Math 131A

Homework Set 5 **Due 2021/07/28 11:59 p.m. PT**

- You are encouraged to discuss the problems with other students. However, you must write the solutions using your own words. Violation of the honor code may void your submissions.
- The assignments must be submitted through Gradescope. **No late homework will be accepted or graded.** Please allow plenty of time to upload your assignments, especially if you are using Gradescope for the first time.
- You should demonstrate your works that lead to the final answers in order to receive full credit.

1. Problems

1. Determine whether each of the following statements is true or false, and justify your answer.

(a) If f and g are uniformly continuous on a set S, then $f+g$ is uniformly continuous on S.

(b) If f and g are uniformly continuous on a set S, then fg is uniformly continuous on S.

(c) If f is uniformly continuous on S and g is uniformly continuous on $f(S)$, then $g \circ f$ is uniformly continuous on S.

2. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that is **periodic**, that is, there exists a constant $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that is **compactly supported**, in the sense that the set $\{x \in \mathbb{R} : f(x) \neq 0\}$ is bounded. Show that f is uniformly continuous.

3. Let f be a real-valued function defined on an open interval I containing $a \in \mathbb{R}$. Show that the followings are equivalent:

(i) f is continuous at a.

(ii) $\displaystyle\lim_{I \ni x \to a} f(x) = f(a)$.

(iii) $\displaystyle\lim_{x \to a} f(x) = f(a)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x + x^2, & \text{if } x \in \mathbb{Q}; \\ x - x^2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Note that f is discontinuous at each non-zero point of $\mathbb{R}$. Show that f is differentiable at 0.

5. Let f be a continuous function defined on an open interval I and c is a point in I. Suppose

(i) f is differentiable at each point of $I \setminus \{c\}$;

(ii) $L = \lim_{x \to c} f'(x)$ exists and is finite.

Show that f is differentiable at c and $f'(c) = L$.

6. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Show that f is Riemann integrable and find $\int_{-1}^{1} f$.

7. **(a)** Let f be a continuous function on $[0, 1]$, and suppose

$$I = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

exists and is finite. Show that $\int_0^1 f = I$.
(*Hint:* If $P_n = \{\frac{k}{n}\}_{k=0}^{n}$, then $L(f, P_n) \leq \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) \leq U(f, P_n)$. Now review the proof of Theorem 33.3 in Note 13 and show that both $L(f, P_n)$ and $U(f, P_n)$ converge to I.)

(b) Use the previous part to show that

$$\int_0^1 x \, \mathrm{d}x = \frac{1}{2}.$$

8. Let $\mathbf{1}_{\mathbb{Q}}$ be the real-valued function defined by

$$\mathbf{1}_{\mathbb{Q}} = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

(a) Find the upper Darboux integral $\overline{\int_0^1} \mathbf{1}_{\mathbb{Q}}$ and the lower Darboux integral $\underline{\int_0^1} \mathbf{1}_{\mathbb{Q}}$.

(b) Is $\mathbf{1}_{\mathbb{Q}}$ Riemann integrable on $[0, 1]$?

9. **(Integral Mean Value Theorem)** Let f and g be real-valued functions on $[a, b]$ such that

(i) f is continuous on $[a, b]$;

(ii) g is bounded, non-negative, and Riemann integrable on $[a, b]$.

Show that there exists a point $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x) \, \mathrm{d}x = f(\xi) \int_a^b g(x) \, \mathrm{d}x.$$

2. More Problems!

Here are some collection of problems for the interested student. They will not be graded and need not be turned in. Some of these problems are challenging than others and are marked with the symbol *.

2.1. Practice Problems

10. Let $f : (a, b) \to \mathbb{R}$ be increasing. For each $c \in (a, b)$, show that

$$\lim_{x\to c^+} f(x) = \inf_{x\in(c,b)} f(x) \qquad \text{and} \qquad \lim_{x\to c^-} f(x) = \sup_{x\in(a,c)} f(x).$$

11. Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by

$$f(x) = \frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Show that $f(x)$ is continuous but not uniformly continuous.

12. Let $-\infty \leq a < b < c \leq +\infty$, and suppose f is a real-valued function which is uniformly continuous both on $(a, b]$ and on $[b, c)$. Show that f is also uniformly continuous on (a, c).

***13.** Let S be a countably infinite subset of $\mathbb{R}$, and enumerate the elements of S by $(r_1, r_2, r_3, \dots)$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n\,:\,r_n < x} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbf{1}_{(r_n, \infty)}(x), \quad \text{where} \quad \mathbf{1}_A = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Show that f is continuous at x if and only if $x \in \mathbb{R} \setminus S$.

14. Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show that f uniformly continuous but not Lipschitz continuous.

15. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f maps Cauchy sequences to Cauchy sequences, but such that f is not uniformly continuous.

16. Let f be a continuous function on $[a, b]$, and define the function f^* on $[a, b]$ by

$$f^*(x) = \sup_{y \in [a,x]} f(y) = \sup\{f(y) : a \leq y \leq x\}, \quad x \in [a, b].$$

Show that f^* is a continuous function on $[a, b]$.

17. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant function.

18. Give an example of a function f on $[0, 1]$ such that f is not Riemann integrable on $[0, 1]$ but such that $|f|$ is Riemann integrable on $[0, 1]$.

2.2. Interesting Topics

The problems below are not strongly connected to the main topics of the course. However, they are intended for demonstrating interesting topics and applications of the materials covered in class.

19. For non-empty $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, define the distance $\operatorname{dist}(x, S)$ between x and S by

$$\operatorname{dist}(x, S) = \inf\{|x - y| : y \in S\}.$$

Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \operatorname{dist}(x, S)$ is Lipschitz continuous for each non-empty subset S of $\mathbb{R}$.

20. Let f be a function defined on an open interval I containing a.

(a) If f is differentiable at a, show that f is **symmetrically differentiable at** a, in the sense that

$$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a).$$

(b) Give an example of a function f which is symmetrically differentiable at a but not differentiable at a.

***(c)** Let $0 < r < 1$, and suppose

$$\lim_{h\to 0} \frac{f(a+h)-f(a-rh)}{(1+r)h} = L$$

for some real number L. Show that f is differentiable at a and $f'(a) = L$.

21. **(Exponential)** Define the function $\exp : \mathbb{R} \to \mathbb{R}$ by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Note that the right-hand side converges for each $x \in \mathbb{R}$.

(a) For each $n \geq 2$ and $h \neq 0$, show that

$$\left| \frac{(x+h)^n - x^n}{h} - nx^{n-1} \right| \leq \frac{(|x|+|h|)^{n-2}}{(n-2)!}|h|.$$

(*Hint*: Use the Mean Value Theorem twice.)

(b) Conclude that $\exp$ is differentiable on $\mathbb{R}$ and $\exp' = \exp$.

(c) By differentiating $x \mapsto \exp(x+y)\exp(-x)$, show that $\exp(x+y) = \exp(x)\exp(y)$ for any $x, y \in \mathbb{R}$.

(d) Show that $\exp(x) > 0$ for any $x \in \mathbb{R}$. In particular, $\exp$ is strictly increasing on $\mathbb{R}$.

22. **(Logarithm)** Let $\exp$ be the function defined in the previous problem. It is easy to verify that $\exp(\mathbb{R}) = (0, \infty)$. Now define the function $\log : (0, \infty) \to \mathbb{R}$ as the inverse of $\exp$.

(a) Show that $\log$ is differentiable and satisfies $\log'(x) = \frac{1}{x}$.

(b) Show that $\log(xy) = \log x + \log y$ for any $x, y > 0$.

23. **(Exponentiation)** Let $\exp$ and $\log$ be the functions defined in the previous problems. Now define the function $\mathrm{pow} : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{pow}(x, s) = \exp(s \log(x)).$$

(a) $\mathrm{pow}(x, s+t) = \mathrm{pow}(x, s)\,\mathrm{pow}(x, t)$ for all $x > 0$ and $s, t \in \mathbb{R}$.

(b) $\mathrm{pow}(x, 0) = 1$ and $\mathrm{pow}(x, k+1) = x \cdot \mathrm{pow}(x, k)$ for any $x > 0$ and $k \in \mathbb{Z}$. In particular, $\mathrm{pow}(x, k) = x^k$ for integer exponents k.

In light of this, we extend the exponentiation to real exponents by setting $x^s = \mathrm{pow}(x, s)$.

(c) $x^s > 0$ for all $x > 0$ and $s \in \mathbb{R}$.

(d) $(x^s)^t = x^{st}$ for all $x > 0$ and $s, t \in \mathbb{R}$.

(e) $(xy)^s = x^s y^s$ for all $x, y > 0$ and $s \in \mathbb{R}$.

(f) $(x^s)' = s x^{s-1}$ for each $s > 0$.

***24.** (Thomae's Function Revisited) Recall that the Thomae's function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = p/q \text{ for some } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ with } \gcd(p,q) = 1; \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

This function is continuous at each irrational number and discontinuous at each rational number. Show that f is Riemann integrable on $[0,1]$ and find $\int_0^1 f$.

***25.** A funciton on $[a,b]$ is called a **step function** if it is of the form

$$f(x) = \sum_{k=1}^{m} c_k \mathbf{1}_{I_k}(x)$$

for some $m \in \mathbb{N}$, real numbers $c_1, \ldots, c_m$, and intervals $I_1, \ldots, I_m$ contained in $[a,b]$. Note that f is Riemann integrable and

$$\int_a^b f = \sum_{k=1}^{n} c_k \cdot \mathrm{length}(I_k).$$

Show that

$$\overline{\int_a^b} f = \inf\left\{ \int_a^b \varphi : \varphi \text{ is a step function on } [a,b] \text{ and } f \leq \varphi \right\},$$

$$\underline{\int_a^b} f = \sup\left\{ \int_a^b \psi : \psi \text{ is a step function on } [a,b] \text{ and } f \geq \psi \right\}.$$

(*Note:* The significance of this exercise is that it gives a recipe for building up a more general theory of integration by replacing "step function" by another family of functions.)