Homework Midterm, MATH 5010

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1 Question 1

1.1 Solve it Using Matlab

Matlab option model. Download from Courseworks matlab option model files `BlackScholesStocks.m` and `BlackScholesGraph.m` and put them in the same directory. `BlackScholesStocks.m` contains the function that calculates the Black Scholes price for options on non-dividend paying stocks. `BlackScholesGraph.m` is a script that makes a graph of option price as a function of stock price. Type at Matlab prompt `>BlackScholesGraph` and the script will be executed, and the graph will appear. Now modify the file `BlackScholesStocks.m` so that the function now calculates the price of options on stocks paying continuous dividends at a rate q. Modify the file `BlackScholesGraph.m` so that it now plots graph of a call with the same parameters as before but with the dividend yield q = 2% and the new strike price 11. Submit printouts of code and graph.

Answer The modified `BlackScholesStocks.m` file is:

```matlab
function Price = BlackScholesStocks(callput, S, K, r, q, sigma, T)

% Adjusted d1 and d2 formulas with continuous dividend yield q
d1 = (log(S/K) + (r - q + 0.5 * sigma^2) * T) / (sigma * sqrt(T));
d2 = d1 - sigma * sqrt(T);

% Define normcdf alternative using erf
function N = norm_cdf_approx(x)
N = 0.5 * (1 + erf(x / sqrt(2)));
end

if callput == 'c'
% for call with dividends
N1 = norm_cdf_approx(d1);
N2 = norm_cdf_approx(d2);
Price = S * exp(-q * T) * N1 - K * exp(-r * T) * N2;

elseif callput == 'p'
% for put with dividends
N1 = norm_cdf_approx(-d1);
N2 = norm_cdf_approx(-d2);
Price = K * exp(-r * T) * N2 - S * exp(-q * T) * N1;

else
error('Invalid option type. Use ''c'' for call or ''p'' for put.');
end
end
```

And the modified `BlackScholesGraph.m` file is:

```matlab
clear all;
clc;

% Parameters
dx = 0.1;
maxX = 20;
X = dx:dx:maxX;
```

```matlab
    Strike = 11; % Updated strike price
    Rate = 0.01;
    Time = 1;
    Volatility = 0.3;
    DividendYield = 0.02; % q = 2%

    Y = max(X - Strike, 0); % Call payoff at expiration

    % Preallocate Z
    Z = zeros(1, length(X));

    % Compute option price with dividends
    i = 0;
    for xVal = X  % Using xVal instead of X to avoid variable conflict
    i = i + 1;
    Z(i) = BlackScholesStocks('c', xVal, Strike, Rate, DividendYield, Volatility, Time);
    end

    % Plot
    plot(X, Z, X, Y);
    xlabel('Stock Price');
    ylabel('Option Value');
    legend('Black-Scholes Call Price', 'Payoff at Expiration');
    grid on;
```

The print-out will be:

