UCLA Math151A
Fall 2021
Lecture 24
2021/11/17

Gaussian Elimination

Equivalent statements

- A is invertible
- Determinant of A is nonzero
- Null space of A is $\{0\}$ (trivial null space)
- Columns of A are linearly independent
- Rows of A are linearly independent
- and so on

We study direct methods for solving the matrix equation $Ax = b$ for x.

Two basic issues in direct methods are:

- How do we construct a solution method for solving $Ax = b$?

- What is the computational complexity (is the method efficient)?

$\boxed{\textbf{Gaussian Elimination}}$

To find x s.t. $Ax = b$.

1. Form augmented matrix : $[A|b]$

2. Use row reduction to transform into upper triangular form
$$[A|b] \rightarrow [U|y]$$
where U is upper triangular.

3. Solve $Ux = y$ using back substitution.

Example 23.1. $$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(1)*(-1/4)+(2)

(1)*(-1/4)+(3)

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 1 & 1 & 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & \frac{3}{4} & \frac{11}{4} & \frac{-1}{4} \end{pmatrix}$$

(2)*(-1/5)+(3)

$$Ux = y$$

$$\begin{pmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} \\ 0 & 0 & \frac{13}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{4} \\ -\frac{1}{5} \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & \frac{13}{5} & -\frac{1}{5} \end{pmatrix}$$

$$Ux = y$$

$$\begin{pmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} \\ 0 & 0 & \frac{13}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{4} \\ -\frac{1}{5} \end{pmatrix}$$

$$x_3 = \frac{y_3}{U_{33}}$$

$$x_2 = \frac{1}{U_{22}} \left(y_2 - U_{23} x_3\right)$$

$$x_1 = \frac{1}{U_{11}} \left(y_1 - U_{12} x_2 - U_{13} x_3\right)$$

General description:

n rows, $n+1$ columns.

(1) Forming augmented matrix.

If $A \in \mathbb{R}^{n\times n}$, and $x, b \in \mathbb{R}^n$, then $[A|b] \in \mathbb{R}^{n\times(n+1)}$

(2) Use row reduction to transform into upper triangular form.

- (non-zero) scalar multiplication:

  if we scale with zero the matrix is no longer invertible.
- scalar multiplictation plus row additoon.
- row swap.

These elemenmtrary row operations are invertible.

Because they are reversible!

$-1/4E_1 + E_2 \to E_2$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 15/4 & 3/4 \\ 1 & 1 & 3 \end{pmatrix} = PA$$

Swap(i,j)

$T_{ij}^{-1} = T_{ij}.$

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

$$D_i(m) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & m & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

So $D_i(m)A$ is the matrix produced from A by multiplying row i by m.

So , row reduction on augmented matrix can be represetned as

$$P_1(Ax) = P_1b$$

$$P_2(P_1Ax) = P_2P_1b$$

$$\dots$$

$$P_nP_{n-1}\dots P_2P_1Ax = P_nP_{n-1}\dots P_2P_1b$$

$$Ux = y$$

(3) Solve $Ux = y$ using back substitution:

Solution has analytic form:

$$Ux = y$$

$$\begin{pmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} \\ 0 & 0 & \frac{13}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{4} \\ -\frac{1}{5} \end{pmatrix}$$

$$x_3 = \frac{y_3}{U_{33}}$$

$$x_2 = \frac{1}{U_{22}} \left(y_2 - U_{23} x_3\right)$$

$$x_1 = \frac{1}{U_{11}} \left(y_1 - U_{12} x_2 - U_{13} x_3\right)$$

$$x_n = y_n / u_{nn}$$

$$x_{n-1} = \frac{1}{u_{n-1,n-1}} (y_{n-1} - u_{n-1,n} x_n)$$

$$x_i = \frac{1}{u_{ii}} (y_i - \sum_{j=i+1}^{n} u_{i,j} x_j)$$

$\boxed{\text{how to choose the elementary operations.}}$

For a 3 by 3 A, “knock out“ a_{21} and a_{31}.

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The coresponding E.R.O. s are

$$\lambda E_1 + E_2 \to E_2$$

$$\mu E_1 + E_3 \to E_3,$$

where we chosoe λ, μ to “knock out” a_{21} and a_{31}.

To do taht we can do $\lambda = -\dfrac{1}{a_{11}} a_{21}, \mu = -\dfrac{1}{a_{11}} a_{31}$

The resulting matrix is $P_1 A$

$$\underbrace{\left(\begin{array}{ccc:c} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right)}_{A} \qquad \underbrace{\left(\begin{array}{ccc:c} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & \frac{3}{4} & \frac{11}{4} & \frac{-1}{4} \end{array}\right)}_{P_1 A}$$

P_1 is a matrix that includes multiple E.R.Os.

Then we just need to knock out the element below the diagonal 22,

Failure

Clearly it can fail if elements along diagonal are zero.

$$\underbrace{\left(\begin{array}{ccc|c} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right)}_{A} \qquad \underbrace{\begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & \frac{3}{4} & \frac{11}{4} & \frac{-1}{4} \end{pmatrix}}_{P_1 A}$$

what if we got a zero here?

In this case we can pivot to a new diagonal element by performing row swapping (which is an E.R.O.).

Fancy word for row swapping: **pivoting**.

Gauss-Jordan Elimination

Use the ith equation to eliminate
not only x_i from the equations $E_{i+1}, E_{i+2}, \ldots, E_n$,
but also from $E_1, E_2, \ldots, E_{i-1}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]}$$