CS 146 - Homework 7

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June 2, 2021

Question 1.

Base case: The problem gives us the base case for dimension M=1. S is the covariance matrix with one entry which is the variance of the feature x. The single eigenvalue of this matrix is simply the variance of the feature x as it satisfies $Sx = \lambda x$.

Induction step: Suppose we have that the vectors $u_1, u_2, ..., u_M$ are the solutions to the maximisation problem in M-space and are the unit length orthogonal eigenvectors corresponding to the largest M eigenvalues of the MxM matrix S_M. We wish to find the next basis vector u_{M+1}.

Constraint 1: The new vector must be orthogonal to the previous vectors.

$$0 = \sum_{i=1}^{M} u_i^T u_{M+1}$$
$$0 = \sum_{i=1}^{M} \alpha_i u_i^T u_{M+1}$$

Constraint 2: The new vector is of unit length.

$$0 = u_{M+1}^T u_{M+1} - 1$$
$$0 = \beta(u_{M+1}^T u_{M+1} - 1)$$

Define the Lagrangian that we want to maximise, which is the variance of all the data projected onto the M+1 axis plus our multiplier quantities.

$$L = \frac{1}{M} \sum_n (x_n \cdot u_{M+1})^2 - \sum_i \alpha_i u_i^T u_{M+1} - \beta(u_{M+1}^T u_{M+1} - 1)$$
$$L = \frac{1}{M} (x_n u_{M+1})^T (x_n u_{M+1}) - \sum_i \alpha_i u_i^T u_{M+1} - \beta(u_{M+1}^T u_{M+1} - 1)$$
$$L = u_{M+1}^T S u_{M+1} - \sum_i \alpha_i u_i^T u_{M+1} - \beta(u_{M+1}^T u_{M+1} - 1)$$

Now take the derivative with respect to u_{M+1}, α_i, β.

$$\frac{\mathrm{d}L}{\mathrm{d}u_{M+1}} = 2Su_{M+1} - 2\sum_i \alpha_i I u_{M+1} - 2\beta I u_{M+1}$$
$$\frac{\mathrm{d}L}{\mathrm{d}u_{M+1}} = 2(S - (\sum_i \alpha_i + \beta)I)u_{M+1}$$
$$\frac{\mathrm{d}L}{\mathrm{d}\alpha_i} = -\sum_i u_i^T u_{M+1}$$
$$\frac{\mathrm{d}L}{\mathrm{d}\beta} = 1 - u_{M+1}^T u_{M+1}$$

Set all derivatives to zero and get some equations.

$$
\begin{aligned}
0 &= 1 - u_{M+1}^T u_{M+1} \\
1 &= u_{M+1}^T u_{M+1} && u_{M+1} \text{ is unit} \\
0 &= \sum_i u_i^T u_{M+1} && u_{M+1} \text{ is orthogonal} \\
0 &= 2(S - (\sum_i \alpha_i + \beta)I)u_{M+1} \\
Su_{M+1} &= (\sum_i \alpha_i + \beta)Iu_{M+1} \\
Su_{M+1} &= (\sum_i \alpha_i + \beta)u_{M+1} \\
Su_{M+1} &= \lambda_{M+1} u_{M+1}
\end{aligned}
$$

We get that u_{M+1} is an eigenvector of the covariance matrix S. But what's its eigenvalue λ_{M+1}? Now we maximise the variance, left multiply both sides.

$$
\begin{aligned}
\max_{u_{M+1}} u_{M+1}^T S u_{M+1} &= u_{M+1}^T \lambda u_{M+1} \\
\max_{u_{M+1}} u_{M+1}^T S u_{M+1} &= \lambda_{M+1} u_{M+1}^T u_{M+1} \\
\max_{u_{M+1}} u_{M+1}^T S u_{M+1} &= \lambda_{M+1}
\end{aligned}
$$

The maximal u_{M+1}, which we already know is an eigenvector with eigenvalue λ, is achieved when its eigenvalue λ is maximised.

Question 2.

Center data to get data matrix.

$$\begin{aligned}
\mu_1 &= 0 \\
\mu_2 &= 0 \\
\mu_3 &= 0 \\
X &= \begin{pmatrix} 2 & 2 & 0 \\ 0 & -2 & 2 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned}$$

Get covariance matrix.

$$\begin{aligned}
S &= \frac{1}{N} X^T X \\
S &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}
\end{aligned}$$

Find largest two eigenvalues from the characteristic equation of S and solve for the largest two eigenvalues.

$$\begin{aligned}
\det\{S\} &= \| \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} \| \\
0 &= -\lambda^3 + 6\lambda^2 - 10\lambda + 4 \\
\lambda_1 &= 2 + \sqrt{2} \\
\lambda_2 &= 2 \\
\lambda_3 &= 2 - \sqrt{2}
\end{aligned}$$

First principal component.

