28. Basic Properties of the Derivative

In this section we review the definition of derivative and its basic properties.

Definition 28.1. Let f be a real-valued function defined on an open interval containing a point a. We say f is **differentiable at** a if the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists and is finite. In this case, the limit is called the **derivative of** f **at** a and denoted by $f'(a)$.

The next criterion is easy to prove but nonetheless useful for establishing differentiability of complicated expressions.

Theorem 28.2. **(Weierstrass–Carathéodory Formulation)** Let f be a real-valued function defined on an open interval containing a point a. Then the followings are equivalent:

(a) f is differentiable at a.

(b) There exists a function $\phi = \phi_{f,a}$ on $\operatorname{dom}(f)$ which is continuous at a and satisfies

$$f(x) = f(a) + \phi(x)(x-a) \quad \text{for all} \quad x \in \operatorname{dom}(f). \tag{1}$$

Moreover, in this case, $\phi(a) = f'(a)$.

Proof. (a) $\Rightarrow$ (b) : Define the function ϕ on $\operatorname{dom}(f)$ by

$$\phi(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a}, & \text{if } x \neq a; \\ f'(a), & \text{if } x = a. \end{cases}$$

This obviously satisfies (1). Moreover,

$$\lim_{x\to a}\phi(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a) = \phi(a)$$

shows that ϕ is continuous at a.

(b) $\Rightarrow$ (a) : From (1) and by the continuity of ϕ at a, we have

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\phi(x) = \phi(a).$$

So it follows that f is differentiable at a and $f'(a) = \phi(a)$. $\square$

The next observation tells that differentiability is at least as strong as continuity.

Theorem 28.3. If f is differentiable at a point a, then f is continuous at a.

Proof. By Theorem 28.2, the right-hand side of the identity (1) is continuous at a and therefore the claim follows.

Theorem 28.4. Let f and g be differentiable at a. Then the followings hold:

(a) For any constant $c \in \mathbb{R}$, cf is differentiable at a and
$$(cf)'(a) = c \cdot f'(a).$$

(b) $f + g$ is differentiable at a and
$$(f+g)'(a) = f'(a) + g'(a).$$

(c) *(Product rule)* fg is differentiable at a and
$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

(d) *(Quotient rule)* If $g(a) \neq 0$, then f/g is differentiable at a and
$$(f/g)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2},$$

Proof. In each item, let $\phi_f = \phi_{f,a}$ and $\phi_g = \phi_{g,a}$ be functions as in the Weierstrass–Caratheodory formulation, Theorem 28.2. That is, they are continuous at a and satisfy the identities
$$f(x) = f(a) + \phi_f(x)(x-a) \qquad \text{and} \qquad g(x) = g(a) + \phi_g(x)(x-a). \tag{$*$}$$

(a) For any constant c, by $(*)$
$$\begin{aligned}(cf)(x) &= c[f(a) + \phi_f(x)(x-a)] \\ &= (cf)(a) + c\phi_f(x)(x-a).\end{aligned}$$
Moreover, $c\phi_f(x)$ is continuous at a. So by Theorem 28.2 again, cf is differentiable at a and
$$(cf)'(a) = c\phi_f(a) = cf'(a).$$

(b) By $(*)$,
$$\begin{aligned}(f+g)(x) &= [f(a) + \phi_f(x)(x-a)] + [g(a) + \phi_g(x)(x-a)] \\ &= (f+g)(a) + [\phi_f(x) + \phi_g(x)](x-a).\end{aligned}$$
Since $\phi_f(x) + \phi_g(x)$ is continuous at a, by Theorem 28.2, $f+g$ is differentiable at a and
$$(f+g)'(a) = \phi_f(a) + \phi_g(a) = f'(a) + g'(a).$$

(c) By $(*)$,
$$\begin{aligned}(fg)(x) &= [f(a) + \phi_f(x)(x-a)] \cdot [g(a) + \phi_g(x)(x-a)] \\ &= (fg)(a) + \underbrace{[\phi_f(x)g(a) + \phi_g(x)f(a) + \phi_f(x)\phi_g(x)(x-a)]}_{\text{call this } \tilde{\phi}(x)}(x-a).\end{aligned}$$

Since $\tilde{\phi}$ is continuous at a, it follows from Theorem 28.2 that fg is differentiable at a. Moreover,

$$(fg)'(a) = \tilde{\phi}(a) = f'(a)g(a) + g'(a)f(a) + f'(a)g'(a)(a-a)$$

and this reduces to the desired formula.

