UCLA Math151A Fall 2021
Lecture 13
20211022

More on High Order Interpolation Issues, Piecewise Polynomails

Issues with High Order Polynomial Interpolations

Theorem 10.2 (Error of Lagrangian Polynomial Interpolation).

Let $\{x_0, x_1, \ldots, x_n\} \in [a, b]$ be distinct. Let $f \in C^{n+1}([a, b])$, $P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$, then $\forall x \in [a, b], \exists \xi(x) \in (a, b)$ s.t.

$$
\begin{aligned}
f(x) &= P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n) \\
&= P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k) \qquad \square
\end{aligned}
$$

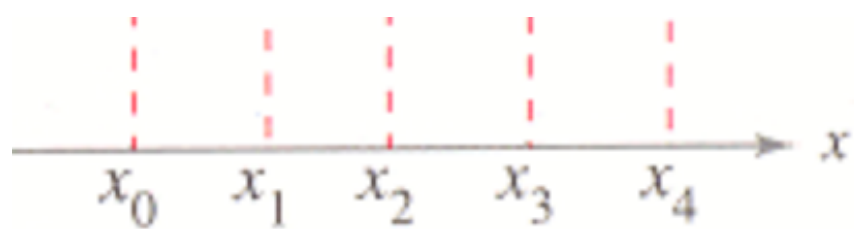
$$\boxed{f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k).}$$

Remark

if we can bound the $n+1$ derivative, $\exists M > 0$ s.t. $\max_{a \le x \le b} |f^{(n+1)}(x)| \le M$, then

$$|f(x) - P(x)| \le \frac{M}{(n+1)!}|\Pi_{k=0}^{n}(x - x_k)|$$

Suppose now $x_i's$ are equispaced. E.g., $x_i = x_0 + ih, h = \frac{b-a}{n}$.



Then we can bound $\displaystyle\max_{a \le x \le b} |\Pi_{k=0}^{n}(x - x_k)| \le \frac{1}{4}h^{n+1} \cdot n!$

We'll skip the proof. See extra reading (not required).

$$\Rightarrow \max_{a \le x \le b} |f(x) - P(x)| \le \frac{M}{(n+1)!}\frac{1}{4}h^{n+1} \cdot n! = \frac{1}{4}\frac{M}{n+1}h^{n+1}$$

$$\max_{a\le x\le b}|f(x)-P(x)| \le \frac{M}{(n+1)!}\frac{1}{4}h^{n+1}\cdot n! = \frac{1}{4}\frac{M}{n+1}h^{n+1}$$

What's the point?

For some nice functions, decreasing h (i.e., increasing n) will decrease the error.

Example 13.1. $f(x)=e^{-x}, x\in[0,1]$.

If we want $\max_{0\le x\le 1}|f(x)-P(x)|<10^{-6}$,

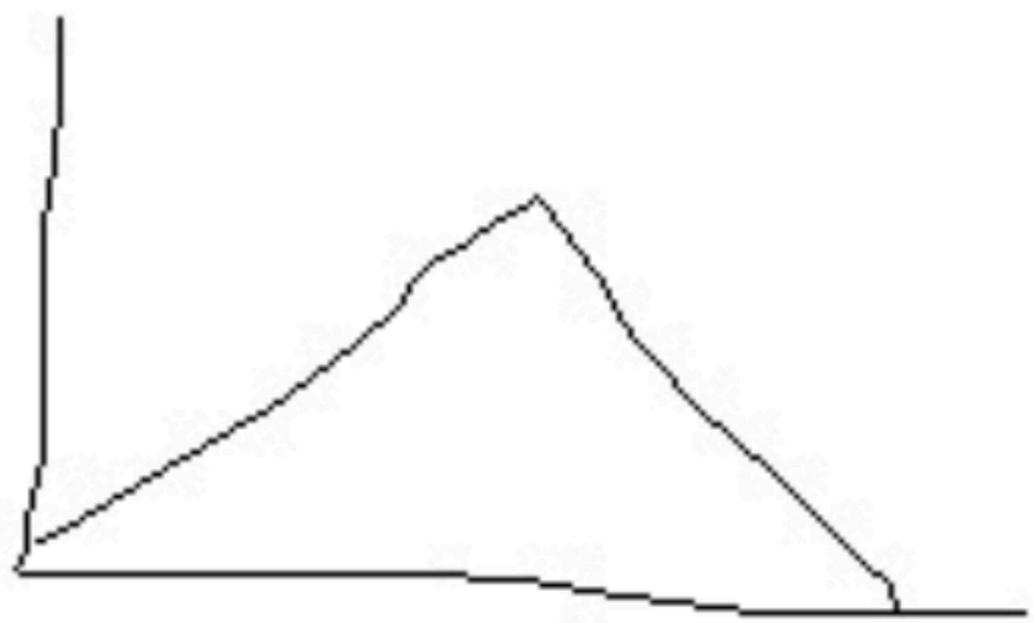
One can show that, using the bound, we need at least n + 1 = 7 data points (n = 6)

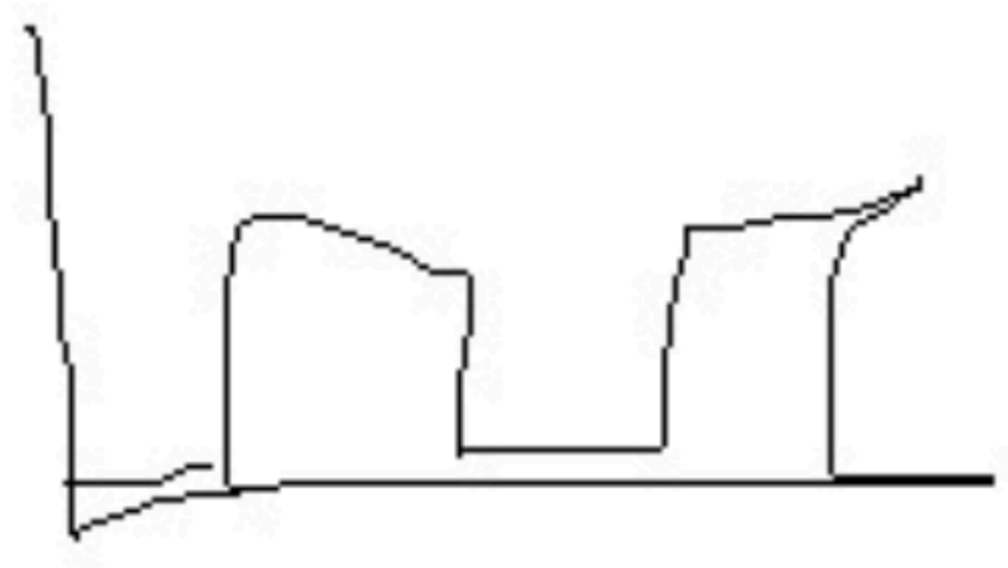
Increasing the points will further decrease the error.

