Math 131A - Homework 3

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Question 1.

(a) We wish to prove that $s_1 > s_3 > s_5 > \dots$. We have the following.

$$s_3 = \frac{\alpha + s_2}{1 + s_2} = \frac{\alpha + \frac{\alpha + s_1}{1+s_1}}{1 + \frac{\alpha+s_1}{1+s_1}} = \frac{s_1 + \alpha s_1 + 2\alpha}{1 + \alpha + 2s_1} = \frac{s_1(1 + \alpha + 2s_1) + 2\alpha - 2s_1^2}{1 + \alpha + 2s_1} = s_1 - 2\frac{s_1^2 - \alpha}{1 + \alpha + 2s_1}$$
$$s_1 > \sqrt{\alpha} \to s_1^2 > \alpha \to s_1^2 - \alpha > 0$$
$$s_3 \leq s_1 - 2\frac{s_1^2 - \alpha}{1 + \alpha + 2s_1} < s_1$$

Next suppose $s_n > s_{n+2}$ holds for some odd $n \geq 1$. Then we have the following.

$$s_{n+4} = \frac{\alpha + s_{n+2}}{1 + s_{n+2}} = \frac{\alpha + \frac{\alpha + s_n}{1+s_n}}{1 + \frac{\alpha+s_n}{1+s_n}} = \frac{s_n + \alpha s_n + 2\alpha}{1 + \alpha + 2s_n} = \frac{s_n(1 + \alpha + 2s_n) + 2\alpha - 2s_n^2}{1 + \alpha + 2s_n} = s_n - \frac{s_n^2 - \alpha}{\frac{1}{2} + \frac{\alpha}{2} + s_n}$$
$$\alpha > 1 \to \frac{\alpha}{2} > \frac{1}{2} \to \frac{1}{2} + \frac{\alpha}{2} + s_n > 1 + s_n$$
$$s_{n+2} = s_n - \frac{s_n^2 - \alpha}{1 + s_n} > s_n - \frac{s_n^2 - \alpha}{\frac{1}{2} + \frac{\alpha}{2} + s_n} = s_{n+4}$$

Therefore by PMI we have that $s_n > s_{n+2}$ holds for all odd $n \geq 1$, or $s_1 > s_3 > s_5 > \dots$.

(b) We wish to prove that $s_2 < s_4 < s_6 < \dots$. We have the following.

$$s_2 = s_1 + \frac{\alpha - s_1^2}{1 + s_1} < s_1$$
$$s_4 = \frac{\alpha + s_3}{1 + s_3} = \frac{s_2 + \alpha s_2 + 2\alpha}{1 + \alpha + 2s_2} = \frac{(1+\alpha)\frac{\alpha+s_1}{1+s_1} + 2\alpha}{1 + \alpha + \frac{\alpha+s_1}{1+s_1}}$$

(sorry idk) Therefore $s_2 < s_4$. Suppose $s_n < s_{n+2}$ holds for some even $n \geq 2$. Then we have the following.

$$=$$

Therefore by PMI we have that $s_n < s_{n+2}$ holds for all even $n \geq 2$, or $s_2 < s_4 < s_6 < \dots$.

(c) We wish to prove that $\lim s_n = \sqrt{\alpha}$. The limit is where the following equality holds:

$$\begin{aligned} s_{n+1} &= s_n \\ l &= \frac{\alpha + l}{1 + l} \\ l + l^2 &= \alpha + l \\ l^2 &= \alpha \\ l &= \sqrt{\alpha} \end{aligned}$$

Question 2.

First prove $l = \limsup s_n \Rightarrow (i), (ii)$.

Then prove $(i), (ii) \Rightarrow l = \limsup s_n$.

Question 3.

We wish to show that $\limsup(-s_n) = -\liminf s_n$. Let $\liminf s_n = m$, then for some N we have that $n > N \Rightarrow s_n \geq m$. Equivalently this means that $n > N \Rightarrow -s_n \leq -m$. Thus for the sequence $-s_n$, we have $\limsup(-s_n) = -m$. Therefore $\limsup(-s_n) = -\liminf s_n$.

Question 4.

