UCLA Math151A Fall 2021
Lecture 8
20211011

Convergence Order Theorem

Optional reading: book 2.4

Last time

Newton's Method

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

Newton Convergence Theorem

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

The idea of the proof: apply F.P.I. Convergence Theorem

Define $g(x) := x - \frac{f(x)}{f'(x)}$.

$(I)[\hat{a}, \hat{b}] \to [\hat{a}, \hat{b}]$ (II) g is C^1;

(III) g has bounded derivative with bound in $(0,1)$.

$\boxed{\text{Today}}$

Evaluate the convergence order of N.M. / F.P.I. on important cases.

Recall the following from lecture 2:

Order of Convergence for Sequences

Definition 2.3 (Convergence order of **convergent** sequences)**.** Let $(p_n)_{n \in \mathbb{N}}$ ($\mathbb{N}$ is natural numbers)

$$= (p_1, p_2, p_3, \dots)$$

be a **convergent** sequence in $\mathbb{R}$. Let $p_n \to p$ as $n \to \infty$. And assume $p_n \neq p$ for each n. Then, if $\exists \lambda, \alpha$ with $0 < \lambda < \infty$ (a finite and positive λ) and $\alpha > 0$ s.t.

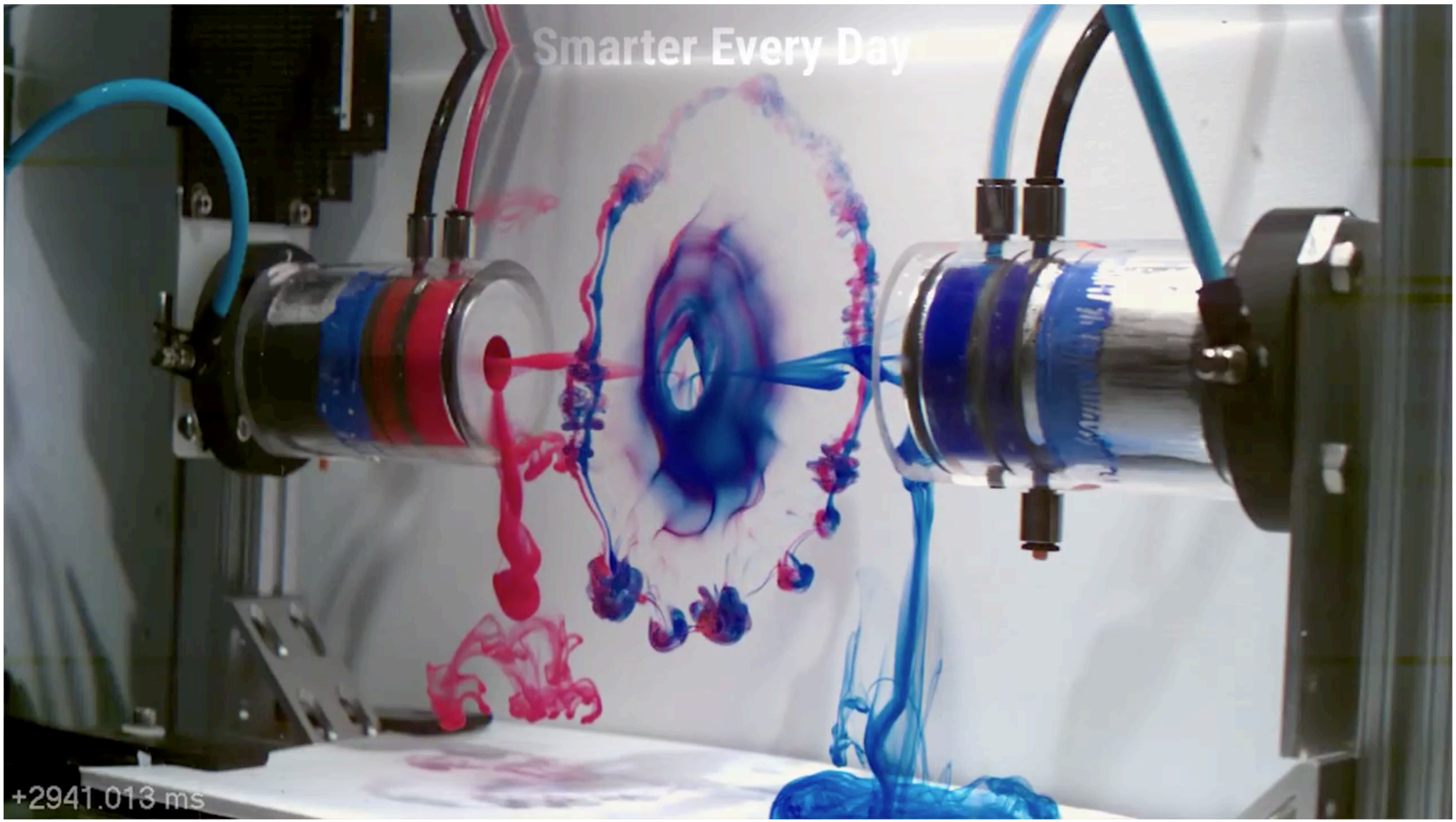
$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then we say p_n converges to p with **order** α. $\square$

