12. Limsup's and Liminf's

In this section, we study the basic properties of limsup/liminf.

Theorem 12.1. Let (s_n) and (t_n) be sequences in $\mathbb{R}$.

(a) The inequalities

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$$

and

$$\liminf(s_n + t_n) \geq \liminf s_n + \liminf t_n$$

hold whenever the right-hand side is not of the form $\infty + (-\infty)$.

(b) If both (s_n) and (t_n) are sequences of non-negative numbers, then the inequalities

$$\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n)$$

and

$$\liminf s_n t_n \geq (\liminf s_n)(\liminf t_n)$$

hold provided the right-hand side is not of the form $0 \cdot \infty$.

(c) If $s_n \leq t_n$ for all but finitely many n's, then

$$\liminf s_n \leq \liminf t_n \quad \text{and} \quad \limsup s_n \leq \limsup t_n.$$

Theorem 12.2. Let (s_n) and (t_n) be sequences in $\mathbb{R}$.

(a) If $\lim s_n$ exists, the equalities

$$\limsup(s_n + t_n) = \lim s_n + \limsup t_n$$

and

$$\liminf(s_n + t_n) = \lim s_n + \liminf t_n$$

hold whenever the right-hand side is not of the form $\infty + (-\infty)$.

(b) If (s_n) is a sequence of non-negative numbers and if $\lim s_n$ exists, then the equalities

$$\limsup s_n t_n = (\lim s_n)(\limsup t_n)$$

and

$$\liminf s_n t_n = (\lim s_n)(\liminf t_n)$$

hold provided the right-hand side is not of the form $0 \cdot (\pm\infty)$.

The next result shows that the geometric mean can only decrease the “fluctuation” of the sequence.

Theorem 12.3. Let (s_n) be a sequence of non-negative real numbers. Then we have

$$\liminf s_n \leq \liminf (s_1 s_2 \dots s_n)^{1/n} \leq \limsup (s_1 s_2 \dots s_n)^{1/n} \leq \limsup s_n.$$

Corollary 12.4. Let (s_n) be a sequence of non-zero real numbers such that $\lim s_n$ exists. Then

$$\lim\,(s_1 s_2 \ldots s_n)^{1/n} = \lim s_n.$$

Proof. If $\alpha = \lim s_n$ exists, then

$$\liminf s_n = \limsup s_n = \alpha$$

and hence all four values in Theorem 12.3 are equal to α. In particular,

$$\liminf\,(s_1 s_2 \ldots s_n)^{1/n} = \limsup\,(s_1 s_2 \ldots s_n)^{1/n} = \alpha$$

and the conclusion follows.

Corollary 12.5. Let (s_n) be a sequence of non-zero real numbers. Then we have

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

Proof. Apply Theorem 12.3 to the sequence (t_n) defined by

$$t_1 = |s_1| \qquad \text{and} \qquad t_n = \left|\frac{s_n}{s_{n-1}}\right| \quad \text{for} \quad n \ge 2.$$