UCLA Math151A Fall 2021
Lecture 10
20211015

Theoretical Results for Lagrangian Polynomials

Optional reading: book 3.1

Lagrangian Polynomial Usage

Recall: given input data points $\{x_i, f(x_i)\}_{i=0}^n$, we say

$$L_i(x) = \Pi_{j=0,j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \qquad P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

$P(x)$ is a degree n polynomial.

Example 10.1. Let $f(x) = e^x$,

$$x_0 = 0,\ x_1 = 1/2,\ x_2 = 1,$$
$$f(x_0) = 1,\ f(x_1) = \sqrt{e},\ f(x_2) = e.$$

$$P(x) = 1 \cdot L_0(x) + \sqrt{e} \cdot L_1(x) + e \cdot L_2(x)$$

$$= 1 \cdot \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \sqrt{e} \cdot \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + e \cdot \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

summing up degree 2 polynomials,

the result is a degree 2 polynomial.

$$P(1/4) \approx 1.2717$$
$$f(1/4) \approx 1.2840.$$

Roughly 1% error.

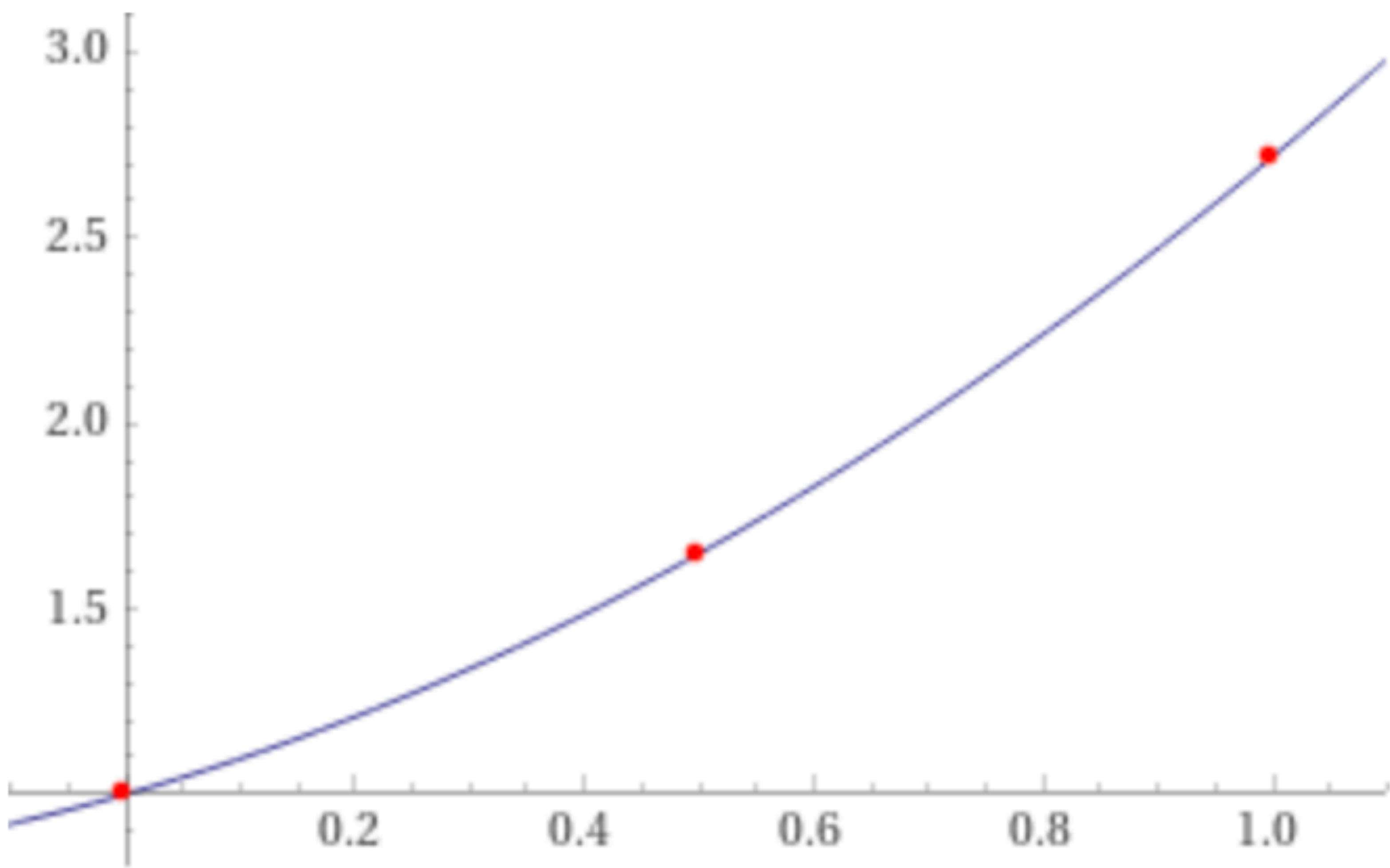
Input interpretation

interpolating polynomial	$\{\{0, 1\}, \{0.5, \sqrt{e}\}, \{1, e\}\}$

Interpolating polynomial

$$0.841679\,x^2 + 0.876603\,x + 1$$

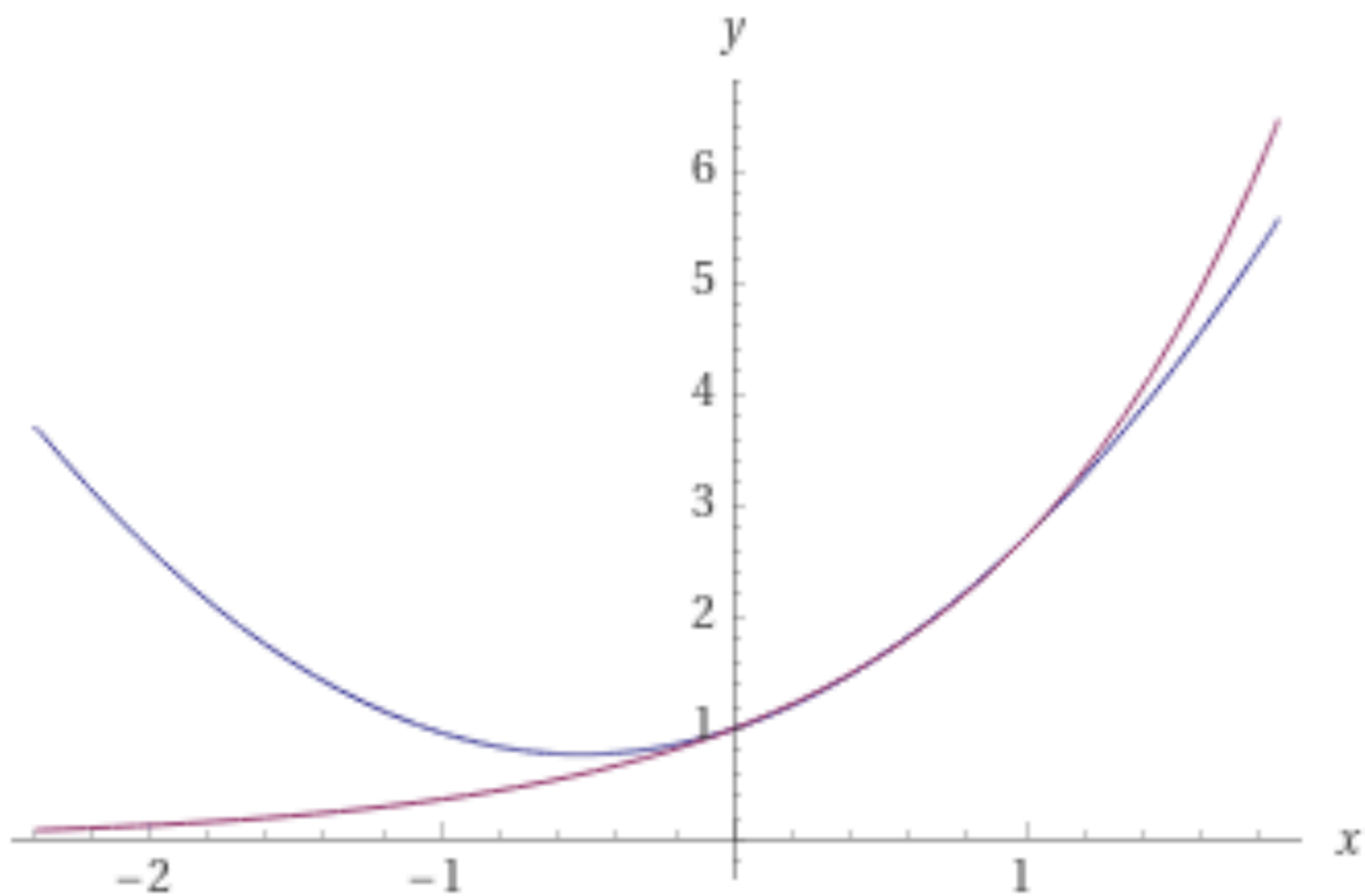
Plot of the interpolating polynomial



Input interpretation

plot	$1 + 0.876603\,x + 0.841679\,x^2$
	e^x

Plots



In the above example, using more points than $n + 1 = 3$ will result in a better approximation.

how do we measure error?

