Lecture 2, IEOR 4732

Fourier Transform and COS Method

傅立葉變換和 COS 方法

Zongyi Liu

Thu, Mar 27, 2025

Contents

1 Agenda

- Recap of FFT
- Fractional FFT (FrFFT)
- COS method

2 FFT

2.1 Implementation

Having $\Phi(\nu)$, choose $\eta, N=2^n$, and β, calculate $\lambda=\frac{2\pi}{N/\eta}$, $\nu_j=(j-1)\eta$, and set $\alpha>0$. Form vector $\mathbf{x}$.

$$\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}=\begin{pmatrix} \frac{\eta}{2}\frac{e^{-rT}}{(\alpha+i\nu_1)(\alpha+i\nu_1+1)}e^{-i\beta\nu_1}\Phi\left(\nu_1-(\alpha+1)i\right) \\ \frac{\eta}{(\alpha+i\nu_2)(\alpha+i\nu_2+1)}e^{-rT}e^{-i\beta\nu_2}\Phi\left(\nu_2-(\alpha+1)i\right) \\ \vdots \\ \frac{\eta}{(\alpha+i\nu_N)(\alpha+i\nu_N+1)}e^{-rT}e^{-i\beta\nu_N}\Phi\left(\nu_N-(\alpha+1)i\right) \end{pmatrix}$$

$$\mathbf{y}=\mathrm{fft}(\mathbf{x})$$

Call prices at strike k_m for $m=1,\ldots,N$

$$\begin{pmatrix} C_T(k_1) \\ C_T(k_2) \\ \vdots \\ C_T(k_N) \end{pmatrix}=\begin{pmatrix} \frac{e^{-\alpha k_1}}{\pi}Re(y_1) \\ \frac{e^{-\alpha k_2}}{\pi}Re(y_2) \\ \vdots \\ \frac{e^{-\alpha k_N}}{\pi}Re(y_N) \end{pmatrix}$$

where $k_m=\beta+(m-1)\lambda$

2.2 Choice of β

$k_m=\beta+(m-1)\lambda$ for $m=1,\ldots,N$

- Two common choices:
  - Middle of the range corresponds to at-the-money: set $\beta=\ln\left(S_0\right)-\frac{N}{2}\lambda$.
  - The first call corresponds to a specific strike K : set $\beta=\ln(K)$.
- Have in mind that the stepsize λ could introduce interpolation error.

2.3 Constraint

- The following relationship between λ and η

$$\lambda\eta=\frac{2\pi}{N}$$

- For $N=2^{12}=4096$ and $\eta=0.25$ we have

$$\lambda=\frac{2\pi}{N/\eta}=0.0061$$

i.e. calculated es options roughly 67 $\left(2\times\frac{20\%}{0.61\%}+1\approx 67\right)$ will fall within the 20% log-strike interval.

- for $N=2^8=256$ and $\eta=0.25$ we have:

$$\lambda=\frac{2\pi}{N/\eta}=0.0981$$

3 Fractional FFT

- The goal is to eliminate dependency between λ and η.
- Fractional FFT does that.
- The fractional FFT procedure computes a sum of the form

$$\sum_{\gamma=1}^{N} e^{ei2\pi\gamma(j-1)(m-1)} x(j)$$

for any value of γ

- A special case: $\gamma = \frac{1}{N}$

3.1 Formation

Define the N-long complex sequence x as:

$$G_m(x,\gamma) = \sum_{j=1}^{N} e^{-i2\pi\gamma(j-1)(m-1)} x(j)$$

γ any complex rational number. The sum can be implemented via three $2N$-point FFT steps. For an N-point fractional FFT on the vector $x(j)$, we define the following $2N$-long sequences:

$$\begin{aligned}
y_j &= x_j e^{-i\pi(j-1)^2\gamma} && 1 \le j \le N \\
y_j &= 0 && N < j \le 2N \\
z_j &= e^{i\pi(j-1)^2\gamma} && 1 \le j \le N \\
z_j &= e^{i\pi(2N-j)^2\gamma} && N < j \le 2N
\end{aligned}$$

$$\text{where} \quad \gamma = \frac{\lambda\eta}{2\pi}.$$

3.2 Formation

It is shown that:

$$G_m(x,\gamma) = \left(e^{-i\pi(m-1)^2}\gamma\right) \odot \mathrm{D}_m^{-1}(\mathrm{D}(\mathbf{y}) \odot \mathrm{D}(\mathbf{z})) \quad 1 \le m \le N$$

