Homework 8, STAT 5261

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1 Question 1

Let X be a random variable and let $\mathrm{VaR}(\alpha)(X)$ be the value at risk corresponding to X. Show that

(a) Translation Invariance:

$$\mathrm{VaR}(\alpha)(X+a) = \mathrm{VaR}(\alpha)(X) + a, \quad \forall a \in \mathbb{R} \tag{1}$$

(b) Positive Homogeneity:

$$\mathrm{VaR}(\alpha)(\lambda X) = \lambda\, \mathrm{VaR}(\alpha)(X), \quad \forall \lambda \geq 0 \tag{2}$$

Answer

In the book def 19.2, we have that for a random variable X, the Value at Risk at level $\alpha \in (0,1)$ is defined by $\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$, where F_X is the distribution function of X.

Part a

Let $a \in \mathbb{R}$. The distribution function of $X + a$ satisfies $F_{X+a}(x) = P(X + a \leq x) = P(X \leq x - a) = F_X(x - a)$. Thus $\mathrm{VaR}_\alpha(X + a) = \inf\{x : F_{X+a}(x) \geq \alpha\} = \inf\{x : F_X(x - a) \geq \alpha\}$. Let $t = x - a$ so that $x = t + a$. Then $\mathrm{VaR}_\alpha(X + a) = \inf\{t + a : F_X(t) \geq \alpha\} = a + \inf\{t : F_X(t) \geq \alpha\}$. Hence $\mathrm{VaR}_\alpha(X + a) = \mathrm{VaR}_\alpha(X) + a$.

Part b

Let $\lambda \geq 0$. For $\lambda > 0$, $F_{\lambda X}(x) = P(\lambda X \leq x) = P\big(X \leq x/\lambda\big) = F_X(x/\lambda)$. Therefore $\mathrm{VaR}_\alpha(\lambda X) = \inf\{x : F_{\lambda X}(x) \geq \alpha\} = \inf\{x : F_X(x/\lambda) \geq \alpha\}$. Let $t = x/\lambda$ so $x = \lambda t$. Then $\mathrm{VaR}_\alpha(\lambda X) = \inf\{\lambda t : F_X(t) \geq \alpha\} = \lambda \inf\{t : F_X(t) \geq \alpha\}$. Thus $\mathrm{VaR}_\alpha(\lambda X) = \lambda\, \mathrm{VaR}_\alpha(X)$.

If $\lambda = 0$, then $0X = 0$ almost surely, hence $\mathrm{VaR}_\alpha(0) = 0 = 0 \cdot \mathrm{VaR}_\alpha(X)$. Therefore the result holds for all $\lambda \geq 0$.

2 Question 2

Suppose the return R on a stock satisfies

$$R = \mu + \lambda Y$$

where μ and λ are fixed and Y has a t-distribution with ν degrees of freedom.

(a) If you hold a position of size S_0 in this stock, show that for one day

$$\mathrm{VaR}(\alpha) = -S_0 \left(\mu + \lambda t_{\alpha,\nu}\right)$$

where $t_{\alpha,\nu}$ is the α th quantile of a t-distribution with ν degrees of freedom. Hint: Recall that $\Pr(L > \mathrm{VaR}(\alpha)) = \alpha$ and $L = -S_0 R$.

(b) If $S_0 = 100,000, \mu = 0.4$, and $\lambda = 0.01$, compute VaR(0.05) when $\nu = 10$.

Answer

Part a

One-day return is given by $R = \mu + \lambda Y$ with $Y \sim t_\nu$. The one-day loss is therefore $L = -S_0 R = -S_0(\mu + \lambda Y)$. Assuming $S_0 > 0$ and $\lambda > 0$, the function L is strictly decreasing in Y.

By definition, $\mathrm{VaR}(\alpha)$ satisfies the equation $\Pr(L > \mathrm{VaR}(\alpha)) = \alpha$. Start by expanding the event: $\Pr(L > \mathrm{VaR}(\alpha)) = \Pr\bigl(-S_0(\mu+\lambda Y) > \mathrm{VaR}(\alpha)\bigr)$. Since $S_0 > 0$, dividing both sides by $-S_0$ reverses the inequality, giving $\Pr\bigl(-S_0(\mu+\lambda Y) > \mathrm{VaR}(\alpha)\bigr) = \Pr\bigl(\mu + \lambda Y < -\mathrm{VaR}(\alpha)/S_0\bigr)$.

