General properties of Value at Risk

0.1 Ordering properties

Pflug (2000) and Jadhav and Ramanathan (2009) established several ordering properties of VaR_p. Some of them are:

(i) VaR_p is translation equivariant, that is $\mathrm{VaR}_p(Y + c) = \mathrm{VaR}_p(Y) + c$;

(ii) VaR_p is positively homogeneous, that is $\mathrm{VaR}_p(cY) = c\mathrm{VaR}_p(Y)$ for $c > 0$;

(iii) $\mathrm{VaR}_p(Y) = \mathrm{VaR}_{1-p}(-Y)$;

(iv) VaR_p is monotonic with respect to stochastic dominance of order 1 (a random variable Y_1 is less than a random variable Y_2 with respect to stochastic dominance of order 1 if $E[\psi(Y_1)] \leq E[\psi(Y_2)]$ for all monotonic integrable functions ψ); that is, Y_1 is less than a random variable Y_2 with respect to stochastic dominance of order 1 then $\mathrm{VaR}_p(Y_1) \leq \mathrm{VaR}_p(Y_2)$;

(v) VaR_p is comonotone additive, that is if Y_1 and Y_2 are comonotone then $\mathrm{VaR}_p(Y_1 + Y_2) = \mathrm{VaR}_p(Y_1) + \mathrm{VaR}_p(Y_2)$. Two random variables Y_1 and Y_2 defined on the same probability space $(\Omega, \mathcal{A}, P)$ are said to be comonotone if for all $w, w^{'} \in \Omega$, $[Y_1(w) - Y_2(w)][Y_1(w^{'}) - Y_2(w^{'})] \geq 0$ almost surely;

(vi) if $X \geq 0$ then $\mathrm{VaR}_p(X) \leq 0$;

(vii) VaR_p is monotonic, that is if $X \geq Y$ then $\mathrm{VaR}_p(X) \leq \mathrm{VaR}_p(Y)$.

Let F denote the joint cdf of (X_1, X_2) with marginal cdfs F_1 and F_2. Write $F \equiv (F_1, F_2, C)$ to mean $F(X_1, X_2) \equiv C\left(F_1\left(X_1\right), F_2\left(X_2\right)\right)$ where C is known as the copula (Nelsen, 1999), a joint cdf of uniform marginals. Let (X_1, X_2) have the joint cdf $F \equiv (F_1, F_2, C)$, $\left(X_1^{'}, X_2^{'}\right)$ have the joint cdf $F^{'} \equiv \left(F_1, F_2, C^{'}\right)$, $X = wX_1 + (1-w)X_2$, and $X^{'} = wX_1 + (1-w)X_2$. Then Tsafack (2009) shows that if $C^{'}$ is stochastically less than C then $\mathrm{VaR}_p\left(X^{'}\right) \geq \mathrm{VaR}_p(X)$ for $p \in (0, 1)$.

0.2 Multivariate extension

Here, we shall concentrate on univariate VaR estimation. Multivariate VaR is a much more recent topic.

Let $\mathbf{X}$ be a random vector in $\mathbb{R}^r$ with joint cdf F. Prékopa (2012) gave the following definition of multivariate VaR:

$$\mathrm{MVaR}_p = \left\{\mathbf{u} \in \mathbb{R}^r : F(\mathbf{u}) = p\right\}. \tag{1}$$

Note that MVaR may not be a single vector. It will often take the form of a set of vectors.

Prékopa (2012) gave the following motivation for multivariate VaR: “A finance company generally faces the problem of constructing different portfolios that they can sell to customers. Each portfolio produces a random total return and it is the objective of the company to have them above given levels, simultaneously, with large probability. Equivalently, the losses should be below given levels, with large probability. In order to ensure it we look at the total losses as components of a random vector and find a multivariate p-quantile or MVaR to know what are those points in the r-dimensional space (r being the number of portfolios), that should surpass the vector of total losses, to guarantee the given reliability".

Cousin and Bernardinoy (2011) provide another definition of multivariate VaR:

$$\mathrm{MVaR}_p = E\left[\mathbf{X} \mid \mathbf{X} \in \partial L(p)\right] = \begin{pmatrix} E\left[X_1 \mid \mathbf{X} \in \partial L(p)\right] \\ E\left[X_2 \mid \mathbf{X} \in \partial L(p)\right] \\ \vdots \\ E\left[X_r \mid \mathbf{X} \in \partial L(p)\right] \end{pmatrix}$$

or equivalently

$$\mathrm{MVaR}_p = E\left[\mathbf{X} \mid F(\mathbf{X}) = p\right] = \begin{pmatrix} E\left[X_1 \mid F(\mathbf{X}) = p\right] \\ E\left[X_2 \mid F(\mathbf{X}) = p\right] \\ \vdots \\ E\left[X_r \mid F(\mathbf{X}) = p\right] \end{pmatrix},$$

where $\partial L(p)$ is the boundary of the set $\{\mathbf{x} \in \mathbb{R}_+^r : F(\mathbf{x}) \geq p\}$. Cousin and Bernardinoy (2011) establish various properties of MVaR similar to those in the univariate case. For instance,

(i) the translation equivariant property holds, that is

$$\mathrm{MVaR}_p\left(\mathbf{c} + \mathbf{X}\right) = \mathbf{c} + \mathrm{MVaR}_p\left(\mathbf{X}\right) = \begin{pmatrix} c_1 + E\left[X_1 \mid F(\mathbf{X}) = p\right] \\ c_2 + E\left[X_2 \mid F(\mathbf{X}) = p\right] \\ \vdots \\ c_r + E\left[X_r \mid F(\mathbf{X}) = p\right] \end{pmatrix};$$

(ii) the positively homogeneous property holds, that is

$$\mathrm{MVaR}_p\left(\mathbf{c}\mathbf{X}\right) = \mathbf{c}\mathrm{MVaR}_p\left(\mathbf{X}\right) = \begin{pmatrix} c_1 E\left[X_1 \mid F(\mathbf{X}) = p\right] \\ c_2 E\left[X_2 \mid F(\mathbf{X}) = p\right] \\ \vdots \\ c_r E\left[X_r \mid F(\mathbf{X}) = p\right] \end{pmatrix};$$

1. if F quasi-concave (Nelson, 1999) then

   $$\mathrm{MVaR}_p^i(\mathbf{X}) \geq \mathrm{VaR}_p\left(X_i\right)$$

   for $i = 1, 2, \ldots, r$, where $\mathrm{MVaR}_p^i(\mathbf{X})$ denotes the ith component of $\mathrm{MVaR}_p(\mathbf{X})$;

2. if $\mathbf{X}$ is a comonotone non-negative random vector then if F quasi-concave (Nelson, 1999) then

   $$\mathrm{MVaR}_p^i(\mathbf{X}) = \mathrm{VaR}_p\left(X_i\right)$$

   for $i = 1, 2, \ldots, r$;

3. if $X_i = Y_i$ in distribution for every $i = 1, 2, \ldots, s$ then

$$\mathrm{MVaR}_p(\mathbf{X}) = \mathrm{MVaR}_p(\mathbf{Y})$$

for all $p \in (0, 1)$;

4. if X_i is stochastically less than Y_i for every $i = 1, 2, \ldots, s$ then

$$\mathrm{MVaR}_p(\mathbf{X}) \leq \mathrm{MVaR}_p(\mathbf{Y})$$

for all $p \in (0, 1)$.

Bivariate value-at-risk in the context of a bivariate normal distribution has been considered much earlier by Arbia (2002).

A matric variate extension of VaR and its application for power supply networks are discussed in Chang (2011).

