UCLA Math151A Fall 2021
Lecture 14
20211027

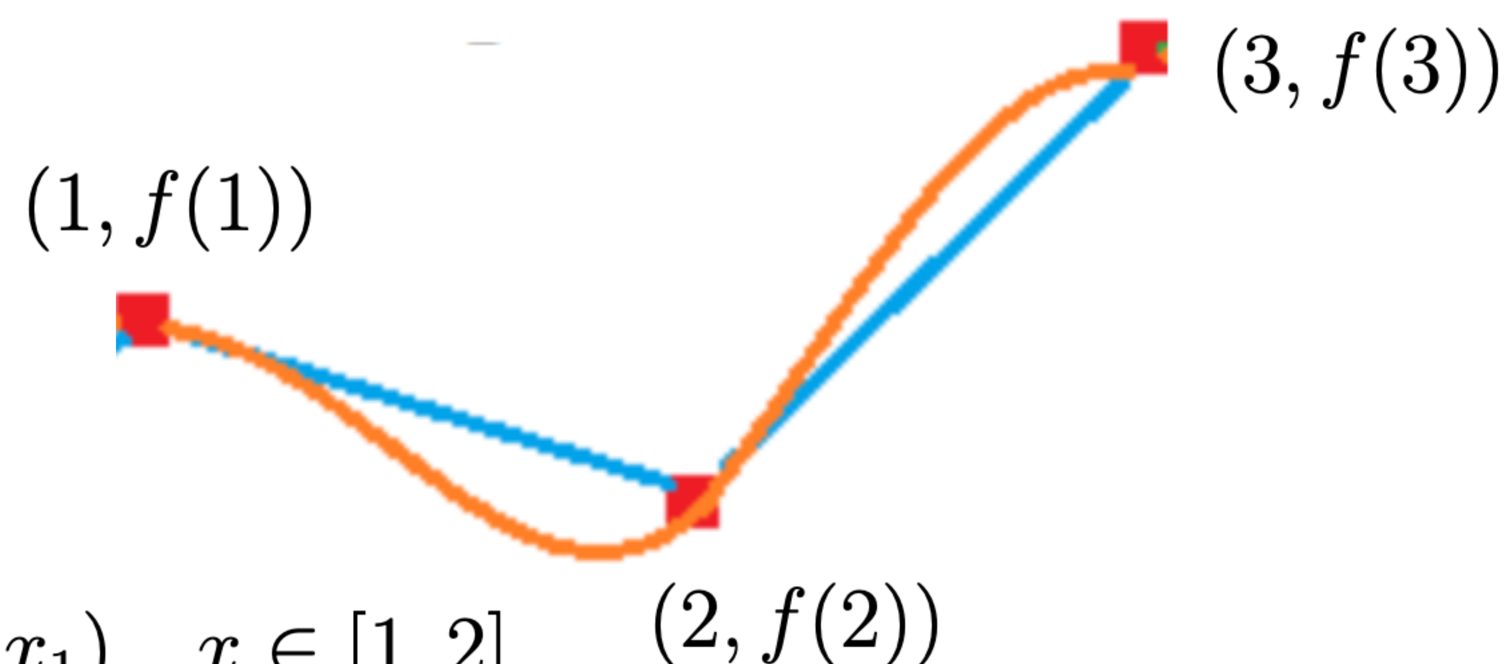
Cubic Splines: Properties and Facts

Introductory Example

A spline is a piecewise defined polynomial.

First let's look at an example of piecewise linear functions.

