

Homework 2, MATH 5010

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1 Question 1

What is the difference between historic and implied volatilities? Which one is higher for Microsoft stock MSFT? Go to the Bloomberg terminal and type: MSFT <Equity> HIVG <Go>. Print the output.

Answer

1.1 Differences Between HV and IV

The Historical Volatility is also called Realized Volatility or Statistical Volatility, and it measures the actual volatility of an asset's price over a specific past period. It is calculated using historical price data (e.g., daily, weekly, or monthly returns).

Historic Volatility is typically calculated as the standard deviation of the asset's logarithmic returns over a given time frame. The formula is:

$$\text{Historic Volatility} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (r_i - \bar{r})^2}$$

where:

- r_i : logarithmic return at time i ,
- \bar{r} : average return over the period,
- N : number of observations.

As for the Implied Volatility (IV), it is a forward-looking measure of volatility derived from the market price of an option. It represents the market's expectation of future volatility over the option's life.

Implied Volatility is not directly calculated but is inferred using an options pricing model (e.g. the Black-Scholes model). It is the value of volatility that, when plugged into the model, matches the market price of the option. The Black-Scholes formula for a call option is:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where:

- C : market price of the call option,
- S_0 : current price of the underlying asset,
- K : strike price,
- r : risk-free rate,
- T : time to expiration,
- $N(d)$: cumulative distribution function of the standard normal distribution,
- d_1 and d_2 : functions of volatility (σ).

Implied Volatility is the value of σ that satisfies this equation.

Finally, for their relationship, over time, implied volatility tends to converge toward historic volatility as the future becomes the past. Implied volatility can be higher or lower than historic volatility depending on market expectations (e.g., during periods of uncertainty or calm).

We can use a table to illustrate this:

Aspect	Historic Volatility	Implied Volatility
Time Frame	Based on past price movements.	Based on future expectations.
Data Used	Historical price data.	Market price of options.
Calculation	Calculated using statistical methods (e.g., standard deviation).	Derived from options pricing models (e.g., Black-Scholes).
Purpose	Measures past price fluctuations.	Predicts future price fluctuations.
Market Sentiment	Does not reflect market sentiment.	Reflects market sentiment and expectations.
Use in Trading	Used for risk analysis and benchmarking.	Used for options pricing and trading strategies.

1.2 MSFT HIVG

Here is the plot provided by Bloomberg Terminal, and the timespan I set was Sept 16, 2020 to Feb 20, 2025.

Figure 1: The Results after using MSFT <Enquiry> HIVG <GO>



Take the date Feb 28, 2025 as the last date for consideration, for a 30-days range, the historical volatility (HV) is 25.99%, which is higher than the implied volatility (IV), which was around 24.51%. The exact data can be found here.

However, if we reconsider it in a large time span, the overall Historical Volatility is smaller than the Implied Volatility. It can be easily seen from the graph, and we can also scrutinize the 180-day-data.

Figure 2: A Simple Plot Showing HV and IV of MSFT



2 Question 2

What is the volatility smile? Using Bloomberg terminal function SKEW (CLJ5 Comdty SKEW <Go>) plot the implied volatility smiles for TYH5 Comdty, CLJ5 Comdty, NGJ25 Comdty, ESH5 Index, SPY Equity. What are these securities? For description type CLJ5 Comdty DES. Compare their smiles. Submit printouts.

Answer

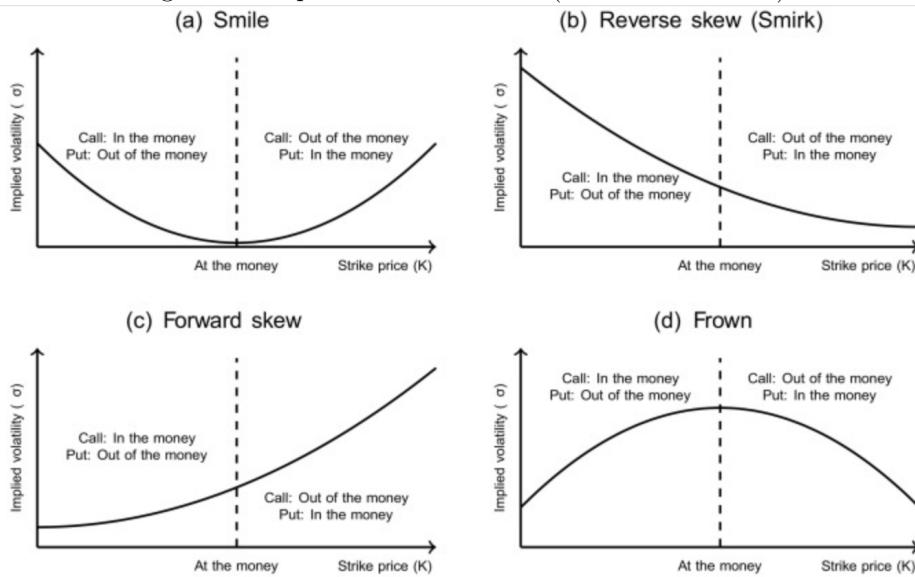
2.1 Definition of Volatility Smile

In the Textbook, the author defines Volatility Smile as "A plot of the implied volatility of an option with a certain life as a function of its strike price." (Hull, 409)

In my understanding, Volatility Smile is a pattern observed in the implied volatility of options across different strike prices, where implied volatility is higher for both out-of-the-money (OTM) puts and calls compared to at-the-money (ATM) options, creating a "smile" shape when plotted. This phenomenon contradicts the assumption of constant volatility in models like Black-Scholes and arises because markets often price in higher uncertainty for extreme price movements (both up and down).

The x-axis is Strike Price, whereas the y-axis is Implied Volatility.

Figure 3: Simple Illustration of VS (cred to Soini et al.)



2.2 Bloomberg Comparison

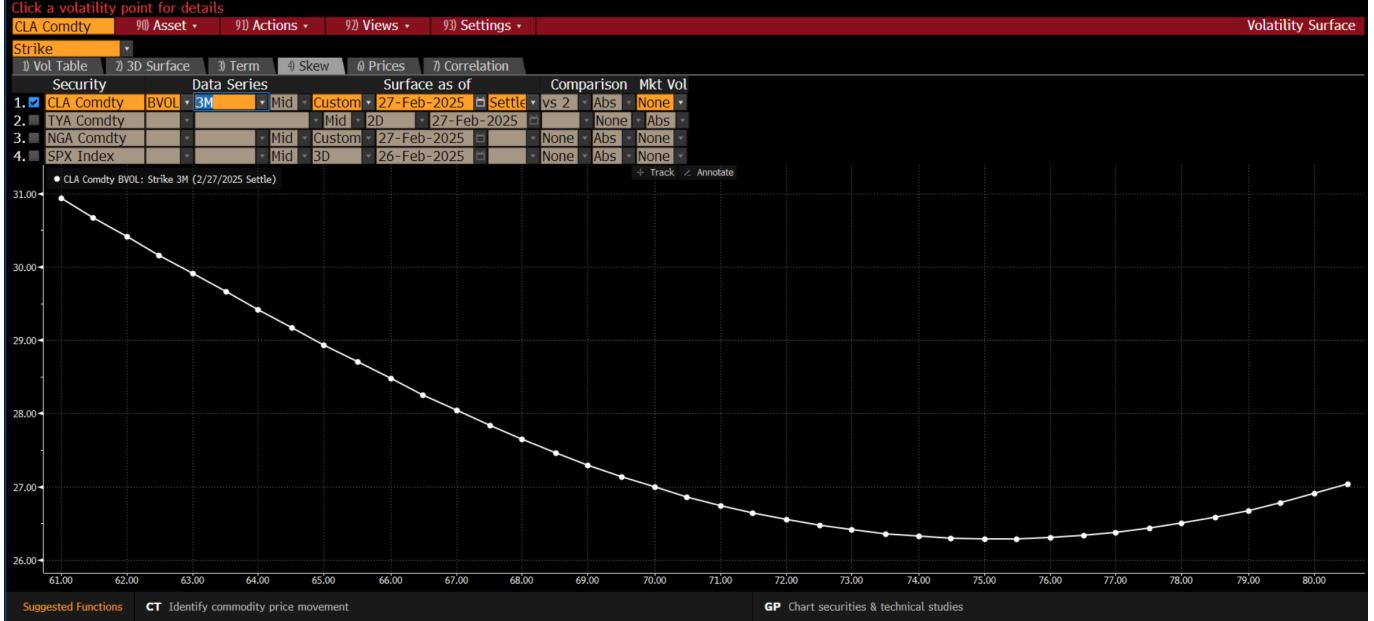
In Bloomberg, they represent:

