Homework 5

1. ***Number of thermal photons.*** Show that the number of photons $\sum\langle s_n\rangle$ in equilibrium at temperature τ in a cavity of volume V is

$$N = 2.404\pi^{-2}V(\tau/\hbar c)^3. \tag{48}$$

From (23) the entropy is $\sigma = (4\pi^2 V/45)(\tau/\hbar c)^3$, whence $\sigma/N \simeq 3.602$. It is believed that the total number of photons in the universe is 10^8 larger than the total number of nucleons (protons, neutrons). Because both entropies are of the order of the respective number of particles (see Eq. 3.76), the photons provide the dominant contribution to the entropy of the universe, although the particles dominate the total energy. We believe that the entropy of the photons is essentially constant, so that the entropy of the universe is approximately constant with time.

6. ***Pressure of thermal radiation.*** Show for a photon gas that:

(a)
$$p = -(\partial U/\partial V)_\sigma = -\sum_j s_j\hbar(d\omega_j/dV) \ , \tag{50}$$

where s_j is the number of photons in the mode j;

(b)
$$d\omega_j/dV = -\omega_j/3V; \tag{51}$$

(c)
$$p = U/3V. \tag{52}$$

Thus the radiation pressure is equal to $\frac{1}{3}\times$ (energy density).

(d) Compare the pressure of thermal radiation with the kinetic pressure of a gas of H atoms at a concentration of 1 mole cm^{-3} characteristic of the Sun. At what temperature (roughly) are the two pressures equal? The average temperature of the Sun is believed to be near 2×10^7 K. The concentration is highly nonuniform and rises to near 100 mole cm^{-3} at the center, where the kinetic pressure is considerably higher than the radiation pressure.

9. Photon gas in one dimension. Consider a transmission line of length L on which electromagnetic waves satisfy the one-dimensional wave equation $v^2\partial^2 E/\partial x^2 = \partial^2 E/\partial t^2$, where E is an electric field component. Find the heat capacity of the photons on the line, when in thermal equilibrium at temperature

τ. The enumeration of modes proceeds in the usual way for one dimension: take the solutions as standing waves with zero amplitude at each end of the line.

17. Entropy and occupancy. We argued in this chapter that the entropy of the cosmic black body radiation has not changed with time because the number of photons in each mode has not changed with time, although the frequency of each mode has decreased as the wavelength has increased with the expansion of the universe. Establish the implied connection between entropy and occupancy of the modes, by showing that for one mode of frequency ω the entropy is a function of the photon occupancy $\langle s \rangle$ only:

$$\sigma = \langle s + 1 \rangle \log\langle s + 1 \rangle - \langle s \rangle \log\langle s \rangle. \tag{58}$$

It is convenient to start from the partition function.

Chapter 5

1. Centrifuge. A circular cylinder of radius R rotates about the long axis with angular velocity ω. The cylinder contains an ideal gas of atoms of mass M at temperature τ. Find an expression for the dependence of the concentration $n(r)$ on the radial distance r from the axis, in terms of $n(0)$ on the axis. Take μ as for an ideal gas.

6. *Gibbs sum for a two level system.* (a) Consider a system that may be unoccupied with energy zero or occupied by one particle in either of two states, one of energy zero and one of energy ε. Show that the Gibbs sum for this system is

$$\mathcal{Z} = 1 + \lambda + \lambda \exp(-\varepsilon/\tau). \tag{76}$$

Our assumption excludes the possibility of one particle in each state at the same time. Notice that we include in the sum a term for $N = 0$ as a particular state of a system of a variable number of particles.

(b) Show that the thermal average occupancy of the system is

$$\langle N \rangle = \frac{\lambda + \lambda \exp(-\varepsilon/\tau)}{\mathcal{Z}}. \tag{77}$$

(c) Show that the thermal average occupancy of the state at energy ε is

$$\langle N(\varepsilon) \rangle = \lambda \exp(-\varepsilon/\tau)/\mathcal{Z}. \tag{78}$$

(d) Find an expression for the thermal average energy of the system.

(e) Allow the possibility that the orbital at 0 and at ε may be occupied each by one particle at the same time; show that