# Homework 5, MATH 4061

Zongyi Liu

Mon, Oct 13, 2025

**1.** Prove that the empty set is a subset of every set.

**2.** A complex number $z$ is said to be *algebraic* if there are integers $a_0, \dots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer $N$ there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

**3.** Prove that there exist real numbers which are not algebraic.

**4.** Is the set of all irrational real numbers countable?

**5.** Construct a bounded set of real numbers with exactly three limit points.

**6.** Let $E'$ be the set of all limit points of a set $E$. Prove that $E'$ is closed. Prove that $E$ and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do $E$ and $E'$ always have the same limit points?

**7.** Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \ldots$.

(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$. Show, by an example, that this inclusion can be proper.

**8.** Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of $E$? Answer the same question for closed sets in $\mathbb{R}^2$.

**9.** Let $E^\circ$ denote the set of all interior points of a set $E$. [*See Definition 2.18(e); $E^\circ$ is called the* interior of $E$.]

(a) Prove that $E^\circ$ is always open.

(b) Prove that $E$ is open if and only if $E^\circ = E$.

(c) If $G \subset E$ and $G$ is open, prove that $G \subset E^\circ$.

(d) Prove that the complement of $E^\circ$ is the closure of the complement of $E$.

(e) Do $E$ and $\bar{E}$ always have the same interiors?

(f) Do $E$ and $E'$ always have the same closures?

[12pt]article amsmath, amssymb

**10.** Let $X$ be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1, & \text{if } p \neq q, \\ 0, & \text{if } p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

**11.** For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2 - y^2|, \\ d_4(x,y) &= |x - 2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

**12.** Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that $K$ is compact directly from the definition (without using the HeineBorel theorem).

**13.** Construct a compact set of real numbers whose limit points form a countable set.

**14.** Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

**15.** Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for example) if the word compact is replaced by closed or by bounded.

**16.** Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let $E$ be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that $E$ is closed and bounded in $\mathbb{Q}$, but that $E$ is not compact. Is $E$ open in $\mathbb{Q}$?

**17.** Let $E$ be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is $E$ countable? Is $E$ dense in $[0,1]$? Is $E$ compact? Is $E$ perfect?

**18.** Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

**19.** (a) If $A$ and $B$ are disjoint closed sets in some metric space $X$, prove that they are separated. (b) Prove the same for disjoint open sets. (c) Fix $p \in X$, $\delta > 0$, define $A$ to be the set of all $q \in X$ for which $d(p,q) < \delta$, define $B$ similarly, with $>$ in place of $<$. Prove that $A$ and $B$ are separated. (d) Prove that every connected metric space with at least two points is uncountable. *Hint:* Use (c).

**20.** Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

**21.** Let $A$ and $B$ be separated subsets of some $\mathbb{R}^k$, suppose $a \in A$, $b \in B$, and define

$$p(t) = (1-t)a + tb$$

for $t \in \mathbb{R}^1$. Put $A_0 = p^{-1}(A)$, $B_0 = p^{-1}(B)$. [Thus $t \in A_0$ if and only if $p(t) \in A$.]