Key observation

$e^{-x}\in C^{\infty}([0,1])$ and its derivatives are bounded.

Recall in previous lecture



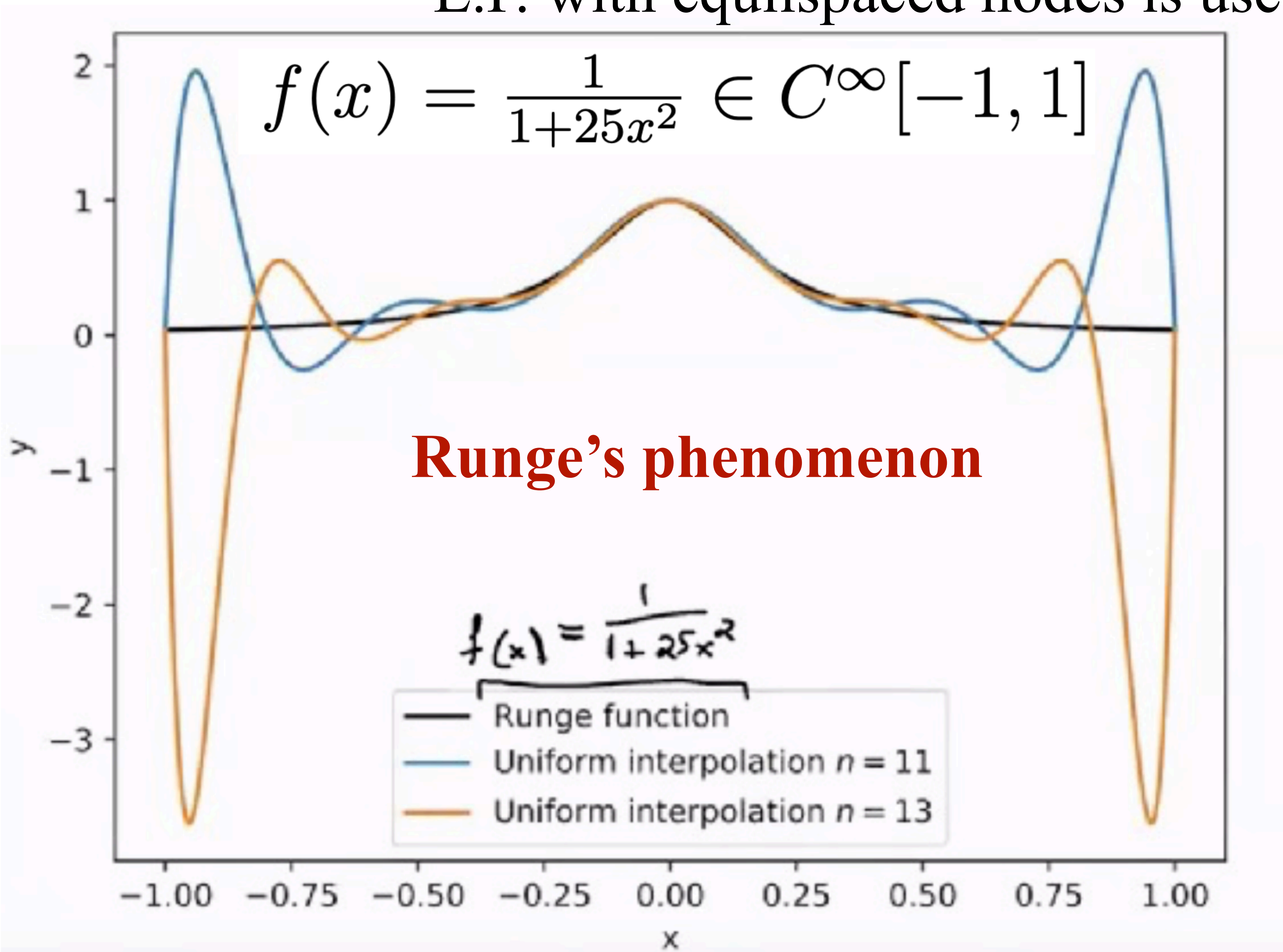


theorem requires $f \in C^{n+1}([a,b]) \quad \Rightarrow \quad$ using high order polynomials will fail.

$\boxed{\text{Runge function.}}$

L.P. with equilspaced nodes is used.

$$f(x) = \frac{1}{1+25x^2} \in C^\infty[-1,1]$$



Runge's phenomenon

If we increase n further, oscillation will have higher magnitude.

Runge's phenomenon: Why do they occur?

3 slides ago: for equispaced nodes,

$$\max_{a \le x \le b} |f(x) - P(x)| \le \frac{M}{(n+1)!} \frac{1}{4} h^{n+1} \cdot n! = \frac{1}{4} \frac{M}{n+1} h^{n+1}$$

(let $[a, b] = [-1, 1]$)

$$\max_{-1 \le x \le 1} |f(x) - P(x)| \le \frac{1}{4} \frac{M}{n+1} h^{n+1} \qquad M = \max_{-1 \le x \le 1} |f^{(n+1)}(x)|$$

The trouble is that for $f(x) = \frac{1}{1+25x^2}$, $M \to \infty$ as $n \to \infty$.

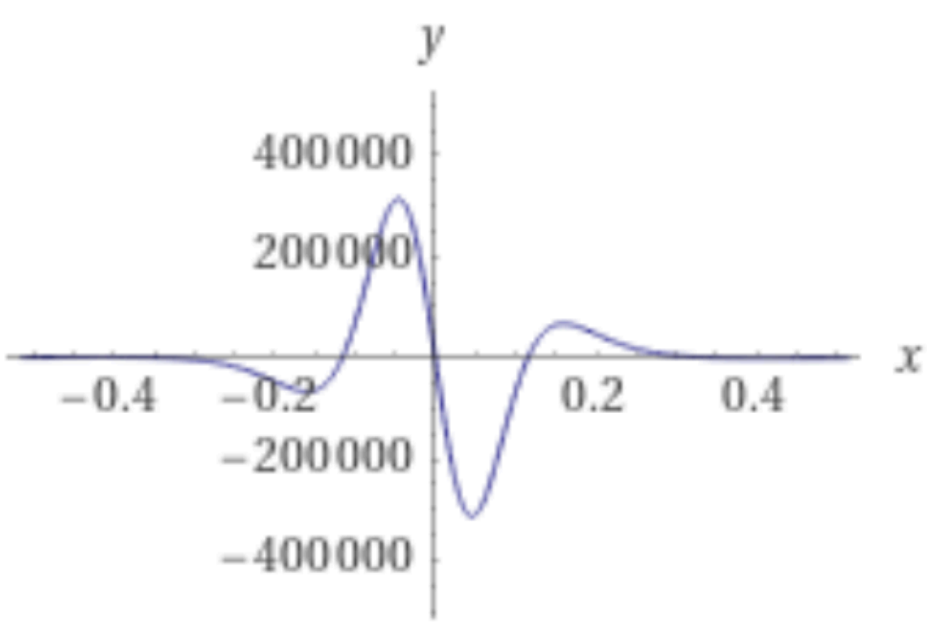
$$\max_{-1\le x\le 1} |f(x) - P(x)| \le \frac{1}{4}\frac{M}{n+1}h^{n+1}$$

The trouble is that for $f(x) = \frac{1}{1+25x^2}$, $M \to \infty$ as $n \to \infty$.