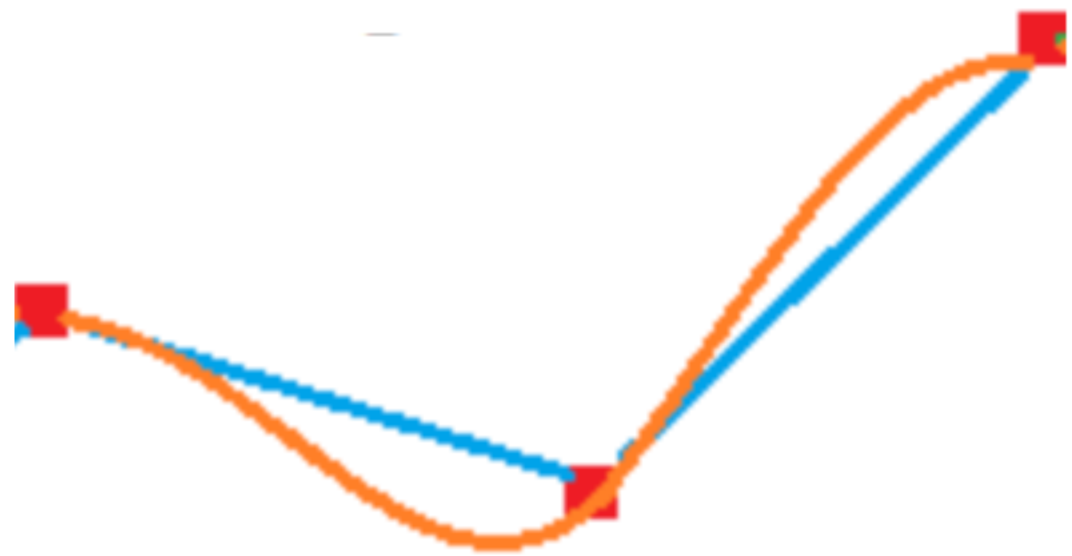
Example 14.1.



piecewise linear function $L(x)$

$$L(x) = \begin{cases} \frac{f(2)-f(1)}{x_2-x_1}(x - x_1) + f(x_1) & x \in [1,2] \\ \dots & x \in [2,3] \end{cases}$$

More points will lead to better approximation.



In general, piecewise linear splines L(x) are simple and powerful.

But, they are not smooth (or "regular"). I.e.,

$$L(x) \in C([a,b]) \text{ but } L(x) \neq C^1([a,b]).$$

Note that each piece is a line (y = mx + b). In particular, it has two degrees of freedom: slope m and b.

Cubic Splines Degrees of Freedom

Cubic splines have more degrees of freedom.

This also implies more regularity (better differentiability properties).

Recall the definition of cubic splines from last lecture:

Definition 13.1 (Cubic Spline Interpolant).

Given f defined on $[a, b]$, $\{x_j\}_{j=0}^{n} \in [a, b]$
$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
The **spline** is a function $S(x)$ that satisfies:

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:
$$S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$
2. $S(x)$ interpolates f at each x_j. I.e., $S(x_j) = f(x_j)$.
3. Conituity: $S \in C([a, b])$.
4. Differentiability: $S \in C^2([a, b])$.

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Theorem 14.1. $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
$f(x)$ has a *unique* "natural" spline interpolant on $[a, b]$ for the points $\{x_j\}_{j=0}^n$.

$$S''(a) = S''(b) = 0.$$

Proof. Totally we have $4n$ unknowns.

(1) Firstly, by property 1 and 2, we have

$$S_0(x_0) = a_0 = f(x_0)$$

$$S_1(x_1) = a_1 = f(x_1)$$

$$\dots$$

$$S_{n-1}(x_{n-1}) = a_{n-1} = f(x_{n-1})$$

$$\boxed{S_j(x_h) = a_j = f(x_j), \quad j = 0, 1, \dots, n-1}$$

introduce notation $a_n := f(x_n)$

$$\boxed{a_j = f(x_j), \quad j = 0, 1, 2, \dots, n}$$

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \dots, n-1$, $S(x)$ is a cubic polynomal:
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(1) $\boxed{S_j(x_h) = a_j = f(x_j), \quad j = 0, 1, \ldots, n-1}$ $\boxed{a_j = f(x_j), \quad j = 0, 1, 2, \ldots, n}$

