Math 151A - Homework 1

Zooey Nguyen

zooeyn@ucla.edu

October 4, 2021

Question 1.

Absolute and relative error.

$$e_{abs} = |14 - 3.7| = \boxed{10.30000}$$
$$e_{rel} = |14 - 3.7|/|14| = \boxed{0.73571}$$

Question 2.

Taylor polynomial.

$$\begin{aligned} P_3(x \approx 0) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}}{6}x^3 \\ &= \sqrt{1+x}|_0 + \frac{1}{2\sqrt{1+x}}|_0 x - \frac{1}{8(1+x)^{3/2}}|_0 x^2 + \frac{1}{16(1+x)^{5/2}}|_0 x^3 \\ &= \boxed{1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}} \end{aligned}$$

Taylor approximations.

$$\begin{aligned} \sqrt{0.5} &\approx P_3(-0.5) = 0.71093 & e_{abs}(\sqrt{0.5}) &= 0.00382 \\ \sqrt{0.75} &\approx P_3(-0.25) = 0.86621 & e_{abs}(\sqrt{0.75}) &= 0.00018 \\ \sqrt{1.25} &\approx P_3(0.25) = 1.11816 & e_{abs}(\sqrt{1.25}) &= 0.00012 \\ \sqrt{1.5} &\approx P_3(0.5) = 1.22656 & e_{abs}(\sqrt{1.5}) &= 0.00181 \end{aligned}$$

Question 3.

We wish to prove that $\exists c \in (0.2, 0.3)$ s.t. $x \cos x - 2x^2 + 3x - 1 = 0$ for $x = c$. Let the left-hand side be denoted $f(x)$. First note that $x \cos x$ is continuous on $[0.2, 0.3]$, as are the polynomial terms, so $f(x)$ is continuous. Next, we have that $f(0.2) = -0.28398 < 0$, and $f(0.3) = 0.00660 > 0$. By IVT, we have that $\exists c \in (0.2, 0.3)$ s.t. $f(c) = 0$, so we have a solution to the equation given. Note that the interval $(0.2, 0.3) \subset [0.2, 0.3]$, therefore $x \in (0.2, 0.3) \Rightarrow x \in [0.2, 0.3]$, so the solution c exists on the interval $[0.2, 0.3]$.

Question 4.

Note this requires 15 FLOPs. The common factor is e^x so we can nest it this way, requiring only 9 FLOPs.

$$f(x) = (((((1.01e^x - 4.62)e^x) - 3.11)e^x) + 12.2)e^x - 1.99$$

Estimating $f(1.53)$ using the naive form.

$$\begin{aligned}
f(1.53) &= 1.01(4.62)(4.62)(4.62)(4.62) - 4.62(4.62)(4.62)(4.62) - 3.11(4.62)(4.62) + 12.2(4.62) - 1.99 \\
&= 1.01(21.344)(4.62)(4.62) - 4.62(21.344)(4.62) - 3.11(21.344) + 56.364 - 1.99 \\
&= 1.01(98.609)(4.62) - 4.62(98.609) - 66.380 + 56.364 - 1.99 \\
&= 1.01(455.574) - 455.574 - 66.380 + 56.364 - 1.99 \\
&= 460.130 - 455.574 - 66.380 + 56.364 - 1.99 \\
&= -7.45
\end{aligned}$$

Estimating $f(1.53)$ using the nested form.

$$\begin{aligned}
f(1.53) &= (((((1.01(4.62) - 4.62)(4.62)) - 3.11)(4.62)) + 12.2)(4.62) - 1.99 \\
&= (((((4.666 - 4.62)(4.62)) - 3.11)(4.62)) + 12.2)(4.62) - 1.99 \\
&= (((((0.046)(4.62)) - 3.11)(4.62)) + 12.2)(4.62) - 1.99 \\
&= ((((0.212) - 3.11)(4.62)) + 12.2)(4.62) - 1.99 \\
&= ((-13.389) + 12.2)(4.62) - 1.99 \\
&= (-1.189)(4.62) - 1.99 \\
&= -5.493 - 1.99 \\
&= -7.483
\end{aligned}$$

Comparing the errors. The error of the nested result is lower..

$$\begin{aligned}
e_{rel,naive} &= |-7.483 + 7.61|/7.61 = 0.021 \\
e_{rel,nested} &= |-7.45 + 7.61|/7.61 = 0.017
\end{aligned}$$

Question 5.