Derivative

$$\frac{d^5}{dx^5}\left(\frac{1}{1+25x^2}\right) = -\frac{11\,250\,000\,x}{(25x^2+1)^4} - \frac{37\,500\,000\,000\,x^5}{(25x^2+1)^6} + \frac{1\,500\,000\,000\,x^3}{(25x^2+1)^5}$$

Plots



(x from -0.5 to 0.5)

Dealing with Runge's Phenomenon

In general, one can :

- Avoid using equispaced points.

  We can cleverly choose $\{x_i\}$ to minimize error,

Chebyshev polynomials of the first kind:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2x\,T_n(x) - T_{n-1}(x)\,. \end{aligned}$$

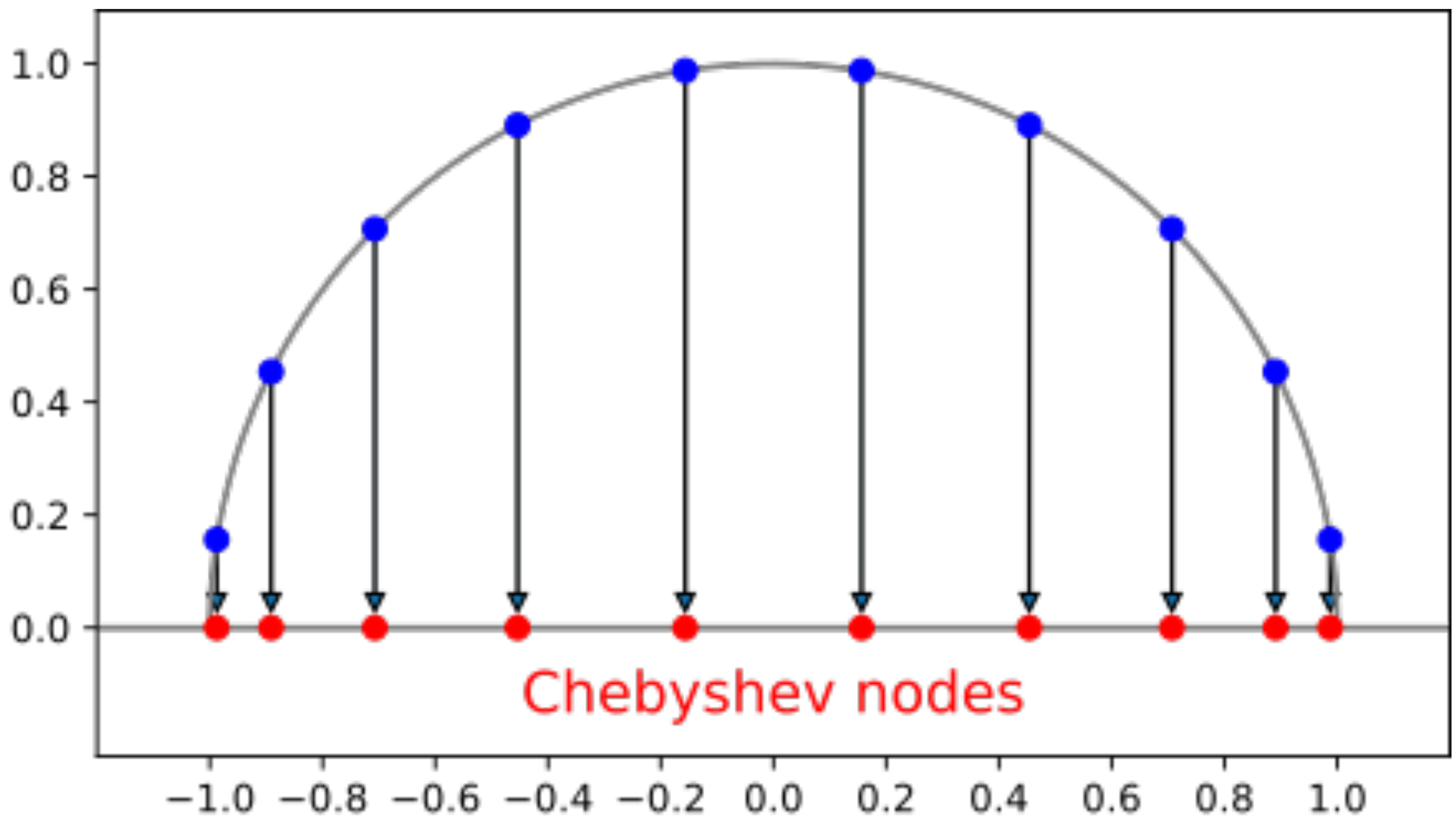
For a given positive integer n the Chebyshev nodes in the interval (−1, 1) are

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \ldots, n.$$

These are the roots of the Chebyshev polynomial of the first kind of degree n.

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \ldots, n.$$

The roots of the polynomails are the projection onto x axis of equal pieces on the circle.



The resulting interpolation polynomial minimizes the effect of Runge's phenomenon.