0.3 Aggregate risks

Let $M_n = \max(X_1, X_2, \ldots, X_n)$ denote the partial maximum. Limiting distributions of M_n under linear normalization are well known, see Resnick (1987), de Haan and Ferrieira (2006) and Falk *et al.* (2011). The notation $F \in D(G)$ means that there exist some suitable normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\Pr\left\{M_n \leq a_n x + b_n\right\} \to G(x)$$

as $n \to \infty$ for all continuity points x of G, where G is a nondegenerate cdf. It is well known that $G(x)$ must belong to one type of the following three classes of extreme value distributions:

$$\begin{array}{lll} \text{Type I} \quad \text{Gumbel:} & \Lambda(x) = \exp\left\{-\exp(-x)\right\}, & x \in \mathbb{R}; \\ \text{Type II} \quad \text{Fréchet:} & \Phi_\alpha(x) = \begin{cases} 0, & x < 0, \\ \exp\left\{-x^{-\alpha}\right\}, & x \geq 0 \end{cases} & \qquad (2) \\ & \text{for some} \quad \alpha > 0; & \qquad (3) \\ \text{Type III} \quad \text{Weibull:} & \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^\alpha\right\}, & x < 0, \\ 1, & x \geq 0 \end{cases} & \qquad (4) \\ & \text{for some} \quad \alpha > 0. & \end{array}$$

Moreover, the three classes of extreme value distributions just mentioned can be rewritten in the universal form:

$$G(x) = G_\kappa(x) = \exp\left\{-(1 + \kappa x)^{-1/\kappa}\right\}, \qquad 1 + \kappa x > 0 \tag{5}$$

for $\kappa \in \mathbb{R}$. For details on necessary and sufficient conditions for $F \in D(G)$ and the choices of normalizing constants a_n and b_n, see Resnick (1987) and de Haan and Ferrieira (2006).

Chen *et al.* study additive properties of VaR for aggregate risks. They assume that $\mathbf{X} = (X_1, \ldots, X_n)$ has an Archimedean copula with generator ψ, which is regularly varying at zero with index $-\alpha < 0$. They derive limiting behavior of VaR for aggregate risks under the three domains of attraction.

If $F \in D(\Phi_\beta)$ then it is shown that

$$\lim_{p \in 1^-} \frac{\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right]}{\mathrm{VaR}_p(X_1)} = \left[q_n^F(\alpha, \beta)\right]^{1/\beta},$$

where

$$q_n^F(\alpha, \beta) = \int_{\mathbb{R}_+^n} I\left\{\sum_{i=1}^n x_i^{-1} \geq 1\right\} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta}\right)^{-1/\alpha} dx_1 \cdots dx_n. \tag{6}$$

Furthermore, for all $\beta > 1$, there exists $p_0 > 0$ such that

$$\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right] < \sum_{i=1}^n \mathrm{VaR}_p(X_i)$$

for all $p_0 < p < 1$. For all $\beta < 1$, there exists $p_1 > 0$ such that

$$\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right] > \sum_{i=1}^n \mathrm{VaR}_p(X_i)$$

for all $p_1 < p < 1$.

Embrechts *et al.* (2009) have studied the analytical shape of (6) with respect to n, α and β. They have shown that

(i) $q_n(\alpha, \beta)$ is increasing in α for $\beta > 1$ and $n \geq 2$;

(ii) $q_n(\alpha, \beta) = n$ for $\beta = 1$ and $n \geq 2$;

(iii) $q_n(\alpha, \beta)$ is decreasing in α for $\beta < 1$ and $n \geq 2$;

(iv) $q_n(\cdot)/n$ is a pdf on $\mathbb{N}_+$;

(v) $q_n(\alpha, \beta)$ is strictly increasing in β;

(vi) $q_n(\alpha, 1) = n$;

(vii) $\min\left(n^\beta, n\right) \leq q_n(\alpha, \beta) \leq \max\left(n^\beta, n\right)$.

If $F \in D(\Psi_\beta)$ then it is shown that

$$\lim_{p \in 1^-} \frac{\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right]}{\mathrm{VaR}_p(X_1)} = \begin{cases} \left[q_n^W(\alpha, \beta)\right]^{1/\beta}, & \text{if } \omega(F) = 0, \\ n, & \text{if } \omega(F) \neq 0, \end{cases}$$

where $\omega(F) = \sup\{x \mid F(x) < 1\}$ and

$$q_n^W(\alpha,\beta) = \int_{\mathbb{R}_+^n} I\left\{\sum_{i=1}^n x_i^{-1} \le 1\right\} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta}\right)^{-1/\alpha} dx_1 \cdots dx_n.$$

Furthermore, if $\omega(F) > 0$, there exists $p_0 > 0$ such that

$$\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right] < \sum_{i=1}^n \mathrm{VaR}_p\left(X_i\right)$$

for all $p_0 < p < 1$. If $\omega(F) \le 0$, there exists $p_0 > 0$ such that

$$\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right] > \sum_{i=1}^n \mathrm{VaR}_p\left(X_i\right)$$

for all $p_0 < p < 1$.

If $F \in D(\Lambda)$ then it is shown that

$$\lim_{p \in 1^-} \frac{\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right]}{\mathrm{VaR}_p\left(X_1\right)} = n.$$

Furthermore, if $\omega(F) > 0$, there exists $p_0 > 0$ such that

$$\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right] < \sum_{i=1}^n \mathrm{VaR}_p\left(X_i\right)$$

for all $p_0 < p < 1$. If $\omega(F) \le 0$, there exists $p_0 > 0$ such that

$$\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right] > \sum_{i=1}^n \mathrm{VaR}_p\left(X_i\right)$$

for all $p_0 < p < 1$.

0.4 Risk concentration

Let $X_1, X_2, \ldots, X_n$ denote future losses, assumed to be non-negative independent random variables with common cdf F and survival function $\overline{F}$. Degen *et al.* (2010) define *risk concentration* as

$$C(\alpha) = \frac{\mathrm{VaR}_\alpha\left[\sum_{i=1}^n X_i\right]}{\sum_{i=1}^n \mathrm{VaR}_\alpha\left(X_i\right)}.$$

If $\overline{F}$ is regularly varying with index $-1/\xi$, $\xi > 0$ (Bingham *et al.*, 1989), meaning that $\overline{F}(tx)/\overline{F}(t) \to x^{-1/\xi}$ as $t \to \infty$, then it is shown that

$$C(\alpha) \to n^{\xi - 1} \tag{7}$$

as $\alpha \to 1$. Degen *et al.* (2010) also study the rate of convergence in (7).

Suppose X_i, $i = 1, 2, \ldots, n$ is regularly varying with index $-\beta$, $\beta > 0$. According to Jang and Jho (2007), for $\beta > 1$,

$$C(\alpha) < 1$$

for all $\alpha \in [\alpha_0, 1]$ for some $\alpha_0 \in (0, 1)$. This property is referred to as subadditivity. If $C(\alpha) < 1$ holds as $\alpha \to 1$ then the property is referred to as asymptotic subadditivity. For $\beta = 1$,

$$C(\alpha) \to 1$$

as $\alpha \to 1$. This property is referred to as asymptotic comonotonicity. For $0 < \beta < 1$,

$$C(\alpha) > 1$$

for all $\alpha \in [\alpha_0, 1]$ for some $\alpha_0 \in (0, 1)$. If $C(\alpha) > 1$ holds as $\alpha \to 1$ then the property is referred to as asymptotic superadditivity. This property is referred to as superadditivity.

Let $N(t)$ denote a counting process with $E[N(t)] < \infty$ for $t > 0$. According to Jang and Jho (2007), in the case of subadditivity,

$$\mathrm{VaR}_\alpha \left[\sum_{i=1}^n X_i \right] \leq E\left[N(t)\right] \sum_{i=1}^n \mathrm{VaR}_\alpha \left(X_i\right)$$

for all $\alpha \in [\alpha_0, 1]$ for some $\alpha_0 \in (0, 1)$. In the case of asymptotic comonotonicity,

$$\mathrm{VaR}_\alpha \left[\sum_{i=1}^n X_i \right] \sim E\left[N(t)\right] \sum_{i=1}^n \mathrm{VaR}_\alpha \left(X_i\right)$$

as $\alpha \to 1$. In the case of superadditivity,

$$\mathrm{VaR}_\alpha \left[\sum_{i=1}^n X_i \right] \geq E\left[N(t)\right] \sum_{i=1}^n \mathrm{VaR}_\alpha \left(X_i\right)$$

for all $\alpha \in [\alpha_0, 1]$ for some $\alpha_0 \in (0, 1)$.

Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ is multivariate regularly varying with index β according to Definition 2.2 in Embrechts *et al.* (2009). If $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{R}$ is a measureable function such that

$$\lim_{x \to \infty} \frac{\Pr\left(\Psi(\mathbf{X}) > x\right)}{\Pr\left(X_1 > x\right)} \to q \in (0, \infty)$$

then it is shown

$$\lim_{\alpha \to 1} \frac{\mathrm{VaR}_\alpha \left(\Psi(\mathbf{X})\right)}{\mathrm{VaR}_\alpha \left(X_1\right)} \to q^{1/\beta},$$

see Lemma 2.3 in Embrechts *et al.* (2009).

