In this chapter, we provide a rigorous development of a theory of integration. Unlike differentiation, defining an integral turns out to be fairly delicate, primarily because the idea of limit involved in the construction is not directly related to the limits of sequences or functions and hence requires some technical notation and terminology.

Another comment is that there are several, often inequivalent, ways of constructing an integral. In this course, we will develop two versions of integrals, the Darboux integral and the Riemann integral. In both versions, the integral will be defined by approximating the (signed) area enclosed by the graph of f by rectangles. On the other hand, the way the rectangles are chosen differs for the two integrals. Despite this difference, however, we will find that they lead to the same theory of integration, at least within the setting of this course.

32. Part 1: The Darboux Integral

In this section, we develop the Darboux integral and study its properties. How the Darboux integral is formulated will be quite similar to how the limit of a sequence is characterized via its limsup and liminf.

Definition 32.1. (Partition)

(a) A **partition** P of the closed interval $[a, b]$ is a finite subset of $[a, b]$ of the form

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

We will also write $P = \{x_k\}_{k=0}^n$ whenever it is convenient.

(b) The **mesh** of a partition $P = \{x_k\}_{k=0}^n$ is the maximum length of the subintervals comprising P. That is,

$$\operatorname{mesh}(P) = \max\{x_k - x_{k-1} : k = 1, 2, \ldots, n\}.$$

Definition 32.2. (Darboux Sum/Integral) Let f be a bounded function on $[a, b]$.

(a) For each $S \subseteq [a, b]$, we adopt the following abbreviation whenever it is convenient:

$$\sup_S f = \sup_{x\in S} f(x) = \sup f(S) = \sup\{f(x) : x \in S\},$$
$$\inf_S f = \inf_{x\in S} f(x) = \inf f(S) = \inf\{f(x) : x \in S\}$$

(b) Let $P = \{x_k\}_{k=0}^n$ be a partition of $[a, b]$. Then the **upper Darboux sum** $U(f, P)$ of f with respect to P is the sum

$$U(f, P) = \sum_{k=1}^{n} \Big(\sup_{[x_{k-1}, x_k]} f \Big) \cdot (x_k - x_{k-1})$$

and the **lower Darboux sum** $L(f, P)$ is

$$L(f, P) = \sum_{k=1}^{n} \Big(\inf_{[x_{k-1}, x_k]} f \Big) \cdot (x_k - x_{k-1})$$

(c) The **upper Darboux integral** $\overline{\int_a^b} f$ of f over $[a, b]$ is defined by

$$\overline{\int_a^b} f = \overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and the **lower Darboux integral** $\underline{\int_a^b} f$ is

$$\underline{\int_a^b} f = \underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$$

(d) If $\underline{\int_a^b} f = \overline{\int_a^b} f$, then f is said to be **Darboux integrable** on $[a, b]$ and we write

$$\int_a^b f = \int_a^b f(x)\,\mathrm{d}x = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

This common value is called the **Darboux integral** of f over $[a, b]$

The next result collects some basic properties of the Darboux sums.

Lemma 32.3. (Properties of Darboux Sums) Let f be a bounded function on $[a, b]$, and let P and Q be any partitions of $[a, b]$. Then

(a) $L(f, P) \leq U(f, P)$;

(b) If $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$ and $U(f, Q) \leq U(f, P)$;

(c) $L(f, P) \leq U(f, Q)$.

(a) This is immediate by noting that $\inf_S f \leq \sup_S f$ for any non-empty $S \subseteq [a, b]$.

(b) We first consider the case where $Q = P \cup \{y_*\}$. Write $P = \{x_k\}_{k=0}^n$ and assume that $y_* \in (x_{l-1}, x_l)$. Then by writing $J_1 = [x_{l-1}, y_*]$ and $J_2 = [y_*, x_l]$ for simplicity,

$$\begin{aligned}
U(f, P) - U(f, Q) &= \left(\sup_{J_1 \cup J_2} f\right)(x_l - x_{l-1}) - \left(\sup_{J_1} f\right)(x_l - y_*) - \left(\sup_{J_2} f\right)(y_* - x_{l-1}) \\
&= \left(\sup_{J_1 \cup J_2} f - \sup_{J_1} f\right)(x_l - y_*) + \left(\sup_{J_1 \cup J_2} f - \sup_{J_2} f\right)(y_* - x_{l-1}) \\
&\geq 0.
\end{aligned}$$

So it follows that $U(f, Q) \leq U(f, P)$ in this case. Now the general case follows by introducing a chain $P = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_m = Q$ of partitions such that P_k contains one more points than P_{k-1} for each $k = 1, \ldots, m$ and applying the above inequality repeatedly:

$$U(f, Q) = U(f, P_m) \leq U(f, P_{m-1}) \leq \cdots \leq U(f, P_1) \leq U(f, P_0) = U(f, P).$$

The other inequality follows in a similar way.

