Homework 2, MATH 5010

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1 Question 1

What is the difference between historic and implied volatilities? Which one is higher for Microsoft stock MSFT? Go to the Bloomberg terminal and type: `MSFT <Equity> HIVG <Go>`. Print the output.

Answer

1.1 Differences Between HV and IV

The Historical Volatility is also called Realized Volatility or Statistical Volatility, and it measures the actual volatility of an asset's price over a specific past period. It is calculated using historical price data (e.g., daily, weekly, or monthly returns).

Historic Volatility is typically calculated as the standard deviation of the asset's logarithmic returns over a given time frame. The formula is:

$$\text{Historic Volatility} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2}$$

where:

- r_i: logarithmic return at time i,
- $\bar{r}$: average return over the period,
- N: number of observations.

As for the Implied Volatility (IV), it is a forward-looking measure of volatility derived from the market price of an option. It represents the market's expectation of future volatility over the option's life.

Implied Volatility is not directly calculated but is inferred using an options pricing model (e.g. the Black-Scholes model). It is the value of volatility that, when plugged into the model, matches the market price of the option. The Black-Scholes formula for a call option is:

$$C = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

where:

- C: market price of the call option,
- S_0: current price of the underlying asset,
- K: strike price,
- r: risk-free rate,
- T: time to expiration,
- $N(d)$: cumulative distribution function of the standard normal distribution,
- d_1 and d_2: functions of volatility (σ).

Implied Volatility is the value of σ that satisfies this equation.

Finally, for their relationship, over time, implied volatility tends to converge toward historic volatility as the future becomes the past. Implied volatility can be higher or lower than historic volatility depending on market expectations (e.g., during periods of uncertainty or calm).

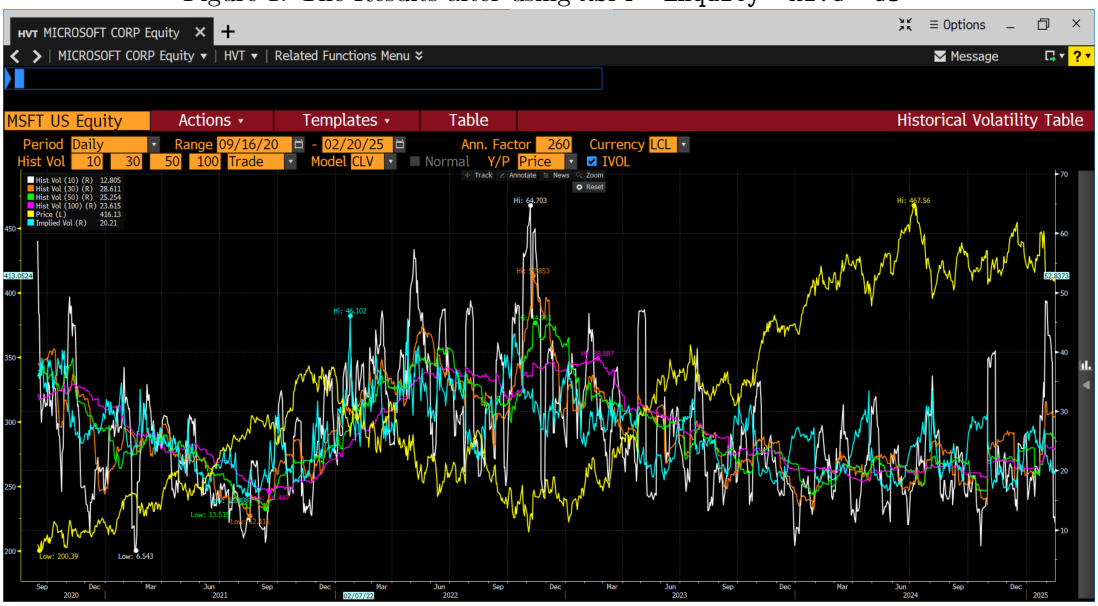
We can use a table to illustrate this:

Aspect	Historic Volatility	Implied Volatility
Time Frame	Based on past price movements.	Based on future expectations.
Data Used	Historical price data.	Market price of options.
Calculation	Calculated using statistical methods (e.g., standard deviation).	Derived from options pricing models (e.g., Black-Scholes).
Purpose	Measures past price fluctuations.	Predicts future price fluctuations.
Market Sentiment	Does not reflect market sentiment.	Reflects market sentiment and expectations.
Use in Trading	Used for risk analysis and benchmarking.	Used for options pricing and trading strategies.

1.2 MSFT HIVG

Here is the plot provided by Bloomberg Terminal, and the timespan I set was Sept 16, 2020 to Feb 20, 2025.

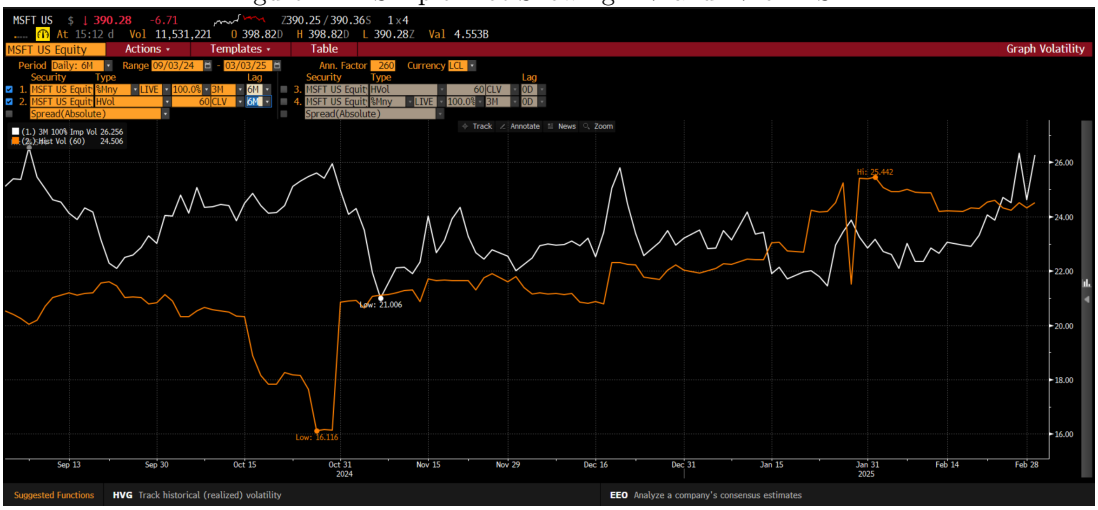
Figure 1: The Results after using `MSFT <Enquity> HIVG <GO>`



Take the date Feb 28, 2025 as the last date for consideration, for a 30-days range, the historical volatility (HV) is 25.99%, which is higher than the implied volatility (IV), which was around 24.51%. The exact data can be found here.

However, if we reconsider it in a large time span, the overall Historical Volatility is smaller than the Implied Volatility. It can be easily seen from the graph, and we can also scrutinize the 180-day-data.

Figure 2: A Simple Plot Showing HV and IV of MSFT



2 Question 2

What is the volatility smile? Using Bloomberg terminal function SKEW (`CLJ5 Comdty SKEW <Go>`) plot the implied volatility smiles for `TYH5 Comdty, CLJ5 Comdty, NGJ25 Comdty, ESH5 Index, SPY Equity`. What are these securities? For description type `CLJ5 Comdty DES`. Compare their smiles. Submit printouts.