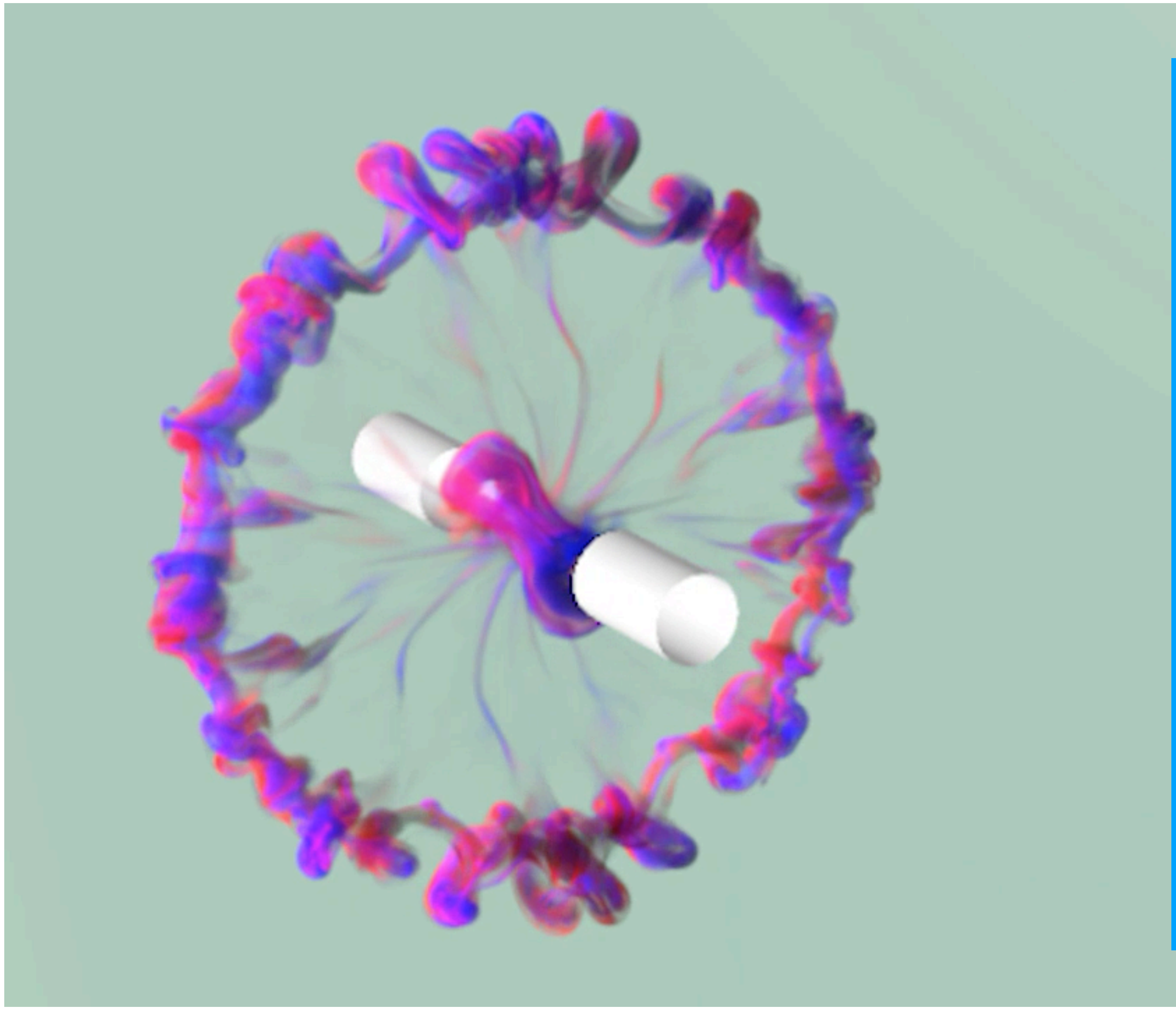
Note that the denominator is not zero because we required that $p_n \neq p$.

