# UCLA Math151A Fall 2021
# Lecture 10
# 20211015

# Neville's Method, Divided Differences

Optional reading: book 3.2, 3.3

## Preliminaries for Neville's Method

- Suppose we have a Lagrangian polynomial from k data points. But now we obtain more information and we want to update P(x)'s approximation to some number x.

- **Neville's Method** lets us re-use our previous work to update the interpolant.

- It lets us generate polynomial approximations recursively.

**Example 11.1.**

Given $\{(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))\}$,

$$P(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \qquad (*)$$

But we can also build it recursively!

Let $P_0 = f(x_0), P_1 = f(x_1), P_2 = f(x_2)$ be 0th degree polynomials. Then define

$$P_{01}(x) := \frac{1}{x_1 - x_0}\left((x-x_0)P_1 - (x-x_1)P_0\right) = f(x_1)\frac{x-x_0}{x_1-x_0} + f(x_0)\frac{x-x_1}{x_0-x_1}.$$

This is just the L.P. formed from $x_0$ and $x_1$.

Similarly $P_{12}(x) := \dfrac{1}{x_2 - x_1}\left((x-x_1)P_2 - (x-x_2)P_1\right)$,

this will equal the L.P. formed from $x_1$ and $x_2$.

CLAIM: $P_{01}$ and $P_{12}$ can be combined to form $(*)$ using $x_0, x_1, x_2$.

$$P(x) = P_{012}(x) := \frac{1}{x_2 - x_0}\left((x-x_0)P_{12} - (x-x_2)P_{01}\right)$$

we'll skip the algebra.

$$P_0 = f(x_0), P_1 = f(x_1), P_2 = f(x_2)$$

$$P_{01}(x) := \frac{1}{x_1 - x_0}((x - x_0)P_1 - (x - x_1)P_0)$$

$$P_{12}(x) := \frac{1}{x_2 - x_1}((x - x_1)P_2 - (x - x_2)P_1),$$

$$P(x) = P_{012}(x) := \frac{1}{x_2 - x_0}((x - x_0)\ P_{12} - (x - x_2)P_{01})$$

To generalize these, let's introduce the formal theorems.

**Definition 11.1.**

Let $f$ be defined at points $\{x_i | 0 \le i \le n\}$ and let $m_1, m_2, \ldots, m_k \subseteq \{0, 1, 2, \ldots, n\}$ be distinct. Then $P_{m_1 m_2 \ldots m_k}(x)$ is the Lagrangian Polynomial formed by interpolating $f(x)$ at the points $\{x_{m_1}, x_{m_2}, \ldots, x_{m_k}\}$.

Can verify the convention in the previous example.

$$P_0 = f(x_0), P_1 = f(x_1), P_2 = f(x_2)$$

$$P_{01}(x) := \frac{1}{x_1 - x_0}\left((x - x_0)P_1 - (x - x_1)P_0\right)$$

$$P_{12}(x) := \frac{1}{x_2 - x_1}\left((x - x_1)P_2 - (x - x_2)P_1\right),$$

$$P(x) = P_{012}(x) := \frac{1}{x_2 - x_0}\left((x - x_0)\ P_{12} - (x - x_2)P_{01}\right)$$

**Theorem 11.1.**

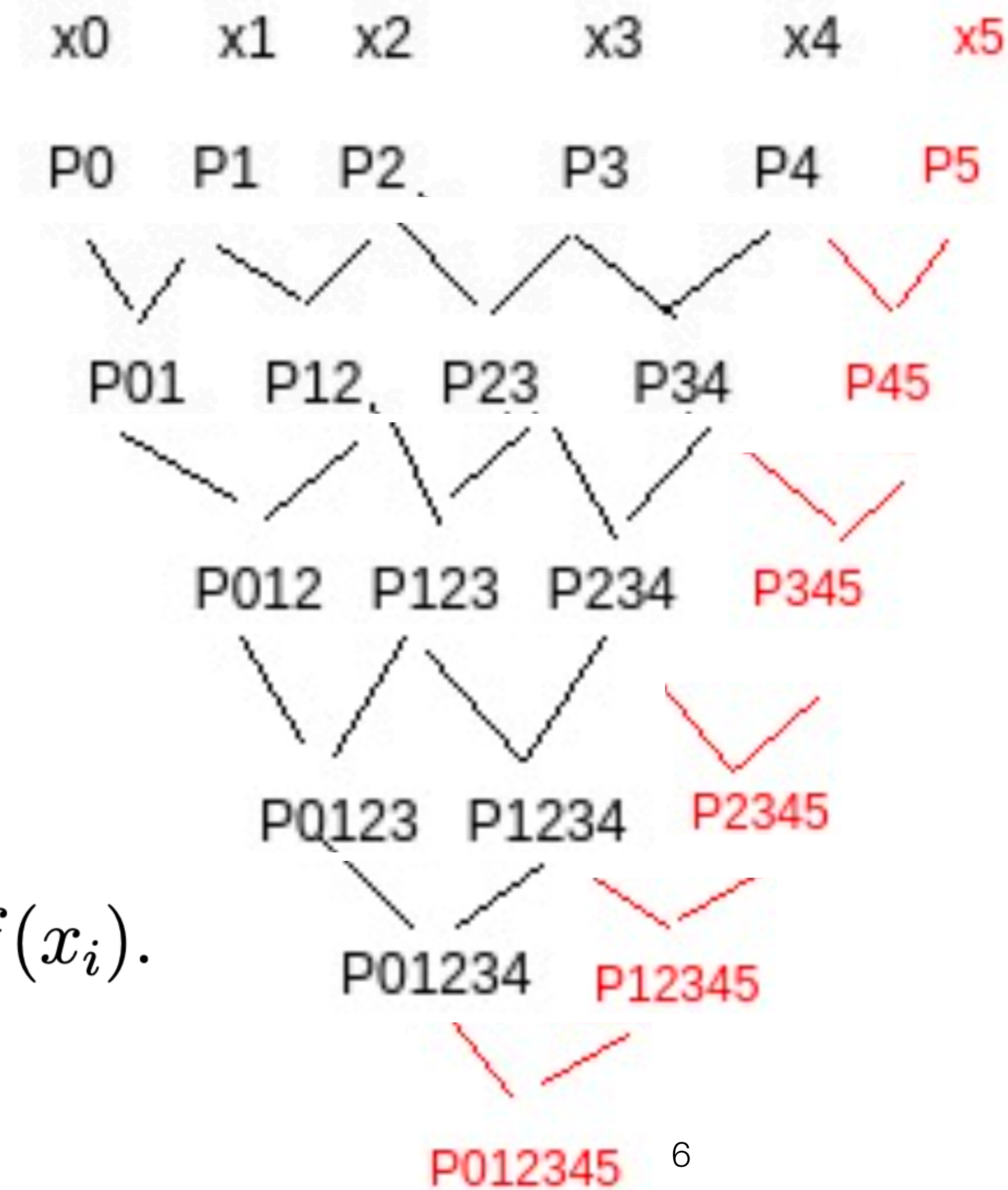
Let $f$ be defined at points $x_0, x_1, x_2, \ldots, x_k$ and let $x_i$ and $x_j$ be distinct points in this set. Then the L.P. that interpolates $f$ at all the $k+1$ points is