0.5 Hürlimann's inequalities

Let X denote a random variable defined over $[A, B]$, $-\infty \leq A < B \leq \infty$ with mean μ, and variance σ. Hürlimann (2002) provides various upper bounds for $\text{VaR}_p(X)$: for $p \leq \sigma^2/\{\sigma^2 + (B-\mu)^2\}$ then

$$\text{VaR}_p(X) \leq B;$$

for $\sigma^2/\{\sigma^2 + (B-\mu)^2\} \leq p \leq (\mu - A)^2/\{\sigma^2 + (\mu - A)^2\}$ then

$$\text{VaR}_p(X) \leq \mu + \sqrt{\frac{1-p}{p}}\sigma;$$

for $p \geq (\mu - A)^2/\{\sigma^2 + (\mu - A)^2\}$ then

$$\text{VaR}_p(X) \leq \mu + \frac{(\mu - A)(B - A)(1-p) - \sigma^2}{(B-A)p - (\mu - A)}.$$

Now suppose X is a random variable defined over $[A, B]$, $-\infty \leq A < B \leq \infty$ with mean μ, variance σ, skewness γ and kurtosis γ_2. In this case, Hürlimann (2002) provides the following upper bound for $\text{VaR}_p(X)$:

$$\text{VaR}_p(X) \leq \mu + x_p\sigma,$$

where x_p is the $100(1-p)$ percentile of the standardized Chebyshev-Markov maximal distribution. The latter is defined as the root of

$$p\left(x_p\right) = p$$

if $p \leq (1/2)\{1 - \gamma/\sqrt{4+\gamma^2}\}$ and as the root of

$$p\left(\psi\left(x_p\right)\right) = p$$

if $p > (1/2)\{1 - \gamma/\sqrt{4+\gamma^2}\}$, where

$$\begin{aligned} p(u) &= \frac{\Delta}{q^2(u) + \Delta\left(1+u^2\right)}, \\ \psi(u) &= \frac{1}{2}\left[\frac{A(u) - \sqrt{A^2(u) + 4q(u)B(u)}}{q(u)}\right], \end{aligned}$$

where $\Delta = \gamma_2 - \gamma^2 + 2$, $A(u) = \gamma q(u) + \Delta u$, $B(u) = q(u) + \Delta$ and $q(u) = 1 + \gamma u - u^2$.

0.6 Ibragimov and Walden's inequalities

Let $X(\mathbf{w}) = \sum_{i=1}^N w_i X_i$ denote a portfolio return made up of N asset returns, X_i, and the non-negative weights w_i. Ibragimov (2009) provides various inequalities for the VaR of $X(\mathbf{w})$. They suppose that X_i are independent and identically distributed and belong to either $\underline{CS}$, the class of distributions which are convolutions of symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in (0, 1]$ and $\sigma > 0$ or $\overline{CSLC}$, convolutions of distributions from the class of symmetric log-concave distributions and the class of distributions which are convolutions of symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in [1, 2]$ and $\sigma > 0$.

Here, $S_\alpha(\beta, \gamma, \mu)$ denotes a stable distribution specified by its characteristic function

$$\phi(t) = \begin{cases} \exp\left\{ \mathrm{i}\mu t - \gamma^\alpha |t|^\alpha \left[1 - \mathrm{i}\beta \tan\left(\pi \frac{\alpha}{2}\right) \mathrm{sign}(t)\right]\right\}, & \alpha \neq 1, \\ \\ \exp\left\{ \mathrm{i}\mu t - \gamma |t| \left(1 + \mathrm{i}\beta \frac{2}{\pi} \ln t\right)\right\}, & \alpha = 1, \end{cases}$$

where $\mathrm{i} = \sqrt{-1}$, $\alpha \in (0, 2]$, $|\beta| \leq 1$, $\gamma > 0$ and $\mu \in \mathbb{R}$. The stable distribution contains as particular cases: the Gaussian distribution for $\alpha = 2$; the Cauchy distribution for $\alpha = 1$, and $\beta = 0$; the Lévy distribution for $\alpha = 1/2$ and $\beta = 1$; the Landau distribution for $\alpha = 1$ and $\beta = 1$; the dirac delta distribution for $\alpha \downarrow 0$ and $\gamma \downarrow 0$.

Furthermore, let $\mathcal{I}_N = \left\{(w_1, \ldots, w_N) \in \mathbb{R}_+^N : w_1 + \cdots + w_N = 1\right\}$. Write $\mathbf{a} \prec \mathbf{b}$ to mean that $\sum_{i=1}^k a_{[i]} \leq \sum_{i=1}^k b_{[i]}$ for $k = 1, \ldots, N-1$ and $\sum_{i=1}^N a_{[i]} = \sum_{i=1}^N b_{[i]}$, where $a_{[1]} \geq \cdots a_{[N]}$ and $b_{[1]} \geq \cdots b_{[N]}$ denote the components of $\mathbf{a}$ and $\mathbf{b}$ in descending order. Let $\underline{\mathbf{w}}_N = (1/N, 1/N, \ldots, 1/N)$ and $\overline{\mathbf{w}}_N = (1, 0, \ldots, 0)$.

With these notation, Ibragimov (2009) provides the following inequalities for $\mathrm{VaR}_q(X(\mathbf{w}))$. Suppose first that $q \in (0, 1/2)$ and X_i belong to $\overline{CSLC}$. Then

(i) $\mathrm{VaR}_q[X(\mathbf{v})] \leq \mathrm{VaR}_q[X(\mathbf{w})]$ if $\mathbf{v} \prec \mathbf{w}$;

(i) $\mathrm{VaR}_q[X(\underline{\mathbf{w}}_N)] \leq \mathrm{VaR}_q[X(\mathbf{w})] \leq \mathrm{VaR}_q[X(\overline{\mathbf{w}}_N)]$ for all $\mathbf{w} \in \mathcal{I}_N$.

Suppose now that $q \in (0, 1/2)$ and X_i belong to $\underline{CS}$. Then

(i) $\mathrm{VaR}_q[X(\mathbf{v})] \geq \mathrm{VaR}_q[X(\mathbf{w})]$ if $\mathbf{v} \prec \mathbf{w}$;

(i) $\mathrm{VaR}_q[X(\overline{\mathbf{w}}_N)] \leq \mathrm{VaR}_q[X(\mathbf{w})] \leq \mathrm{VaR}_q[X(\underline{\mathbf{w}}_N)]$ for all $\mathbf{w} \in \mathcal{I}_N$.

Further inequalities for VaR are provided in Ibragimov and Walden (2011) when a portfolio return, say Y, is made up of a two dimensional array of asset returns say Y_{ij}. That is,

$$\begin{aligned} Y(\mathbf{w}) &= \sum_{i=1}^r \sum_{j=1}^c w_{ij} Y_{ij} \\ &= \sum_{i=1}^r w_{i0} R_i + \sum_{i=1}^r w_{0j} C_j + \sum_{i=1}^r \sum_{j=1}^c w_{ij} U_{ij} \\ &= R\left(\mathbf{w}_0^{(\mathrm{row})}\right) + C\left(\mathbf{w}_0^{(\mathrm{col})}\right) + U(\mathbf{w}), \end{aligned}$$

where $R_i, i = 1, \ldots, r$ are referred to as "row effects", $C_j, j = 1, \ldots, c$ are referred to as "column effects", and $U_{ij}, i = 1, \ldots, r, j = 1, \ldots, c$ are referred to as "idiosyncratic components".

Let $\underline{\mathbf{w}}_{rc} = (1/(rc), 1/(rc), \ldots, 1/(rc))$, $\overline{\mathbf{w}}_{rc} = (1, 0, \ldots, 0)$, $\underline{\mathbf{w}}_0^{(\mathrm{row})} = (1/r, 1/r, \ldots, 1/r)$, $\overline{\mathbf{w}}_0^{(\mathrm{row})} = (1, 0, \ldots, 0)$, $\underline{\mathbf{w}}_0^{(\mathrm{col})} = (1/c, 1/c, \ldots, 1/c)$, and $\overline{\mathbf{w}}_0^{(\mathrm{col})} = (1, 0, \ldots, 0)$.

