19. Uniform Continuity

Recall the ε–δ property for the continuity of a function at a point:

$$\begin{gathered}\text{For each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ x \in \mathrm{dom}(f) \text{ and } |x - a| < \delta \text{ implies } |f(x) - f(a)| < \varepsilon.\end{gathered} \tag{1}$$

Here, it is usually the case that the choice of δ depends on both ε and the point a. Often, it is very useful if we can choose a δ that only depends on ε and not on the point you are considering. However, we will see that this is not always possible. So introduce the following definition:

Definition 19.1. Let f be a real-valued function defined on a subset of $\mathbb{R}$.

(a) Let $S \subseteq \mathrm{dom}(f)$. Then f is said to be **uniformly continuous on** S if

$$\begin{gathered}\text{For each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ x, y \in S \text{ and } |x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon.\end{gathered} \tag{2}$$

(b) f is said to be **uniformly continuous** if it is uniformly continuous on $\mathrm{dom}(f)$.

We emphasize that uniform continuity is a property concerning a function and a set. So there is no point of speaking of uniform continuity at a point.

Example 19.1. (Lipschitz Continuity) A real-valued function f is said to be **Lipschitz continuous on** S if there exists a constant $L \in [0, \infty)$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all} \quad x, y \in S.$$

It is easy to check that Lipschitz continuity implies uniform continuity. Indeed, if f is Lipschitz continuous with the constant L, then for each given $\varepsilon > 0$ we may set $\delta = \frac{\varepsilon}{L+1}$ in (2).

Many natural examples of uniformly continuous functions are Lipschitz continuous, although they are not the same notions and we actually have exceptions.

(a) Consider the function $f(x) = \sin x$ on $\mathbb{R}$. Then by the Mean Value Theorem, for any distinct $x, y \in \mathbb{R}$, we can find ξ between x and y such that

$$|f(x) - f(y)| = |f'(\xi)(x - y)| = |\cos(\xi)| \cdot |x - y| \leq |x - y|.$$

So the sine function is Lipschitz continuous and hence uniformly continuous on $\mathbb{R}$.

(b) For each given $a > 0$, the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$. Indeed,

$$x, y \in [a, \infty) \qquad \Longrightarrow \qquad |f(x) - f(y)| = \frac{|x - y|}{xy} \leq \frac{|x - y|}{a^2}.$$

So f is Lipschitz continuous and hence uniformly continuous on $[a, \infty)$.

For the continuity, we noted that the sequential characterization allowed us to borrow many powerful machineries proved for the limit of sequences. The next result shows that it is also possible to characterize uniform continuity using sequences.

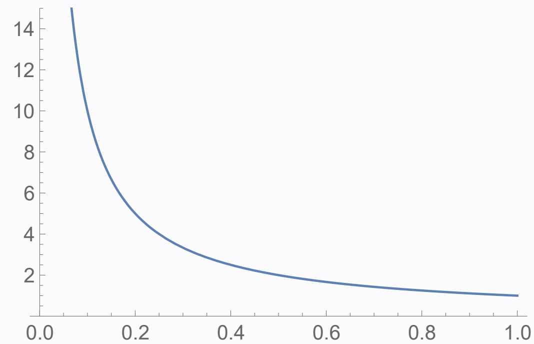
Theorem 19.2. Let f be a real-valued function defined on a subset of $\mathbb{R}$, and let $S \subseteq \mathrm{dom}(f)$. Then the followings are equivalent:

(a) f is uniformly continuous on S.

(b) For any sequences (x_n) and (y_n) in S satisfying $|x_n - y_n| \to 0$, we have $|f(x_n) - f(y_n)| \to 0$.

Example 19.2. (Not Uniformly Continuous)

(a) The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous. The intuition is that, as the point gets closer to the origin, the smaller the value of δ is.

