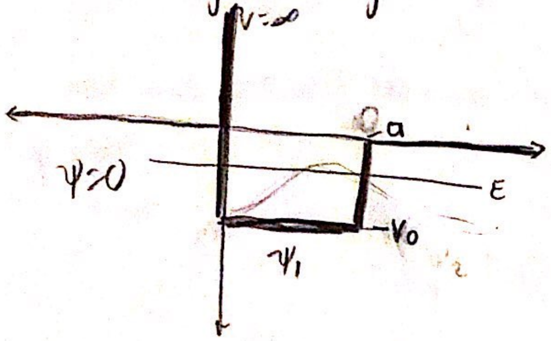
actual hw 7

oopsies

Problem 1

a This is a one-sided finite square well. Since it must be 0 at the origin it will look like an odd state of the finite square well with $V_0 = +32\hbar^2/ma^2$ with only half being there.



- $\psi_1 = A\sin lx$ $\quad l^2 = 2m(E+V_0)/\hbar^2$
  $\psi_2 = Be^{-kx}$ $\quad k^2 = -2mE/\hbar^2$
- $\psi_1(a) = \psi_2(a)$ $\quad$ • $\psi_1'(a) = \psi_2'(a)$
  $A\sin la = Be^{-ka}$ $\quad lA\cos la = -kBe^{-ka}$
- combine

$-k = l\cot la \rightarrow ka = -(la)\cot(la)$

note $k^2a^2 = -\frac{2mE}{\hbar^2}a^2$

$= \left[\frac{2mV_0}{\hbar^2} - \left(\frac{2mE}{\hbar^2} + \frac{2mV_0}{\hbar^2}\right)\right]a^2$

$= \frac{2mV_0}{\hbar^2}a^2 - (la)^2$

let $z = la = \frac{\sqrt{2m(E+V_0)}}{\hbar}a$

$z_0 = \frac{\sqrt{2mV_0}}{\hbar}a$

then $\sqrt{z_0^2 - z^2} = -z\cot z$

- since $V_0 = 32\hbar^2/ma^2$ we get

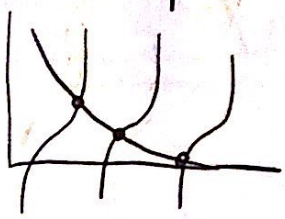
$z_0 = \frac{\sqrt{2m}}{\hbar}a\sqrt{\frac{32\hbar^2}{ma^2}} = \sqrt{64} = 8$

- eq becomes

$\sqrt{8 - z^2} = -z\cot z$

$\sqrt{\left(\frac{8}{z}\right)^2 - 1} = -\cot z$

- solutions at 3 points $\rightarrow$ $\boxed{\text{3 bound states}}$



b idk lol

Problem 2

a Case $E < V_0$.

- $\psi_1 = Ae^{ikx} + Be^{-ikx}$
  $\psi_2 = Ce^{lx} + De^{-lx}$
  $\psi_3 = Fe^{ikx}$ $\rightarrow$ no init right travel
  $k^2 = 2mE/\hbar^2$
  $l^2 = 2m(V_0 - E)/\hbar^2$
- [1] $Ae^{-ika} + Be^{ika} = Ce^{-la} + De^{la}$
  [2] $ik(Ae^{-ika} - Be^{ika}) = l(Ce^{-la} - De^{la})$
  [3] $Ce^{la} + De^{-la} = Fe^{ika}$
  [4] $l(Ce^{la} - De^{-la}) = ikFe^{ika}$
- (1)(2) $2Ae^{-ika} = \left(1 + \frac{l}{ik}\right)Ce^{-la} + \left(1 - \frac{l}{ik}\right)De^{la}$
  $2Ae^{-ika} = \left(1 - i\frac{l}{k}\right)Ce^{-la} + \left(1 + i\frac{l}{k}\right)De^{la}$

(3)(4) $2Ce^{la} = \left(1 + \frac{ik}{l}\right)Fe^{ika}$
$2De^{-la} = \left(1 - \frac{ik}{l}\right)Fe^{ika}$

$2Ae^{-ika} = \left[\left(1 - \frac{il}{k}\right)\left(1 + \frac{ik}{l}\right)e^{-2la} + \left(1 + \frac{il}{k}\right)\left(1 - \frac{ik}{l}\right)e^{2la}\right]Fe^{ika}$

$\frac{A}{F} = \frac{e^{2ika}}{4}\left[4\cosh(2a) + i\frac{2(l^2 - k^2)}{lk}\sinh(la)\right]$