Dealing with Runge's Phenomenon

If we cannot easily pick points (e.g., doing an experiment).

another option is to approximate f(x) with something other than polynomials

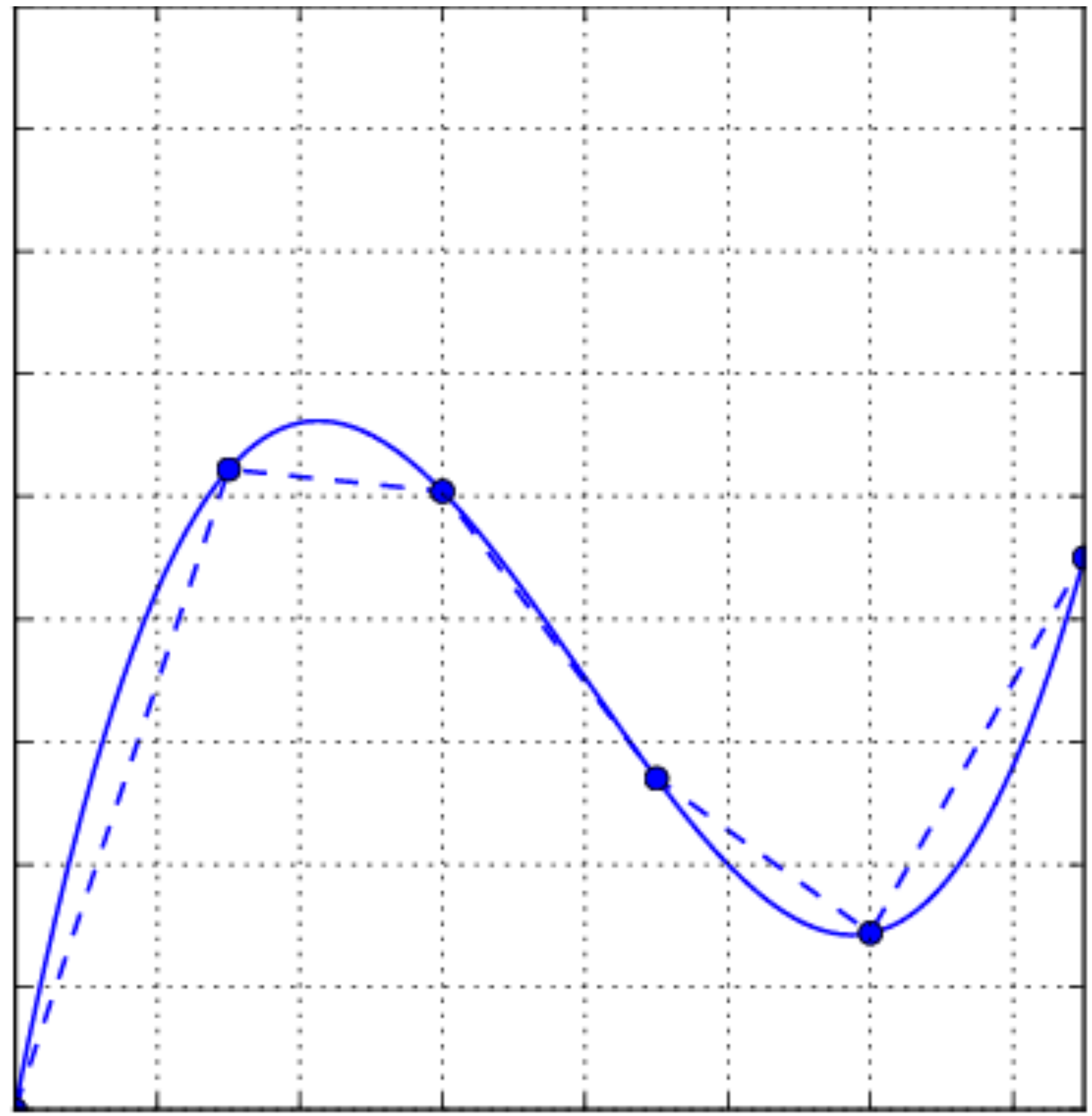
e.g., Fourier (approximating a function with sines and cosines at different frequencies).

Another option:

Use pieceiwse polynomial apprioxmation.

Piecewise Polynomails

E.g., piecewise linear.