(c) By (a) and (b),

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

□

Theorem 32.4. Let f be a bounded function on $[a, b]$. Then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof. Let P be a partition of $[a, b]$. By Theorem 32.3(c), $L(f, P)$ is a lower bound of the set of all Darboux upper sums of f, and so,

$$L(f, P) \leq \inf\{U(f, P) : P \text{ is a partition of } [a, b]\} = \overline{\int_a^b} f.$$

This then shows that $\overline{\int_a^b} f$ is an upper bound of the set of all Darboux lower sums of f, and so,

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\} \leq \overline{\int_a^b} f,$$

completing the proof. □

The next theorem gives a "Cauchy criterion" for Darboux integrability.

Theorem 32.5. (Cauchy-like Criterion for Darboux Integral) For a bounded function f on $[a, b]$, the followings are equivalent.

(a) f is Darboux integrable on $[a, b]$.

(b) For each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon. \tag{1}$$

Proof. (a) $\implies$ (b) : Let $I = \int_a^b f$. Then for each $\varepsilon > 0$, there exist partitions P_1 and P_2 of the interval $[a, b]$ such that

$$U(f, P_1) - I < \varepsilon/2 \qquad \text{and} \qquad I - L(f, P_2) < \varepsilon/2.$$

Then with $P = P_1 \cup P_2$,

$$U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_2) = [U(f, P_1) - I] + [I - L(f, P_2)] < \varepsilon.$$

(b) $\implies$ (a) : For each $\varepsilon > 0$, pick a partition P such that $U(f, P) - L(f, P) < \varepsilon$. Then

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(f, P) - L(f, P) < \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we get $\overline{\int_a^b} f = \underline{\int_a^b} f$ and hence f is Darboux integrable. □

This concludes the first half of Section 32 about the Darboux integral. In the latter half, we will define the Riemann integral and then show that they are equivalent, which is the assertion of Theorem 32.9. However, because the development of the Riemann integral is a bit more technical and most of the result regarding Riemann integral can be proved using the Darboux's formulation of integral, we will postpone this topic to the end.

For the rest of this chapter, we will take it for granted that the notion of Darboux integral and Riemann integral coincide. Then we are left with a single version of integration theory, which we will simply call the "Riemann integral".[1]

33. Properties of the Riemann Integral

In this section, we establish some basic properties of the Riemann integral and verify that many familiar functions are Riemann integrable. To make statements short, we will introduce the following notation.

Definition 33.1. The set of all Riemann integrable functions on $[a, b]$ will be denoted by $\mathcal{R}[a, b]$.

33.1. Examples of Riemann Integrable Functions

Theorem 33.2. Every monotone function f on $[a, b]$ is Riemann integrable.

Proof. By replacing f by $-f$ if necessary, we may assume that f is increasing.

Let $\varepsilon > 0$ be arbitrary, and choose any partition $P = \{x_k\}_{k=0}^{n}$ so that $\operatorname{mesh}(P) < \frac{\varepsilon}{f(b)-f(a)+1}$. Then by noting that

$$\inf_{[x_{k-1},x_k]} f = f(x_{k-1}) \qquad \text{and} \qquad \sup_{[x_{k-1},x_k]} f = f(x_k),$$

we get

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) \operatorname{mesh}(P) \\ &= (f(b) - f(a)) \operatorname{mesh}(P) \\ &< \varepsilon. \end{aligned}$$

In light of our "Cauchy criterion", this implies that f is Riemann integrable on $[a, b]$. □

Theorem 33.3. Every continuous function f on $[a, b]$ is Riemann integrable.

[1] So, you might ask why we are not calling them the same name in the first place. It might be because the authors often want to credit Darboux's contribution to an alternative definition of Riemann integral. It might be also because their respective generalizations, the Darboux–Stieltjes integral and Riemann–Stieltjes integral, are actually different.

