UCLA Math151A Fall 2021
Lecture 1
20210924

See Canvas/syllabus for course info and policies

Lecture 1. Calculus Review

Optional reading: textbook 1.1

This lecture talks about brief calculus review.

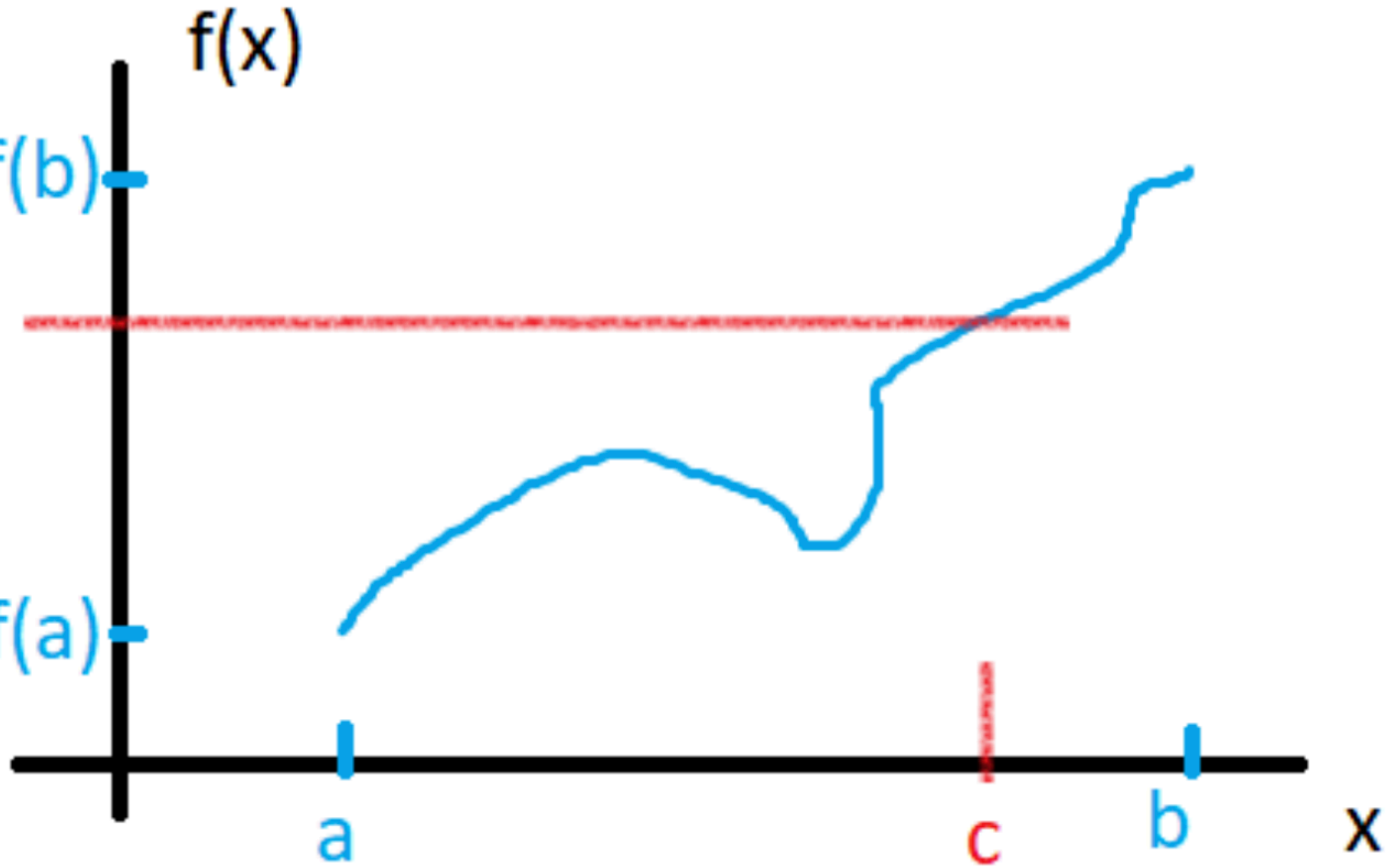
- I.V.T
- Taylor's theorem :fundamental threoreom of numerical analysys. Nice function (continuous, smooth) are well approxiamted by polinomials

