11. Subsequences

Definition 11.1. Consider a sequence $(s_n)_{n\in\mathbb{N}}$. A **subsequence** of this sequence is a sequence $(t_k)_{k\in\mathbb{N}}$ of the form

$$t_k = s_{n(k)}$$

for some strictly increasing sequence $(n(k))_{k\in\mathbb{N}}$ of indices, i.e., $n(k)$'s are indices of $(s_n)_{n\in\mathbb{N}}$ and

$$n(1) < n(2) < n(3) < \cdots < n(k) < n(k+1) < \ldots.$$

Here are some comments:

- Loosely speaking, a subsequence of (s_n) is a sequence obtained by leaving out some of the terms of (s_n) while keeping others without changing their order.
- *Convention for Indices.* By the usual convention, we will also write n_k for $n(k)$. The choice of notation is just a matter of convenience and one's own preference, and we will use whichever convention that is easier to read.
- *Equivalent Formulation.* By recalling that sequences are just special cases of functions, we may reformulate the above definition in function language:

  A sequence $t : \mathbb{N} \to \mathbb{R}$ is a subsequence of $s : \mathbb{N} \to \mathbb{R}$ if there exists a strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $t = s \circ \sigma$.

- *Notation for Subsequence?* Surprisingly, there is no universally adopted notation for the relation of

  '(t_n) being a subsequence of (s_n)'

  in the literature. Instead of devising an idiosyncratic notation for this, we will take an indirect approach. Note that a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ is uniquely determined by its set of values $S = \{n_k : k \in \mathbb{N}\}$, which is always an infinite subset of $\mathbb{N}$. Consequently, we have the following observation:

  Any subsequence of $(a_n)_{n\in\mathbb{N}}$ takes the form $(a_n)_{n\in S}$ for some infinite subset S of $\mathbb{N}$.

  This convention will be particularly useful when a 'further subsequence' (subsequence of subsequence) is considered.

11.1. Subsequential Limits

The next definition captures the clustering behavior of a sequence:

Definition 11.2. Let (s_n) be a sequence in $\mathbb{R}$. A **subsequential limit** of (s_n) is any element in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ that is the limit of a subsequence of (s_n).

The next result characterizes the sequential limits of a sequence in terms of the long-term behavior:

Theorem 11.3. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

(a) A real number ℓ is a subsequential limit of (s_n) if and only if the following condition holds:

$$\text{for any } \varepsilon > 0, \quad |s_n - \ell| < \varepsilon \text{ holds for infinitely many } n\text{'s}.$$

(b) $+\infty$ is a sequential limit of (s_n) if and only if (s_n) is not bounded above.

(c) $-\infty$ is a sequential limit of (s_n) if and only if (s_n) is not bounded below.

In the proof of Theorem (11.3), the following algorithm of extracting subsequences comes handy:

Lemma 11.4. Consider a family $(P_{n,k})_{n,k\in\mathbb{N}}$ of sentences satisfying the following condition:

$$\text{for each } k \in \mathbb{N}, \quad \text{the statement } P_{n,k} \text{ is true for infinitely many } n\text{'s}.$$

Then there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $P_{n_k,k}$ holds for all $k \in \mathbb{N}$.

Proof. We define $(n_k)_{k\in\mathbb{N}}$ recursively as follows:

- Set $n_0 = 0$ for convenience.
- Suppose n_{k-1} has been defined. Then choose $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $P_{n_k,k}$ is true. This is possible because $P_{n,k}$ holds for infinitely many n's.

By the mathematical induction, $(n_k)_{k\in\mathbb{N}}$ is well-defined and satisfies the desired properties. □

Now we come back to the proof of Theorem 11.3. We will only prove part (a), and the rest is left to an exercise.

Proof of Theorem 11.3.

Example 11.1. Let $(s_n)_{n\in\mathbb{N}}$ be defined by

$$s_n = n^{(-1)^n}.$$

This sequence begins with:

$$(s_n)_{n\in\mathbb{N}} = \left(1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, 8, \dots\right)$$

- If we collect the even terms (i.e., if we take $n(k) = 2k$), then we get a subsequence

  $$(s_{2k})_{k\in\mathbb{N}} = (2, 4, 6, 8, \dots), \quad \text{or simply} \quad s_{2k} = 2k.$$

  This subsequence diverges to $+\infty$.
- If we collect the odd terms, then the corresponding subsequence is

  $$(s_{2k-1})_{k\in\mathbb{N}} = \left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right), \quad \text{or simply} \quad s_{2k-1} = \frac{1}{2k-1}.$$

  This subsequence converges to 0.
- The above observations show that $+\infty$ and 0 are subsequential limits of $(s_n)_{n\in\mathbb{N}}$. Moreover, it is not hard (albeit somewhat tedious) to prove that these are the only subsequential limits.