- TYH5 Comdty: U.S. 10-Year Treasury Futures Contract (April 2025).
- CLJ5 Comdty: Crude Oil Futures (April 2025).
- NGJ25 Comdty: Natural Gas Futures (April 2025).
- ESH5 Index: S&P 500 E-mini Futures (March 2025).
- SPY Equity: SPDR S&P 500 ETF Trust.

Here is the graph of volatility smile in Bloomberg Terminal. The model I set was **Strike** instead of **Delta** or so, because in **Delta** mode, there are only four comparable niches, and it is more standard to show their volatility smiles with strike price; moreover, the overall trend will not be impacted by different x-axes we chose.

Take **Strike** as the x-axis, the Volatility Surface has a Smirk Shape, or Reverse Skew for CLJ5 Comdty, and it reflects a market expectation of greater downside risk than upside potential. It is characterized by higher implied

Figure 4: Volatility Surface of CLJ5 Comdty



volatility for OTM puts compared to OTM calls and is often driven by bearish sentiment, economic uncertainty, or fear of sharp declines. In such scenarios, OTM puts become more expensive due to higher implied volatility, while OTM calls are relatively cheaper. Traders may use strategies like put spreads or risk reversals to capitalize on this skew, while investors might hedge against potential price declines.

For both TYH5 Comdty and NGJ5 Comdty, we have a shape as Forward Skew, meaning its implied volatility is higher for out-of-the-money (OTM) calls (higher strike prices) compared to out-of-the-money (OTM) puts (lower strike prices). This creates an upward-sloping asymmetry in the volatility curve. This pattern indicates that the market expects greater upside potential (price increases) than downside risk (price decreases), often driven by bullish sentiment, anticipated positive events (e.g., earnings reports, supply constraints), or speculative demand. In such scenarios, OTM calls become more expensive due to higher implied volatility, while OTM puts are relatively cheaper. Traders may use strategies like call spreads or risk reversals to capitalize on this skew, while investors might hedge against potential price increases.

Figure 5: Volatility Surface of TYH5 Comdty

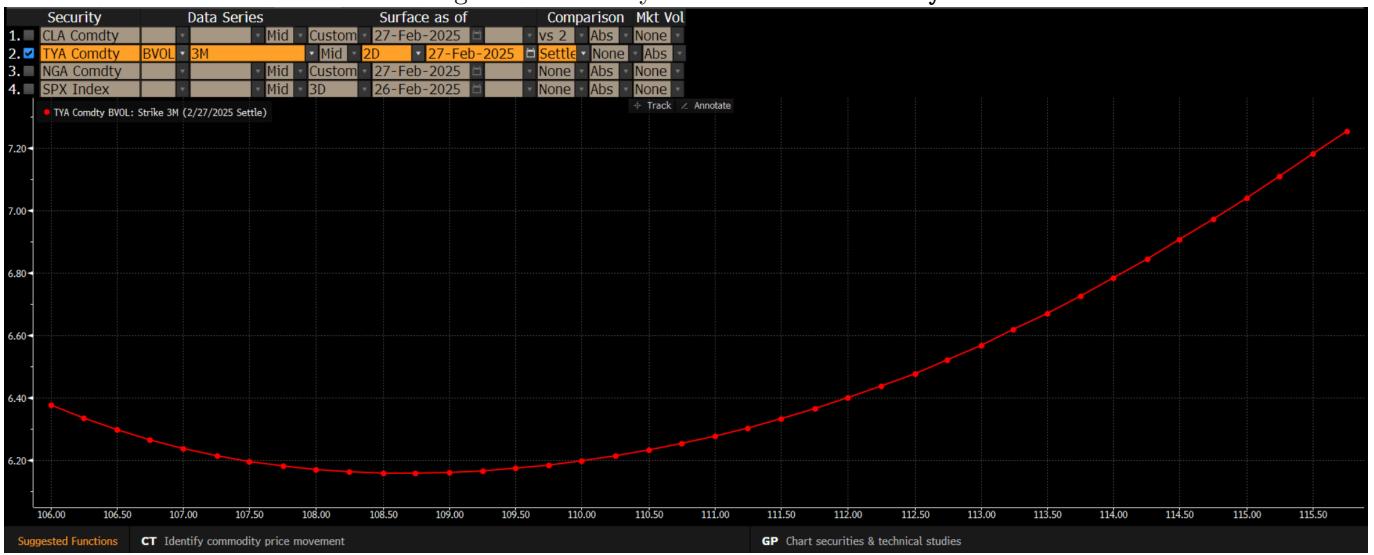
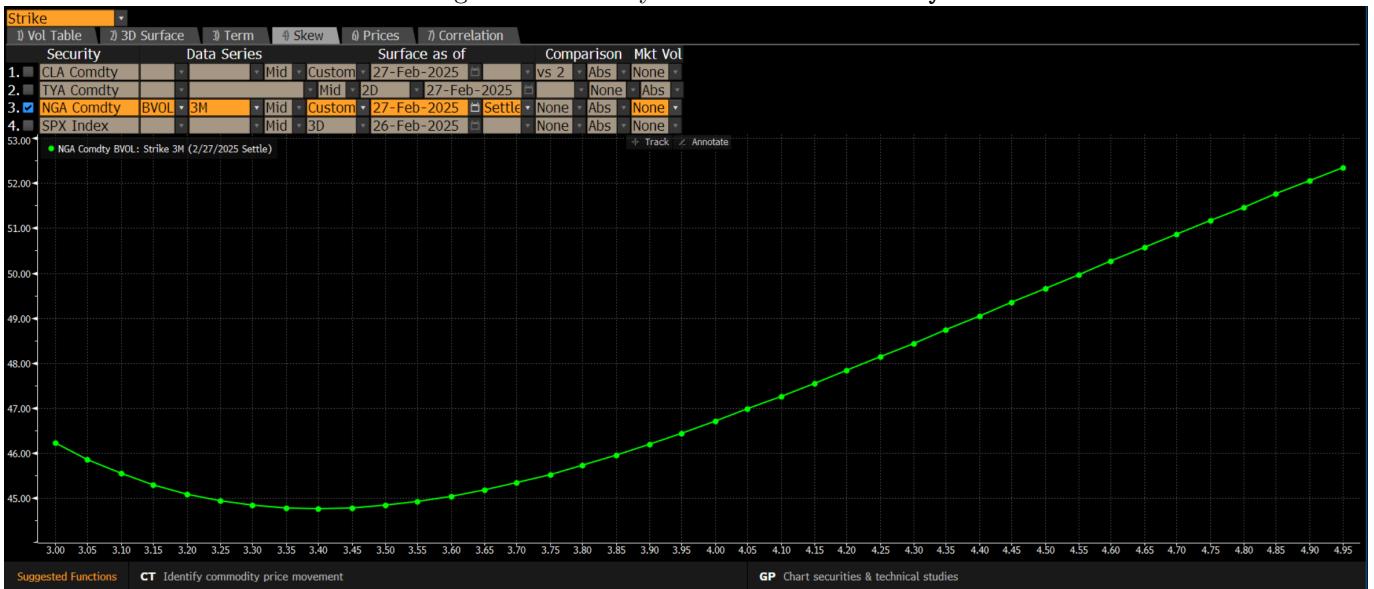


