Math 131A - Homework 1

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Question 1.

By Corollary 2.3 we know that any rational solution of this equation must be an integer that divides the bias c_0. Since $c_0 = -1$ we have only the options ± 1, so let's check both.

$$\begin{aligned} P_1 &: (1)^4 - 2(1)^3 + 3(1)^2 + 5(1) - 1 \\ &= 1 - 2 + 3 + 5 - 1 \\ &\neq 0 \\ P_{-1} &: (-1)^4 - 2(-1)^3 + 3(-1)^2 + 5(-1) - 1 \\ &= 1 + 2 + 3 - 5 - 1 \\ &= 0 \end{aligned}$$

Thus the only rational solution to this is $\boxed{\text{-1}}$.

Question 2.

Proof of the properties of a field.

1. $a + c = b + c \Rightarrow a = b$. Assume $a + c = b + c$.

$$\begin{aligned} a &= a + 0 \\ a &= a + (c + (-c)) \\ a &= (a + c) + (-c) \\ a &= (b + c) + (-c) \\ a &= b + (c + (-c)) \\ a &= b + 0 \\ a &= b \end{aligned}$$

2. $a \cdot 0 = 0$.

$$\begin{aligned} a \cdot 0 &= a(0 + 0) \\ a \cdot 0 &= a \cdot 0 + a \cdot 0 \\ a \cdot 0 - a \cdot 0 &= a \cdot 0 + (a \cdot 0 - a \cdot 0) \\ 0 &= a \cdot 0 + 0 \\ 0 &= a \cdot 0 \end{aligned}$$

3. $(-a)b = -(ab)$

$$
\begin{aligned}
(-a)b &= (-a)b + 0 \\
(-a)b &= (-a)b + (ab + -(ab)) \\
(-a)b &= (-a)b + ab + -(ab) \\
(-a)b &= (-a + a)b + -(ab) \\
(-a)b &= (0)b + -(ab) \\
(-a)b &= 0 + -(ab) \\
(-a)b &= -(ab)
\end{aligned}
$$

4. $(-a)(-b) = ab$

$$
\begin{aligned}
(-a)(-b) &= (-a)(-b) + ab + -(ab) \\
(-a)(-b) &= (-a)(-b) + ab + (-a)b \\
(-a)(-b) &= (-a)(-b + b) + ab \\
(-a)(-b) &= (-a)(0) + ab \\
(-a)(-b) &= 0 + ab \\
(-a)(-b) &= ab
\end{aligned}
$$

5. $ac = bc \wedge c \neq 0 \Rightarrow a = b$. Let $ac = bc$ and $c \neq 0$.

$$
\begin{aligned}
ac &= bc \\
ac - bc &= 0 \\
(a - b)c &= 0 \\
a - b &= 0 \\
a &= b
\end{aligned}
$$

6. $ab = 0 \Rightarrow a = 0 \vee b = 0$. Go over all possible values of a. First assume $a = 0$, then $a = 0 \vee b = 0$ is true. Second assume $a \neq 0$. Then it has inverse a^{-1}.

$$
\begin{aligned}
ab &= 0 \\
ab(a^{-1}) &= 0(a^{-1}) \\
(a \cdot a^{-1})b &= 0 \\
1b &= 0 \\
b &= 0
\end{aligned}
$$

So that $a = 0 \vee b = 0$ is true since b must be 0.

Question 3.

$$\begin{aligned}
\frac{a^2+b^2}{2} &= \frac{a^2+b^2-2ab+2ab}{2} \\
\frac{a^2+b^2}{2} &= \frac{(a-b)^2+2ab}{2} \\
\frac{a^2+b^2}{2} &= \frac{(a-b)^2}{2}+ab \\
(a-b)^2 &\geq 0 \\
\frac{a^2+b^2}{2} &\geq ab
\end{aligned}$$

Question 4.

Use triangle inequality that $|a-c| \leq |a-b| + |b-c|$.

$$\begin{aligned}
a &= a+0 \\
a &= a+((-b)+b) \\
a &= (a-b)+b \\
|a| &= |(a-b)+b| \\
|(a-b)+b| &\leq |a-b|+|b| \\
|a| &\leq |a-b|+|b| \\
|a|-|b| &\leq |a-b| \\
||a|-|b|| &\leq |a-b|
\end{aligned}$$

Question 5.

Let the set $[a,b) = S$, and $S \subset R$.

$\inf[a,b) = a$ because

- a is a lower bound of S since $a \in \mathbb{R}$ and for all $x \in S, x \geq a$.
- Suppose we have a different lower bound m of S, then for all $x \in S, x \geq m$. Since $a \in S$, $a \geq m$.

$\sup[a,b) = b$ because

- b is an upper bound of S since $b \in \mathbb{R}$ and for all $x \in S, x \leq b \Rightarrow x \leq b$.
- Suppose we have a different minimal upper bound $M < b$ of S, then for all $x \in S, x \leq M$ and $M \in S$. But $b \notin S$, so we can choose $c \in S : M < c < b$. Thus M is not an upper bound, let alone the minimal upper bound.

Question 6.

If T is bounded above that means that it has some upper bound M such that $\forall t \in T : t \leq M$. Since $S \subseteq T$ by definition of subset $\forall s \in S : s \in T$. Therefore $\forall s \in S : s \leq M$ so M is an upper bound of S and S is bounded above.

Let $\sup T = m$. Then $\forall t \in T : t \leq m$ and any other upper bound M of T has to be either outside T or equal to m, that is, $m \leq M$. Let $\sup S = n$. Note from before $\forall s \in S : s \leq M$ for an upper bound M of T. Then we have $\forall s \in S : s \leq m$. Since $n \in S$ we have $n \leq m$, or $\sup S \leq \sup T$.

Question 7.

Note that since $a \leq \sup A$ and $b \leq \sup B$ then $a + b \leq \sup A + \sup B$. Since this is the case for all $a + b$ then $\sup(A + B) \leq \sup A + \sup B$.

Next note that we can choose some $\epsilon > 0$ so that we can represent an a, b as $\sup A - \epsilon, \sup B - \epsilon$. Then $a + b = \sup A + \sup B - 2\epsilon$ so $a + b \geq \sup A + \sup B$. We also have $\sup(A + B) \geq a + b$ so $\sup(A + B) \geq \sup A + \sup B$.

Since we have the two inequalities $\sup(A+B) \leq \sup A+\sup B$ and $\sup(A+B) \geq \sup A+\sup B$, it must be the case that $\sup(A + B) = \sup A + \sup B$.

Question 8.

Define the set S to be $r \in \mathbb{Q} : r < a$ for some $a \in \mathbb{R}$. First, a is an upper bound of S since $\forall r \in S : r < a$ by definition of S. Suppose we have a different minimal upper bound M of S where $M < a$, then $\forall r \in S : r \leq M$ and $M \in S$. But $a \notin S$. The question is, can we choose a $c \in S : M < c < a$? Note that M and c are rational numbers, while a may be rational or irrational. However, because $\mathbb{Q}$ is dense in $\mathbb{R}$, this means that between *any* two real numbers there will always be a rational number. Thus, there would be a rational $c \in S : M < c < a$, which means that M cannot be the minimal upper bound of S. Thus a is the minimal upper bound and therefore $\sup r \in \mathbb{Q} : r < a = a$ for all $a \in \mathbb{R}$.