$\odot$ element componentwise vector multiplication

$$\mathrm{D}(\xi) = \begin{pmatrix} \mathrm{D}_1(\xi) \\ \mathrm{D}_2(\xi)) \\ \vdots \\ \mathrm{D}_{2N}(\xi) \end{pmatrix}$$

With

$$\eta_j = \mathcal{D}_j(\xi) = \sum_{m=1}^{2N} \exp\left(-i\frac{2\pi}{2N}(j-1)(m-1)\right) \xi(m), \quad 1 \le j \le 2N$$

and

$$\xi_m = \mathcal{D}_m^{-1}(\eta) = \frac{1}{2N}\sum_{j=1}^{2N} \exp\left(i\frac{2\pi}{2N}(j-1)(m-1)\right)\eta(j), \quad 1 \le m \le 2N$$

- The remaining N results of the final inverse discrete Fourier transform are discarded
- exponential quantities do not depend on the actual function that is integrated and therefore can be precomputed and stored

3.3 Implementation

Having the characteristic function of the log of the underlying process X_t that is $\Phi(\nu)$, we choose η, λ (independently) and $N = 2^n$. Calculate $\gamma = \frac{\eta\lambda}{2\pi}, \nu_j = (j-1)\eta$ and set α. Form vector $\mathbf{x}$.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \frac{\eta}{2}\frac{C}{(\alpha+i\nu_1)(\alpha+i\nu_1+1)}e^{-i\beta\nu_1}\Phi(\nu_1-(\alpha+1)i) \\ \frac{\eta C}{(\alpha+i\nu_2)(\alpha+i\nu_2+1)}e^{-i\beta\nu_2}\Phi(\nu_2-(\alpha+1)i) \\ \vdots \\ \frac{\eta C}{(\alpha+i\nu_N)(\alpha+i\nu_N+1)}e^{-i\beta\nu_N}\Phi(\nu_N-(\alpha+1)i) \end{pmatrix}$$

Form $\mathbf{y}$ and $\mathbf{z}$:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \\ y_{N+1} \\ y_{N+2} \\ \vdots \\ y_{2N} \end{pmatrix} = \begin{pmatrix} x_1 \\ \exp(-i\pi\gamma)x_2 \\ \vdots \\ \exp(-i\pi\gamma(N-1)^2)x_N \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \\ z_{N+1} \\ z_{N+2} \\ \vdots \\ z_{2N} \end{pmatrix} = \begin{pmatrix} 1 \\ \exp(i\gamma\pi) \\ \vdots \\ \exp(i\gamma\pi(N-1)^2) \\ \exp(i\gamma\pi(N-1)^2) \\ \exp(i\gamma\pi(N-2)^2) \\ \vdots \\ 1 \end{pmatrix}$$

$\mathbf{y}$ and $\mathbf{z}$ are the input to FFT, and its output issivectors $\widehat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ of the same size. Form ξ by multiplying vectors $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ element-wise.

