Physics 115A - Homework 2

Zooey Nguyen

zooeyn@ucla.edu

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Question 1.

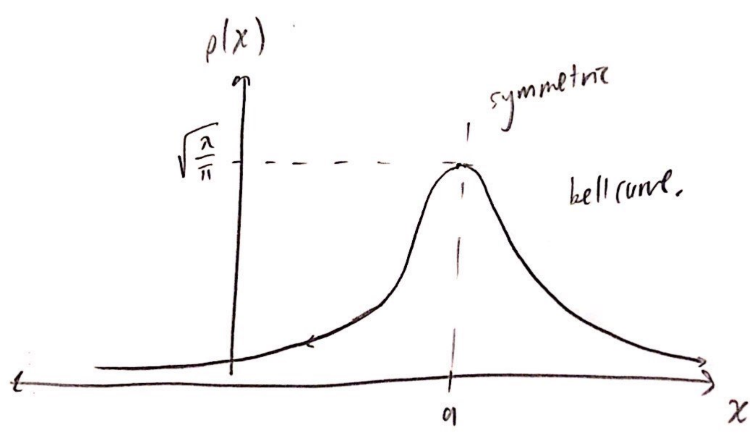
Normalising with A.

$$1 = \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}\,\mathrm{d}x = A\sqrt{\frac{\pi}{\lambda}}$$
$$A = \boxed{\sqrt{\frac{\lambda}{\pi}}}$$

Finding expectation values.

$$E(x) = \sqrt{\frac{\lambda}{\pi}}\int xe^{-\lambda(x-a)^2}\,\mathrm{d}x = \sqrt{\frac{\lambda}{\pi}}\left[\int (x-a)e^{-\lambda(x-a)^2}\,\mathrm{d}x + \int ae^{-\lambda(x-a)^2}\,\mathrm{d}x\right]$$
$$= \sqrt{\frac{\lambda}{\pi}}\left[0 + a\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{a}$$
$$E(x^2) = \sqrt{\frac{\lambda}{\pi}}\int x^2e^{-\lambda(x-a)^2}\,\mathrm{d}x = \sqrt{\frac{\lambda}{\pi}}\int (u+a)^2e^{-\lambda u^2}\,\mathrm{d}u$$
$$= \sqrt{\frac{\lambda}{\pi}}\left[\int u^2e^{-\lambda u^2}\,\mathrm{d}u + 2a\int ue^{-\lambda u^2}\,\mathrm{d}u + a^2\int e^{-\lambda u^2}\,\mathrm{d}u\right]$$
$$= \sqrt{\frac{\lambda}{\pi}}\left[\frac{\sqrt{\pi}}{2\lambda^{3/2}} - 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{\frac{1}{2\lambda} - a^2}$$
$$\sigma_x = \sqrt{E(x^2) - E^2(x)} = \sqrt{\frac{1}{2\lambda} - a^2 - (0)^2} = \boxed{\sqrt{\frac{1}{2\lambda} - a^2}}$$

Graph of $\rho(x)$.



Question 2.

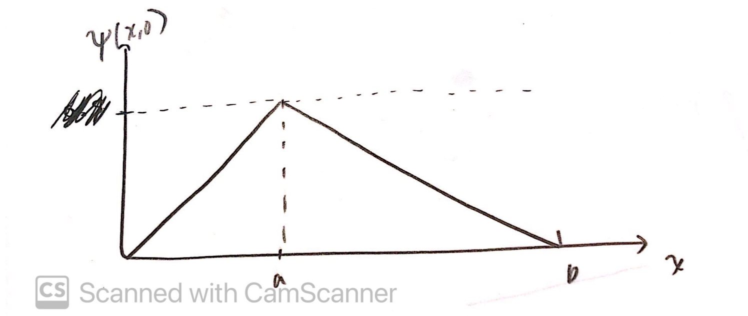
Normalising with A.

$$1 = \int_0^a (A\frac{x}{a})^2 \, \mathrm{d}x + \int_a^b (A\frac{b-x}{b-a})^2 \, \mathrm{d}x$$

$$\frac{1}{A^2} = \int_0^a \frac{x^2}{a^2} \, \mathrm{d}x + \int_a^b \frac{x^2 - 2bx + b^2}{(b-a)^2} \, \mathrm{d}x = \frac{b}{3}$$

$$A = \boxed{\sqrt{\frac{3}{b}}}$$

Graph of $\Psi(x, 0)$. The particle is most likely to be found at $\boxed{x = a}$ at $t = 0$.