$$P(x) = \left[(x - x_j)P_{012\ldots(j-1)(j+1)\ldots k}(x) - (x - x_i)P_{012\ldots(i-1)(i+1)\ldots k}(x)\right]\frac{1}{x_i - x_j}$$

*Proof:* verify the interpolation property, the degree, and use the uniqueness of L.P.

# Neville's Method

**Example 11.2.** Let's be given $x_0, x_1, \ldots, x_4$, and $P_0, P_1, \ldots, P_4$ be the constant degree-0 polynomials.




Note that we just need to save the values of some intermediate values (rather than the actual intermediate polynomials).

$$P(x_i) = f(x_i).$$

$$P(x) = \left[(x - x_j)P_{012\ldots(j-1)(j+1)\ldots k}(x) - (x - x_i)P_{012\ldots(i-1)(i+1)\ldots k}(x)\right]\frac{1}{x_i - x_j}$$

**Neville's method** is useful when we want to successively generate higher degree polynomial approximations at a specific point.

$\boxed{\text{Example}}$

Values of various interpolating polynomials at x = 1.5

$$x_0 = 1.0,\ x_1 = 1.3,\ x_2 = 1.6,$$

| | | | | |
|---|---|---|---|---|
| 1.0 | 0.7651977 P0 | | | |
| | | 0.5233449 P01(1.5) | | |
| 1.3 | 0.6200860 P1 | | 0.5124715 P012(1.5) | |
| | | 0.5102968 P12(1.5) | | 0.51181 |
| 1.6 | 0.4554022 P2 | | $\boxed{0.5112857}$ P123(1.5) | |
| | | $\boxed{0.5132634}$ P23(1.5) | | |
| $\boxed{1.9}$ | $\boxed{0.2818186}$ P3 | | | |

$$x_3 = 1.9$$

# Divided Differences

**Neville's method** is useful when we want to successively generate higher degree polynomial approximations at a specific point.

**Divided difference method** is useful for successively generating higher degree polynomial expressions (as a function of x).

Let $\{x_0, x_1, \ldots, x_n\}$ be distinct and $P(x)$ is the L.P. of $f(x)$.

$$L_i(x) = \Pi_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j} \qquad P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

We know P(x) is unique (see homework), but it can be written in many different ways.

One of these ways is called **"Newton's Divided Differences".** it defines a function looking like:

$$P_n(x) := a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}),$$

$$P_n(x) := a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}),$$

Observation:

$$P_n(x_0) = a_0$$

$$P_n(x_1) = a_0 + a_1(x_1 - x_0)$$

$$\dots$$

$P_n(x_k)$ contains the first $k+1$ terms of $P_n(x)$.

**Theorem 11.2.** $P_n(x) = P(x)$ if $a_j$'s are chosen correctly.

For example, if we want $P_n(x_0) = P(x_0) = f(x_0)$, then $a_0 = f(x_0)$.

If we want $P_n(x_1) = P(x_1) = f(x_1)$, then

$$f(x_1) = P_n(x_1) = a_0 + a_1(x_1 - x_0) \longrightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

(this is a divided difference)

$$P_n(x) := a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}),$$

$$a_0 = f(x_0). \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

**Definition 11.2.** We can introduce notation:

$$f[x_i] = f(x_i) \quad \text{(0th divided differences)}$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \quad \text{(first divided differences)}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_{i+1}, x_i]}{x_{i+2} - x_i} \quad \text{(second divided differences)}$$

The kth divided differences is

$$\boxed{f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}}$$

$$P_n(x) := a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}),$$

$$a_0 = f(x_0). \qquad a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k}-x_i}$$

it turns out that $P_n(x) = P(x)$ can be achieved by choosing

$$a_k = f[x_0, x_1, x_2, \dots, x_k],$$

therefore, the the *Newton's Divided Difference* way of writing the L.P. is:

$$\boxed{\begin{aligned} P_n(x) = P(x) = &f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots \\ &+ f[x_0,x_1,\dots,x_n](x-x_0)(x-x_1)\dots(x-x_{n-1}). \end{aligned}}$$