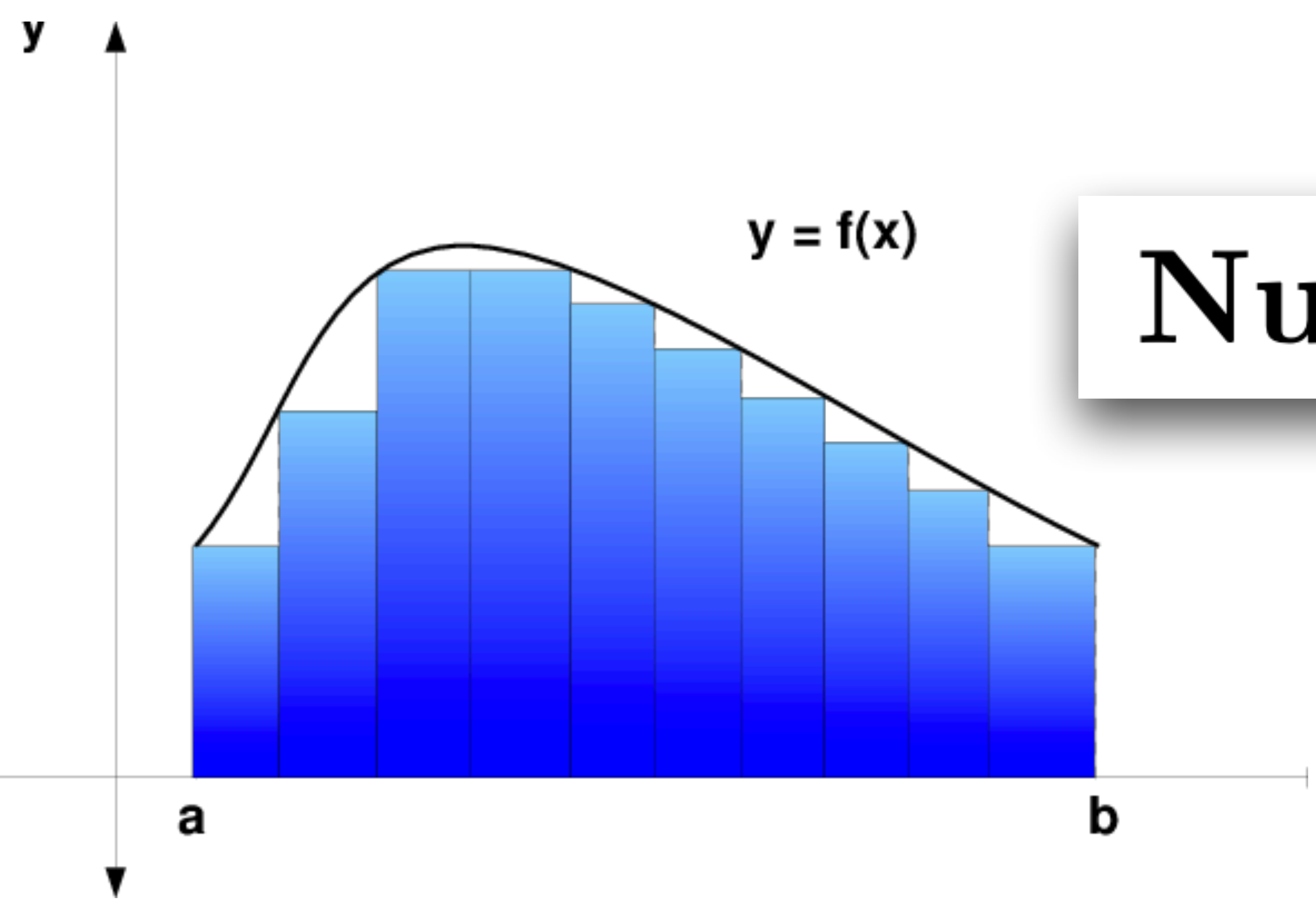
UCLA Math151A
Fall 2021
Lecture 17
2021/11/03



Numerical Integration

[Book:4.3]

A function F is an *antiderivative* of f on an interval I if

$$F'(x) = f(x)$$

for all x in I.

Ex. $F(x) = 3x^2 + 2$ is an *antiderivative* of $f(x) = 6x$ Since $F'(x) = f(x)$.

