Physics 112 - Homework 2

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Question 1.

Free energy of two-state system.

$$
\begin{aligned}
F &= -\tau \ln Z \\
&= \boxed{-\tau \ln 1 + e^{-\epsilon/\tau}}
\end{aligned}
$$

Energy from F.

$$
\begin{aligned}
U &= -\tau^2 \frac{\partial (F/\tau)}{\partial \tau} \\
&= -\tau^2 \frac{\partial}{\partial \ln 1 + e^{-\epsilon/\tau}} \\
&= \frac{\tau^2 \frac{\epsilon}{\tau^2} e^{-\epsilon/\tau}}{1 + e^{-\epsilon/\tau}} \\
&= \boxed{\frac{\epsilon e^{-\epsilon/\tau}}{1 + e^{-\epsilon/\tau}}}
\end{aligned}
$$

Entropy from F.

$$
\begin{aligned}
\sigma &= \frac{\partial F}{\partial \tau} \\
&= \ln\left(1 + e^{-\epsilon/\tau}\right) + \frac{\tau \frac{\epsilon}{\tau^2} e^{-\epsilon/\tau}}{1 + e^{-\epsilon/\tau}} \\
&= \boxed{\ln\left(1 + e^{-\epsilon/\tau}\right) + \frac{\epsilon e^{-\epsilon/\tau}}{\tau(1 + e^{-\epsilon/\tau})}}
\end{aligned}
$$

Question 2.

Partition function.

$$
\begin{aligned}
Z &= \sum_s \binom{N}{N+} e^{2smB/\tau} \\
&= \sum_s \frac{N!}{s!(N-s)!} (e^{2mB/\tau})^{s-N/2} \\
&= \sum_s \binom{N}{s} (e^{2mB/\tau})^s \cdot e^{-mBN/\tau} \\
&= (1+e^{2mB/\tau})^N (e^- mBN/\tau) \\
&= (1+e^{2mB/\tau})^N (e^- mB/\tau)^N \\
&= (e^{-mB/\tau} + e^{mB/\tau})^N \\
&= \boxed{\cosh^N (mB/\tau)}
\end{aligned}
$$

Magnetisation.

$$
\begin{aligned}
M &= -\tau^2 \frac{\partial}{\partial \tau} \ln Z \\
&= -\tau^2 \frac{\partial}{\partial \tau} \ln \cosh^N (mB/\tau) \\
&= -N\tau^2 \frac{\partial}{\partial \tau} \ln \cosh (mB/\tau) \\
&= -N\tau^2 \frac{\sinh (mB/\tau) \frac{-m}{\tau^2}}{\cosh (mB/\tau)} \\
&= \boxed{Nm \tanh (mB/\tau)}
\end{aligned}
$$

Magnetic susceptibility.

$$
\begin{aligned}
\chi &= \frac{\partial M}{\partial B} \\
&= Nm \operatorname{sech}^2 (mB/\tau) \frac{m}{\tau} \\
&= \boxed{\frac{Nm^2}{\tau} \operatorname{sech}^2 (mB/\tau)}
\end{aligned}
$$

Free energy.

$$\begin{aligned} F &= -\tau \ln Z \\ &= -\tau \ln \cosh^N (mB/\tau) \\ &= -N\tau \ln \cosh(mB/\tau) \\ &= -N\tau \ln \frac{1}{\sqrt{1-\tanh^2(mB/\tau)}} \\ &= \boxed{-N\tau \ln \left(\frac{1}{\sqrt{1-x^2}}\right)} \end{aligned}$$

Magnetic susceptibility in the limit.

$$\begin{aligned} \lim_{mB \ll \tau} \chi &= \lim_{mB/\tau \to 0} \frac{Nm^2}{\tau} \operatorname{sech}^2 (mB/\tau) \\ &= \frac{Nm^2}{\tau} \operatorname{sech}^2 0 \\ &= \boxed{\frac{Nm^2}{\tau}} \end{aligned}$$

Question 3.

Partition function first.

$$\begin{aligned} Z &= \sum_s e^{-s\hbar\omega/\tau} \\ &= (1 - e^{-\hbar\omega/\tau})^{-1} \end{aligned}$$

Free energy.

$$\begin{aligned} F &= -\tau \ln\Big(1 - e^{-\hbar\omega/\tau}\Big)^{-1} \\ &= \boxed{\tau \ln\Big(1 - e^{-\hbar\omega/\tau}\Big)} \end{aligned}$$

Entropy.

$$\begin{aligned} \sigma &= \frac{\partial F}{\partial \tau} \\ &= -\left(\ln\Big(1 - e^{-\hbar\omega/\tau}\Big) + \frac{\tau e^{-\hbar\omega/\tau}(\hbar\omega/\tau)}{1 - e^{-\hbar\omega/\tau}}\right) \\ &= -\frac{(\hbar\omega/\tau)e^{-\hbar\omega/\tau}}{1 - e^{-\hbar\omega/\tau}} - \ln\Big(1 - e^{-\hbar\omega/\tau}\Big) \\ &= \frac{\hbar\omega}{\tau}\frac{1 - e^{-\hbar\omega/\tau} - 1}{1 - e^{-\hbar\omega/\tau}} - \ln\Big(1 - e^{-\hbar\omega/\tau}\Big) \\ &= \boxed{\frac{\hbar\omega/\tau}{e^{-\hbar\omega/\tau} - 1} - \ln\Big(1 - e^{-\hbar\omega/\tau}\Big)} \end{aligned}$$

Question 4.

Get energy in terms of partition function, note $\tau = 1/\beta$ so $\frac{\partial}{\partial \tau} = -\frac{1}{\tau^2}\frac{\partial}{\partial \beta}$.

$$\begin{aligned}
Z &= \sum_s e^{\epsilon_s \beta} \\
U &= \frac{1}{Z}\sum_s \epsilon_s e^{\epsilon_s \beta} \\
&= \frac{\frac{\partial}{\partial B} Z}{Z} \\
\frac{\partial U}{\partial \tau} &= -\frac{1}{\tau^2}\frac{\partial U}{\partial \beta} \\
&= -\frac{1}{\tau^2}\frac{\partial}{\partial \beta}\left(\frac{\frac{\partial}{\partial \beta} Z}{Z}\right) \\
&= -\frac{1}{\tau^2 Z^2}\left(\frac{\partial^2 Z}{\partial \beta^2}\cdot Z - (\frac{\partial Z}{\partial \beta})^2\right) \\
&= -\frac{1}{\tau^2}\left(\frac{\frac{\partial^2 Z}{\partial \beta^2}}{Z} - (\frac{\frac{\partial Z}{\partial \beta}}{Z})^2\right) \\
\tau^2 \frac{\partial U}{\partial \tau} &= \boxed{\langle \epsilon^2 \rangle - \langle \epsilon \rangle^2}
\end{aligned}$$