Answer

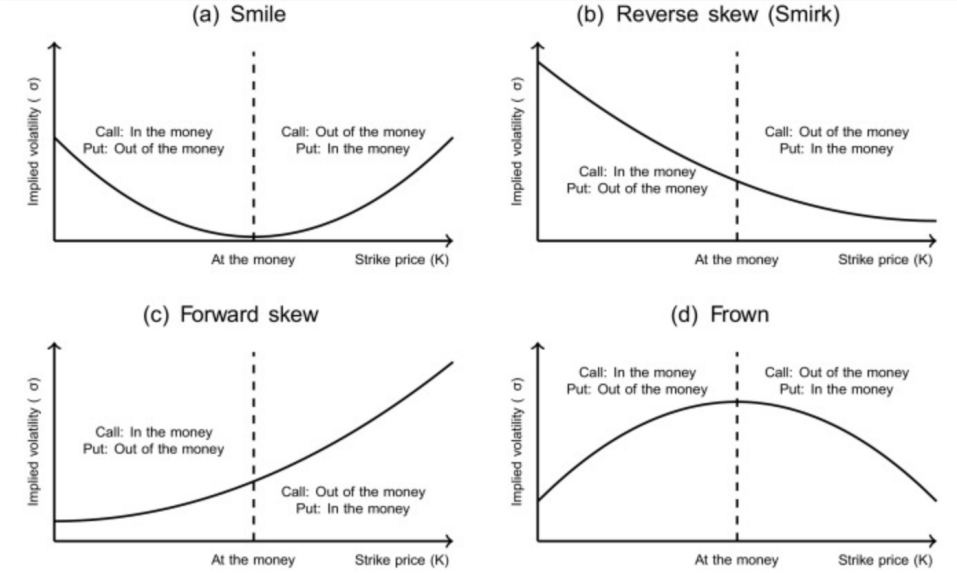
2.1 Definition of Volatility Smile

In the Textbook, the author defines Volatility Smile as "A plot of the implied volatility of an option with a certain life as a function of its strike price." (Hull, 409)

In my understanding, Volatility Smile is a pattern observed in the implied volatility of options across different strike prices, where implied volatility is higher for both out-of-the-money (OTM) puts and calls compared to at-the-money (ATM) options, creating a "smile" shape when plotted. This phenomenon contradicts the assumption of constant volatility in models like Black-Scholes and arises because markets often price in higher uncertainty for extreme price movements (both up and down).

The x-axis is Strike Price, whereas the y-axis is Implied Volatility.

Figure 3: Simple Illustration of VS (cred to Soini et al.)



2.2 Bloomberg Comparison

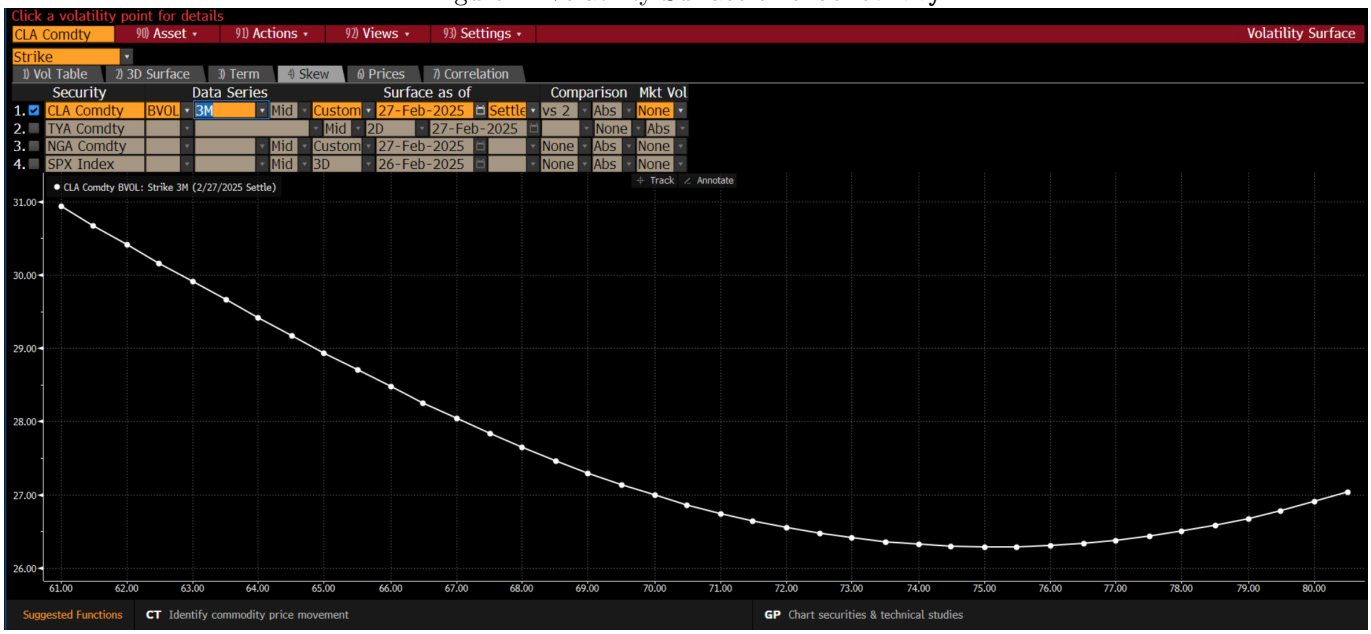
In Bloomberg, they represent:

- `TYH5 Comdty`: U.S. 10-Year Treasury Futures Contract (April 2025).
- `CLJ5 Comdty`: Crude Oil Futures (April 2025).
- `NGJ25 Comdty`: Natural Gas Futures (April 2025).
- `ESH5 Index`: S&P 500 E-mini Futures (March 2025).
- `SPY Equity`: SPDR S&P 500 ETF Trust.

Here is the graph of volatility smile in Bloomberg Terminal. The model I set was `Strike` instead of `Delta` or so, because in `Delta` mode, there are only four comparable niches, and it is more standard to show their volatility smiles with strike price; moreover, the overall trend will not be impacted by different x-axises we chose.

Take `Strike` as the x-axis, the Volatility Surface has a Smirk Shape, or Reverse Skew for `CLJ5 Comdty`, and it reflects a market expectation of greater downside risk than upside potential. It is characterized by higher implied

Figure 4: Volatility Surface of `CLJ5 Comdty`



volatility for OTM puts compared to OTM calls and is often driven by bearish sentiment, economic uncertainty, or fear of sharp declines. In such scenarios, OTM puts become more expensive due to higher implied volatility, while OTM calls are relatively cheaper. Traders may use strategies like put spreads or risk reversals to capitalize on this skew, while investors might hedge against potential price declines.

For both `TYH5 Comdty` and `NGJ5 Comdty`, we have a shape as Forward Skew, meaning its implied volatility is higher for out-of-the-money (OTM) calls (higher strike prices) compared to out-of-the-money (OTM) puts (lower strike prices). This creates an upward-sloping asymmetry in the volatility curve. This pattern indicates that the market expects greater upside potential (price increases) than downside risk (price decreases), often driven by bullish sentiment, anticipated positive events (e.g., earnings reports, supply constraints), or speculative demand. In such scenarios, OTM calls become more expensive due to higher implied volatility, while OTM puts are relatively cheaper. Traders may use strategies like call spreads or risk reversals to capitalize on this skew, while investors might hedge against potential price increases.

