UCLA Math151A Fall 2021
Lecture 4
20211001

Remarks about B.M.
Fixed Point Iteration

Optional reading: book 2.2.

last time…

Root Finding with Bisection

Algorithm 1: Bisection Method (given $f(x) \in C([a,b])$, with $f(a)f(b) < 0$)

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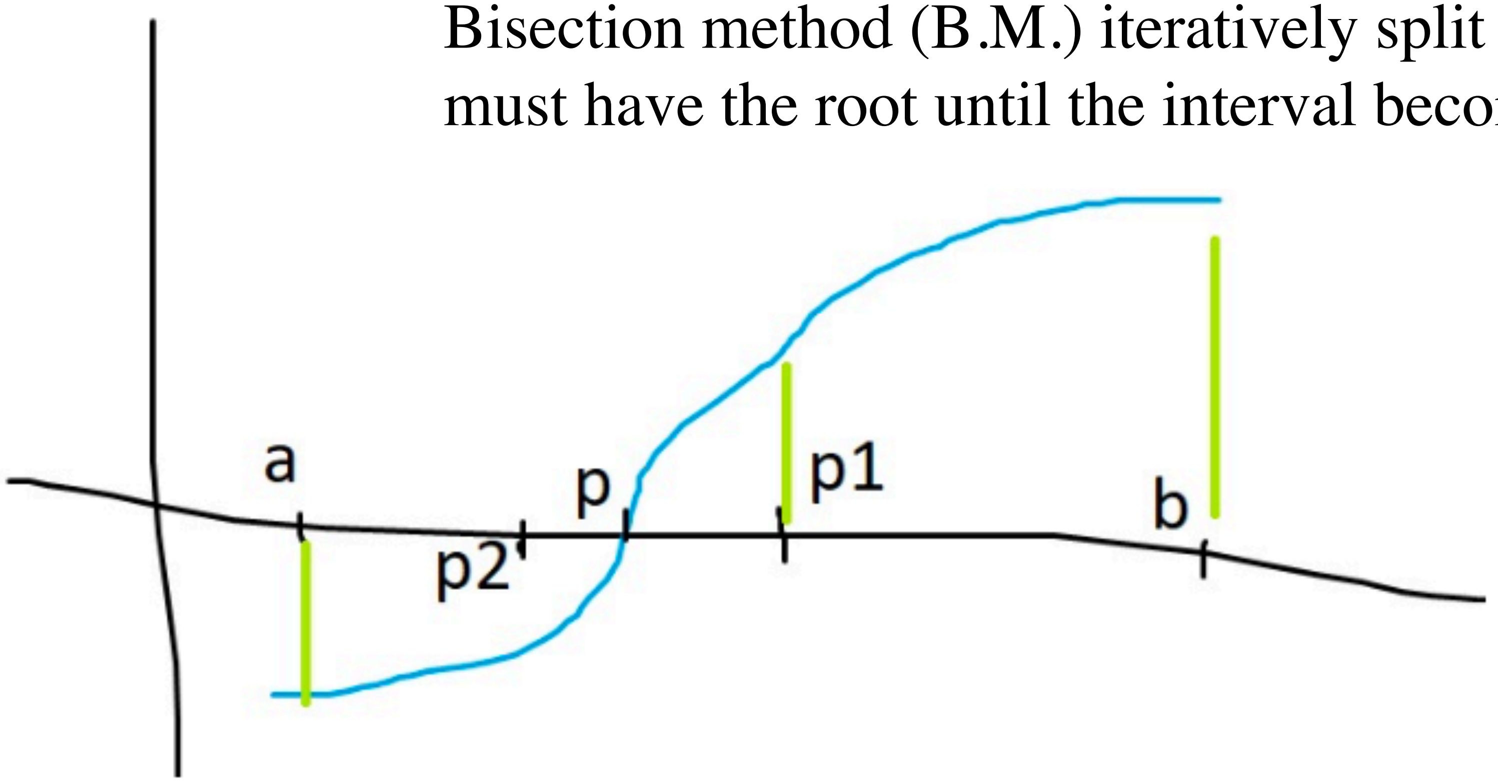
set $a_1 = a$, $b_1 = b$;
set $p_1 = \frac{a_1+b_1}{2}$;
if $f(p_1) == 0$ **then**
  We are done;
else if $f(p_1)$ *has same sign as* $f(a_1)$ **then**
  $p \in (p_1, b_1)$;
  set $a_2 = p_1$, $b_2 = b_1$
else if $f(p_1)$ *has same sign as* $f(b_1)$ **then**
  $p \in (a_1, p_1)$;
  set $a_2 = a_1$, $b_2 = p_1$.
end
set $p_2 = \frac{a_2+b_2}{2}$;
Repeat

Bisection method (B.M.) iteratively split the interval and check which one must have the root until the interval becomes narrow enough.



