- You are encouraged to discuss the problems with other students. However, you must write the solutions using your own words. Violation of the honor code may void your submissions.
- The assignments must be submitted through Gradescope. **No late homework will be accepted or graded.** Please allow plenty of time to upload your assignments, especially if you are using Gradescope for the first time.
- You should demonstrate your works that lead to the final answers in order to receive full credit.

Problems

1. Fix $\alpha > 1$. Take $s_1 > \sqrt{\alpha}$, and define the sequence $(s_n)_{n\in\mathbb{N}}$ recursively by

$$s_{n+1} = \frac{\alpha + s_n}{1 + s_n} = s_n + \frac{\alpha - s_n^2}{1 + s_n}.$$

(a) Prove that $s_1 > s_3 > s_5 > \dots$.

(b) Prove that $s_2 < s_4 < s_6 < \dots$.

(c) Prove that $\lim s_n = \sqrt{\alpha}$.

2. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ that is bounded above. Show that a real number ℓ is equal to $\limsup s_n$ if and only if

(i) For any $\varepsilon > 0$, $s_n < \ell + \varepsilon$ holds for all but finitely many n's, and

(ii) For any $\varepsilon > 0$, $s_n > \ell - \epsilon$ holds for infinitely many n's.

Conclude that

$$\limsup s_n = \inf\{M \in \mathbb{R} : s_n < M \text{ holds for all but finitely many } n\text{'s}\}.$$

3. Let (s_n) be a sequence in $\mathbb{R}$. Show that $\limsup(-s_n) = -\liminf s_n$.

4. Let (s_n) and (t_n) be sequences in $\mathbb{R}$ such that $s_n \le t_n$ for all but finitely many n's. Show that

$$\liminf s_n \le \liminf t_n \quad \text{and} \quad \limsup s_n \le \limsup t_n.$$

5. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Show that $+\infty$ is a subsequential limit of (s_n) if and only if (s_n) is not bounded above.

6. In each of the following subproblems, give an example of a sequence (s_n) satisfying the given condition.

(a) $\limsup s_n = -\infty$.

(b) $\liminf s_n = -1$ and $\limsup s_n = 1$, but $s_n \notin [-1, 1]$ for any n.

(c) The set of subsequential limits of (s_n) is $[0, 1]$.
Hint: Example 3 of Section 11 in the textbook might be helpful.

7. Let $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be the following sequences that repeat in cycles of four:

$$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \dots),$$
$$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \dots).$$

(a) Find $\liminf(s_n + t_n)$ and $\liminf s_n + \liminf t_n$.

(b) Find $\limsup(s_n + t_n)$ and $\limsup s_n + \limsup t_n$.

(c) Find $\limsup s_n t_n$ and $(\limsup s_n)(\limsup t_n)$.

(Note: This problem demonstrates an example for which the inequalities in Theorem 12.2 of Note 8 can be strict.)

8. **(a)** Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ satisfying

$$|s_{n+1} - s_n| \le 2^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Show that (s_n) is a Cauchy sequence and hence converges in $\mathbb{R}$.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| \le \frac{1}{n}$ for all $n \in \mathbb{N}$?

More Practice Problems

Here are some more practice problems for the interested student. However, these problems will not be graded and need not be turned in.

9. (Existence of e) In this exercise, we show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists in $\mathbb{R}$.

(a) Define $(A_n)_{n\in\mathbb{N}}$ by $A_n = \sum_{k=0}^{n} \frac{1}{k!}$. By noting that $\frac{1}{k!} \le \frac{1}{k-1} - \frac{1}{k}$ for $k \ge 2$, show that (A_n) converges.

(b) Show that

$$\frac{(n-k)^k}{k!} \le \binom{n}{k} \le \frac{n^k}{k!} \quad \text{for all} \quad 0 \le k \le n,$$

where we adopt the convention that $0^0 = 1$.

(c) Fix $N \in \mathbb{N}$. Show that for $n \ge N$,

$$\sum_{k=0}^{N} \frac{1}{k!} \left(1 - \frac{k}{n}\right)^k \le \left(1 + \frac{1}{n}\right)^n \le A_n.$$

(d) By taking liminf and limsup to the inequalities in part (c), deduce that for any $N \in \mathbb{N}$,

$$A_N \le \liminf \left(1 + \frac{1}{n}\right)^n \le \limsup \left(1 + \frac{1}{n}\right)^n \le \lim A_n.$$

(e) Conclude that $\lim \left(1 + \frac{1}{n}\right)^n = \lim A_n$.

10. (Cesàro Mean) Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ by

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} s_k = \frac{s_1 + s_2 + \cdots + s_n}{n} \quad \text{for each} \quad n \in \mathbb{N}.$$

(a) Show

$$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n.$$

Hint: For the last inequality, show first that $M, N \in \mathbb{N}$ and $M \ge N \ge 1$ implies

$$\sup\{\sigma_n : n \ge M\} \le \frac{s_1 + \cdots + s_N}{M} + \left(1 - \frac{N}{M}\right) \sup\{s_n : n \ge N\}.$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and is equal to $\lim s_n$.

(c) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

11. Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that

$$\lim_{n\to\infty} (2s_{n+1} - s_n) = \ell$$

for some real number ℓ.

(a) Let M be an upper bound of $\{|s_1|\} \cup \{|2s_{n+1} - s_n| : n \in \mathbb{N}\}$. (Such M exists since $2s_{n+1} - s_n$ converges.) Show that $|s_n| \le M$ for all $n \in \mathbb{N}$.

(b) Show that $\limsup s_n \le \ell$ and $\ell \le \liminf s_n$. *Hint:* The following equality might be helpful:

$$s_{n+1} = \frac{1}{2}(2s_{n+1} - s_n) + \frac{1}{2}s_n.$$

(c) Conclude that $\lim s_n = \ell$.