Keep in mind: Lambda < 1 is implicitly needed for linear convergence. Why?

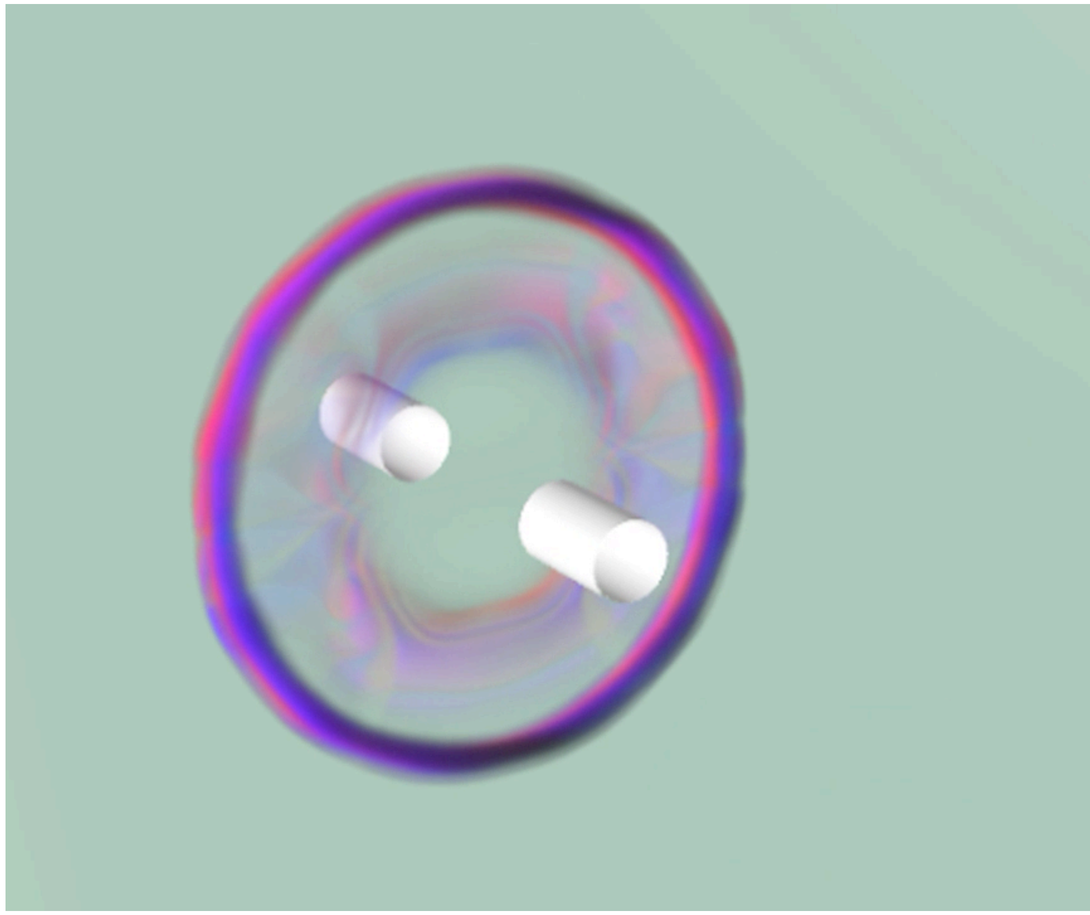
Importance of doing high order



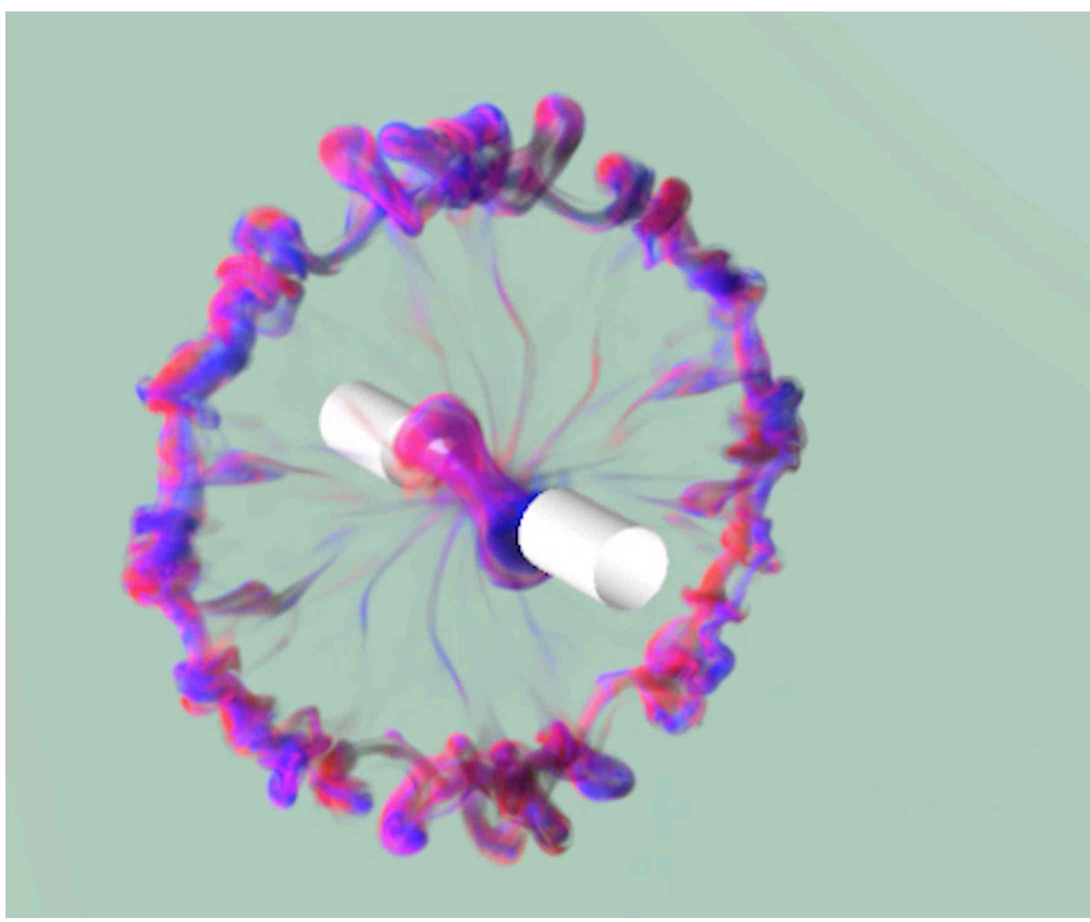
A first order method.
Run N iterations.
Solve the equations inaccurately.



A second order method.
Run same #N iterations.
Much higher accuracy!



A first order method.
Run N iterations



A second order method.
Run N iterations

Start with some handy results…

Fact: Let $p = g(p)$ be a fixed point,

F.P.I. $g(p_n) = p_{n+1}$,

if $g'(p) \neq 0$, we get linear convergence (order of convergence $\alpha = 1$).

if $g'(p) = 0$, we get quadratic convergence ($\alpha = 2$)

Let's start with studying the linear case

Theorem 8.1 (The case where we get linear convergence.).

Let $g \in C^1([a,b])$ with $|g'(x)| \leq k$ for some $0 < k < 1$.
If $g'(p) \neq 0$, then F.P.I. converges to p linearly. $\square$

Proof. From lecture 5's FPI convergence theorem, we know that F.P.I. converges in this case. So we just need to prove the linear order.