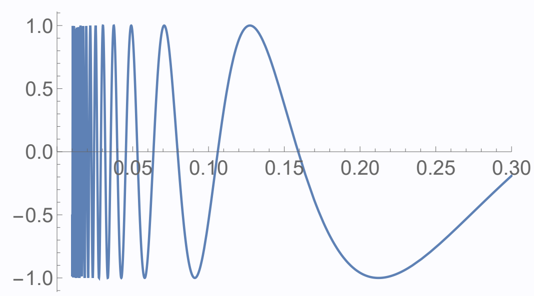


For a proof, choose $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Then

$$|x_n - y_n| = \frac{1}{n(n+1)} \to 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| = 1 \not\to 0.$$

So by Theorem 19.2, f cannot be uniformly continuous. This contrasts with Example 19.1, in which $f(x) = \frac{1}{x}$ is shown to be uniformly continuous on $[a, \infty)$ for any $a > 0$. This demonstrates that uniform continuity depends both on the function and the set considered.

(b) The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \cos(\frac{1}{x})$ is not uniformly continuous. The intuition is that its value fluctuates more abruptly as the point approaches the origin.



To prove this, choose $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+1)\pi}$. Then

$$|x_n - y_n| = \frac{1}{2n(2n+1)\pi} \to 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| = 2 \not\to 0.$$

So by Theorem 19.2, f cannot be uniformly continuous.

(c) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous. The intuition is that the function grows more rapidly as the point moves away towards $\pm\infty$.

Indeed, choose $x_n = n$ and $y_n = n + \frac{1}{n}$. Then

$$|x_n - y_n| = \frac{1}{n} \to 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| = 2 + \frac{1}{n} \not\to 0.$$

So by Theorem 19.2, f cannot be uniformly continuous.

The next theorem shows that the notion of uniform continuity coincides with that of continuity on closed bounded intervals.

Theorem 19.3. If f is continuous on a closed bounded interval $[a, b]$ of $\mathbb{R}$, then f is uniformly continuous on $[a, b]$.

Example 19.3. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Then f is uniformly continuous on $[0, 1]$, since it is continuous on the closed bounded interval $[0, 1]$.

The next result shows that uniformly continuous function preserves "Cauchy-ness".

Theorem 19.4. If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.

The next result shows that uniform continuity naturally allows to extend a function to a larger domain as if by "filling in missing values". This is possible because the uniform continuity prevents the function from wildly varying near each point.

Theorem 19.5. Suppose f is a function defined on a subset S of $\mathbb{R}$, and let $\overline{S}$ be the set of the limits of convergent sequences in S. Then the followings are equivalent:

(a) f is uniformly continuous on S.

(b) f extends to a uniformly continuous function $\tilde{f}$ on $\overline{S}$.

For instance, this result allows to extend any uniformly continuous function on $S = (a, b)$ to a continuous function on $\overline{S} = [a, b]$.

Proof. The direction (b) $\Rightarrow$ (a) is trivial from the definition, so we only prove (a) $\Rightarrow$ (b).

Define $\tilde{f} : \overline{S} \to \mathbb{R}$ by the following rule:

$$\text{if} \quad (x_n) \text{ in } S \text{ is convergent,} \quad \text{then} \quad \tilde{f}(\lim x_n) = \lim f(x_n).$$

We have to resolve several questions:

(i) Is $\tilde{f}$ well-defined? More specifically, why does $\lim f(x_n)$ always exist, and why does $\lim x_n = \lim y_n$ imply $\lim f(x_n) = \lim f(y_n)$?

A: Let (x_n) be a sequence in S that converges. Then (x_n) is a Cauchy sequence, hence by Theorem 19.4, $(f(x_n))$ is also a Cauchy sequence. So $\lim f(x_n)$ exists in $\mathbb{R}$.

Moreover, if (x_n) and (y_n) are convergent sequences in S and if $\lim x_n = \lim y_n$, then by Theorem 19.2, $|x_n - y_n| \to 0$ and hence $|f(x_n) - f(y_n)| \to 0$. So $\lim f(x_n) = \lim f(y_n)$.

Altogether, we have verified that the above rule yields a well-defined function $\tilde{f}$ on $\overline{S}$.

(ii) Is $\tilde{f}$ uniformly continuous on $\overline{S}$?