With these notation, Ibragimov and Walden (2011) provide the following inequalities for $q \in (0, 1/2)$:

(i) if R_i, C_j, U_{ij} belong to $\overline{CSLC}$ then $\mathrm{VaR}_q[Y(\underline{\mathbf{w}}_{rc})] \leq \mathrm{VaR}_q[Y(\mathbf{w})] \leq \mathrm{VaR}_q[Y(\overline{\mathbf{w}}_{rc})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$;

(ii) if R_i, C_j, U_{ij} belong to $\overline{CS}$ then $\mathrm{VaR}_q[Y(\underline{\mathbf{w}}_{rc})] \geq \mathrm{VaR}_q[Y(\mathbf{w})] \geq \mathrm{VaR}_q[Y(\overline{\mathbf{w}}_{rc})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$;

(iii) if U_{ij} belong to $\overline{CSLC}$ then $\mathrm{VaR}_q[U(\underline{\mathbf{w}}_{rc})] \leq \mathrm{VaR}_q[U(\mathbf{w})] \leq \mathrm{VaR}_q[U(\overline{\mathbf{w}}_{rc})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$;

(iv) if U_{ij} belong to $\overline{CS}$ then $\mathrm{VaR}_q[U(\underline{\mathbf{w}}_{rc})] \geq \mathrm{VaR}_q[U(\mathbf{w})] \geq \mathrm{VaR}_q[U(\overline{\mathbf{w}}_{rc})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$;

(v) if R_i belong to $\overline{CSLC}$ then $\mathrm{VaR}_q[R(\underline{\mathbf{w}}_{r})] \leq \mathrm{VaR}_q[R(\mathbf{w}_0^{(\mathrm{row})})] \leq \mathrm{VaR}_q[R(\overline{\mathbf{w}}_{r})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$;

(vi) if R_i belong to $\overline{CS}$ then $\mathrm{VaR}_q[R(\underline{\mathbf{w}}_{r})] \geq \mathrm{VaR}_q[R(\mathbf{w}_0^{(\mathrm{row})})] \geq \mathrm{VaR}_q[R(\overline{\mathbf{w}}_{r})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$;

(vii) if C_j belong to $\overline{CSLC}$ then $\mathrm{VaR}_q[C(\underline{\mathbf{w}}_{c})] \leq \mathrm{VaR}_q[C(\mathbf{w}_0^{(\mathrm{col})})] \leq \mathrm{VaR}_q[C(\overline{\mathbf{w}}_{c})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$;

(viii) if C_j belong to $\overline{CS}$ then $\mathrm{VaR}_q[C(\underline{\mathbf{w}}_{c})] \geq \mathrm{VaR}_q[C(\mathbf{w}_0^{(\mathrm{col})})] \geq \mathrm{VaR}_q[C(\overline{\mathbf{w}}_{c})]$ for all $\mathbf{w} \in \mathcal{I}_{rc}$.

Ibragimov and Walden (2011, Section 4) discuss an application of these inequalities to portfolio component value at risk analysis.

0.7 Denis *et al.*'s inequalities

Let $\{X_t\}$ denote a process defined by

$$X_t = m + \int_0^t \sigma_s dB_s + \int_0^t b_s ds + \sum_{i=1}^{N_t} \gamma_{T_i^-} Y_i,$$

where B is a Brownian motion, $\widetilde{N}$ is a compound Poisson process independent of B, $T_1, T_2, \ldots$ are jump times for $\widetilde{N}$, b is an adapted integrable process, and σ, γ are certain random variables. Denis *et al.* (2009) derive various bounds for the VaR of the process

$$X_t^* = \sup_{0 \leq u \leq t} X_u.$$

The following assumptions are made:

(i) for all $t > 0$, $E\left(\int_0^t \sigma_s^2 ds\right) < \infty$;

(ii) jumps of the compound Poisson process are non-negative and Y_1 is not identically equal to zero;

(iii) the process $\sum_{i=1}^{N_t} \gamma_{T_i^-} Y_i$ for $t > 0$ is well defined and integrable;

(iv) the jumps have a Laplace transform, $L9x) = E\left[\exp\left(xY_1\right)\right]$, $x < c$ for c a positive constant.

(v) there exists $\gamma^* > 0$ such that $\gamma_s \leq \gamma*$ almost surely for all $s \in [0, t]$.

(vi) there exists constant $b^*(t) \geq 0$ and $a^*(t) \geq 0$ such that

$$\int_0^t \sigma_u^2 du \leq a^*(t), \quad \int_0^s b_u du \leq b^*(t)$$

almost everywhere for all $s \in [0, t]$. In this case, let

$$K_t(\delta) = \delta b^*(t) + \delta^2 \frac{a^*(t)}{2} + \lambda t \left[L\left(\delta \gamma^*\right) - 1 \right]$$

for $0 < \delta < c/\gamma^*$.

With these assumptions, Denis *et al.* (2009) show that

$$\mathrm{VaR}_\alpha \left(X_t^* \right) \leq \inf_{\delta < c/\gamma^*} \left\{ m + \frac{K_t(\delta) - \ln \alpha}{\delta} \right\},$$

$$\mathrm{VaR}_\alpha \left(X_t^* \right) \leq \inf_{0 < \delta < c/\gamma^*} \left\{ m + b^*(t) + \frac{a^*(t)\delta}{2} + \lambda t \frac{L\left(\delta \gamma^*\right) - 1}{\delta} - \frac{\ln \alpha}{\delta} \right\}.$$

For $\gamma \leq 0$, Denis *et al.* (2009) show that

$$\mathrm{VaR}_\alpha \left(X_t^* \right) \leq m + b^*(t) + \sqrt{-2 a^*(t) \ln \alpha}.$$

If the jumps follow a simple Poisson process, Denis *et al.* (2009) show that

$$\mathrm{VaR}_\alpha \left(X_t^* \right) \leq \inf_{0 < \delta < \infty} \left\{ m + b^*(t) + \frac{a^*(t)\delta}{2} + \lambda t \frac{\exp\left(\delta \gamma^*\right) - 1}{\delta} - \frac{\ln \alpha}{\delta} \right\}.$$

If the jumps follow an exponential distribution with parameter $\nu > 0$, Denis *et al.* (2009) show that

$$\mathrm{VaR}_\alpha \left(X_t^* \right) \leq \inf_{0 < \delta < \nu/\gamma^*} \left\{ m + b^*(t) + \frac{a^*(t)\delta}{2} + \frac{\lambda t}{\nu/\gamma^* - \delta} - \frac{\ln \alpha}{\delta} \right\}.$$

0.8 Jaworski's inequalities

Jaworski (2008, 2009) considers the following situation: suppose s_i, $i = 1, \ldots, d$ are the quotients of the currency rates at the end and at the beginning of an investment; suppose that the joint cdf of $(s_1, \ldots, s_d)$ is $C\left(F_1\left(s_1\right), \ldots, F_d\left(s_d\right)\right)$, where C is a copula (Nelsen, 1999) and F_i is the marginal cdf of s_i; suppose w_i is the part of the capital invested in the ith currency, where w_i are non-negative and sum to one. Then the final investment value is

$$W_1(\mathbf{w}) = \left(w_1 s + 1 + \cdots + w_d s_d\right) W_0,$$

where $\mathbf{w} = (w_1, \ldots, w_d)$. Jaworski (2008, 2009) defines the value of risk for a given $\mathbf{w}$ and a probability α as

$$\mathrm{VaR}_\alpha(\mathbf{w}) = \sup \left\{ V : \Pr\left(W_0 - W_1(\mathbf{w}) \leq V\right) \leq \alpha \right\}.$$

Jaworski (2009) shows this VaR can be bounded as

$$\sum_{i=1}^d \mathrm{VaR}_{\alpha'}\left(\mathbf{e}_i\right) \leq \mathrm{VaR}_\alpha \leq \sum_{i=1}^d \mathrm{VaR}_\alpha\left(\mathbf{e}_i\right),$$

where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ and $\alpha' = \alpha^2 / C(\alpha, \ldots, \alpha)$.