Figure 6: Volatility Surface of NGJ5 Comdty



Here we get a downward sloping graph of SPY Equity and ESH5 Index; typically a downward-sloping volatility surface in the 3-month range indicates that implied volatility decreases as strike prices increase, reflecting a negative volatility skew. This pattern often signals bearish market sentiment, with investors more concerned about downside risks (e.g., potential price drops) than upside potential. Higher implied volatilities for lower strike prices (out-of-the-money puts) suggest increased demand for downside protection, possibly due to market stress, upcoming events like earnings announcements, or historical patterns of sharp declines. For traders, this skew implies that put options are more expensive, while call options are cheaper, influencing strategies such as put spreads or risk reversals.

Figure 7: Volatility Surface of SPY Equity

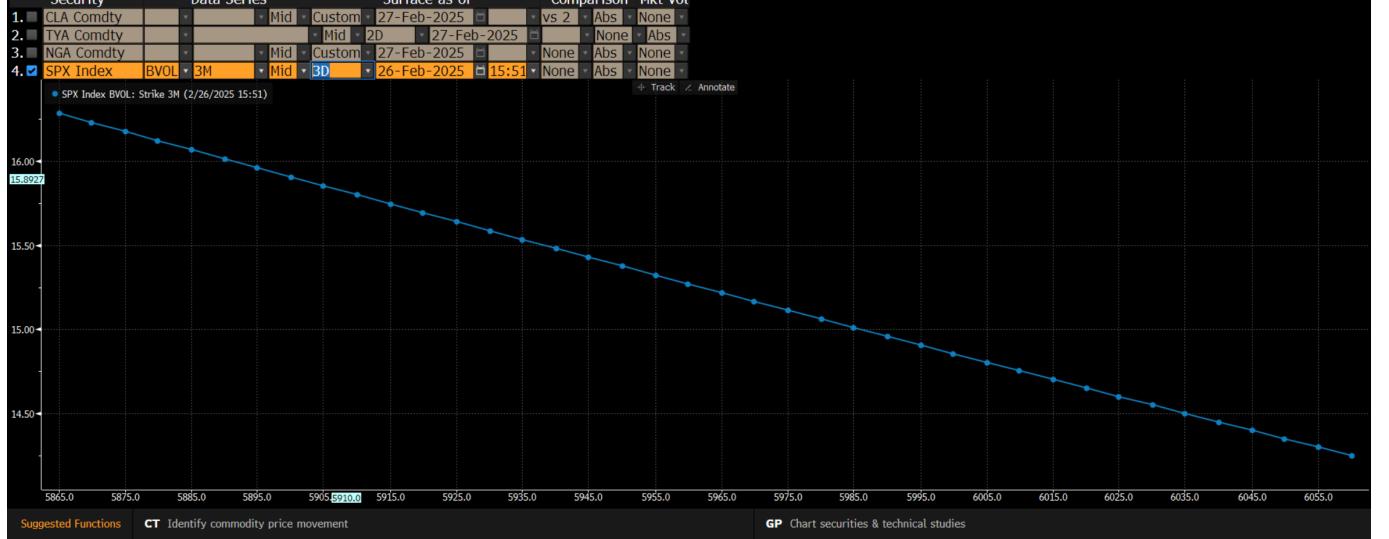
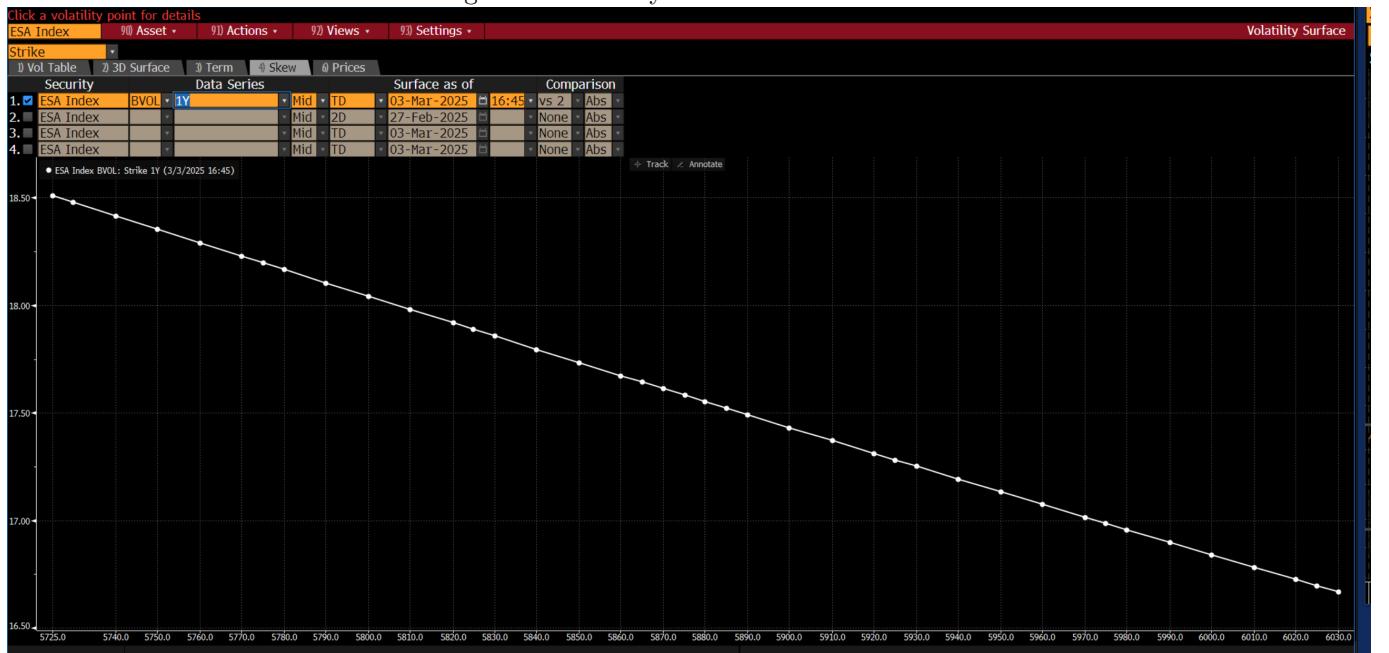


Figure 8: Volatility Surface of ESH5 Index



3 Question 3

Suppose the stock price is 70, the risk-free rate is 5% continuously compounded. What is the price of a 1 year call struck at 70 if the volatility is 0. How would you hedge the call. Check your answer with the option calculator making volatility smaller and smaller.

