Homework 5, MATH 4061

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1 Question 1

Set of subsequential limits

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$ and denote with $\mathcal{A}$ the set of subsequential limits. Show that $\mathcal{A}$ is closed.

Answer

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$, and let:

$$\mathcal{A} = \{a \in \mathbb{R} : \exists (x_{n_k}) \text{ a subsequence of } (x_n) \text{ with } x_{n_k} \to a\}$$

be the set of subsequential limits. We will show that $\mathcal{A}$ is closed.

Suppose (a_m) is a sequence in $\mathcal{A}$ such that $a_m \to a$ for some $a \in \mathbb{R}$. We must prove that $a \in \mathcal{A}$.

For each $m \in \mathbb{N}$, since $a_m \in \mathcal{A}$, there exists a subsequence $(x_{n_j^{(m)}})_{j\geq 1}$ of (x_n) such that:

$$x_{n_j^{(m)}} \to a_m \quad \text{as } j \to \infty.$$

We now construct a single subsequence (x_{k_m}) of (x_n) that converges to a, then proceed inductively:

- Choose j_1 such that $|x_{n_{j_1}^{(1)}} - a_1| < 1$, and set $k_1 = n_{j_1}^{(1)}$.
- Having chosen k_{m-1}, choose j_m large enough so that:

$$|x_{n_{j_m}^{(m)}} - a_m| < \frac{1}{m} \quad \text{and} \quad n_{j_m}^{(m)} > k_{m-1}.$$

  Then set $k_m = n_{j_m}^{(m)}$.

This defines an increasing sequence of indices (k_m), hence (x_{k_m}) is a subsequence of (x_n). We claim that $x_{k_m} \to a$. Indeed, for each m,

$$|x_{k_m} - a| \leq |x_{k_m} - a_m| + |a_m - a| < \frac{1}{m} + |a_m - a|.$$

Taking limits as $m \to \infty$, both terms on the right tend to 0, so $x_{k_m} \to a$. Therefore $a \in \mathcal{A}$, which shows that every limit of a sequence from $\mathcal{A}$ belongs to $\mathcal{A}$. Hence $\mathcal{A}$ is closed.

2 Question 2

Limit inferior and limit superior

Let $y \in \mathbb{R}$ with $y > \limsup_{n\to\infty} x_n$. Show that there is a $N \in \mathbb{N}$ such that $x_n < y$ for all $n \geq N$. Moreover, show that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ converges if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

Answer

Define the tail suprema and infima:

$$s_n := \sup_{k\geq n} x_k, \qquad i_n := \inf_{k\geq n} x_k \qquad (n \in \mathbb{N}).$$

Then (s_n) is decreasing and (i_n) is increasing, and:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} s_n, \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} i_n.$$

<u>Part 1. If $y > \limsup_{n\to\infty} x_n$, then $x_n < y$ eventually</u>

Since $y > \lim_{n\to\infty} s_n$, there exists N such that $s_n < y$ for all $n \geq N$. But $x_n \leq s_n$ for every n, hence for all $n \geq N$ we have $x_n \leq s_n < y$, as required.

<u>Part 2. Convergence $\iff$ $\limsup = \liminf$</u>

(⇒) If $x_n \to L \in \mathbb{R}$, then for every $\varepsilon > 0$ there exists N such that $L - \varepsilon < x_n < L + \varepsilon$ for all $n \geq N$. Taking suprema and infima over $k \geq n$ yields, for $n \geq N$,

$$L - \varepsilon \leq i_n \leq s_n \leq L + \varepsilon.$$

Letting $n \to \infty$ gives:

$$L - \varepsilon \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq L + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\limsup x_n = \liminf x_n = L$.

(⇐) Suppose $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n =: L \in \mathbb{R}$. Then $s_n \downarrow L$ and $i_n \uparrow L$. Given $\varepsilon > 0$, choose N_1, N_2 such that for $n \geq N_1$, $s_n < L + \varepsilon$, and for $n \geq N_2$, $i_n > L - \varepsilon$. Set $N = \max\{N_1, N_2\}$. For all $n \geq N$ we have:

$$L - \varepsilon < i_n \leq x_n \leq s_n < L + \varepsilon,$$

so $|x_n - L| < \varepsilon$. Hence $x_n \to L$.

Thus a real sequence converges iff its limsup and liminf are equal.

