Physics 112 - Homework 3

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Question 1.

Effective Boltzmann factor.

$$\begin{aligned}
P(\epsilon) &\propto \frac{P(E_{n+1})}{P(E_n)} \\
&\propto \frac{e^{-E_{n+1}/\tau}}{e^{-E_n/\tau}} \\
&\propto \frac{e^{(-n\epsilon-(\alpha-1)\epsilon)/\tau}}{e^{-n\epsilon/\tau}} \\
&\propto e^{-(\alpha-1)\epsilon/\tau} \\
&\propto \boxed{\exp\left\{-\frac{\epsilon}{\tau}(1-\alpha)\right\}}
\end{aligned}$$

Question 2.

Partition function.

$$Z = \boxed{\sum_{j=0}^{n} (2j+1) e^{-(j^2+j)\epsilon_0/\tau}}$$

Partition function in the limit $\tau \gg \epsilon_0$. Need to get the differential out.

$$\begin{aligned}
Z &= -\frac{\tau}{\epsilon_0} \sum_{j=0}^{n} \frac{\mathrm{d}}{\mathrm{d}j} e^{-(j^2+j)\epsilon_0/\tau} \\
&= -\frac{\tau}{\epsilon_0} \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}j} e^{-j(^2+j)\epsilon_0/\tau} dj \\
&= -\frac{\tau}{\epsilon_0} e^{-(j^2+j)\epsilon_0/\tau} |_0^{\infty} \\
&= -\frac{\tau}{\epsilon_0} [e^{-\infty} - e^{0}] \\
&= \boxed{\frac{\tau}{\epsilon_0}}
\end{aligned}$$

Partition function in the limit $\epsilon_0 \gg \tau$.

$$\begin{aligned}
Z &= (2(0)+1) e^{-(0^2+0)\epsilon_0/\tau} + (2(1)+1) e^{-(1^2+1)\epsilon_0/\tau} \\
&= \boxed{1 + 3e^{-2\epsilon_0/\tau}}
\end{aligned}$$

Energy in the limit $\tau \gg \epsilon_0$.

$$\begin{aligned}
U &= \tau^2 \frac{\partial \ln Z}{\partial \tau} \\
&= \tau^2 \frac{\partial}{\partial \tau} \frac{\tau}{\epsilon_0} \\
&= \boxed{\tau}
\end{aligned}$$

Energy in the limit $\epsilon_0 \gg \tau$.

$$\begin{aligned}
U &= \tau^2 \frac{\partial}{\partial \tau} \left(1 + 3e^{-2\epsilon_0/\tau}\right) \\
&= \tau^2 \frac{(6\epsilon_0/\tau^2) e^{-2\epsilon_0/\tau}}{(1 + 3e^{-2\epsilon_0/\tau})^2} \\
&= \boxed{\frac{6\epsilon_0 e^{-2\epsilon_0/\tau}}{(1 + 3e^{-2\epsilon_0/\tau})^2}}
\end{aligned}$$

Heat capacity in the limit $\tau \gg \epsilon_0$.

$$\begin{aligned}
C_V &= \left(\frac{\partial U}{\partial \tau}\right)_V \\
&= \frac{\partial}{\partial \tau}\tau \\
&= \boxed{1}
\end{aligned}$$

Heat capacity in the limit $\epsilon_0 \gg \tau$.

$$\begin{aligned}
C_V &= \frac{\partial}{\partial \tau}\frac{6\epsilon_0 e^{-2\epsilon_0/\tau}}{(1+3e^{-2\epsilon_0/\tau})^2} \\
&= \frac{\epsilon_0^2}{\tau^2}\frac{12(e^{-2\epsilon_0/\tau}-3e^{-4\epsilon_0/\tau})}{(1+3e^{-2\epsilon_0/\tau})^3} \\
&\approx \frac{\epsilon_0^2}{\tau^2}\frac{12e^{-2\epsilon_0/\tau}}{1+9e^{-2\epsilon_0/\tau}} \\
&\approx \frac{\epsilon_0^2}{\tau^2}\frac{12}{e^{2\epsilon_0/\tau}+9} \\
&\approx \frac{\epsilon_0^2}{\tau^2}\frac{12}{e^{2\epsilon_0/\tau}} \\
&\approx 12\frac{\epsilon_0^2}{\tau^2 e^{2\epsilon_0/\tau}} \\
&\approx \boxed{\frac{12e^{-2\epsilon_0/\tau}}{(\tau/\epsilon)^2}}
\end{aligned}$$

Question 3.

Every state n corresponds to an energy of $E_n = n\epsilon$ exactly up to N, and the system occupies one state at a time.

$$
\begin{aligned}
Z &= \sum_{n=0}^{N} e^{-E_i/\tau} \\
&= \sum_{n=0}^{N} e^{-n\epsilon/\tau} \\
&= \sum_{n=1}^{N} (e^{-\epsilon/\tau})^n \\
&= \frac{1-(e^{-\epsilon/\tau})^{(N+1)}}{1-e^{-\epsilon/\tau}} \\
&= \boxed{\frac{1-\exp\{-(N+1)\epsilon/tau\}}{1-\exp\{-\epsilon/\tau\}}}
\end{aligned}
$$

Expected value of open links in the limit $\epsilon \gg \tau$.

$$
\begin{aligned}
\bar{n} &= \sum_n nP(n) \\
&= \sum_n \frac{ne^{-n\epsilon/\tau}}{\frac{1-e^{-(N+1)\epsilon/tau}}{1-e^{-\epsilon/\tau}}} \\
&= \sum_n \frac{ne^{-n\epsilon/\tau}(1-e^{-\epsilon/\tau})}{1-e^{-(N+1)\epsilon/\tau}} \\
&\approx \sum_n \frac{n}{1-e^{-(N+1)\epsilon/\tau}} \\
&\approx \frac{N(N+1)}{2(1-e^{-(N+1)\epsilon/\tau})}
\end{aligned}
$$

Ok I don't know this one.

Question 4.

Proving partition function of sum of two systems is the product of partition function of the systems.

$$
\begin{aligned}
Z_{1+2} &= \sum_i \sum_j e^{-(E_i+E_j)/\tau} \\
&= \sum_i \sum_j e^{-E_i/\tau} e^{-E_j/\tau} \\
&= \sum_i e^{-E_i/\tau} \sum_j e^{-E_j/\tau} \\
&= \boxed{Z_1 * Z_2}
\end{aligned}
$$