Fundamental theorem of calculus (FTC 2)

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

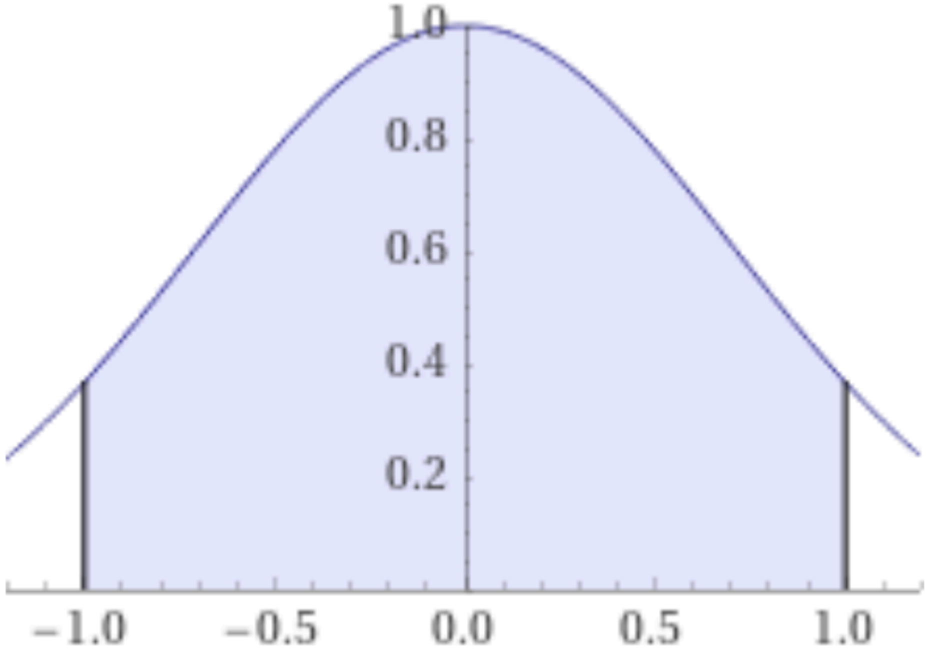
Numerical Quadrature

More or less another word for "numerical integration"…

GOAL: approximate $\int_a^b f(x)dx$ when no explicit antiderivative $F(x)$ is known.