3 Question 3

Convergence of sequences

Calculate $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$. Moreover, define iteratively, $s_1=\sqrt{2}$ and $s_{n+1}=\sqrt{2+s_n}$. Show that $(s_n)_{n\in\mathbb{N}}$ converges and that $s_n<2$ for all $n\in\mathbb{N}$. What is the limit $\lim_{n\to\infty} s_n$?

Answer

Part 1. Compute $\lim\limits_{n\to\infty}(\sqrt{n^2+n}-n)$

Multiply numerator and denominator by the conjugate:

$$\sqrt{n^2+n}-n=\frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}=\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\frac{n}{\sqrt{n^2+n}+n}.$$

Divide numerator and denominator by n:

$$\frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

Letting $n\to\infty$ gives:

$$\lim_{n\to\infty}(\sqrt{n^2+n}-n)=\frac{1}{1+1}=\frac{1}{2}.$$

Part 2. Show convergence

Analyze the sequence (s_n) defined by:

$$s_1=\sqrt{2},\qquad s_{n+1}=\sqrt{2+s_n}.$$

Firstly , we show that $s_n<2$ for all n. Here we use induction.

Base case is $s_1=\sqrt{2}<2$.

For inductive step we assume $s_n<2$. Then:

$$s_{n+1}=\sqrt{2+s_n}<\sqrt{2+2}=2.$$

Hence, by induction, $s_n<2$ for all n.

Then we need to show that (s_n) is increasing.

We check that $s_{n+1}>s_n$. Since $s_{n+1}=\sqrt{2+s_n}$, we compare s_{n+1}^2 and s_n^2:

$$s_{n+1}^2=2+s_n>s_n^2\quad\text{if and only if}\quad s_n^2-s_n-2<0.$$

The inequality $s_n^2-s_n-2<0$ holds whenever $-1<s_n<2$. Since $s_n>0$ and $s_n<2$, it follows that $s_{n+1}>s_n$. Thus (s_n) is increasing.

Part 3. Find the limit

Finally we need to show the can confirm the convergence. The sequence (s_n) is increasing and bounded above by 2, so by the Monotone Convergence Theorem it converges.

Let $\ell=\lim_{n\to\infty}s_n$. Taking limits in the recurrence,

$$\ell=\sqrt{2+\ell}.$$

Square both sides:

$$\ell^2=2+\ell\quad\Rightarrow\quad\ell^2-\ell-2=0.$$

Solving gives $\ell=2$ or $\ell=-1$. Since $s_n>0$, we have $\ell=2$.

Therefore, we have:

$$\lim_{n\to\infty}(\sqrt{n^2+n}-n)=\tfrac{1}{2},\qquad\lim_{n\to\infty}s_n=2.$$

4 Question 4

Limits

Analyze whether the sequences below converge or diverge. In case the sequence converges, find its limit.

- The sequence $x_n = \frac{1}{2n+3}$ in $\mathbb{R}$ equipped with the standard metric.
- The sequence $y_n = \frac{1}{2n+3}$ in $\mathbb{R}$ equipped with the discrete metric.

Compare this to the statement of the Bolzano-Weierstrass theorem.

Moreover, show that the following sequences in $\mathbb{R}$ equipped with the standard metric diverge and evaluate their limes superior and limes inferior.

- The sequence $z_n = n^2 - 100n$.
- The sequence $w_n = (-1)^n \left(1 + \frac{1}{2n+3}\right)$.

Answer

Part 1. Find the limit properties

$$x_n = \frac{1}{2n+3} \xrightarrow[n\to\infty]{} 0, \qquad \text{since } \lim_{n\to\infty}(2n+3) = +\infty.$$

Hence (x_n) converges with $\lim x_n = 0$.

In the discrete metric $d(x,y) = \mathbf{1}_{\{x\neq y\}}$, a sequence (a_n) converges to L iff it is eventually constant equal to L.

Here $y_n = \frac{1}{2n+3}$ is strictly decreasing and never 0, so it is not eventually constant. Thus (y_n) diverges in the discrete metric (it does not converge to 0 nor to any other point).

Part 2. Comparison with Bolzano–Weierstrass theorem

In $\mathbb{R}$, every bounded sequence has a convergent subsequence (by B-W theorem). Our (x_n) is bounded and indeed converges.

In the discrete metric, every sequence is bounded (diameter ≤ 1), but B-W theorem fails as (y_n) is bounded yet has no convergent subsequence, since any convergent subsequence would have to be eventually constant, which cannot happen because all y_n are distinct.

