UCLA Math151A
Fall 2021
Lecture 24
2021/11/19

Gaussian Elimination with Pivoting
Computational Complexity
Partial pivoting
Matrix Decomposition (starting)

$\boxed{\text{No failure}}$

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(1)*(-1/4)+(2)

(1)*(-1/4)+(3)

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 1 & 1 & 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & \frac{3}{4} & \frac{11}{4} & \frac{-1}{4} \end{pmatrix}$$

$$Ux = y$$

$$\begin{pmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} \\ 0 & 0 & \frac{13}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{4} \\ -\frac{1}{5} \end{pmatrix}$$

(2)*(-1/5)+(3)

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & \frac{13}{5} & -\frac{1}{5} \end{pmatrix}$$

$\boxed{\text{Failure}}$

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & \frac{1}{4} & 1 \\ 1 & 1 & 3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 4 & 1 & 1 \\ 0 & 0 & \frac{3}{4} \\ 0 & \frac{3}{4} & \frac{11}{4} \end{pmatrix}$$

In this case, we can just swap row 2 and row 3.

In general, swapping rows to avoid division by 0 is called pivoting.

stuck with zero diagonals

Theorem 24.1. Let $A \in \mathbb{R}^{n\times n}$, then

$\det(A) \neq 0 \quad \Leftrightarrow \quad$ Gaussian elimination with row interchanges can be performed on A without failure.

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & \frac{1}{4} & 1 \\ 1 & 1 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 & 1 \\ 0 & 0 & \frac{3}{4} \\ 0 & \frac{3}{4} & \frac{11}{4} \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 & 1 \\ 0 & \frac{3}{4} & \frac{11}{4} \\ 0 & 0 & \frac{3}{4} \end{pmatrix}$$

invertible

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & \frac{1}{4} & 1 \\ 1 & \frac{1}{4} & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 & 1 \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{11}{4} \end{pmatrix}$$

stuck with zero diagonals.
failure/singular

Gaussian Elimination with Pivoting.

INPUT: invertible matrix $A \in \mathbb{R}^{n \times n}$

for $i = 1, 2, \ldots, n-1$

let $p (i \leq p \leq n)$ be the smallest interger s.t. $a_{pi} \neq 0$

if $p \neq i$, perform E.R.O. $E_i \leftrightarrow E_p$

for $j = i+1, i+2, \ldots, n$

set $\lambda_{ji} = -a_{ji}/a_{ii}$

perform E.R.O. $E_j + \lambda_{ji} E_i \rightarrow E_j$

OUTPUT: Upper triangular matrix $U \in \mathbb{R}^{n \times n}$

Computational Complexity of G.E.

Cost of upper-triangulazation:

To transfform A to U, the answer is: $\frac{n^3}{3} + n^2 - \frac{n}{3}$ multiplications/divisions, $\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$ additions/subtractions.

What the take-away

(that you should remember) is the order $\frac{2}{3}n^3$ FLOPs for large n. Don't worry about remembering the lower order coefficients.

Cost of back-substitution:

Fact: Let U be upper traingular, then solving $Ux = y$ with back substitution requires $\approx n^2$ FLOPs.

Upper-triangularization: n^3

Back substitution: n^2

Takeaway: The cost to solve $Ax = b$ for x is dominated by the row reduction process as apposed to back substitution.

Floating Point Considerations

Everything we've done assumed exact arithmetic. What happens when numerical roundoff error is present?

Recall there are two types of red flags in floating point operations that can cause problems, something we want to avoid in computations, they amplify roundoff errors.

Red flags:

1. Division by small numbers.
2. Subtracting two numbers that are close.

In Gaussian elimination, we are potentially doing both.

for $i = 1, 2, \ldots, n-1$

let $p (i \leq p \leq n)$ be the smallest interger s.t. $a_{pi} \neq 0$

if $p \neq i$, perform E.R.O. $E_i \leftrightarrow E_p$

for $j = i+1, i+2, \ldots, n$

set $\lambda_{ji} = -a_{ji}/a_{ii}$

perform E.R.O. $E_j + \lambda_{ji} E_i \rightarrow E_j$

How to mitigate?