Answer

3.1 Solve it Using Fast Fourier Transform

Here I used the integration of Fast Fourier Transform (FFT) and Fractional Fast Fourier Transform (FrFFT) for this problem; there are 4 methods to be chosen, GBM, Heston, Variance Gamma, and Black-Scholes. Here I used Black-Scholes.

If $\sigma = 0$ and there is no dividend in this scenario, the stock price is deterministic and grows at rate r . In one year, it is thus worth $70 \cdot e^{0.05} \approx 73.5889$. The strike is $K = 70$. Our payoff is thus 3.588. Discounting at rate r , we get as today's fair option price $3.588 \cdot e^{-0.05} \approx 3.4139$. When I set $r = 0.05$, $q = 0$, $\sigma = 0$, the printout is:

```
Model is BlackScholes
```

```
-----
```

```
Option via Integration: for strike 70.0000 the option premium is 3.4155
```

```
FrFFT execution time was 0.0215478
```

The choice of damping factors, grid truncation error, and numerical integration error may all cause small differences when using Fast Fourier Transformation, but overall the result corresponds with the result we calculated from the Option Pricing Calculator.

My python codes are as follows:

```
1 import warnings
2 warnings.filterwarnings("ignore")
3
4
5 import numpy as np
6 import cmath
7 import math
8 import time
9
10 # Fixed Parameters
11 S0 = 70
12 K = 70
13 k = math.log(K)
14 r = 0.05
15 q = 0
16
17 # Parameters for FFT and FrFFT
18
19 n_FFT = 9
20 N_FFT = 2**n_FFT
21
22 n_FrFFT = 9
23 N_FrFFT = 2**n_FrFFT
24
25 N = 1000 # it doesn't impact the numerical results if the code is correct
26
27 #step-size
28 eta = 0.25
29 # damping factor
30 alpha = 0.4
31
32 # step-size in log strike space
33 lda_FFT = (2*math.pi/N_FFT)/eta # lda is fixed under FFT
34 lda_FrFFT = 0.001 # lda is an adjustable parameter under FrFFT,
```

```

36 #Choice of beta
37 beta = np.log(S0)-N*lida_FFT/2
38 #beta = np.log(S0)-N*lida_FrFFT/2
39 #beta = np.log(K)
40
41 #model-specific Parameters
42 model = 'BlackScholes'
43
44 params = []
45 if (model == 'GBM'):
46
47     sig = 0.3
48     params.append(sig);
49
50 elif model == 'BlackScholes':
51     sigma = 0 # Volatility
52     params.append(sigma)
53
54 elif (model == 'VG'):
55
56     sig = 0.3
57     nu = 0.5
58     theta = -0.4
59     #
60     params.append(sig);
61     params.append(nu);
62     params.append(theta);
63
64
65
66 elif (model == 'Heston'):
67
68     kappa = 2.0
69     theta = 0.05
70     sig = 0.30
71     rho = -0.70
72     v0 = 0.04
73     #
74     params.append(kappa)
75     params.append(theta)
76     params.append(sig)
77     params.append(rho)
78     params.append(v0)
79
80 def generic_CF(u, params, S0, r, q, T, model):
81
82 if (model == 'GBM'):
83
84     sig = params[0]
85     mu = np.log(S0) + (r-q-sig**2/2)*T
86     a = sig*np.sqrt(T)
87     phi = np.exp(1j*mu*u-(a*u)**2/2)
88
89 elif (model == 'BlackScholes'):
90     sigma = params[0] # Volatility
91
92     mu = np.log(S0) + (r - q - 0.5 * sigma**2) * T # Drift
93     a = sigma * np.sqrt(T) # Standard deviation over the maturity period
94     phi = np.exp(1j * mu * u - 0.5 * a**2 * u**2) # Characteristic function for Black-
95     Scholes
96
97

```

```

98 elif(model == 'Heston'):
99
100    kappa = params[0]
101    theta = params[1]
102    sigma = params[2]
103    rho = params[3]
104    v0 = params[4]
105
106    tmp = (kappa-1j*rho*sigma*u)
107    g = np.sqrt((sigma**2)*(u**2+1j*u)+tmp**2)
108
109    pow1 = 2*kappa*theta/(sigma**2)
110
111    numer1 = (kappa*theta*T*tmp)/(sigma**2) + 1j*u*T*r + 1j*u*math.log(S0)
112    log_denum1 = pow1 * np.log(np.cosh(g*T/2)+(tmp/g)*np.sinh(g*T/2))
113    tmp2 = ((u*u+1j*u)*v0)/(g/np.tanh(g*T/2)+tmp)
114    log_phi = numer1 - log_denum1 - tmp2
115    phi = np.exp(log_phi)
116
117    #g = np.sqrt((kappa-1j*rho*sigma*u)**2+(u*u+1j*u)*sigma*sigma)
118    #beta = kappa-rho*sigma*1j*u
119    #tmp = g*T/2
120
121    #temp1 = 1j*(np.log(S0)+(r-q)*T)*u + kappa*theta*T*beta/(sigma*sigma)
122    #temp2 = -(u*u+1j*u)*v0/(g/np.tanh(tmp)+beta)
123    #temp3 = (2*kappa*theta/(sigma*sigma))*np.log(np.cosh(tmp)+(beta/g)*np.sinh(tmp))
124
125    #phi = np.exp(temp1+temp2-temp3);
126
127
128 elif (model == 'VG'):
129
130    sigma = params[0];
131    nu = params[1];
132    theta = params[2];
133
134    if (nu == 0):
135        mu = np.log(S0) + (r-q - theta - 0.5*sigma**2)*T
136        phi = np.exp(1j*u*mu) * np.exp((1j*theta*u-0.5*sigma**2*u**2)*T)
137    else:
138        mu = np.log(S0) + (r-q + np.log(1-theta*nu-0.5*sigma**2*nu)/nu)*T
139        phi = np.exp(1j*u*mu)*((1-1j*nu*theta*u+0.5*nu*sigma**2*u**2)**(-T/nu))
140
141    return phi
142    def evaluateIntegral(params, S0, K, r, q, T, alpha, eta, N, model):
143
144        # Just one strike at a time
145        # no need for Fast Fourier Transform
146
147        # discount factor
148        df = math.exp(-r*T)
149
150        sum1 = 0
151        for j in range(N):
152            nuJ = j*eta
153            psi_nuJ = df*generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*
154                nuJ)*(alpha+1+1j*nuJ))
155            if j == 0:
156                wJ = (eta/2)
157            else:
158                wJ = eta
159            sum1 += np.exp(-1j*nuJ*k)*psi_nuJ*wJ