Part 3. Divergence and limsup/liminf in the standard metric

- $z_n = n^2 - 100n = n(n-100) \xrightarrow[n\to\infty]{} +\infty$. Hence it diverges and lim sup and inf are positive infinities.

$$\limsup_{n\to\infty} z_n = +\infty, \qquad \liminf_{n\to\infty} z_n = +\infty.$$

- $w_n = (-1)^n\left(1 + \frac{1}{2n+3}\right)$. For even $n = 2k$,

$$w_{2k} = 1 + \frac{1}{4k+3} \downarrow 1; \quad \text{for odd } n = 2k+1, \quad w_{2k+1} = -\left(1 + \frac{1}{4k+5}\right) \uparrow -1.$$

Thus (w_n) diverges (two distinct subsequential limits), and

$$\limsup_{n\to\infty} w_n = 1, \qquad \liminf_{n\to\infty} w_n = -1.$$

5 Question 5

Definition of limits

Given a metric space (X, d), a sequence $(x_n)_{n\in\mathbb{N}}$, and point $x \in X$, show that the following statements are equivalent:

1. $\forall\varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) \leq \varepsilon$,
2. $\forall\varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) < \varepsilon$,
3. $\forall\varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) \leq 2\varepsilon$.

Does this equivalence still hold if the term 2ε in the third item is replaced by an arbitrary function $f(\varepsilon)$ as long as $f(0) = 0$?

Answer

Part 1. Proof of equivalency

Proof of (1) $\Leftrightarrow$ (2)

- (1) $\implies$ (2): Assume (1) holds. Let $\varepsilon > 0$ be given for (2). Choose $\varepsilon' = \varepsilon/2$. Since $\varepsilon' > 0$, by (1), there exists $N \in \mathbb{N}$ such that for all $n \geq N$:
$$d(x_n, x) \leq \varepsilon' = \frac{\varepsilon}{2}$$
Since $\frac{\varepsilon}{2} < \varepsilon$, we have $d(x_n, x) < \varepsilon$. Thus, (2) holds.
- (2) $\implies$ (1): Assume (2) holds. Let $\varepsilon > 0$ be given for (1). By (2), there exists $N \in \mathbb{N}$ such that for all $n \geq N$:
$$d(x_n, x) < \varepsilon$$
The strict inequality $d(x_n, x) < \varepsilon$ implies $d(x_n, x) \leq \varepsilon$. Thus, (1) holds.

Proof of (1) $\Leftrightarrow$ (3)

- (1) $\implies$ (3): Assume (1) holds. Let $\varepsilon > 0$ be given for (3). Choose $\varepsilon' = 2\varepsilon$. Since $\varepsilon' > 0$, by (1), there exists $N \in \mathbb{N}$ such that for all $n \geq N$:
$$d(x_n, x) \leq \varepsilon' = 2\varepsilon$$
Thus, (3) holds.
- (3) $\implies$ (1): Assume (3) holds. Let $\varepsilon > 0$ be given for (1). Choose $\varepsilon' = \varepsilon/2$. Since $\varepsilon' > 0$, by (3), there exists $N \in \mathbb{N}$ such that for all $n \geq N$:
$$d(x_n, x) \leq 2\varepsilon' = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$
Thus, (1) holds.

Since (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3), all three statements are equivalent.

Part 2. Replacement by $f(\varepsilon)$

The equivalence still holds if the term 2ε in statement (3) is replaced by an arbitrary function $f(\varepsilon)$ if and only if $f(\varepsilon)$ satisfies the following essential condition (we assume $f(\varepsilon) > 0$ for $\varepsilon > 0$):

$$\lim_{\varepsilon\to 0^+} f(\varepsilon) = 0$$

The requirement $f(0) = 0$ alone is insufficient.

Let (4) be the modified statement: $\forall\varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) \leq f(\varepsilon)$. We prove (4) $\Leftrightarrow$ (1) under the limit condition.

- (1) $\implies$ (4):

  Let $\varepsilon_0 > 0$ be given for (4). We choose $\varepsilon' = f(\varepsilon_0)$. Since $\varepsilon' > 0$, statement (1) guarantees $\exists N$ such that $d(x_n, x) \leq \varepsilon' = f(\varepsilon_0)$. This holds trivially.

- (4) $\implies$ (1):

  Let $\varepsilon_0 > 0$ be given for (1). We need $d(x_n, x) \leq \varepsilon_0$. Since $\lim_{\varepsilon \to 0^+} f(\varepsilon) = 0$, by the definition of the limit, for the desired bound ε_0, $\exists \delta > 0$ such that if $0 < \varepsilon < \delta$, then $f(\varepsilon) \leq \varepsilon_0$. We choose $\varepsilon' = \delta/2$ for statement (4). Statement (4) guarantees $\exists N$ such that:

  $$d(x_n, x) \leq f(\varepsilon') \leq \varepsilon_0$$

  Thus, (1) holds.