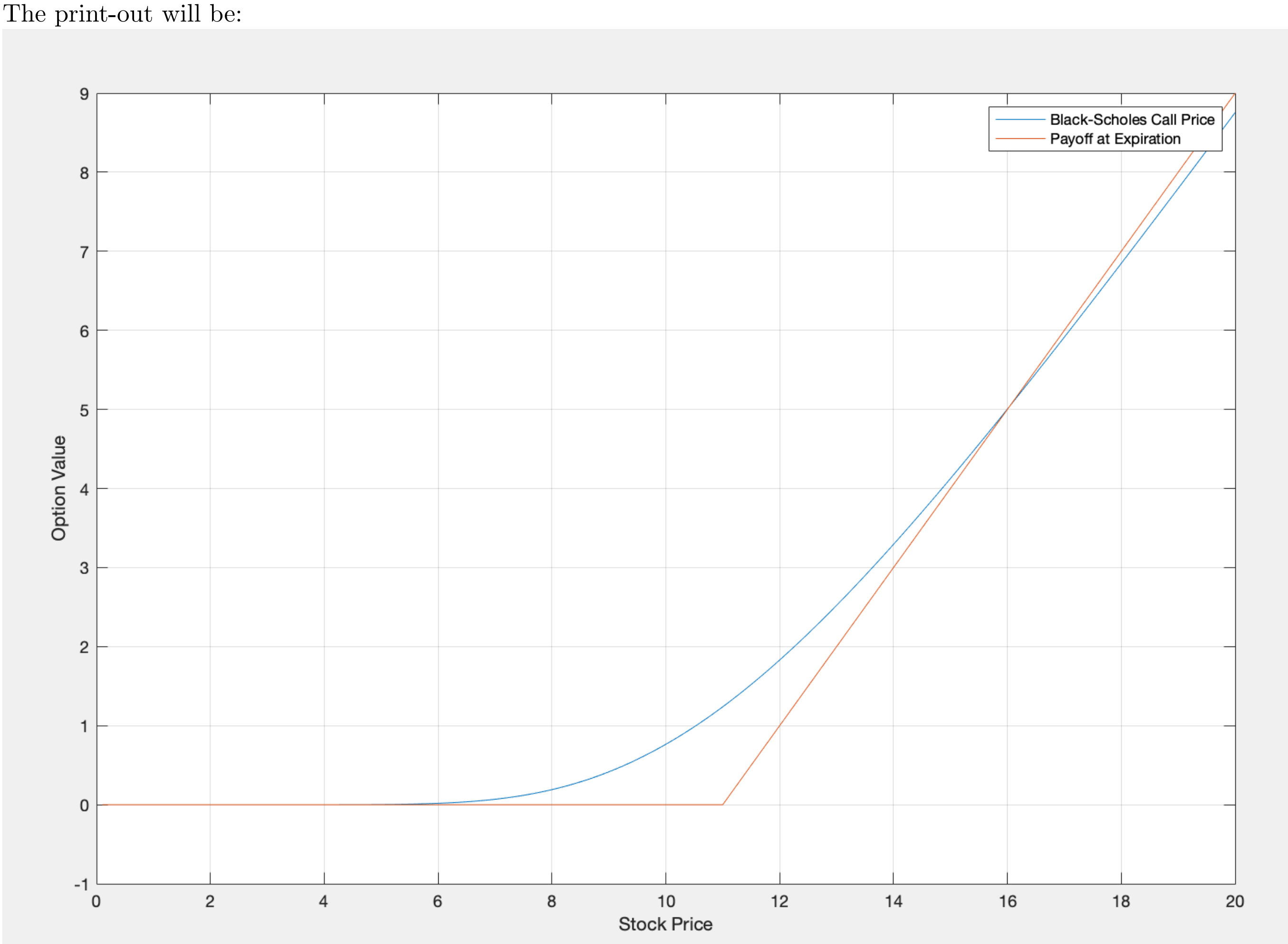

Figure 1: Black-Scholes Call

1.2 Solve it Using R

I also tried to solve it using R.

```r
# Parameters
dx <- 0.1
maxX <- 20
X <- seq(dx, maxX, by=dx)  # Stock price range
Strike <- 11  # Updated strike price
Rate <- 0.01  # Risk-free rate
Time <- 1  # Time to maturity
Volatility <- 0.3  # Volatility
DividendYield <- 0.02  # Dividend yield

# Black-Scholes formula for call option price
d1 <- function(S) {
   (log(S / Strike) + (Rate - DividendYield + 0.5 * Volatility^2) * Time) / (
       Volatility * sqrt(Time))
}
d2 <- function(S) {
   d1(S) - Volatility * sqrt(Time)
}
CallPrice <- function(S) {
   S * exp(-DividendYield * Time) * pnorm(d1(S)) - Strike * exp(-Rate * Time) * pnorm
       (d2(S))
}

# Payoff at expiration (European call option payoff)
Payoff <- function(S) {
   pmax(S - Strike, 0)
}

# Calculate stock price and corresponding call option price and payoff
callOptionPrice <- CallPrice(X)
payoffAtExpiration <- Payoff(X)

# Plotting both the Black-Scholes Call Price and Payoff at Expiration
plot(X, callOptionPrice, type="l", col="blue", lwd=2,
xlab="Stock Price", ylab="Option Value",
ylim=c(0, max(c(callOptionPrice, payoffAtExpiration))))
lines(X, payoffAtExpiration, col="red", lwd=2)
legend("topleft", legend=c("Black-Scholes Call Price", "Payoff at Expiration"),
col=c("blue", "red"), lwd=c(2, 2))
grid()
```

And the printout is as below:

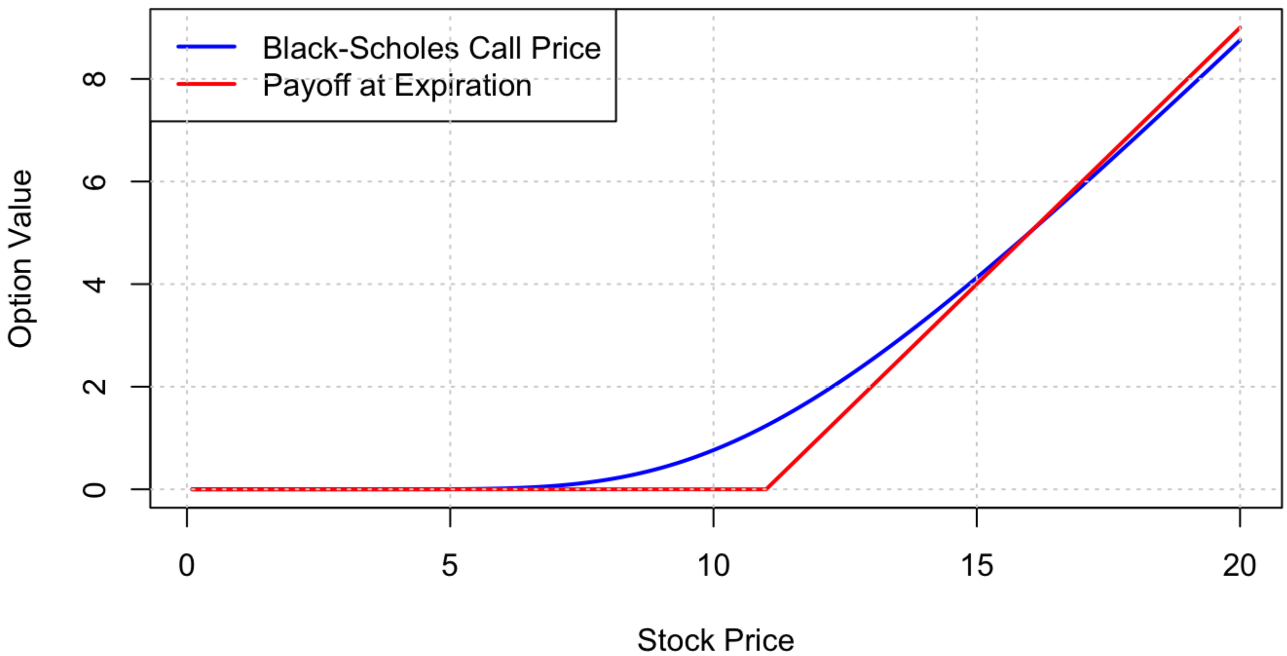

Figure 2: Black-Scholes Call

2 Question 2

Download matlab Brownian motion model from the courseworks. Modify it to Geometric Brownian motion with starting value $X_0 = 100$, growth rate $\mu = 0.14$, volatility $\sigma = 0.28$ and 5000 trajectories. Check that the code works. Try out 50,000 trajectories. Try out 100,000 trajectories. Submit the code printout and the graph printout for 5000 trajectories.

Answer

2.1 Solve it Using Matlab

The modified code will be:

```
     % Parameters
     M = 100000;   % Number of trajectories
     N = 250;    % Number of steps in each trajectory
     X0 = 100;   % Initial value
     T = 1;      % Final time (years)
     mu = 0.14;  % Growth rate (drift)
     sigma = 0.28; % Volatility

     dt = T / N;          % Time step
     Sqrtdt = sqrt(dt);  % Square root of time step

     % Preallocate matrix for efficiency
     X = zeros(M, N + 1);
     X(:, 1) = X0; % Set initial value for all trajectories

     % Generate M trajectories
     for j = 1:M
     for i = 2:N+1
     dW = randn(1,1) * Sqrtdt; % Brownian increment
     X(j, i) = X(j, i-1) * exp((mu - 0.5 * sigma^2) * dt + sigma * dW);
     end
     end

     % Time vector
     t = linspace(0, T, N + 1);

     % Plot multiple trajectories
     figure;
     plot(t, X');
␣␣␣␣␣xlabel('Time (years)');
␣␣␣␣␣ylabel('Stock Price');
␣␣␣␣␣grid␣on;␣
```

