# UCLA Math151A
# Fall 2021
# Lecture 21
# 2021/11/12

Gaussian Quadrature [continued]

Let $\{q_i\}_{i=0}^{n}$ are orthonormal, $q_i$ has degree $i$

**Lemma 1** $\{q_i\}_{i=0}^{n}$ form a basis for $\mathbb{P}^n$ **Lemma 2** $\mathbb{P}^{n-1} \perp q_n$

**Theorem**

$x_i : q_n(x_i) = 0 \quad w_i = \int_{-1}^{1} L_i(x)dx \Rightarrow DOE\left(\sum_i w_i f(x_i)\right) = 2n-1$

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(1) First suppose $f \in \mathbb{P}^{n-1}$

there are n nodes $\{x_i\}_{i=1}^{n} \Rightarrow \exists P(x)$ to interpolate $f(x)$

$$P(x) = \sum_{i=1}^{n} f(x_i)L_i(x)$$

Both $P(x)$ and $f(x)$are degree $n-1$ polynomails for $\{x_i\}_{i=1}^{n}$

$$\Rightarrow \quad f(x) = \sum_{i=1}^{n} f(x_i)L_i(x) \quad \text{(uniqueness of L.I.P.)}$$

$$\Rightarrow \quad \int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Exact!

Let $\{q_i\}_{i=0}^{n}$ are orthonormal, $q_i$ has degree $i$

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(1) First suppose $f \in \mathbb{P}^{n-1}$ Exact!

(2) Next assume $f$ is a polynomial of degree $n \leq d \leq 2n-1$

Polynomial Long Division implies that quotient remainder

$$f(x) = Q(x)q_n(x) + R(x),$$

degree: $f : [n, 2n-1], \quad q_n : n, \quad Q : [0, n-1], \quad R : [0, n-1]$

$$f(x_i) = Q(x_i)q_n(x_i) + R(x_i) = R(x_i). \qquad (*)$$

by part (1) $$\int_{-1}^{1} R(x)dx = \sum_{i=1}^{n} w_i R(x_i). \qquad (**)$$

Let $\{q_i\}_{i=0}^{n}$ are orthonormal, $q_i$ has degree $i$

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$$f(x_i) = Q(x_i)q_n(x_i) + R(x_i) = R(x_i). \qquad (*)$$

$$\int_{-1}^{1} R(x)dx = \sum_{i=1}^{n} w_i R(x_i). \qquad (**)$$

$$\int_{-1}^{1} f(x)dx = \cancel{\int_{-1}^{1} Q(x)q_n(x)dx} + \int_{-1}^{1} R(x)dx$$

$$= \sum_{i=1}^{n} w_i R(x_i) \qquad = \sum_{i=1}^{n} w_i f(x_i) \qquad \blacksquare$$

Let $\{q_i\}_{i=0}^{n}$ are orthonormal, $q_i$ has degree $i$

Theorem

$$x_i : q_n(x_i) = 0 \quad w_i = \int_{-1}^{1} L_i(x)dx \Rightarrow DOE\left(\sum_i w_i f(x_i)\right) = 2n - 1$$

