Math 131A - Homework 5

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Question 1.

(a) $\boxed{\text{True.}}$ Given $\epsilon > 0$, there are $\delta_f > 0$ and $\delta_g > 0$ such that for some $x, y \in S$, $|x - y| < \delta_f \Rightarrow |f(x) - f(y)| < \epsilon/2$ and $|x - y| < \delta_g \Rightarrow |g(x) - g(y)| < \epsilon/2$. If $|x - y| < \min(\delta_f, \delta_g)$, then $|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon$. Thus $f + g$ is uniformly continuous on S for $\delta = \min(\delta_f, \delta_g)$.

(b) $\boxed{\text{False.}}$ Let $f(x) = g(x) = x$, then $(fg)(x) = x^2$ which is not uniformly continuous. Consider $\epsilon = 1$, then there must be δ such that $|x - y| < \delta \Rightarrow |x^2 - y^2| < 1$, so if we take $x = x, y = x + \frac{\delta}{2}$ it needs to be true that $|x^2 - (x + \frac{\delta}{2})^2| = |\delta x + \frac{\delta^2}{2}| < 1$. But this does not hold for every x, simply choose $x = 1/\delta$ and the left side becomes greater than one.

(c) $\boxed{\text{True.}}$ Given $\epsilon > 0$, we have $|x - y| < \delta_g \Rightarrow |g(x) - g(y)| < \epsilon$. Given δ_g, we have $|x - y| < \delta_f \Rightarrow |f(x) - f(y)| < \delta_g$. Thus for any $\epsilon > 0$, we have δ_f such that $|x - y| < \delta_f \Rightarrow |g(f(x)) - g(f(y))| < \epsilon$.

Question 2.

Question 3.

If f is continuous at a, then for any $\epsilon > 0$, there is δ so that $|x - a| < \delta \Rightarrow |f(x) - f(a) < \epsilon|$. By definition of a limit we get that $\lim_{I \ni x \to a} = f(a)$. This then implies for some, since for some $\eta > 0, a \in I = (a + \eta, a - \eta) \in \mathbb{R}$, that $\lim_{x \to a} = f(a)$.

Question 4.

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f is differentiable at 0 if the limit exists: $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$. If $x \in \mathbb{Q}$ then $\frac{f(x)}{x} = 1 + x$, and $\lim_{x \to 0} 1 + x = 1$. If $x \in \mathbb{R} \setminus \mathbb{Q}$ then $\frac{f(x)}{x} = 1 - x$, and $\lim_{x \to 0} 1 - x = 1$. Thus $\lim_{x \to 0} \frac{f(x)}{x} = 1$ exists, so f is differentiable at 0.

Question 5.

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f is differentiable at c if the limit exists: $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) = \lim_{x \to c} f'(x) = L$. The limit exists, so f is differentiable at c and $f'(c) = L$.

Question 6.

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f is Riemann integrable if the limit of the lower and upper Riemann sums equal each other. Let S_l denote the lower sum and S_u denote the upper. Partitioning the interval into n pieces gives us pieces of $2/n$ since the interval is on $[-1, 1]$.

$$\begin{aligned}
S_l &= \sum_{i=1}^{n} \underline{f(x_n)}\frac{2}{n} = \sum_{i=1}^{n/2} 0\frac{2}{n} + \sum_{i=1}^{n/2} 1\frac{2}{n} = \sum_{i=1}^{n/2} \frac{2}{n} = 1 \\
S_u &= \sum_{i=1}^{n} \overline{f(x_n)}\frac{2}{n} = \sum_{i=1}^{n/2} 0\frac{2}{n} + \sum_{i=1}^{n/2} 1\frac{2}{n} = \sum_{i=1}^{n/2} \frac{2}{n} = 1 \\
\lim_{n\to\infty} S_l &= 1 = \lim_{n\to\infty} S_u
\end{aligned}$$

So f is Riemann integrable and $\int_{-1}^{1} f = 1$.