Homework 3, MATH 5261

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1 Question 1

1.1 Problem 1

Suppose that there are two risky assets, A and B, with expected returns equal to 2.3% and 4.5%, respectively. Suppose that the standard deviations of the returns are 6% and 11% and that the returns on the assets have a correlation of 0.17.

- (a) What portfolio of A and B achieves a 3 % rate of expected return?
- (b) What portfolios of A and B achieve a 5.5% standard deviation of return? Among these, which has the largest expected return?

Answer
We have ω be the fraction of the portfolio to invest in the A, that is, $E(R_p) = \omega\mu_A + (1-\omega)\mu_B$.
In this case, we have the rate of expected return to be 3%, which is $E(R_P) = 3$, thus $\omega\mu_A + (1-\omega)\mu_B = 3$

$$\omega\, 0.023 + (1-\omega)\, 0.045 = 0.03$$
$$-0.22\omega = -0.015$$

$$\omega = 0.06818$$

thus we have $\omega = 0.681818$, the total portfolio we invest in asset A, and $1-\omega = 0.318182$, the total portfolio we invest in asset B.

Part b

For two risky assets, the variance of the portfolio return is given by:

$$\sigma_R^2 = \omega^2\sigma_A^2 + (1-\omega)^2\sigma_B^2 + 2\omega(1-\omega)\rho_{AB}\sigma_A\sigma_B.$$

This combines the variances of each asset with their covariance.
Using the problems data (standard deviations and correlation), this becomes

$$\sigma_R^2 = 6\omega^2 + 11(1-\omega)^2 + 2.762173\,\omega(1-\omega).$$

Then we solve the quadratic equation

$$\sigma_R^2 = 5.5.$$

The quadratic equation in ω yields two possible solutions:

$$\omega_1 = 0.4107773, \quad \omega_2 = 0.9403999.$$

These represent two possible allocations to asset A (and $1-\omega$ to asset B) that achieve the same target variance.
The portfolios expected return is:

$$E(R_p) = \omega\mu_A + (1-\omega)\mu_B.$$

For $\omega_1 \approx 0.4108$:

$$E(R_p) \approx 3.5963\%.$$

For $\omega_2 \approx 0.9404$:

$$E(R_p) \approx 2.4311\%.$$

1.2 Problem 2

Suppose there are two risky assets, C and D, the tangency portfolio is 65% C and 35% D, and the expected return and standard deviation of the return on the tangency portfolio are 5% and 7%, respectively. Suppose also that the risk-free rate of return is 1.5%. If you want the standard deviation of your return to be 5%, what proportions of your capital should be in the risk-free asset, asset C, and asset D?

Answer

Let y be the share put in the tangency portfolio P and $1 - y$ in the risk-free asset. Because $\sigma_{rf} = 0$, the portfolio volatility is

$$\sigma = y\,\sigma_P \quad \Rightarrow \quad y = \frac{\sigma}{\sigma_P} = \frac{0.05}{0.07} = \frac{5}{7} \approx 0.7142857.$$

Inside P, the risky weights are 65% in C and 35% in D. Therefore the overall allocations are:

$$\begin{aligned} w_{rf} &= 1 - y = \frac{2}{7} \approx 28.571\%, \\ w_C &= y \times 0.65 = \frac{13}{28} \approx 46.429\%, \\ w_D &= y \times 0.35 = \frac{1}{4} = 25.000\%. \end{aligned}$$

2 Question 2

Let $f(x, y)$ be a continuous and differentiable function of x and y. The function f is said to be homogeneous of degree one if

$$f(cx, cy) = cf(x, y).$$

Eulers theorem states that if f is homogeneous of degree one, then

$$f(x, y) = x\frac{\partial}{\partial x}f(x, y) + y\frac{\partial}{\partial y}f(x, y).$$

Let

$$\mu_P(x, y) = x\mu_A + y\mu_B, \quad \sigma_P(x, y) = \sqrt{x^2\sigma_A^2 + y^2\sigma_B^2 + 2xy\sigma_{AB}}.$$