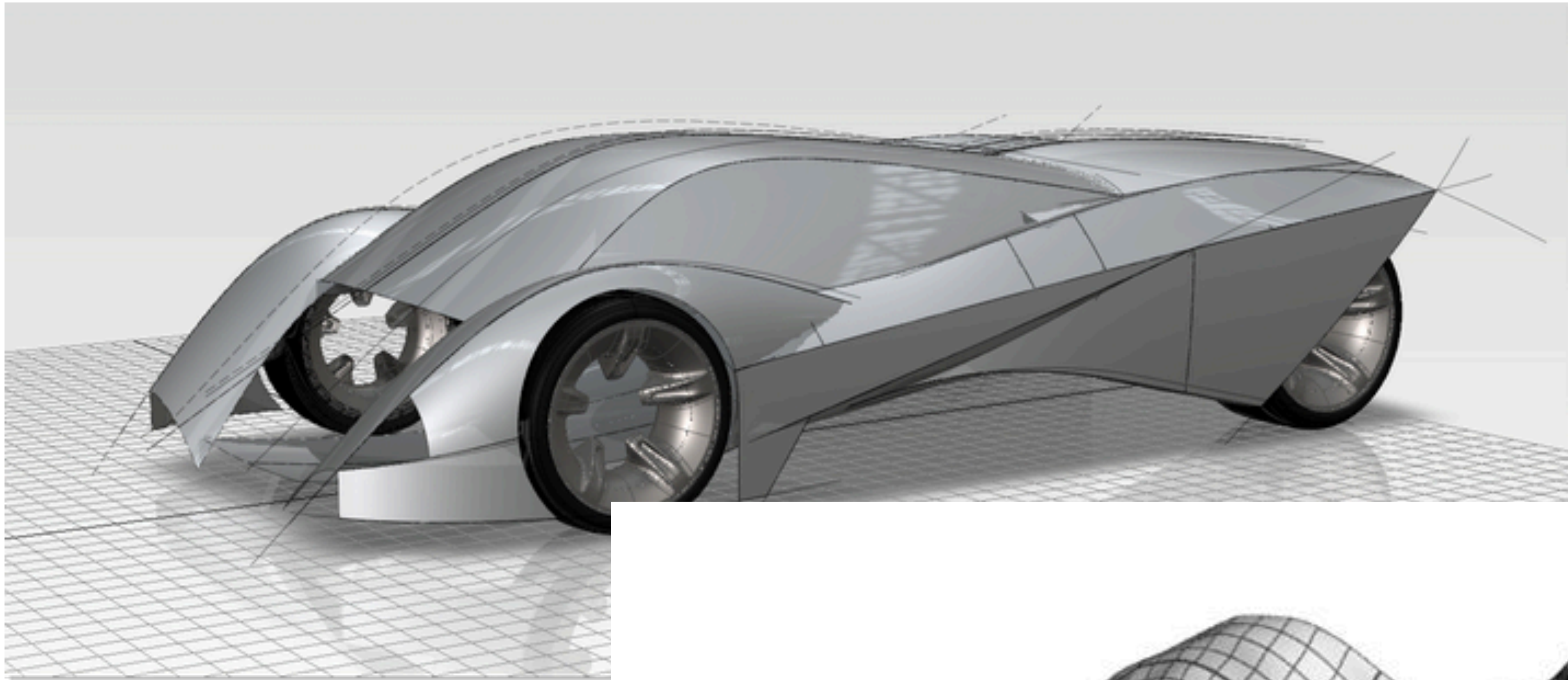


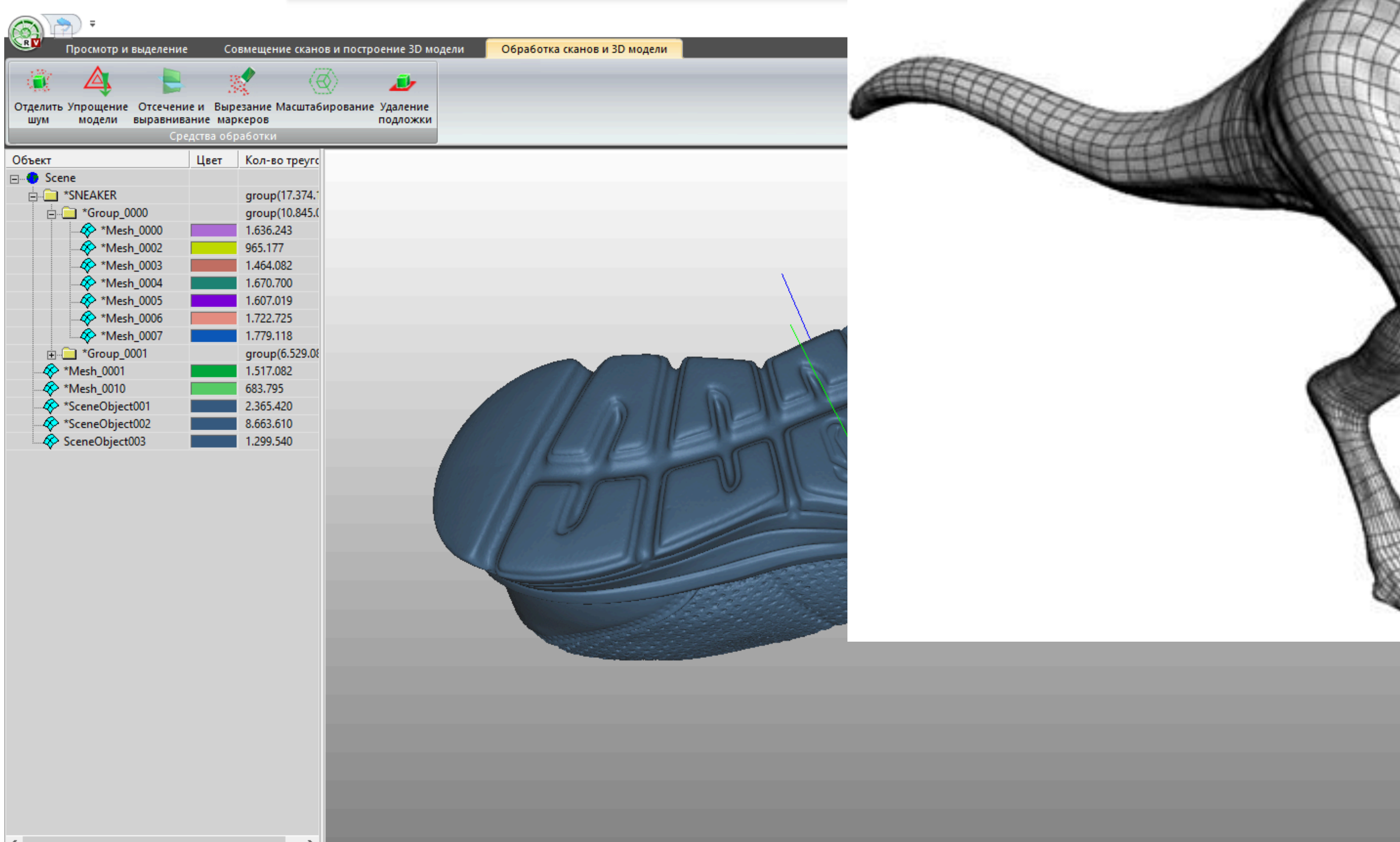
Or, can use higher order piecewise polynomials.

Cubic polynomials are popular.

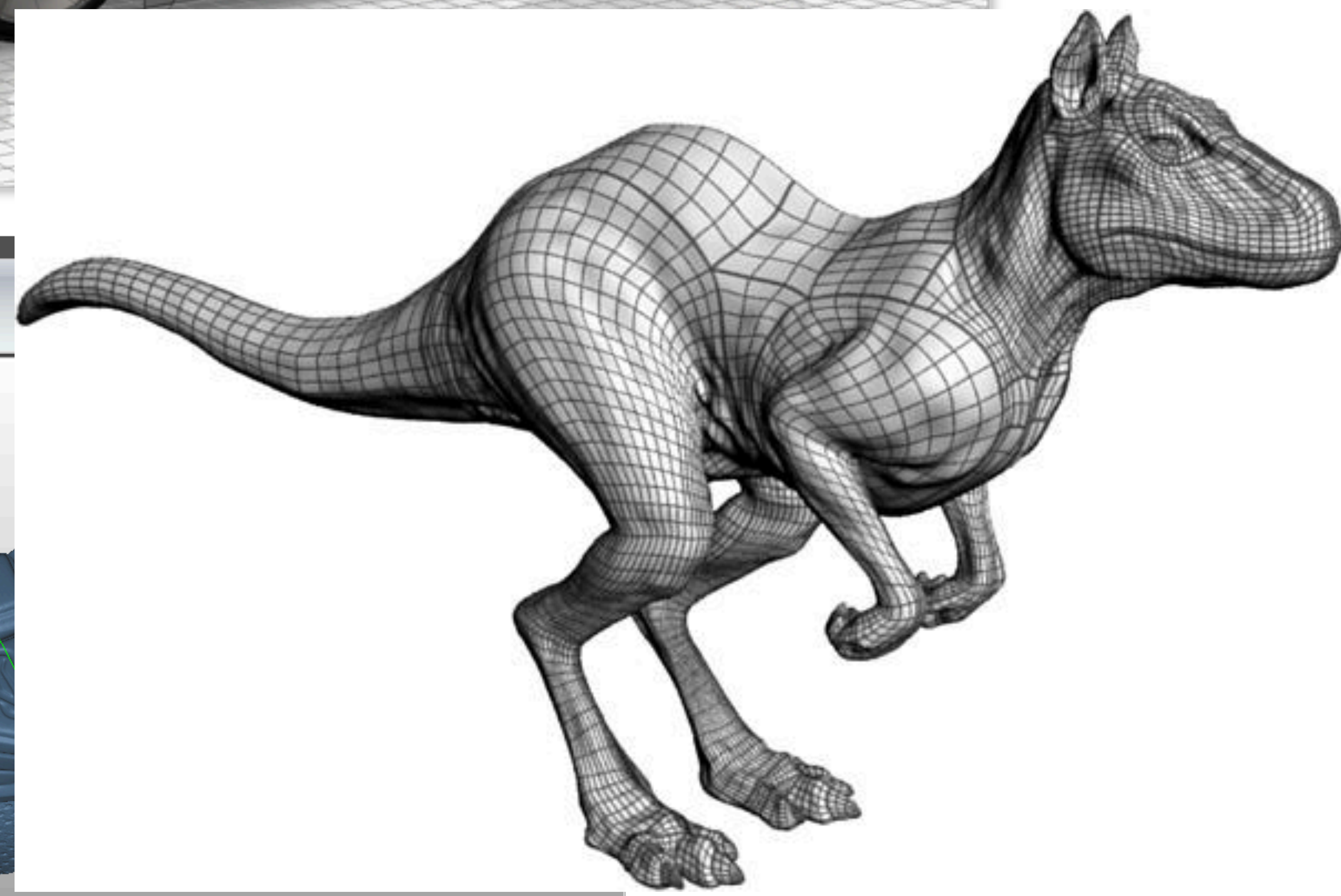
These are often called splines and are very powerful.