(d) For each x in the set $\operatorname{dom}(f/g) = \operatorname{dom}(f) \cap \{x \in \operatorname{dom}(g) : g(x) \neq 0\}$, the identities $(*)$ yield

$$\begin{aligned}(f/g)(x) &= \frac{f(a)}{g(a)} + \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)} \\ &= \frac{f(a)}{g(a)} + \frac{[f(a) + \phi_f(x)(x-a)]g(a) - f(a)[g(a) + \phi_g(x)(x-a)]}{g(x)g(a)} \\ &= \frac{f(a)}{g(a)} + \underbrace{\left[\frac{\phi_f g(a) - \phi_g(x)f(a)}{g(x)g(a)}\right]}_{\text{call this } \tilde{\phi}}(x-a).\end{aligned}$$

By the assumption, $\tilde{\phi}$ is defined on $\operatorname{dom}(f/g)$ and is continuous at a. So by Theorem 28.2, f/g is differentiable at a. Also,

$$(f/g)'(a) = \tilde{\phi}(a) = \frac{f'(a)g(a) - g'(a)f(a)}{g(a)^2}.$$

□

Theorem 28.5. (Chain Rule) If f is differentiable at a and g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof. By the assumption and Theorem 28.2, there exist functions ϕ_f and ϕ_g such that

- ϕ_f is continuous at a and $f(x) = f(a) + \phi_f(x)(x-a)$ for all $x \in \operatorname{dom}(f)$;
- ϕ_g is continuous at $b = f(a)$ and $g(y) = g(b) + \phi_g(y)(y-b)$ for all $y \in \operatorname{dom}(g)$.

Now by substituting $y = f(x)$ to the second identity,

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(b) + \phi_g(f(x))[f(x) - b] \\ &= g(b) + \phi_g(f(x))[f(a) + \phi_f(x)(x-a) - b] \\ &= g(f(a)) + [\phi_g(f(x))\phi_f(x)](x-a).\end{aligned}$$

Since this holds true for all $x \in \operatorname{dom}(f)$ and $\phi_g(f(x))\phi_f(x)$ is continuous at a, it follows that $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = \phi_g(f(a))\phi_f(a) = g'(f(a))f'(a).$$

□

Theorem 28.6. (Inverse Function Theorem) Consider a one-to-one continuous function f on an open interval I such that f is differentiable at a point $x_0 \in I$ and $f'(x_0) \neq 0$. Then f^{-1} is also differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Remark. Assuming the differentiability of f^{-1} at y_0, it is easy to deduce the form of $(f^{-1})'(y_0)$. Indeed, differentiating both sides of the identity $f^{-1}(f(x)) = x$ at x_0 and applying the chain rule,

$$(f^{-1})'(f(x_0)) \cdot f'(x_0) = 1.$$

This shows that $(f^{-1})'(y_0)$, if exists, must be reciprocal to $f'(x_0)$. So the key part of the proof of Theorem 28.6 is to establish the differentiability of f^{-1} at y_0.

Proof. Let $\phi = \phi_{f,x_0}$ be as in Theorem 28.2. That is, ϕ is continuous at x_0 and satisfies (1). Also, note that

- For $x \neq x_0$, we have $f(x) \neq f(x_0)$ and hence $\phi(x) \neq 0$;
- $\phi(x_0) = f'(x_0) \neq 0$.

Altogether, we know that $\phi(x) \neq 0$ for any $x \in I$. Now by plugging $x = f^{-1}(y)$ to (1) and utilizing $x_0 = f^{-1}(y_0)$,

$$y = y_0 + \phi(f^{-1}(y))[f^{-1}(y) - f^{-1}(y_0)], \quad \forall y \in f(I).$$

Solve this for $f^{-1}(y)$, which is possible because $\phi(f^{-1}(y)) \neq 0$, we get

$$f^{-1}(y) = f^{-1}(y_0) + \frac{1}{\phi(f^{-1}(y))}(y - y_0).$$

Since f^{-1} is continuous by Corollary 18.6 of Note 10, $\phi(f^{-1}(y))$ is continuous at y_0 and hence f^{-1} is differentiable at y_0. Moreover,

$$(f^{-1})'(y_0) = \frac{1}{\phi(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

□

Example 28.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Compute $f'(x)$ for $x \in \mathbb{R} \setminus \{0\}$.