1. (3 pt) Show that $\mu_P(x, y)$ and $\sigma_P(x, y)$ are homogeneous of degree 1 (for $\sigma_P(x, y)$ assume $c \geq 0$).
2. (3 pt) In a portfolio where the portion x is invested in asset A and the portion y is invested in asset B, the partial derivatives

$$\frac{\partial}{\partial x}\sigma_P(x, y) \quad \text{and} \quad \frac{\partial}{\partial y}\sigma_P(x, y),$$

are called the marginal contributions to risk of A and B, respectively. The contributions to risk of assets A and B are given by

$$x\frac{\partial}{\partial x}\sigma_P(x, y) \quad \text{and} \quad y\frac{\partial}{\partial y}\sigma_P(x, y),$$

respectively. Find the expression of the marginal risks and the contribution to risk of assets A and B.

Answer
Part (a)
Claim
$\mu_P(x, y)$ and $\sigma_P(x, y)$ are homogeneous of degree 1 (for σ_P assume $c \geq 0$).

Proof
For any scalar c,

$$\mu_P(cx, cy) = cx\,\mu_A + cy\,\mu_B = c(x\mu_A + y\mu_B) = c\,\mu_P(x, y),$$

so μ_P is homogeneous of degree 1.
For σ_P,

$$\sigma_P(cx, cy) = \sqrt{(cx)^2\sigma_A^2 + (cy)^2\sigma_B^2 + 2(cx)(cy)\sigma_{AB}} = \sqrt{c^2\left(x^2\sigma_A^2 + y^2\sigma_B^2 + 2xy\sigma_{AB}\right)}.$$

Hence $\sigma_P(cx, cy) = |c|\,\sigma_P(x, y)$; in particular, if $c \geq 0$ then $\sigma_P(cx, cy) = c\,\sigma_P(x, y)$. Thus σ_P is (positively) homogeneous of degree 1.
Part (b)
Let: $\sigma_P(x, y) = \sqrt{x^2\sigma_A^2 + y^2\sigma_B^2 + 2xy\,\sigma_{AB}}$.
Marginal contributions to risk

$$\frac{\partial\sigma_P}{\partial x} = \frac{x\sigma_A^2 + y\sigma_{AB}}{\sigma_P}, \qquad \frac{\partial\sigma_P}{\partial y} = \frac{y\sigma_B^2 + x\sigma_{AB}}{\sigma_P}.$$

Risk contributions
They can be expressed as below:

$$\mathrm{RC}_A = x\,\frac{\partial\sigma_P}{\partial x} = \frac{x^2\sigma_A^2 + xy\,\sigma_{AB}}{\sigma_P}, \qquad \mathrm{RC}_B = y\,\frac{\partial\sigma_P}{\partial y} = \frac{y^2\sigma_B^2 + xy\,\sigma_{AB}}{\sigma_P}.$$

3 Question 3

The annual estimates of the parameters for Boeing (B) and Microsoft (M) stocks are given below:

$$\mu_B = 0.1492, \qquad \mu_M = 0.3308, \qquad \sigma_B^2 = 0.0695, \qquad \sigma_M^2 = 0.1369, \qquad \rho_{BM} = -0.0083.$$

Assume a risk–free rate of 6% per year for the T–bill (risk–free rate).

1. (3 pt) Use the Lagrange multiplier method to derive the minimum variance portfolio.
2. (3 pt) Find the tangency portfolio and compute its mean and risk.
3. (4 pt) Suppose you desire a portfolio with an expected return of 20%. What should be the weights of this portfolio if you only use risky assets? What is its risk equal to?
4. (4 pt) Suppose you desire a portfolio with a risk of 20%. What should be the weights of this portfolio if you use risky and risk–free assets? What is its expected return equal to?