Proof. Since f is continuous on a closed bounded interval, it is uniformly continuous. So, for each $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$x, y \in [a, b] \text{ and } |x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Now let $P = \{x_k\}_{k=0}^n$ be any partition of $[a, b]$ satisfying $\mathrm{mesh}(P) < \delta$. Then for each subinterval I of P, the Extreme Value Theorem tells that there exist $x_{\min}, x_{\max} \in I$ such that $\min_I f = f(x_{\min})$ and $\max_I f = f(x_{\max})$. Since $|x_{\max} - x_{\min}| < \delta$, we get

$$\sup_I f - \inf_I f = f(x_{\max}) - f(x_{\min}) < \frac{\varepsilon}{b - a}.$$

So it follows that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^n \left(\sup_{[x_{k-1}, x_k]} f - \inf_{[x_{k-1}, x_k]} f \right)(x_k) - x_{k-1}) \\ &\leq \sum_{k=1}^n \frac{\varepsilon}{b - a}(x_k - x_{k-1}) \\ &= \varepsilon \end{aligned}$$

and therefore f is Riemann integrable on $[a, b]$ by the "Cauchy criterion". $\square$

The next result shows that modifying a given function at finitely many points does not alter the value of the Riemann integral.

Theorem 33.4. If f is a function on $[a, b]$ such that $f(x) = 0$ for all but finitely many points $x \in [a, b]$, then $f \in \mathcal{R}[a, b]$ and $\int_a^b f = 0$.

Proof. Suppose there are exactly m points, $c_1, c_2, \ldots, c_m$, in $[a, b]$ at which f takes non-zero value. Let M be a bound of f. Given $\varepsilon > 0$, let P be any partition with $\mathrm{mesh}(P) < \varepsilon/2mM$. Then each c_i lies in at most two subintervals comprising P. Moreover, for each subinterval I of P not containing any of c_i, we have $\sup_I f = 0 = \inf_I f$. So

$$\overline{\int_a^b} f \leq 2m \left(\sup_{[a,b]} f \right) \mathrm{mesh}(P) < 2mM \cdot \frac{\varepsilon}{2mM} = \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we get $\overline{\int_a^b} f \leq 0$. A similar reasoning shows that $\underline{\int_a^b} f \geq 0$, and so, the desired conclusion follows. $\square$

Corollary 33.5. If $f, g \in \mathcal{R}[a, b]$ such that $f(x) = g(x)$ for all but finitely many points $x \in [a, b]$, then $\int_a^b f = \int_a^b g$.

Proof. Apply Theorem 33.4 to $f - g$. $\square$

33.2. Operations and Comparison on Riemann Integral

Theorem 33.6. Let f and g be bounded functions on $[a, b]$.

(a) $\overline{\int_a^b} cf = c\overline{\int_a^b} f$ and $\underline{\int_a^b} cf = c\underline{\int_a^b} f$ for any real number $c \geq 0$.

(b) $\overline{\int_a^b} (-f) = -\underline{\int_a^b} f$.

(c) $\overline{\int_a^b} (f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$ and $\underline{\int_a^b} (f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g$.

(a) By using the fact that $\sup(cA) = c \sup A$ and $\inf(cA) = c \inf A$ for any non-empty bounded set $A \subseteq \mathbb{R}$ and $c \geq 0$, for each partition $P = \{x_k\}_{k=0}^n$ of $[a, b]$ we get

$$U(cf, P) = \sum_{k=1}^n \left(\sup_{[x_{k-1}, x_k]} (cf) \right) (x_k - x_{k-1}) = c \sum_{k=1}^n \left(\sup_{[x_{k-1}, x_k]} f \right) (x_k - x_{k-1}) = cU(f, P)$$

and by taking infimum over P, we get

$$\overline{\int_a^b} cf = \inf_P U(cf, P) = \inf_P cU(f, P) = c \inf_P U(f, P) = c \overline{\int_a^b} f.$$

A similar argument shows that $\underline{\int_a^b} cf = c\underline{\int_a^b} f$.