Example 17.1. $\int_{-1}^{1} e^{-x^2} dx$

There is no explicit antiderivative!



Definition 17.1.

A quadrature formula to approximate $\int_a^b f(x)dx$

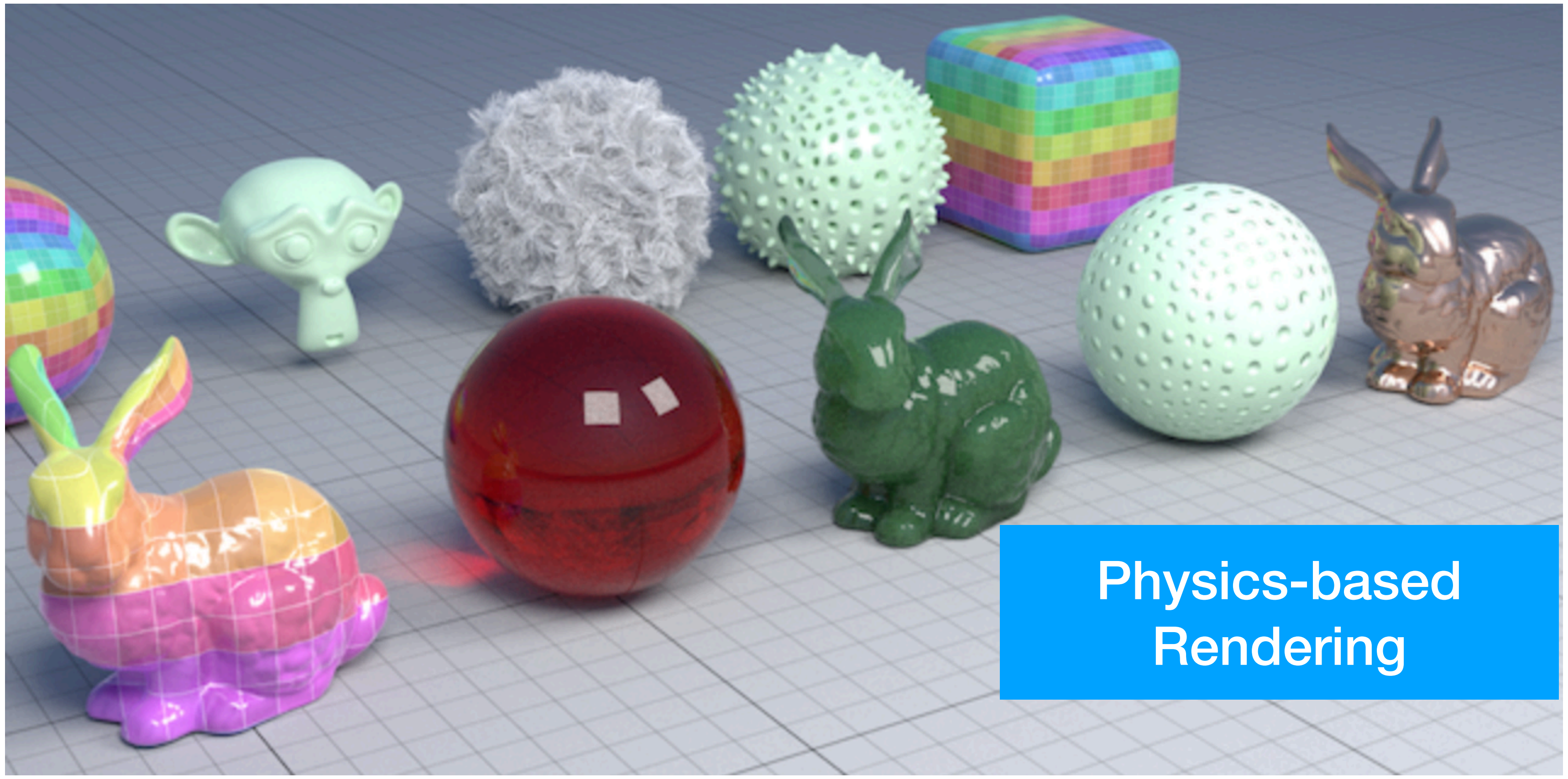
$$\begin{cases} \text{nodes } \{x_i\}_{i=0}^n \\ \text{weights } \{w_i\}_{i=0}^n \end{cases}$$