Figure 5: Volatility Surface of `TYH5 Comdty`

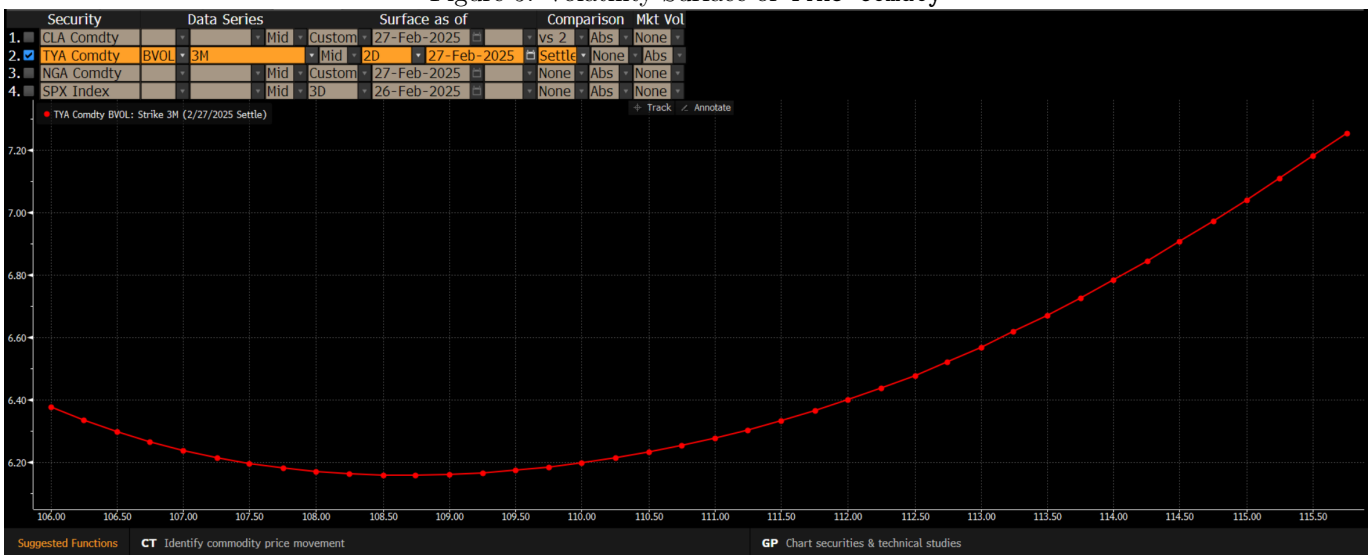
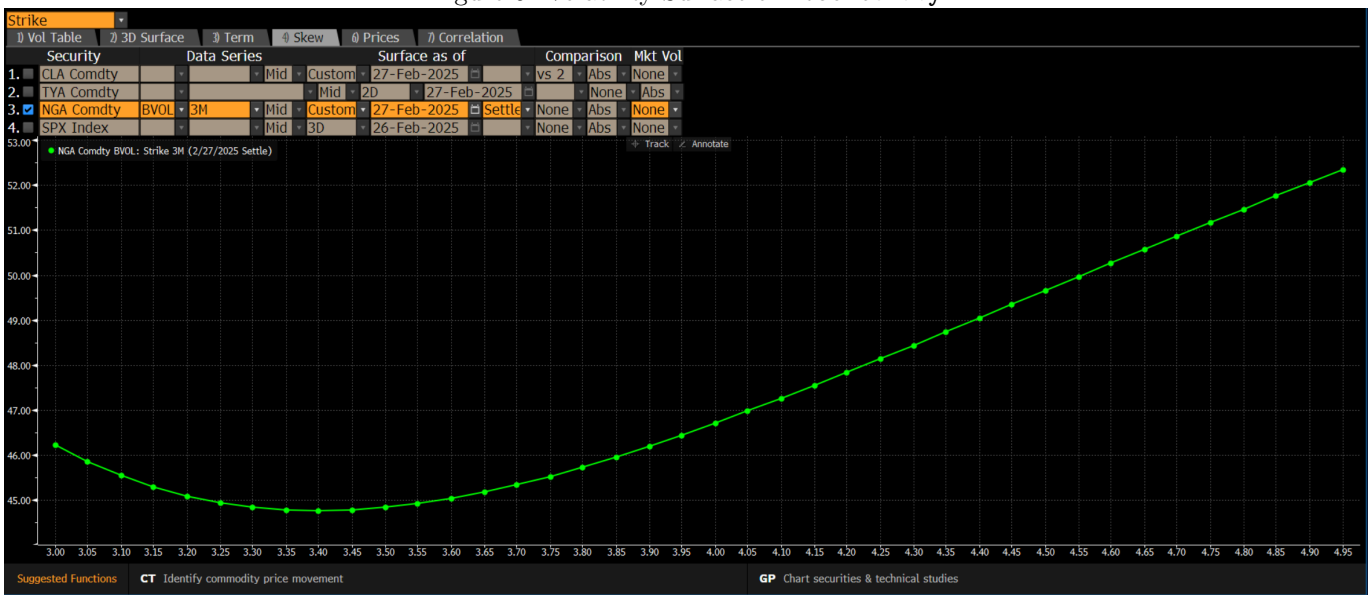


Figure 6: Volatility Surface of `NGJ5 Comdty`



Here we get a downward sloping graph of `SPY Equity` and `ESH5 Index`; typically a downward-sloping volatility surface in the 3-month range indicates that implied volatility decreases as strike prices increase, reflecting a negative volatility skew. This pattern often signals bearish market sentiment, with investors more concerned about downside risks (e.g., potential price drops) than upside potential. Higher implied volatilities for lower strike prices (out-of-the-money puts) suggest increased demand for downside protection, possibly due to market stress, upcoming events like earnings announcements, or historical patterns of sharp declines. For traders, this skew implies that put options are more expensive, while call options are cheaper, influencing strategies such as put spreads or risk reversals.

Figure 7: Volatility Surface of `SPY Equity`

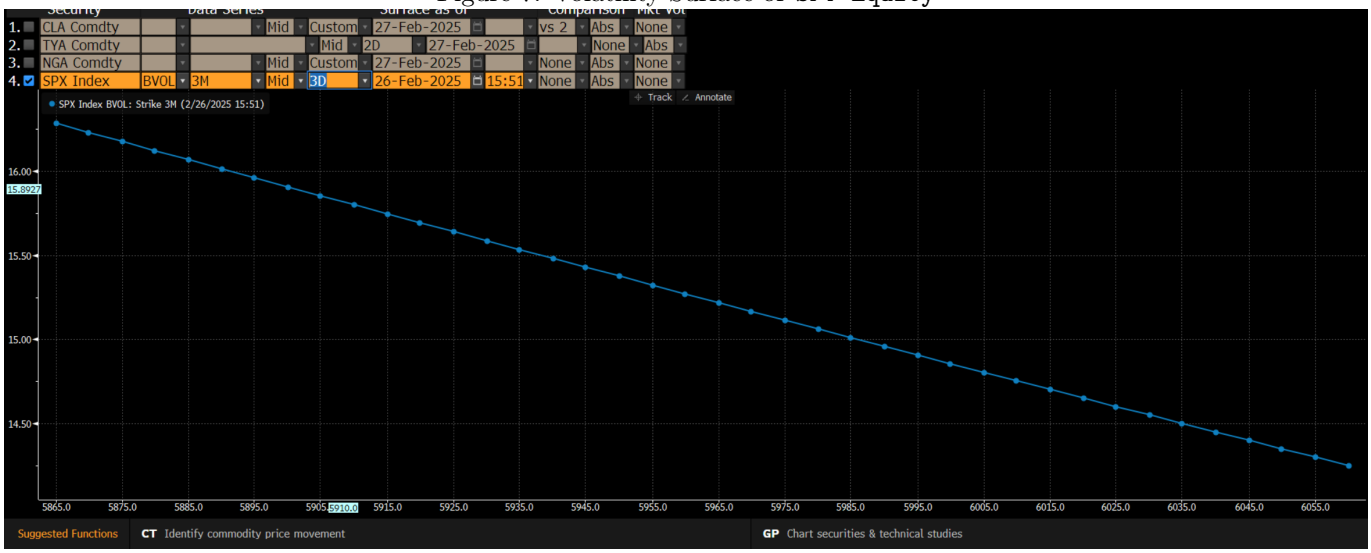
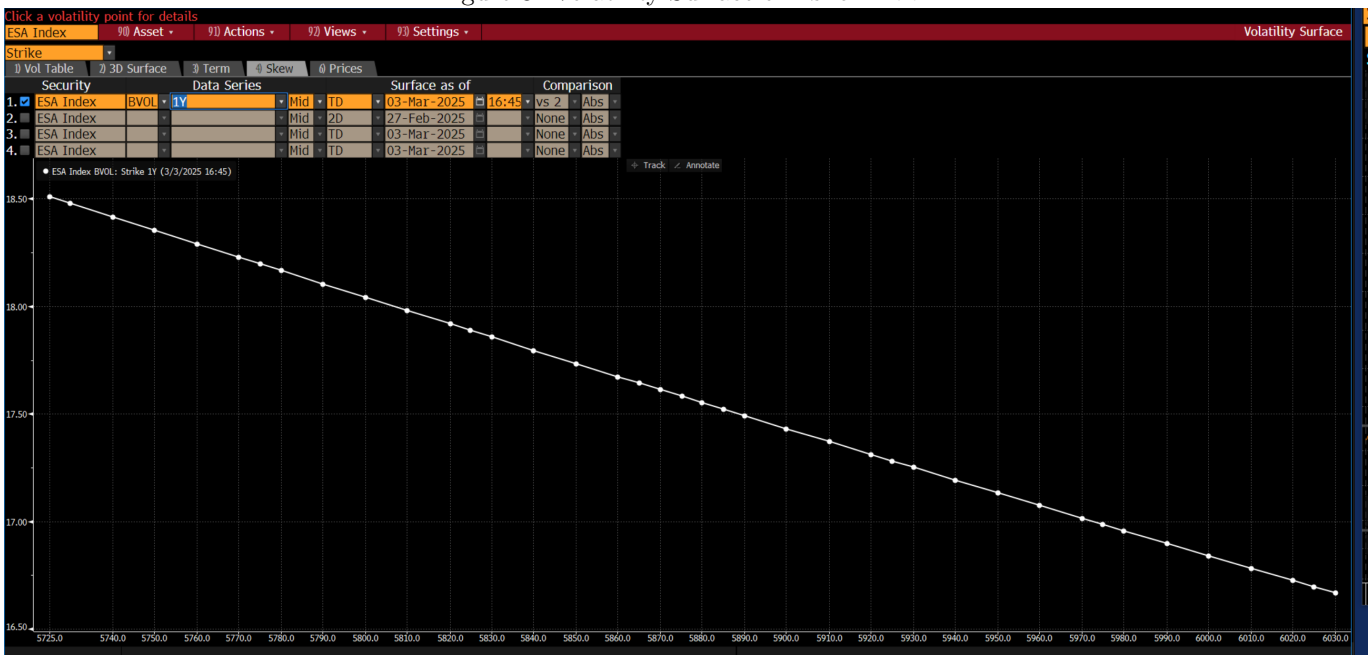