INPUT: invertible matrix $A \in \mathbb{R}^{n\times n}$

for $i = 1, 2, \ldots, n-1$ modify this step

$\boxed{\text{let } p(i \le p \le n) \text{ be the smallest interger s.t. } a_{pi} \neq 0}$

if $p \neq i$, perform E.R.O. $E_i \leftrightarrow E_p$

for $j = i+1, i+2, \ldots, n$

set $\lambda_{ji} = -a_{ji}/a_{ii}$

perform E.R.O. $E_j + \lambda_{ji} E_i \to E_j$

OUTPUT: Upper triangular matrix $U \in \mathbb{R}^{n\times n}$

INPUT: invertible matrix $A \in \mathbb{R}^{n \times n}$

for $i = 1, 2, \ldots, n-1$

$\boxed{\text{let } p (i \leq p \leq n) \text{ be the smallest interger s.t. } a_{pi} \neq 0}$

if $p \neq i$, perform E.R.O. $E_i \leftrightarrow E_p$

modify this step into

$\boxed{\text{Search for the maximum } |a_{pi}|.}$

for $j = i+1, i+2, \ldots, n$

set $\lambda_{ji} = -a_{ji}/a_{ii}$

perform E.R.O. $E_j + \lambda_{ji} E_i \rightarrow E_j$

OUTPUT: Upper triangular matrix $U \in \mathbb{R}^{n \times n}$

will be largest possible.

$$\left[\begin{array}{cccc|c} 0.02 & 0.01 & 0 & 0 & 0.02 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0.02 & 0.01 & 0 & 0 & 0.02 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0.02 & 0.01 & 0 & 0 & 0.02 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\leftarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0.02 & 0.01 & 0 & 0 & 0.02 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0.02 & 0.01 & 0 & 0 & 0.02 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\uparrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\uparrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0.04 & 0.03 & 0.12 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 0 & 1 \\
0 & 1 & 2 & 1 & 4 \\
0 & 0 & 0.04 & 0.03 & 0.12 \\
0 & 0 & 100 & 200 & 800
\end{array}\right]
\longleftarrow
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 0 & 1 \\
0 & 1 & 2 & 1 & 4 \\
0 & 0 & 0.04 & 0.03 & 0.12 \\
0 & 0 & 100 & 200 & 800
\end{array}\right]
$$

$$
\downarrow
$$

$$
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 0 & 1 \\
0 & 1 & 2 & 1 & 4 \\
0 & 0 & 100 & 200 & 800 \\
0 & 0 & 0.04 & 0.03 & 0.12
\end{array}\right]
\longrightarrow
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 0 & 1 \\
0 & 1 & 2 & 1 & 4 \\
0 & 0 & 100 & 200 & 800 \\
0 & 0 & 0 & -0.05 & -0.2
\end{array}\right]
$$

Remark 25.1. This strategy is called partial pivoting. The exists other strategies such as scaled partial pivoting and complete pivoting. $\square$

"normalize" each row before comparing

do column swaps too for pivoting

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 99 & -1 & 7 & 1 \\ 0 & 0 & 33 & 2 & 15 & 2 \\ 0 & -4 & 5 & 6 & 1 & 3 \\ 0 & 6 & 25 & 88 & 2 & 4 \\ 0 & -8 & 5 & 0 & 10 & 5 \end{array}\right]$$

$$\boxed{\text{Matrix Decompositions}}$$

Eigen Value Decomposition

Until specified later, we assume now we are using exact arithmetic.

Recall from linear algebra, $A \in \mathbb{R}^{n\times n}$ is called normal if it commutes with its transpose:

$$AA^T = A^T A.$$

(Remember that A^T is a matrix made with entries a_{ji}.)

Theorem If $A \in \mathbb{R}^{n \times n}$ is normal, then

$$A = UDU^T, \qquad (*)$$

where $D = \mathrm{diag}(\lambda_i)$ is diagonal and U is orthogonal ($U^{-1} = U^T$). $\square$

$(*)$ is called a matrix factorization (or decomposition): because it decomposes a matrix into three pieces.

LU Decomposition

Recall that for Gaussian Elimination, row reduction can be represented by E.R.O.s that can be represented as multiplications of matricies. Thus G.E. is equivalent to doing:

$$P_{n-1}P_{n-2}\ldots P_3P_2P_1A = U,$$

where P_j's are called E.R.O. matrices.

$$P_{n-1}P_{n-2}\dots P_3P_2P_1A = U,$$

Here are some facts about the E.R.O. matrices:

- Fact 1: each P_j is invertible, because you can always undo E.R.O.s.
- If no row swapping is performed, then each P_j is lower triangular.
- The inverse of a lower triangular matrix is lower triangular.
- If L_1 and L_2 are both lower triangular, then their product is also lower triangular.

Therefore $P_{n-1}P_{n-2}\dots P_3P_2P_1$ is lower triangular.

$L^{-1} = P_{n-1}P_{n-2}\dots P_3P_2P_1,$ so $A = LU$.