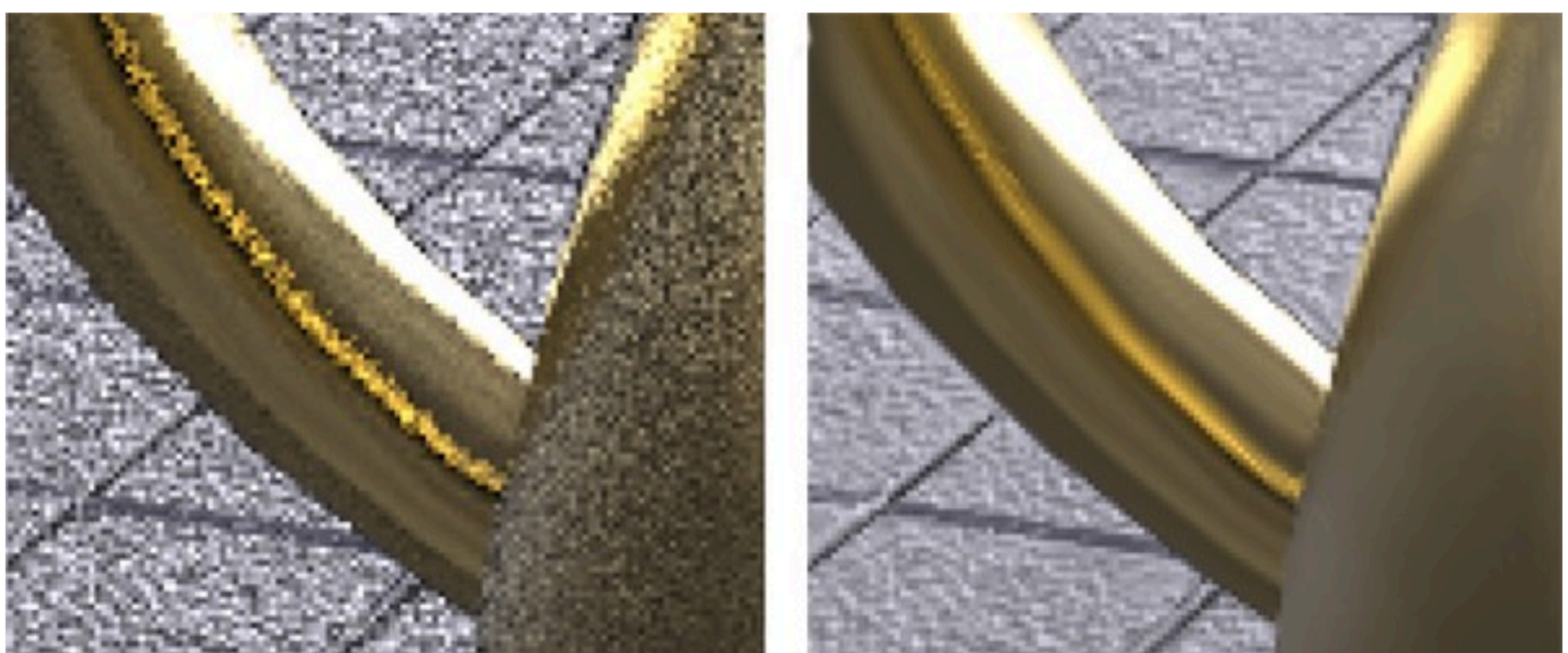
and is given by

$$\sum_{i=0}^n w_i f(x_i)$$

$\square$

Using properly chosen nodes and weights…





Monte-Carlo Sampling

To quantify accuracy,

Definition 17.2 (The Degree of Exactness (D.O.E.))

D.O.E. of a quadrature formula is the largest non-zero integer N s.t. the quadrature formula is exact for

$$f(x) = x^k, k = 0, 1, \ldots, N.$$

I.e., reproducing up to degree N polynomials. $\square$

$$\int_a^b f(x)dx = \sum_{i=0}^{n} w_i f(x_i) \quad ?$$

$$f(x) = a_N x^N + a_{N-1} x^{N-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

$$f(x) = a_{N+1} x^{N+1} + a_N x^N + a_{N-1} x^{N-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

How to derive quadrature formulas?

It has something to do with integrating polynomials…

And we do know how to approximate functions with polynomials.

function	$\approx$	polynomial
		+ some error terms
function integration	$\approx$	polynomial integration
		+ some error terms

One way to derive quadrature formulas is to approximate f(x) with polynomial interpolants:

$$f(x) = P(x) + E(x)$$

Thus, $$\int_a^b f(x)dx = \int_a^b P(x)dx + \int_a^b E(x)dx,$$

We know how to find antiderivatives for polynomials!

