10. Monotone Sequences and Cauchy sequences

Even though we have defined the notion of convergence and developed various limit laws, we have not utilized the completeness of $\mathbb{R}$ in a critical way. As such, all those developments also apply to $\mathbb{Q}$ or other ordered fields with due modifications. So, what makes $\mathbb{R}$ special in terms of limits?

We will answer this question by establishing a result that critically depends on the completeness of $\mathbb{R}$. Also, the result will turn out to be practical as well, allowing us to study the convergence of certain sequences without knowing the limit in advance.

10.1. Limits of monotone sequences

Definition 10.1. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.

- **(a)** (s_n) is said to be **increasing** (or **nondecreasing**) if $s_n \leq s_{n+1}$ holds for all n.
- **(b)** (s_n) is said to be **decreasing** (or **nonincreasing**) if $s_n \geq s_{n+1}$ holds for all n.
- **(c)** (s_n) is said to be **monotone** if it is either increasing or decreasing.

The next theorem is one of the main results of this section.

Theorem 10.2.[1] Every bounded monotone sequence in $\mathbb{R}$ converges. More precisely,

- **(a)** If (s_n) is a bounded increasing sequence in $\mathbb{R}$, then $\lim s_n = \sup\{s_n : n \in \mathbb{N}\}$.
- **(b)** If (s_n) is a bounded decreasing sequence in $\mathbb{R}$, then $\lim s_n = \inf\{s_n : n \in \mathbb{N}\}$.

[1] This theorem is sometimes called the monotone convergence theorem, although this name is more often saved for a more advance theorem in measure theory.

- Note that the completeness axiom plays a key role in proving this theorem, and in fact, it can be proved that this theorem is equivalent to the completeness axiom.

Example 10.1. Consider the sequence $(s_n)_{n\in\mathbb{N}}$ defined recursively by

$$s_1 = 10 \qquad \text{and} \qquad s_{n+1} = \frac{1}{2}\left(s_n + \frac{7}{s_n}\right) \quad \text{for} \quad n \geq 2.$$

(a) Show that $\sqrt{7} \leq s_{n+1} \leq s_n \leq 10$ holds for all $n \geq 1$.

(b) Prove that $(s_n)_{n\in\mathbb{N}}$ converges and find the limit.

Theorem 10.3. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.

(a) If (s_n) is unbounded and increasing, then $\lim s_n = +\infty$.

(b) If (s_n) is unbounded and decreasing, then $\lim s_n = -\infty$.

- Consequently, the limit always exists in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ for monotone sequences.
- Convention: In this course, the phrase '$\lim s_n$ exists' will always mean that $\lim s_n$ exists in $\overline{\mathbb{R}}$ (that is, either (s_n) converges to a real number or it diverges to $\pm\infty$).

Example 10.2. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function, in the sense that

$$f(x) \le f(y) \quad \text{whenever} \quad x \le y.$$

Now for each $x_0 \in \mathbb{R}$, define the sequence $(x_n)_{n=0}^{\infty}$ recursively by the formula

$$x_{n+1} = f(x_n) \quad \text{for } n \ge 0.$$

Show that $\lim x_n$ always exists.

10.2. Limsups and liminfs

In this part, we introduce useful substitutes for the limits of sequences.

Definition 10.4. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.

(a) The **limit superior** of (s_n) is defined by

$$\limsup_{n\to\infty} s_n = \begin{cases} \lim\limits_{N\to\infty} \sup\{s_n : n \geq N\}, & \text{if } (s_n) \text{ is bounded above;} \\ +\infty, & \text{if } (s_n) \text{ is unbounded above.} \end{cases}$$

(b) The **limit inferior** of (s_n) is defined by

$$\liminf_{n\to\infty} s_n = \begin{cases} \lim\limits_{N\to\infty} \inf\{s_n : n \geq N\}, & \text{if } (s_n) \text{ is bounded below;} \\ -\infty, & \text{if } (s_n) \text{ is unbounded below.} \end{cases}$$

Here are some comments:

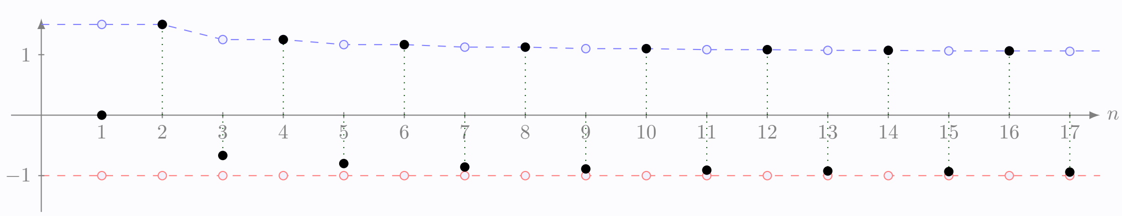
- Both $N \mapsto \sup\{s_n : n \geq N\}$ and $N \mapsto \inf\{s_n : n \geq N\}$ are monotone sequences in $\overline{\mathbb{R}}$. So we know that $\limsup s_n$ and $\liminf s_n$ always exist in $\overline{\mathbb{R}}$. Moreover, by Theorem 10.2, we get

$$\limsup_{n\to\infty} s_n = \inf_{N\geq 1}\left(\sup_{n\geq N} s_n\right) \qquad \text{and} \qquad \liminf_{n\to\infty} s_n = \sup_{N\geq 1}\left(\inf_{n\geq N} s_n\right)$$

- From the inequality $\inf\{s_n : n \geq N\} \leq \sup\{s_n : n \geq N\}$ we immediately obtain:

$$\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n.$$

Example 10.3. Consider the sequence $s_n = (-1)^n + \frac{1}{n}$ for $n \geq 1$. Plotting the points (n, s_n) on the coordinate plain gives



From this, it is not hard to deduce that

$$\sup\{s_n : n \geq N\} = 1 + \frac{1}{2\lceil N/2\rceil} \qquad \text{and} \qquad \inf\{s_n : n \geq N\} = -1$$

for each positive integer N. Therefore $\limsup s_n = 1$ and $\liminf s_n = -1$.

Theorem 10.5. Consider a sequence (s_n) of real numbers. Then the followings are equivalent:

(a) $\lim s_n$ exists in $\overline{\mathbb{R}}$.

(b) $\liminf s_n = \limsup s_n$.

Moreover, in this case, we have $\liminf s_n = \lim s_n = \limsup s_n$.