If $f(\varepsilon)$ does not tend to zero, the equivalence fails. For the function $f(\varepsilon) = 1$ for $\varepsilon > 0$ (with $f(0) = 0$). Statement (4) becomes: $\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d\left(x_n, x\right) \leq 1$.

This condition is satisfied by every sequence that is merely eventually bounded (e.g., $x_n = 1 + (-1)^n$ with $x = 1$). Such a sequence does not converge to $x = 1$ (statement 1 fails), even though (4) is true.

6 Question 6

Limits in Euclidean space

Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be two sequences in Euclidean space $\mathbb{R}^k$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Show that $(x_n)_{n\in\mathbb{N}}$ converges to $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ if and only if the component sequences $(x_{j,n})_{n\in\mathbb{N}}$ converge to x_j for $j = 1, \dots, k$. Moreover, assuming that $x_n \to x, y_n \to y$ and $\lambda_n \to \lambda$ as $n \to \infty$, show that $x_n + y_n \to x + y$, $\langle x_n, y_n\rangle \to \langle x, y\rangle$, and $\lambda_n x_n \to \lambda x$ as $n \to \infty$. Finally, if one additionally assumes that $\lambda_n \neq 0$ and $\lambda \neq 0$, show that $\lim_{n\to\infty} \frac{x_n}{\lambda_n} = \frac{x}{\lambda}$.

Answer

Part 1. Convergence in $\mathbb{R}^k$ via component convergence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^k$, where $x_n = (x_{1,n}, x_{2,n}, \dots, x_{k,n})$. Let $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$. The metric is the standard Euclidean metric:

$$d(x_n, x) = \|x_n - x\| = \sqrt{\sum_{j=1}^{k} (x_{j,n} - x_j)^2}$$

We aim to show the equivalence:

$$\lim_{n\to\infty} x_n = x \quad \iff \quad \lim_{n\to\infty} x_{j,n} = x_j \quad \text{for all } j = 1, \dots, k$$

($\Rightarrow$) *Convergence in $\mathbb{R}^k$ Implies Component Convergence*

Assume $\lim_{n\to\infty} x_n = x$. By the definition of convergence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$\|x_n - x\| < \varepsilon$$

We relate the distance in the j-th component to the total Euclidean distance. For any $j \in \{1, \dots, k\}$:

$$|x_{j,n} - x_j| = \sqrt{(x_{j,n} - x_j)^2} \leq \sqrt{\sum_{i=1}^{k} (x_{i,n} - x_i)^2} = \|x_n - x\|$$

Combining the inequalities, for all $n \geq N$:

$$|x_{j,n} - x_j| \leq \|x_n - x\| < \varepsilon$$

Since $|x_{j,n} - x_j| < \varepsilon$ for all $n \geq N$, the j-th component sequence converges to its limit: $\lim_{n\to\infty} x_{j,n} = x_j$. Since j was arbitrary, this holds for all components.

($\Leftarrow$) *Component Convergence Implies Convergence in $\mathbb{R}^k$*

Assume $\lim_{n\to\infty} x_{j,n} = x_j$ for all $j = 1, \dots, k$. Let $\varepsilon > 0$ be given. We want to find N such that $\|x_n - x\| < \varepsilon$.

For each component j, we choose the individual error bound to be $\varepsilon_j = \frac{\varepsilon}{\sqrt{k}}$. Since $\lim_{n\to\infty} x_{j,n} = x_j$, for this ε_j, there exists $N_j \in \mathbb{N}$ such that for all $n \geq N_j$:

$$|x_{j,n} - x_j| < \frac{\varepsilon}{\sqrt{k}}$$

. Let $N = \max\{N_1, N_2, \dots, N_k\}$. For all $n \geq N$, the inequality holds for every component j. For $n \geq N$:

$$\|x_n - x\|^2 = \sum_{j=1}^{k} (x_{j,n} - x_j)^2$$

Since $|x_{j,n} - x_j| < \frac{\varepsilon}{\sqrt{k}}$, we have $(x_{j,n} - x_j)^2 < \left(\frac{\varepsilon}{\sqrt{k}}\right)^2 = \frac{\varepsilon^2}{k}$. Substituting this into the sum:

$$\|x_n - x\|^2 < \sum_{j=1}^{k} \frac{\varepsilon^2}{k} = k \cdot \frac{\varepsilon^2}{k} = \varepsilon^2$$

Taking the square root, we get:

$$\|x_n - x\| < \varepsilon$$

Since this holds for an arbitrary $\varepsilon > 0$, we conclude that $\lim_{n\to\infty} x_n = x$.