(b) By using the fact that $\sup(-A) = -\inf A$ for any non-empty bounded set $A \subseteq \mathbb{R}$, for each partition $P = \{x_k\}_{k=0}^n$ of $[a, b]$ we get

$$U(-f, P) = \sum_{k=1}^n \left(\sup_{[x_{k-1}, x_k]} (-f) \right) (x_k - x_{k-1}) = -\sum_{k=1}^n \left(\inf_{[x_{k-1}, x_k]} f \right) (x_k - x_{k-1}) = -L(f, P).$$

So by taking infimum over P,

$$\overline{\int_a^b} cf = \inf_P U(-f, P) = \inf_P(-L(f, P)) = -\sup_P L(f, P) = -\underline{\int_a^b} f.$$

(c) By using the fact that $\sup_S(f + g) \leq \sup_S f + \sup_S g$, for any partitions P and Q of $[a, b]$, we get

$$U(f, P) + U(g, Q) \geq U(f, P \cup Q) + U(g, P \cup Q) \geq U(f + g, P \cup Q) \geq \overline{\int_a^b} (f + g).$$

So by taking infimum uver P and Q, we get

$$\overline{\int_a^b} f + \overline{\int_a^b} g \geq \overline{\int_a^b} (f + g).$$

The other inequality follows in a similar way. $\square$

Corollary 33.7. (Linearity of Integral) Let $f, g \in \mathcal{R}[a,b]$. Then

(a) $cf \in \mathcal{R}[a,b]$ and $\int_a^b cf = c\int_a^b f$ for any real number c;

(b) $f + g \in \mathcal{R}[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Proof. The assumption tells that $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$ and $\overline{\int_a^b} g = \underline{\int_a^b} g$.

(a) If $c \geq 0$ then we have $\overline{\int_a^b} cf = c\int_a^b f = \underline{\int_a^b} cf$. Since the upper and lower Darboux integrals of cf coincide with the value $c\int_a^b f$, we have $cf \in \mathcal{R}[a,b]$ and the desired equality follows.

If $c = -1$, then $\overline{\int_a^b}(-f) = -\int_a^b f = \underline{\int_a^b}(-f)$ and the claim follows.

Finally, the case $c \leq 0$ follows by combining the above two observations.

(b) Since $\int_a^b f + \int_a^b g \leq \underline{\int_a^b}(f+g) = \overline{\int_a^b}(f+g) \leq \int_a^b f + \int_a^b g$, the upper and lower Darboux integrals of $f + g$ coincide with the value $\int_a^b f + \int_a^b g$. Therefore the conclusion follows. □

Theorem 33.8. (Comparison) Let f and g be bounded functions on $[a,b]$ such that $f(x) \leq g(x)$ for all $x \in [a,b]$.

(a) $\overline{\int_a^b} f \leq \overline{\int_a^b} g$ and $\underline{\int_a^b} f \leq \underline{\int_a^b} g$.

(b) If $f, g \in \mathcal{R}[a,b]$, then $\int_a^b f \leq \int_a^b g$.

Proof.

(a) The inequality implies that $\sup_S f \leq \sup_S g$ and $\inf_S f \leq \inf_S g$ for any non-empty subset S of $[a,b]$. So, for any partition P of $[a,b]$,

$$\overline{\int_a^b} f \leq U(f,P) \leq U(g,P).$$

This inequality tells that $\overline{\int_a^b} f$ is a lower bound for the values of $U(g,P)$, and so, $\overline{\int_a^b} f \leq \overline{\int_a^b} g$ follows. Likewise, a similar reasoning applied to the inequality

$$\underline{\int_a^b} g \geq L(g,P) \geq L(f,P),$$

which holds for any partition P of $[a,b]$, proves $\underline{\int_a^b} g \geq \underline{\int_a^b} f$.

(b) This is a direct consequence of the previous step. □

Theorem 33.9. If $f \in \mathcal{R}[a,b]$ satisfies $f(x) \in [m,M]$ for all $x \in [a,b]$, and if ϕ is a Lipschitz continuous function on $[m,M]$, then $\phi \circ f \in \mathcal{R}[a,b]$.