Linear convergence.

$$\begin{aligned}
\lim \frac{|p_{n+1} - p|}{|p_n - p|} &= \lim \frac{|p_{n+1}|}{|p_n|} \\
&= \lim \frac{|p_{n+1}|}{|p_n|} \\
&= \lim \frac{(1/10)^{n+1}}{(1/10)^n} \\
&= \lim \frac{1}{10}
\end{aligned}$$

Quadratic convergence.

$$\begin{aligned}
\lim \frac{|p_{n+1} - p|}{|p_n - p|^2} &= \lim \frac{|p_{n+1}|}{|p_n|^2} \\
&= \lim \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} \\
&= \lim \frac{10^{-2^{n+1}}}{10^{-2^n * 2}} \\
&= \lim \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} \\
&= 1
\end{aligned}$$

Question 6.

We wish to prove that a function being L-Lipschitz on $[a, b]$ implies continuity. Choose any $\epsilon > 0$. We have by the Lipschitz condition that $\forall x_0 \in [a, b]$, we can get $|f(x) - f(x_0)| \leq L|x - x_0|$. We can choose $\delta = \epsilon/L$. Then we get that if $|x - x_0| < \delta = \epsilon/L$, this implies that $|f(x) - f(x_0)| \leq L|\epsilon/L| = \epsilon$. Therefore we have that $f(x) \in C([a, b])$.

We wish to prove that if the derivative of f is bounded on $[a, b]$ by L, then f is L-Lipschitz. By MVT, $\exists c \in (a, b)$ s.t. $\forall x, y \in [a, b], f'(c) = \frac{f(x)-f(y)}{x-y}$. Since $|f'(c)| < L$, we have that $\frac{|f(x)-f(y)|}{|x-y|} < L$, so $|f(x) - f(y)| \leq L|x - y|$, meaning that f is L-Lipschitz on $[a, b]$.

Consider the function $f(x) = \sqrt{(}x)$ on $[0, a]$ for some real number a. It is not Lipschitz because nearing 0, the change in $f(x)$ becomes very great compared to the change in x. Say we choose the range $[0, x]$. Then the Lipschitz inequality becomes $\sqrt{x} \leq Lx$, or $\frac{1}{\sqrt{x}} \leq L$. There is no constant L that satisfies this for all x, since as $x \to 0$, $\frac{1}{\sqrt{x}}$ goes to infinity.

Question 7.

(a) False. It cannot be the case that $3x^2 \leq cx^4$ for any value of c because this inequality is equivalent to $\frac{3}{x^2} \leq c$, and the left hand side is not bounded above as $x \to 0$.

(b) True. We can choose $c = 5$ so that we get $x^10 + 4x^3 \leq 5x^3$, or equivalently $x^10 \leq x^3$, which is true as $x \to 0$.

(c) True. We need c s.t. $4x^4 + 3x^3 + 2x^2 \leq cx$, or equivalently, $4x^3 + 3x^2 + 2x \leq c$. When $x \to 0$ and $x \in [0, 1]$ we have that $x^3, x^2, x \leq 1$. Thus we can choose $c = 4 + 3 + 2 = 9$ to satisify the inequality.

(d) True. Note that the Taylor expansion of e^x as $x \to 0$ is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$. So the problem is to see if $\frac{x^3}{6} + \frac{x^4}{24} + \ldots = O(x^3)$. It is, because all the following exponentials rapidly decrease faster than x^3 as $x \to 0$.