0.9 Mesfioui and Quessy's inequalities

Suppose a portfolio is made up of n assets and let $X_1, X_2, \ldots, X_n$ denote the losses for the n assets. Suppose also that the joint cdf of $(X_1, \ldots, X_n)$ is $C\left(F_1\left(x_1\right), \ldots, F_n\left(x_n\right)\right)$, where C is a copula (Nelsen, 1999), and F_i is the marginal cdf of X_i. Furthermore, define the dual of a given copula C (Definition 2.4, Mesfioui and Quessy, 2005) as

$$C^d\left(u_1, \ldots, u_n\right) = \Pr\left(U(0,1) \leq u_1 \text{ or } \cdots \text{ or } U(0,1) \leq u_n\right).$$

With thesee notation, Mesfioui and Quessy (2005) derive various inequalities for the value at risk of $S = X_1 + \cdots + X_n$. If C is such that $C \geq qC_L$ and $C \leq C_U^d$ for some copulas C_L and C_U then

$$\underline{VaR}_\alpha \leq \mathrm{VaR}_\alpha(S) \leq \overline{VaR}_\alpha,$$

where

$$\underline{VaR}_\alpha = \sup_{C_U^d(u_1,\ldots,u_n)=\alpha} \sum_{i=1}^n F_i^{-1}\left(u_i\right)$$

and

$$\overline{VaR}_\alpha = \inf_{C_L(u_1,\ldots,u_n)=\alpha} \sum_{i=1}^n F_i^{-1}\left(u_i\right).$$

If $X_1, X_2, \ldots, X_n$ are identical random variables with common cdf F and if $x^* \in \mathbb{R}$ is such that $f(x) = dF(x)/dx$ is non-increasing for $x \geq x^*$ then it is shown under certain conditions that

$$\mathrm{VaR}_\alpha(S) \leq nF^{-1}\left(\delta_{C_L}^{-1}(\alpha)\right),$$

where $\delta_{C_L}(t) = C_L(t, \ldots, t)$ is the diagonal section of C_L.

Mesfioui and Quessy (2005) also show that if X is a random variable with mean μ and variance σ^2 then

$$g_{\mu,\sigma}(\alpha) \leq \mathrm{VaR}_\alpha(X) \leq h_{\mu,\sigma}(\alpha),$$

where

$$g_{a,b}(u) = \{a - bq(1-u)\} I\left(u \geq \frac{b^2}{a^2+b^2}\right)$$

and

$$gh_{a,b}(u) = a + aq^2(u) I\left(u \leq \frac{b^2}{a^2+b^2}\right) + bq(u) I\left(u > \frac{b^2}{a^2+b^2}\right),$$

where $q(u) = \sqrt{u/(1-u)}$. If X_i, $i = 1, \ldots, n$ have means μ_i, $i = 1, \ldots, n$ and variances σ_i^2, $i = 1, \ldots, n$ then it is shown that

$$g_{\mu,\sigma}(\alpha) \leq \mathrm{VaR}_\alpha(S) \leq h_{\mu,\sigma}(\alpha),$$

where $\mu = \mu_1 + \cdots + \mu_n$ and $\sigma = \sigma_1 + \cdots + \sigma_n$.

0.10 Slim *et al.*'s inequalities

Suppose a portfolio is made up of d assets. Let $X_1, X_2, \ldots, X_d$ denote the losses for the d assets. Let F_i and f_i denote the cdf and the pdf of X_i. Let x_i^* denote the values for which $f_i(x)$ is non-increasing for all $x \leq x_i^*$. Given the notation, the total portfolio loss can be expressed as $S = w_1 X_1 + w_2 X_2 + \cdots + w_d X_d$ for some non-negative weights w_i summing to one. Slim *et al.* (2010) show that the VaR of S can be bounded as follows:

$$\underline{VaR}_p \leq \mathrm{VaR}_p(S) \leq \overline{VaR}_p,$$

where

$$\overline{VaR}_p = \inf_{u_1+\cdots+u_d=\alpha+d-1} \sum_{i=1}^{d} F_i^{-1}\left(u_i\right)$$

and

$$\underline{VaR}_p = \max_{1 \leq i \leq d} \left\{ F_i^{-1}(\alpha) + \sum_{1 \leq j \neq i \leq d} F_j^{-1}(d) \right\}$$

for $\alpha \leq \min\left\{F_1\left(x_1^*\right), \ldots, F_d\left(x_d^*\right)\right\}$.

0.11 Kaas *et al.*'s inequalities

Kaas *et al.* (2009) establish various inequalities for the VaR of the sum of two uniform random variables. Let U and V be two uniform random variables on the unit interval. Let (U_i, V_i), $i = 1, 2, 3$ be independent copies of (U, V). Define

$$\begin{aligned}
\tau &= \Pr\left[\left(U_1 - U_2\right)\left(V_1 - V_2\right) > 0\right] - \Pr\left[\left(U_1 - U_2\right)\left(V_1 - V_2\right) < 0\right], \\
\rho &= 3 \Pr\left[\left(U_1 - U_2\right)\left(V_1 - V_3\right) > 0\right] - \Pr\left[\left(U_1 - U_2\right)\left(V_1 - V_3\right) < 0\right], \\
\beta &= \Pr\left[\left(U - \mathrm{Median}(U)\right)\left(V - \mathrm{Median}(V)\right) > 0\right] \\
&\qquad - \Pr\left[\left(U - \mathrm{Median}(U)\right)\left(V - \mathrm{Median}(V)\right) < 0\right].
\end{aligned}$$

With this notation, Kaas *et al.* (2009) establish various inequalities for $\mathrm{VaR}_(U+V)$. If τ, ρ or β is given then they show that

$$\begin{aligned}
\mathrm{VaR}_p(U+V) &\geq \begin{cases} p, & p \in \left(0, \sqrt{1-\tau}\right), \\ 2p - \sqrt{1-\tau}, & p \in \left(\sqrt{1-\tau}, 1\right], \end{cases} \\
&\leq \begin{cases} 2p + \sqrt{1-\tau}, & p \in \left(0, 1 - \sqrt{1-\tau}\right], \\ 1 + p, & p \in \left(1 - \sqrt{1-\tau}, 1\right], \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathrm{VaR}_p(U+V) &\geq \begin{cases} p, & p \in \left(0, 2\frac{(1-\rho)^{1/3}}{12^{1/3}}\right], \\ 2p - 2\frac{(1-\rho)^{1/3}}{12^{1/3}}, & p \in \left(2\frac{(1-\rho)^{1/3}}{12^{1/3}}, 1\right], \end{cases} \\
&\leq \begin{cases} 2p + 2\frac{(1-\rho)^{1/3}}{12^{1/3}}, & p \in \left(0, 1 - 2\frac{(1-\rho)^{1/3}}{12^{1/3}}\right], \\ 1 + p, & p \in \left(1 - 2\frac{(1-\rho)^{1/3}}{12^{1/3}}, 1\right], \end{cases}
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{VaR}_p(U+V) &\geq \begin{cases} p, & p \in \left(0, 1-\frac{\beta+1}{4}\right], \\ p+\frac{\beta+1}{4}, & p \in \left(1-\frac{\beta+1}{4}, 1\right], \end{cases} \\
&\leq \begin{cases} 1+p-\frac{\beta+1}{4}, & p \in \left(0, 1-\frac{\beta+1}{4}\right], \\ 1+p, & p \in \left(1-\frac{\beta+1}{4}, 1\right]. \end{cases}
\end{aligned}
$$

If (U,V) are positively quadrant dependent and τ, ρ or β is given then they show that

$$
\begin{aligned}
\mathrm{VaR}_p(U+V) &\geq \max\left[2\left(1-\sqrt{1-p}\right), 2p-\sqrt{1-\tau}\right] \\
&\leq \min\left[2\rho+\sqrt{1-\tau}, 2\sqrt{p}\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{VaR}_p(U+V) &\geq \max\left[2\left(1-\sqrt{1-p}\right), 2p-2\frac{(1-\rho)^{1/3}}{12^{1/3}}\right] \\
&\leq \min\left[2p+2\frac{(1-\rho)^{1/3}}{12^{1/3}}, 2\sqrt{p}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{VaR}_p(U+V) &\geq \max\left[2\left(1-\sqrt{1-p}\right), \left(p+\frac{\beta+1}{4}\right) I\left\{p \in \left(1-\frac{\beta+1}{4}, 1\right]\right\}\right] \\
&\leq \min\left[\left(1+p-\frac{\beta+1}{4}\right) J\left\{p \in \left(0, \frac{\beta+1}{4}\right]\right\}, 2\sqrt{p}\right],
\end{aligned}
$$

where $J(A)=1$ if A is true and $J(A)=\infty$ if A is false. If τ and β are given then they show that

$$
\begin{aligned}
& \mathrm{VaR}_p(U+V) \\
\geq\ & 2p-\sqrt{4\left(\xi-\xi^2\right)-\tau},\ p \in \left(p^*, 1\right] \\
\leq\ & \begin{cases} 2p+\sqrt{4\left(\xi-\xi^2\right)-\tau}, & p \in \left(p^{**}, 1-\sqrt{4\left(\xi-\xi^2\right)-\tau}\right], \\ 1+p, & p \in \left(p^{**} \vee \left(1-\sqrt{4\left(\xi-\xi^2\right)-\tau}\right), 1\right], \end{cases}
\end{aligned}
$$

where $\xi=(1+\beta)/4$, and

$$
\begin{aligned}
p^* &= \min\left[1, \frac{1}{2}\left(\frac{3}{2}+\sqrt{4\left(\xi-\xi^2\right)-\tau}\right)\right], \\
p^{**} &= \min\left[1, \frac{1}{2}\left(\frac{3}{2}-\sqrt{4\left(\xi-\xi^2\right)-\tau}\right)\right].
\end{aligned}
$$