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ c_2 \\ \vdots \\ \xi_{2N} \end{pmatrix} = \begin{pmatrix} \widehat{y}_1 \widehat{z}_1 \\ \widehat{y}_2 \widehat{z}_2 \\ \vdots \\ \widehat{y}_{2N} \widehat{z}_{2N} \end{pmatrix}$$

ξ is the input IFFT, and its output is $\widehat{\xi}$. Using vector $\hat{\xi}$, call prices at strike k_m for $m = 1, \ldots, N$ are:

$$\begin{pmatrix} C_T(k_1) \\ C_T(k_2) \\ \vdots \\ C_T(k_N) \end{pmatrix} = \begin{pmatrix} \frac{e^{-\alpha\beta\lambda}}{\pi} \operatorname{Re}\left(\hat{\xi}_1\right) \\ \frac{e^{-\alpha(\beta+1)\lambda}}{\pi} \operatorname{Re}\left(\exp(-i\pi\gamma)\hat{\xi}_2\right) \\ \vdots \\ \frac{e^{-\alpha(\beta+(N-1))\lambda}}{\pi} \operatorname{Re}\left(\exp(-i\pi\gamma(N-1)^2)\hat{\xi}_N\right) \end{pmatrix}$$

wheree as before $\operatorname{Re}(z)$ is the real part of z

3.4 FrFFT vs. FFT

關於傅立葉變換和分數階傅立葉變換的的區別.

- Note that the last N elements of $\widehat{\xi}$ are never used and are discarded
- Considering λ and η are independent, we can choose λ that would yield a range around $\operatorname{tn} X_0$ with desired money-ness (for example, for 25% money-ness we get $\lambda = \frac{2(0.25)}{N}$
- Also typically N $\because= 2^n$ is much smaller than the fast Fourier technique (e.g., 2^7 as opposed to 2^{14})
  Q: Ask yourself about FFT vs. FrFFT

4 Cosine Method

4.1 Cosine Series Expansion

The Fourier cosine series expansion of a function $f(\theta)$ on $[0, \pi]$ is

$$\begin{aligned} f(\theta) &= \frac{1}{2} A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k_k \theta\right) \\ &= \sum_{in}^{\infty} {}_{k=0}^{\infty} A_k \cos(k\theta) \end{aligned}$$

with the Fourier cosine coefficient

$$A_k = \frac{2}{\pi} \int_0^\pi f(\theta) \cos(k\theta) d\theta$$

where $\bar{\sum}$ indicates the first term in the summation is weighted by one-half.
For functions on $[a, b]$, make the following change of variable that maps a to 0 and b to π :

$$\theta = \frac{\pi - 0}{b - a}(x - a) = \frac{x - a}{b - a}\pi$$

x in terms of θ

$$x = \frac{b-a}{\pi}\theta + a$$

substitute

$$f(x) = \sum\nolimits_{k=0}^{\infty} A_k \cos\left(k\frac{x-a}{b-a}\pi\right)$$

with

$$A_k = \frac{2}{b-a}\int_a^b f(x)\cos\left(k\frac{x-a}{b-a}\pi\right)dx$$

4.2 Linking it to CF

$$\mathbb{E}\left(e^{i\nu x}\right) = \phi(\nu) = \int_{-\infty}^{\infty} e^{i\nu x} f(x)dx$$

Evaluate it at $\nu = \frac{k\pi}{b-a}$

$$\phi\left(\frac{k\pi}{b-a}\right) = \int_{-\infty}^{\infty} e^{i\left(\frac{k\pi}{b-a}\right)x} f(x)\,dx$$

$$\hat{\phi}\left(\frac{k\pi}{b-a}\right) = \int_{a}^{b} e^{i\left(\frac{k\pi}{b-a}\right)x} f(x)\,dx$$

Multiplying it by $e^{-i\frac{k\pi a}{b-a}}$

$$\hat{\phi}\left(\frac{k\pi}{b-a}\right)e^{-i\frac{k\pi a}{b-a}} = \int_{a}^{b} e^{ik\pi\left(\frac{x-a}{b-a}\right)} f(x)\,dx$$

$$= \int_a^b \left(\cos\left(k\pi\frac{x-a}{b-a}\right) + i\sin\left(k\pi\frac{x-a}{b-a}\right)\right)f(x)\,dx$$

Thus
$\mathrm{Re}\left\{\hat{\phi}\left(\frac{k\pi}{b-a}\right)\exp\left(-i\frac{ka\pi}{b-a}\right)\right\} = \int_a^b \cos\left(k\pi\left(\frac{x-a}{b-a}\right)\right)f(x)dx$
If we assume $[a,b]$ is s.t.