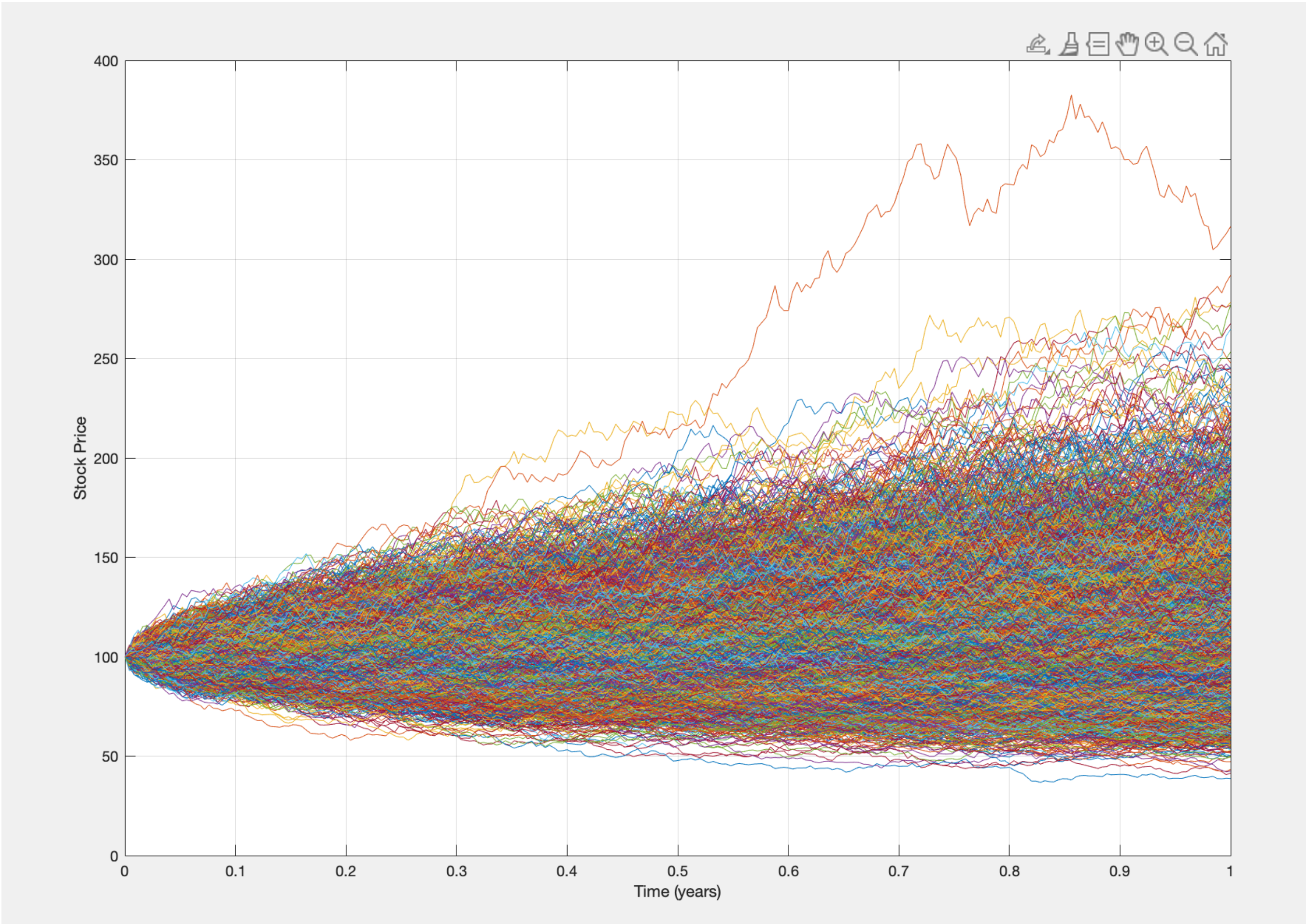

Figure 3: Geometric Brownian Motion with 5,000 Trajectories in MATLAB

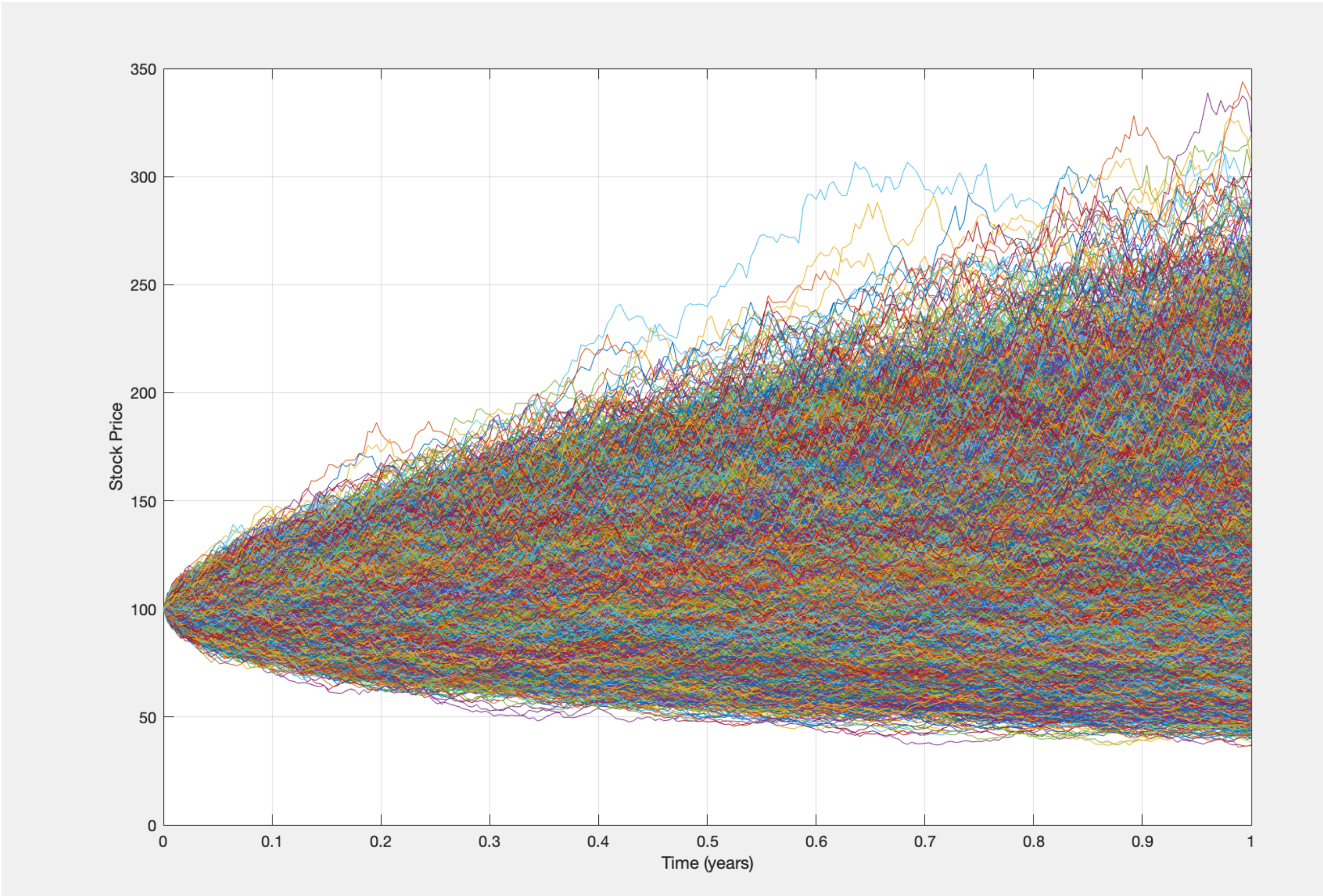

Figure 4: Geometric Brownian Motion with 50,000 Trajectories in MATLAB

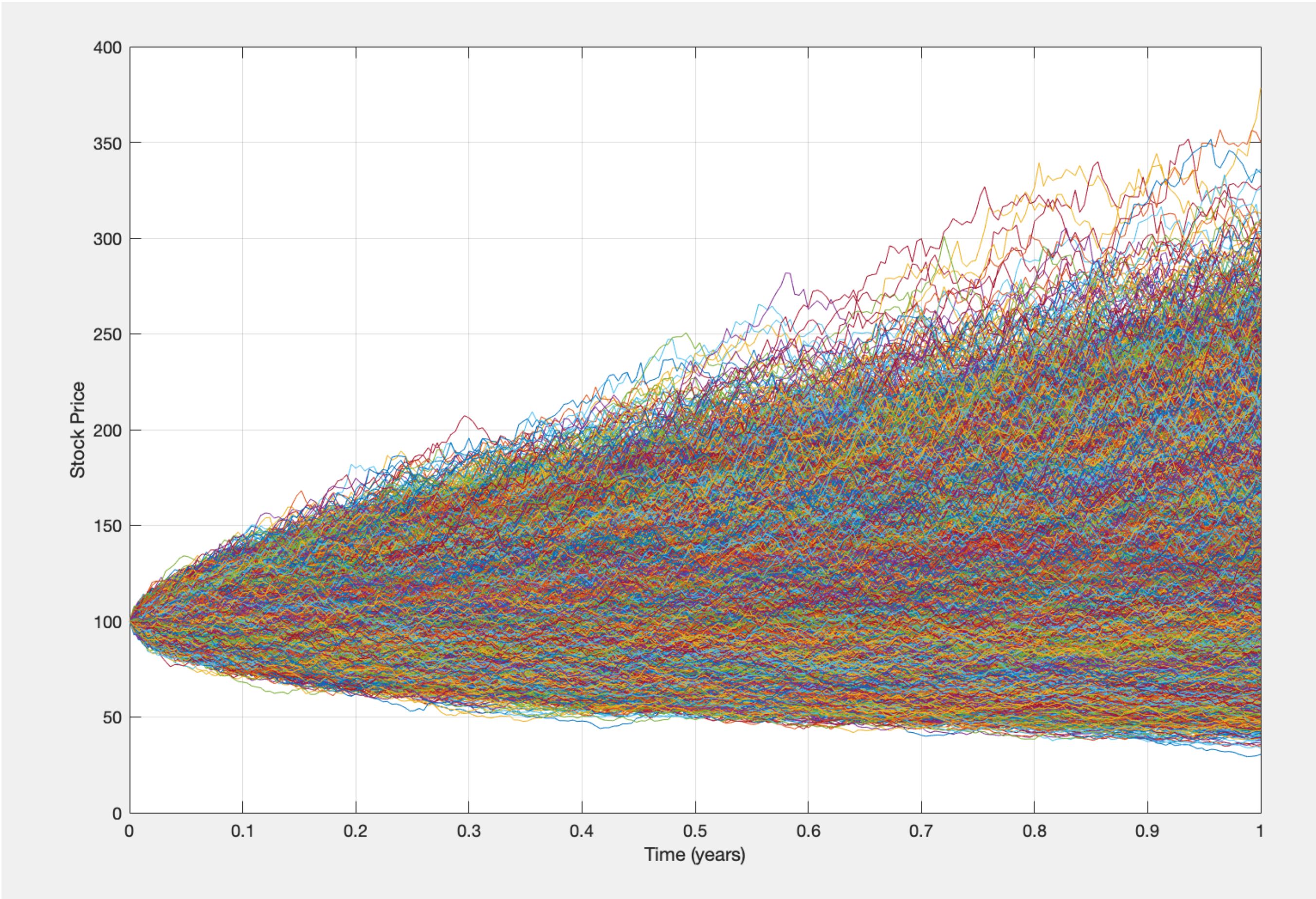

Figure 5: Geometric Brownian Motion with 100,000 Trajectories in MATLAB

2.2 Solve it Using R

I also did it in R, and the codes are as below:

```
library(ggplot2)
library(tidyr)

# Parameters
X0 <- 1000
mu <- 0.14
sigma <- 0.20
n <- 50000    # Number of paths
T <- 1      # Time horizon
N <- 250    # Number of time steps
dt <- T / N # Time increment

# Generate time vector
time <- seq(0, T, by = dt)

# Initialize matrix to store trajectories
X <- matrix(NA, nrow = N + 1, ncol = n)
X[1, ] <- X0

# Simulate GBM
set.seed(123)  # For reproducibility
for (i in 1:n) {
   for (j in 2:(N + 1)) {
```

```
                dW <- rnorm(1, mean = 0, sd = sqrt(dt))
                X[j, i] <- X[j - 1, i] * exp((mu - 0.5 * sigma^2) * dt + sigma * dW)
        }
    }

    # Convert matrix to data frame for ggplot
    X_df <- as.data.frame(X)
    X_df$time <- time

    # Reshape data from wide to long format using tidyr's pivot_longer
    X_long <- X_df %>%
    pivot_longer(cols = -time, names_to = "trajectory", values_to = "price")

    # Plot the trajectories with different colors
    ggplot(X_long, aes(x = time, y = price, color = trajectory)) +
    geom_line(alpha = 0.7) +  # Set transparency for better visibility
    labs(
    x = "Time",
    y = "Price",
    ) +
    theme_minimal() +
    scale_color_viridis_d()+  # Use a perceptually uniform color palette
    theme(legend.position = "none")  # Remove the legend
```