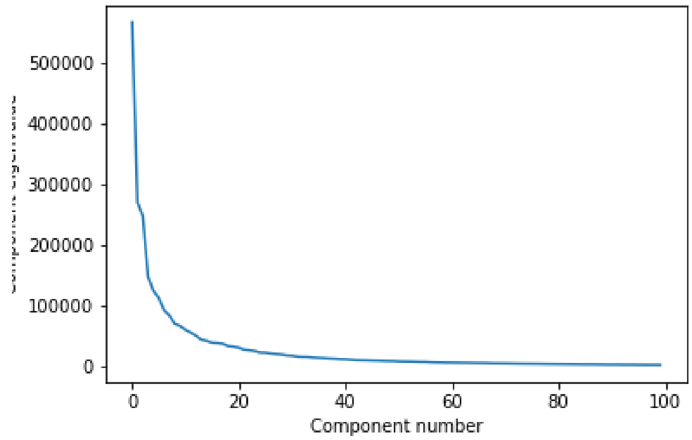
$$\begin{aligned}
Sw_1 &= \lambda_1 w_1 \\
(S - \lambda_1 I) w_1 &= 0 \\
0 &= \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \\
0 &= b - \sqrt{2}a \\
&= a - \sqrt{2}b - c \\
&= -b - \sqrt{2}c \\
(a, b, c) &= \boxed{(1, \sqrt{2}, -1)}
\end{aligned}$$

Second principal component.

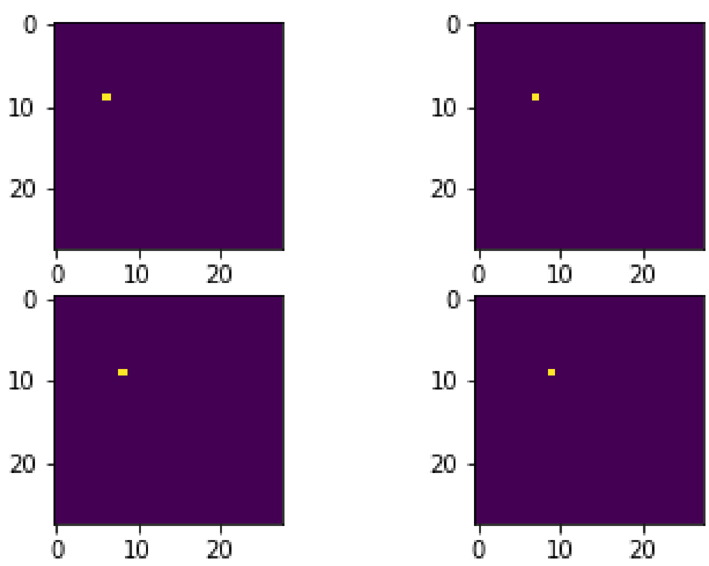
$$
\begin{aligned}
Sw_2 &= \lambda_2 w_2 \\
(S - \lambda_2 I)w_2 &= 0 \\
0 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \\
0 &= b \\
&= a - c \\
&= -b \\
(a, b, c) &= \boxed{(1, 0, -1)}
\end{aligned}
$$

Question 3.

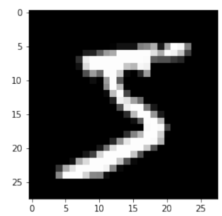
Plot of the 100 largest eigenvalues.