A: Given $\varepsilon > 0$, we use the condition (2) to choose $\delta > 0$ such that

$$x, y \in S \text{ and } |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \tfrac{1}{2}\varepsilon.$$

Now let $x, y \in \overline{S}$ satisfy $|x - y| < \delta$, and choose (x_n) and (y_n) in S such that $\lim x_n = x$ and $\lim y_n = y$. Then $|x_n - y_n| < \delta$ holds for all sufficiently large n's, and so, $|f(x_n) - f(y_n)| < \frac{1}{2}\varepsilon$ holds for all sufficiently large n's. Then by taking limit as $n \to \infty$,

$$|\tilde{f}(x) - \tilde{f}(y)| = \lim_{n\to\infty} |f(x_n) - f(y_n)| \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

This shows that $\tilde{f}$ is uniformly continuous.

(iii) Is $\tilde{f}$ the extension of the original function f?

A: For each $a \in S$, consider the constant sequence $x_n = a$. Then $\tilde{f}(a) = \lim f(x_n) = f(a)$ and hence $\tilde{f}$ extends f.

Therefore the proof is complete. □

Example 19.4. Consider the function $f(x) = x\sin(1/x)$ on $(0, 1]$. Then we can extend this to a continuous function $\tilde{f}$ on $[0, 1]$ as

$$\tilde{f}(x) = \begin{cases} x\sin(1/x), & \text{if } x \in (0, 1]; \\ 0, & \text{if } x = 0. \end{cases}$$

Since $\tilde{f}$ is continuous on $[0, 1]$, it Is uniformly continuous on $[0, 1]$. This proves that f is uniformly continuous on $(0, 1]$.

20. Limits of Functions

In this section we develop the notion of limit of function.

Definition 20.1. Let S be a subset of $\mathbb{R}$, let $a \in \overline{\mathbb{R}}$ be a limit of some sequence in S, and let $L \in \overline{\mathbb{R}}$. Then we write

$$\lim_{S \ni x \to a} f(x) = L$$

if the following conditions are satisfied.

(i) f is a function defined on S;

(ii) For every sequence (x_n) in S with limit a, we have $\lim_{n\to\infty} f(x_n) = L$.

- The expression "$\lim_{S\ni x\to a} f(x)$" is read "limit, as x tends to a along S, of $f(x)$".
- In light of the definition of continuity, we find that a function f is continuous at $a \in \operatorname{dom}(f)$ if and only if $\lim_{\operatorname{dom}(f)\ni x\to a} f(x) = f(a)$.
- Limits of a function are unique, because the same is true for limits of a sequence.

Next we connect Definition 20.1 to the traditional notation of function limits.

Definition 20.2. In each of the following cases, the limit $\lim_{S\ni x\to a} f(x) = L$ will be denoted by the given expression:

Expression	Setting
$\lim_{x\to a} f(x) = L$	• $a \in \mathbb{R}$; • $S = (a-\delta, a+\delta) \setminus \{a\}$ for some $\delta > 0$;
$\lim_{x\to a^+} f(x) = L$	• $a \in \mathbb{R}$; • $S = (a, a+\delta)$ for some $\delta > 0$;
$\lim_{x\to a^-} f(x) = L$	• $a \in \mathbb{R}$; • $S = (a-\delta, a)$ for some $\delta > 0$;
$\lim_{x\to+\infty} f(x) = L$	• $S = (c, +\infty)$ for some $c \in \mathbb{R}$;
$\lim_{x\to-\infty} f(x) = L$	• $S = (-\infty, c)$ for some $c \in \mathbb{R}$;

Here are some basic examples.

Example 20.1. In each of the following problems, determine whether the given limit exists. Also, if the limit exists, then find its value.

(a) $\displaystyle\lim_{x\to 0^+} \frac{1}{x}$ **(b)** $\displaystyle\lim_{x\to 0^-} \frac{1}{x}$ **(c)** $\displaystyle\lim_{x\to 0} \frac{1}{x}$

(d) $\displaystyle\lim_{x\to+\infty} \frac{1}{x}$ **(e)** $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x-2}$ **(f)** $\displaystyle\lim_{\mathbb{R}\ni x\to 2} \frac{x^2-4}{x-2}$

The next theorem is almost immediate from the definition and limit laws for sequences.