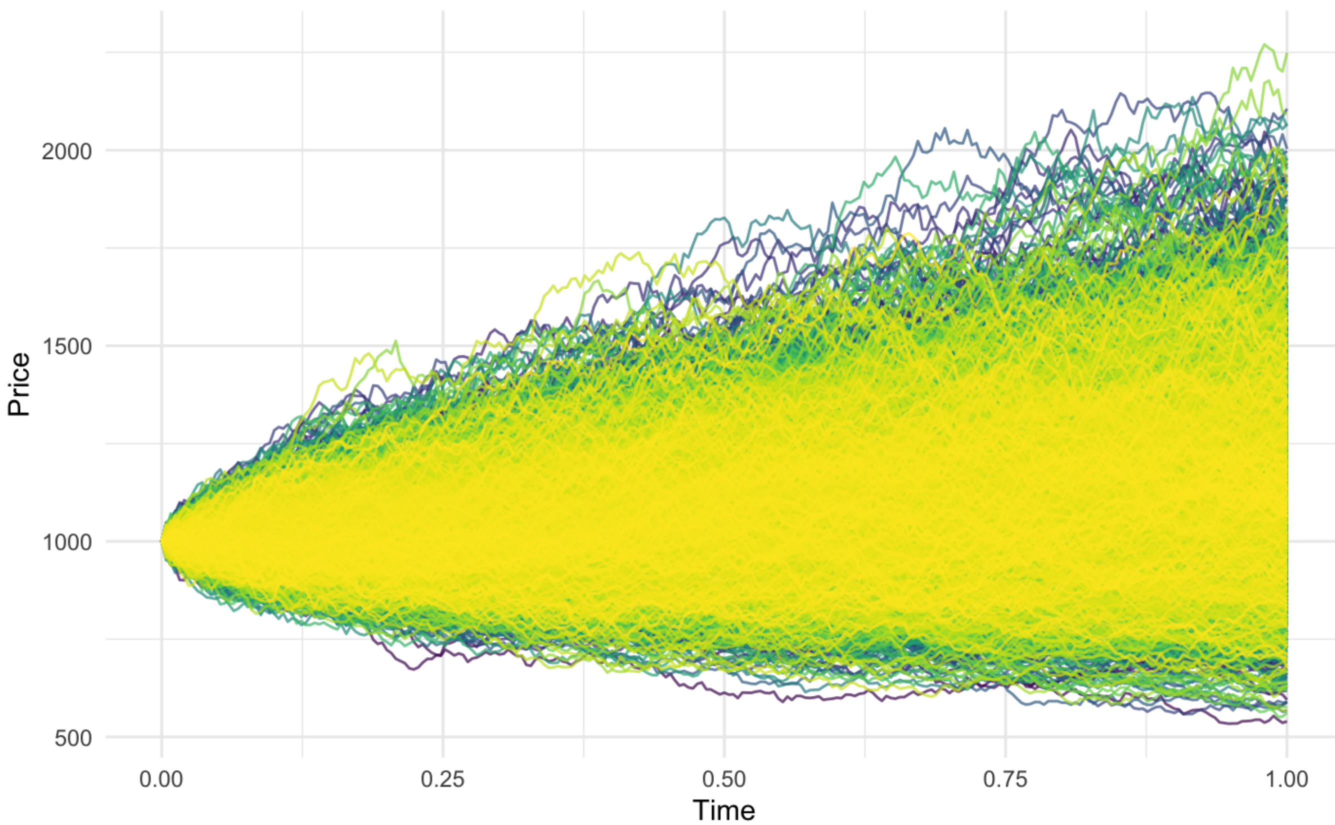

Figure 6: Geometric Brownian Motion with 5,000 Trajectories in R

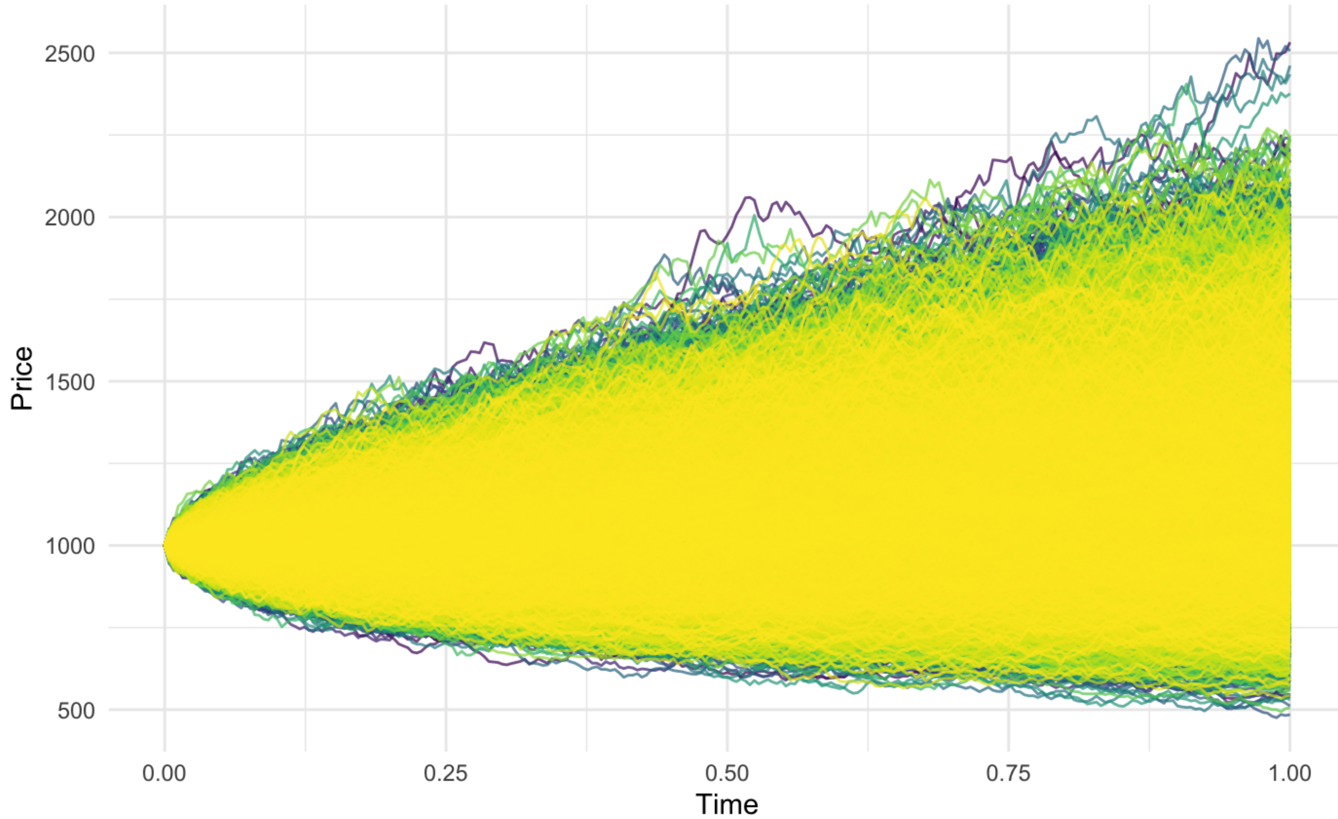

Figure 7: Geometric Brownian Motion with 50,000 Trajectories in R

3 Question 3

Using arbitrage arguments explain why the price of an American call option on a stock paying no dividends should be the same as the price of a corresponding European call. Why American calls on a non-dividend paying stock should not be exercised early.

Answer

For an American call option on a non-dividend-paying stock, the price should be the same as the corresponding European call option. We can show this using an arbitrage argument.

Let C_E be the price of a European call and C_A be the price of a American call with the same strike price and maturity. Since an American call option has at least the same flexibility as a European call, we must have that $C_A \geq C_E$.

Now, suppose $C_A > C_E$. An arbitrageur could execute the following strategy:

- Sell the American call at a higher price C_A.
- Buy the European call at a lower price C_E.
- Since both options have the same payoff at expiration, this results in a risk-free profit.

Since arbitrage opportunities cannot exist in an efficient market, we conclude that:

$$C_A = C_E$$

for an American call on a non-dividend-paying stock.