$T^{-1} = \left|\frac{A}{F}\right|^2 = \cosh^2(la) + \frac{(l^2 - k^2)^2}{(lk)^2}\sinh^2(la)$

$$T^{-1} = 1 + \left(1 + \frac{(\ell^2 - k^2)^2}{(2\ell k)^2}\right) \sinh^2(2\ell a)$$

$$= 1 + \frac{4\ell^2k^2 + \ell^4 + k^4 - 2\ell^2k^2}{(2\ell k)^2} \sinh^2(2\ell a)$$

$$= 1 + \frac{(\ell^2 + k^2)^2}{(2\ell k)^2} \sinh^2(2\ell a)$$

$$= 1 + \frac{(2m(V_0 - E) + 2mE)^2}{4(2mE)(2m(V_0 - E))} = \frac{(2mV_0)^2}{16m^2 E(V_0 - E)} \sinh^2$$

$$\boxed{= 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2\left(\frac{2\sqrt{2m(V_0 - E)}}{\hbar} a\right)}$$

b Case $E = V_0$. Here

- TISE for ψ_2: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = 0$

  $\psi_2(x) = C + Dx$

- $Ae^{-ika} + Be^{ika} = C - Da$

  $Fe^{ika} = C + Da$

  $ik(Ae^{-ika} - Be^{-ika}) = D$

  $ikFe^{ika} = D$

- $2C = Fe^{ika} + Ae^{ika} + Be^{ika} = Ae^{-ika} + (F+B)e^{ika}$

  $2Da = Fe^{ika} - Ae^{-ika} - Be^{ika} = -Ae^{-ika} + (F-B)e^{ika}$

  $Ae^{-2ika} - B = F$

  $2Ae^{-2ika} = 2F(1 - ika)$

  $\left|\frac{A}{F}\right| = 1 - ika, \quad T^{-1} = 1 + (ka)^2$

$$\boxed{T^{-1} = 1 + \frac{2mE}{\hbar^2} a^2}$$

c case $E > V_0$. Same as finite square well

- with $E > V_0$ but now V_0 sign flips.

$$T^{-1} = \boxed{1 + \frac{V_0^2}{4E(E - V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E - V_0)}\right)}$$

Problem 3

a Delta well S-matrix. In text we are given.

- $F + G = A + B$,

  $F - G = A(1 + 2i\beta) - B(1 - 2i\beta)$

  for $\beta = m\alpha/\hbar^2 k$

- Eliminate F to get B

  $2G = -2i\beta A + 2B - 2i\beta B$

  $B = \frac{1}{1 - i\beta}(G + i\beta A)$

- eliminate B to get F

  $F(1 - 2i\beta) + G(1 - 2i\beta) = A(1 - 2i\beta) + B(1 - 2i\beta)$

  $F(2 - 2i\beta) + G(-2i\beta) = 2A$

  $F = \frac{1}{1 - i\beta}(A + i\beta G)$

- $$\frac{1}{1 - i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}\begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} B \\ F \end{pmatrix}$$

$$\boxed{S = \frac{1}{1 - i\beta}\begin{bmatrix} i\beta & 1 \\ 1 & i\beta \end{bmatrix}}$$

b Finite square well S-matrix, with two-direction scattering.

- symmetry: switching dirs of left/right gives us that coefficient orders

  left – A, B, F, G

  right – G, F, B, A

- left

  $B = S_{11}A + S_{12}G$

  $F = S_{21}A + S_{22}G$

- right

  $F = S_{11}G + S_{12}A$

  $B = S_{21}G + S_{22}A$

  $S_{11} = S_{22}, \quad S_{12} = S_{21}$

- In text we are given

$$F = \frac{e^{-2ika}}{\cos(2\ell a) - i\frac{\ell^2 + k^2}{2\ell k}\sin(2\ell a)} A \rightarrow S_{12}$$

$$B = \frac{i\frac{\sin(2\ell a)}{2\ell k}(\ell^2 - k^2)e^{-2ika}}{\cos(2\ell a) - i\left(\frac{\ell^2 + k^2}{2\ell k}\right)\sin(2\ell a)} A \rightarrow S_{11}$$

$$\boxed{S=\frac{e^{-2ika}}{\cos(2la)-i\frac{l^2+k^2}{2lk}\sin(2la)}\begin{pmatrix}\frac{i\sin(2la)(l^2-k^2)}{2lk} & 1\\ 1 & \frac{i\sin(2la)(l^2+k^2)}{2lk}\end{pmatrix}}$$

Problem 4

Note for square well we have

$$z_0=\frac{a}{\hbar}\sqrt{2mV_0}$$

and delta well has $\alpha = 2aV_0$ is constant

$$\lim_{a\to 0} z_0 = \lim_{a\to 0}\frac{a}{\hbar}\sqrt{\frac{m\alpha}{a}} = \frac{\sqrt{m\alpha}}{\sqrt{\hbar}}\lim_{a\to 0}\sqrt{a}$$

$$=0 \text{ so } \boxed{z_0\to 0}$$

Bound state energy from well equation for E:

$$\tan z = \sqrt{\left(\frac{z_0}{z}\right)^2-1} \begin{cases}\text{small} & z = la\\ \text{smaller} & z_0\end{cases}$$

$$\tan z \simeq z = \frac{1}{z}\sqrt{z_0^2-z^2}$$

$z_0^2-z^2 \approx \cancel{z^4}$ → very small, ~0

$z_0 \simeq z$ →

Note also $\cancel{z_0^2}-z^2 = k^2a^2 \to z^2 \simeq ka$??

$$z_0^2 \simeq ka \to k\cancel{a} = \frac{m\alpha\cancel{a}}{\hbar^2}$$

$$\to -\frac{2mE}{\cancel{\hbar^2}} = \left(\frac{m\alpha}{\cancel{\hbar^2}}\right)^2$$

$$\to -2\cancel{m}E = \frac{m^{\cancel{2}}\alpha^2}{\hbar^2}$$

$$\to \boxed{E=-\frac{m\alpha^2}{2\hbar^2}}$$

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