Lecture H. Homework 8

Covered Contents: Gaussian Elimination, LU and Cholesky Factorization (Lec 23 - 27)
Deadline: 12/10/2021, 23:59 PST
Total points: Pen-and-Paper (10 + 15 + 20 + 15 + 10 = 70) + Coding (30) = 100.
Submit "hw8.zip"

Pen and Paper

H.1. Use Gaussian elimination with back substitution to solve the linear system of equations

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 2x_1 + 5x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 + 4x_3 &= 11 \end{aligned}$$

H.2. Consider the two matrices

$$A = \begin{pmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 7 & 2 & -1 \end{pmatrix} \quad \tilde{A} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 5 & 5/2 \\ 0 & -5 & -23/2 \end{pmatrix}$$

(a) Write down two elementary row operations to transform A to $\tilde{A}$.
(b) For each elementary row operation, construct a 3×3 matrix P such that multiplication of A on the left by P is identical to performing the elementary row operation. Are the two matricies P_1 and P_2 lower triangular?
(c) Multiply the two matricies P_1 and P_2 from part (b) together to produce one matrix P. Is P also lower triangular?

H.3. (a) Suppose L is a nonsingular 4×4 lower triangular matrix

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}$$

Show that if

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} \end{pmatrix}$$

is such that $L\Gamma = Id$ where Id is the 4×4 identify matrix, then Γ must also be lower triangular.

(*Note:* this shows that the 'right-inverse' of L must be lower triangular; a nearly identical computation shows its 'left-inverse' also must be lower triangular, which then implies L^{-1} must be lower triangular.)

(b) Suppose that L_1 and L_2 are both 4×4 lower triangular matrices. Show that their product $L_1 L_2$ is also lower triangular.

H.4. Recall problem 2 from HW5, which asked you to construct the natural cubic spline $s(x)$ for the data in the table below; there are 8 unknowns to determine: 4 coefficients for each cubic function s_1 and s_2.

x	$f(x)$
0.1	− 0.29004996
0.2	− 0.56079734
0.3	− 0.81401972

(a) Using the constraints coming from the definition of the cubic spline, derive a matrix equation $Ax = b$ for the 8×8 linear system that the spline coefficients must satisfy.

(b) Use MATLAB (or Octave) to solve the linear system you constructed in (a), and then write down the resulting cubic spline. (Note: the command is quite simple: $x = A\backslash b$.)

H.5. A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric positive-definite if (i) it is equal to its own transpose $A = A^T$ and (ii) for any nonzero $x \in \mathbb{R}^n$

$$\langle x, Ax \rangle > 0.$$

Show that every eigenvalue of A is both real and positive.

Coding

What to submit: (X points).

Download the functions 'HW8.m' and 'cholesky.m', and implement the Cholesky factorization algorithm from class. Given a matrix A that is symmetric and positive-definite, your function should return a lower triangular matrix L such that

$$A = LL^T.$$