Only question left: how to construct

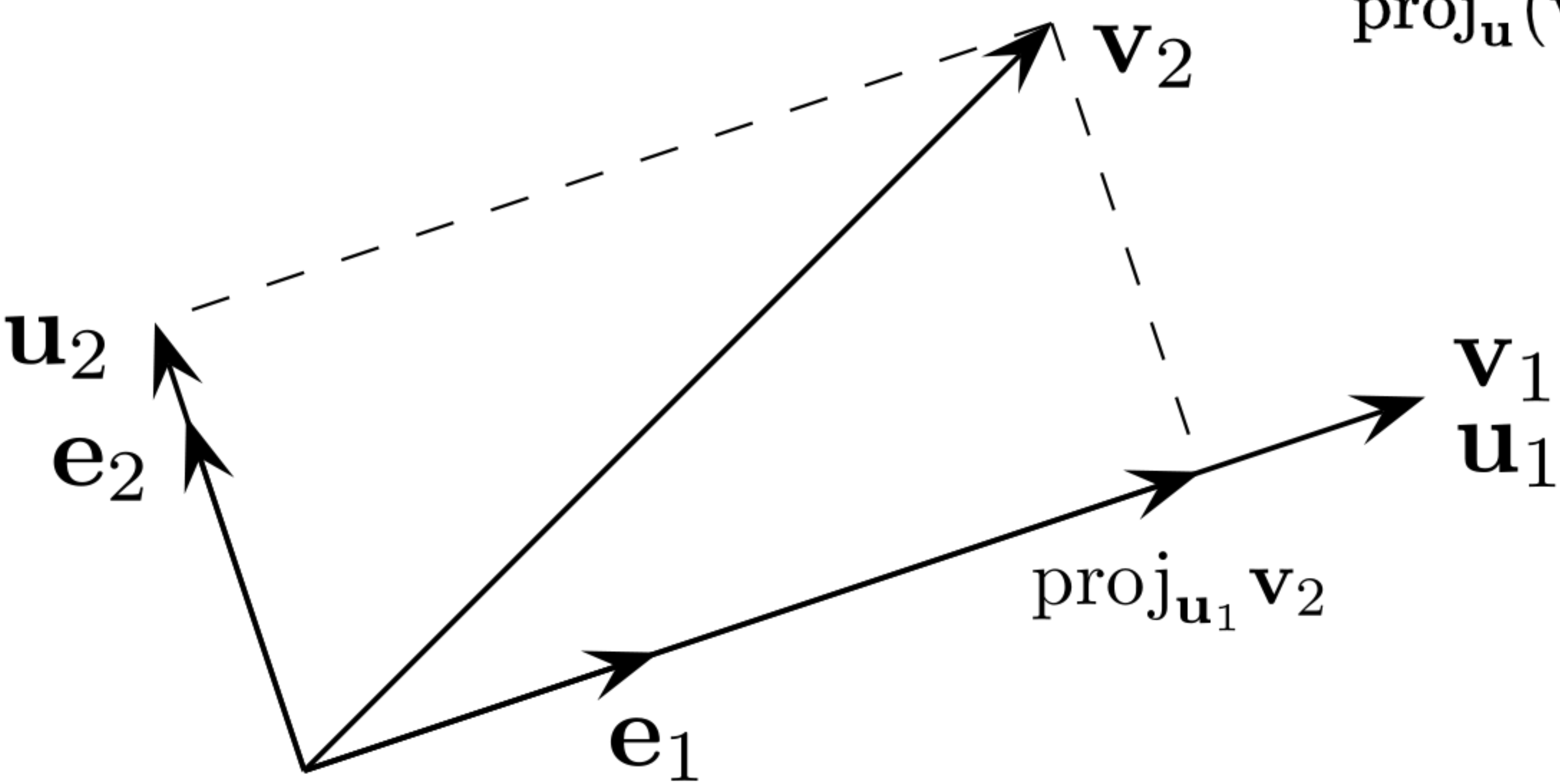
$$\{q_1, q_2, \ldots, q_n\}$$

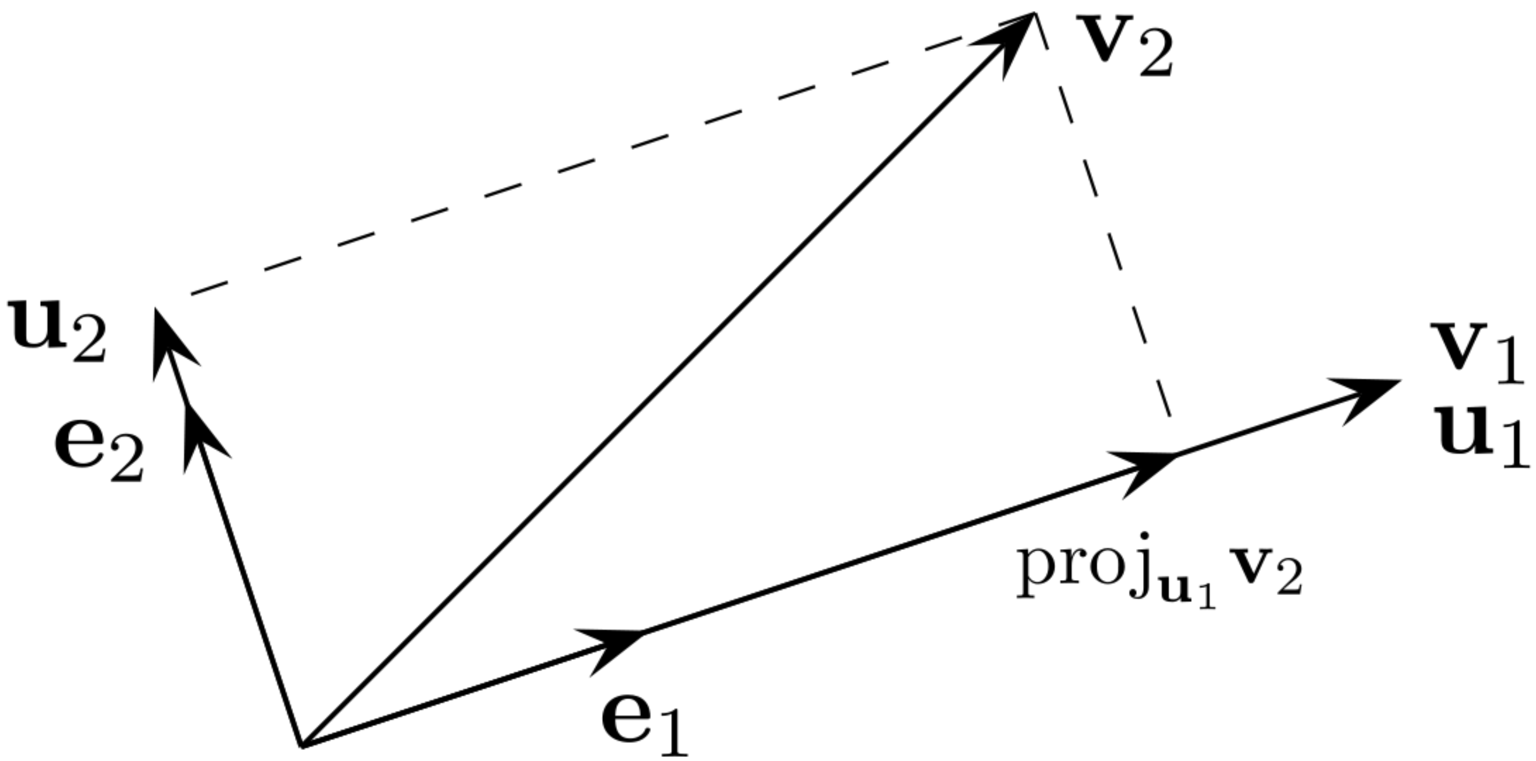
How to construct an orthonormal basis for a vector space?

# Construct Orthonormal Basis

The **Gram–Schmidt** process is a method for orthonormalizing a set of vectors in an inner product space.

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$






$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

Consider the following set of vectors in $\mathbf{R}^2$ (with the conventional inner product)

$$S = \left\{ \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}.$$

Now, perform Gram–Schmidt, to obtain an orthogonal set of vectors:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \mathrm{proj}_{\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{8}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 6/5 \end{bmatrix}.$$

# Gram–Schmidt process

- Let $\{x_1, x_2, \ldots, x_n\}$ be linearly indepdendent, this is the input.
- Set $v_1 = x_1$
- For $i = 2, \ldots, n$ set $v_i = x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, v_j \rangle}{\langle v_j, v_j \rangle} v_j$
- For $i = 1, \ldots, n$ normalize $q_i = \frac{v_i}{\|v_i\|}$ where $\|v_i\| = (\langle v_i, v_i \rangle)^{1/2}$
- Output is $\{q_1, q_2, \ldots, q_n\}$ orthonormal.

