Physics 115A - Homework 9

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Question 1.

Commutator distributive identity.

$$\begin{aligned}[AB,C] &= ABC - CAB \\ &= ABC - ACB + ACB - CAB \\ &= A(BC - CB) + (AC - CA)B \\ &= \boxed{A[B,C] + [A,C]B}\end{aligned}$$

Exponentiated position and momentum commutator.

$$\begin{aligned}[x^n,p] &= x^n p - p x^n \\ &= -x^n i\hbar \frac{\mathrm{d}}{\mathrm{d}x} + i\hbar \frac{\mathrm{d}}{\mathrm{d}x} x^n \\ &= -i\hbar x^n \frac{\mathrm{d}}{\mathrm{d}x} + i\hbar n x^{n-1} + i\hbar x^n \frac{\mathrm{d}}{\mathrm{d}x} \\ &= \boxed{i\hbar n x^{n-1}}\end{aligned}$$

Function and momentum commutator.

$$\begin{aligned}[f(x),p] &= f(x)p - pf(x) \\ &= -f(x) i\hbar \frac{\mathrm{d}}{\mathrm{d}x} + i\hbar \frac{\mathrm{d}}{\mathrm{d}x} f(x) \\ &= -i\hbar f(x) \frac{\mathrm{d}}{\mathrm{d}x} + i\hbar f'(x) + i\hbar f(x) \frac{\mathrm{d}}{\mathrm{d}x} \\ &= \boxed{i\hbar \frac{\mathrm{d}f}{\mathrm{d}x}}\end{aligned}$$

Question 2.

Uncertainty in position vs energy.

$$\begin{aligned}
\sigma_x \sigma_H &= \frac{1}{2i} |\langle [A, B] \rangle| \\
&= \frac{1}{2i} |\langle [x, \frac{p^2}{2m} + V] \rangle| \\
&= \frac{1}{2i} |\langle \frac{1}{2m} [x, p^2] + [x, V] \rangle| \\
&= \frac{1}{2i} |\langle \frac{1}{2m} (xp^2 - p^2 x) \rangle| \\
&= \frac{1}{2i} |\langle \frac{1}{2m} ((xp - px)p + p(xp - px)) \rangle| \\
&= \frac{1}{2i} |\langle \frac{1}{2m} (2i\hbar p) \rangle| \\
&= \boxed{\frac{\hbar}{2m} |\langle p \rangle|}
\end{aligned}$$

In stationary states, uncertainty in energy is 0, as is expected value of momentum on the other side of the inequality, so it doesn't give us information on uncertainty in position.

Question 3.

Normalising with A.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \left| \frac{A}{x^2 + a^2} \right|^2 \mathrm{d}x \\
\frac{1}{A^2} &= \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2} \mathrm{d}x \\
\frac{1}{A^2} &= \frac{\pi}{2a^3} \\
A &= \boxed{\sqrt{\frac{2}{\pi}} a^{3/2}}
\end{aligned}$$

Expectation values of position.

$$\begin{aligned}
E(x) &= \frac{2a^3}{\pi} \int_{-\infty}^{\infty} \frac{x}{(x^2 + a^2)^2} \mathrm{d}x \\
&= \boxed{0} \\
E(x^2) &= \frac{2a^3}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2} \mathrm{d}x \\
&= \frac{2a^3}{\pi} \frac{\pi}{2a} \\
&= \boxed{a^2} \\
\sigma_x &= \sqrt{E(x^2) - E(x)^2} \\
&= \boxed{a}
\end{aligned}$$

Momentum space wavefunction.

$$\begin{aligned}
\phi(p, 0) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x, 0) \, \mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} a^{3/2} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{x^2 + a^2} \mathrm{d}x \\
&= \frac{2a^{3/2}}{\pi\sqrt{\hbar}} \int_{0}^{\infty} \frac{\cos(px/\hbar) - i\sin(px/\hbar)}{x^2 + a^2} \mathrm{d}x \\
&= \frac{2a^{3/2}}{\pi\sqrt{\hbar}} \frac{\pi}{2a} e^{-|p|a/\hbar} \\
&= \boxed{\sqrt{\frac{a}{\hbar}} e^{-|p|a/\hbar}}
\end{aligned}$$

Expectation values of momentum.

$$\begin{aligned}
E(p) &= \frac{a}{\hbar}\int_{-\infty}^{\infty} pe^{-2|p|a/\hbar}\,\mathrm{d}p \\
&= \boxed{0} \\
E(p^2) &= \frac{a}{\hbar}\int_{-\infty}^{\infty} p^2 e^{-2|p|a/\hbar}\,\mathrm{d}p \\
&= \frac{2a}{\hbar}\int_{0}^{\infty} p^2 e^{-2|p|a/\hbar}\,\mathrm{d}p \\
&= \frac{2a}{\hbar}\frac{\hbar^3}{4a^3} \\
&= \boxed{\frac{\hbar^2}{2a^2}} \\
\sigma_p &= \sqrt{E(p^2) - E(p)^2} \\
&= \frac{\hbar}{a\sqrt{2}}
\end{aligned}$$

Uncertainty principle check.

$$\begin{aligned}
\sigma_x\sigma_p &= a\frac{\hbar}{a\sqrt{2}} \\
&= \frac{\hbar}{\sqrt{2}} \\
&> \frac{\hbar}{2}
\end{aligned}$$