$$\int_a^b f(x)dx = \int_a^b P(x)dx + \int_a^b E(x)dx$$

$$P(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$$

$$E(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)\dots(x - x_n)$$

$$\int_a^b P(x)dx = \sum_{i=0}^{n} f(x_i)\int_a^b L_i(x)dx$$

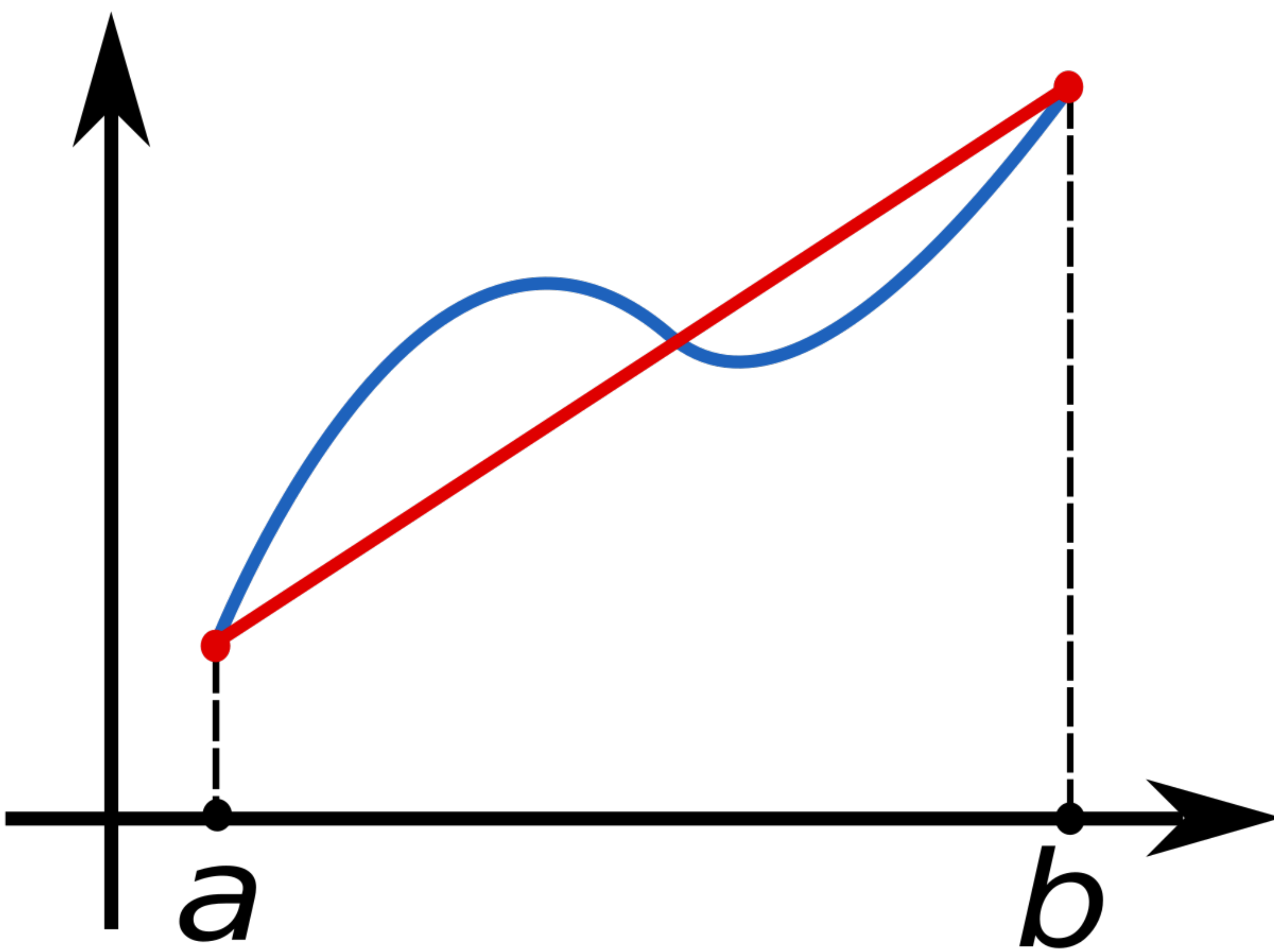
Compare with $\sum_{i=0}^{n} w_i f(x_i)$

$$\boxed{\Rightarrow w_i := \int_a^b L_i(x)dx.}$$

We can also compute the error (integral of E(x)).

Traperzoidal Rule

Idea: Approximate f with a line



$$n = 1$$

$$x_0, x_1$$

linear interpolation.

The area under a linear interpolant is a trapezoid

$f(x)$ defined in $[a, b]$, let $x_0 = a$ and $x_1 = b$

Then we know $P(x) = f(x_0)\dfrac{x - x_1}{x_0 - x_1} + f(x_1)\dfrac{x - x_0}{x_1 - x_0}$

$$\int_a^b f(x)dx \approx \int_a^b P(x)dx = \sum_{i=0}^{1} w_i f(x_i), \qquad \text{(quadrature formula)}$$

$$\text{where } w_i = \int_{x_0}^{x_1} L_i(x)dx.$$

We can do the integrals (we know how to integrate linear functions!): we get $w_0 = w_1 = \frac{x_1 - x_0}{2} = \frac{h}{2}$

Therefore finally $\boxed{\displaystyle\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}\left(f(x_0) + f(x_1)\right)}$

Traperzoidal rule: it is the area of the trapezoid below the line.

$$\int_a^b f(x)dx = \int_a^b P(x)dx + \int_a^b E(x)dx$$

$$P(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$$

$$E(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\dots(x-x_n)$$

The error in the Trapezoidal Rule is

$$\int_{x_0}^{x_1} E(x)dx = \int_{x_0}^{x_1} \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1)dx$$

How do we understand this error?

$$\int_{x_0}^{x_1} E(x)dx = \int_{x_0}^{x_1} \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1)dx$$

Theorem 17.1 (Weighted Mean Value Theorem)**.** Suppose $f \in C([a,b])$ and the integral of $g(x)$ exists on $[a,b]$. Suppose further that the sign of $g(x)$ does not change on $[a,b]$. Then $\exists c \in (a,b)$ s.t.