Preparation. The proof hinges on the following general observation:

- **Lemma.** For any bounded function f on a non-empty set S,

$$\sup_{x,y\in S} |f(x) - f(y)| = \sup_S f - \inf_S f.$$

  Indeed, write

$$\alpha = \inf_S f, \qquad \beta = \sup_S f, \qquad M = \sup_{x,y\in S} |f(x) - f(y)|.$$

  By adding $\alpha \le f(x) \le \beta$ and $-\beta \le -f(y) \le -\alpha$, we get $|f(x) - f(y)| \le \beta - \alpha$ and hence $M \le \beta - \alpha$. For the opposite direction, note that we can choose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S such that $\lim f(x_n) = \alpha$ and $\lim f(y_n) = \beta$. Then $\beta - \alpha = \lim |f(x_n) - f(y_n)| \le M$. □

Proof of Theorem. Since ϕ is Lipschitz continuous on $[m, M]$, there exists $C \in (0, \infty)$ such that

$$|\phi(y_1) - \phi(y_2)| \le C|y_1 - y_2| \quad \text{for any} \quad y_1, y_2 \in [m, M].$$

Since $f([a, b]) \subseteq [m, M]$, this implies that

$$|\phi(f(x_1)) - \phi(f(x_2))| \le C|f(x_1) - f(x_2)| \quad \text{for any} \quad x_1, x_2 \in [a, b].$$

Now let $\varepsilon > 0$ be arbitrary, and choose a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon/C$. Then for any subinterval I in P,

$$\begin{aligned}
\sup_{x\in I} \phi(f(x)) - \inf_{x\in I} \phi(f(x)) &= \sup_{x_1,x_2\in I} |\phi(f(x_1)) - \phi(f(x_2))| \\
&\le \sup_{x_1,x_2\in I} C|f(x_1) - f(x_2)| \\
&= C \sup_{x_1,x_2\in I} |f(x_1) - f(x_2)| \\
&= C\left(\sup_{x\in I} f(x) - \inf_{x\in I} f(x)\right).
\end{aligned}$$

This then implies that

$$U(\phi \circ f, P) - L(\phi \circ f, P) \le C[U(f, P) - L(f, P)] < \varepsilon.$$

Therefore $\phi \circ f$ is also Riemann integrable on $[a, b]$.

Corollary 33.10. If $f, g \in \mathcal{R}[a, b]$. Then $fg \in \mathcal{R}[a, b]$.

Proof. Since both f and g are bounded, we can choose M such that $f(x), g(x) \in [-M, M]$ for all $x \in [a, b]$. Now let $\phi(x) = x^2$. Then $\phi'(x) = 2x$ is bounded on $[-2M, 2M]$, and so, ϕ is Lipschitz continuous on $[-2M, 2M]$ by a simple application of the Mean Value Theorem. Then by Theorem 33.9 and the linearity of integrals, both $(f \pm g)^2$ are Riemann integrable and hence

$$fg = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2$$

is also Riemann integrable as desired. □

Corollary 33.11. Let f be a Riemann integrable function on $[a, b]$. Then $|f|$ is Riemann integrable on $[a, b]$, and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. By the reverse triangle inequality, $||x| - |y|| \leq |x - y|$ and hence the absolute value function $|\cdot|$ is Lipschitz continuous. So $|f|$ is also Riemann integrable by Theorem 33.9. Moreover, the inequality $-|f| \leq f \leq |f|$ gives

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|,$$

which then implies the desired inequality. □

Corollary 33.12. (Additivity of Integrals) Let f be a bounded function on $[a, b]$ and $c \in (a, b)$.

(a) We have

$$\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f \quad \text{and} \quad \underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f.$$

(b) If f is Riemann integrable on both $[a, c]$ and $[c, b]$, then f is Riemann integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof.

(a) Let P_1 be any partition of $[a, c]$ and P_2 any partition of $[c, b]$. Then $P_1 \cup P_2$ is a partition of $[a, b]$, and so,

$$\overline{\int_a^b} f \leq U_a^b(f, P_1 \cup P_2) = U_a^c(f, P_1) + U_c^b(f, P_2).$$

Taking infimum over P_1 and P_2, we get $\overline{\int_a^b} f \leq \overline{\int_a^c} f + \overline{\int_c^b} f$.

For the other direction, let P be any partition of $[a, b]$. Then $P_1 = (P \cup \{c\}) \cap [a, c]$ is a partition of $[a, c]$ and $P_2 = (P \cup \{c\}) \cap [c, b]$ is a partition of $[c, b]$. So

$$U_a^b(f, P) \geq U_a^b(f, P \cup \{c\}) = U_a^c(f, P_1) + U_c^b(f, P_2) \geq \overline{\int_a^c} f + \overline{\int_c^b} f.$$

Taking infimum over P, we get $\overline{\int_a^b} f \geq \overline{\int_a^c} f + \overline{\int_c^b} f$. This and the previous inequality combined then proves the first equality in the assertion. The second equality can be proved in a similar way.