We can use M.V.T.: $p_{n+1} - p = g(p_n) - g(p) = g'(\xi)(p_n - p)$
where ξ is between p_n and p.

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|} = \lim_{n\to\infty} |g'(\xi)| = |g'(p)| = \text{a positive number that is smaller than 1}$$

It's also easy to see that it ***only*** has linear convergence, e.g.,

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|^2} = \lim_{n\to\infty} |g'(\xi)| \frac{1}{|p_n - p|} = \infty.$$

$\blacksquare$

$\boxed{\textbf{Theorem 8.1} \text{ (The case where we get linear convergence.).}}$

Let $g \in C^1([a,b])$ with $|g'(x)| \le k$ for some $0 < k < 1$.

If $g'(p) \neq 0$, then F.P.I. converges to p linearly. $\square$

$g'(p) \neq 0$ has linear convergence order

How about other cases?

consider cases where $g'(p) = 0$.

A general theorem for other cases:

Theorem 8.2 (Convergence Order Theorem of FPI).

Let $g \in C^{\alpha}([a,b])$, $\boxed{\alpha \geq 2}$ is an interger,

If (i) $g(p) = p$, and

(ii) $g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0$,

and (iii) $g^{(\alpha)} \neq 0$.

Then

F.P.I. converges $\forall p_0$ sufficiently close to p with order α.

$\square$

Convergence Order Theorem

$g \in C^\alpha([a,b]) \qquad \alpha \geq 2$

(i) $g(p) = p$, (ii) $g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0$, (iii) $g^{(\alpha)} \neq 0$.

$\Rightarrow$ F.P.I. converges $\forall p_0$ sufficiently close to p with order α.

(a) First let's prove that $p_n \to p$.

We can follow the procedure in the proof in lecture 7.

Sketch of proof:

$g'(p) = 0$ and $g' \in C([a,b])$,

$$\Rightarrow \qquad |g'(x)| \leq k \text{ in } [p-\delta, p+\delta] \text{ for any } k \in (0,1).$$

$$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)||x-p| \leq k|x-p| < |x-p|.$$

$$\Rightarrow \qquad p - \delta \leq g(x) \leq p + \delta$$

these conditions guarantee convergence for $p_n \to p$ by F.P.I. Theorem.

Convergence Order Theorem

$g \in C^\alpha([a,b]) \qquad \alpha \geq 2$

(i) $g(p) = p$, (ii) $g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0$, (iii) $g^{(\alpha)} \neq 0$.

$\Rightarrow$ F.P.I. converges $\forall p_0$ sufficiently close to p with order α.

(b) Next let's prove that the order is α. Let $n = \alpha - 1$,

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0)\frac{(x-x_0)^2}{2!}$$
$$+ \cdots + g^{(\alpha-1)}(x_0)\frac{(x-x_0)^{\alpha-1}}{(\alpha-1)!} + g^{(\alpha)}(\xi(x))\frac{(x-x_0)^\alpha}{\alpha!}$$

where $\xi(x)$ is betwee x_0 and x is a general unknown .

Theorem 6.1 (Taylor's theorem). Let $f \in C^n([a,b])$, let $x_0 \in [a,b]$, and let $f^{(n+1)}$ exists on (a,b). Then $\forall x \in [a,b]$, $\exists$ some $\xi(x) \in \mathbb{R}$ s.t. $x_0 < \xi < x$ and

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2! + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

(Taylor's polynomial)

$$R_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)^{n+1} \qquad \text{(The remainder term)}$$

Convergence Order Theorem

$g \in C^{\alpha}([a,b]) \qquad \alpha \geq 2$

(i) $g(p) = p$, (ii) $g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0$, (iii) $g^{(\alpha)} \neq 0$.

$\Rightarrow$ F.P.I. converges $\forall p_0$ sufficiently close to p with order α.

(b) Next let's prove that the order is α. Let $n = \alpha - 1$,

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0)\frac{(x-x_0)^2}{2!}$$
$$+ \cdots + g^{(\alpha-1)}(x_0)\frac{(x-x_0)^{\alpha-1}}{(\alpha-1)!} + g^{(\alpha)}(\xi(x))\frac{(x-x_0)^{\alpha}}{\alpha!}$$

where $\xi(x)$ is betwee x_0 and x is a general unknown .