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

$\square$

Proof. Not covered in this course.

$$\int_{x_0}^{x_1} E(x)dx = \int_{x_0}^{x_1} \frac{f''(\xi(x))}{2} \boxed{(x-x_0)(x-x_1)} dx$$

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

$f(x)$ $g(x)$ clearly it does not change sign

$$= \frac{f''(c)}{2}\int_{x_0}^{x_1}(x-x_0)(x-x_1)dx \;\; = -\frac{f''(c)}{2}\frac{h^3}{6}$$

$$\Rightarrow \quad \boxed{\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f(x_0)+f(x_1)) - \frac{f''(c)}{2}\frac{h^3}{6}}$$

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{f''(c)}{2}\frac{h^3}{6}$$

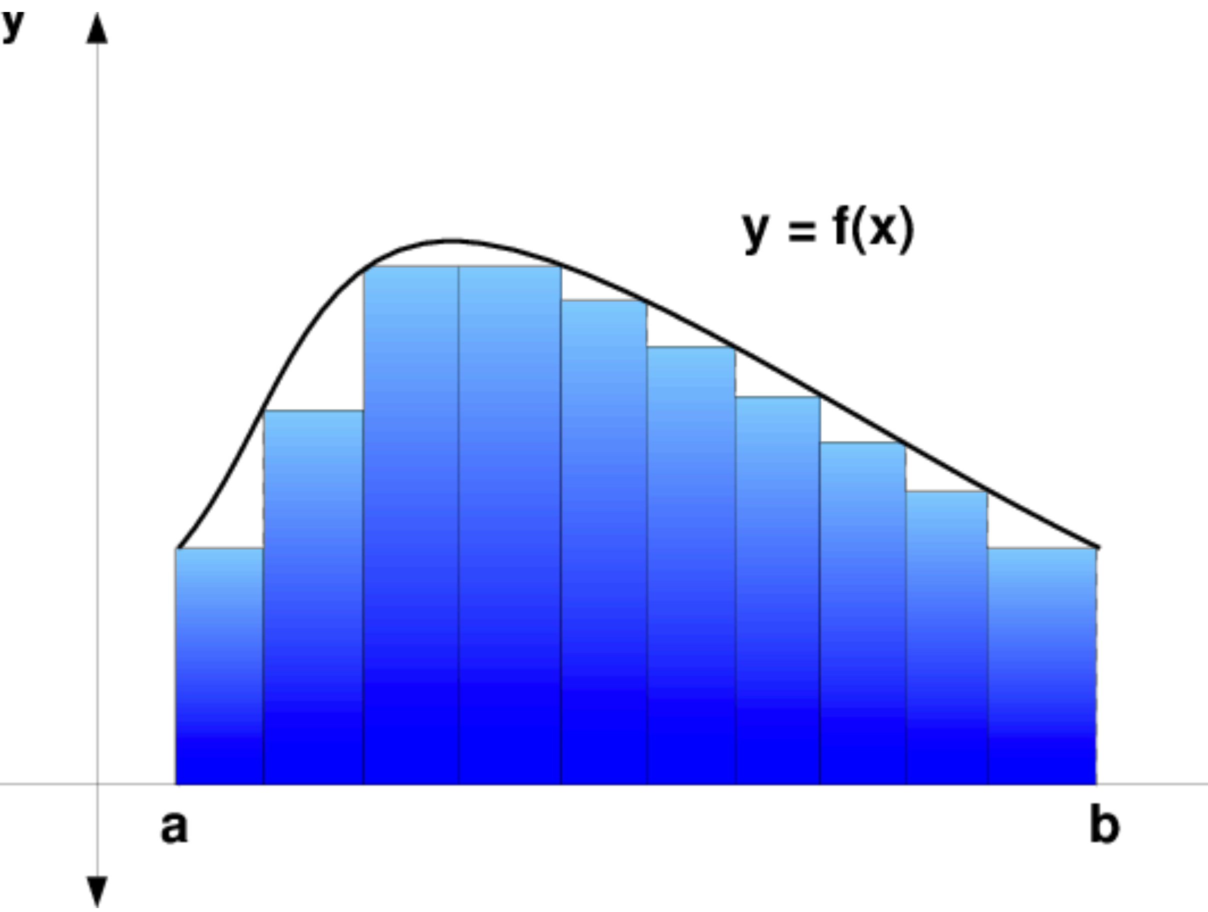
So what is the D.O.E. for this?

only when f is a linear polynomial $f''(c)$ vanishes.