Remarks about the B.M.

Remark 4.1. B.M. a **global** method (in contrast to a local one).

as long as the assumptions are satisfied: $f \in C([a,b])$, $f(a)f(b) < 0$ the B.M. will converge.

In particular it will converge to some p s.t. $f(p) = 0$.

Here "global" means the algorithm doesn't need a good initial guess p_0 unlike some other "local" methods that we will cover in this course.

Remark 4.2. (Repeating what we had last time.)

If f has multiple roots on $[a, b]$, the B.M. will only find **one** of them.

Further, there is no guarantees on which one it will find.

Remark 4.3. The B.M. won't work for functions like

$$f(x) = x^2$$

even though it a root at $p = 0$.

This is because we couldn't find any $[a, b]$ satisfying the opposite sign property.

$\boxed{\text{Convergence order of B.M.}}$

Theorem 4.1 (Convergence order of B.M.)**.**

The sequence provided by B.M. satisfies

$$|p_n - p| \le \frac{b-a}{2^n}.$$

This error thus $\to 0$ as $n \to \infty$.

Proof. See homework 2.

$$\boxed{\frac{b-a}{2^n}}$$

This further tells us that **the error bound of B.M.** converges linearly.

To see that, recall from previous previous lectures that linear convergence for a convergent sequence (p_n) means that

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^1} = \lambda \quad \text{for some finite positive } \lambda.$$

$$p_n = \frac{b-a}{2^n}, \quad p = 0 \qquad \text{We can easily show that } \lambda = \tfrac{1}{2}.$$

Remark 4.4. The B.M. converges **slowly** compared to other methods. We will soon see that Newton's method has quadratic order of convergence. □

Fixed Point

Definition 4.1 (Fixed point of a function)**.**

Let function g be $\quad g : [a, b] \to \mathbb{R}$,

let $p \in [a, b]$ s.t. $g(p) = p$.

Then p is a **fixed point** of g.

there is a close connection between fixed point and roots of a function.

Theorem 4.2. Let p be a fixed point of g,

then also p is a root of $G(x) := g(x) - x$.

Proof. By definition.

Converting a root-finding problem to a fixed-point problem

Given a root-finding problem $f(p) = 0$,

we can define functions g with a fixed point at p in a number of ways,

for example, as $\quad g(x) = x - f(x)$

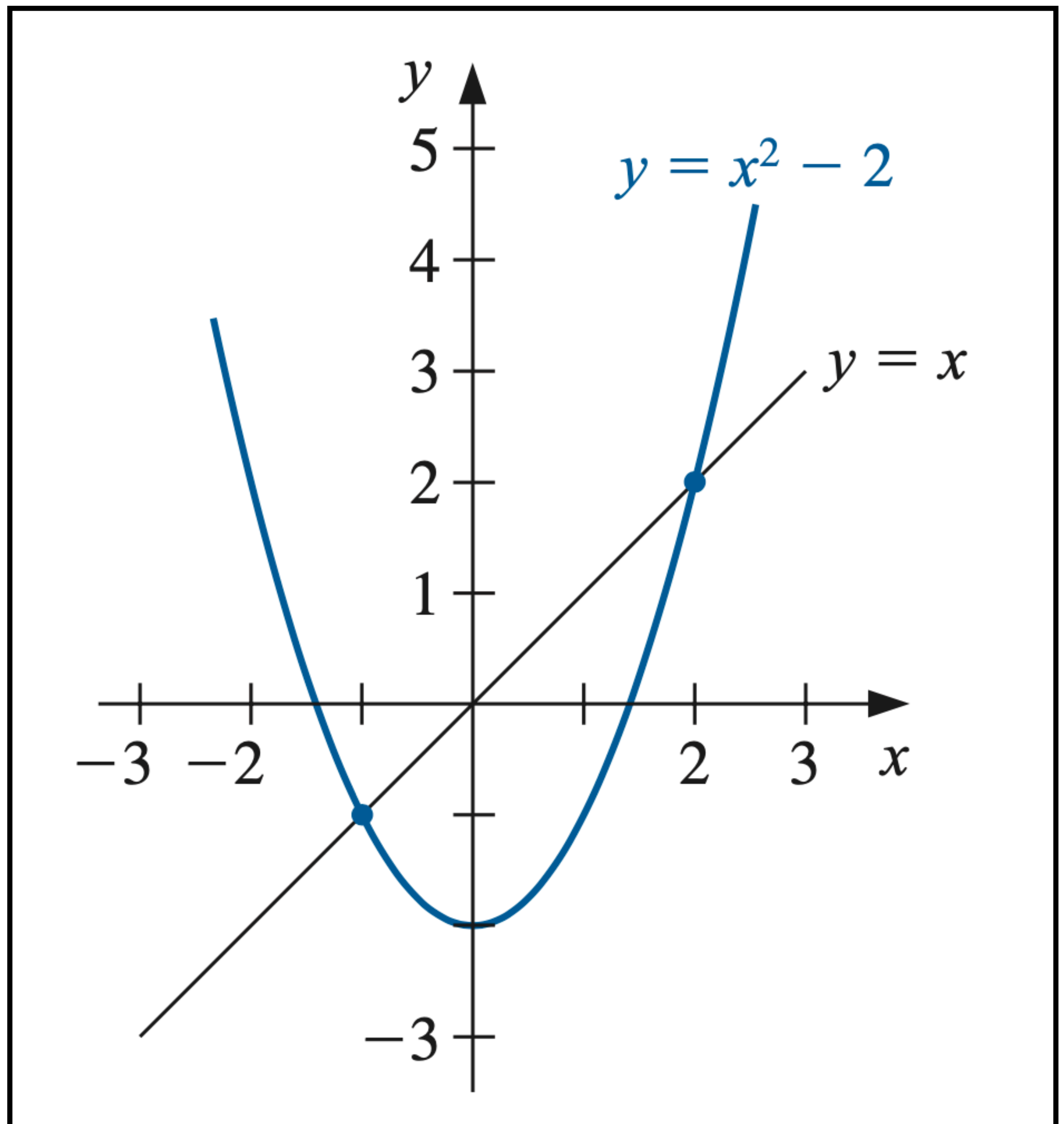
or as $\quad g(x) = x + 3f(x)$.