Subtract μ from the right-hand side: $\Pr\bigl(\mu + \lambda Y < -\mathrm{VaR}(\alpha)/S_0\bigr) = \Pr\bigl(\lambda Y < -\mathrm{VaR}(\alpha)/S_0 - \mu\bigr)$. Because $\lambda > 0$, dividing by λ preserves the direction of the inequality: $\Pr\bigl(\lambda Y < -\mathrm{VaR}(\alpha)/S_0 - \mu\bigr) = \Pr\bigl(Y < (-\mathrm{VaR}(\alpha)/S_0 - \mu)/\lambda\bigr)$. Returning to the defining condition $\Pr(L > \mathrm{VaR}(\alpha)) = \alpha$, we now have $\Pr\bigl(Y < (-\mathrm{VaR}(\alpha)/S_0 - \mu)/\lambda\bigr) = \alpha$. Since $Y \sim t_\nu$, let $t_{\alpha,\nu}$ denote the α-quantile of the t_ν distribution, so that $\Pr(Y < t_{\alpha,\nu}) = \alpha$. Matching the two expressions yields $(-\mathrm{VaR}(\alpha)/S_0 - \mu)/\lambda = t_{\alpha,\nu}$.

Now solve for $\mathrm{VaR}(\alpha)$: $-\mathrm{VaR}(\alpha)/S_0 - \mu = \lambda t_{\alpha,\nu}$, so $-\mathrm{VaR}(\alpha)/S_0 = \mu + \lambda t_{\alpha,\nu}$ and multiplying by $-S_0$ gives $\mathrm{VaR}(\alpha) = -S_0(\mu + \lambda t_{\alpha,\nu})$.

Thus the Value at Risk is $\mathrm{VaR}(\alpha) = -S_0(\mu + \lambda t_{\alpha,\nu})$.

Part b

For $S_0 = 100000$, $\mu = 0.4$, $\lambda = 0.01$, and $\nu = 10$, we use $t_{0.05,10} \approx -1.812$. Substitution into the expression from part (a) gives $\mathrm{VaR}(0.05) = -100000(0.4 + 0.01\, t_{0.05,10}) = -100000(0.4 - 0.01812) \approx -38188$. Thus the 5% one-day VaR is approximately -3.82×10^4, meaning a potential loss of about \$38,200 at the 5% tail.

3 Question 3

Suppose the daily returns (R_A, R_B) on Stocks A and B have a bivariate normal distribution with

$$\boldsymbol{\mu} = \begin{pmatrix} 0.0002 \\ 0.0003 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.0003 & 0.0002 \\ 0.0002 & 0.0004 \end{pmatrix}.$$

This implies that

$$R_A \sim N(0.0002, 0.0003), \quad R_B \sim N(0.0003, 0.0004),$$

and for any a, b,

$$aR_A + bR_B \sim N\left(0.0002a + 0.0003b, 0.0003a^2 + 0.0004b^2 + 0.0004ab\right)$$

(a) Suppose that you hold a \$1000 position in Stock A (i.e. $S_0 = 1000$). Compute $\mathrm{VaR}_A(0.05)$.

(b) Suppose that you hold a \$1000 position in Stock B (i.e. $S_0 = 1000$). Compute $\mathrm{VaR}_B(0.05)$.

(c) What is VaR(0.05) of a portfolio holding \$500 in Stock A and \$500 in Stock B?

Answer

By definition, the (oneday) $\mathrm{VaR}_{0.05}$ is defined as the 5% lower quantile of the portfolio value change (reported as a positive loss). Let R be the (random) return and S_0 the initial position. Then we have:

$$S_1 = S_0(1+R), \qquad \Delta S = S_1 - S_0 = S_0 R, \qquad L = -\Delta S = -S_0 R.$$

If $q_{0.05}(R)$ is the 5%-quantile of R, then:

$$\mathrm{VaR}_{0.05} = -S_0\, q_{0.05}(R).$$

Throughout the question, use $z_{0.05} \approx -1.645$ for the standard normal quantile.