The next result is sometimes useful:

Theorem 11.5. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and let ℓ be a real number. Then the followings are equivalent:

(a) (s_n) converges to ℓ.

(b) Every subsequence of (s_n) converges to ℓ.

(c) Every subsequence of (s_n) has a further subsequence that converges to ℓ.

11.2. Limsups and liminfs revisited

In this part, we reinterpret limsups and liminfs in terms of subsequential limits.

Theorem 11.6. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Then both $\limsup s_n$ and $\liminf s_n$ are subsequential limits of (s_n).

An immediate consequence of the above result is the Bolzano–Weierstrass theorem, which asserts that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.[1]

Theorem 11.7 (Bolzano–Weierstrass Theorem). Every bounded sequence in $\mathbb{R}$ has a subsequence that converges to some real number.

Theorem 11.8. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, and let E denote the set of all subsequential limits of (s_n).

(a) E is a non-empty subset of $\overline{\mathbb{R}}$.

(b) $\sup E = \limsup s_n$ and $\inf E = \liminf s_n$.

(c) $\lim s_n$ exists in $\overline{\mathbb{R}}$ if and only if E has exactly one element.

[1] In the language of topology, this theorem essentially characterizes the sequential compactness in $\mathbb{R}$.

We conclude this section with the result showing that the set of all subsequential limits is "closed under taking limit".[2]

Theorem 11.9. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, and let E denote the set of all subsequential limits of (s_n). Suppose $(\ell_k)_{k\in\mathbb{N}}$ is a sequence in $E\cap\mathbb{R}$ and that $\lim \ell_k$ exists in $\overline{\mathbb{R}}$. Then $\lim \ell_k$ belongs to E.

[2] In metric topology, such sets are called closed sets. See Section 13 of the textbook for a brief introduction to this sort of topics.

Now we return to Section 10 of the textbook and develop the notion of the Cauchy sequences, which turns out to be very useful in analysis.

10. Monotone Sequences and Cauchy sequences (continued)

10.1. Cauchy sequences

Definition 10.6. A sequence (s_n) of real numbers is called a **Cauchy sequence** if

$$\begin{pmatrix} \text{for each } \varepsilon > 0, \quad \text{there exists a number } N \text{ such that} \\ m, n > N \text{ implies } |s_m - s_n| < \varepsilon. \end{pmatrix} \tag{1}$$

- Loosely speaking, a sequence is Cauchy if its terms become arbitrarily close to each other as the sequence progresses.
- The idea of the Cauchy sequences and its variants turn out to be quite useful for various theoretical developments throughout this course.

Lemma 10.7. Convergent sequences are Cauchy sequences.

Lemma 10.8. Cauchy sequences are bounded.

Lemma 10.9. If a subsequence of a Cauchy sequence converges to $\ell \in \mathbb{R}$, then the sequence itself converges to ℓ.

- The above results show that the notion of Cauchy sequences is at most as powerful as that of convergent sequences. This naturally leads to a question: Are they the same concept?
- In order to answer this question, we note that the notion of convergent and Cauchy sequences can be defined for a wider range of mathematical structures where the notion of distance is defined.[3] For example, Lemmas 10.7–10.9 will hold true even for $\mathbb{Q}$. However, we can find Cauchy sequences in $\mathbb{Q}$ that do not have a limit in $\mathbb{Q}$, providing another hint that $\mathbb{Q}$ suffers from "gaps".
- The next result show that the completeness of $\mathbb{R}$ ensures that Cauchy sequences are convergent.

Theorem 10.10. Every Cauchy sequence in $\mathbb{R}$ converges to an element of $\mathbb{R}$.

[3]A set endowed with a distance function is called a metric space. Section 13 of the textbook provides a crash course on some basic topics about metric spaces.

Example 10.4. (Convergence of Contractive Sequences)

(a) A sequence (s_n) in $\mathbb{R}$ is called **contractive** if there exists $r \in [0,1)$ such that

$$|s_{n+2} - s_{n+1}| \leq r\,|s_{n+1} - s_n| \quad \text{for all } n.$$

Show that every contractive sequence in $\mathbb{R}$ converges.

(b) Let $(x_n)_{n=0}^{\infty}$ be defined recursively by

$$x_0 \geq 0 \qquad \text{and} \qquad x_{n+1} = \frac{2}{2x_n + 3}.$$

Show that (x_n) converges and find the limit.