0.12 de Schepper and Heijnen's inequalities

de Schepper and Heijnen (2010) establish various inequalities for the VaR depending on moments and the mode. Let X be a random variable with mode m and the first three moments μ_1, μ_2, μ_3. Let $\nu_1=2\mu_1-m$, $\nu_2=3\mu_2-2m\mu_1$, $o^{'}=\nu_2/\nu_1$, $\alpha=\left[b\left(2o^{'}-m\right)+\left(o^{'}-m\right)^2\right]$, $\beta=$

$\nu_2 \left(\nu_2 - m\nu_1\right) \left(3o^{'} - 2m\right)$, $\gamma = \left(3b - 2m - b^{'}\right) \left(b^2 - 2\nu_1 b + \nu_2\right)$, $\delta = b^2 + \left(m - b^{'}\right)^2 + b \left(m - 2b^{'}\right)$, and $b^{'} = \left(b\nu_1 - \nu_2\right) / \left(b - \nu_1\right)$. Let

$$\begin{aligned}
&f_t = t^{-1} \left[t\left(b+m\right) - bm - \sqrt{bm(b-t)(m-t)} \right], \\
&g_t = (b-t)^{-1} \left[t\left(b-m\right) + \sqrt{bm(b-m)(t-m)} \right], \\
&A_t = -2^{-1}\left(2\nu_1 + m + 3t\right), \ B_t = \left(2\nu_1 + m\right)t, \ C_t = 2^{-1}\left(m\nu_2 - t\nu_2 - 2mt\nu_1\right), \\
&s_t = \frac{\left(b\mu_2 - \mu_3\right) - t\left(b\mu_1 - \mu_2\right)}{\left(b\mu_1 - \mu_2\right) - t\left(b - \mu_1\right)}, \\
&\alpha_p = -\mu_1 \left[(1-p)\mu_2 - \mu_1^2\right], \ \beta_p = (1-p)\left(\mu_1\mu_3 + 2\mu_2^2\right) - 3\mu_1^2\mu_2, \\
&\gamma_p = -3\mu_2\left[(1-p)\mu_3 - \mu_1\mu_2\right], \ \delta_p = (1-p)\mu_3^2 - \mu_2^3, \\
&\widehat{\alpha}_p = (1-p)\mu_2 - \mu_1^2, \ \widehat{\beta}_p = \mu_1\mu_2 - (1-p)\mu_3, \\
&\widehat{\delta}_p = \mu_1\mu_3 - \mu_2^2.
\end{aligned}$$

Let $\kappa_1 \le \kappa_2$ denote the two real roots of

$$\left(\mu_2 - \mu_1^2\right)\kappa^2 + \left(\mu_1\mu_2 - \mu_3\right)\kappa + \left(\mu_1\mu_2 - \mu_2^2\right) = 0.$$

Also let x_t be the unique root of $x^3 + A_t x^2 + B_t x + C = 0$ in the interval $\left[0, \min\left(b^{'}, t\right)\right]$ if $t \le m$, and let y_t be the unique root of $x^3 + A_t x^2 + B_t x + C = 0$ in the interval $\left[\max\left(o^{'}, t\right), b\right]$ if $t > m$.

If X has the support $[0, b]$ then it is shown that

$$L \le \mathrm{VaR}_p \le U,$$

where

$$L = \begin{cases}
0, & \text{if } 0 < p \le 1 - \dfrac{\mu_1\mu_3 - \mu_2^2}{b\left(b^2\mu_1 - 2b\mu_2 + \mu_3\right)} \\
& \quad - \dfrac{\left(b\mu_1 - \mu_2\right)^3}{\left(b\mu_2 - \mu_3\right)\left(b^2\mu_1 - 2b\mu_2 + \mu_3\right)}, \\
\text{solution for } t \text{ of} & \\
1 - p = \dfrac{\mu_2 - \mu_1(t+b) + tb}{\left(t - s_t\right)\left(b - s_t\right)} & \\
\quad + \dfrac{\mu_2 - \mu_1\left(s_t + t\right) + s_t b}{\left(b - s_t\right)\left(b - t\right)}, & \text{if } 1 - \dfrac{\mu_1\mu_3 - \mu_2^2}{b\left(b^2\mu_1 - 2b\mu_2 + \mu_3\right)} \\
& \quad - \dfrac{\left(b\mu_1 - \mu_2\right)^3}{\left(b\mu_2 - \mu_3\right)\left(b^2\mu_1 - 2b\mu_2 + \mu_3\right)} < p \le 1 \\
& \quad - \dfrac{\left(\mu_2 - \kappa_1\mu_1\right)^3}{\left(\mu_3 - \kappa_1\mu_2\right)\left(\mu_3 - 2\kappa_1\mu_2 + \kappa_1^2\mu_1\right)}, \\
\text{solution for } t \text{ of} & \\
\alpha_p + \beta_p t^2 + \gamma_p t + \delta_p = 0, & \text{if } 1 - \dfrac{\left(\mu_2 - \kappa_1\mu_1\right)^3}{\left(\mu_3 - \kappa_1\mu_2\right)\left(\mu_3 - 2\kappa_1\mu_2 + \kappa_1^2\mu_1\right)} \\
& \quad < p \le \dfrac{\mu_1\mu_3 - \mu_2^2}{b\left(b^2\mu_1 - 2b\mu_2 + \mu_3\right)}, \\
\text{solution for } t \text{ of} & \\
1 - p = \dfrac{\mu_2 - \mu_1\left(s_t + t\right) + s_t t}{\left(b - s_t\right)\left(b - t\right)}, & \text{if } 1 - \dfrac{\mu_1\mu_3 - \mu_2^2}{b\left(b^2\mu_1 - 2b\mu_2 + \mu_3\right)} < p < 1,
\end{cases}$$

and

$$U=\begin{cases}\text{solution for } t \text{ of} & \\ \widehat{\alpha}_p t^2+\widehat{\beta}_p t+\widehat{\gamma}_p=0, & \text{if } 0<p\le 1-\dfrac{\mu_1\left(b\mu_1-\mu_2\right)}{b\mu_2-\mu_3} \\ & +\dfrac{\mu_1\mu_3-\mu_2^2}{b\left(b\mu_2-\mu_3\right)} \\ \text{solution for } t \text{ of} & \\ 1-p=\dfrac{\mu_2-\mu_1\left(s_t+t\right)+s_t t}{\left(b-s_t\right)\left(b-t\right)} & \\ +\dfrac{\mu_2-\mu_1\left(s_t+b\right)+s_t b}{\left(s_t-t\right)\left(b-t\right)}, & \text{if } 1-\dfrac{\mu_1\left(b\mu_1-\mu_2\right)}{b\mu_2-\mu_3} \\ & +\dfrac{\mu_1\mu_3-\mu_2^2}{b\left(b\mu_2-\mu_3\right)}<p\le 1 \\ & -\dfrac{\mu_3\mu_1-\mu_2^2}{\kappa_2\left(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1\right)}, \\ \text{solution for } t \text{ of} & \\ \mu_1 t^3-2\mu_2 t^2+\mu_3 t=\dfrac{\mu_3\mu_1-\mu_2^2}{1-p}, & \\ & \text{if } 1-\frac{\mu_3\mu_1-\mu_2^2}{\kappa_1\left(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1\right)} \\ & <p\le 1-\dfrac{\mu_3\mu_1-\mu_2^2}{b\left(\mu_3-2b\mu_2+b^2\mu_1\right)}, \\ b, & \text{if } 1-\frac{\mu_3\mu_1-\mu_2^2}{b\left(\mu_3-2b\mu_2+b^2\mu_1\right)}<p<1.\end{cases}$$

The second equation in L has a unique root for t in $(\kappa_1, (b\mu_2-\mu_3)/(b\mu_1-\mu_2)]$. The third equation in L has a unique root for t in $((b\mu_2-\mu_3)/(b\mu_1-\mu_2), \kappa_2)$. The first equation in U has a unique root for t in $(\kappa_1, (b\mu_2-\mu_3)/(b\mu_1-\mu_2)]$. The second equation in U has a unique root for t in $((b\mu_2-\mu_3)/(b\mu_1-\mu_2), \kappa_2]$. The third equation in U has a unique root for t in $(\kappa_2, b]$.