<u>Part a</u>

We have

$$R_A \sim N(\mu_A, \sigma_A^2), \quad \mu_A = 0.0002, \quad \sigma_A^2 = 0.0003, \quad \sigma_A = \sqrt{0.0003} \approx 0.01732.$$

Thus

$$q_{0.05}(R_A) = \mu_A + z_{0.05}\sigma_A \approx 0.0002 + (-1.645) \cdot 0.01732 \approx 0.0002 - 0.02849 \approx -0.02829.$$

Therefore

$$\mathrm{VaR}_A(0.05) = -1000\, q_{0.05}(R_A) \approx -1000 \cdot (-0.02829) \approx 28.3.$$

<u>Part b</u>

We have

$$R_B \sim N(\mu_B, \sigma_B^2), \quad \mu_B = 0.0003, \quad \sigma_B^2 = 0.0004, \quad \sigma_B = \sqrt{0.0004} = 0.02.$$

Thus

$$q_{0.05}(R_B) = \mu_B + z_{0.05}\sigma_B \approx 0.0003 + (-1.645) \cdot 0.02 \approx 0.0003 - 0.0329 \approx -0.0326.$$

Hence

$$\mathrm{VaR}_B(0.05) = -1000\, q_{0.05}(R_B) \approx 32.6.$$

<u>Part c</u>

The total initial value is $S_0 = 1000$. Weights in A and B are

$$w_A = \frac{500}{1000} = 0.5, \qquad w_B = 0.5.$$

With means $w = (1/2, 1/2)^T$, and the portfolio return is

$$R_p = w_A R_A + w_B R_B = 0.5R_A + 0.5R_B.$$

From the given joint normal $R_p \sim N(\mu_p, \sigma_p^2)$, with $\mu_p = 0.0002 \cdot 0.5 + 0.0003 \cdot 0.5 = 0.0001 + 0.00015 = 0.00025$ and variance is:

$$\sigma_p^2 = 0.0003a^2 + 0.0004b^2 + 0.0004ab \quad \text{with } a = b = 0.5.$$

So we have:

$$\sigma_p^2 = 0.0003 \cdot 0.25 + 0.0004 \cdot 0.25 + 0.0004 \cdot 0.25 = 0.000075 + 0.0001 + 0.0001 = 0.000275,$$

$$\sigma_p = \sqrt{0.000275} \approx 0.0166.$$

Thus:

$$q_{0.05}(R_p) = \mu_p + z_{0.05}\sigma_p \approx 0.00025 + (-1.645) \cdot 0.0166 \approx 0.00025 - 0.0273 \approx -0.0270.$$

Hence we get:

$$\mathrm{VaR}_p(0.05) = -S_0\, q_{0.05}(R_p) \approx -1000 \cdot (-0.0270) \approx 27.0.$$

4 Question 4

Suppose the distribution of R has a pdf f. Show that

$$\text{ES}(\alpha) = -S_0 \frac{\int_{-\infty}^{q_\alpha} r f(r) dr}{\alpha}$$

where q_α is the α th quantile of the distribution of R.

Answer

Let R be the portfolio return with pdf f and cdf F, and initial portfolio value $S_0 > 0$. Define the loss $L = -S_0 R$. Let q_α be the αth quantile of R, i.e. $F(q_\alpha) = \Pr(R \le q_\alpha) = \alpha$.

The α-VaR of the loss L is $\text{VaR}_\alpha(L) = \inf\{x : \Pr(L \le x) \ge \alpha\}$. Since $L = -S_0 R$ is strictly decreasing in R, we have $\Pr(L \le x) = \Pr(-S_0 R \le x) = \Pr(R \ge -x/S_0)$. Thus the α-quantile of L corresponds to the α-quantile of R, and hence $\text{VaR}_\alpha(L) = -S_0 q_\alpha$.

