UCLA Math151A Fall 2021
Lecture 15
20211029

Continued: Uniqueness of Cubic Splines
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Start Numerical Differentiation

We had this definition and this theorem…

Definition 13.1 (Cubic Spline Interpolant).

Given f defined on $[a,b]$, $\{x_j\}_{j=0}^{n} \in [a,b]$
$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
The **spline** is a function $S(x)$ that satisfies:

1. On each sub-interval $[x_j, x_{j+1}], j = 0, \ldots, n-1$, $S(x)$ is a cubic polynomal:

$$S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

2. $S(x)$ interpolates f at each x_j. I.e., $S(x_j) = f(x_j)$.
3. Conituity: $S \in C([a,b])$.
4. Differentiability: $S \in C^2([a,b])$.

Theorem 14.1. $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
$f(x)$ has a *unique* "natural" spline interpolant on $[a,b]$ for the points $\{x_j\}_{j=0}^{n}$.

$$S''(a) = S''(b) = 0.$$

In lecture 14, we utilized the definition to get many equations:

$$h_j = x_{j+1} - x_j$$

$$\begin{aligned}
&a_j = f(x_j), \quad j = 0, 1, 2, \ldots, n \\
&b_n = S'(x_n) \\
&c_n = S''(x_n)/2 \\
&a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \quad j = 0, 1, 2, \ldots n-1 \\
&b_j + 2c_j h_j + 3d_j h_j^2 = b_{j+1}, \quad j = 0, 1, 2, \ldots n-1 \\
&c_j + 3d_j h_j = c_{j+1}, \quad j = 0, 1, 2, \ldots n-1 \\
&c_0 = c_n = 0
\end{aligned}$$

The variables for these equations include:

$$a_j, b_j, c_j, d_j, \quad j = 0, 1, 2, 3, \ldots, n-1$$

$$S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

$$a_n, b_n, c_n$$

We defined them (introduced them into the system)

$$a_j = f(x_j), \quad j = 0, 1, 2, \ldots, n \qquad\qquad h_j = x_{j+1} - x_j$$
$$b_n = S'(x_n)$$
$$c_n = S''(x_n)/2$$
$$a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$
$$b_j + 2c_j h_j + 3d_j h_j^2 = b_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$
$$c_j + 3d_j h_j = c_{j+1}, \quad j = 0, 1, 2, \ldots n-1$$
$$c_0 = c_n = 0$$

These tell us what a is, and allow us to express d and b using c.

We did many manipulations in lec14, in the end we got:

for $j = 0, \ldots, n-2,$

$$\frac{1}{h_{j+1}}(a_{j+2} - a_{j+1}) - \frac{h_{j+1}}{3}(2c_{j+1} + c_{j+2}) = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) + h_j(c_j + c_{j+1})$$

$$\boxed{\text{for } j = 1, 2, \ldots, n-1, \quad h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})}$$

n-1 equations, but we have n+1 c variables.

2+!

for $j = 1, 2, \ldots, n-1$,

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$

$$c_0 = c_n = 0$$

rewrite this into a matrix equation $Mc = b,$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & \ldots & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_3 \\ \ldots \\ c_{n-1} \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{pmatrix}$$

M is tridiagonal and strictly diagonal dominant.

$$|M_{ii}| > \sum_{j \neq i} |M_{ij}|$$

Lemma 15.1. If a square $n \times n$ matrix M satisfies $|M_{ii}| > \sum_{j \neq i} |M_{ij}|$ then M is invertible.

Proof

Suppose M is non-invertible.

$\Rightarrow$ not full rank $\Rightarrow$ null space is at least dimension 1.

$\Rightarrow$ some non-zero vector v such that $Mv = 0$.

let's assume $v_i > 0$ has the largest magnitude in v.

This can always be chosen, because otherwise we could just use −v instead as our v. The i'th row of $Mv = 0$ is then

$$\sum_j M_{ij} v_j = 0 \Leftrightarrow M_{ii} v_i = -\sum_{j \neq i} M_{ij} v_j \Leftrightarrow M_{ii} = -\sum_{j \neq i} M_{ij} \frac{v_j}{v_i}$$

$$\Rightarrow |M_{ii}| \leq \sum_{j \neq i} |M_{ij} \frac{v_j}{v_i}| \leq \sum_{j \neq i} |M_{ij}|.$$

Contradiction!

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 \\
h_0 & 2(h_0+h_1) & h_1 & 0 & \dots & 0 \\
0 & h_1 & 2(h_1+h_2) & h_2 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \\
0 & 0 & \dots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\
0 & 0 & \dots & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_0 \\ c_1 \\ c_3 \\ \dots \\ c_{n-1} \\ c_n
\end{pmatrix}
=
\begin{pmatrix}
0 \\
\frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\
\vdots \\
\frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\
0
\end{pmatrix}
$$

invertible

$\Rightarrow$ The process is deterministic and gives a unique solution to c.

Theorem proved.

NEW TOPIC in the COURSE

Numerical Differentiation

Goal:
- find approximations to derivatives of f(x).
- Estimate the error.

Needed for solving ODEs and PDEs (partially in 151B)

ODE

$$y' = F(y(t), t)$$
$$y(0) = y_0.$$

PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Also needed for Stochastic Differential Equations, important in mathematical finance, thermal physics, statistic physics.

Numerical Differentiation: First Order

Recall the definition of derivative:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if h is small, then

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

Recall that this idea was used in the Secant Method for root finding.

Let's make this idea rigorous.

By Taylor's Theorem, if $f \in C^2([a,b])$, and $x_0, x_1 \in [a,b]$, then

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + f''(\xi)\frac{(x_1 - x_0)^2}{2}.$$

Let $x_1 = x_0 + h$, then this becomes

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(\xi)h^2}{2}$$

Forward Difference Formula

$$\Rightarrow \boxed{\frac{f(x_0 + h) - f(x_0)}{h}} = f'(x_0) + h\frac{f''(\xi)}{2}$$

if we used x_0 and $x_0 - h$ instead:

$$\frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0) + \frac{h}{2}f''(\xi).$$

Backward Difference Formula

★ the error is $\frac{h}{2}|f''(\xi)| \le \frac{h}{2}M$ where $M = \max_{a \le x \le b} |f''(x)|$.

$\boxed{\text{the error is of } O(h).}$

How to get $O(h^2)$? Use higher order approximations!

Numerical Differentiation: Second Order

Suppsoe $f \in C^3[a,b]$, $x_0, x_1 \in [a,b]$, then

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + f''(x_0)\frac{(x_1 - x_0)^2}{2} + f'''(\xi)\frac{(x - x_0)^3}{3!}$$

Let $x_1 = x_0 + h$, then

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(\xi_1)\frac{h^3}{3!}$$

Let $x_1 = x_0 - h$, then

$$f(x_0 - h) = f(x_0) - f'(x_0)h + f''(x_0)\frac{h^2}{2} - f'''(\xi_2)\frac{h^3}{3!}$$

$$\Longrightarrow\ f(x_0 + h) - f(x_0 - h) = 2f'(x_0)h + (f'''(\xi_1) + f'''(\xi_2))\frac{h^3}{3!}$$

$$\Rightarrow\quad \frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + (f'''(\xi_1) + f'''(\xi_2))\frac{h^2}{12}$$

Centered Difference Formula