We have that for some N, $n > N \Rightarrow s_n \leq t_n$. Let $\liminf s_n = a$, so for some M we have $n > M \Rightarrow a \leq s_n$. Let $\liminf t_n = b$, so for some L you have $n > L \Rightarrow b \leq t_n$. Then $n > \max(N, M, L) \Rightarrow a \leq s_n, b \leq t_n$. Because we have that $s_n \leq t_n$ for every n here, it is the case that $s_n \leq b$ since b is part of the elements of t_n in this range. Then $a \leq s_n \leq b \leq t_n$, or $a \leq b$, or $\liminf s_n \leq \liminf t_n$.

We have that for some N, $n > N \Rightarrow s_n \leq t_n$. Let $\limsup s_n = a$, so for some M we have $n > M \Rightarrow s_n \leq a$. Let $\limsup t_n = b$, so for some L you have $n > L \Rightarrow t_n \leq b$. Then $n > \max(N, M, L) \Rightarrow s_n \leq a, t_n \leq b$. Because we have that $s_n \leq t_n$ for every n here, it is the case that $a \leq t_n$ since a is part of the elements of s_n in this range. Then $s_n \leq a \leq t_n \leq b$, or $a \leq b$, or $\limsup s_n \leq \limsup t_n$.

Question 5.

First prove that for a subsequence (t_k) of (s_n), $\lim t_k = +\infty \Rightarrow (s_n)$ not bounded above.

Then prove (s_n) not bounded above $\Rightarrow$ for a subsequence (t_k) of (s_n), $\lim t_k = +\infty$.

Question 6.

(a) $s_n = -n^2$. By Theorem 10.5, if the limit of s_n exists in $\bar{\mathbb{R}}$ then $\limsup s_n = \liminf s_n = \lim s_n$, and $\lim -n^2 = -\infty$.

(b) $s_n = \begin{cases} 1 - \frac{1}{1+n^2} & \text{even } n \\ -1 + \frac{1}{1+n^2} & \text{odd } n \end{cases}$

(c) $s_n = \sin^2 n$.

Question 7.

(a) $(s_n + t_n) = (2, 2, 3, 1, 2, 2, 3, 1, \dots)$, and the minimum as $n \to \infty$ is 1, so $\liminf(s_n + t_n = 1$. Thus $\liminf(s_n + t_n) = 1$. However, $\liminf s_n + \liminf t_n = 0 + 0 = 0$.

(b) From before we have the sequence of values in $(s_n + t_n)$ and note that the maximum as $n \to \infty$ is 3, so $\limsup(s_n + t_n) = 3$. However, $\limsup s_n + \limsup t_n = 2 + 2 = 4$.

(c) $(s_n t_n) = (0, 1, 2, 0, 0, 1, 2, 0, \dots)$, so the maximum as $n \to \infty$ is 2, so $\limsup s_n t_n = 2$. However, $(\limsup s_n)(\limsup t_n) = (2)(2) = 4$.

Question 8.

We wish to show that (s_n) is a Cauchy sequence. A Cauchy sequence is a sequence for which $\forall \epsilon > 0, \exists N : m, n > N \Rightarrow |s_m - s_n| < \epsilon$. Suppose we have any $\epsilon > 0$ and let $m = n + k$. Then we have the following.

$$\begin{aligned}
|s_m - s_n| &= |s_{n+k} - s_n| \\
&= |(s_{n+k} - s_{n+k-1}) + (s_{n+k-1} - s_{n+k-2}) + \cdots + (s_{n+1} - s_n)| \\
&\leq |s_{n+k} - s_{n+k-1}| + \cdots + |s_{n+1} - s_n| \\
&\leq \sum_{i=n}^{n+k-1} (\frac{1}{2})^i = \sum_{i=1}^{k-1} ((\frac{1}{2})^n)^i = \frac{1 - ((\frac{1}{2})^n)^k}{1 - (\frac{1}{2})^n}
\end{aligned}$$

Thus you can take N which satisfies $\epsilon = \frac{1-(\frac{1}{2})^{k+N}}{1-(\frac{1}{2})^N}$. Therefore (s_n) is Cauchy.