Lecture G. Homework 7

Covered Contents: Composite Quadrature Rules and Gaussian Quadrature (Lec 19 - 21)
Deadline: 11/22/2021, 23:59 PST
Total points: Pen-and-Paper (10 + 10 + 10 + 10 + 15 + 20 = 75) + Coding (25) = 100.
Submit "hw7.zip"

Pen and Paper

G.1. Use the Composite Trapezoidal and Simpson's rules to approximate the integral

$$\int_1^2 x \ln(x) dx$$

with $n = 4$ subintervals.

G.2. Determine the values of n and h required to approximate

$$\int_1^2 x \ln(x) dx$$

to within an error tolerance of $\tau = 10^{-5}$. Use
(a) Composite Trapezoidal Rule.
(b) Composite Simpson's Rule.

G.3. Determine the constants a, b, c, d that will produce a quadrature formula

$$\int_{-1}^1 f(x) dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

with degree of exactness $N = 3$.

G.4. Use the 5-point Gaussian quadrature formula to approximate

$$\frac{1}{\sqrt{\pi}} \int_{-1}^1 e^{-x^2} dx.$$

The nodes and weights are $(x_i, w_i) =$

$$\begin{aligned} -0.906179845938664, 0.236926885056182 \\ -0.538469310105683, 0.478628670499366 \\ 0.0, 0.568888888888888 \\ 0.538469310105683, 0.478628670499366 \\ 0.906179845938664, 0.236926885056182 \end{aligned}$$

G.5. In this exercise you will derive a change of variables so that the technique of Gaussian quadrature discussed in lecture can be applied to integrate a function f over an arbitrary interval $\int_a^b f(x)dx$.
(a) Define a *linear* function $u(x)$ that maps $[a, b]$ onto $[-1, 1]$. The map should have the property that $u(a) = -1$ and $u(b) = 1$.
(b) Use your answer from (a), as well as the change of variables formula

$$\int_a^b f(x)dx = \int_{u(a)}^{u(b)} f(x(u))\frac{dx}{du}du$$

to express the original integral over $[a, b]$ as an integral over $[-1, 1]$. You will need to invert your expression $u(x)$ from (a) to get $x(u)$. (note: this is simply the 'u-substitution' rule from calculus).
(c) The nodes and weights for the two-point Gaussian quadrature rule were derived using the 'brute-force' method in HW6, exercise 5; the application was given by

$$\int_{-1}^{1} g(x)dx \approx g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) \qquad (*)$$

Suppose $[a, b] = [1, 2]$. Use your answer from part (b), as well as (*) to approximate

$$\int_1^2 \log(x)dx$$

and report the relative error.

G.6. Recall the Graham-Schmidt (G-S) procedure from linear algebra; given n linearly independent vectors $\{x_1, x_2, \ldots, x_n\}$ and an inner product $\langle \cdot, \cdot \rangle$ on that vector space, G-S produces a new set of mutually orthonormal vectors $\{q_1, q_2, \ldots, q_n\}$, i.e.,

$$\langle q_i, q_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

The G-S process can be summarized by: For $i = 1, 2, \ldots, n$:

- Set $v_1 = x_1$
- For $i = 2, \ldots, n$ set

$$v_i = x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, v_j \rangle}{\langle v_j, v_j \rangle} v_j$$

- For $i = 1, \dots, n$, normalize:

$$q_i = \frac{v_i}{\|v_i\|}$$

where the norm $\|v_i\| = (\langle v_i, v_i \rangle)^{1/2}$.

(a) Let $x_1 = (0, -1, 2), x_2 = (1, 0, -1)$, and $x_3 = (-3, 1, 0)$. Apply the G-S procedure to produce three orthonormal vectors $\{q_1, q_2, q_3\}$ in $\mathbb{R}^3$.
(b) Consider the inner-product defined in lecture on functions:

$$\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) dx.$$

Consider the 'vectors' in this function space $\{y_1(x), y_2(x), y_3(x)\} = \{1, x, x^2\}$. Perform the G-S procedure to produce three orthonormal polynomials $\{q_1(x), q_2(x), q_3(x)\}$; this will produce the first three Legendre polynomials.

Coding

What to submit: (25 points).
Consider the nonlinear equation for x:

$$\int_0^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 0.45$$

(a) Define

$$f(x) := \int_0^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt - 0.45.$$

Using the Fundamental Theorem of Calculus, write down Newton's method applied to f.
(b) Implement in MATLAB Newton's method to find the solution x to the equation $f(x) = 0$; terminate the iteration when the residual is smaller than $\tau = 10^{-5}$. Use $x_0 = 0.5$ as an initial guess. Since each step of Newton's method consists of an evaluation $f(x_k)$, use the Composite Trapezoidal Rule to estimate the value.
Notes:

- It is probably easiest to write a separate function to estimate $f(x_k)$ based on the Composite Trapezoidal Rule and then call that function at each iteration k. This has been partially pre-written for you in 'trap_rule.m' file; you need to implement the quadrature formula.
- You are free to select the number of subintervals N in the quadrature rule; in general, using N will give you a more accurate answer, but it will imply that each iteration of Newton's method will take longer. You should experiment to find a reasonable compromise.