(b) Show that f is differentiable at 0 and find $f'(0)$. Explain why this does not contradict the formula found in (a).

(c) Is f' continuous at 0?

Solution.

(a) Since $x \mapsto \frac{1}{x}$ is differentiable and $\sin(\cdot)$ is differentiable on $\mathbb{R}$, the chain rule tells that the composition function $x \mapsto \sin(1/x)$ is differentiable on $\mathbb{R} \setminus \{0\}$ with the derivative

$$(\sin(1/x))' = -(1/x^2)\cos(1/x).$$

Also, since $x \mapsto x^2$ is differentiable on $\mathbb{R}$, the product rule shows that $f(x)$ is differentiable on $\mathbb{R} \setminus \{0\}$ and

$$f'(x) = (x^2)' \sin(1/x) + x^2(\sin(1/x))' = 2x\sin(1/x) - \cos(1/x).$$

(b) For $x \neq 0$, we have

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = |x \sin(1/x)| \leq |x|.$$

So by the Squeeze Theorem (for limit of function) applied to the inequality

$$-|x| \leq \frac{f(x) - f(0)}{x - 0} \leq |x|, \qquad \forall x \in \mathbb{R} \text{ with } x \neq 0,$$

we find that the defining limit for $f'(0)$ exists and has the value 0. So f is differentiable at 0 and $f'(0) = 0$.

This result does not contradict the formula found in (a), because that formula is obtained via the rules of differentiation and they fail to apply for f at 0.

(c) Consider the sequence $x_n = \frac{1}{2\pi n}$. Then $x_n \to 0$ but $f'(x_n) \to -1 \neq 0 = f'(0)$. Therefore f' is not continuous at 0.

29. The Mean Value Theorem

The next result shows that the derivative at each interior extremum point is zero. This is sometimes called the Fermat's theorem and is often useful for locating candidates for the extremum points of a differentiable function.

Theorem 29.1. Consider a real-valued function f on an open interval containing x_0. Suppose

(i) f assumes its maximum or minimum at x_0;

(ii) f is differentiable at x_0.

Then $f'(x_0) = 0$.

Solution. Let (a, b) be the open interval on which f is defined. Suppose x_0 is a maximum point for f on (a, b); the other case is proved by considering $-f$ instead. Also, let $\phi = \phi_{f,x_0}$ be as in Theorem 28.2. Then

$$f(x_0) \geq f(x) = f(x_0) + \phi(x)(x - x_0),$$

and so, we get

$$\phi(x)(x - x_0) \leq 0$$

for all $x \in (a, b)$. Now by dividing both sides by $x - x_0$ in each of the cases $x > x_0$ and $x < x_0$,

- $\phi(x) \leq 0$ for any $x \in (x_0, b)$, and by taking limit as $x \to x_0^+$, we get $f'(a) = \phi(a) \leq 0$;
- $\phi(x) \geq 0$ for any $x \in (a, x_0)$, and by taking limit as $x \to x_0^-$, we get $f'(a) = \phi(a) \geq 0$.

Therefore $f'(a) = 0$. □

An easy consequence of the above theorem is:

Theorem 29.2. (Rolle's Theorem) Consider a real-valued function f such that

(i) f is continuous on $[a, b]$;

(ii) f is differentiable on (a, b);

(iii) $f(a) = f(b)$.

Then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Proof. By the Extreme Value Theorem, there exist points $x_{\min}, x_{\max} \in [a, b]$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in [a, b]$.

- If $f(a) = f(b) < f(x_{\max})$, then the maximum of f is achieved at a point $x_{\max} \in (a, b)$. So by Theorem 29.1, $f'(x_{\max}) = 0$.
- If $f(a) = f(b) > f(x_{\min})$, then by a similar reasoning as above, we get $f'(x_{\max}) = 0$.
- Otherwise, $f(x_{\max}) \leq f(a) = f(b) \leq f(x_{\min})$ and this forces that f is a constant function. In this case, $f'(x) = 0$ for all $x \in (a, b)$.

Therefore the desired conclusion holds in any cases and we are done. □

Now we are in a position to state and prove one of the most important theorem regarding derivative:

Theorem 29.3. (Mean Value Theorem) Consider a real-valued function f such that

(i) f is continuous on $[a, b]$;

(ii) f is differentiable on (a, b).

Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider the function

$$g(x) = f(x) - L(x),$$

where

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

is the function whose graph is the line joining the two points $(a, f(a))$ and $(b, f(b))$. Then it is obvious that g satisfies all the conditions for the Rolle's Theorem. So there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$. Now the desired conclusion follows from the equalities

$$g'(\xi) = f'(\xi) - L'(\xi), \qquad L'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

□

29.1. Application: When do the derivatives coincide?

Corollary 29.4. Consider a differentiable function f on an open interval I. Suppose $f'(x) = 0$ at each point $x \in I$. Then f is a constant function on I.

Proof. Let $x, y \in I$. Without loss of generality, we may assume $x < y$. Then $[x, y] \subseteq I$, so by the Mean Value Theorem there exists $\xi \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(\xi) = 0.$$

This implies that $f(x) = f(y)$ for all $x, y \in I$ and therefore f is a constant function. □

As an immediate consequence, we obtain the result that justifies the notion of "constant of integration".

Corollary 29.5. Consider differentiable functions f and g on an open interval I. Suppose $f'(x) = g'(x)$ at each point $x \in I$. Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in I$.

Proof. Apply Corollary 29.4 to the function $f - g$.

29.2. Application: Monotonicity and Derivatives

In order to state the next result, we review the types of monotonicity.

Definition 29.6. Consider a real-valued function f defined on an interval I. Then we say f is

strictly increasing on I		$f(x_1) < f(x_2)$;
strictly decreasing on I	if $x_1, x_2 \in I$ and $x_1 < x_2$ imply	$f(x_1) > f(x_2)$;
increasing on I		$f(x_1) \le f(x_2)$;
decreasing on I		$f(x_1) \ge f(x_2)$.

The next result associate the monotonicity of a differentiable function to its derivative.

Corollary 29.7. Let f be a differentiable function on an open interval I. Then

(a) f is strictly increasing if $f'(x) > 0$ for all $x \in I$;

(b) f is strictly decreasing if $f'(x) < 0$ for all $x \in I$;

(c) f is increasing if and only if $f'(x) \ge 0$ for all $x \in I$;

(d) f is decreasing if and only if $f'(x) \le 0$ for all $x \in I$;

Proof. We will only prove part (c); all the other assertions can be proved in similar ways.

$(\Rightarrow)$: Suppose f is increasing. Then for any $x_0, x \in I$ such that $x > x_0$, we have $f(x) - f(x_0) \ge 0$ and $x - x_0 > 0$, hence

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \ge 0.$$

$(\Leftarrow)$: Suppose $f' \ge 0$. Then for any $x < y$, we have

$$\frac{f(y) - f(x)}{y - x} = f'(\xi) \ge 0$$

for some $\xi \in (x, y)$ by the Mean Value Theorem. This then implies $f(y) \ge f(x)$ as required. $\square$

29.3. (Optional) Application: Intermediate Value Property for Derivatives

We have examined an example of differentiable function whose derivative is not continuous. Nevertheless, the derivative on an interval always enjoys the intermediate value property.

Theorem 29.8. (Darboux's Theorem) Let f be a differentiable function on an open interval I. Suppose

(i) $a, b \in I$ and $a < b$;

(ii) m lies between $f'(a)$ and $f'(b)$ [i.e., either $f'(a) < m < f'(b)$ or $f'(b) < m < f'(a)$].

Then there exists $c \in (a, b)$ such that $f'(c) = m$.

Proof. We may assume $f'(a) < m < f'(b)$. Consider the function

$$g(x) = f(x) - mx.$$

Then we have

$$g'(a) = f'(a) - m < 0 \qquad \text{and} \qquad g'(b) = f'(b) - m > 0.$$

Since g is continuous on $[a, b]$, by the Extreme Value Theorem, g assumes its minimum on $[a, b]$ at some point $c \in [a, b]$. Now we claim that this c satisfies the desired condition.

Indeed, by Theorem 28.2 there is a function ϕ, continuous at a and $\phi(a) = g'(a) < 0$, such that

$$g(x) = g(a) + \phi(x)(x - a).$$

So by continuity, $\phi(x) < 0$ for x close to a, and so, $g(x) < g(a)$ for $x > a$ close to a. This shows that a cannot be a minimum point of g on $[a, b]$ and hence $c \neq a$. By a similar reasoning, $b \neq c$. Altogether, $c \in (a, b)$. Then by Theorem 29.1, $g'(c) = 0$ and therefore $f'(c) = m$.