(b) Part (a) and the assumption tell that both the upper and lower Darboux integrals of f on $[a, b]$ are precisely $\int_a^c f + \int_c^b f$. Therefore f is Riemann integrable on $[a, b]$ and the desired equality holds. □

Example 33.1. Let J be a subinterval of $[a, b]$. Show that the indicator function $\mathbf{1}_J : [a, b] \to \mathbb{R}$ defined by

$$\mathbf{1}_J(x) = \begin{cases} 1, & \text{if } x \in J; \\ 0, & \text{if } x \notin J; \end{cases}$$

is Riemann integrable on $[a, b]$ and $\int_a^b \mathbf{1}_J$ is equal to the length of J.

Example 33.2. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

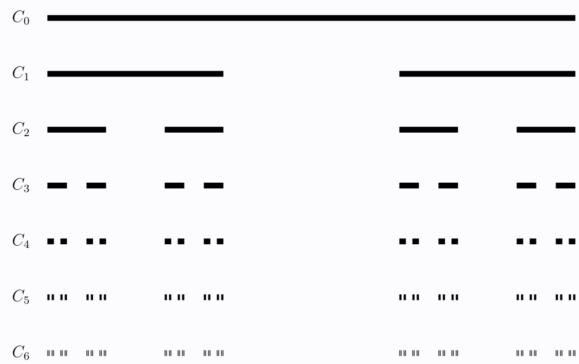
$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is Riemann integrable on $[-1, 1]$.

Example 33.3. Define the sequence $(C_n)_{n=0}^{\infty}$ of subsets of $[0,1]$ as follows:

$$C_0 = [0,1] \qquad \text{and} \qquad C_{n+1} = \frac{C_n}{3} \cup \frac{2+C_n}{3}.$$

Equivalently, C_{n+1} is obtained from C_n by excising the open middle third of each closed interval comprising C_n:



Then the intersection $C = \bigcap_{n=0}^{\infty} C_n$ of all C_n's turns out to be a non-empty subset of $[0,1]$ called the **Cantor set**.

Show that the indicator function $\mathbf{1}_C : [0,1] \to \mathbb{R}$ defined by

$$\mathbf{1}_C(x) = \begin{cases} 1, & \text{if } x \in C; \\ 0, & \text{if } x \notin C; \end{cases}$$

is Riemann integrable on $[0,1]$ and find the value of $\int_0^1 \mathbf{1}_C$.

34. Fundamental Theorem of Calculus

Theorem 34.1. (Fundamental Theorem of Calculus I) If F is a continuous function on $[a,b]$ that is differentiable on (a,b), and if F' is Riemann integrable on $[a,b]$, then

$$\int_a^b F' = F(b) - F(a).$$

Here is a technical point of this statement: The assumption only guarantees that F' is defined on the open interval (a,b), hence we cannot directly discuss the integrability of F'. In light of Corollary 33.5, however, modifying a given function at finitely many points preserves integrability. So we may use any extension of F' onto the interval $[a,b]$ for defining its Riemann integral. This is the convention that we will use throughout.

Proof. For each $\varepsilon > 0$, choose a partition $P = \{x_k\}_{k=0}^n$ such that $U(F',P) - L(F',P) < \varepsilon$. Also, for each k, apply the Mean Value Theorem to choose $\xi_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = F'(\xi_k)(x_k - x_{k-1}).$$

Then by noting that

$$F(b) - F(a) = \sum_{k=1}^n (F(x_k) - F(x_{k-1})) = \sum_{k=1}^n F'(\xi_k)(x_k - x_{k-1})$$

and that

$$\inf_{[x_{k-1},x_k]} F' \leq F'(\xi_k) \leq \sup_{[x_{k-1},x_k]} F',$$

we get

$$L(F',P) \leq F(b) - F(a) \leq U(F',P).$$

Moreover, we also have

$$L(F',P) \leq \int_a^b F' \leq U(F',P).$$

Combining altogether, we find that

$$\left| \int_a^b F' - (F(b) - F(a)) \right| \leq U(F',P) - L(F',P) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies $\int_a^b F' = F(b) - F(a)$ as desired. $\square$