```

```

160 cT_k = (np.exp(-alpha*k)/math.pi)*sum1
161
162     return np.real(cT_k)
163
164 def genericFFT(params, S0, K, r, q, T, alpha, eta, n, model):
165
166 N = 2**n
167
168 # step-size in log strike space
169 lda = (2*np.pi/N)/eta
170
171 #Choice of beta
172 #beta = np.log(S0)-N*lda/2
173 #beta = np.log(K)
174
175 # forming vector x and strikes km for m=1,...,N
176 km = np.zeros((N))
177 xX = np.zeros((N))
178
179 # discount factor
180 df = math.exp(-r*T)
181
182 nuJ = np.arange(N)*eta
183 psi_nuJ = generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*nuJ)*(alpha+1+1j*nuJ))
184
185 for j in range(N):
186     km[j] = beta+j*lda
187     if j == 0:
188         wJ = (eta/2)
189     else:
190         wJ = eta
191
192 xX[j] = np.exp(-1j*beta*nuJ[j])*df*psi_nuJ[j]*wJ
193
194 yY = np.fft.fft(xX)
195 cT_km = np.zeros((N))
196 for i in range(N):
197     multiplier = np.exp(-alpha*km[i])/math.pi
198     cT_km[i] = multiplier*np.real(yY[i])
199
200
201 return km, cT_km
202
203 def genericFrFFT(params, S0, K, r, q, T, alpha, eta, n, lda, model):
204
205 N = 2**n
206 gamma = eta*lda/(2*math.pi)
207
208 #Choice of beta
209 #beta = np.log(S0)-N*lda/2
210 beta = np.log(K)
211
212 # initialize x, y, z, and cT_km
213 km = np.zeros((N))
214 x = np.zeros((N))
215 y = np.zeros((2*N), dtype=np.complex128)
216 z = np.zeros((2*N), dtype=np.complex128)
217 cT_km = np.zeros((N))
218
219 # discount factor
220 df = math.exp(-r*T)
221
222 # compute x

```

```

222 nuJ = np.arange(N)*eta
223 psi_nuJ = generic_CF(nuJ-(alpha+1)*1j, params, S0, r, q, T, model)/((alpha + 1j*nuJ)
224   *(alpha+1+1j*nuJ))
225
226 for j in range(N):
227     km[j] = beta+j*lida
228     if j == 0:
229         wJ = (eta/2)
230     else:
231         wJ = eta
232     x[j] = np.exp(-1j*beta*nuJ[j])*df*psi_nuJ[j]*wJ
233
234 # set up y
235 for i in range(N):
236     y[i] = np.exp(-1j*math.pi*gamma*i**2)*x[i]
237     y[N:] = 0
238
239 # set up z
240 for i in range(N):
241     z[i] = np.exp(1j*math.pi*gamma*i**2)
242     z[N:] = z[:N][::-1]
243
244 # compute xi_hat
245 xi_hat = np.fft.ifft(np.fft.fft(y) * np.fft.fft(z))
246
247 # compute call prices
248 for i in range(N):
249     cT_km[i] = np.exp(-alpha*(beta + i*lida))/math.pi * (np.exp(-1j*math.pi*gamma*i**2)*
250       xi_hat[i]).real
251
252 return km, cT_km
253
254 print('')
255 print('=====')
256 print('Model is %s' % model)
257 print('-----')
258
259 T = 1
260
261 # FFT
262 print('')
263 start_time = time.time()
264 km, cT_km = genericFFT(params, S0, K, r, q, T, alpha, eta, n_FFT, model)
265 #cT_k = cT_km[0]
266 cT_k = np.interp(k, km, cT_km)
267
268 elapsed_time = time.time() - start_time
269
270 #cT_k = np.interp(np.log(k), km, cT_km)
271 print("Option via FFT: for strike %s the option premium is %6.4f" % (np.exp(k), cT_k))
272
273 #print("Option via FFT: for strike %s the option premium is %6.4f" % (np.exp(k),
274 #      cT_km[0]))
275 print('FFT execution time was %0.7f' % elapsed_time)
276
277 # FrFFT
278 print('')
279 start_time = time.time()
280 km, cT_km = genericFrFFT(params, S0, K, r, q, T, alpha, eta, n_FrFFT, lda_FrFFT,
281   model)
282 #cT_k = cT_km[0]
283 cT_k = np.interp(k, km, cT_km)

```

```

280 elapsed_time = time.time() - start_time
281
282 #cT_k = np.interp(np.log(), km, cT_km)
283 print("Option via FrFFT: for strike %s the option premium is %6.4f" % (np.exp(k),
284     cT_k))
285 #print("Option via FFT: for strike %s the option premium is %6.4f" % (np.exp(k),
286     cT_km[0]))
287 print('FrFFT execution time was %0.7f' % elapsed_time)
288
289 # Integral
290 print(' ')
291 start_time = time.time()
292 cT_k = evaluateIntegral(params, S0, K, r, q, T, alpha, eta, N, model)
293 elapsed_time = time.time() - start_time
294 print("Option via Integration: for strike %s the option premium is %6.4f" % (np.exp(k),
295     cT_k))
296 print('Evaluation of integral time was %0.7f' % elapsed_time)

```

3.2 Solve it Using Option Calculator

Then I double checked it from the option price calculator:

Figure 9: Results from Option Calculator

Option		Stock		Market	
Strike	70	Price	70	Interest Rate	5%
Expiration (years)	1	Volatility	0	Dividend	0%
		Settings			
		Precision			
European Call	0.00000	European Put	0.00000	Forward	3.41394
Price	0.00000	Delta	-0.50000	Binary Call	0.50000
Gamma	0.50000	Vega	1.00000	Binary Put	-0.50000
Rho	0.00000	Rho	0.00000	Theta	0.00000
Theta	-0.50000	Theta	-0.50000	Rho	0.00000

Option		Stock		Market	
Strike	70	Price	70	Interest Rate	5%
Expiration (years)	1	Volatility	0	Dividend	3%
		Settings			
		Precision			
European Call	0.00000	European Put	0.00000	Forward	1.34513
Price	0.00000	Delta	-0.50000	Binary Call	0.50000
Gamma	0.50000	Vega	0.97045	Binary Put	-0.50000
Rho	0.00000	Rho	0.00000	Theta	0.00000
Theta	-0.00000	Theta	-0.00000	Rho	-0.00000

4 Question 4

Explain why an American option on a stock paying continuous dividend yield is always worth as much as its intrinsic value. Give a numerical example of a situation when European option is worth less than intrinsic value. (Give the numerical value of stock price, strike price, time to expiration, etc.)

Answer

4.1 Concept

Intrinsic value of a call option is the difference between the current price of the underlying asset and the strike price of the option, if the underlying asset's price is above the strike price. If the underlying asset's price is below the strike price, the call option has no intrinsic value. Thus it can be expressed as:

$$\max(S - K, 0)$$

Here we have S to be the current stock price, and K to be the strike price

If the stock pays a continuous dividend yield, the stock price is expected to decrease as dividends are paid. This increases the incentive to exercise the call option early to avoid losing value from future dividends. Since the American option can be exercised early, the holder will exercise it if the intrinsic value exceeds the option's continuation value (the value of holding the option). Therefore, the American option is always worth at least its intrinsic value.