The probability of finding the particle to the left of a.

$$P(x < a) = \frac{3}{b} \int_0^a \frac{x^2}{a^2} \, \mathrm{d}x = \boxed{\frac{a}{b}}$$

$$P(x < a)|_{b=a} = 1$$

$$P(x < a)|_{b=2a} = 0.5$$

Expectation value of x.

$$E(x) = \frac{3}{b} \left(\int_0^a \frac{x^3}{a^2} \, \mathrm{d}x + \int_a^b \frac{x^3 - 2bx^2 + b^2 x}{(b-a)^2} \, \mathrm{d}x \right)$$

$$= \boxed{\frac{2a^3 - 3ba^2 + b^3}{4(b-a)^2}}$$

Question 3.

Normalising with A at $t = 0$.

$$1 = \int |\Psi(x,0)|^2 \, \mathrm{d}x = A^2 \int e^{-2\lambda|x|} \, \mathrm{d}x = \frac{A^2}{\lambda}$$
$$A = \boxed{\sqrt{\lambda}}$$

Expectation values.

$$E(x) = \int (\sqrt{\lambda} e^{-\lambda|x|} e^{-i\omega t})[x](\sqrt{\lambda} e^{-\lambda|x|} e^{i\omega t}) \, \mathrm{d}x = \lambda \int x e^{-2\lambda|x|} \, \mathrm{d}x = \boxed{0}$$
$$E(x^2) = \lambda \int x^2 e^{-2\lambda|x|} \, \mathrm{d}x = \boxed{\frac{1}{2\lambda^2}}$$

Standard deviation.

$$\sigma = \sqrt{E(x^2) - E^2(x)} = \boxed{\frac{1}{\lambda\sqrt{2}}}$$

Question 4.

Normalising with C.

$$
\begin{aligned}
1 &= \int_0^\infty [Ce^{-x}(1-e^{-x]})]^2 \,\mathrm{d}x = C^2 \int_0^\infty e^{-2x}(1-2e^{-x}+e^{-2x})\,\mathrm{d}x \\
&= C^2[-\frac{e^{-2x}}{2} + \frac{2e^{-3x}}{3} - \frac{e^{-4x}}{4}]_0^\infty = \frac{C^2}{12} \\
C &= \boxed{2\sqrt{3}}
\end{aligned}
$$

Most probable position.

$$
\begin{aligned}
P(x) &= 12e^{-2x}(1-2e^{-x}+e^{-2x}) \\
0 &= \frac{\mathrm{d}P}{\mathrm{d}x} = -24e^{-2x} + 72e^{-3x} - 48e^{-4x} \\
0 &= (e^x)^2 - 3e^x + 2 \\
e^x &= (3 \pm \sqrt{9-4*2})/2 = 1, 2 \\
P(x=0) &= 0 \\
P(x=\ln 2) &= 0.6931 \\
x_{max} &= \boxed{\ln 2}
\end{aligned}
$$

Expectation values. It is higher than the most probable value, which makes sense since the distribution is right-skew, so it is more likely to find it to the right of the mode value than the left of it.

$$
\begin{aligned}
E(x) &= \int_0^\infty xP(x)\,\mathrm{d}x = \int_0^\infty 12xe^{-2x}(1-2e^{-x}+e^{-2x})\,\mathrm{d}x \\
&= \left[12\left(-\frac{2x+1}{4}e^{-2x} + \frac{6x+2}{9}e^{-3x} - \frac{4x+1}{16}e^{-4x}\right)\right]_0^\infty = \boxed{\frac{13}{12}}
\end{aligned}
$$