Graphical view of fixed point

A fixed point for g $\Longrightarrow$ $y = g(x)$ intersects $y = x$.

Example

$$g(x) = x^2 - 2.$$



$$p = -1$$

$$p = 2$$

Fixed Point Iteration (F.P.I.)

The F.P.I. method is quite simple.

Method 4.1 (F.P.I).

For $g \in C([a,b])$.

Let $p_0 \in [a,b]$, and set $p_{n+1} = g(p_n)$.

That's the FPI method, which finds a fixed point for $g(x)$.

we also need $g(x) \in [a,b]$ otherwise at some point of the algorithm we won't be able to proceed to evaluate g. $\square$

Note that the initial guess p_0 is arbitrary.

$$p_1 = g(p_0), p_2 = g(p_1), p_3 = g(p_2), \ldots, p_{n+1} = g(p_n)$$

Stopping Criteria

Method 4.1 (F.P.I).

For $g \in C([a,b])$.

Let $p_0 \in [a,b]$,

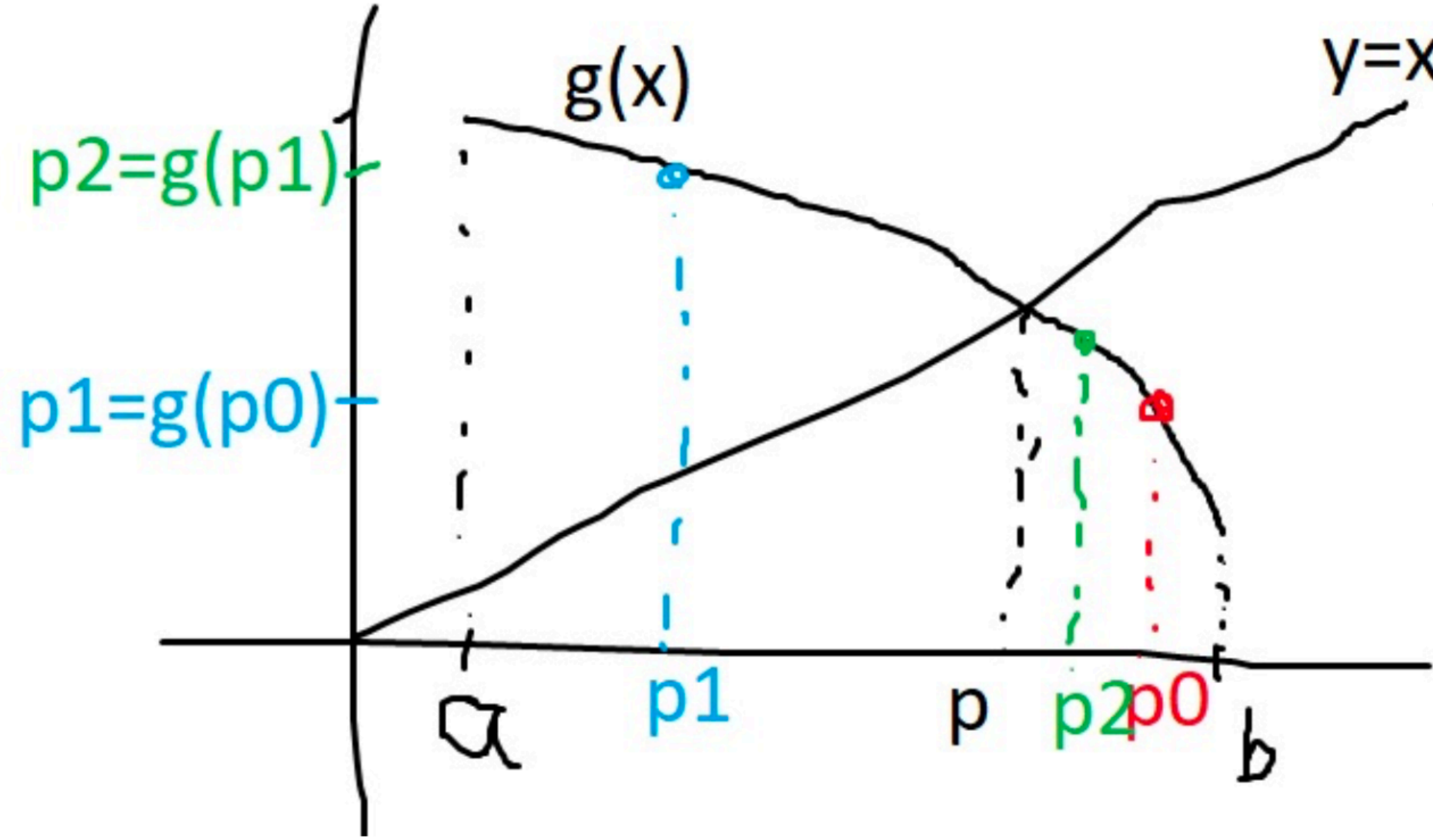
and set $p_{n+1} = g(p_n)$.