Error Measure

First, need two results (we will not prove here) from Calculus:

Theorem 10.1 ((I) Generalized Rolle's Theorem).

Let $f \in C^n([a, b])$. Suppose $\exists\ n+1$ distinct roots of f on $[a, b]$.

Then $\exists\ \xi \in (a, b)$ s.t. $f^{(n)}(\xi) = 0$. $\square$

It basically says zeros in a function implies a zero of the high-order derivative.

Lemma 10.1 ((II) Derivative of Multiplied Monomials).

$$\frac{d^{n+1}}{dt^{n+1}}(t-t_0)(t-t_1)\dots(t-t_n) = (n+1)! \qquad \square$$

E.g., $\dfrac{d}{dt}(t-x_0) = 1 = 1!$ $\quad \dfrac{d^2}{dt^2}(t-x_0)(t-x_1) = 2 = 2!$ Induction

Theorem vs Lemma vs Proposition

The Theorem of Today

Theorem 10.2 (Error of Lagrangian Polynomial Interpolation).

Let $\{x_0, x_1, \dots, x_n\} \in [a, b]$ be distinct. Let $f \in C^{n+1}([a, b])$, $P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$, then $\forall x \in [a, b], \exists \xi(x) \in (a, b)$ s.t.

$$\begin{aligned} f(x) &= P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n) \\ &= P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k) \qquad \square \end{aligned}$$

(Error of Lagrangian Polynomial Interpolation).

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k)$$

Proof. true if $x = x_i$ since $f(x_i) = P(x_i)$ by construction.

So we only deal with $x \neq x_i$.

Let x be fixed and define

$$g(t) := f(t) - P(t) - (f(x) - P(x)) \cdot \Pi_{j=0}^{n}\left(\frac{t - x_j}{x - x_j}\right). \qquad (*)$$

I want to apply Generalized Rolle's Theorem on g(t) to claim:

$g^{(n+1)}(\xi) = 0.$ I need to show g is C(n+1) and has n+2 distinct roots.

Theorem 10.1 ((I) Generalized Rolle's Theorem).

Let $f \in C^n([a, b])$. Suppose $\exists$ $n + 1$ distinct roots of f on $[a, b]$.

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(Error of Lagrangian Polynomial Interpolation).

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x-x_k)$$

Proof. true if $x = x_i$ since f ... n.

So we only c...

Let x be fixed and define

$t = x_i, 0 \le i \le n$

clearly $f(t) - P(t) = 0$

and $x_i - x_j|_{j=i} = 0$;

For $t = x$,

$\Pi_{j=0}^{n} = 1.$

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(Error of Lagrangian Polynomial Interpolation).

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k)$$

because that $f \in C^{n+1}([a,b])$ and $P \in C^{\infty}([a,b])$

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Generalized Rolle's Theorem says $g^{(n+1)}(\xi) = 0.$

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \cancel{P^{(n+1)}(t)} - (f(x) - P(x))\frac{d^{n+1}}{dt^{n+1}}\Pi_{j=0}^{n}\frac{(t - x_j)}{(x - x_j)}$$

Lemma 10.1 $\frac{d^{n+1}}{dt^{n+1}}(t - t_0)(t - t_1)\ldots(t - t_n) = (n+1)!$ $\square$

(Error of Lagrangian Polynomial Interpolation).

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k)$$

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$$= f^{(n+1)}(t) - (f(x) - P(x))(n+1)!\Pi_{j=0}^{n}\frac{1}{(x - x_j)}$$

(Error of Lagrangian Polynomial Interpolation).

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k)$$

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$$= f^{(n+1)}(t) - (f(x) - P(x))(n+1)!\Pi_{j=0}^{n}\frac{1}{(x - x_j)}$$

$$0 = f^{(n+1)}(\xi) - (f(x) - P(x))(n+1)!\Pi_{j=0}^{n}\frac{1}{(x - x_j)}$$

$$\Rightarrow \quad f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k). \qquad \blacksquare$$

$\boxed{\text{Remark}}$

the *pointwise error* $\quad f(x) - P(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k).$

In order for it to be useful, we need a bound on $|f^{n+1}(\xi)|$.

$\boxed{\text{Remark}}$

L.P. is unique. See homework.