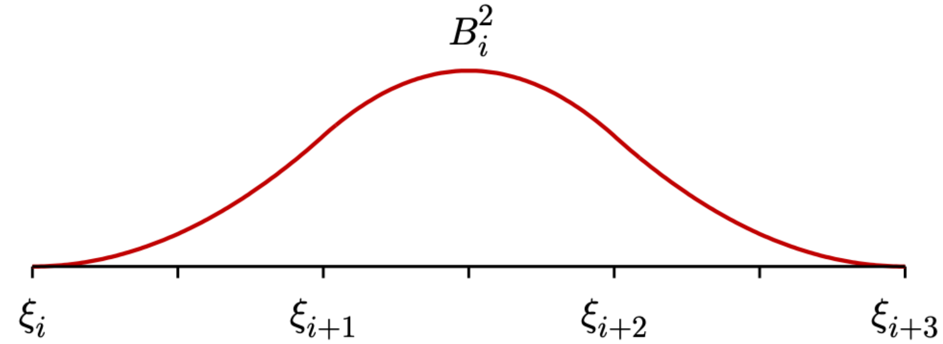
They are useful in computer graphics, and numerical solutions to PDEs.

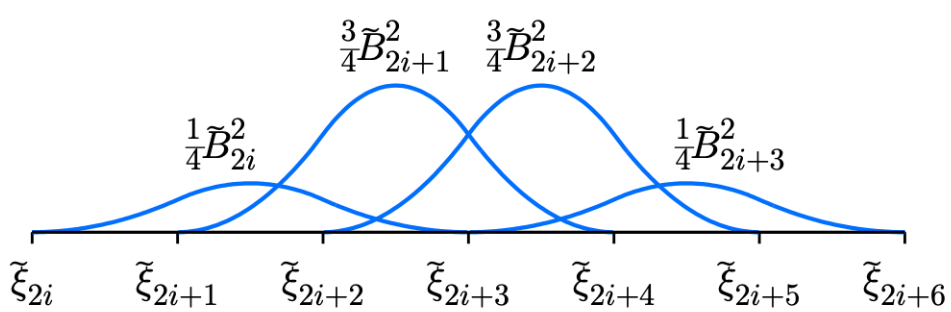




Просмотр и выделение
Совмещение сканов и построение 3D модели
Обработка сканов и 3D модели
Отделить шум
Упрощение модели
Отсечение и выравнивание
Вырезание маркеров
Масштабирование
Удаление подложки
Средства обработки
Объект
Цвет
Scene
*SNEAKER
*Group_0000
*Mesh_0000 1.636.243
*Mesh_0002 965.177
*Mesh_0003 1.464.082
*Mesh_0004 1.670.700
*Mesh_0005 1.607.019
*Mesh_0006 1.722.725
*Mesh_0007 1.779.118
*Group_0001
*Mesh_0001 1.517.082
*Mesh_0010 683.795
*SceneObject001 2.365.420
*SceneObject002 8.663.610
SceneObject003 1.299.540







$$
\begin{aligned}
\frac{\rho^f \mathbf{v}^{f,n+1}}{\Delta t} + \nabla p^{f,n+1} &= \frac{\rho^f \mathbf{v}^{f,n}}{\Delta t} + \mathbf{f}^{sf,n+1} + \mathbf{f}^{w,n+1}, \\
\nabla \cdot \mathbf{v}^{f,n+1} &= 0,
\end{aligned}
$$