Figure 8: Volatility Surface of `ESH5 Index`



3 Question 3

Suppose the stock price is 70, the risk-free rate is 5% continuously compounded. What is the price of a 1 year call struck at 70 if the volatility is 0. How would you hedge the call. Check your answer with the option calculator making volatility smaller and smaller.

Answer

3.1 Solve it Using Fast Fourier Transform

Here I used the integration of Fast Fourier Transform (FFT) and Fractional Fast Fourier Transform (FrFFT) for this problem; there are 4 methods to be chosen, GBM, Heston, Variance Gamma, and Black-Scholes. Here I used Black-Scholes.

If $\sigma = 0$ and there is no dividend in this scenario, the stock price is deterministic and grows at rate r. In one year, it is thus worth $70 \cdot e^{0.05} \approx 73.5889$. The strike is $K = 70$. Our payoff is thus 3.588. Discounting at rate r, we get as today's fair option price $3.588 \cdot e^{-0.05} \approx 3.4139$. When I set `r = 0.05, q = 0, sigma = 0`, the printout is:

```
====================
Model is BlackScholes
--------------------


Option via Integration: for strike 70.0000 the option premium is 3.4155
FrFFT execution time was 0.0215478
```

The choice of damping factors, grid truncation error, and numerical integration error may all cause small differences when using Fast Fourier Transformation, but overall the result corresponds with the result we calculated from the Option Pricing Calculator.

My python codes are as follows:

```
1
2       import warnings
3       warnings.filterwarnings("ignore")
4
5       import numpy as np
6       import cmath
7       import math
8       import time
9
10      # Fixed Parameters
11      S0 = 70
12      K = 70
13      k = math.log(K)
14      r = 0.05
15      q = 0
16
17      # Parameters for FFT and FrFFT
18
19      n_FFT = 9
20      N_FFT = 2**n_FFT
21
22      n_FrFFT = 9
23      N_FrFFT = 2**n_FrFFT
24
25      N = 1000 # it doesn' impact the numerical results if the code is correct
26
27      #step-size
28      eta = 0.25
29      # damping factor
30      alpha = 0.4
31
32      # step-size in log strike space
33      lda_FFT = (2*math.pi/N_FFT)/eta # lda is fixed under FFT
34      lda_FrFFT = 0.001 # lda is an adjustable parameter under FrFFT,
35
```

```
36
37      #Choice of beta
38      beta = np.log(S0)-N*lda_FFT/2
39      #beta = np.log(S0)-N*lda_FrFFT/2
40      #beta = np.log(K)
41
42      #model-specific Parameters
43      model = 'BlackScholes'
44
45      params = []
46      if (model == 'GBM'):
47
48      sig = 0.3
49      params.append(sig);
50
51      elif model == 'BlackScholes':
52      sigma = 0  # Volatility
53      params.append(sigma)
54
55      elif (model == 'VG'):
56
57      sig = 0.3
58      nu = 0.5
59      theta = -0.4
60      #
61      params.append(sig);
62      params.append(nu);
63      params.append(theta);
64
65
66
67      elif (model == 'Heston'):
68
69      kappa = 2.0
70      theta = 0.05
71      sig = 0.30
72      rho = -0.70
73      v0 = 0.04
74      #
75      params.append(kappa)
76      params.append(theta)
77      params.append(sig)
78      params.append(rho)
79      params.append(v0)
80
81      def generic_CF(u, params, S0, r, q, T, model):
82
83      if (model == 'GBM'):
84
85      sig = params[0]
86      mu = np.log(S0) + (r-q-sig**2/2)*T
87      a = sig*np.sqrt(T)
88      phi = np.exp(1j*mu*u-(a*u)**2/2)
89
90      elif (model == 'BlackScholes'):
91      sigma = params[0]  # Volatility
92
93      mu = np.log(S0) + (r - q - 0.5 * sigma**2) * T  # Drift
94      a = sigma * np.sqrt(T)  # Standard deviation over the maturity period
95      phi = np.exp(1j * mu * u - 0.5 * a**2 * u**2)  # Characteristic function for Black-
            Scholes
96
97
```

```
    elif(model == 'Heston'):

    kappa  = params[0]
    theta  = params[1]
    sigma  = params[2]
    rho    = params[3]
    v0     = params[4]

    tmp = (kappa-1j*rho*sigma*u)
    g = np.sqrt((sigma**2)*(u**2+1j*u)+tmp**2)

    pow1 = 2*kappa*theta/(sigma**2)

    numer1 = (kappa*theta*T*tmp)/(sigma**2) + 1j*u*T*r + 1j*u*math.log(S0)
    log_denum1 = pow1 * np.log(np.cosh(g*T/2)+(tmp/g)*np.sinh(g*T/2))
    tmp2 = ((u*u+1j*u)*v0)/(g/np.tanh(g*T/2)+tmp)
    log_phi = numer1 - log_denum1 - tmp2
    phi = np.exp(log_phi)

    #g = np.sqrt((kappa-1j*rho*sigma*u)**2+(u*u+1j*u)*sigma*sigma)
    #beta = kappa-rho*sigma*1j*u
    #tmp = g*T/2

    #temp1 = 1j*(np.log(S0)+(r-q)*T)*u + kappa*theta*T*beta/(sigma*sigma)
    #temp2 = -(u*u+1j*u)*v0/(g/np.tanh(tmp)+beta)
    #temp3 = (2*kappa*theta/(sigma*sigma))*np.log(np.cosh(tmp)+(beta/g)*np.sinh(tmp))

    #phi = np.exp(temp1+temp2-temp3);


    elif (model == 'VG'):

    sigma  = params[0];
    nu     = params[1];
    theta  = params[2];

    if (nu == 0):
    mu = np.log(S0) + (r-q - theta -0.5*sigma**2)*T
    phi  = np.exp(1j*u*mu) * np.exp((1j*theta*u-0.5*sigma**2*u**2)*T)
    else:
    mu  = np.log(S0) + (r-q + np.log(1-theta*nu-0.5*sigma**2*nu)/nu)*T
    phi = np.exp(1j*u*mu)*((1-1j*nu*theta*u+0.5*nu*sigma**2*u**2)**(-T/nu))

    return phi
    def evaluateIntegral(params, S0, K, r, q, T, alpha, eta, N, model):

    # Just one strike at a time
    # no need for Fast Fourier Transform

    # discount factor
    df = math.exp(-r*T)

    sum1 = 0
    for j in range(N):
    nuJ = j*eta
    psi_nuJ = df*generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*
        nuJ)*(alpha+1+1j*nuJ))
    if j == 0:
    wJ = (eta/2)
    else:
    wJ = eta
    sum1 += np.exp(-1j*nuJ*k)*psi_nuJ*wJ

