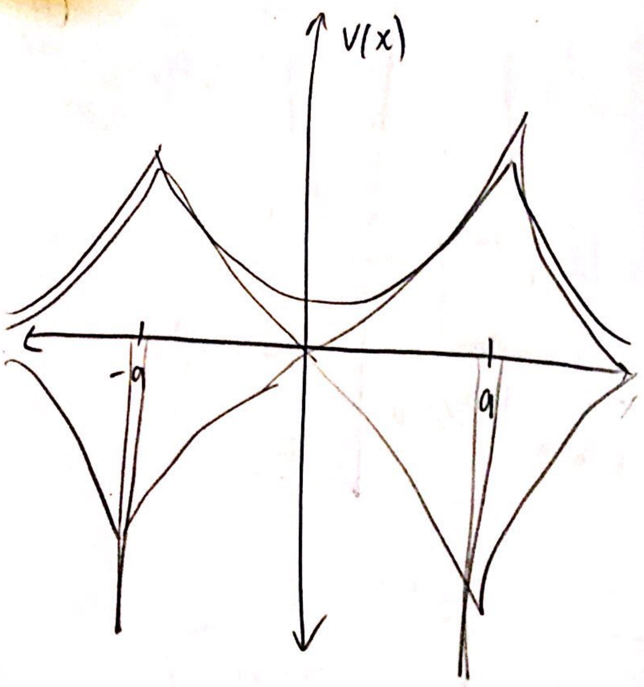
# Problem 1

1)




## Even bound states

$\psi_1 = Ae^{kx}$    $x < -a$

$\psi_2 = B(e^{kx} + e^{-kx})$    $-a < x < a$

$\psi_3 = Ae^{-kx}$    $a < x$

(bc) $\psi_1(-a) = \psi_2(-a) \leftrightarrow \psi_2(a) = \psi_3(a)$

$$Ae^{-ka} = B(e^{-ka} + e^{ka})$$

$$A = B(1 + e^{2ka})$$

(bc) discont at $+a$

$$\Delta \frac{d\psi}{dx}\Big|_a = -\frac{2m\alpha}{\hbar^2}\psi\Big|_a$$

$$-kA\cancel{e^{-ka}} - kB(\cancel{e^{ka}} - \cancel{e^{-ka}}) = -\frac{2m\alpha}{\hbar^2}A\cancel{e^{-ka}}$$

$$A + B(e^{2ka} - 1) = \frac{2m\alpha}{\hbar^2 k}A$$

$$A\left(1 - \frac{2m\alpha}{\hbar^2 k}\right) = B(1 - e^{2ka})$$

$$(1 + e^{2ka})\left(1 - \frac{2m\alpha}{\hbar^2 k}\right) = 1 - e^{2ka}$$

$$e^{-2ka} = \frac{\hbar^2 k}{m\alpha} - 1$$

let $u = 2ak$, $v = \frac{\hbar^2}{2am\alpha} = \frac{1}{u}\frac{\hbar^2 k}{m\alpha}$

$$e^{-u} - vu + 1 = 0$$

has solution for

$$x = \frac{vW\left(\frac{e^{-1/v}}{v}\right) + 1}{v}$$

$\rightarrow$ $\boxed{\text{one even bound state}}$

- energy of $\alpha = \hbar^2/ma$

$$e^{-2ka} = ka - 1 \rightarrow ka = 1.1088$$

$$E = -\frac{\hbar^2 k^2}{2ma} = \boxed{-0.407\frac{\hbar^2}{ma^2}}$$

- energy of $\alpha = \hbar^2/4ma$

$$ka = 0.3694$$

$$E = \boxed{-3.664\frac{\hbar^2}{ma^2}}$$

## Odd bound states

$\psi_1 = Ae^{-kx}$, $\psi_2 = B(e^{kx} - e^{-kx})$, $\psi_3 = -Ae^{kx}$

(bc) $Ae^{-ka} = B(e^{ka} - e^{-ka})$

$$A = B(e^{2ka} - 1)$$

(bc) $-kAe^{-ka} - kB(e^{ka} + e^{-ka}) = -\frac{2m\alpha}{\hbar^2}Ae^{-ka}$

$$A\left(\frac{2m\alpha}{\hbar^2 k} - 1\right) = B(1 + e^{2ka})$$

$$e^{-2ka} = 1 - \frac{\hbar^2 k}{m\alpha}$$

same $u, v$ gives

$$e^{-u} + vu - 1 = 0$$

$\rightarrow$ $\boxed{\text{two odd bound states}}$

- if $\alpha = \hbar^2/ma$, $E = \boxed{-0.787\frac{\hbar^2}{ma^2}}$
- if $\alpha = \hbar^2/4ma$, $E = \boxed{-1.767\frac{\hbar^2}{ma^2}}$