Theorem 34.2. (Fundamental Theorem of Calculus II) Let f be a Riemann integrable function on $[a,b]$. Define the function $F:[a,b]\to\mathbb{R}$ by

$$F(x)=\int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $x_0\in(a,b)$, then F is differentiable at x_0 and

$$F'(x_0)=f(x_0).$$

Proof. Since f is Riemann integrable, it is bounded by some number M, that is, $|f(x)|\le M$ for all $x\in[a,b]$. Then for any $x,y\in[a,b]$ with $x<y$,

$$|F(y)-F(x)|=\left|\int_x^t f(t)\,\mathrm{d}t\right|\le\int_x^y |f(t)|\,\mathrm{d}t\le\int_x^y M\,\mathrm{d}t=M|y-x|$$

and this implies that F is Lipschitz continuous on $[a,b]$.

Now suppose f is continuous at $x_0\in(a,b)$. By the ε–δ property, for each $\varepsilon>0$, we can find $\delta>0$ such that

$$x\in[a,b]\quad\text{and } |x-x_0|<\delta\qquad\Longrightarrow\qquad |f(x)-f(x_0)|<\varepsilon.$$

Now for each $x\in(x_0,x_0+\delta)\cap[a,b]$, we have

$$\begin{aligned}\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|&=\frac{1}{x-x_0}\left|\int_{x_0}^x (f(t)-f(x_0))\,\mathrm{d}t\right|\\&\le\frac{1}{x-x_0}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t\\&\le\frac{1}{x-x_0}\int_{x_0}^x \varepsilon\,\mathrm{d}t=\varepsilon.\end{aligned}$$

This proves that

$$\lim_{x\to x_0^+}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0).$$

A similar argument reveals that

$$\lim_{x\to x_0^-}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0)$$

as well, and therefore F is differentiable at x_0 with $F'(x_0)=f(x_0)$. $\square$

Theorem 34.3. **(Integration by Parts)** If u and v are continuous functions on $[a, b]$ that are differentiable on (a, b), and if u' and v' are Riemann integrable on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,\mathrm{d}x.$$

Proof. The condition guarantees that

- uv is continuous on $[a, b]$;
- uv is differentiable on (a, b);
- $(uv)' = u'v + uv'$ is Riemann integrable on $[a, b]$, since it is the sum of products of Riemann integrable functions.

So by the Fundamental Theorem of Calculus I applied to uv,

$$\int_a^b (u'(x)v(x) + u(x)v'(x))\,\mathrm{d}x = u(b)v(b) - u(a)v(a).$$

Rearranging this equality proves the desired claim.

To state the next result, we extend the definition of Riemann integral by setting

$$\int_b^a f = -\int_a^b f$$

for f Riemann integrable on $[a, b]$. Then the properties of Riemann integral that do not involve inequalities (such as the linearity, additivity, Fundamental Theorems of Calculus, etc) continue to hold for this extended version of integrals.

Theorem 34.4. **(Change of Variable)** Let ϕ be a continuous function on $[a, b]$ that is differentiable on (a, b), and suppose ϕ' is Riemann integrable on $[a, b]$. Let f be a continuous function on the closed interval $\phi([a, b])$.[2] Then $(f \circ \phi)\phi'$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f(\phi(x))\phi'(x)\,\mathrm{d}x = \int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u.$$

Proof. Fix $y_0 \in \phi([a, b])$ and define $F(y) = \int_{y_0}^y f(t)\,\mathrm{d}t$. Then by the Fundamental Theorem of Calculus II, F is differentiable on $\phi([a, b])$ and $F' = f$. Then

- $F \circ \phi$ is continuous on $[a, b]$;
- $F \circ \phi$ is differentiable on (a, b);
- $(F \circ \phi)' = (f \circ \phi)\phi'$ is Riemann integrable on $[a, b]$, since it is the product of Riemann

[2] By the intermediate value theorem, $\phi([a, b])$ is an interval. By the Extreme Value Theorem, this interval contains both of its endpoints and hence is closed.

integrable functions.