$$\mathbb{P}^n \ : \ \{1, x, x^2, \ldots, x^{n-1}, x^n\}$$
$$\langle f, g\rangle = \int_{-1}^{1} fg dx$$

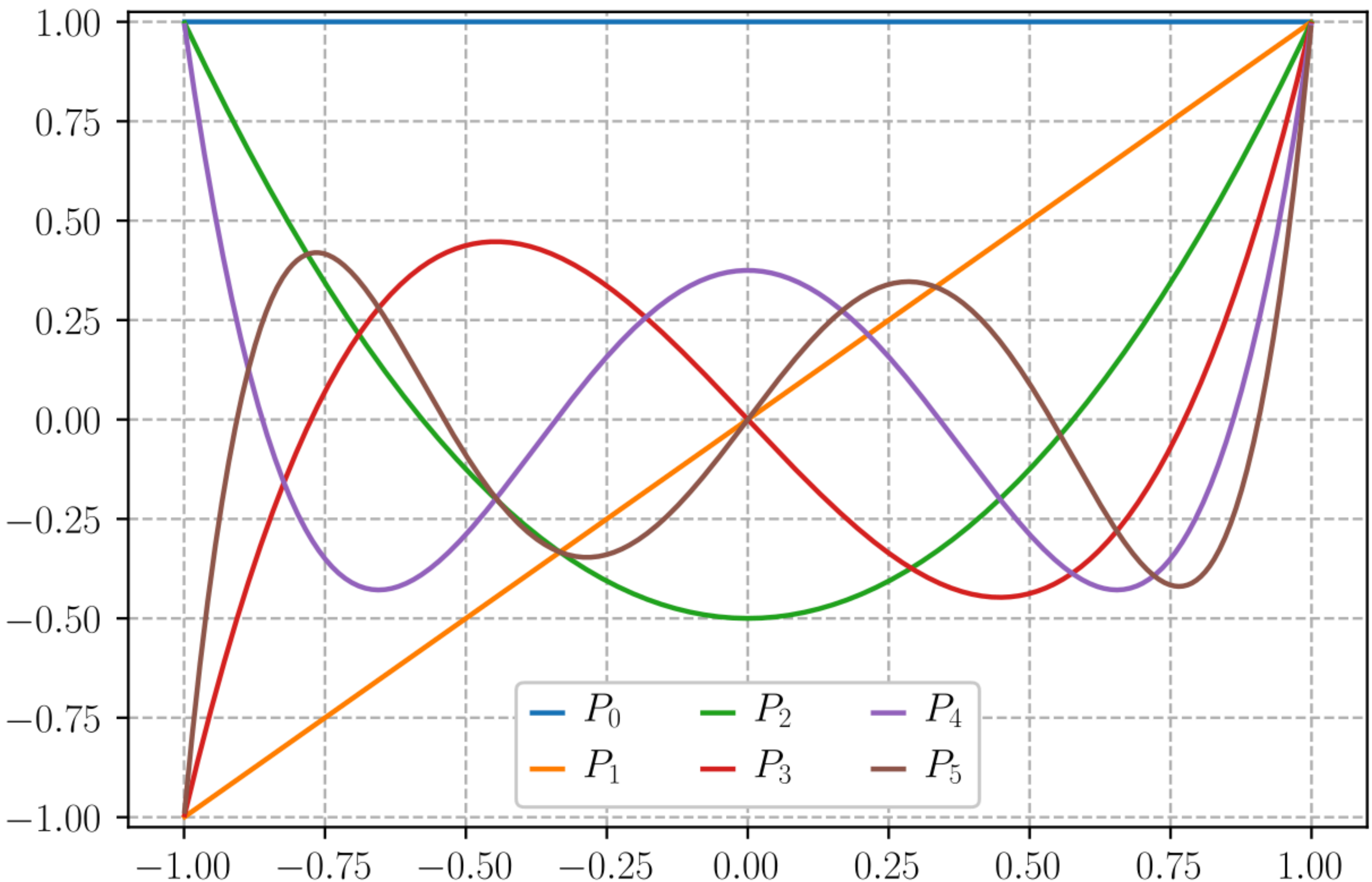
$$P_0(x) = 1 \qquad P_1(x) = x - \frac{< x, 1 >}{< 1, 1 >} 1 = x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} = x$$