```

```
    cT_k = (np.exp(-alpha*k)/math.pi)*sum1

    return np.real(cT_k)

    def genericFFT(params, S0, K, r, q, T, alpha, eta, n, model):

    N = 2**n

    # step-size in log strike space
    lda = (2*np.pi/N)/eta

    #Choice of beta
    #beta = np.log(S0)-N*lda/2
    #beta = np.log(K)

    # forming vector x and strikes km for m=1,...,N
    km = np.zeros((N))
    xX = np.zeros((N))

    # discount factor
    df = math.exp(-r*T)

    nuJ = np.arange(N)*eta
    psi_nuJ = generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*nuJ)
        *(alpha+1+1j*nuJ))

    for j in range(N):
    km[j] = beta+j*lda
    if j == 0:
    wJ = (eta/2)
    else:
    wJ = eta

    xX[j] = np.exp(-1j*beta*nuJ[j])*df*psi_nuJ[j]*wJ

    yY = np.fft.fft(xX)
    cT_km = np.zeros((N))
    for i in range(N):
    multiplier = np.exp(-alpha*km[i])/math.pi
    cT_km[i] = multiplier*np.real(yY[i])

    return km, cT_km

    def genericFrFFT(params, S0, K, r, q, T, alpha, eta, n, lda, model):

    N = 2**n
    gamma = eta*lda/(2*math.pi)

    #Choice of beta
    #beta = np.log(S0)-N*lda/2
    beta = np.log(K)

    # initialize x, y, z, and cT_km
    km = np.zeros((N))
    x = np.zeros((N))
    y = np.zeros((2*N), dtype=np.complex128)
    z = np.zeros((2*N), dtype=np.complex128)
    cT_km = np.zeros((N))

    # discount factor
    df = math.exp(-r*T)

    # compute x
```

```
    nuJ = np.arange(N)*eta
    psi_nuJ = generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*nuJ)
        *(alpha+1+1j*nuJ))

    for j in range(N):
    km[j] = beta+j*lda
    if j == 0:
    wJ = (eta/2)
    else:
    wJ = eta
    x[j] = np.exp(-1j*beta*nuJ[j])*df*psi_nuJ[j]*wJ

    # set up y
    for i in range(N):
    y[i] = np.exp(-1j*math.pi*gamma*i**2)*x[i]
    y[N:] = 0

    # set up z
    for i in range(N):
    z[i] = np.exp(1j*math.pi*gamma*i**2)
    z[N:] = z[:N][::-1]

    # compute xi_hat
    xi_hat = np.fft.ifft(np.fft.fft(y) * np.fft.fft(z))

    # compute call prices
    for i in range(N):
    cT_km[i] = np.exp(-alpha*(beta + i*lda))/math.pi * (np.exp(-1j*math.pi*gamma*i**2)*
        xi_hat[i]).real

    return km, cT_km

    print('␣')
    print('====================')
    print('Model␣is␣%s' % model)
    print('--------------------')

    T = 1

    # FFT
    print('␣')
    start_time = time.time()
    km, cT_km = genericFFT(params, S0, K, r, q, T, alpha, eta, n_FFT, model)
    #cT_k = cT_km[0]
    cT_k = np.interp(k, km, cT_km)

    elapsed_time = time.time() - start_time

    #cT_k = np.interp(np.log(k), km, cT_km)
    print("Option␣via␣FFT:␣for␣strike␣%s␣the␣option␣premium␣is␣%6.4f" % (np.exp(k), cT_k)
        )
    #print("Option via FFT: for strike %s the option premium is %6.4f" % (np.exp(k),
        cT_km[0]))
    print('FFT␣execution␣time␣was␣%0.7f' % elapsed_time)

    # FrFFT
    print('␣')
    start_time = time.time()
    km, cT_km = genericFrFFT(params, S0, K, r, q, T, alpha, eta, n_FrFFT, lda_FrFFT,
        model)
    #cT_k = cT_km[0]
    cT_k = np.interp(k, km, cT_km)

```

```
    elapsed_time = time.time() - start_time

    #cT_k = np.interp(np.log(), km, cT_km)
    print("Option via FrFFT: for strike %s the option premium is %6.4f" % (np.exp(k),
        cT_k))
    #print("Option via FFT: for strike %s the option premium is %6.4f" % (np.exp(k),
        cT_km[0]))
    print('FrFFT execution time was %0.7f' % elapsed_time)


    # Integral
    print(' ')
    start_time = time.time()
    cT_k = evaluateIntegral(params, S0, K, r, q, T, alpha, eta, N, model)
    elapsed_time = time.time() - start_time
    print("Option via Integration: for strike %s the option premium is %6.4f" % (np.exp(k
        ), cT_k))
    print('Evaluation of integral time was %0.7f' % elapsed_time)
```

3.2 Solve it Using Option Calculator

Then I double checked it from the option price calculator:

Figure 9: Results from Option Calculator

Option Strike 70 Expiration (years) 1 | **Stock** Price 70 Volatility 0 Dividend 0% | **Market** Interest Rate 5% | **Settings** Precision 5

	European Call	European Put	Forward	Binary Call	Binary Put
Price	0.00000	0.00000	3.41394	0.50000	-0.50000
Delta	0.50000	-0.50000	1.00000	0.00000	-0.00000
Gamma	0.00000	0.00000	0.00000	0.00000	-0.00000
Vega	0.00000	0.00000	0.00000	0.00000	-0.00000
Rho	0.00000	0.00000	66.58606	0.00000	-0.95123
Theta	-0.00000	-0.00000	3.32930	0.00000	-0.04756

Option Strike 70 Expiration (years) 1 | **Stock** Price 70 Volatility 0 Dividend 3% | **Market** Interest Rate 5% | **Settings** Precision 5

	European Call	European Put	Forward	Binary Call	Binary Put
Price	0.00000	0.00000	1.34513	0.50000	-0.50000
Delta	0.50000	-0.50000	0.97045	0.00000	-0.00000
Gamma	0.00000	0.00000	0.00000	0.00000	-0.00000
Vega	0.00000	0.00000	0.00000	0.00000	-0.00000
Rho	0.00000	0.00000	66.58606	0.00000	-0.95123
Theta	-0.00000	-0.00000	1.29137	0.00000	-0.04756

4 Question 4

Explain why an American option on a stock paying continuous dividend yield is always worth as much as its intrinsic value. Give a numerical example of a situation when European option is worth less than intrinsic value. (Give the numerical value of stock price, strike price, time to expiration, etc.)

Answer

4.1 Concept

Intrinsic value of a call option is the difference between the current price of the underlying asset and the strike price of the option, if the underlying asset's price is above the strike price. If the underlying asset's price is below the strike price, the call option has no intrinsic value. Thus it can be expressed as:

$$\max(S - K, 0)$$

Here we have S to be the current stock price, and K to be the strike price

If the stock pays a continuous dividend yield, the stock price is expected to decrease as dividends are paid. This increases the incentive to exercise the call option early to avoid losing value from future dividends. Since the American option can be exercised early, the holder will exercise it if the intrinsic value exceeds the option's continuation value (the value of holding the option). Therefore, the American option is always worth at least its intrinsic value.

