hw 7

Problem 1

a Hermitian conjugates $\hat{x}^\dagger, \hat{i}^\dagger, \left(\frac{\hat{d}}{dx}\right)^\dagger$.

x Hermitian so $\boxed{x^\dagger = x}$

$$\langle f | i g\rangle = \int f^* (ig)\, dx$$
$$= \int f^* (-i)^* g\, dx$$
$$= \int (-if)^* g\, dx$$
$$= \langle -if | g\rangle, \quad \boxed{i^\dagger = -i}$$

$$\langle f | \tfrac{d}{dx} g\rangle = \int f^* \frac{dg}{dx}\, dx$$
$\to u = f^*, \quad dv = \frac{dg}{dx}dx = dg$
$\to du = \frac{df^*}{dx}dx, \quad v = g$
$$= f^* g\Big|_{-\infty}^{\infty} - \int g \frac{df^*}{dx}\, dx$$
$$= -\int \left(\tfrac{d}{dx}f\right)^* g\, dx$$
$$= \langle -\tfrac{d}{dx} f | g\rangle, \quad \boxed{\left(\tfrac{d}{dx}\right)^\dagger = -\tfrac{d}{dx}}$$

b Hermitian conjugate $(a_+)^\dagger$.

Note $a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - ip)$

$$\langle f | a_+ g\rangle = \frac{1}{\sqrt{2\hbar m\omega}}\left[m\omega \int f^* x g - p\int f^*(ig)\right]$$
$$= \frac{1}{\sqrt{2\hbar m\omega}}\left[m\omega\langle xf|g\rangle - p\langle -if|g\rangle\right]$$
$$= \left\langle \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + ip) f | g\right\rangle$$
$$= \langle a_- f | g\rangle, \quad \boxed{(a_+)^\dagger = a_-}$$

c Hermitian conjugate $(QR)^\dagger$.

$$\langle f | \hat{Q}\hat{R} g\rangle = \langle f | \hat{Q}(\hat{R}g)\rangle = \langle \hat{Q}^\dagger f | \hat{R} g\rangle$$
$$= \langle \hat{R}^\dagger \hat{Q}^\dagger f | g\rangle \to \boxed{(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger \hat{Q}^\dagger}$$

Problem 2

a Check if $\hat{Q}$ is Hermitian.

$$\langle f | \hat{Q} g\rangle = \int_0^{2\pi} f^* \frac{d^2 g}{d\varphi^2}\, d\varphi$$
$\to u = f^* \qquad dv = d^2g/d\varphi^2\, d\varphi$
$\to du = \left(\frac{df}{d\varphi}\right)^* d\varphi \quad v = dg/d\varphi$
$$= \cancel{f^* \frac{dg}{d\varphi}\Big|_0^{2\pi}} \to 0 - \int_0^{2\pi} \frac{dg}{d\varphi}\left(\frac{df}{d\varphi}\right)^* d\varphi$$
$\to$ Eq 3.26: f, g periodic with 2π
$$= -\int_0^{2\pi} \left(\frac{df}{d\varphi}\right)^* dg$$
$\to u = \left(\frac{df}{d\varphi}\right)^* \quad dv = dg$
$\to du = \left(\frac{d^2f}{d\varphi^2}\right)^* d\varphi \quad v = g$
$$= -\cancel{\left(\frac{df}{d\varphi}\right)^* g\Big|_0^{2\pi}} \to 0 + \int_0^{2\pi} g\left(\frac{d^2f}{d\varphi^2}\right)^* d\varphi$$
$$= \int_0^{2\pi} \left(\frac{d^2f}{d\varphi^2}\right)^* g\, d\varphi$$
$$= \langle \hat{Q} f | g\rangle \to \boxed{\hat{Q}\text{ Hermitian}}$$

b $\hat{Q}$'s eigenfunctions, eigenvalues.

$$\hat{Q}f = \lambda f$$
$$\cancel{\frac{d^2f}{d\varphi^2} - \lambda f = 0} \to k = \sqrt{-\lambda}$$
$$\boxed{\frac{d^2 f}{d\varphi^2} + k^2 f = 0}$$
$$\boxed{f(\varphi) = Ae^{\pm ik\varphi}}$$

Find eigenvalues λ:
note $e^{\pm ik\varphi}$ is periodic on 2π for
$k\varphi = 2\pi n \to k = \frac{2\pi n}{\varphi} \to \boxed{\lambda = -\left(\frac{2\pi}{\varphi}\right)^2 n^2}$

$\boxed{\text{Degenerate}}$ because a single eigenvalue has two roots for n, positive and negative.

Problem 3

$$p\psi_1(x) = -i\hbar\frac{d}{dx}\left(\sqrt{\tfrac{2}{a}}\sin\left(\tfrac{\pi x}{a}\right)\right) = -i\hbar\sqrt{\tfrac{2}{a}}\left(\tfrac{\pi}{a}\cos\tfrac{\pi x}{a}\right)$$

$\boxed{\text{NO}}$. For a function to be an eigenfunction of momentum means that it is a determinate state of the momentum, or that momentum is constant on all measurements. This is not the case for an oscillator, where p^2 may be determinate with E, but it has two roots (negative, positive) for, Problem 4

a $\langle Q \rangle = \langle f | \hat{Q} f \rangle = \langle \hat{Q}^{+} f | f \rangle$
$= -\langle \hat{Q} f | f \rangle = -\langle f | \hat{Q} f \rangle^{*}$
$= -\langle Q \rangle^{*} \rightarrow$ $\boxed{\text{imaginary}}$

b $[\hat{Q}, \hat{H}]^{+} = (\hat{Q}\hat{H} - \hat{H}\hat{Q})^{+}$ (Hermitian)
$= \hat{H}^{+}\hat{Q}^{+} - \hat{Q}^{+}\hat{H}^{+}$
$= \hat{H}\hat{Q} - \hat{Q}\hat{H}$
$= -(\hat{Q}\hat{H} - \hat{H}\hat{Q})$
$= -[\hat{Q}, \hat{H}]$

since $[\hat{Q}, \hat{H}]^{+} = -[\hat{Q}, \hat{H}]$,
$\boxed{[\hat{Q}, \hat{H}] \text{ anti-Hermitian}}$

$[\hat{Q}, \hat{H}]^{+} = \hat{H}^{+}\hat{Q}^{+} - \hat{Q}^{+}\hat{H}^{+}$
$= (-\hat{H})(-\hat{Q}) - (-\hat{Q})(-\hat{H})$ (anti Hermitian)
$= \hat{H}\hat{Q} - \hat{Q}\hat{H}$ → as before

so commutators still $\boxed{\text{anti-Hermitian}}$

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