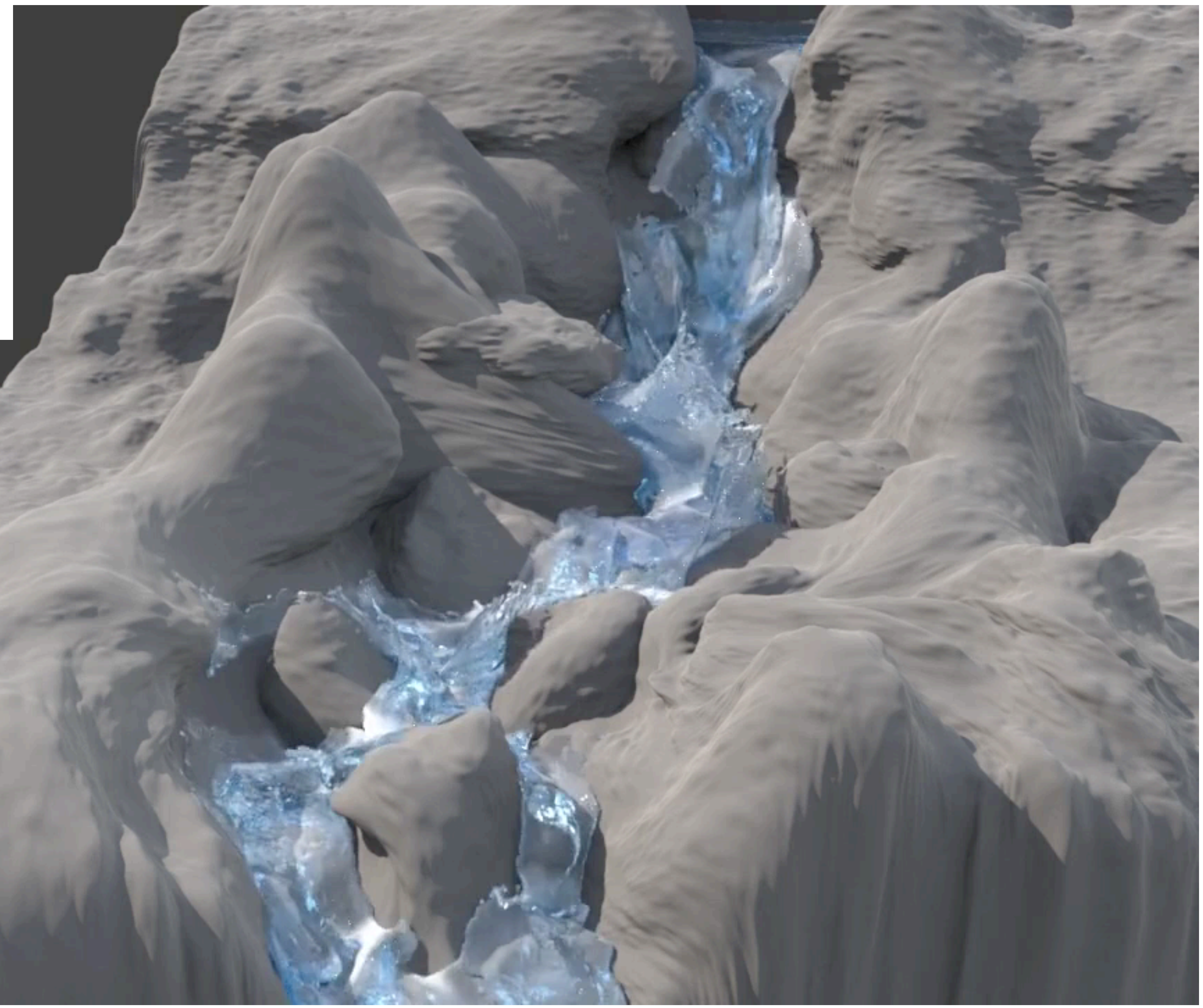
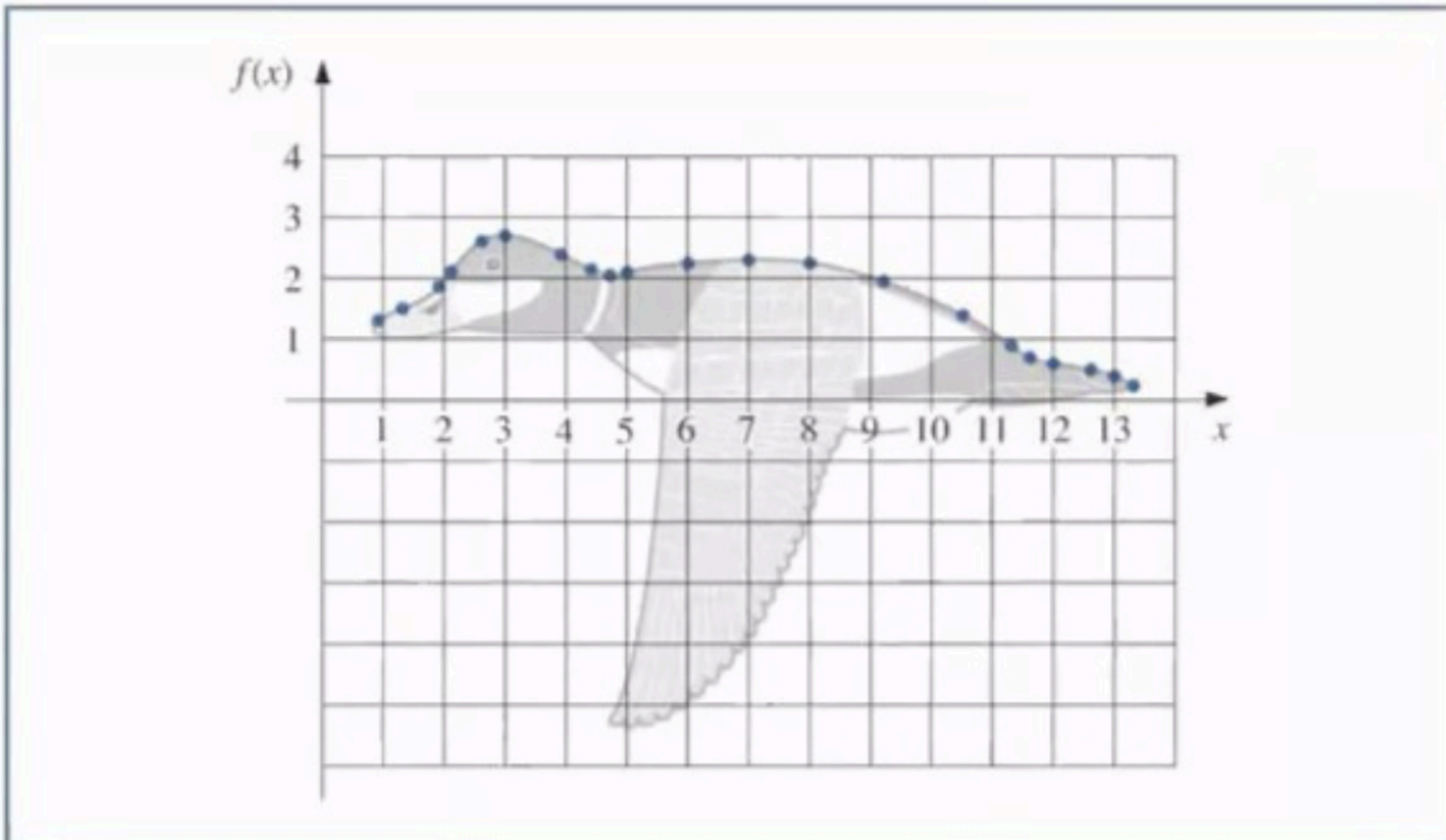


