Math 151A - Homework 5

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Question 1.

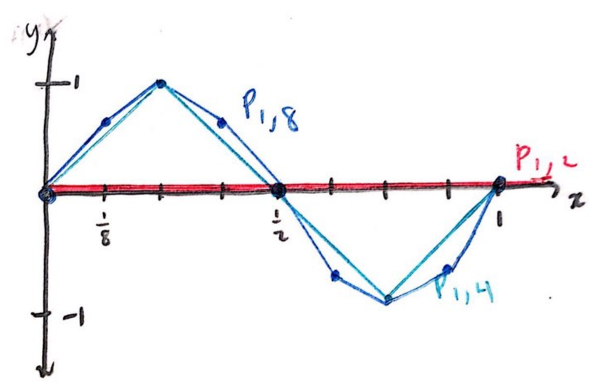
Piecewise linear polynomial for three points.

$$\begin{aligned} f(0) &= 0 \\ f(0.5) &= 0 \\ f(1) &= 0 \\ P_{1,2} &= 0 \end{aligned}$$

Piecewise linear polynomial for five points.

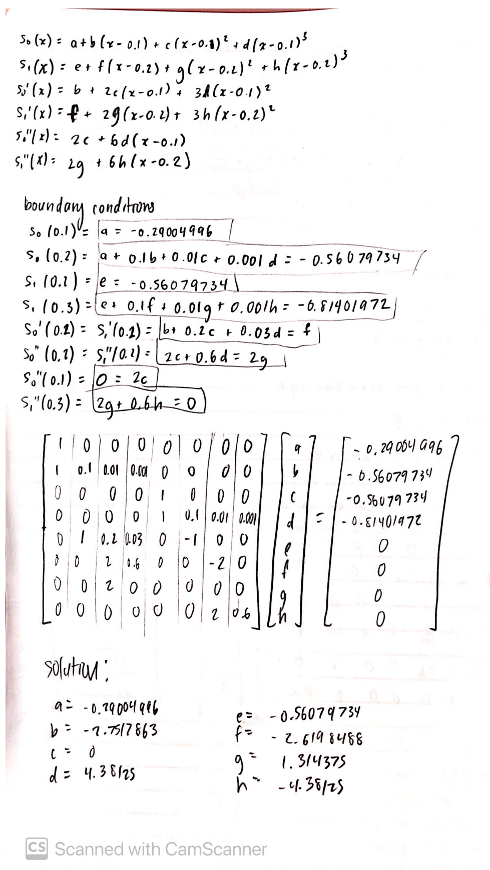
$$\begin{aligned} f(0) &= 0 \\ f(0.25) &= 1 \\ f(0.5) &= 0 \\ f(0.75) &= -1 \\ f(1) &= 0 \\ P_{1,4} &= \begin{cases} 4x & x \in (0, 0.25) \\ -4x + 2 & x \in (0.25, 0.75) \\ 4x - 4 & x \in (0.75, 1) \end{cases} \end{aligned}$$

Graph of piecewise linear interpolants. The pointwise error should go to 0 based on these pictures.



Question 2.

Derivation of cubic spline. This gives the result:



$s_0(x) = a + b(x-0.1) + c(x-0.1)^2 + d(x-0.1)^3$
$s_1(x) = e + f(x-0.2) + g(x-0.2)^2 + h(x-0.2)^3$
$s_0'(x) = b + 2c(x-0.1) + 3d(x-0.1)^2$
$s_1'(x) = f + 2g(x-0.2) + 3h(x-0.2)^2$
$s_0''(x) = 2c + 6d(x-0.1)$
$s_1''(x) = 2g + 6h(x-0.2)$

boundary conditions

$s_0(0.1) = \boxed{a = -0.29004996}$
$s_0(0.2) = \boxed{a + 0.1b + 0.01c + 0.001d = -0.56079734}$
$s_1(0.2) = \boxed{e = -0.56079734}$
$s_1(0.3) = \boxed{e + 0.1f + 0.01g + 0.001h = -0.81401972}$
$s_0'(0.2) = s_1'(0.2) = \boxed{b + 0.2c + 0.03d = f}$
$s_0''(0.2) = s_1''(0.2) = \boxed{2c + 0.6d = 2g}$
$s_0''(0.1) = \boxed{0 = 2c}$
$s_1''(0.3) = \boxed{2g + 0.6h = 0}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.1 & 0.01 & 0.001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.1 & 0.01 & 0.001 \\ 0 & 1 & 0.2 & 0.03 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0.6 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0.6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} -0.29004996 \\ -0.56079734 \\ -0.56079734 \\ -0.81401972 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

solution:

$a = -0.29004996$
$b = -2.7517863$
$c = 0$
$d = 4.38125$
$e = -0.56079734$
$f = -2.6198488$
$g = 1.314375$
$h = -4.38125$

$$s(x) = \begin{cases} -0.29004996 - 2.7512863(x-0.1) + 4.38125(x-0.1)^3 & x \in [0.1, 0.2] \\ -0.56079734 - 2.6198488(x-0.2) + 2.6198488(x-0.2)^2 - 4.38125(x-0.2)^3 & x \in [0.2, 0.3] \end{cases}$$

Approximations at $x = 0.18$, use $s_1(x)$ since $0.18 \in [0.1, 0.2]$.

$$s(0.18) = -0.507909664 \qquad e_{rel} = 0.00042$$
$$s'(0.18) = -2.6671575 \qquad e_{rel} = 0.00586$$

Approximations of $f'(x)$ at $x = 0.2$ comparing $s_1(x), s_2(x)$. They agree exactly. This is because in the definition of a cubic spline we required continuity of the derivative at the in-between points, or at $x = 0.2$

$$s_1'(0.2) = 13.14375(0.2)^2 - 2.62875(0.2) - 2.619848 = -2.619848$$
$$s_2'(0.2) = -13.14375(0.2)^2 + 10.4971976(0.2) - 4.19353832 = -2.619848$$

Question 3.

Third order Taylor expansion.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(\xi)}{4!}(x - x_0)^4$$

Centered difference approximation.

$$\begin{aligned}
f(x_0 + h) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + \frac{f^{(4)}(\xi_1)}{4!}h^4 \\
f(x_0 - h) &= f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f^{(3)}(x_0)}{3!}h^3 + \frac{f^{(4)}(\xi_2)}{4!}h^4 \\
f(x_0 + h) + f(x_0 - h) &= 2f(x_0) + 2\frac{f''(x_0)}{2}h^2 + \frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{4!}h^4 \\
f(x_0 + h) - 2f(x_0) + f(x_0 - h) &= f''(x_0)h^2 + \frac{h^4}{4!}(f^{(4)}(\xi_1) + f^{(4)}(\xi_2)) \\
\frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} &= f''(x_0) + \frac{h^2}{4!}(f^{(4)}(\xi_1) + f^{(4)}(\xi_2))
\end{aligned}$$

Question 4.

Centered difference approximation of $f(x) = 3xe^x - \cos x$ at $x = 1.3$ with $h = 0.1, 0.01$.

$$f''(1.3) \approx \frac{f(1.4) - 2f(1.3) + f(1.2)}{0.01} = \frac{16.86187 - 2(14.04276) + 11.59006}{0.01} = 36.641$$
$$f''(1.3) \approx \frac{f(1.31) - 2f(1.3) + f(1.29)}{0.0001} = \frac{14.30741 - 2(14.04276) + 13.78176}{0.0001} = 36.5$$
$$f''(1.3) = 3((1.3)e^(1.3) + 2e^(1.3)) + \cos(1.3) = 36.593$$