4.2 Numerical Example

For a the numerical example, we can set:

- Stock price: $S = 100$
- Strike price: $K = 90$
- Time to expiration: 1 year
- Risk-free interest rate: 5%
- Continuous dividend yield: 3%
- Volatility: 20%

Using the Black-Scholes formula, the European call option price is approximately:

$$C_{\text{European}} = 12.03$$

Calculation of C_{European}

We have the Black-Scholes Formula:

$$C = Se^{-qt}N(d_1) - Ke^{-rt}N(d_2)$$

Where d_1 and d_2 can be calculated as:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Given what we previously set: $S = 100$, $K = 90$, $r = 0.05$, $q = 0.03$, $\sigma = 0.20$, $t = 1$:

$$d_1 = \frac{\ln(100/90) + (0.05 - 0.03 + 0.5(0.2^2))(1)}{0.2\sqrt{1}} = \frac{\ln(1.1111) + (0.02 + 0.04)}{0.2} = \frac{0.1054 + 0.04}{0.2} = \frac{0.1454}{0.2} = 0.727$$

$$d_2 = 0.727 - 0.2 = 0.527$$

Then we can find $N(d_1)$ and $N(d_2)$ using standard normal distribution tables:

$$N(0.727) \approx 0.7673$$

$$N(0.527) \approx 0.7014$$

Finally we plug them back into the Black-Scholes Formula:

$$C = 100e^{-0.03(1)}(0.7673) - 90e^{-0.05(1)}(0.7014) = 100(0.9704)(0.7673) - 90(0.9512)(0.7014)$$

$$C = 74.43 - 60.91 \approx 12.03$$

The intrinsic value is:

$$S - K = 100 - 90 = 10$$

Here, the European option is worth 12.03, which is greater than the intrinsic value because there is still time value. Then we modify the example:

- Stock price: $S = 100$
- Strike price: $K = 110$ (out of the money)
- Time to expiration: 1 month
- Dividend yield: 3%
- Volatility: 10%

Using Black-Scholes and similar calculation as previous step:

$$C_{\text{European}} \approx 0.25$$

But the intrinsic value is:

$$\max(100 - 110, 0) = 0$$

In this case, the European option is worth less than its intrinsic value, because there is no early exercise opportunity, and the option is out of the money.

5 Question 5

Explain the European call-put parity argument. Why it can not be used for American options

Answer

5.1 Concept

The Put-Call Parity is a fundamental relationship between the prices of European call and put options on the same underlying asset, with the same strike price and expiration date. It defines a relationship between the price of a European call option and European put option, both with the identical strike price and expiry, namely that a portfolio of a long call option and a short put option is equivalent to (and hence has the same value as) a single forward contract at this strike price and expiry

The formula is:

$$C - P = Se^{-qt} - Ke^{-rt}$$

- C : Price of the European call option
- P : Price of the European put option
- S : Current stock price
- K : Strike price
- r : Risk-free interest rate
- q : Continuous dividend yield
- t : Time to expiration
- e^{-qt} : Discounted stock price due to dividend payments
- e^{-rt} : Discounted strike price due to risk-free rate

The parity works well for European options because they can only be exercised at expiration. The key idea behind this formula is that a portfolio consisting of a long European call option and a short European put option is equivalent to holding the underlying stock while borrowing the present value of the strike price.

Both portfolios will yield the same payoff at expiration, which is why they must have the same value before expiration (by the no-arbitrage principle).

However, the Put-Call Parity does not directly apply to American options because:

- American options can be exercised at any time before expiration, which adds an additional layer of complexity. This early exercise possibility makes the payoff structure of American options different from European options.
- For American call options on dividend-paying stocks, early exercise is often optimal just before the ex-dividend date (to avoid losing dividends). This means the American call option might be more valuable than the European call option.
- Unlike European options, American calls and puts do not have a fixed parity relationship because the option holder's decision to exercise early can affect the option's price.
- The strict no-arbitrage condition that holds for European options does not necessarily apply to American options due to the early exercise feature. This creates pricing discrepancies that do not align with put-call parity.

5.2 Numerical Example

A Numerical Example

Suppose we take stock price $S = 100$, strike price $K = 95$, time to expiration as 6 months, risk-free rate as 5%, dividend yield as 2%, European call price 8 and European put price to be 3.

$$8 - 3 = 100e^{-0.02(0.5)} - 95e^{-0.05(0.5)}$$

$$5 = 100(0.9900) - 95(0.9753)$$

$$5 = 99 - 92.65$$

$$5 = 6.35$$

There is a small difference due to rounding, but the parity approximately holds.

6 Question 6

Calculate the implied volatility of Microsoft stock using March 2025 calls expiring March 21, 2025 with strike 410 and with strike 420. Get the quotes from any data provider, for example, finance.yahoo.com and use all other necessary data. On Bloomberg type MSFT Equity CALL (page down as needed).

Answer

6.1 Solve it Using Python

I used python to calculate the implied volatility. Currently the risk-free rate is 4.24% as I got from the source. And the current market price for strike 410 is \$3.04 and 420 \$1.21 separately. Then use the Black-Scholes model to recalculate the implied volatility:

```
1 import numpy as np
2 from scipy.stats import norm
3
4 # Black-Scholes call option pricing formula
5 def black_scholes_call(S, K, T, r, sigma, q=0):
6     d1 = (np.log(S / K) + (r - q + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
7     d2 = d1 - sigma * np.sqrt(T)
8     return np.exp(-q * T) * S * norm.cdf(d1) - K * np.exp(-r * T) * norm.cdf(d2)
9
10 # Implied Volatility Function using Newton-Raphson Method
11 def implied_volatility(S, K, T, r, market_price, q=0, tol=1e-6, max_iter=100):
12     sigma = 0.2 # Initial Guess
13     for i in range(max_iter):
14         price = black_scholes_call(S, K, T, r, sigma, q)
15         vega = S * np.exp(-q * T) * norm.pdf((np.log(S / K) + (r - q + 0.5 * sigma ** 2) * T)
16             / (sigma * np.sqrt(T))) * np.sqrt(T)
17         price_diff = price - market_price
18         if abs(price_diff) < tol:
19             return sigma
20         sigma -= price_diff / vega
21     raise ValueError("Implied volatility did not converge")
22
23 # Example Inputs
24 S = 394.9      # Current stock price (MSFT)
25 K = 420        # Strike price, 410 and 420
26 T = 18 / 365   # Time to expiration in years, March 21-3
27 r = 0.0424     # Risk-free interest rate, get from website
28 q = 0.0085     # Continuous dividend yield
```

```

28 market_price = 1.21 # Call option market price
29
30 try:
31 iv = implied_volatility(S, K, T, r, market_price, q)
32 print(f"Implied Volatility: {iv:.4f}")
33 except ValueError as e:
34     print(e)

```

And the printout for strike price 410 is:

Implied Volatility: 0.2335

And the printout for strike price 420 is:

Implied Volatility: 0.2289

6.2 Solve it Using Calculator

And I also tested it from the calculator:

Figure 10: Results using Calculator

The image shows two identical-looking calculators side-by-side. Both calculators have the following fields and values:

- Option Type:** Call Option
- Underlying Price:** 394.9
- Exercise Price:** 410 (left) or 420 (right)
- Days Until Expiration:** 18
- Interest Rate:** 4.24 %
- Dividend Yield:** 0.85 %
- Market Price:** 3.02 (left) or 1.21 (right)
- Implied Volatility:** 23.35% (left) or 22.89% (right)
- Calculate:** A blue button at the bottom.

This can be reconfirmed with the data from Yahoo Finance:

Figure 11: Strike Price 410, Expire Date 2025-03-21

NasdaqGS - Nasdaq Real Time Price • USD

Microsoft Corporation (MSFT) [Follow](#) [Compare](#) [Time to buy MSFT?](#)

394.90 -2.09 (-0.53%)

As of 12:37:32 PM EST. Market Open.