$$\begin{aligned} P_2(x) &= x^2 - \frac{< x^2, 1 >}{< 1, 1 >} 1 - \frac{< x^2, x >}{< x, x >} x \\ &= x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} x \\ &= x^2 - \frac{2/3}{2} - 0 \\ &= x^2 - \frac{1}{3} \end{aligned}$$

$$P_3(x) = x^3 - \frac{3}{5} x$$

$$P_4(x) = x^4 - \frac{6}{7} x^2 + \frac{3}{35}$$

Legendre polynomials



1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
$P_0$
$P_1$
$P_2$
$P_3$
$P_4$
$P_5$

Finding the roots: Fixed-point?

Bisection?

Newton?

JUST LOOK IT UP!

| $n$ | Roots $r_{n,i}$ |
|---|---|
| 2 | 0.5773502692 |
| | −0.5773502692 |
| 3 | 0.7745966692 |
| | 0.0000000000 |
| | −0.7745966692 |
| 4 | 0.8611363116 |
| | 0.3399810436 |
| | −0.3399810436 |
| | −0.8611363116 |
| 5 | 0.9061798459 |
| | 0.5384693101 |
| | 0.0000000000 |
| | −0.5384693101 |
| | −0.9061798459 |

There is one last thing remaining…

Recall this lengthy version

**Theorem 20.1.** [Gaussian Quadrature Theorem]

Let $\{x_i\}_{i=1}^n$ be the $n$ roots of n degree polynomial $q_n(x)$, where $q_n$ is the last in the set $\{q_i\}_{i=0}^n$.

We'll assume $\boxed{\text{they are real and distinct.}}$

Let

one last thing remaining...

$$w_i = \int_{-1}^{1} \Pi_{j=1,j\neq i}^{n} \frac{x - x_j}{x_i - x_j} dx, \qquad i = 1, 2, \ldots, n,$$

where the integrand equals to $L_i(x)$ from Lagnraigan interpolation polynomial. Then

$$\sum_{i=1}^{n} w_i f(x_i) \quad \text{is exact for any } f \in \mathbb{P}^{2n-1}. \qquad \square$$

**Theorem 21.1.**

Let $\{\Phi_1, \Phi_2, \ldots, \Phi_n\}$ be a set of orthgonal polynomails on $[a, b]$, and let each $\Phi_k$ has degree $k$.

Then each $\Phi_k$ has precisely $k$ real roots which are simple

Proof: not required/covered in 151A.

# Example of using Gaussian Quadrature

Approximate $\int_{-1}^{1} e^x \cos x \, dx$ using Gaussian quadrature with $n = 3$.

$$w_i = \int_{-1}^{1} \Pi_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} dx, \qquad i = 1, 2, \ldots, n,$$

| $n$ | Roots $r_{n,i}$ | |
|---|---|---|
| 2 | 0.5773502692 | 1.0000000000 |
| | −0.5773502692 | 1.0000000000 |
| 3 | 0.7745966692 | 0.5555555556 |
| | 0.0000000000 | 0.8888888889 |
| | −0.7745966692 | 0.5555555556 |
| 4 | 0.8611363116 | 0.3478548451 |
| | 0.3399810436 | 0.6521451549 |
| | −0.3399810436 | 0.6521451549 |
| | −0.8611363116 | 0.3478548451 |
| 5 | 0.9061798459 | [illegible] |
| | 0.5384693101 | [illegible] |
| | 0.0000000000 | 0.5688888889 |
| | −0.5384693101 | [illegible] |
| | −0.9061798459 | 0.2369268850 |

$$0.\overline{5}e^{0.774596692} \cos 0.774596692$$
$$+ 0.\overline{8} \cos 0$$
$$+ 0.\overline{5}e^{-0.774596692} \cos(-0.774596692)$$
$$= 1.9333904$$

true value of the integral is 1.9334214.

the absolute error is less than $3.2 \times 10^{-5}$