4.2 Numerical Example

For a the numerical example, we can set:

- Stock price: $S = 100$
- Strike price: $K = 90$
- Time to expiration: 1 year
- Risk-free interest rate: 5%
- Continuous dividend yield: 3%
- Volatility: 20%

Using the Black-Scholes formula, the European call option price is approximately:

$$C_{\text{European}} = 12.03$$

Calculation of C_{European}

We have the Black-Scholes Formula:

$$C = Se^{-qt}N(d_1) - Ke^{-rt}N(d_2)$$

Where d_1 and d_2 can be calculated as:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Given what we previously set: $S = 100$, $K = 90$, $r = 0.05$, $q = 0.03$, $\sigma = 0.20$, $t = 1$:

$$d_1 = \frac{\ln(100/90) + (0.05 - 0.03 + 0.5(0.2^2))(1)}{0.2\sqrt{1}} = \frac{\ln(1.1111) + (0.02 + 0.02)}{0.2} = \frac{0.1054 + 0.04}{0.2} = \frac{0.1454}{0.2} = 0.727$$

$$d_2 = 0.727 - 0.2 = 0.527$$

Then we can find $N(d_1)$ and $N(d_2)$ using standard normal distribution tables:

$$N(0.727) \approx 0.7673$$

$$N(0.527) \approx 0.7014$$

Finally we plug them back into the Black-Scholes Formula:

$$C = 100e^{-0.03(1)}(0.7673) - 90e^{-0.05(1)}(0.7014) = 100(0.9704)(0.7673) - 90(0.9512)(0.7014)$$

$$C = 74.43 - 60.91 \approx 12.03$$

The intrinsic value is:

$$S - K = 100 - 90 = 10$$

Here, the European option is worth 12.03, which is greater than the intrinsic value because there is still time value. Then we modify the example:

- Stock price: $S = 100$
- Strike price: $K = 110$ (out of the money)
- Time to expiration: 1 month
- Dividend yield: 3%
- Volatility: 10%

Using Black-Scholes and similar calculation as previous step:

$$C_{\text{European}} \approx 0.25$$

But the intrinsic value is:

$$\max(100 - 110, 0) = 0$$

In this case, the European option is worth less than its intrinsic value, because there is no early exercise opportunity, and the option is out of the money.

5 Question 5

Explain the European call-put parity argument. Why it can not be used for American options

Answer

5.1 Concept

The Put-Call Parity is a fundamental relationship between the prices of European call and put options on the same underlying asset, with the same strike price and expiration date. It defines a relationship between the price of a European call option and European put option, both with the identical strike price and expiry, namely that a portfolio of a long call option and a short put option is equivalent to (and hence has the same value as) a single forward contract at this strike price and expiry

The formula is:

$$C - P = Se^{-qt} - Ke^{-rt}$$

- C: Price of the European call option
- P: Price of the European put option
- S: Current stock price
- K: Strike price
- r: Risk-free interest rate
- q: Continuous dividend yield
- t: Time to expiration
- e^{-qt}: Discounted stock price due to dividend payments
- e^{-rt}: Discounted strike price due to risk-free rate

The parity works well for European options because they can only be exercised at expiration. The key idea behind this formula is that a portfolio consisting of a long European call option and a short European put option is equivalent to holding the underlying stock while borrowing the present value of the strike price.

Both portfolios will yield the same payoff at expiration, which is why they must have the same value before expiration (by the no-arbitrage principle).

However, the Put-Call Parity does not directly apply to American options because:

- American options can be exercised at any time before expiration, which adds an additional layer of complexity. This early exercise possibility makes the payoff structure of American options different from European options.
- For American call options on dividend-paying stocks, early exercise is often optimal just before the ex-dividend date (to avoid losing dividends). This means the American call option might be more valuable than the European call option.
- Unlike European options, American calls and puts do not have a fixed parity relationship because the option holder's decision to exercise early can affect the option's price.
- The strict no-arbitrage condition that holds for European options does not necessarily apply to American options due to the early exercise feature. This creates pricing discrepancies that do not align with put-call parity.

5.2 Numerical Example

A Numerical Example

Suppose we take stock price $S = 100$, strike price $K = 95$, time to expiration as 6 months, risk-free rate as 5%, dividend yield as 2%, European call price 8 and European put price to be 3.

$$8 - 3 = 100e^{-0.02(0.5)} - 95e^{-0.05(0.5)}$$

$$5 = 100(0.9900) - 95(0.9753)$$

$$5 = 99 - 92.65$$

$$5 = 6.35$$

There is a small difference due to rounding, but the parity approximately holds.

6 Question 6

Calculate the implied volatility of Microsoft stock using March 2025 calls expiring March 21, 2025 with strike 410 and with strike 420. Get the quotes from any data provider, for example, finance.yahoo.com and use all other necessary data. On Bloomberg type `MSFT Equity CALL` (page down as needed).

Answer

6.1 Solve it Using Python

I used python to calculate the implied volatility. Currently the risk-free rate is 4.24% as I got from the source. And the current market price for strike 410 is $3.04 and 420 $1.21 separately. Then use the Black-Scholes model to recalculate the implied volatility:

```
import numpy as np
from scipy.stats import norm

# Black-Scholes call option pricing formula
def black_scholes_call(S, K, T, r, sigma, q=0):
d1 = (np.log(S / K) + (r - q + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
d2 = d1 - sigma * np.sqrt(T)
return np.exp(-q * T) * S * norm.cdf(d1) - K * np.exp(-r * T) * norm.cdf(d2)

# Implied Volatility Function using Newton-Raphson Method
def implied_volatility(S, K, T, r, market_price, q=0, tol=1e-6, max_iter=100):
sigma = 0.2  # Initial Guess
for i in range(max_iter):
price = black_scholes_call(S, K, T, r, sigma, q)
vega = S * np.exp(-q * T) * norm.pdf((np.log(S / K) + (r - q + 0.5 * sigma ** 2) * T)
    / (sigma * np.sqrt(T))) * np.sqrt(T)
price_diff = price - market_price
if abs(price_diff) < tol:
return sigma
sigma -= price_diff / vega
raise ValueError("Implied volatility did not converge")

# Example Inputs
S = 394.9       # Current stock price (MSFT)
K = 420           # Strike price, 410 and 420
T = 18 / 365      # Time to expiration in years, March 21-3
r = 0.0424        # Risk-free interest rate, get from website
q = 0.0085         # Continuous dividend yield
market_price = 1.21 # Call option market price

try:
iv = implied_volatility(S, K, T, r, market_price, q)
print(f"Implied Volatility: {iv:.4f}")
except ValueError as e:
print(e)
```

And the printout for strike price 410 is:

```
Implied Volatility: 0.2335
```

And the printout for strike price 420 is:

```
Implied Volatility: 0.2289
```

6.2 Solve it Using Calculator

And I also tested it from the calculator:

This can be reconfirmed with the data from Yahoo Finance:

Figure 10: Results using Calculator

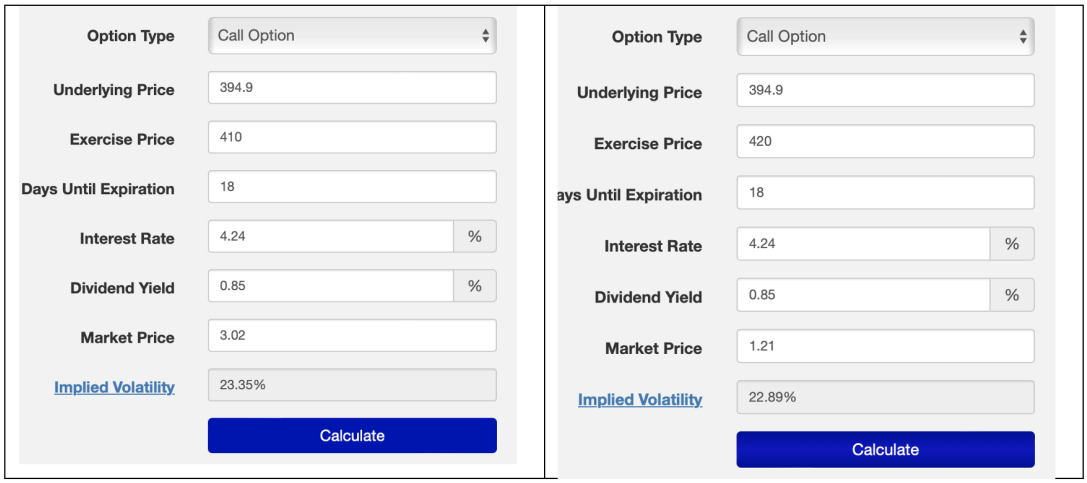


Figure 11: Strike Price 410, Expire Date 2025-03-21

NasdaqGS - Nasdaq Real Time Price • USD

Microsoft Corporation (MSFT) ☆ Follow Compare Time to buy MSFT?