If X has the support $\mathbb{R}^+$ then it is shown that

$$L\le \mathrm{VaR}_p\le U,$$

where

$$L=\begin{cases}0, & \text{if } 0<p\le\dfrac{\mu_2-\mu_1^2}{\mu_2} \\ \mu_1-\sqrt{\dfrac{1-p}{p}\left(\mu_2-\mu_1^2\right)}, & \text{if } \dfrac{\mu_2-\mu_1^2}{\mu_2}<p\le 1 \\ & -\dfrac{\left(\mu_2-\kappa_1\mu_1\right)^3}{\left(\mu_3-\kappa_1\mu_2\right)\left(\mu_3-2\kappa_1\mu_2+\kappa_1^2\mu_1\right)}, \\ \text{solution for } t \text{ of} & \\ \alpha_p t^3+\beta_p t^2+\gamma_p t+\delta_p=0, & 1-\dfrac{\left(\mu_2-\kappa_1\mu_1\right)^3}{\left(\mu_3-\kappa_1\mu_2\right)\left(\mu_3-2\kappa_1\mu_2+\kappa_1^2\mu_1\right)} \\ & <p<1,\end{cases}$$

and

$$U=\begin{cases}\text{solution for } t \text{ of} & \\ \widehat{\alpha}_p t^2+\widehat{\beta}_p t+\widehat{\gamma}_p=0, & \text{if } 0<p\le \dfrac{\mu_2-\mu_1^2}{\mu_2} \\ \mu_1+\sqrt{\dfrac{p}{1-p}\left(\mu_2-\mu_1^2\right)}, & \text{if } \dfrac{\mu_2-\mu_1^2}{\mu_2}<p\le 1 \\ & -\dfrac{\mu_3\mu_1-\mu_2^2}{\kappa_2\left(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1\right)}, \\ \text{solution for } t \text{ of} & \\ \mu_1 t^3-2\mu_2 t^2+\mu_3 t=\dfrac{\mu_3\mu_1-\mu_2^2}{1-p}, & 1-\dfrac{\mu_3\mu_1-\mu_2^2}{\kappa_2\left(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1\right)} \\ & <p<1.\end{cases}$$

The first equation in L has a unique root for t in $(\kappa_1, \mu_2/\mu_1]$. The first equation in U has a unique root for t in $(\kappa_1, \mu_2/\mu_1)$. The second equation in U has a unique root for t in (κ_2, ∞).

If X is a unimodal random variable and has the support $[0, b]$ and $b^{'}>m$ then it is shown that

$$L \le \mathrm{VaR}_p \le U,$$

where

$$L=\begin{cases}\dfrac{m\nu_2 p}{\nu_2-\nu_1^2}, & \text{if } 0<p\le 1-\dfrac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}, \\ \text{solution for } t \text{ of} & \\ 1-p=\dfrac{(m-t)\left(\nu_2-\nu_1^2\right)}{(m-x_t)\left(\nu_2-2\nu_1x_t+x_t^2\right)} & \\ +\dfrac{\left(\nu_1-x_t\right)^2}{\nu_2-2\nu_1x_t+x_t^2}, & \text{if } \dfrac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}<p\le \\ & \dfrac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2}, \\ \nu_1-\dfrac{1}{2}(1-p)\left(\nu_1-m\right) & \\ -\dfrac{1}{2}\sqrt{(1-p)^2\left(\nu_1-m\right)^2+4(1-p)\left(\nu_2-\nu_1^2\right)}, & \text{if } \dfrac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2}< \\ & p\le \\ & \dfrac{\nu_2-\nu_1^2+\left(b^{'}-m\right)\left(\nu_1-b^{'}\right)}{\nu_2-\nu_1^2+\left(\nu_1-m\right)\left(\nu_1-b^{'}\right)}, \\ \dfrac{\nu_2-(1-p)b(b-m)}{\nu_1}, & \text{if} \\ & \dfrac{\nu_2-\nu_1^2+\left(b^{'}-m\right)\left(\nu_1-b^{'}\right)}{\nu_2-\nu_1^2+\left(\nu_1-m\right)\left(\nu_1-b^{'}\right)} \\ & <p<1,\end{cases}$$

and

$$
U=\begin{cases}
\dfrac{b\left(\nu_1-b^{'}\right)\left(b^{'}-m\right)+b^{'}\left(b-\nu_1\right)\left(b-m\right)}{\left(\nu_1-b^{'}\right)\left(b^{'}-m\right)+\left(b-\nu_1\right)\left(b-m\right)} \\ \quad -\dfrac{(1-p)\left(b-b^{'}\right)\left(b-m\right)\left(b^{'}-m\right)}{\left(\nu_1-b^{'}\right)\left(b^{'}-m\right)+\left(b-\nu_1\right)\left(b-m\right)}, & \text{if } 0<p\leq 1-\dfrac{bb^{'}+\nu_1\left(b^{'}-m\right)}{b\left(2b^{'}-m\right)+\left(b^{'}-m\right)^2}, \\
\text{solution for } t \text{ of} \\ 1-p=\dfrac{\left(b\nu_1-\nu_2\right)\left(g_1-t\right)}{g_t\left(b-g_t\right)\left(g_t-m\right)} \\ \quad +\dfrac{\left(\nu_2-\nu_1 g_t\right)\left(b-t\right)}{b\left(b-g_t\right)\left(b-m\right)}, & \text{if } 1-\dfrac{bb^{'}+\nu_1\left(b^{'}-m\right)}{b\left(2b^{'}-m\right)+\left(b^{'}-m\right)^2} \\ & <p\leq 1-\dfrac{\nu_1^2 b\left(2o^{'}-m\right)}{\alpha\left(\nu_2-m\nu_1\right)} \\ & -\dfrac{\nu_1^2\left(o^{'}-m\right)^2}{\alpha\left(\nu_2-m\nu_1\right)} \\ & +\dfrac{\nu_1^3 b o^{'2}}{\nu_2\alpha\left(\nu_2-m\nu_1\right)}, \\
\dfrac{\nu_2}{\nu_1^3}\left[\nu_1^2-(1-p)\left(\nu_2-m\nu_1\right)\right], & \text{if } 1-\dfrac{\nu_1^2 b\left(2o^{'}-m\right)}{\alpha\left(\nu_2-m\nu_1\right)} \\ & -\dfrac{\nu_1^2\left(o^{'}-m\right)^2}{\alpha\left(\nu_2-m\nu_1\right)} \\ & +\dfrac{\nu_1^3 b o^{'2}}{\nu_2\alpha\left(\nu_2-m\nu_1\right)} \\ & <p\leq 1 \\ & -\dfrac{\nu_1^2\left(3o^{'}-2m\right)}{\left(\nu_2-m\nu_1\right)\left(3o^{'}-2m\right)} \\ & +\dfrac{\nu_1^3 o^{'}\left(2o^{'}-m\right)}{\beta}, \\
\text{solution for } t \text{ of} \\ 1-p=\dfrac{\left(y_t-t\right)\left(\nu_2-\nu_1^2\right)}{\left(y_t-m\right)\left(y_t^2-2\nu_1 y_t+\nu_2\right)}, & \text{if } 1-\dfrac{\nu_1^2\nu_2\left(3o^{'}-2m\right)}{\beta} \\ & +\dfrac{\nu_1^3 o^{'}\left(2o^{'}-m\right)}{\beta} \\ & <p\leq 1-\dfrac{\left(\nu_2-\nu_1^2\right)\left(b-b^{'}\right)}{\gamma}, \\
b-(1-p)(b-m)\dfrac{b^2-2\nu_1 b+\nu_2}{\nu_2-\nu_1^2}, & \text{if } 1-\dfrac{\left(\nu_2-\nu_1^2\right)\left(b-b^{'}\right)}{\gamma} \\ & <p<1.
\end{cases}
$$

The first equation in L has a unique root for t in $(m\nu_2/(\nu_2+2m\nu_1), m]$. The first equation in U has a unique root for t in $(m\nu_2/(\nu_2+2m\nu_1), m)$. The second equation in U has a unique root for t in

$$\left(bb^{'2}/\left(b\left(2b^{'}-m\right)+\left(b^{'}-m\right)2\right), bo^{'2}/\left(b\left(2o^{'}-m\right)+\left(o^{'}-m\right)^2\right)\right].$$

