Homework 2, STAT 5261

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1 Question 0

1.1 Part 1

Use the plot to estimate graphically the yield to maturity. Does this estimate agree with that from spline interpolation?

As an alternative to interpolation, the yield to maturity can be found using a nonlinear root finder (equation solver) such as `uniroot()`, which is illustrated here:

`uniroot(function(r) r^2 - .5, c(0.7, 0.8))`

Answer

First we use the sample code to run the result:

```
bondvalue = function(c, T, r, par)
{
   #       Computes bv = bond values (current prices) corresponding
   #       to all values of yield to maturity in the
   #       input vector r
   #
   # INPUT
   #        c = coupon payment (semiannual)
   #        T = time to maturity (in years)
   #        r = vector of yields to maturity (semiannual rates)
   #        par = par value
   #
   bv = c / r + (par - c / r) * (1 + r)^(-2 * T)
   bv
}

price = 1200
C = 40
T= 30
par = 1000
#   current price of the bond
#   coupon payment
#   time to maturity
#   par value of the bond
r = seq(0.02, 0.05, length = 300)
value = bondvalue(C, T, r, par)
yield2M = spline(value, r, xout = price) # spline interpolation
yield2M
```

The answer gives `0.03239813`

And we can see that the price-YTM curve is decreasing and convex; reading the graph should give ≈ 3.2% - 3.3% semiannual, which also matches the spline interpolation output (0.0324).

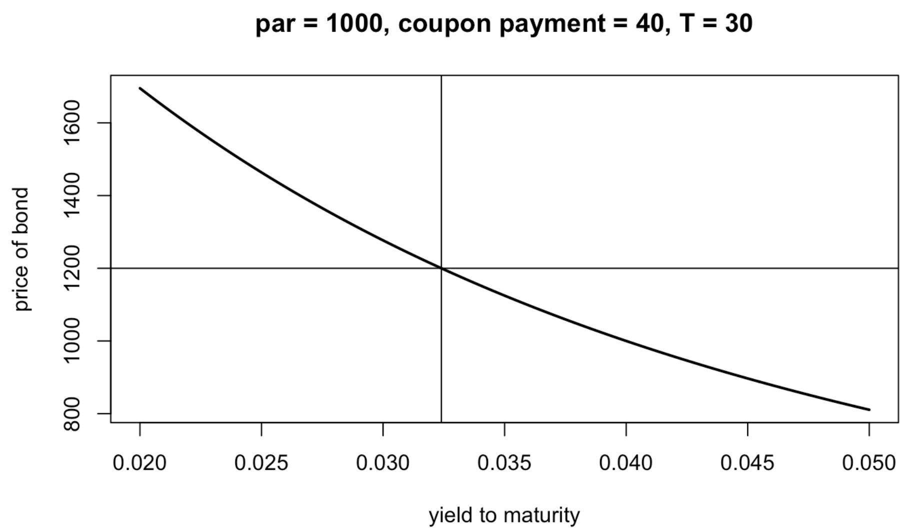

Figure 1: Price of Bond and YTM

1.2 Part 2

What does the code `uniroot(function(r) r^2 - 0.5, c(0.7, 0.8))` do?

Answer

The function being solved in this case is:

$$f(r) = r^2 - 0.5.$$

The command `uniroot` searches for a root of $f(r)$ in the interval $[0.7, 0.8]$. That is, it finds r such that

$$f(r) = 0 \quad \Longrightarrow \quad r^2 - 0.5 = 0.$$

Here it is finding the square root of 0.5.
Solving gives

$$r = \pm\sqrt{0.5} = \pm 0.707106\ldots$$

Since the interval $[0.7, 0.8]$ only contains the positive solution, the output is

$$r \approx 0.7071068.$$

The mechanism of `uniroot` is to use the Intermediate Value Theorem together with the bisection method, gradually narrowing the interval in order to find an approximate solution to the equation $f(r) = 0$.

By switching the interval, we can approximate the result we want to get.

1.3 Part 3

Use `uniroot()` to find the yield to maturity of the 30-year par $1,000 bond with coupon payments of $40 that is selling at $1,200.

