, 0.05, 0.3, -0.7, 0.04\}$$

	$\eta = 0.10$		$\eta = 0.25$	
α	$N = 2^6$	2^{10}	$2^{66} + 0^0$	2^{10}
0.01	139.0174	139.5996	375.2260	375.2224
0.5	24.5194	25.2428	25.2457	25.2432
1.0	24.3968	25.2428	25.2431	25.2428
1.5	24.3160	25.2428	25.2393	25.2428
2.0	24.2985	25.2428	25.2335	25.2428
5.0	26.6605	25.2428	25.1255	25.2428
20.0	48.7592	25.2428	25.3538	25.2428

Table 8: Premiums for various values of α, N, and η

7.4 Example 3: Variance Gamma Model

S_t is given by:

$$S_t = S_0 e^{(r-q+\omega)t + X(t;\partial,\nu,\theta)}$$

where (這裡是隨時間變動的布朗運動)

$$X\left(t;\sigma,\nu,\theta\right) = \theta\gamma(t;1,\nu) + \sigma W(\gamma(t;1,\nu))$$

CF of the log of the stock price:

$$\Phi(u) = \exp\left(iu\left(\ln S_0 + (r-q)T\right)\right)\left(\frac{1}{1 - iu\theta\nu + \frac{1}{2}\sigma^2u^2\nu}\right)^{\frac{T}{\nu}}$$

In this example, we assume:

$$\Theta = \{\sigma, \nu, \theta\} = \{0.3, 0.5, -0.4\}$$

可以發現這些 premium 的差別都不大.

	$\eta = 0.10$		$\eta = 0.25$	
α	$N = 2^6$	2^{10}	2^6	2^{10}
0.01	141.2533	141.4392	374.7132	374.7175
0.5	28.0383	28.2203	28.2153	28.2206
1.0	28.0855	28.2203	28.2139	28.2203
1.5	28.1944	28.2203	28.2129	28.2203
2.0	28.3824	28.2203	28.2124	28.2203
5.0	31.5220	28.2203	28.2570	28.2203
20.0	9.2062	28.2203	29.4317	28.2203

Table 9: Premiums for various values of α, N, and η

7.5 Findings and Observations

Pros

- Model-free setup
- Fast $O(N \ln(N))$

Cons

- Constraint on
- Choice of η, N and α