394.90 -2.09 (-0.53%)

As of 12:37:32 PM EST. Market Open.

Mar 21, 2025 | 410.00 | List | All Options

Calls

In The Money

Contract Name	Last Trade Date (EST)	Expire Date	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
MSFT250307C00410000	3/3/2025 12:06 PM	2025-03-07	0.47	0.47	0.48	-0.40	-45.98%	4,358	6,556	23.98%
MSFT250314C00410000	3/3/2025 12:05 PM	2025-03-14	1.77	1.75	1.79	-1.23	-41.00%	927	2,513	23.47%
MSFT250321C00410000	3/3/2025 12:01 PM	2025-03-21	3.02	2.98	3.05	-0.78	-20.53%	590	7,464	23.33%
MSFT250328C00410000	3/3/2025 11:54 AM	2025-03-28	4.17	4.05	4.15	-1.04	-19.96%	243	860	23.12%
MSFT250404C00410000	3/3/2025 11:54 AM	2025-04-04	5.32	5.15	5.30	-1.20	-18.40%	91	423	23.32%

Figure 12: Strike Price 420, Expire Date 2025-03-21

NasdaqGS - Nasdaq Real Time Price • USD

Microsoft Corporation (MSFT) ☆ Follow Compare Time to buy MSFT?

394.90 -2.09 (-0.53%)

As of 12:37:32 PM EST. Market Open.

Mar 21, 2025 | 420.00 | List | All Options

Calls

In The Money

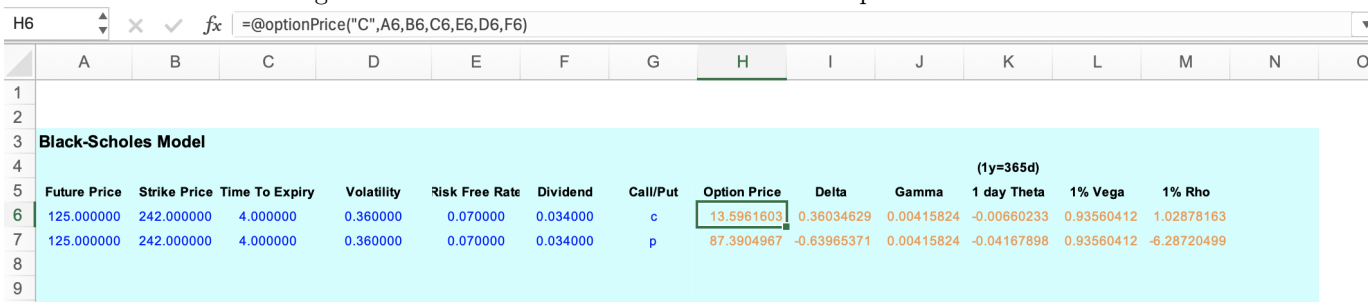
Contract Name	Last Trade Date (EST)	Expire Date	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
MSFT250307C00420000	3/3/2025 12:00 PM	2025-03-07	0.11	0.10	0.11	-0.08	-44.44%	2,693	9,199	26.17%
MSFT250314C00420000	3/3/2025 12:03 PM	2025-03-14	0.58	0.54	0.56	-0.30	-34.09%	458	2,513	23.15%
MSFT250321C00420000	3/3/2025 12:04 PM	2025-03-21	1.21	1.18	1.22	-0.39	-24.38%	1,305	23,553	22.57%
MSFT250328C00420000	3/3/2025 12:00 PM	2025-03-28	1.94	1.86	1.94	-0.52	-21.14%	286	2,729	22.33%
MSFT250404C00420000	3/3/2025 12:01 PM	2025-04-04	2.73	2.67	2.74	-0.80	-22.66%	79	547	22.42%

7 Question 7

Download Excel options model with VBA code from the courseworks . Save as a source. Open in Excel. Modify the Black model for futures to Black-Scholes model for stocks paying dividends at rate q (like in the Hull's book). Make the necessary changes in Visual Basic code. Use Excel help or consult TA's if you do not know what to do. Check that code works and submit the code printout.

Answer

Figure 13: The Printout of the B-S Model Re-implemented in Excel




```
    Function OptionPrice(OptionType As String, S As Double, X As Double, T As Double, r
        As Double, v As Double, d As Double) As Double
    dOne = (Log(S / X) + (r - d + 0.5 * v ^ 2) * T) / (v * (Sqr(T)))
    NdOne = Exp(-(dOne(S, X, T, r, v, d) ^ 2) / 2) / (Sqr(2 *
    dTwo = dOne(S, X, T, r, v, d) - v * Sqr(T)
    NdTwo = Application.NormSDist(dTwo(S, X, T, r, v, d))
    If OptionType = "C" Then
    OptionPrice = Exp(-d * T) * S * Application.NormSDist(dOne(S, X, T, r, v, d)) - X *
        Exp(-r * T) * Application.NormSDist(dOne(S, X, T, r, v, d) - v * Sqr(T))
    ElseIf OptionType = "P" Then
    OptionPrice = X * Exp(-r * T) * Application.NormSDist(-dTwo(S, X, T, r, v, d)) - Exp
        (-d * T) * S * Application.NormSDist(-dOne(S, X, T, r, v, d))
    End If
    End Function

    Function OptionDelta(OptionType As String, S As Double, X As Double, T As Double, r
        As Double, v As Double, d As Double) As Double
    If OptionType = "C" Then
    OptionDelta = Application.NormSDist(dOne(S, X, T, r, v, d))
    ElseIf OptionType = "P" Then
    OptionDelta = Application.NormSDist(dOne(S, X, T, r, v, d)) - 1
    End If
    End Function

    Function OptionTheta(OptionType As String, S As Double, X As Double, T As Double, r
        As Double, v As Double, d As Double) As Double
    dOne = (Log(S / X) + (r - d + 0.5 * v ^ 2) * T) / (v * (Sqr(T)))
    NdOne = Exp(-(dOne(S, X, T, r, v, d) ^ 2) / 2) / (Sqr(2 *
    dTwo = dOne(S, X, T, r, v, d) - v * Sqr(T)
    NdTwo = Application.NormSDist(dTwo(S, X, T, r, v, d))
    If OptionType = "C" Then
    OptionTheta = -((S * v * NdOne(S, X, T, r, v, d)) / (2 * Sqr(T)) - r * X * Exp(-r * (
        T)) * NdTwo(S, X, T, r, v, d)) / 365
    ElseIf OptionType = "P" Then
    OptionTheta = -((S * v * NdOne(S, X, T, r, v, d)) / (2 * Sqr(T)) + r * X * Exp(-r * (
        T)) * (1 - NdTwo(S, X, T, r, v, d))) / 365
    End If
    End Function

    Function Gamma(S As Double, X As Double, T As Double, r As Double, v As Double, d As
        Double) As Double
```

```
    Gamma = NdOne(S, X, T, r, v, d) / (S * (v * Sqr(T)))
    End Function

    Function Vega(S As Double, X As Double, T As Double, r As Double, v As Double, d As
        Double) As Double
    Vega = 0.01 * S * Sqr(T) * NdOne(S, X, T, r, v, d)
    End Function

    Function OptionRho(OptionType As String, S As Double, X As Double, T As Double, r As
        Double, v As Double, d As Double) As Double
    If OptionType = "C" Then
    OptionRho = 0.01 * X * T * Exp(-r * T) * Application.NormSDist(dTwo(S, X, T, r, v, d)
        )
    ElseIf OptionType = "P" Then
    OptionRho = -0.01 * X * T * Exp(-r * T) * (1 - Application.NormSDist(dTwo(S, X, T, r,
         v, d)))
    End If
    End Function
```

8 Question 8

Download Excel Brownian Motion model from the courseworks. Save as a source. Open in Excel. Modify it to Geometric Brownian motion with growth rate $\mu = 0.04$, volatility $\sigma = 0.20$ and 250 trajectories. Submit excel formulas printout. Geometric Brownian motion starts with positive X_0 so you must change a starting value from 0 to a positive number that you may choose (you can choose 1, 100 or other positive number).