Mar 21, 2025 ▾ 410.00 ▾ List ▾ All Options ▾

Calls In The Money

Contract Name	Last Trade Date (EST)	Expire Date	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
MSFT250307C00410000	3/3/2025 12:06 PM	2025-03-07	0.47	0.47	0.48	-0.40	-45.98%	4,358	6,556	23.98%
MSFT250314C00410000	3/3/2025 12:05 PM	2025-03-14	1.77	1.75	1.79	-1.23	-41.00%	927	2,513	23.47%
MSFT250321C00410000	3/3/2025 12:01 PM	2025-03-21	3.02	2.98	3.05	-0.78	-20.53%	590	7,464	23.33%
MSFT250328C00410000	3/3/2025 11:54 AM	2025-03-28	4.17	4.05	4.15	-1.04	-19.96%	243	860	23.12%
MSFT250404C00410000	3/3/2025 11:54 AM	2025-04-04	5.32	5.15	5.30	-1.20	-18.40%	91	423	23.32%

Figure 12: Strike Price 420, Expire Date 2025-03-21

NasdaqGS - Nasdaq Real Time Price • USD

Microsoft Corporation (MSFT) [Follow](#) [Compare](#) [Time to buy MSFT?](#)

394.90 -2.09 (-0.53%)

As of 12:37:32 PM EST. Market Open.

Mar 21, 2025 ▾ 420.00 ▾ List ▾ All Options ▾

Calls In The Money

Contract Name	Last Trade Date (EST)	Expire Date	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
MSFT250307C00420000	3/3/2025 12:00 PM	2025-03-07	0.11	0.10	0.11	-0.08	-44.44%	2,693	9,199	26.17%
MSFT250314C00420000	3/3/2025 12:03 PM	2025-03-14	0.58	0.54	0.56	-0.30	-34.09%	458	2,513	23.15%
MSFT250321C00420000	3/3/2025 12:04 PM	2025-03-21	1.21	1.18	1.22	-0.39	-24.38%	1,305	23,553	22.57%
MSFT250328C00420000	3/3/2025 12:00 PM	2025-03-28	1.94	1.86	1.94	-0.52	-21.14%	286	2,729	22.33%
MSFT250404C00420000	3/3/2025 12:01 PM	2025-04-04	2.73	2.67	2.74	-0.80	-22.66%	79	547	22.42%

7 Question 7

Download Excel options model with VBA code from the courseworks . Save as a source. Open in Excel. Modify the Black model for futures to Black-Scholes model for stocks paying dividends at rate q (like in the Hull's book). Make the necessary changes in Visual Basic code. Use Excel help or consult TA's if you do not know what to do. Check that code works and submit the code printout.

Answer

Figure 13: The Printout of the B-S Model Re-implemented in Excel

H6	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
	X	✓	fx	=@optionPrice("C",A6,B6,C6,E6,D6,F6)											
1															
2															
3	Black-Scholes Model														
4															
5	Future Price	Strike Price	Time To Expiry	Volatility	Risk Free Rate	Dividend	Call/Put	Option Price	Delta	Gamma	1 day Theta	1% Vega	1% Rho		(1y=365d)
6	125.000000	242.000000	4.000000	0.360000	0.070000	0.034000	c	13.5961603	0.36034629	0.00415824	-0.0060233	0.93560412	1.02878163		
7	125.000000	242.000000	4.000000	0.360000	0.070000	0.034000	p	87.3904967	-0.63965371	0.00415824	-0.04167898	0.93560412	-6.28720499		
8															
9															

```

1 Function OptionPrice(OptionType As String, S As Double, X As Double, T As Double, r
2   As Double, v As Double, d As Double) As Double
3   dOne = (Log(S / X) + (r - d + 0.5 * v ^ 2) * T) / (v * (Sqr(T)))
4   NdOne = Exp(-(dOne(S, X, T, r, v, d) ^ 2) / 2) / (Sqr(2 *
5   dTwo = dOne(S, X, T, r, v, d) - v * Sqr(T)
6   NdTwo = Application.NormSDist(dTwo(S, X, T, r, v, d))
7   If OptionType = "C" Then
8     OptionPrice = Exp(-d * T) * S * Application.NormSDist(dOne(S, X, T, r, v, d)) - X *
9       Exp(-r * T) * Application.NormSDist(dOne(S, X, T, r, v, d) - v * Sqr(T))
10  ElseIf OptionType = "P" Then
11    OptionPrice = X * Exp(-r * T) * Application.NormSDist(-dTwo(S, X, T, r, v, d)) - Exp
12      (-d * T) * S * Application.NormSDist(-dOne(S, X, T, r, v, d))
13  End If
14  End Function
15
16 Function OptionDelta(OptionType As String, S As Double, X As Double, T As Double, r
17   As Double, v As Double, d As Double) As Double
18  If OptionType = "C" Then
19    OptionDelta = Application.NormSDist(dOne(S, X, T, r, v, d))
20  ElseIf OptionType = "P" Then
21    OptionDelta = Application.NormSDist(dOne(S, X, T, r, v, d)) - 1
22  End If
23  End Function
24
25 Function OptionTheta(OptionType As String, S As Double, X As Double, T As Double, r
26   As Double, v As Double, d As Double) As Double
27  dOne = (Log(S / X) + (r - d + 0.5 * v ^ 2) * T) / (v * (Sqr(T)))
28  NdOne = Exp(-(dOne(S, X, T, r, v, d) ^ 2) / 2) / (Sqr(2 *
29  dTwo = dOne(S, X, T, r, v, d) - v * Sqr(T)
30  NdTwo = Application.NormSDist(dTwo(S, X, T, r, v, d))
31  If OptionType = "C" Then
32    OptionTheta = -((S * v * NdOne(S, X, T, r, v, d)) / (2 * Sqr(T)) - r * X * Exp(-r * (
33      T)) * NdTwo(S, X, T, r, v, d)) / 365
34  ElseIf OptionType = "P" Then
35    OptionTheta = -((S * v * NdOne(S, X, T, r, v, d)) / (2 * Sqr(T)) + r * X * Exp(-r * (
36      T)) * (1 - NdTwo(S, X, T, r, v, d))) / 365
37  End If
38  End Function
39
40 Function Gamma(S As Double, X As Double, T As Double, r As Double, v As Double, d As
41   Double) As Double

```

```

34 Gamma = NdOne(S, X, T, r, v, d) / (S * (v * Sqr(T)))
35 End Function
36
37 Function Vega(S As Double, X As Double, T As Double, r As Double, v As Double, d As
38     Double) As Double
39 Vega = 0.01 * S * Sqr(T) * NdOne(S, X, T, r, v, d)
40 End Function
41
42 Function OptionRho(OptionType As String, S As Double, X As Double, T As Double, r As
43     Double, v As Double, d As Double) As Double
44 If OptionType = "C" Then
45 OptionRho = 0.01 * X * T * Exp(-r * T) * Application.NormSDist(dTwo(S, X, T, r, v, d))
46 ElseIf OptionType = "P" Then
47 OptionRho = -0.01 * X * T * Exp(-r * T) * (1 - Application.NormSDist(dTwo(S, X, T, r,
48     v, d)))
49 End If
50 End Function

```

8 Question 8

Download Excel Brownian Motion model from the courseworks. Save as a source. Open in Excel. Modify it to Geometric Brownian motion with growth rate $\mu = 0.04$, volatility $\sigma = 0.20$ and 250 trajectories. Submit excel formulas printout. Geometric Brownian motion starts with positive X_0 so you must change a starting value from 0 to a positive number that you may choose (you can choose 1, 100 or other positive number).