Intermediate Value Theorem

Essense of continuous functions. Statemnt about continusou functions: continuous functions let us take advantage of its trend to solve problems efficiently (compared to discontinuous functions/combinatorics)

Recall $C([a,b])$ is a set meaning continuous functions on the closed inteval a,b. Intuitively, a continuous function is a functinon you can draw with a pen without picking up the pen.

Theorem 1.1 (I.V.T.)**.** Let $f \in C([a,b])$ Let $k \in \mathbb{R}$ s.t. k is strictly between $f(a)$ and $f(b)$. Then the theorem says, there $\exists$ some $c \in (a,b)$ (note the now open interval, not at the end points) s.t. $f(c) = k$. $\square$



Note that the theorem does not nessarily $f(b) > f(a)$. (can draw an example with $f(a) > f(b)$)

Example 1.1. Let $f(x) = 4x^2 - e^x$, claim: $\exists \quad x^*$ s.t. $f(x^*) = 0$.

x^* is called the root to the nonlinear equation. Nonlinear because the squared and the exponential are both nonlinear functions.

This is a common situation in applied science and engineering, you want to find the rooth to some nonlinear equations. The I.V.T can guarentee the existance of root provided you know the function is continuous.

First, $f(0) = 0 - e^0 = -1$.

Next, $f(1) = 4 - e = 4 - 2.7 \cdots > 0$

Therefore by I.V.T., $\exists x^* \in (0, 1)$ s.t. $f(x^*) = 0$. $\square$

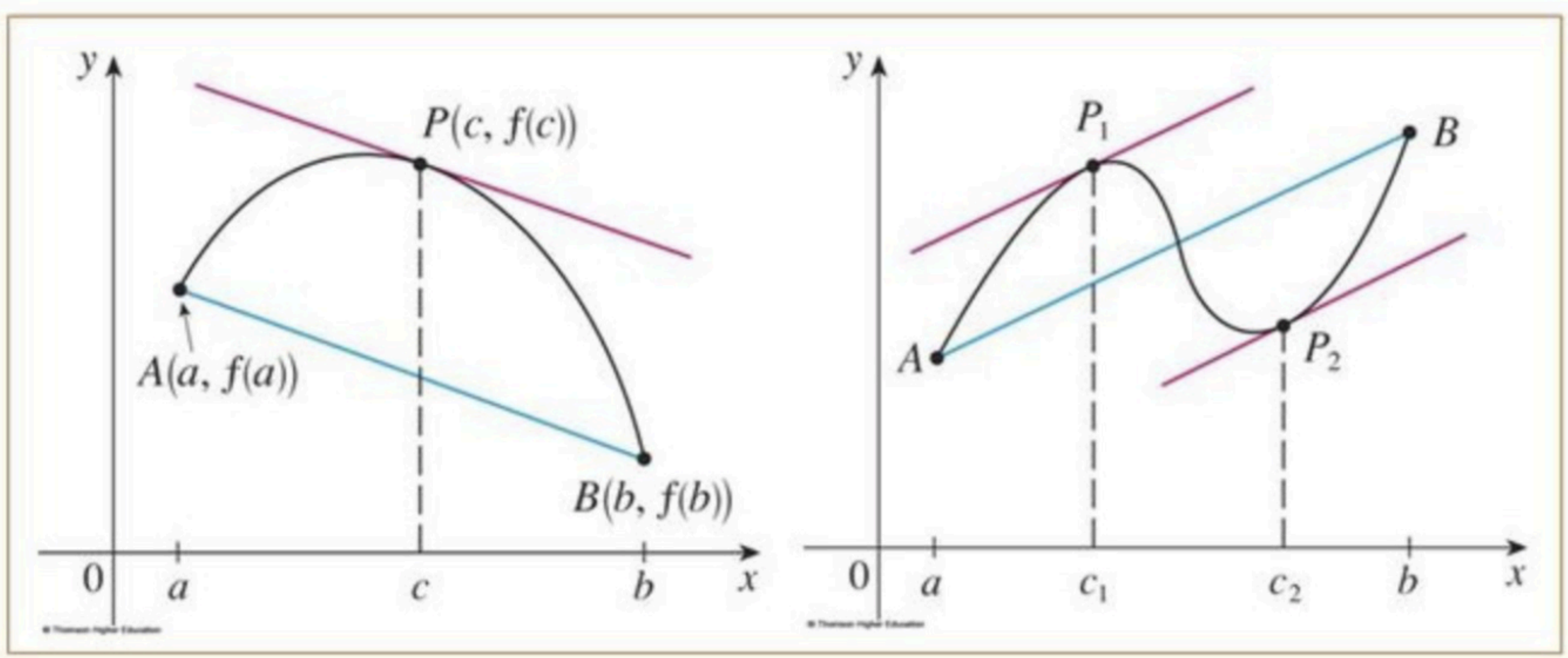
Remark 1.1. The proof of I.V.T. is covered in the Analysis course. Not part of this course. It's not difficult, if interested should take a look. This course needs to know how you can apply the theorem. $\square$