Theorem 20.3. Let f_1 and f_2 be functions for which both the limits $L_1 = \lim_{S \ni x \to a} f_1(x)$ and $L_2 = \lim_{S \ni x \to a} f_2(x)$ exist. Then

(a) $\lim_{S \ni x \to a}(f_1 + f_2)(x) = L_1 + L_2$ provided the right-hand side exists.

(b) $\lim_{S \ni x \to a}(f_1 f_2)(x) = L_1 L_2$ provided the right-hand side exists.

(c) $\lim_{S \ni x \to a}(f_1/f_2)(x) = L_1/L_2$ provided the right-hand side exists and $f_2(x) \neq 0$ for $x \in S$.

Theorem 20.4. Let f be a function for which the limit $b = \lim_{S \ni x \to a} f(x)$ exists, and let g be a function for which $\lim_{f(S) \ni y \to b} g(y) = L$ exists. Then $\lim_{S \ni x \to a}(g \circ f)(x) = L$.

20.1. Alternative criteria of limits of functions

In Section 17, we proved that the definition of continuity using sequences is equivalent to an ε–δ property. By imitating the proof therein, we can formulate the definition of limit of function in a similar fashion:

Theorem 20.5. Let f be a function defined on a subset S of $\mathbb{R}$, let $a \in \mathbb{R}$ be a limit of some sequence in S, and let $L \in \mathbb{R}$. Then $\lim_{S \ni x \to a} f(x) = L$ if and only if

$$\begin{gathered} \text{For each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ x \in S \text{ and } |x-a| < \delta \text{ imply } |f(x) - L| < \varepsilon. \end{gathered} \tag{3}$$

- *Generalization.* Similar statements can be obtained for the case $a = \pm\infty$ and/or $L = \pm\infty$. Indeed, $\lim_{S \ni x \to a} f(x) = L$ will be equivalent to the statement of the form

$$\begin{gathered} \text{For each } \underline{\quad \text{(A)} \quad}, \text{ there exists } \underline{\quad \text{(B)} \quad} \text{ such that} \\ x \in S \text{ and } \underline{\quad \text{(C)} \quad} \text{ imply } \underline{\quad \text{(D)} \quad} \end{gathered}$$

  where the choices of (A)–(D) are determined according to the tables

	$L = -\infty$	$L \in \mathbb{R}$	$L = +\infty$
(A)	$M \in \mathbb{R}$	$\varepsilon > 0$	$M \in \mathbb{R}$
(D)	$f(x) < M$	$\lvert f(x) - L\rvert < \varepsilon$	$f(x) > M$

  and

	$a = -\infty$	$a \in \mathbb{R}$	$a = +\infty$
(B)	$R \in \mathbb{R}$	$\delta > 0$	$R \in \mathbb{R}$
(C)	$x < R$	$\lvert x - a\rvert < \delta$	$x > R$

- *Further Generalization.* More formally, for each $a \in \overline{\mathbb{R}}$, define the collection $\mathcal{N}_a$ of subsets of $\mathbb{R}$ by

$$\mathcal{N}_a = \begin{cases} \{U \subseteq \mathbb{R} : \text{there exists } r \in \mathbb{R} \text{ s.t. } (-\infty, r) \subseteq U\}, & \text{if } a = -\infty; \\ \{U \subseteq \mathbb{R} : \text{there exists } \delta > 0 \text{ s.t. } (a-\delta, a+\delta) \subseteq U\}, & \text{if } a \in \mathbb{R}; \\ \{U \subseteq \mathbb{R} : \text{there exists } r \in \mathbb{R} \text{ s.t. } (r, +\infty) \subseteq U\}, & \text{if } a = +\infty; \end{cases}$$

  Each element of the collection $\mathcal{N}_a$ is called a **neighborhood of** a. Using these collections of neighborhoods, we can provide a unified statement for a criterion of limit:

  Observation. $\lim_{S \ni x \to a} f(x) = L$ if and only if

$$\begin{gathered} \text{For any } U \in \mathcal{N}_L, \text{ there exists } V \in \mathcal{N}_a \text{ such that} \\ x \in S \cap V \text{ implies } f(x) \in U \end{gathered} \tag{4}$$

  Now by playing with the choice of the collections of neighborhoods, we can produce a myriad of different types of limits. This is also how the limit is usually defined in general topology.