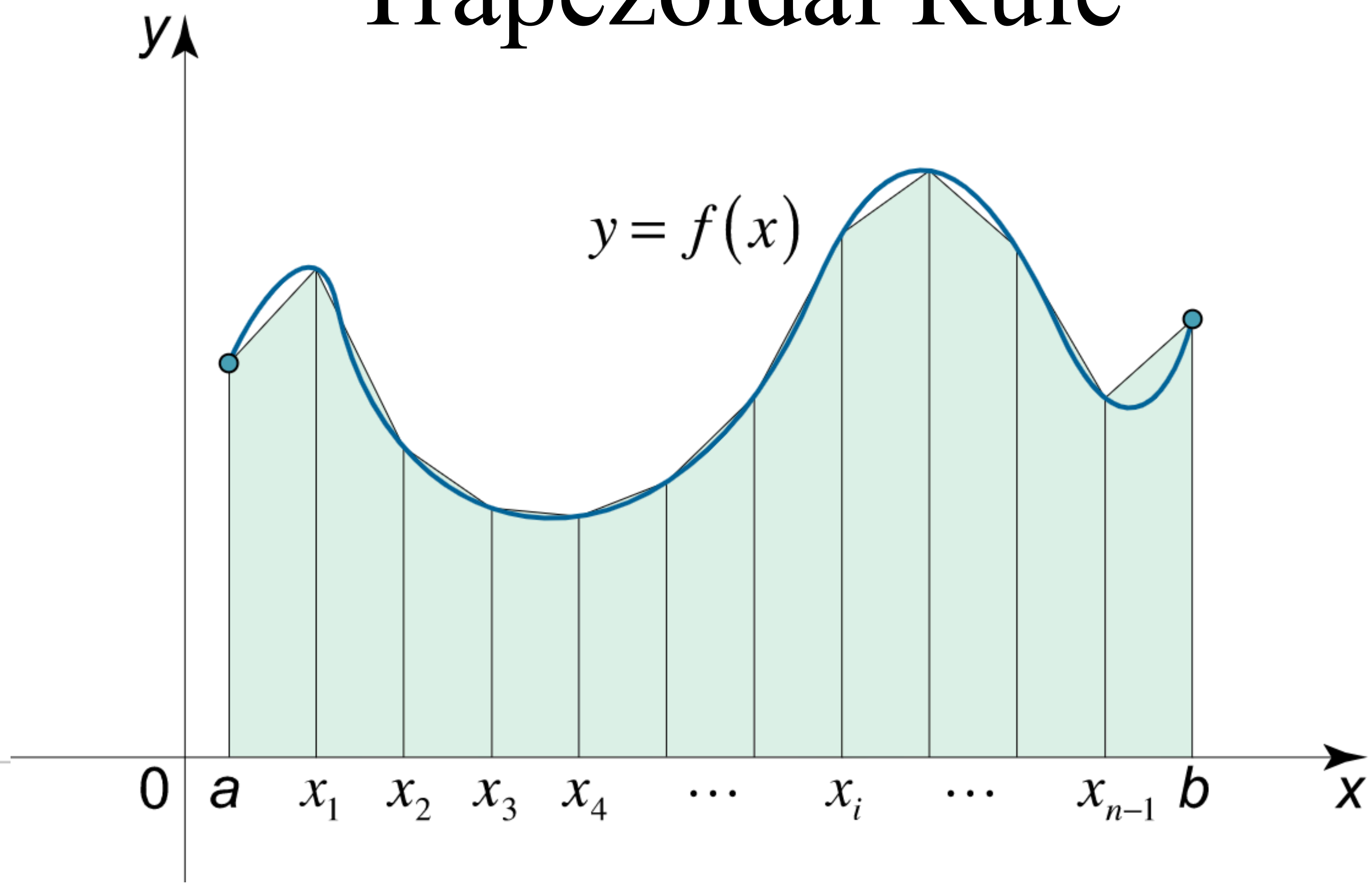
Therefore D.O.E. for Traperzoidal Rule is 1.