Table 3.18

x	0.9	1.3	1.9	2.1	2.6	3.0	3.9	4.4	4.7	5.0	6.0	7.0	8.0	9.2	10.5	11.3	11.6	12.0	12.6	13.0	13.3
$f(x)$	1.3	1.5	1.85	2.1	2.6	2.7	2.4	2.15	2.05	2.1	2.25	2.3	2.25	1.95	1.4	0.9	0.7	0.6	0.5	0.4	0.25

piecewise cubic polynomials



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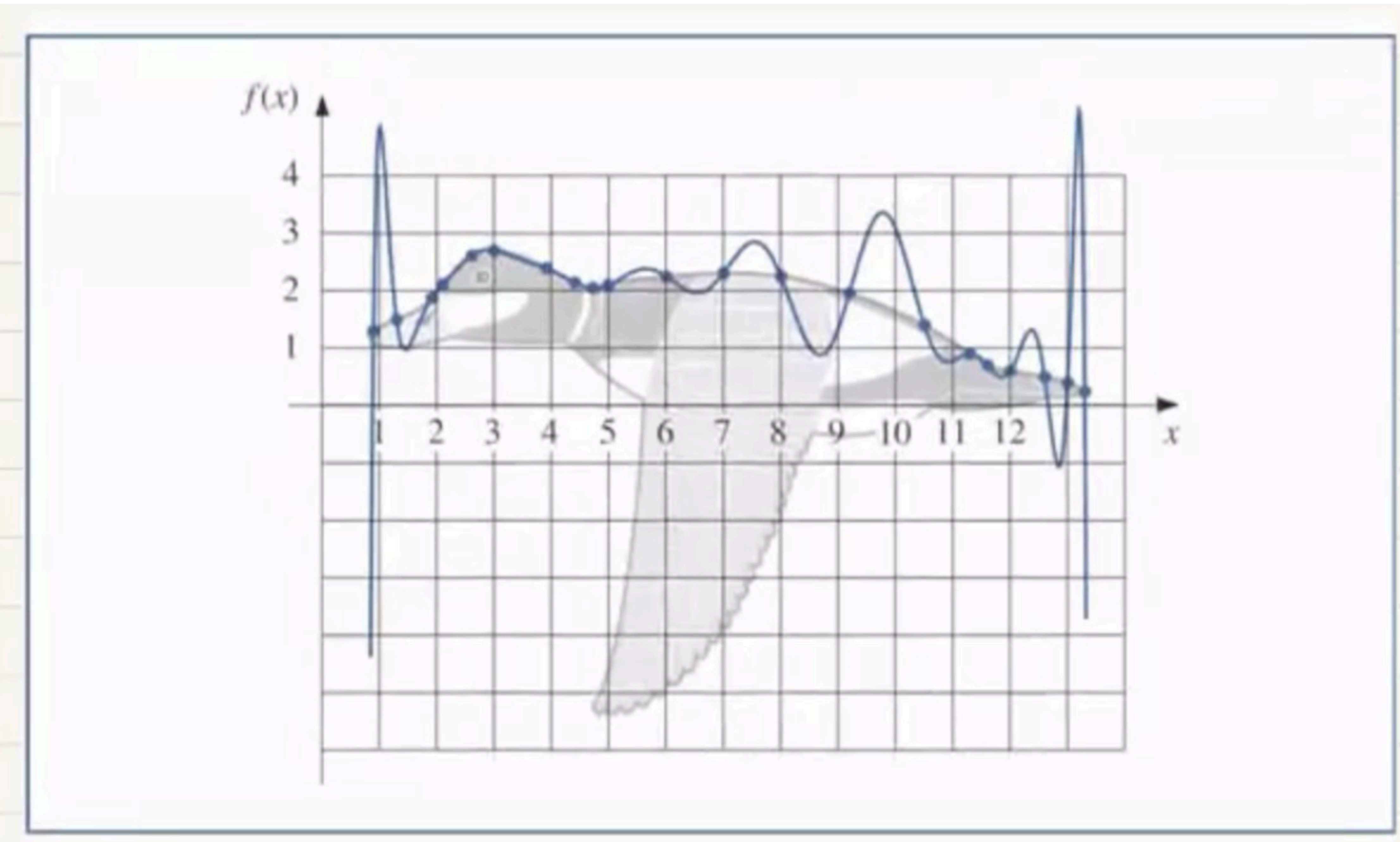
x	0.9	1.3	1.9	2.1	2.6	3.0	3.9	4.4	4.7	5.0	6.0	7.0	8.0	9.2	10.5	11.3	11.6	12.0	12.6	13.0	13.3
$f(x)$	1.3	1.5	1.85	2.1	2.6	2.7	2.4	2.15	2.05	2.1	2.25	2.3	2.25	1.95	1.4	0.9	0.7	0.6	0.5	0.4	0.25

High order polynomial



Definition 13.1 (Cubic Spline Interpolant).

Given f defined on $[a,b]$, $\{x_j\}_{j=0}^{n} \in [a,b]$
$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
The **spline** is a function $S(x)$ that satisfies:

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:
$$S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$
2. $S(x)$ interpolates f at each x_j. I.e., $S(x_j) = f(x_j)$.
3. Conituity: $S \in C([a,b])$.
4. Differentiability: $S \in C^2([a,b])$.

We can use the properties 2,3,4 (and some extra conditions) to determine coefficients in 1.

Example (1, 2), (2, 3), and (3, 5).

[1, 2]: $S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3,$

8 unknowns

[2, 3] $S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3.$

$2 = f(1) = a_0, \quad 3 = f(2) = a_0 + b_0 + c_0 + d_0, \quad 3 = f(2) = a_1, \quad \text{and}$

$5 = f(3) = a_1 + b_1 + c_1 + d_1.$ property 2 and 3

$S_0'(2) = S_1'(2): \quad b_0 + 2c_0 + 3d_0 = b_1$

$S_0''(2) = S_1''(2): \quad 2c_0 + 6d_0 = 2c_1$ property 4

$S_0''(1) = 0: \quad 2c_0 = 0$

$S_1''(3) = 0: \quad 2c_1 + 6d_1 = 0.$

natural boundary condition

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:

$$S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

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3. Conituity: $S \in C([a, b])$.
4. Differentiability: $S \in C^2([a, b])$.

Example $(1,2)$, $(2,3)$, and $(3,5)$.

$[1,2]$: $S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3,$

$[2,3]$ $S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3.$

8 unknowns

$2 = f(1) = a_0, \quad 3 = f(2) = a_0 + b_0 + c_0 + d_0, \quad 3 = f(2) = a_1,$ and

$5 = f(3) = a_1 + b_1 + c_1 + d_1.$ property 2 and 3

$S_0'(2) = S_1'(2): \quad b_0 + 2c_0 + 3d_0 = b_1$

$S_0''(2) = S_1''(2): \quad 2c_0 + 6d_0 = 2c_1$ property 4

$S_0''(1) = 0: \quad 2c_0 = 0$

$S_1''(3) = 0: \quad 2c_1 + 6d_1 = 0.$

natural boundary condition

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, \text{ for } x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, \text{ for } x \in [2,3] \end{cases}$$

Next time:

General ways of cubic spline construction, and some theoretical results (uniqueness etc.)