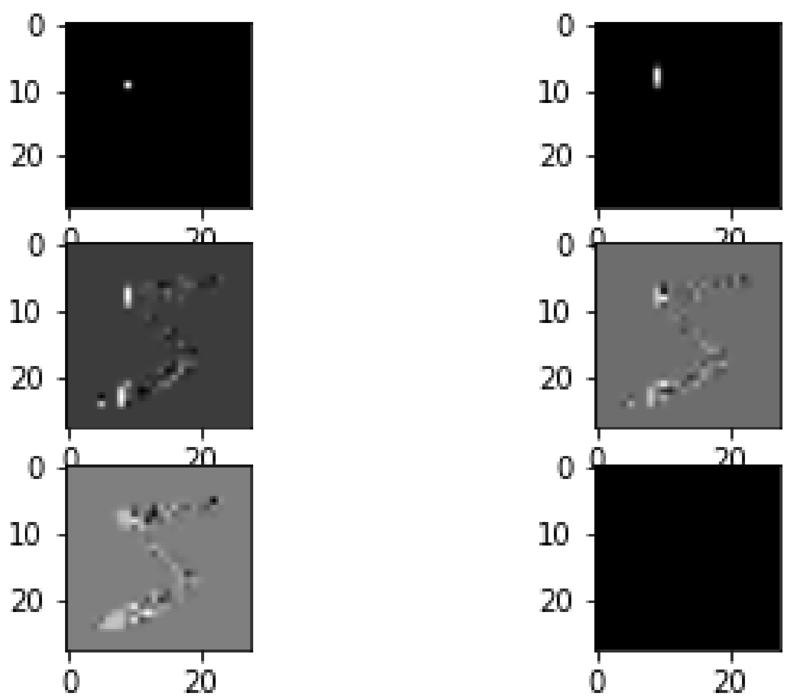
Plots of the four eigenvectors with highest eigenvalues. They each have only one nonzero value, corresponding to the pixel they represent.



Original image.



Plots of the compressed images. From left-right and top-down we have M = 1, 10, 50, 250, 784, and 0 dimensions. The quality increases as M increases. The basis vectors don't perfectly recover the original image but we get the clear shape of the number we are recovering. When M is 0, there are no eigen-pixels there to represent any of the image, so it's completely blank.



Question 4.

Probability that ball 1 is never picked when we pick N times from a pile of N balls without replacement is zero. At some point you run out of balls so you have to have picked up ball 1.

Probability that ball 1 is never picked when we pick with replacement is $\left(\frac{N-1}{N}\right)^N$, which is definitely higher than zero.

For N = 1000 and limit to infinity.

$$
\begin{aligned}
\left(\frac{1000-1}{1000}\right)^{1000} &= (0.999)^{1000} \\
&= 0.36769542477 \\
1/e &= 0.36787944117 \\
A &= \lim_{N\to\infty} \left(\frac{N-1}{N}\right)^N \\
\ln A &= \lim_{N\to\infty} \ln \left(\frac{N-1}{N}\right)^N \\
\ln A &= \lim_{N\to\infty} \frac{\ln\left(\frac{N-1}{N}\right)}{1/N} \\
\ln A &= \lim_{N\to\infty} \frac{1/N(N-1)}{-1/N^2} \\
\ln A &= \lim_{N\to\infty} -\frac{N^2}{N^2-N} \\
\ln A &= \lim_{N\to\infty} -\frac{2N}{2N} \\
\ln A &= -1 \\
A &= e^{-1} \\
A &= \frac{1}{e}
\end{aligned}
$$

Question 5.

Find π_k.

$$
\begin{aligned}
L &= \sum_n \ln \sum_j \pi_j P(x_n|\mu_j, \Sigma_j) + \lambda(1 - \sum_j \pi_j) \\
\frac{\mathrm{d}L}{\mathrm{d}\pi_k} &= \sum_n \frac{P(x_n|\mu_k, \Sigma_k)}{\sum_j \pi_j P(x_n|\mu_j, \Sigma_j)} - \lambda \\
\frac{\mathrm{d}L}{\mathrm{d}\lambda} &= \sum_j \pi_j - 1 \\
\pi_k \lambda &= \sum_n \frac{\pi_k P(x_n|\mu_k, \Sigma_k)}{\sum_j \pi_j P(x_n|\mu_j, \Sigma_j)} \\
\pi_k &= \frac{1}{\lambda} \sum_n \frac{\pi_k P(x_n|\mu_k, \Sigma_k)}{\sum_j \pi_j P(x_n|\mu_j, \Sigma_j)} \\
1 &= \sum_j \pi_j \\
1 &= \sum_j \frac{1}{\lambda} \sum_n \frac{\pi_k P(x_n|\mu_k, \Sigma_k)}{\sum_j \pi_j P(x_n|\mu_j, \Sigma_j)} \\
\frac{\lambda}{K} &= \sum_n \frac{\pi_k P(x_n|\mu_k, \Sigma_k)}{\sum_j \pi_j P(x_n|\mu_j, \Sigma_j)}
\end{aligned}
$$

???? don't know how to solve this one

Question 6.

This dataset is not linear separable and we could not use a single layer to perfectly classify this set.