So by the Fundamental Theorem of Calculus I,

$$\int_a^b f(\phi(x))\phi'(x)\,\mathrm{d}x = F(\phi(b)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u.$$

$\square$

Example 34.1. (Integral Test) Suppose f is a decreasing function on $[1,\infty)$ such that $f(x) \geq 0$ for all $x \in [1,\infty)$. Show that the followings are equivalent:

(i) $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges.

(ii) $\displaystyle\lim_{R\to\infty}\int_1^R f(x)\,\mathrm{d}x$ converges.

Example 34.2. Let g be a strictly increasing, differentiable function on an open interval $I = (a, b)$. Then $g(I) = (g(a), g(b))$ is an open interval, and g^{-1} is differentiable on $g(I)$. Show that

$$\int_a^b g(x)\,\mathrm{d}x + \int_{g(a)}^{g(b)} g^{-1}(y)\,\mathrm{d}y = bg(b) - ag(a).$$

Example 34.3. Suppose f is a continuous function on $[a, b]$.

(a) Show that if $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)\,\mathrm{d}x = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) Show that if $\int_a^b f(x)g(x)\,\mathrm{d}x = 0$ for any continuous function g on $[a, b]$ such that $g(a) = g(b) = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(c) Show that if $\int_a^b f(x)g(x)\,\mathrm{d}x = 0$ for any continuous function g on $[a, b]$ such that $\int_a^b g(x)\,\mathrm{d}x = 0$, then f is a constant function.

Example 34.4. Let f be a continuous function on $[0,1]$, and let g be a continuous function on $\mathbb{R}$ such that $g(x+1) = g(x)$ for all $x \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)\,\mathrm{d}x = \left(\int_0^1 f(x)\,\mathrm{d}x\right)\left(\int_0^1 g(x)\,\mathrm{d}x\right).$$

This tells that high-frequency factors and low-frequency factors are almost "independent".

32. Part 2: The Riemann Integral

Now we return to Section 32 and develop the idea of Riemann integral. This time, the integral will be formulated via an ε–δ property for a family of approximate (signed) areas called the Riemann sums. As a starting point, we first establish an ε–δ property for the upper/lower Darboux integrals.

Theorem 32.6. (Darboux Integrals as Limit) Let f be a bounded function on $[a, b]$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathrm{mesh}(P) < \delta \quad \text{implies} \quad U(f, P) - \overline{\int_a^b} f < \varepsilon \quad \text{and} \quad \underline{\int_a^b} f - L(f, P) < \varepsilon \tag{2}$$

for any partition P of $[a, b]$.

Definition 32.7. (Tagged Partition)

(a) A **tagged partition** $\dot{P}$ is a partition $P = \{x_k\}_{k=0}^n$ of $[a,b]$ furnished with a choice of points $\{\xi_k\}_{k=1}^n$ satisfying $\xi_k \in [x_{k-1}, x_k]$ for each $k = 1, 2, \ldots, n$ and is denoted by

$$\dot{P} = \{\xi_k \in [x_{k-1}, x_k]\}_{k=1}^n.$$

The points $\{\xi_k\}_{k=1}^n$ are called **tags**.

(b) The mesh of a tagged partition $\dot{P}$ is the mesh of the partition P.

Definition 32.8. (Riemann Sum/Integral) Let f be a bounded function on $[a,b]$.

(a) Let $\dot{P} = \{\xi_k \in [x_{k-1}, x_k]\}_{k=0}^n$ be a tagged partition of $[a,b]$. Then the **Riemann sum** $S(f, \dot{P})$ of f with respect to $\dot{P}$ is the sum

$$S(f, \dot{P}) = \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}).$$

(b) f is said to be **Riemann integrable** on $[a,b]$ if there exists a real number I with the following property: For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathrm{mesh}(\dot{P}) < \delta \quad \text{implies} \quad |S(f, \dot{P}) - I| < \varepsilon \tag{3}$$

for any tagged partition $\dot{P}$ of $[a,b]$. In this case, the number I is called the **Riemann integral** of f over $[a,b]$.

The next result shows that the Darboux integral and the Riemann integral coincide.

Theorem 32.9. (Equivalence of Darboux and Riemann integrals) For a bounded function f on $[a,b]$ and a real number I, the followings are equivalent.

(a) f is Darboux integrable on $[a,b]$ with the Darboux integral I.

(b) f is Riemann integrable on $[a,b]$ with the Riemann integral I.