(2) Next, property 3 tells us that

$$h_j = x_{j+1} - x_j$$

$$S_0(x_1) = S_1(x_1)$$

$$S_1(x_2) = S_2(x_2) \qquad \ldots$$

$$S_{n-2}(x_{n-1}) = S_{n-1}(x_{n-1})$$

$$S_{n-1}(x_n) = f(x_n) =: a_n$$

$$a_0 + b_0 h_0 + c_0 h_0^2 + d_0 h_0^3 = S_0(x_1) = a_1$$

$$a_1 + b_1 h_1 + c_1 h_1^2 + d_1 h_1^3 = S_1(x_2) = a_2$$

$$\ldots$$

$$a_{n-2} + b_{n-2} h_{n-2} + c_{n-2} h_{n-2}^2 + d_{n-2} h_{n-2}^3 = S_{n-2}(x_{n-1}) = a_{n-1}$$

$$a_{n-1} + b_{n-1} h_{n-1} + c_{n-1} h_{n-1}^2 + d_{n-1} h_{n-1}^3 = f(x_n) = a_n$$

$$\boxed{a_{j-1} + b_{j-1} h_{j-1} + c_{j-1} h_{j-1}^2 + d_{j-1} h_{j-1}^3 = a_j, \quad j = 1, 2, \ldots n}$$

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:
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(2) $\boxed{a_{j-1} + b_{j-1}h_{j-1} + c_{j-1}h_{j-1}^2 + d_{j-1}h_{j-1}^3 = a_j, \quad j = 1, 2, \ldots n}$

(3) Similarly, property 4 on first derivative tells us that

$$\begin{aligned} S_0'(x_1) &= S_1'(x_1) \\ S_1'(x_2) &= S_2'(x_2) \\ &\cdots \\ S_{n-2}'(x_{n-1}) &= S_{n-1}'(x_{n-1}) \\ S_{n-1}'(x_n) &=: b_n \end{aligned}$$

$$\begin{aligned} b_0 + 2c_0h_0 + 3d_0h_0^2 &= b_1 \\ b_1 + 2c_1h_1 + 3d_1h_1^2 &= b_2 \\ &\cdots \\ b_{n-2} + 2c_{n-2}h_{n-2} + 3d_{n-2}h_{n-2}^2 &= b_{n-1} \\ b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}^2 &= b_n \end{aligned}$$

$$\boxed{b_{j-1} + 2c_{j-1}h_{j-1} + 3d_{j-1}h_{j-1}^2 = b_j, \quad j = 1, 2, \ldots n}$$

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:
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(1) $\boxed{S_j(x_h) = a_j = f(x_j), \quad j = 0, 1, \ldots, n-1}$ $\boxed{a_j = f(x_j), \quad j = 0, 1, 2, \ldots, n}$

(2) $\boxed{a_{j-1} + b_{j-1}h_{j-1} + c_{j-1}h_{j-1}^2 + d_{j-1}h_{j-1}^3 = a_j, \quad j = 1, 2, \ldots n}$

(3) $\boxed{b_{j-1} + 2c_{j-1}h_{j-1} + 3d_{j-1}h_{j-1}^2 = b_j, \quad j = 1, 2, \ldots n}$

(4) Similarly, property 4 on second derivative tells us that

$$
\begin{aligned}
S_0''(x_1) &= S_1''(x_1) & \qquad 2c_0 + 6d_0h_0 &= 2c_1 \\
S_1''(x_2) &= S_2''(x_2) & \qquad 2c_1 + 6d_1h_1 &= 2c_2 \\
&\ldots & & \boxed{c_{j-1} + 3d_{j-1}h_{j-1} = c_j, \quad j = 1, 2, \ldots n} \\
S_{n-2}''(x_{n-1}) &= S_{n-1}''(x_{n-1}) & \qquad 2c_{n-2} + 6d_{n-2}h_{n-2} &= 2c_{n-1} \\
S_{n-1}''(x_n) &=: 2c_n & \qquad 2c_{n-1} + 6d_{n-1}h_{n-1} &= 2c_n
\end{aligned}
$$

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:

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(2) $\boxed{a_{j-1} + b_{j-1}h_{j-1} + c_{j-1}h_{j-1}^2 + d_{j-1}h_{j-1}^3 = a_j, \quad j = 1, 2, \ldots n}$

(3) $\boxed{b_{j-1} + 2c_{j-1}h_{j-1} + 3d_{j-1}h_{j-1}^2 = b_j, \quad j = 1, 2, \ldots n}$