If X is a unimodal random variable and has the support $[0,b]$ and $b^{'}\leq m$ then it is shown that

$$L \leq \mathrm{VaR}_p \leq U,$$

where

$$L = \begin{cases} \dfrac{m\nu_2 p}{\nu_2-\nu_1^2}, & \text{if } 0<p\leq \dfrac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}, \\ \text{solution for } t \text{ of} & \\ 1-p=\dfrac{(m-t)\left(\nu_2-\nu_1^2\right)}{(m-x_t)\left(\nu_2-2\nu_1x_t+x_t^2\right)} & \\ +\dfrac{(\nu_1-x_t)^2}{\nu_2-2\nu_1x_t+x_t^2}, & \text{if } \dfrac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}<p< \\ & \dfrac{b-\nu_1}{b+2m-3b^{'}}, \\ m-\dfrac{m-b^{'}}{b-\nu_1}\left[(1-p)\left(b-b^{'}\right)-\nu_1+b^{'}\right], & \text{if } \dfrac{b-\nu_1}{b+2m-3b^{'}}< \\ & p<\dfrac{b^2-\nu_1\left(m-b^{'}\right)}{\delta} \\ & -\dfrac{b\left(\nu_1-m+b^{'}\right)}{\delta}, \\ \text{solution for } t \text{ of} & \\ 1-p=\dfrac{(m-t)\left[\nu_2-\nu_1(b+f_t)+bf_t\right]}{mbf_t} & \\ +\dfrac{(m-t)(b\nu_1-\nu_2)}{(m-f_t)f_t(b-f_t)} & \\ +\dfrac{\nu_2-\nu_1f_t}{b(b-f_t)}, & \text{if } \dfrac{b^2-\nu_1\left(m-b^{'}\right)}{\delta} \\ & -\dfrac{b\left(\nu_1-m+b^{'}\right)}{\delta}< \\ & p<1-\dfrac{\nu_2-m\nu_1}{b(b-m)}, \\ \dfrac{\nu_2-(1-p)b(b-m)}{\nu_1}, & \text{if } 1-\dfrac{\nu_2-m\nu_1}{b(b-m)}<p<1, \end{cases}$$

and

$$U=\begin{cases} \dfrac{b\left(m+\nu_1\right)-\nu_2-(1-p)mb}{b-\nu_1}, & \text{if } 0<p\leq 1-\dfrac{(b+m)\nu_1-\nu_2}{mb}, \\ \text{solution for } t \text{ of} & \\ 1-p=\dfrac{\left(b\nu_1-\nu_2\right)\left(g_1-t\right)}{g_t\left(b-g_t\right)\left(g_t-m\right)} & \\ +\dfrac{\left(\nu_2-\nu_1 g_t\right)\left(b-t\right)}{b\left(b-g_t\right)\left(b-m\right)}, & \text{if } 1-\dfrac{(b+m)\nu_1-\nu_2}{mb} \\ & <p\leq 1-\dfrac{\nu_1^2 b\left(2o^{'}-m\right)}{\alpha\left(\nu_2-m\nu_1\right)} \\ & -\dfrac{\nu_1^2\left(o^{'}-m\right)^2}{\alpha\left(\nu_2-m\nu_1\right)} \\ & +\dfrac{\nu_1^3 bo^{'2}}{\nu_2\alpha\left(\nu_2-m\nu_1\right)}, \\ \dfrac{\nu_2}{\nu_1^3}\left[\nu_1^2-(1-p)\left(\nu_2-m\nu_1\right)\right], & \text{if } 1-\dfrac{\nu_1^2 b\left(2o^{'}-m\right)}{\alpha\left(\nu_2-m\nu_1\right)} \\ & -\dfrac{\nu_1^2\left(o^{'}-m\right)^2}{\alpha\left(\nu_2-m\nu_1\right)} \\ & +\dfrac{\nu_1^3 bo^{'2}}{\nu_2\alpha\left(\nu_2-m\nu_1\right)} \\ & <p\leq 1 \\ & -\dfrac{\nu_1^2\left(3o^{'}-2m\right)}{\left(\nu_2-m\nu_1\right)\left(3o^{'}-2m\right)} \\ & +\dfrac{\nu_1^3 o^{'}\left(2o^{'}-m\right)}{\beta}, \\ \text{solution for } t \text{ of} & \\ 1-p=\dfrac{\left(y_t-t\right)\left(\nu_2-\nu_1^2\right)}{\left(y_t-m\right)\left(y_t^2-2\nu_1 y_t+\nu_2\right)}, & \text{if } 1-\dfrac{\nu_1^2\nu_2\left(3o^{'}-2m\right)}{\beta} \\ & +\dfrac{\nu_1^3 o^{'}\left(2o^{'}-m\right)}{\beta} \\ & <p\leq 1-\dfrac{\left(\nu_2-\nu_1^2\right)\left(b-b^{'}\right)}{\gamma}, \\ b-(1-p)(b-m)\dfrac{b^2-2\nu_1 b+\nu_2}{\nu_2-\nu_1^2}, & \text{if } 1-\dfrac{\left(\nu_2-\nu_1^2\right)\left(b-b^{'}\right)}{\gamma} \\ & <p<1. \end{cases}$$

The first equation in L has a unique root for t in

$$\left(o^{'}\left(2o^{'}-m\right)/\left(3o^{'}-2m\right),\left(b\left(2b-m\right)-b^{'}m\right)/\left(3b-2m-b^{'}\right)\right].$$

The second equation in L has a unique root for t in

$$\left(m\nu_2/\left(\nu_2+2m\nu_1\right),\left(mb+mb^{'}-2b^{'2}\right)/\left(3b-2m-b^{'}\right)\right].$$

The first equation in U has a unique root for t in

$$\left(bm\left(b+m-2b^{'}\right)/\left(b\left(b+m-2b^{'}\right)+\left(m-b^{'}\right)^{2}\right),m\right].$$

The second equation in U has a unique root for t in

$$\left(m,bo^{'2}/\left(b\left(2o^{'}-m\right)+\left(o^{'}-m\right)^{2}\right)\right].$$

If X is a unimodal random variable and has the support $\mathbb{R}_+$ then it is shown that

$$L \leq \mathrm{VaR}_p \leq U,$$

where

$$L = \begin{cases} \dfrac{m\nu_2 p}{\nu_2-\nu_1^2}, & \text{if } 0 < p \leq \dfrac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}, \\ \text{solution for } t \text{ of} & \\ 1-p = \dfrac{(m-t)\left(\nu_2-\nu_1^2\right)}{(m-x_t)\left(\nu_2-2\nu_1 x_t+x_t^2\right)} & \\ +\dfrac{(\nu_1-x_t)^2}{\nu_2-2\nu_1 x_t+x_t^2}, & \text{if } \dfrac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1} < p < \\ & \dfrac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2}, \\ \nu_1-\dfrac{1}{2}(1-p)(\nu_1-m) & \\ \dfrac{1}{2}\sqrt{(1-p)^2\left(\nu_1-m\right)^2+4(1-p)\left(\nu_2-\nu_1^2\right)}, & \text{if } \dfrac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2} \\ & < p < 1, \end{cases}$$

and

$$U = \begin{cases} p\nu_1+(1-p)m, & \text{if } 0 < p \leq \dfrac{\nu_1-m}{2\nu_1-m}, \\ \dfrac{\nu_1^2+(1-p)^2m^2+2m\nu_1(1-p)}{4\nu_1(1-p)}, & \text{if } 1-\dfrac{\nu_1^2}{2\nu_2-m\nu_1} < p \\ & \leq 1-\dfrac{\nu_1^2}{3\nu_2-2m\nu_1}, \\ \text{solution for } t \text{ of} & \\ 1-p = \dfrac{(y_t-t)\left(\nu_2-\nu_1^2\right)}{(y_t-m)\left(y_t^2-2\nu_1 y_t+\nu_2\right)}, & 1-\dfrac{\nu_1^2}{3\nu_2-2m\nu_1} < p < 1. \end{cases}$$

The first equation in L has a unique root for t in $(m\nu_2/(\nu_2+2m\nu_1),m]$. The first equation in U has a unique root for t in

$$(\nu_2(2\nu_2-m\nu_1)/(\nu_1(3\nu_2-2m\nu_1)),\infty).$$