Integrate by summing through subintervals

"Naive Rule"

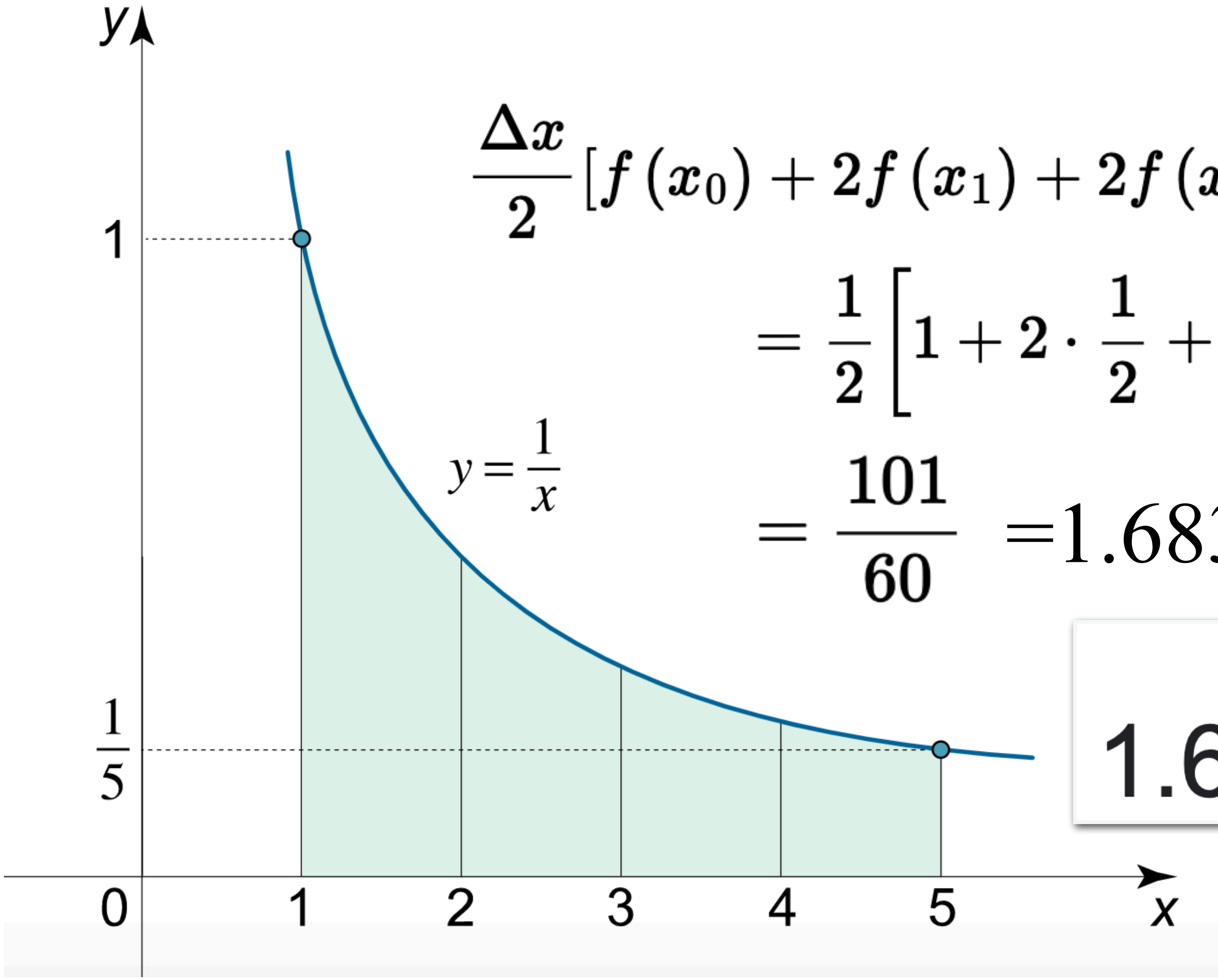


Trapezoidal Rule



$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Example $\displaystyle\int_1^5 \frac{1}{x}\,dx,$ use 4 subintervals.



$$\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right].$$

$$= \frac{1}{2}\left[1 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 2\cdot\frac{1}{4} + \frac{1}{5}\right]$$

$$= \frac{101}{60} = 1.68333333333\ldots$$

ln(5) - ln(1) =

1.60943791243

Simpson's Rule

Let's next derive Simpson's Rule with 3 points (n = 2):

$$a = x_0, x_1 = \frac{a+b}{2}, x_2 = b.$$

$$\int_a^b f(x)dx = \int_a^b P(x)dx + \int_a^b E(x)dx,$$

$$\int_a^b P(x)dx = \sum_{i=0}^{2} w_i f(x_i), \qquad w_i = \int_a^b L_i(x)dx$$

$$\int_a^b E(x)dx = \int_a^b \frac{f'''(\xi(x))}{3!} \boxed{(x - x_0)(x - x_1)(x - x_2)} dx$$

$(x - x_0)(x - x_1)(x - x_2)$ changes sign

we cannot use Weighted M.V.T. here