Answer

Part (a)

Given

$$\mu_B = 0.1492, \quad \mu_M = 0.3308, \quad \sigma_B^2 = 0.0695, \quad \sigma_M^2 = 0.1369, \quad \rho_{BM} = -0.0083, \quad r_f = 0.06.$$

We can calculate:

$$\sigma_B = \sqrt{0.0695} \approx 0.26363, \quad \sigma_M = \sqrt{0.1369} = 0.37, \quad \Sigma = \begin{pmatrix} 0.0695 & \sigma_{BM} \\ \sigma_{BM} & 0.1369 \end{pmatrix}, \ \sigma_{BM} = \rho_{BM}\sigma_B\sigma_M \approx -0.0008096.$$

By Lagrange multipliers, we can calculate the MVP as below:

$$\min_{w_B, w_M} \ w^\top \Sigma w \quad \text{s.t.} \quad \mathbf{1}^\top w = 1.$$

Lagrangian $\mathcal{L} = w^\top \Sigma w - \lambda(\mathbf{1}^\top w - 1)$. FOC $\Rightarrow 2\Sigma w - \lambda \mathbf{1} = 0 \Rightarrow w = \frac{\lambda}{2}\Sigma^{-1}\mathbf{1}$. Impose $\mathbf{1}^\top w = 1$ to get

$$w^{\text{MVP}} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}.$$

In this case we have:

$$w_B^{\text{MVP}} = \frac{\sigma_M^2 - \sigma_{BM}}{\sigma_B^2 + \sigma_M^2 - 2\sigma_{BM}}, \qquad w_M^{\text{MVP}} = 1 - w_B^{\text{MVP}}.$$

$$w_B^{\text{MVP}} \approx 0.6620, \qquad w_M^{\text{MVP}} \approx 0.3380.$$

Mean and risk:

$$\mu_{\text{MVP}} = w_B^{\text{MVP}}\mu_B + w_M^{\text{MVP}}\mu_M = 0.662 \cdot 0.1492 + 0.338 \cdot 0.3308 \approx 0.21058$$

$$\sigma_{\text{MVP}} = \sqrt{(w_B^{\text{MVP}})^2\sigma_B^2 + (w_M^{\text{MVP}})^2\sigma_M^2 + 2w_B^{\text{MVP}}w_M^{\text{MVP}}\sigma_{BM}} \approx 0.2139$$

Part (b)

With excess returns $\mu - r_f\mathbf{1}$, tangency weights (over risky assets) are

$$w^T = \frac{\Sigma^{-1}(\mu - r_f\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r_f\mathbf{1})}.$$

Then we have:

$$w_B^T \approx 0.39685, \qquad w_M^T \approx 0.60315.$$

Mean and risk:

$$\mu_T = (w^T)^\top \mu \approx 0.2587, \qquad \sigma_T = \sqrt{(w^T)^\top \Sigma w^T} \approx 0.2457$$

Sharpe $\approx (\mu_T - r_f)/\sigma_T \approx 0.809$.

Part (c)

Solve the two-asset mean line:

$$w_B = \frac{\mu_M - \mu^*}{\mu_M - \mu_B}, \quad w_M = 1 - w_B, \quad \mu^* = 0.20.$$

$$w_B \approx 0.72026, \quad w_M \approx 0.27974,$$

$$\sigma_p = \sqrt{w_B^2 \sigma_B^2 + w_M^2 \sigma_M^2 + 2 w_B w_M \sigma_{BM}} \approx 0.2155.$$

Part (d)

Mix the tangency portfolio P with T-bill. Let y be the share in P (so $1 - y$ in T-bill). Since $\sigma_{rf} = 0$:

$$y = \frac{\sigma^*}{\sigma_T} = \frac{0.20}{0.2457} \approx 0.81405, \qquad w_{rf} = 1 - y \approx 0.18595.$$

Overall weights:

$$w_B = y\, w_B^T \approx 0.81405 \times 0.39685 \approx 0.32306, \qquad w_M = y\, w_M^T \approx 0.81405 \times 0.60315 \approx 0.490996.$$

Expected return:

$$E[R] = r_f + y(\mu_T - r_f) = 0.06 + 0.81405\,(0.2587 - 0.06) \approx 0.22178.$$