Answer

8.1 Solve it Using Excel

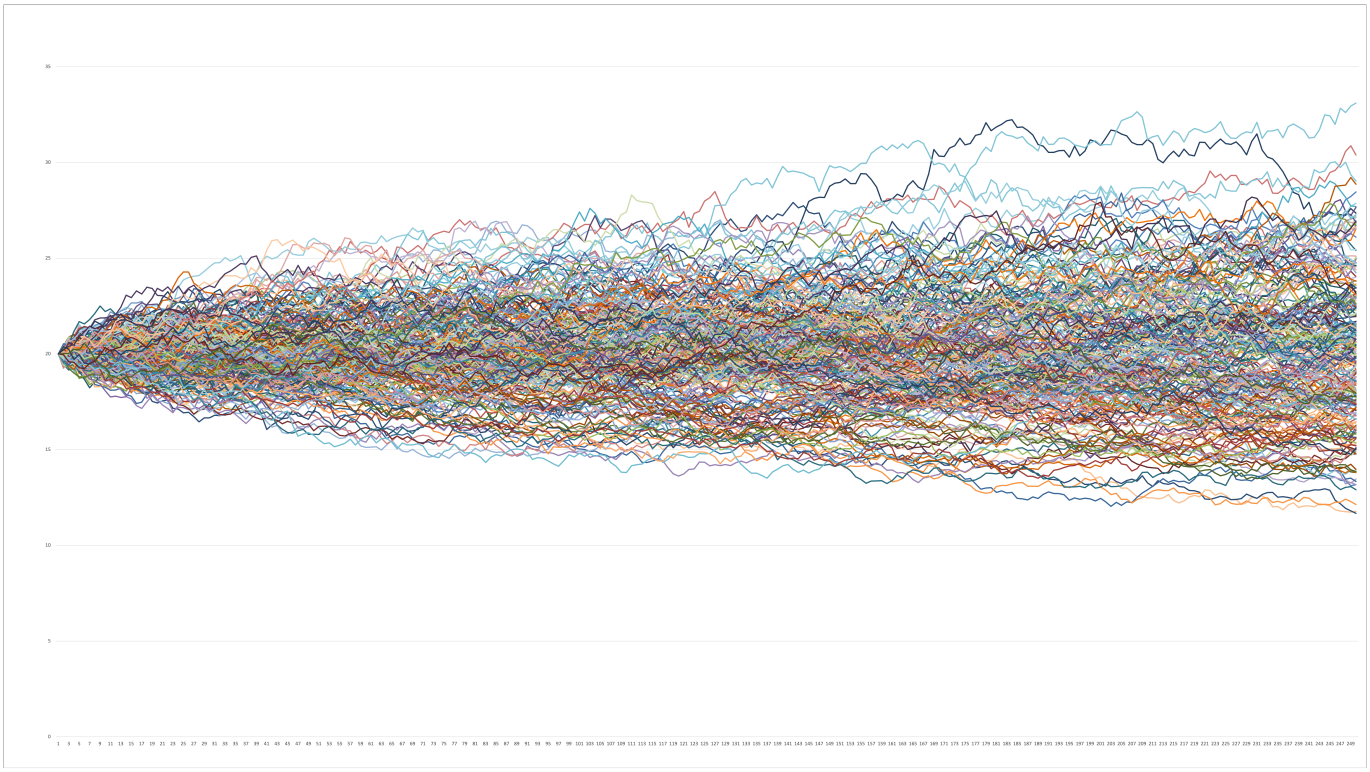
The number for X_0 I chose was 20, I firstly put several parameters in the `A` column (for time step, it is 250).

Figure 14: Settings in my Excel

	A	B	C	D	E	F	G
1	Growth Rate	Time	1	2	3	4	5
2	0.04	Trac 1	20	19.82077317	19.54878179	19.43056657	19.52799716
3	Volatility	Trac 2	20	19.65231767	19.9391074	20.203435	20.29430839
4	0.2	Trac 3	20	20.14019724	19.95438317	19.9537862	20.0414594
5	Initial Value	Trac 4	20	19.92875626	19.8819482	19.5855312	19.54197732
6	20	Trac 5	20	19.88932391	20.11611406	19.41433972	19.18248718
7	Num of Trajectories	Trac 6	20	20.38122799	20.34695327	20.71934936	21.04401308
8	250	Trac 7	20	20.08282482	19.96230487	19.99068615	19.46131656
9	Time Steps	Trac 8	20	20.11316508	20.37823955	20.01866757	19.76659571
10	250	Trac 9	20	19.87982115	19.70980455	19.64561904	19.62167339
11		Trac 10	20	20.14935925	20.04433286	19.50385732	19.30481408
12		Trac 11	20	20.10647884	20.30952821	20.26797587	20.13457117
13		Trac 12	20	20.11397681	20.43318372	20.64539772	20.69600439
14		Trac 13	20	20.0551359	20.15186673	19.8341527	20.02963292
15		Trac 14	20	19.653336	19.69143688	19.35136806	19.14842257
16		Trac 15	20	19.8889085	19.64843999	19.58420851	19.57516713
17		Trac 16	20	19.85366181	19.6011544	19.48604228	19.54749401
18		Trac 17	20	20.07770051	20.30712651	20.54204803	20.33692895
19		Trac 18	20	19.87541073	19.77064086	19.52381052	19.52223837
20		Trac 19	20	19.86821155	19.95924734	20.28252577	20.40764293
21		Trac 20	20	20.1672283	20.1779516	20.54243567	20.09745128
22		Trac 21	20	20.28688007	20.14439704	19.73582509	20.07415532

In `D2`, I set the formula to be `=C2*EXP(($A$2-0.5*$A$4^2)*(1/250)+$A$4*SQRT(1/250)*NORMSINV(RAND()))`. The final printout is:

Figure 15: Geometric Brownian Motion with 250 Trajectories



8.2 Solve it Using R

Out of interest, I redid it in R:

```r
X0 <- 20
mu <- 0.04
sigma <- 0.20
n <- 250
T <- 1
N <- 250
dt <- T / N

# Generate time vector
time <- seq(0, T, by = dt)

# Initialize matrix to store trajectories
X <- matrix(NA, nrow = N + 1, ncol = n)
X[1, ] <- X0

# Simulate GBM
set.seed(123)  # For reproducibility
for (i in 1:n) {
   for (j in 2:(N + 1)) {
            dW <- rnorm(1, mean = 0, sd = sqrt(dt))
            X[j, i] <- X[j - 1, i] * exp((mu - 0.5 * sigma^2) * dt + sigma * dW)
   }
}

# Convert to data frame for plotting
X_df <- as.data.frame(X)
X_df$time <- time
X_long <- melt(X_df, id.vars = "time", variable.name = "trajectory", value.name = "
    price")

# Plot the trajectories
ggplot(X_long, aes(x = time, y = price, group = trajectory)) +
```

```
    geom_line(alpha = 0.5) +
    labs(
    x = "Time",
    y = "Price") +
    theme_minimal()
```

One interesting thing I noticed is that the set of trajectories tend to move upward, that's because there exists a drift term μ, which is typically positive in financial implications (and also in this case). Thus we can notice as time goes on, the trajectories get out of 30 in both plots.

Figure 16: Geometric Brownian Motion with 250 Trajectories Generated in R