- $|p_n - p_{n-1}| < \epsilon$

- $\frac{|p_n - p_{n-1}|}{|p_n|} < \epsilon$ (assumes $p_n \neq 0$)

- $|f(p_n)| < \epsilon \Leftrightarrow |g(p_n) - p_n| < \epsilon$

$$p_1 = g(p_0), p_2 = g(p_1), p_3 = g(p_2), \ldots, p_{n+1} = g(p_n)$$



We have straight line $y = x$, function is $g(x)$. Fixed point p is when the two are equal (thus the intersection point).

We start with an arbitrary initial guess p_0, and go on to perform the iterations.

One may wonder when does FPI converge and when does it fail.

We will introduce a theorem for it soon.

Example 4.1 (F.P.I. failure case). To solve $x^2 - 7 = 0$,

it is equivalent to

$$x = \frac{7}{x}, \qquad \text{Note that } \sqrt{7} = 2.6457\ldots.$$

A straightforward option to do it, if you want to use the F.P.I. to find $p = \sqrt{7}$, we can set

$$g_1(x) = \frac{7}{x},$$

then the goal is to find p s.t. $p = g_1(p)$.

Another option is to use

$$x = \frac{x + \frac{7}{x}}{2} =: g_2(x).$$

Let $p_0 = 3$, we can show that

option 1: $p_0 = 3, \quad p_1 = \frac{7}{3}, \quad p_2 = 3, \quad \ldots,$ oscillates between 2 numbers!

option 2: $p_0 = 3, p_1 = 2.666\ldots, p_2 = 2.645833\ldots, \ldots$

In fact option 2 will converge!

Example $x^3+4x^2-10=0$ has a unique root in $[1,2]$. $\boxed{1.365230013}$

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\dfrac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \dfrac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\dfrac{10}{4+x}\right)^{1/2}$

(e) $x = g_5(x) = x - \dfrac{x^3+4x^2-10}{3x^2+8x}$

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5		$\boxed{\text{undefined}}$	1.360094193	1.365225594	$\boxed{\text{excellent}}$
6	$\boxed{\text{divergent}}$		1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236	$\boxed{\text{excellent}}$	
25			1.365230006		
30			1.365230013	$\boxed{\text{excellent}}$	

Clearly, by this example we see F.P.I does not always converge. It dependes on the function $g(x)$.

How to characterize this? We'll show next time.

But let's first establish some theorems for the existance of the solution before worrying about whether F.P.I. finds a solution.

Existence of a fixed point

Theorem 4.3 (Existence).

Let $g \in \mathbb{C}([a,b])$ with $a \leq g(x) \leq b \quad \forall x \in [a,b]$,

then $\exists$ at least one fixed point p s.t. $g(p) = p$.

Proof. Next time. By I.V.T.