Part 2. Algebra of limits

Assume $x_n \to x$, $y_n \to y$, $\lambda_n \to \lambda$.

Sum: $\|(x_n + y_n) - (x + y)\|_2 \le \|x_n - x\|_2 + \|y_n - y\|_2 \to 0$, hence $x_n + y_n \to x + y$.

Inner product: By adding and subtracting $\langle x, y_n\rangle$ and using Cauchy–Schwarz inequality,

$$\big|\langle x_n, y_n\rangle - \langle x, y\rangle\big| \le \big|\langle x_n - x, y_n\rangle\big| + \big|\langle x, y_n - y\rangle\big| \le \|x_n - x\|_2 \, \|y_n\|_2 + \|x\|_2 \, \|y_n - y\|_2.$$

Since $y_n \to y$, the sequence $(\|y_n\|_2)$ is bounded; both terms tend to 0, so $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

Scalar product: $\|\lambda_n x_n - \lambda x\|_2 \le |\lambda_n| \, \|x_n - x\|_2 + |\lambda_n - \lambda| \, \|x\|_2 \to 0$, so $\lambda_n x_n \to \lambda x$.

Part 3. Quotients

If additionally $\lambda_n \ne 0$ for all n and $\lambda \ne 0$, then $\lambda_n \to \lambda \ne 0$ implies $\frac{1}{\lambda_n} \to \frac{1}{\lambda}$ (continuity of $t \mapsto 1/t$ on $\mathbb{R} \setminus \{0\}$). Thus

$$\frac{x_n}{\lambda_n} = x_n \cdot \frac{1}{\lambda_n} \;\longrightarrow\; x \cdot \frac{1}{\lambda} = \frac{x}{\lambda},$$

using the scalar product limit with scalars $1/\lambda_n$.

7 Question 7

Monotone convergence theorem

Show that a monotone decreasing sequence is convergent if and only if it is bounded.

Answer

Let $(a_n)_{n\geq 1}$ be a monotone decreasing sequence in $\mathbb{R}$, i.e.

$$a_{n+1} \leq a_n \quad \text{for all } n \in \mathbb{N}.$$

(⇒) Suppose that (a_n) converges. Then there exists $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$. Hence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - L| < 1,$$

so $|a_n| \leq |L| + 1$ for $n \geq N$. The finitely many terms $a_1, \ldots, a_{N-1}$ are also bounded, hence the entire sequence (a_n) is bounded.

(⇐) Conversely, suppose that (a_n) is bounded. Since (a_n) is decreasing, it is automatically bounded above by a_1. If it is bounded below, define

$$L := \inf\{a_n : n \in \mathbb{N}\}.$$

We claim that $a_n \to L$.

Because L is the infimum, we have $L \leq a_n$ for all n. Given $\varepsilon > 0$, by the definition of infimum there exists $N \in \mathbb{N}$ such that

$$a_N < L + \varepsilon.$$

Since (a_n) is decreasing, for all $n \geq N$ we have $a_n \leq a_N < L + \varepsilon$. Thus,

$$L \leq a_n < L + \varepsilon \quad \text{for all } n \geq N,$$

which implies $|a_n - L| < \varepsilon$. Hence $a_n \to L$ as $n \to \infty$. Therefore, a monotone decreasing sequence is convergent if and only if it is bounded.

8 Question 8

Cauchy sequences

Give an example of a metric space (X, d) together with a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X which does not converge.

Answer

By definition, this metric space must not be a complete metric space.

Let $X = \mathbb{Q}$ with the metric $d(x, y) = |x - y|$. Define a sequence (x_n) of rational numbers converging to $\sqrt{2}$ in $\mathbb{R}$, for example

$$x_1 = 1, \quad x_2 = 1.4, \quad x_3 = 1.41, \quad x_4 = 1.414, \ \ldots$$

Then (x_n) is Cauchy in $(\mathbb{Q}, d)$, since it converges to $\sqrt{2}$ in $\mathbb{R}$. However, $\sqrt{2} \notin \mathbb{Q}$, so (x_n) does not converge in $\mathbb{Q}$.