Next, let $x = p_n$ and $x_0 = p$,

$$g(p_n) = p + g^{(\alpha)}(\xi_n)\frac{(p_n - p)^{\alpha}}{\alpha!},$$

where $\xi_n := \xi(p_n)$ is between p_n and p.

Why evaluating g(p_n)? To relate to p_{n+1}!

p_{n+1}

Convergence Order Theorem

$g \in C^{\alpha}([a,b]) \qquad \alpha \geq 2$

(i) $g(p) = p$, (ii) $g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0$, (iii) $g^{(\alpha)} \neq 0$.

$\Rightarrow$ F.P.I. converges $\forall p_0$ sufficiently close to p with order α.

… (proof continued)

$$p_{n+1} = p + g^{(\alpha)}(\xi_n)\frac{(p_n - p)^{\alpha}}{\alpha!}, \qquad p_{n+1} - p = g^{(\alpha)}(\xi_n)\frac{(p_n - p)^{\alpha}}{\alpha!}.$$

$$\frac{p_{n+1} - p}{(p_n - p)^{\alpha}} = \frac{g^{(\alpha)}(\xi_n)}{\alpha!} \qquad \lim_{n\to\infty}\frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lim_{n\to\infty}\left|\frac{g^{(\alpha)}(\xi_n)}{\alpha!}\right|.$$

$g \in C^{\alpha}([a,b])$, lim for continuous functions can go inside/outside a function

$$\lim_{n\to\infty}\frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \frac{1}{\alpha!}|g^{(\alpha)}(\lim_{n\to\infty}\xi_n)|.$$

Recall $\xi_n \in [p_n, p]$ or $\in [p, p_n]$. $\quad p_n \to p \quad \Rightarrow \quad \xi_n$ also converges to p.

$$\Rightarrow \quad \lim_{n\to\infty}\frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \frac{1}{\alpha!}|g^{(\alpha)}(p)| := \lambda \in (0,\infty) \qquad \blacksquare$$

Note that from Extreme Value Theorem we know that continuous function in a bounded interval is bounded.

Application of the Theorem

Convergence Order Theorem

$g \in C^\alpha([a,b]) \qquad \alpha \geq 2$

(i) $g(p) = p$, (ii) $g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0$, (iii) $g^{(\alpha)} \neq 0$.

$\Rightarrow$ F.P.I. converges $\forall p_0$ sufficiently close to p with order α.

For Newton's Method, let $f(p) = 0$ and $f'(p) \neq 0$.

List facts:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} := g(p_n)$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g(p) = p$$

$$g'(p) = \frac{f(p)f''(p)}{(f'(p))^2} = 0.$$

$g(p_n) = p_{n+1}$ converges with order 2

(or better, if $g''(p) = 0$).

I.e. N.M. has **<u>at least</u>** quadratic convergence for such functions.

Some Remarks about N.M.

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} := g(p_n)$$

Suppose that derivative vanishes at p, i.e, $f'(p) = 0$,
then N.M. may 1) not converge at all, or 2) converge very slowly (only linearly), depending on the initial guess.

Intuitively, if $p_n \to p$ and $f'(p) = 0$,
that implies $f'(p_n) \approx 0$ for n large

So N.M. has division by a very small number at each iteration.

Recall the graphical derivation of Newton's method, this easily causes shooting far away.

$\boxed{\text{Example}}$

$$f(x) = x^2$$

$$f(0) = 0 \quad f'(0) = 0$$

0 is a double root of f.

$$f'(x) = 2x$$

$$f'(p) = 0 \text{ and } f(p) = 0,$$

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2}{2x} = \frac{x}{2}$$

$$g(p) = p$$

$$g'(x) = 1 - \left(\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}\right) = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(x) = \frac{1}{2}$$

Theorem 8.1 (The case where we get linear convergence.).

Let $g \in C^1([a,b])$ with $|g'(x)| \leq k$ for some $0 < k < 1$.
If $g'(p) \neq 0$, then F.P.I. converges to p linearly. $\square$

Next time:

Finish Newton

Start new topic: polynomial interpolation