Answer

```
    bondvalue <- function(c, T, r, par) c/r + (par - c/r)*(1+r)^(-2*T)

    price <- 1200; C <- 40; T <- 30; par <- 1000
    f <- function(r) bondvalue(C, T, r, par) - price
    out <- uniroot(f, interval = c(0.02, 0.05))
    out$root
```

And the result is `0.03238059`, which is basically the same as we got from Part 1; using those two methods would not yield too many differences.

```
uniroot( function(r) bondvalue(C, T, r, par) - price, c(0.03,0.04) )
```

We can get result as

```
$root
[1] 0.03241618
$f.root
[1] -0.5497406
$iter
[1] 3
$estim.prec
[1] 6.103516e-05
```

1.4 Part 4

Find the yield to maturity of a par $10,000 bond selling at $9,800 with semiannual coupon payments equal to $280 and maturing in 8 years.

Answer

Here we use:

```
    uniroot( function(r) bondvalue(280, 8, r, 10000) - 9800, c(0.01,0.04) )
```

This gives the value of r=0.29564.

1.5 Part 5

Use `uniroot()` to find the yield to maturity of the 20-year par $1,000 bond with semiannual coupon payments of $35 that is selling at $1,050.

Answer

Here we use:

```
uniroot( function(r) bondvalue(35, 20, r, 1000) - 1050.0, c(0.01, 0.04) )
```

This gives the value of r=0.060413.

2 Question 1

Suppose that the forward rate is r(t) = 0.028 + 0.00042t.

- (a) What is the yield to maturity of a bond maturing in 20 years?
- (b) What is the price of a par $1,000 zero-coupon bond maturing in 15 years?

Answer
For (a), we have

$$y_{20} = \frac{1}{20}\int_0^{20} \left(0.028 + 0.00042\, t\right) dt = \frac{1}{20}\left(0.028 \cdot 20 + \frac{0.00042}{2} \cdot 20^2\right) = 0.0322.$$

For (b), we can find that $y_{15} = 0.03115$, thus the price is given by

$$P(T) = \mathrm{PAR}\ \exp\{-T\, y_T\}.$$

Then we have

$$P(15) = 1000\, e^{-0.03115 \cdot 15} = 626.7234.$$

3 Question 3

A coupon bond has a coupon rate of 3 % and a current yield of 2.8 %.

- (a) Is the bond selling above or below par? Why or why not?
- (b) Is the yield to maturity above or below 2.8 %? Why or why not?

Answer
For (a), it is selling above the par, as the current yield rate is lower than the coupon rate.
For (b), the yield to maturity (YTM) is below the current yield (2.8%), because the bond is selling above par.

4 Question 4

Suppose that the forward rate is $r(t) = 0.032 + 0.001t + 0.0002t^2$.

- (a) What is the 5-year continuously compounded spot rate?
- (b) What is the price of a zero-coupon bond that matures in 5 years?

Answer

For (a), the 5-year spot rate (= yield) is 0.0362, plugging $T = 5$ back into the formula, we have $0.032 + 0.001 \times 2.5 + 0.0002 \times 5^2/3 = 0.036167$

For (b), we have $e^{(-T \times yield)} = e^{-5 \times 0.036167} = 0.83457$

5 Question 7

One year ago a par \$1,000 20-year coupon bond with semiannual coupon payments was issued. The annual interest rate (that is, the coupon rate) at that time was 8.5 %. Now, a year later, the annual interest rate is 7.6 %.

- (a) What are the coupon payments?
- (b) What is the bond worth now? Assume that the second coupon payment was just received, so the bondholder receives an additional 38 coupon payments, the next one in 6 months.
- (c) What would the bond be worth if instead the second payment were just about to be received?

Answer
For (a), the coupon payment is 100*8.5%/2=\$42.5.
For (b), calculate as below

$$P = 42.50 \cdot \frac{1-(1+0.038)^{-38}}{0.038} + 1000 \cdot (1+0.038)^{-38}$$
$$P = 1089.718$$

For (c), the price of bond is $42.5 + 1089.718 = 1132.218$, so it is \$ 1132.218.