For this kind of settings, early exercise is never optimal due to the following three reasons:

- The first is that there are no additional cash flow before expiration. When an option holder exercises early, they pay K (the strike price) and receive the stock worth S_t. However, the holder loses the time value of the option (Hull, Ch. 10.5, pg 225, 8th Edition). Instead of exercising, they could sell the option at a higher price in the market. we can see this mathematically:

  $$\begin{aligned} E &: \text{one American call} c, Xe^{-r(T-t)} \text{cash} \\ F &: \text{one share} S. \end{aligned}$$

  If the exercise time is $\tau < T$, the value of E is

  $$\begin{aligned} E &= (S - X) + Xe^{-r(T-t)} \\ &< S = F \end{aligned}$$

  If the exercise is at $t = T$, then

  $$\begin{aligned} E &= \max(S - X, 0) + X \\ &= \max(S, X) \\ &\geq S = F \end{aligned}$$

  It follows that $E \geq F$ for all times, so that one should never take $t < T$.

- The second reason is that holding the call option means postponing the payment of K. Since money has time value, it is better to delay payment as long as possible. Exercising early forces the holder to pay K sooner, which is suboptimal.
- And the last reason is that stock price can increase further. If the option is exercised early, the holder locks in an immediate profit of $S_t - K$. However, if the option is held longer, the stock price could increase further, making the option even more valuable. In Hull's book, he argues: "relates to the insurance that it provides. A call option, when held instead of the stock itself, in effect insures the holder against the stock price falling below the strike price. Once the option has been exercised and the strike price has been exchanged for the stock price, this insurance vanishes." (Ch.10.5, pg 226, 8th Edition) So we can say that early exercise limits the upside potential and cut the insurance.

Since early exercise never benefits the holder of an American call on a non-dividend-paying stock, its price must be the same as the corresponding European call. American call holders should never exercise early because holding the option provides greater flexibility and potential value than exercising and holding the stock.

Thus, we conclude that $C_A = C_E$, for an American call on a non-dividend-paying stock, and early exercise is never optimal.

4 Question 4

Why when the stock pays dividends the argument of the problem 3 can not be used. Give a numerical example (choosing $x,\ k,\ r,\ T-t, \sigma$) in which it is obvious (without any formulas) that American put price on a nondividend paying stock is larger than the corresponding European put price.

Answer

4.1 Dividends-Paying Scenario

When there are dividends, the arugement in Problem 3 is not valid, its price typically drops by the amount of the dividend on the ex-dividend date. For call options, a lower stock price reduces the option's value. For American call options, the holder has the right to exercise the option at any time before expiration. If a dividend is paid, the holder may choose to exercise the option early to capture the stock's value before the price drops due to the dividend. whereas for European call options, the holder can only exercise the option at expiration. They cannot avoid the stock price drop caused by the dividend payment.

If we expand further, when dividends are paid, mostly it can only be optimal to exercise at a time immediately before the stock goes ex-dividend for an American call option (Hull, Ch. 14.12, pg 320, 8th Edition, and the mathematical demonstrations are derived from this chapter). Assuming that n ex-dividend dates are anticipated and that they are at times $t_1, t_2, \ldots, t_n$, with $t_1 < t_2 < \cdots < t_n$. The dividends corresponding to these times will be denoted by $D_1, D_2, \ldots, D_n$.

We can start by considering the possibility of early exercise just prior to the final ex-dividend date (i.e., at time t_n). If the option is exercised at time t_n, the investor receives$S\left(t_n\right)-K$

Here $S(t)$ denotes the stock price at time t. If the option is not exercised, the stock price drops to $S\left(t_n\right)-D_n$. The value of the option is then greater than

$$S\left(t_n\right)-D_n-K e^{-r\left(T-t_n\right)}$$

It follows that, if

$$S\left(t_n\right)-D_n-K e^{-r\left(T_n t_n\right)} \geqslant S\left(t_n\right)-K$$

that is,

$$D_n \leqslant K\left[1-e^{-r\left(T-t_n\right)}\right]$$

it cannot be optimal to exercise at time t_n. On the other hand, if $D_n > K\left[1-e^{-n\left(T-t_n\right)}\right]$

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S\left(t_n\right)$. The inequality in equation above will tend to be satisfied when the final e: dividend date is fairly close to the maturity of the option (i.e., $T-t_n$ is small) and th dividend is large.

For the next time t_{n-1}, the penultimate ex-dividend date. If the option is exercise immediately prior to time t_{n-1}, the investor receives $S\left(t_{n-1}\right)-K$. If the option is n, exercised at time t_{n-1}, the stock price drops to $S\left(t_{n-1}\right)-D_{n-1}$ and the carlie subsequent time at which exercise could take place is t_n. A lower bound to the option price if it is not exercised at time t_{n-1} is:

$$S\left(t_{n-1}\right)-D_{n-1}-K e^{-r\left(t_n-t_{n-1}\right)}$$

If

$$S\left(t_{n-1}\right)-D_{n-1}-K e^{-\left(\left(t_n-t_{n-1}\right)\right.} \geqslant S\left(t_{n-1}\right)-K$$

or

$$D_{n-1} \leqslant K\left[1-e^{-\int\left(t_n-t_{n-1}\right)}\right]$$

it is not optimal to exercise immediately prior to time t_{n-1}. Similarly, for any $i<n$,

$$D_i \leqslant K\left[1-e^{-r\left(t_{i+1}-t_i\right)}\right]$$

it is not optimal to exercise immediately prior to time t_i. The above inequality is approximately equivalent to:

$$D_i \leqslant K r\left(t_{i+1}-t_i\right)$$

Assuming that K is fairly close to the current stock price, this inequality is satisfies when the dividend yield on the stock is less than the risk-free rate of interest, and this is usually the case.

We can conclude from the derivations above that in most of cases, doing an early exercise for American call option is optimal, thus it will not stay the same as the European call option as did in Problem 3, and it changes the argument.

4.2 Put Option on Nondividend Paying Stock

We can get a simple numerical example to show that the American put option on a non-dividend-paying stock can be more valuable than the corresponding European put option. This happens because the American put option allows for early exercise, which can be advantageous in certain scenarios.

- Stock price $S = 100$,
- Strike price $K = 110$,
- Risk-free rate $r = 5\%$,
- Time to maturity $T - t = 1$ year,
- Volatility $\sigma = 20\%$,
- Dividends: None.

Suppose during the life of the option, the stock price drops significantly (e.g., due to a market crash or bad news). Let's say the stock price drops to $S = 80$ at some point before expiration.

The holder of the American put option can exercise early when the stock price is $S = 80$. The payoff from early exercise is $K - S = 110 - 80 = 30$. By exercising early, the holder locks in a profit of 30 and avoids the risk of the stock price recovering later.

Whereas the holder of the European put option cannot exercise early and must wait until expiration. If the stock price recovers to $S = 90$ by expiration, the payoff is $K - S = 110 - 90 = 20$. The European put option holder receives only 20, which is less than the American put option holder's payoff of 30.

We can say that the American put option allows the holder to capture the maximum payoff when the stock price drops significantly, even if the stock price later recovers. The European put option holder is stuck waiting until expiration and may miss the opportunity to lock in a higher payoff.

Thus, it is clear that the American put option is more valuable than the European put option when the stock price drops significantly, even if the stock pays no dividends. This demonstrates the value of the early exercise feature in American options.