(4) $\boxed{c_{j-1} + 3d_{j-1}h_{j-1} = c_j, \quad j = 1, 2, \ldots n}$

(5) The natural boundary condition

$$S_0''(x_0) = 0 \qquad 2c_0 = 0$$

$$S_{n-1}''(x_n) = 0 \qquad 2c_n = 0$$

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:

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Summarizing the above

$$a_j = f(x_j), \quad j = 0, 1, 2, \ldots, n$$

$$b_n = S'(x_n)$$

$$c_n = S''(x_n)/2$$

$$a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

$$b_j + 2c_j h_j + 3d_j h_j^2 = b_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

$$c_j + 3d_j h_j = c_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

$$c_0 = c_n = 0$$

Variables:

$$a_j, b_j, c_j, d_j, \quad j = 0, 1, 2, 3, \ldots, n-1$$

$$a_n, b_n, c_n$$

Goal: eliminate to equations of c only

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$$c_j + 3d_j h_j = c_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

$$c_0 = c_n = 0$$

- All a_j values for $j = 0, 1, \ldots, n$ can be determined

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$$c_n = S''(x_n)/2$$

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$$c_j + 3d_j h_j = c_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

$$c_0 = c_n = 0$$

$$d_j = \frac{c_{j+1} - c_j}{3h_j} \text{ for } j = 0, 1, \ldots, n-1.$$

$$a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}) \quad (*)$$

$$b_{j+1} = b_j + h_j(c_j + c_{j+1}) \quad (**)$$

for $j = 0, 1, \ldots, n-1.$

Summarizing the above

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \quad j = 0, 1, \ldots, n-1$$

$$a_j = f(x_j), \quad j = 0, 1, 2, \ldots, n$$

$$b_n = S'(x_n)$$

$$b_{j+1} = \frac{1}{h_{j+1}}(a_{j+2} - a_{j+1}) - \frac{h_{j+1}}{3}(2c_{j+1} + c_{j+2}), \quad j = 0, \ldots, n-2.$$

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$$a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

$$b_j + 2c_j h_j + 3d_j h_j^2 = b_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

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$$\begin{aligned} a_{j+1} &= a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}) \quad (*) \\ b_{j+1} &= b_j + h_j(c_j + c_{j+1}) \quad (**) \end{aligned} \quad \text{for } j = 0, 1, \ldots, n-1.$$

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$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \quad j = 0, 1, \ldots, n-1$$

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$$b_n = S'(x_n)$$

$$b_{j+1} = \frac{1}{h_{j+1}}(a_{j+2} - a_{j+1}) - \frac{h_{j+1}}{3}(2c_{j+1} + c_{j+2}), \quad j = 0, \ldots, n-2.$$

$$c_n = S''(x_n)/2$$

$$a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

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$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \quad j = 0, 1, \ldots, n-1$$

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$$b_{j+1} = \frac{1}{h_{j+1}}(a_{j+2} - a_{j+1}) - \frac{h_{j+1}}{3}(2c_{j+1} + c_{j+2}), \quad j = 0, \ldots, n-2.$$

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$$a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

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for $j = 0, 1, \ldots, n-1.$

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$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \quad j = 0, 1, \ldots, n-1$$

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$$b_{j+1} = \frac{1}{h_{j+1}}(a_{j+2} - a_{j+1}) - \frac{h_{j+1}}{3}(2c_{j+1} + c_{j+2}), \quad j = 0, \ldots, n-2.$$

$$c_n = S''(x_n)/2$$

$$a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

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$$\frac{1}{h_{j+1}}(a_{j+2} - a_{j+1}) - \frac{h_{j+1}}{3}(2c_{j+1} + c_{j+2}) = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) + h_j(c_j + c_{j+1}).$$

for $j = 0, \ldots, n-2$,

$$a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}) \quad (*)$$

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for $j = 0, 1, \ldots, n-1$.

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$$c_j + 3d_j h_j = c_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$

$$\frac{1}{h_{j+1}}(a_{j+2} - a_{j+1}) - \frac{h_{j+1}}{3}(2c_{j+1} + c_{j+2}) = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) + h_j(c_j + c_{j+1}).$$

for $j = 0, \ldots, n-2$,

Shifting index by 1 and simplify, we get for $j = 1, 2, \ldots, n-1$,

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}).$$