6 Question 15

Suppose that a bond pays a cash flow C_i at time T_i for $i = 1, \dots, N$. Then the net present value (NPV) of cash flow C_i is

$$\mathrm{NPV}_i \;=\; C_i \exp(-T_i\, y_{T_i}).$$

Define the weights

$$\omega_i \;=\; \frac{\mathrm{NPV}_i}{\sum_{j=1}^N \mathrm{NPV}_j}$$

and define the duration of the bond to be

$$\mathrm{DUR} \;=\; \sum_{i=1}^N \omega_i T_i,$$

which is the weighted average of the times of the cash flows. Show that

$$\frac{d}{d\delta} \sum_{i=1}^N C_i \exp\big\{ -T_i\,(y_{T_i} + \delta)\big\}\bigg|_{\delta=0} = -\,\mathrm{DUR} \sum_{i=1}^N C_i \exp\{-T_i\, y_{T_i}\},$$

and use this result to verify Eq. (3.31).

Answer

Proof

Define the (shifted) present value as a function of a parallel shift δ:

$$P(\delta) \;=\; \sum_{i=1}^N C_i \exp\big\{ -T_i\,(y_{T_i} + \delta)\big\}.$$

Differentiate with respect to δ:

$$\frac{dP}{d\delta} = \sum_{i=1}^N C_i \frac{d}{d\delta} \exp\big\{ -T_i\,(y_{T_i} + \delta)\big\} = \sum_{i=1}^N C_i \big(-T_i\big) \exp\big\{ -T_i\,(y_{T_i} + \delta)\big\}.$$

Evaluating at $\delta = 0$ gives

$$\frac{dP}{d\delta}\bigg|_{\delta=0} = -\sum_{i=1}^N T_i\, C_i \exp(-T_i y_{T_i}) = -\sum_{i=1}^N T_i\, \mathrm{NPV}_i.$$

Using the definitions $\omega_i = \dfrac{\mathrm{NPV}_i}{\sum_j \mathrm{NPV}_j}$ and $\mathrm{DUR} = \sum_i \omega_i T_i$, we can rewrite the last line as

$$\frac{dP}{d\delta}\bigg|_{\delta=0} = -\Big(\sum_{i=1}^N \mathrm{NPV}_i\Big)\Big(\sum_{i=1}^N \omega_i T_i\Big) = -\,\mathrm{DUR} \sum_{i=1}^N C_i \exp(-T_i y_{T_i}),$$

which proves the displayed identity.

Verification of Eq. (3.31).

Since $P(0) = \sum_{i=1}^N C_i \exp(-T_i y_{T_i})$, the above yields

$$\frac{dP}{d\delta}\bigg|_{\delta=0} = -\,\mathrm{DUR}\, P(0), \qquad \text{so} \qquad \frac{d\ln P}{d\delta}\bigg|_{\delta=0} = \frac{1}{P(0)} \frac{dP}{d\delta}\bigg|_{\delta=0} = -\,\mathrm{DUR}.$$

Also, we have:

$$\Delta P = \frac{d}{d\delta}\left(\sum_{i=1}^N C_i \exp\{-T_i\,(y_{T_i} + \delta)\}\bigg|_{\delta=0}\right) \delta y = -\,\mathrm{DUR}\, P\, \delta y.$$

Therefore, for a small parallel shift δ,

$$\frac{\Delta P}{P} \;\approx\; -\,\mathrm{DUR}\,\delta,$$

which is the standard first-order duration relation (Eq. (3.31)).

7 Question 21

A par \$1,000 bond matures in 4years and pays semiannual coupon payments of \$25. The price of the bond is \$1,015. What is the semiannual yield to maturity of this bond?

Answer

We want to get the result of r in:

$$1015 = \frac{C}{r} + \left(\text{PAR} - \frac{C}{r}\right)(1+r)^{-2T} = \frac{25}{r} + \left(1000 - \frac{25}{r}\right)(1+r)^{-8}.$$

Let

$$f(r) = \frac{25}{r} + \left(1000 - \frac{25}{r}\right)(1+r)^{-8} - 1015.$$

Using the `bondvalue` function with `uniroot` from R lab, we can get:

```
uniroot( function(r) bondvalue(280, 4, r, 10000) - 9800, c(0.01,0.04) )
```

$$r = 0.02291701 \quad \text{(per half-year)}.$$