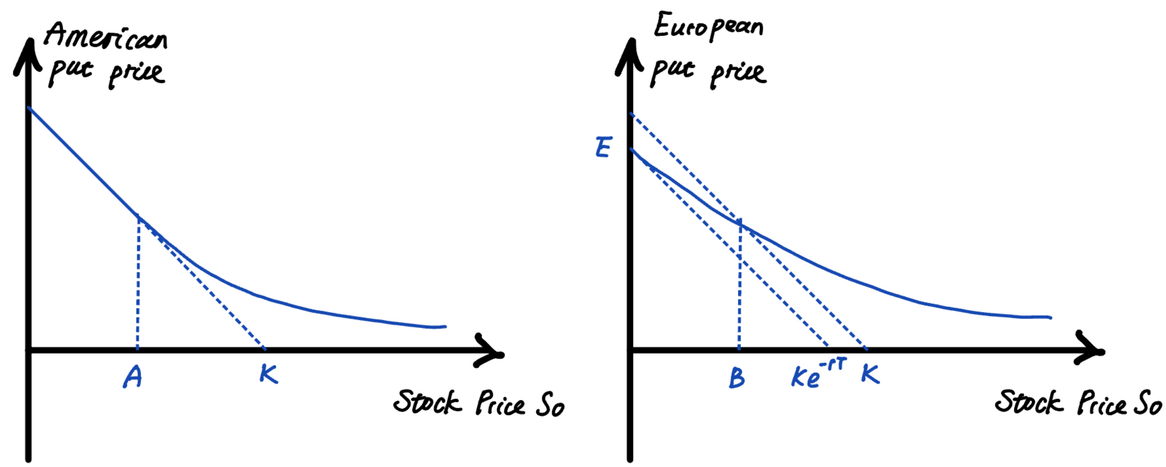

Figure 8: Comparison of American and European put Options with the Stock Price

5 Question 5

5.1 Part a

The stock price is 40 the volatility of the stock is 20%. Assuming that the time to expiration is 3 months and the interest rate is 1% per annum calculate the price P of the European call option with strike 41.

Answers

[Black Scholes Calculator]

Option		Stock		Market	
Strike	41	Price	40	Interest Rate	1%
Expiration (years)	0.25	Volatility	20%	**Settings**	
		Dividend	0%	Precision	5

	European Call	European Put	Forward	Binary Call	Binary Put
Price	1.20371	2.10134	-0.89763	0.39186	-0.39186
Delta	0.43175	-0.56825	1.00000	0.09588	-0.09588
Gamma	0.09827	0.09827	0.00000	0.00412	-0.00412
Vega	7.86179	7.86179	0.00000	0.32967	-0.32967
Rho	4.01655	-6.20786	10.22441	0.86079	-1.11017
Theta	3.30538	2.89640	0.40898	-0.11240	0.10243

Figure 9: Calculator Results

This gives the result to be \$ 1.20371.

And I also tried to calculate using Black-Scholes Model, in which the European call option price is calculated as:

$$C = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

and for we can calculate $d_1 = \frac{ln(S_0/K)+(r-q+0.5\sigma^2)T}{\sigma\sqrt{T}}$, and it equals to $\frac{ln(40/41)+(0.01+0.5*0.04)*0.25}{0.2*\sqrt{0.25}} = -0.1719261259$. For d_2, we have $d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.1 = -0.2719261259$.

Using the Z-table, we can find that:

- $-0.17 \to 0.4325$
- $-0.27 \to 0.3936$

Plugging back, we can get $C = S_0 * N_{d1} - Ke^{-rT} * N_{d2} = 40 * 0.4325 - 41 * e^{-0.01*0.25} * 0.3936 = 1.202693$. It is roughly \$ 1.2, as the Z-table as not extremely precise.

5.2 Part b

Calculate Δ, Γ, ρ, Vega using formulas for these parameters. Calculate the same parameters approximately using the options calculator.

Answers

As seen from the graph, we have the value of those Greeks to be:

- $\Delta = 0.43175$
- $\Gamma = 0.09827$

- $\rho = 4.01655$
- $Vega = 7.86179$
- $\Theta = 3.30538$
- Price = 1.20371

Because calculating Greeks requires more precision, I can not simply using Z-table, here is a chunk of code for R

```
# Parameters
S0 <- 40  # Stock price
K <- 41   # Strike price
T <- 0.25  # Time to expiration in years (3 months)
sigma <- 0.2  # Volatility
r <- 0.01  # Risk-free rate

# Black-Scholes formula components
d1 <- (log(S0 / K) + (r + 0.5 * sigma^2) * T) / (sigma * sqrt(T))
d2 <- d1 - sigma * sqrt(T)

# CDF of d1 and d2 using normal distribution
N_d1 <- pnorm(d1)
N_d2 <- pnorm(d2)
N_prime_d1 <- dnorm(d1)  # PDF of the normal distribution

# Delta
Delta <- N_d1

# Gamma
Gamma <- N_prime_d1 / (S0 * sigma * sqrt(T))

# Vega
Vega <- S0 * sqrt(T) * N_prime_d1

# Theta
Theta <- -(S0 * N_prime_d1 * sigma) / (2 * sqrt(T)) - r * K * exp(-r * T) * N_d2

# Rho
Rho <- K * T * exp(-r * T) * N_d2
```

And its printout is:

```
Delta: 0.4317478
Gamma: 0.09827239
Vega: 7.861791
Theta: -3.305378
Rho: 4.01655
```

The same as above.

5.3 Part c

Check that following relationship holds:

$$\Theta + rx\Delta + \frac{1}{2}\sigma^2 x^2 \Gamma = rP$$

Answer

$\Theta + r \cdot S \cdot \Delta + \frac{1}{2}\sigma^2 \cdot S^2 \cdot \Gamma = -3.30538 + 0.01 \cdot 40 \cdot 0.431748 + 0.5 \cdot 0.2^2 40^2 \cdot 0.098272 = 0.0120232$, and $rP = 0.0120371$. Those two sides are the same.

6 Question 6

What are the parameters affecting prices European and American calls and puts. How do the prices change when one of the parameters changes with all the others remaining the same?

