Lecture C. Homework 3

Covered Contents: Newton's Method, Secant Method, Convergence Order Theorem (Lec 6-8)
Deadline: 10/18/2021, 23:59 PST
Total points: Pen-and-Paper (10+10+20+15+20=75) + Coding (25) = 100.
Submit "hw3.zip"

Pen and Paper

What to submit: 'hw3pen.pdf'

C.1. Let $f(x) = -x^3 - \cos(x)$ and $p_0 = -1$.
(a) Use Newton's method to find p_2.
(b) Could $p_0 = 0$ be used? Show what happens.

C.2. Let $f(x) = -x^3 - \cos(x)$, $p_0 = -1$, $p_1 = 0$. Use the Secant Method to find p_3.

C.3. Let $p \in [a, b]$ be the root of $f \in C^1([a.b])$, and assume $f'(p) \neq f'(p_0)$ for some $p_0 \in [a, b]$. Consider the following iteration scheme (note that it is slightly different from Newton's method):

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_0)}, \qquad \forall n \geq 0.$$

We will assume that the iterative scheme does converge, i.e., that $p_n \to p$ as $n \to \infty$. Please show that this method has order of convergence $\alpha = 1$. (Hint: The proof more or less follows our proof for the Convergence Order Theorem which was covered in the lectures).

(*Note: to actually show that indeed the scheme converges, one needs to assume $f \in C^2([a, b])$. This is a sufficient, but not necessary condition*).

C.4. Use the secant method to find a solution to the equation

$$e^x + 2^{-x} + 2\cos(x) - 6 = 0$$

on the interval $[1, 2]$ that is accurate to within 10^{-5} (here accuracy should be measured by the residual).

C.5. Recall the definition of order of convergence for a sequence $(p_n)_n$ that converges to p as $n \to \infty$: if $\exists \lambda \in (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda,$$

then $(p_n)_n$ converges with order α. When n is sufficiently large, this means

$$e_{n+1} \approx \lambda e_n^\alpha, \qquad (*)$$

where $e_n := |p_n - p|$ is the error at step n.

Assume $(*)$ holds true *exactly*. Show that

$$\alpha = \frac{\log(e_{n+1}/e_n)}{\log(e_n/e_{n-1})}. \qquad (**)$$

Coding

What to submit: Your code for each sub-problem, and a PDF file that contains the screenshots and your answer to the question in (b). In summary, submit "hw3a.m", "hw3b.m", and "hw3result.pdf".

This problem requires you to work with the sequence of approximate roots generated by a root-finding method. You may use the script "newtonRoot.m" as a starting point for your work. You will empirically determine the order of convergence for two different root finding methods. The two methods are

- Newton's method
- A modified Newton's method where $f'(p_0)$ is used instead of $f'(p_n)$ for each step n (p_0 is the initial iterate)

(a) For each method, compute the positive root of $f(x) = x^2 - 3$. Use a starting guess of $p_0 = 5$. Print out the error in the approximate root for each iteration and record the results with a screenshot. Stop your computation when the residual is less than 10^{-12}.
(b) Using the values of the error for each iteration, estimate the order of convergence α of each method using the result $(**)$ from the Pen-and-paper exercise 5. Print out the results and record them with a screenshot. Do your results agree with the theoretical predictions?

(*Suggestion:* use the computer to do the computation to estimate the order of convergence. For example, you could store the root approximations in a vector and then use a loop to perform the computation.)