Problem 2

a) $1 = A^2 \int_{-\infty}^{\infty} e^{-ax^2}\,dx$

$\frac{1}{A^2} = \sqrt{\frac{\pi}{2a}} \rightarrow A = \boxed{\left(\frac{2a}{\pi}\right)^{1/4}}$

b) $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{-i(kx-\omega t)}\,dk$

$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\,dx$

$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-ax^2-ikx}\,dx$

$= \frac{1}{\sqrt{2\cancel{\pi}}} \left(\frac{2a}{\cancel{\pi}}\right)^{1/4} \sqrt{\frac{\cancel{\pi}}{a}}\, e^{-k^2/4a}$

$= \frac{2^{1/4}}{2^{1/2}} \frac{1}{\pi^{1/4}} \frac{a^{1/4}}{a^{1/2}} e^{-k^2/4a}$

$= (2\pi a)^{-\frac{1}{4}} e^{-\frac{k^2}{4a}}$, note $\omega = \frac{\hbar k^2}{2m}$

$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a}} e^{-i(kx - \frac{\hbar k^2}{2m}t)}\,dk$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\left[\frac{1}{4a} + \frac{i\hbar t}{2m}\right]k^2 - [ix]k}\,dk$

$= \frac{1}{\sqrt{2\cancel{\pi}}} \frac{1}{(2\pi a)^{1/4}} \sqrt{\frac{\cancel{\pi}}{\frac{1}{4a} + \frac{i\hbar t}{2m}}}\; e^{\frac{(ix)^2}{4\left(\frac{1}{4a} + \frac{i\hbar t}{2m}\right)}}$

$= \frac{1}{\sqrt{2}} \frac{2}{(2\pi a)^{1/4}} \frac{e^{-x^2/\left(\frac{1}{a} + 2i\hbar t/m\right)}}{2\sqrt{\frac{1}{4a} + i\hbar t/2m}}$

$= \boxed{\left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+2i\hbar a t/m)}}{\sqrt{1+2i\hbar a t/m}}}$

c) $|\Psi(x,t)|^2 =$

$= \sqrt{\frac{2a}{\pi}} \left| \frac{e^{\frac{-ax^2}{1+2i\hbar a t/m}}}{\sqrt{1+2i\hbar a t/m}} \right|^2$ let $u = \frac{2\hbar a t}{m}$

$= \sqrt{\frac{2a}{\pi}} e^{-2ax^2} \left( \frac{e^{1/(1+iu)(1-iu)}}{\sqrt{(1+iu)(1-iu)}} \right)$

$= \sqrt{\frac{2a}{\pi}} e^{-2ax^2/(1+u^2)} \frac{1}{\sqrt{1+u^2}}$

$w = \sqrt{\frac{a}{1+u^2}}$ so

$\boxed{|\Psi|^2 = \sqrt{\frac{2}{\pi}}\, w e^{-2w^2x^2}}$

as t increases, w decreases, so $|\Psi|^2$ starts out narrow Gaussian and flattens out.

d) $\boxed{\langle x \rangle = 0}$, Gaussian is symmetric

$\boxed{\langle p \rangle = 0}$

$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}} \int x^2 w e^{-2w^2 x^2}\,dx$

$= \sqrt{\frac{2}{\pi}}\, w \frac{1}{4w^2} \sqrt{\frac{\pi}{2w^2}} = \frac{1}{4w^2}$

$= \frac{1+u^2}{4a} = \boxed{\frac{1 + 4\hbar^2 a^2 t^2/m^2}{4a}}$

$\langle p^2 \rangle = -\hbar^2 \int \Psi^* \frac{d^2\Psi}{dx^2}\,dx$

uhhside. $\langle p^2 \rangle = \boxed{a\hbar^2}$

e) $\sigma_x \sigma_p = \sqrt{\frac{1}{4w^2}\, a\hbar^2} = \frac{\hbar}{2}\sqrt{1+u^2}$

$\geq \hbar/2$, $\boxed{\text{holds}}$

closest for $u=0$ or $\boxed{t=0}$