Answer

The prices of European and American call and put options depend on several factors, including the underlying asset price, strike price, time to expiration, volatility, interest rate, and dividends. Their effects can be seen in the table below (considering parameters to be "higher", to impose effects on prices):

Parameter	**Effect on Call Price**	**Effect on Put Price**
Underlying Price (S)	Increases (↑)	Decreases (↓)
Strike Price (K)	Decreases (↓)	Increases (↑)
Time to Expiration (T)	Usually Increases (↑)	Usually Increases (↑)
Volatility (σ)	Increases (↑)	Increases (↑)
Risk-Free Rate (r)	Increases (↑)	Decreases (↓)
Dividends (D)	Decreases (↓)	Increases (↑)

Table 1: Impact of Various Parameters on Call and Put Prices

Some explanations:

1. Underlying Asset Price (S)
   - **Call Options**: Higher S increases the call price, as the right to buy at K is more valuable.
   - **Put Options**: Higher S decreases the put price, as the right to sell at K is less valuable.

$$\frac{\partial C}{\partial S} > 0, \quad \frac{\partial P}{\partial S} < 0$$

2. Strike Price (K)
   - **Call Options**: Higher K decreases the call price.
   - **Put Options**: Higher K increases the put price.

$$\frac{\partial C}{\partial K} < 0, \quad \frac{\partial P}{\partial K} > 0$$

3. Time to Expiration (T)
   - **European Options**: Longer T usually increases both call and put prices.
   - **American Options**: Longer T generally increases option prices, but deep in-the-money puts may behave differently due to early exercise potential.

$$\frac{\partial C}{\partial T} > 0, \quad \frac{\partial P}{\partial T} > 0$$

4. Volatility (σ)
   - Higher **volatility** increases both call and put prices since it increases the probability of large price movements.

$$\frac{\partial C}{\partial \sigma} > 0, \quad \frac{\partial P}{\partial \sigma} > 0$$

5. Risk-Free Interest Rate (r)
   - **Call Options**: Higher r increases call prices because the present value of K decreases.
   - **Put Options**: Higher r decreases put prices for the same reason.

$$\frac{\partial C}{\partial r} > 0, \quad \frac{\partial P}{\partial r} < 0$$

6. Dividends (D)

- **Call Options**: Higher dividends reduce call prices since they lower S.
- **Put Options**: Higher dividends increase put prices as S declines.

$$\frac{\partial C}{\partial D} < 0, \quad \frac{\partial P}{\partial D} > 0$$

7 Question 7

Suppose that we have three European calls with strikes 60, 65, and 70 and the same maturity 1 month. Their prices are 9.00, 7.00, 4.00. Is it possible to do an arbitrage?

Answer

To determine if there is an arbitrage opportunity, we need to check whether the given call option prices satisfy the conditions of a butterfly spread, ensuring that no arbitrage violation occurs.

For European call options with the same expiry, the call option prices must be non-increasing in strike prices:

$$C(K_1) \geq C(K_2) \geq C(K_3)$$

Given prices:

$$C(60) = 9.00, \quad C(65) = 7.00, \quad C(70) = 4.00$$

This satisfies the basic monotonicity condition:

$$9.00 \geq 7.00 \geq 4.00$$

Thus, there is no immediate arbitrage due to mis-pricing of monotonicity.

A butterfly spread using these strikes consists of:

- Long 1 Call at $K_1 = 60$
- Short 2 Calls at $K_2 = 65$
- Long 1 Call at $K_3 = 70$

The cost of setting up the butterfly spread is $C(60) - 2C(65) + C(70)$, thus substituting the given prices $9.00 - 2(7.00) + 4.00 = 9.00 - 14.00 + 4.00 = -1.00$

Since the total cost of the butterfly spread is negative, it means that an arbitrage opportunity exists. We are receiving 1.00 upfront while ensuring a non-negative payoff at expiration.

To exploit this arbitrage, we should:

- Buy 1 Call at $K_1 = 60$ for 9.00.
- Sell 2 Calls at $K_2 = 65$ for 7.00 each (receive 14.00).
- Buy 1 Call at $K_3 = 70$ for 4.00.

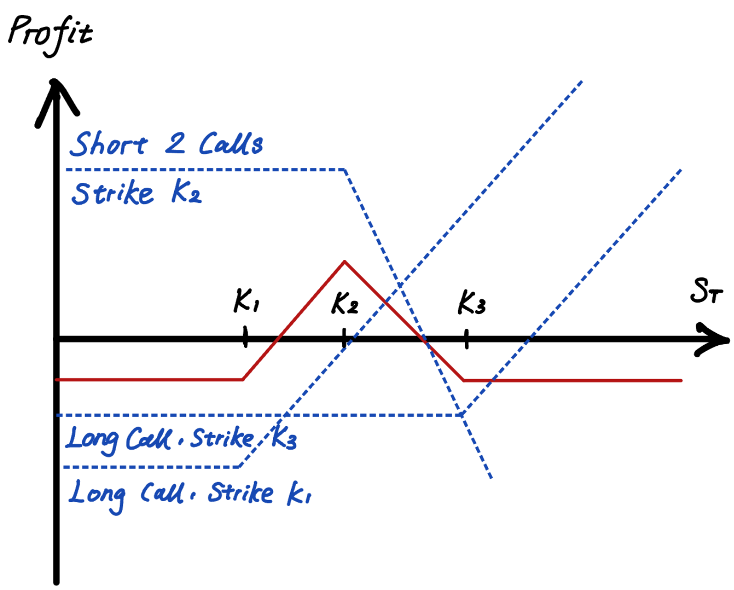

Figure 10: Butterfly Spread Using Call Option

Total cost = -1.00.

At expiration, the payoff of the butterfly spread is always non-negative, so this setup guarantees a risk-free profit.

There exists an arbitrage opportunity exists because the cost of the butterfly spread is negative, and we can execute this arbitrage to receive 1.00 upfront with no risk.

8 Question 8

Suppose that current stock price is \$50. Its annualized volatility is 30% and annualized return 10% i.e. we assume that the stock price follows $dX_t = 0.1X_t dt + 0.3X_t dW$. Write the probability density function for the stock in 1 year. What is the mean and standard deviation of the terminal stock price? (standard deviation of price, not of return)

Answer

(This number is not doubtful)

The stock price X_1 at time $t = 1$ follows a log-normal distribution:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}\sqrt{t}} \exp\left(-\frac{\left(\ln(x) - ln(X_0) - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2}\right)$$

Here:

- $\mu = 0.1$ is the drift (annualized return),
- $\sigma = 0.3$ is the volatility (annualized standard deviation of return),
- x is the value of the stock price X_1 at time $t = 1$.

Thus, the PDF for X_1 becomes:

$$f(x) = \frac{1}{x(0.3)\sqrt{2\pi}} \exp\left(-\frac{(\ln(x) - 3.967023005)^2}{2(0.3)^2}\right)$$

We can get that the <u>mean</u> of X_1 is $\mathbb{E}[X_0] = X_0 e^{(\mu t)}$, and substituting the given values $\mathbb{E}[X_0] = 50e^{(0.1*t)} = 50e^{0.1}$. Using $e^{0.1} \approx 1.10517$, we get $\mathbb{E}[X_0] \approx 50 \times 1.10517 = 55.2585$.

The <u>variance</u> of X_0 is $\mathrm{SD}(X_0) = X_0^2 \exp(2\mu t) * (\exp(\sigma^2 t) - 1)$, and substituting the values we have $Var(X_1) = 50^2 * e^{2*0.1*t}(e^{0.09} - 1) = 287.5618$. Then take its square, we can get 16.9576.

- The <u>mean</u> of the terminal stock price is approximately $\mathbb{E}[X_1] \approx 55.2585$.
- The <u>standard deviation</u> of the terminal stock price is approximately $\mathrm{SD}(X_1) \approx 16.9576$.