The Expected Shortfall at level α is $\text{ES}(\alpha) = E[L \mid L \ge \text{VaR}_\alpha(L)]$. Using $L = -S_0 R$ and $\text{VaR}_\alpha(L) = -S_0 q_\alpha$, the event $\{L \ge \text{VaR}_\alpha(L)\}$ becomes $\{-S_0 R \ge -S_0 q_\alpha\} = \{R \le q_\alpha\}$. Hence:

$$\text{ES}(\alpha) = E[-S_0 R \mid R \le q_\alpha] = -S_0\, E[R \mid R \le q_\alpha].$$

Now we have:

$$E[R \mid R \le q_\alpha] = \frac{E[R\, \mathbf{1}_{\{R \le q_\alpha\}}]}{\Pr(R \le q_\alpha)} = \frac{\int_{-\infty}^{q_\alpha} r\, f(r)\, dr}{F(q_\alpha)}.$$

Since $F(q_\alpha) = \alpha$, we obtain $E[R \mid R \le q_\alpha] = \left(\int_{-\infty}^{q_\alpha} r\, f(r)\, dr\right)/\alpha$.

Therefore:

$$\text{ES}(\alpha) = -S_0\, E[R \mid R \le q_\alpha] = -S_0\, \frac{\int_{-\infty}^{q_\alpha} r\, f(r)\, dr}{\alpha},$$

thus we prove that:

$$\text{ES}(\alpha) = -S_0\, \frac{\int_{-\infty}^{q_\alpha} r\, f(r)\, dr}{\alpha}.$$

5 Question 5

Assume $R \sim N\left(\mu, \sigma^2\right)$. Show that

$$\mathrm{ES}(\alpha) = -S_0\left[\mu - \sigma \frac{1}{\alpha\sqrt{2\pi}} e^{-z_\alpha^2/2}\right]$$

where z_α is the α th quantile of the standard normal distribution $N(0,1)$.

Answer

Assume that $R \sim N(\mu, \sigma^2)$ and $S_0 > 0$. From the previous result, the Expected Shortfall at level α is $\mathrm{ES}(\alpha) = -S_0\big(\int_{-\infty}^{q_\alpha} r f_R(r)\, dr\big)/\alpha$, where q_α is the α-quantile of R.

Since $R = \mu + \sigma Z$ with $Z \sim N(0,1)$, the α-quantile of R is $q_\alpha = \mu + \sigma z_\alpha$, where z_α is the α-quantile of $N(0,1)$. Let the standard normal pdf and cdf be $\varphi(z) = (1/\sqrt{2\pi})e^{-z^2/2}$ and $\Phi(z) = \int_{-\infty}^{z} \varphi(x)\, dx$.

The pdf of R is $f_R(r) = (1/\sigma)\varphi((r-\mu)/\sigma)$. Consider the integral $I = \int_{-\infty}^{q_\alpha} r f_R(r)\, dr$. With the change of variables $x = (r-\mu)/\sigma$, so that $r = \mu + \sigma x$, $dr = \sigma\, dx$, the upper limit becomes $x = z_\alpha$. Thus $I = \int_{-\infty}^{z_\alpha} (\mu + \sigma x)\varphi(x)\, dx$. Splitting the integral gives $I = \mu \int_{-\infty}^{z_\alpha} \varphi(x)\, dx + \sigma \int_{-\infty}^{z_\alpha} x\varphi(x)\, dx$.

The first term equals $\mu\Phi(z_\alpha) = \mu\alpha$. For the second, using $d(-\varphi(x))/dx = x\varphi(x)$, we obtain $\int_{-\infty}^{z_\alpha} x\varphi(x)\, dx = -\varphi(z_\alpha)$. Hence $I = \mu\alpha - \sigma\varphi(z_\alpha)$. Dividing by α gives $(1/\alpha)I = \mu - \sigma\varphi(z_\alpha)/\alpha$. Substituting $\varphi(z_\alpha) = (1/\sqrt{2\pi})e^{-z_\alpha^2/2}$ yields that:

$$\mathrm{ES}(\alpha) = -S_0\left[\mu - \sigma \frac{1}{\alpha\sqrt{2\pi}} e^{-z_\alpha^2/2}\right].$$

Thus the identity $\mathrm{ES}(\alpha) = -S_0\left[\mu - \sigma(\alpha^{-1}\sqrt{2\pi}^{-1})e^{-z_\alpha^2/2}\right]$ is established.