

Homework 4, MATH 5261

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1 Question 1

Suppose that you have an asset that pays a dividend at the end of each period and let $1 + R_t(k)$ be the multiple gross return of k periods. Show that

$$1 + R_t(k) = \left(\frac{P_t + D_t}{P_{t-1}} \right) \left(\frac{P_{t-1} + D_{t-1}}{P_{t-2}} \right) \cdots \left(\frac{P_{t-k+1} + D_{t-k+1}}{P_{t-k}} \right)$$

That is, show that it is equal to the product of single period gross returns.

Answer

Let P_t denote the ex-dividend price of the asset at the end of period t , and D_t the dividend paid at the end of period t .

The single-period gross return is defined as

$$1 + R_t(1) = \frac{P_t + D_t}{P_{t-1}}.$$

The k -period gross return, with dividends reinvested each period, is the product of the single-period gross returns:

$$1 + R_t(k) = \prod_{j=0}^{k-1} (1 + R_{t-j}(1)).$$

Substituting the definition of the single-period return, we obtain

$$\begin{aligned} 1 + R_t(k) &= \prod_{j=0}^{k-1} \frac{P_{t-j} + D_{t-j}}{P_{t-j-1}} \\ &= \left(\frac{P_t + D_t}{P_{t-1}} \right) \left(\frac{P_{t-1} + D_{t-1}}{P_{t-2}} \right) \cdots \left(\frac{P_{t-k+1} + D_{t-k+1}}{P_{t-k}} \right). \end{aligned}$$

Hence, the k -period gross return is equal to the product of the single-period gross returns:

$$1 + R_t(k) = \left(\frac{P_t + D_t}{P_{t-1}} \right) \left(\frac{P_{t-1} + D_{t-1}}{P_{t-2}} \right) \cdots \left(\frac{P_{t-k+1} + D_{t-k+1}}{P_{t-k}} \right).$$

Alternative proof (by induction):

For $k = 1$, the formula holds trivially. Assume it holds for some k . Then,

$$\begin{aligned} 1 + R_t(k+1) &= (1 + R_t(1))(1 + R_{t-1}(k)) \\ &= \frac{P_t + D_t}{P_{t-1}} \prod_{j=1}^k \frac{P_{t-j} + D_{t-j}}{P_{t-j-1}} \\ &= \prod_{j=0}^k \frac{P_{t-j} + D_{t-j}}{P_{t-j-1}}, \end{aligned}$$

so the formula holds for $k+1$. By induction, it is true for all $k \geq 1$.

2 Question 2

Do problems 7 and 8 on page 17.

2.1 Problem 7

Let r_t be a log return. Suppose that r_1, r_2, \dots are i.i.d. $N(0.06, 0.47)$.

- (a) What is the distribution of $r_t(4) = r_t + r_{t-1} + r_{t-2} + r_{t-3}$?
- (b) What is $P\{r_1(4) < 2\}$?
- (c) What is the covariance between $r_2(1)$ and $r_2(2)$?
- (d) What is the conditional distribution of $r_t(3)$ given $r_{t-2} = 0.6$?

2.2 Problem 8

Suppose that X_1, X_2, \dots is a lognormal geometric random walk with parameters (μ, σ^2) . More specifically, suppose that

$$X_k = X_0 \exp(r_1 + \dots + r_k),$$

where X_0 is a fixed constant and r_1, r_2, \dots are i.i.d. $N(\mu, \sigma^2)$.

- (a) Find $P(X_2 > 1.3X_0)$.
- (b) Use (A.4) to find the density of X_1 .
- (c) Find a formula for the 0.9 quantile of X_k for all k .
- (d) What is the expected value of X_k^2 for any k ? (Find a formula giving the expected value as a function of k .)
- (e) Find the variance of X_k for any k .

3 Question 3

If X is a continuous random variable with a strictly increasing distribution function F , find the distribution of $U = F(X)$ (show all your work to get a full credit)

Answer

Because F is continuous and strictly increasing on \mathbb{R} , it is bijective from \mathbb{R} onto $(0, 1)$ with a (continuous, strictly increasing) inverse $F^{-1} : (0, 1) \rightarrow \mathbb{R}$.

For any $u \in \mathbb{R}$, the cumulative distribution function (CDF) of U is

$$F_U(u) = \mathbb{P}(U \leq u) = \mathbb{P}(F(X) \leq u).$$

We consider three cases.

(i) If $u \leq 0$: Since $F(X) \in (0, 1)$ almost surely, $\mathbb{P}(F(X) \leq u) = 0$, so $F_U(u) = 0$.

(ii) If $0 < u < 1$: Because F is strictly increasing,

$$\{F(X) \leq u\} \iff \{X \leq F^{-1}(u)\}.$$

Hence

$$F_U(u) = \mathbb{P}(X \leq F^{-1}(u)) = F(F^{-1}(u)) = u.$$

(iii) If $u \geq 1$: Again $F(X) \in (0, 1)$ a.s., so $\mathbb{P}(F(X) \leq u) = 1$, i.e. $F_U(u) = 1$.

Combining, we have

$$F_U(u) = \begin{cases} 0, & u \leq 0, \\ u, & 0 < u < 1, \\ 1, & u \geq 1, \end{cases}$$

which is exactly the CDF of the $\text{Unif}(0, 1)$ distribution. Therefore,

$$U = F(X) \sim \text{uniform}(0, 1).$$

4 Question 4

Let X have a normal distribution with mean μ and variance σ^2 and let $Y = e^X$. Y is said to have a lognormal distribution with parameters μ and σ^2 (since $X = \log(Y)$ has a normal distribution). Find the density $f_Y(y)$. (Hint: compute $F_Y(y) = P(Y \leq y)$)

Answer

For $y > 0$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = \mathbb{P}(X \leq \ln y) = \Phi\left(\frac{\ln y - \mu}{\sigma}\right),$$

where Φ is the standard normal CDF. For $y \leq 0$, clearly $F_Y(y) = 0$.

Differentiate for $y > 0$ to obtain the PDF:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = \phi\left(\frac{\ln y - \mu}{\sigma}\right) \cdot \frac{1}{\sigma} \cdot \frac{1}{y} \\ &= \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma^2}\right), \quad y > 0, \end{aligned}$$

and $f_Y(y) = 0$ for $y \leq 0$, where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard normal PDF.

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma^2}\right) \text{ for } y > 0; \quad f_Y(y) = 0 \text{ for } y \leq 0.$$