Answer

8.1 Solve it Using Excel

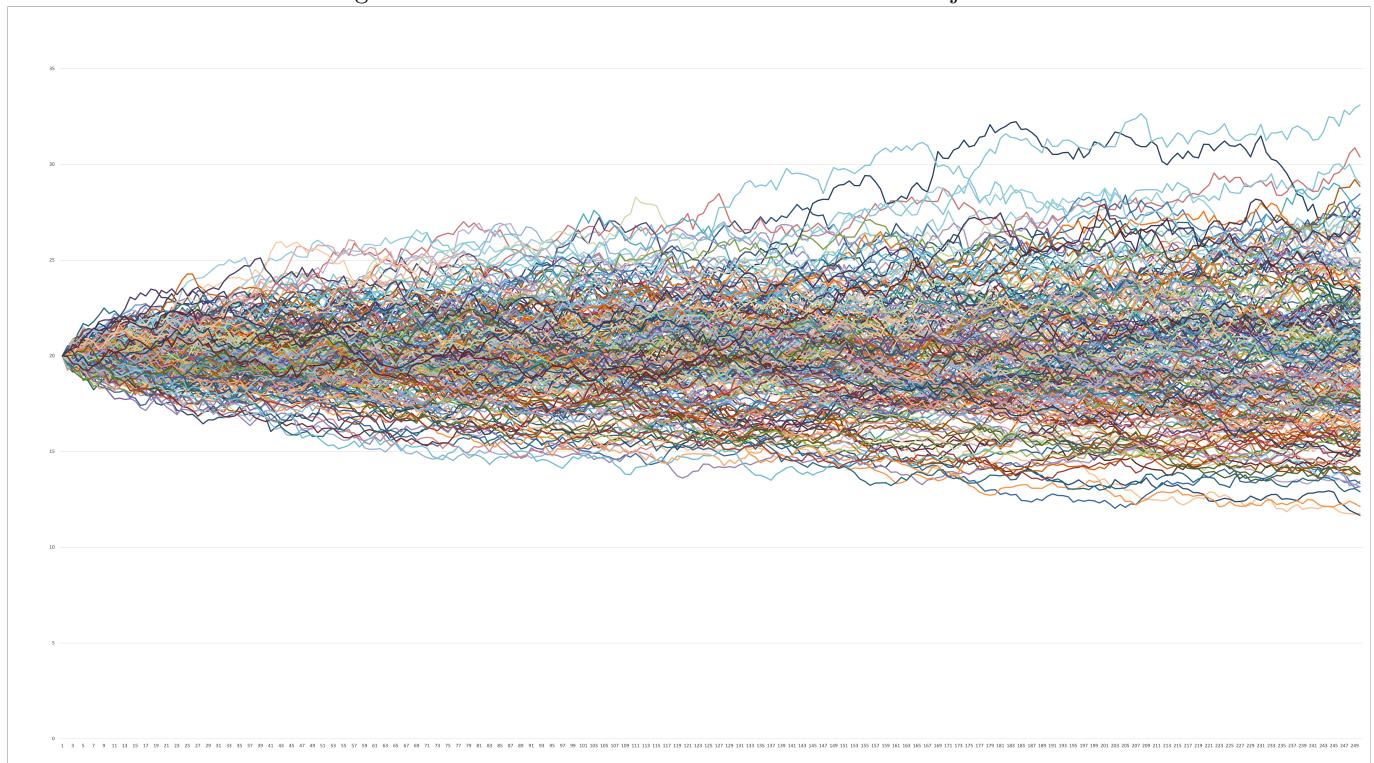
The number for X_0 I chose was 20, I firstly put several parameters in the A column (for time step, it is 250).

Figure 14: Settings in my Excel

	A	B	C	D	E	F	G
1	Growth Rate	Time		1	2	3	4
2		0.04	Trac 1	20	19.82077317	19.54878179	19.43056657
3	Volatility		Trac 2	20	19.65231767	19.9391074	20.203435
4		0.2	Trac 3	20	20.14019724	19.95438317	19.9537862
5	Initial Value		Trac 4	20	19.92875626	19.8819482	19.5855312
6		20	Trac 5	20	19.88932391	20.11611406	19.41433972
7	Num of Trajectories		Trac 6	20	20.38122799	20.34695327	20.71934936
8		250	Trac 7	20	20.08282482	19.96230487	19.99068615
9	Time Steps		Trac 8	20	20.11316508	20.37823955	20.01866757
10		250	Trac 9	20	19.87982115	19.70980455	19.64561904
11			Trac 10	20	20.14935925	20.04433286	19.50385732
12			Trac 11	20	20.10647784	20.30952821	20.26797587
13			Trac 12	20	20.11397681	20.43318372	20.64539772
14			Trac 13	20	20.0551359	20.15186673	19.8341527
15			Trac 14	20	19.653336	19.69143688	19.35136806
16			Trac 15	20	19.8889085	19.64843999	19.58420851
17			Trac 16	20	19.85366181	19.6011544	19.48604228
18			Trac 17	20	20.07770051	20.30712651	20.54204803
19			Trac 18	20	19.87541073	19.77064086	19.52381052
20			Trac 19	20	19.86821155	19.95924734	20.28252577
21			Trac 20	20	20.1672283	20.1779516	20.54243567
22			Trac 21	20	20.28688007	20.14439704	20.73582509

In D2, I set the formula to be $=C2*EXP((\$A\$2-0.5*\$A\$4^2)*(1/250)+\$A\$4*SQRT(1/250)*NORMSINV(RAND()))$. The final printout is:

Figure 15: Geometric Brownian Motion with 250 Trajectories



8.2 Solve it Using R

Out of interest, I redid it in R:

```
1 X0 <- 20
2 mu <- 0.04
3 sigma <- 0.20
4 n <- 250
5 T <- 1
6 N <- 250
7 dt <- T / N

8
9 # Generate time vector
10 time <- seq(0, T, by = dt)

11
12 # Initialize matrix to store trajectories
13 X <- matrix(NA, nrow = N + 1, ncol = n)
14 X[1, ] <- X0

15
16 # Simulate GBM
17 set.seed(123) # For reproducibility
18 for (i in 1:n) {
19   for (j in 2:(N + 1)) {
20     dW <- rnorm(1, mean = 0, sd = sqrt(dt))
21     X[j, i] <- X[j - 1, i] * exp((mu - 0.5 * sigma^2) * dt + sigma * dW)
22   }
23 }

24
25 # Convert to data frame for plotting
26 X_df <- as.data.frame(X)
27 X_df$time <- time
28 X_long <- melt(X_df, id.vars = "time", variable.name = "trajectory", value.name = "price")

29
30 # Plot the trajectories
31 ggplot(X_long, aes(x = time, y = price, group = trajectory)) +
32   geom_line(alpha = 0.5) +
33   labs(
34     x = "Time",
35     y = "Price") +
36   theme_minimal()
```

One interesting thing I noticed is that the set of trajectories tend to move upward, that's because there exists a drift term μ , which is typically positive in financial implications (and also in this case). Thus we can notice as time goes on, the trajectories get out of 30 in both plots.

Figure 16: Geometric Brownian Motion with 250 Trajectories Generated in R