Mean Value Theorem

Theorem 1.2. If $f \in C[a,b]$ and f is differentiable on (a,b), then $c \in (a,b)$ exists with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- $f'(c)$ is the slope of the tangent line at $(c, f(c))$.
  - So, the Mean Value Theorem—in the form given by Equation 1—states that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB.



Taylor's Theorem

It's a statement of functions $f \in C^n([a,b])$, this means f is n times continously differentiable.

Note $C^0 = C$.

Intuition: If f is smooth, then **locally** it looks like a polynomial.

Theorem 1.3 (Taylor's theorem)**.** Let $f \in C^n([a,b])$, let $x_0 \in [a,b]$, and let $f^{(n+1)}$ exists on (a,b). Then $\forall x \in [a,b]$, $\exists$ some $\xi(x) \in \mathbb{R}$ s.t. $x_0 < \xi < x$ or $x < \xi < x_0$, and

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2! + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

(Taylor's polynomial)

$$R_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)^{n+1} \qquad \text{(The remainder term)}$$

Example 1.2. Let $f(x) = \cos(x)$ and $x_0 = 0$, then

$$\begin{aligned} f(x) = \cos(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3 \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin(\xi(x)) \end{aligned}$$

$\square$

Remark 1.3. Mean Value Theorem is Taylor's theorem with $n = 0$. $\square$

Extra material: $f \in C^1$ is different from saying $f'(x)$ exists

A must-remember differentiable function with discontinous derivative:

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It's easy to see that

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} h \sin(\frac{1}{h})$$

Since $-1 \leq \sin(\frac{1}{h}) \leq 1$, we know

$$-h \leq h \sin(\frac{1}{h}) \leq h.$$

Thus as $h \to 0$ it goes to 0. Thus $f'(0) = 0$, it exists! Thus we have

$$f'(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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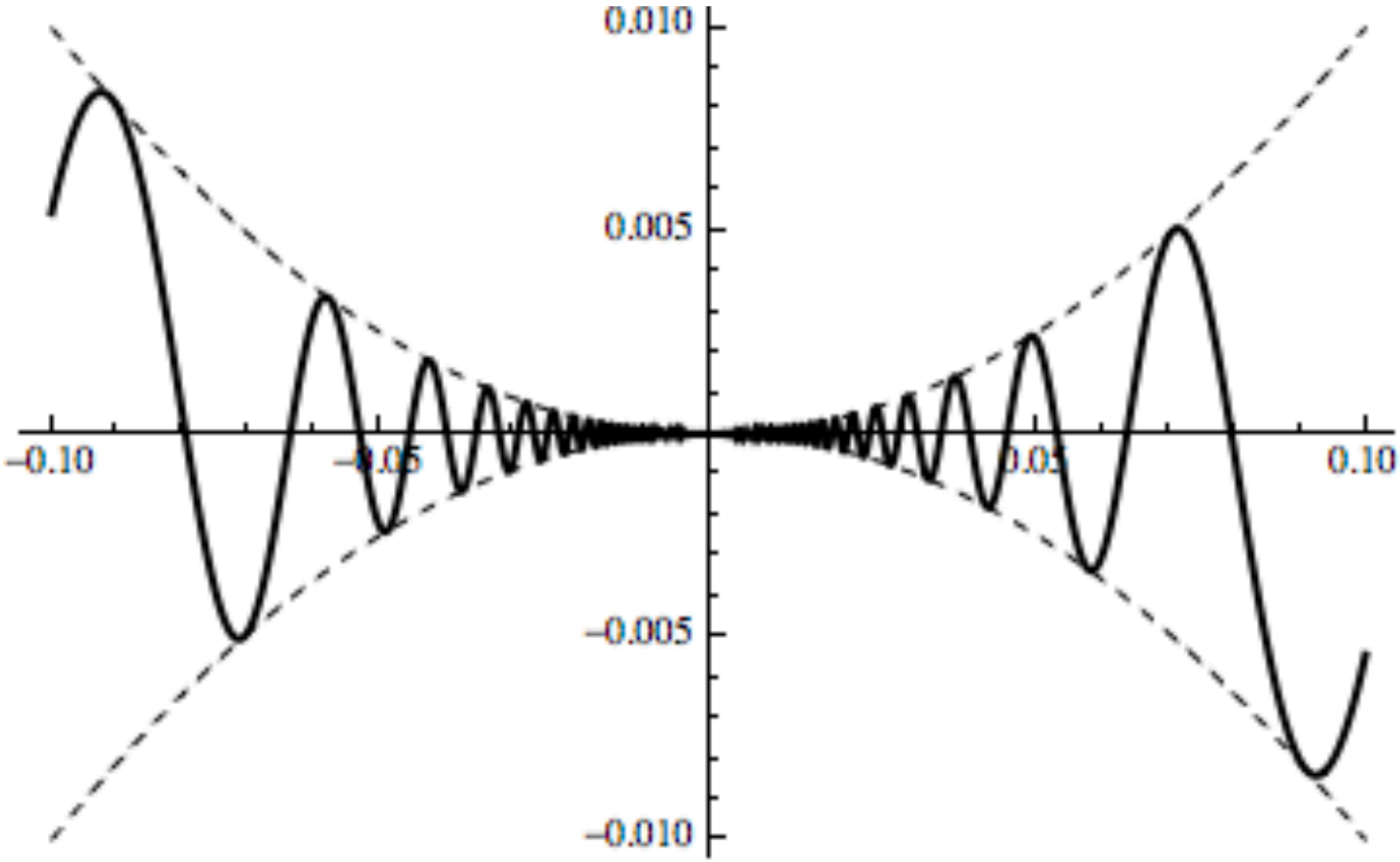
Is it continuous at 0?

Here we show that $f'(x)$ for $x \neq 0$ doesn't have a limit as x goes to 0: because if you look at the sequence $\frac{1}{2k\pi}$, $f' \to -1$, but sequence $\frac{1}{(2k+1)\pi}$ lets $f' \to 1$.

In other words, $\lim_{x\to 0} f'(x)$ does not exist. Thus $f'(x)$ is not continous.



0.010
0.005
−0.005
−0.010
−0.10
−0.05
0.05
0.10

Here is what $f'(x)$ looks like (crazy!):