$$\widehat{\phi}(\nu) = \int_a^b e^{ivx} f(x)dx \approx \int_{-\infty}^{+\infty} e^{i\nu x} f(x)dx = \phi(\nu)$$

$$A_k = \frac{2}{b-a}\,\mathrm{Re}\left\{\hat{\phi}\left(\frac{k\pi}{b-a}\right)\exp\left(-i\frac{ka\pi}{b-a}\right)\right\}$$

with $A_k \approx F_k$ where

$$F_k = \frac{2}{b-a}\,\mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a}\right)\exp\left(-i\frac{ka\pi}{b-a}\right)\right\}$$

$$\hat{f}(x) = \sum_{k=0}^{\infty} F_k\cos\left(k\frac{x-a}{b-a}\pi\right)$$

truncating it further

$$\tilde{f}(x) = \bar{\sum}_{k=0}^{N-1} F_k \cos\left(k \frac{x-a}{b-a} \pi\right)$$

原文有個 long bar, 在 sum 符號上, 不知道是幹嘛的

4.3 COS Option Pricing

- x be the modeled quantity at $t, \ln X_t$
- y be the modeled quantity at $T, \ln X_T$
- $f(y \mid x)$ be the probability density function under the pricing measure
- $v(x, t)$ be the option value at t
- $v(y, T)$ be the option value at T, the payoff at expiration

Then the option value at time t can be written as

$$v(x, t) = C \int_a^b v(y, T) f(y \mid x) dy$$

for an appropriate value of C.

$$\begin{aligned} v(x, t) &= C \int_a^b v(y, T) \bar{\sum}_{k=0}^{\infty} A_k \cos\left(k \frac{y-a}{b-a} \pi\right) dy \\ &= C \sum_{k=0}^{\infty} A_k \left(\int_a^b v(y, T) \cos\left(k \frac{y-a}{b-a} \pi\right) dy\right) \end{aligned}$$

Define

$$V_k = \frac{2}{b-a} \int_a^b v(y, T) \cos\left(k\pi \frac{y-a}{b-a}\right) dy$$

$$v(x, t) = \frac{b-a}{2} C \sum_{k=0}^{\infty} A_k V_k$$

another approximation:

$$\approx C \sum_{k=0}^{N-1} \operatorname{Re}\left\{\phi\left(\frac{k\pi}{b-a}; x\right) \exp\left(-ik\pi \frac{a}{b-a}\right)\right\} V_k$$

4.4 Vanilla Option Price

- X_t is the current price of the underlying security
- X_T is the T-time price of the underlyingosecurity
- K is the strike of the option
- $x = \ln\left(X_t / K\right)$
- $y = \ln\left(X_T / K\right)$

Vanilla European options expressed as

$$v(y,T) = [\alpha K\,(e^y - 1)]^+$$

with $\alpha = 1$ for a call and $\alpha = -1$ for a put.

V_k are known analytically for vanilla options. Define

$$\chi_k(c,d) = \int_c^d e^y \cos\left(k\pi\frac{y-a}{b-a}\right) dy$$
$$\varphi_k(c,d) = \int_c^d \cos\left(k\pi\frac{y-a}{b-a}\right) dy$$

For a vanilla call and put we obtain

$$\begin{aligned} V_k^{\text{call}} &= \frac{2}{b-a}\int_a^b K(e^y-1)^+ \cos\left(k\pi\frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b) - \varphi_k(0,b)\right) \end{aligned}$$

$$\begin{aligned} V_k^{\text{put}} &= \frac{2}{b-a}\int_a^b K(1-e^y)^+ \cos\left(k\pi\frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a}K\left(-\chi_k(a,0) + \varphi_k(a,0)\right) \end{aligned}$$

4.5 Pros and Cons

Q: how to find the truncation range $[a, b]$

A: not easy, one of disadvantages of this method

- One advantage of the COS method is its flexibility in switching to different payoffs