UCLA Math151A Fall 2021
Lecture 9
20211013

Multiple Roots, Modified Newton, New topic: Interpolation

Optional reading: book 2.4, 3.1

Example: Difficulty with Newton

$$f(x) = x^2$$

$$f(0) = 0 \quad f'(0) = 0$$

0 is a double root of f.

$$f'(x) = 2x$$

$f'(p) = 0$ and $f(p) = 0$,

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2}{2x} = \frac{x}{2}$$

$$g(p) = p$$

$$g'(x) = 1 - \left(\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} \right) = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(x) = \frac{1}{2}$$

Theorem 8.1 (The case where we get linear convergence.).

Let $g \in C^1([a,b])$ with $|g'(x)| \leq k$ for some $0 < k < 1$.
If $g'(p) \neq 0$, then F.P.I. converges to p linearly. $\square$

If $f'(p) = 0$ and $f(p) = 0$, then p is called a **multiple root** of f.

Definition (Multiple Root).

A root of $f(x) = 0$, p, is called a root of multiplicity m of f $\Leftrightarrow$ for $x \neq p$, there exists decomposition

$$f(x) = (x-p)^m q(x) \text{ where } \lim_{x\to p} q(x) \neq 0.$$

If the multiplicity of a root p is 1, then p is called a simple zero.

Theorem Let $f \in C^m([a,b])$, $p \in [a,b]$,
then p is a root of multiplicity m $\Leftrightarrow$
$f(p) = f'(p) = f''(p) = \cdots = f^{m-1}(p) = 0$ but $f^{(m)}(p) \neq 0$.

Proof: see extra reading material

Theorem Let $f \in C^m([a,b])$, $p \in [a,b]$,
then p is a root of multiplicity $m \Leftrightarrow$
for $x \neq p$, there exists decomposition

$$f(x) = (x-p)^m q(x) \text{ where } \lim_{x \to p} q(x) \neq 0. \quad \Leftrightarrow$$

$$f(p) = f'(p) = f''(p) = \cdots = f^{m-1}(p) = 0 \text{ but } f^{(m)}(p) \neq 0.$$

Example

$$f(x) = x^2, \qquad f'(x) = 2x, \qquad f''(x) = 2 \neq 0,$$

$$p = 0, m = 2.$$

$$f(x) = (x-0)^2 \cdot 1, \quad q(x) = 1$$

Theorem Let $f \in C^m([a,b])$, $p \in [a,b]$,
then p is a root of multiplicity $m \Leftrightarrow$
for $x \neq p$, there exists decomposition

$$f(x) = (x-p)^m q(x) \text{ where } \lim_{x\to p} q(x) \neq 0. \quad \Leftrightarrow$$

$$f(p) = f'(p) = f''(p) = \cdots = f^{m-1}(p) = 0 \text{ but } f^{(m)}(p) \neq 0.$$

Example $f(x) = e^{x^2} - 1$

$$f(0) = 0$$

$$f'(x) = 2xe^{x^2}$$

$$f'(0) = 0$$

$$f''(x) = 2e^{x^2} + 4x^2 e^{x^2}$$

$$f''(0) = 2$$

$$p = 0, m = 2$$

$$f(x) = (x - 0)^2 \frac{e^{x^2} - 1}{x^2}$$

$$q(x) = \frac{e^{x^2} - 1}{x^2}$$

$$\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + O(x^8) - 1}{x^2} = 1$$

Theorem Let $f \in C^m([a,b])$, $p \in [a,b]$,
then p is a root of multiplicity $m \Leftrightarrow$

for $x \neq p$, there exists decomposition

$$f(x) = (x-p)^m q(x) \text{ where } \lim_{x \to p} q(x) \neq 0. \quad \Leftrightarrow$$

$$f(p) = f'(p) = f''(p) = \cdots = f^{m-1}(p) = 0 \text{ but } f^{(m)}(p) \neq 0.$$

How does this relate to N.M.?

We know N.M. suffers when

$$f(p) = 0, f'(p) = 0.$$

I.e., when we have a root of multiplicity m larger than 1.

Theorem Let $f \in C^m([a, b])$, $p \in [a, b]$,
then p is a root of multiplicity $m \Leftrightarrow$
for $x \neq p$, there exists decomposition

$$f(x) = (x-p)^m q(x) \text{ where } \lim_{x \to p} q(x) \neq 0. \quad \Leftrightarrow$$

$$f(p) = f'(p) = f''(p) = \cdots = f^{m-1}(p) = 0 \text{ but } f^{(m)}(p) \neq 0.$$

What if we create another function that also has root at p, but with multiplicity 1?

Theorem Let $f \in C^m([a,b])$, $p \in [a,b]$,
then p is a root of multiplicity $m \Leftrightarrow$

for $x \neq p$, there exists decomposition

$$f(x) = (x-p)^m q(x) \text{ where } \lim_{x\to p} q(x) \neq 0. \quad \Leftrightarrow$$

$$f(p) = f'(p) = f''(p) = \cdots = f^{m-1}(p) = 0 \text{ but } f^{(m)}(p) \neq 0.$$

let's introduce a very cool function $\mu(x) := \dfrac{f(x)}{f'(x)}$.

$$f'(x) = m(x-p)^{m-1}q(x) + (x-p)^m q'(x),$$

$$\Rightarrow \quad \mu(x) = (x-p)\frac{q(x)}{mq(x) + (x-p)q'(x)}.$$

$$\mu(p) = 0 \qquad \frac{q(p)}{mq(p)+(p-p)q'(p)} = \frac{1}{m} \neq 0$$

$\Rightarrow$ $\mu(x)$ has root p with multiplicity 1! $\boxed{\mu'(p) \neq 0.}$

Method 9.1 (Modifeid N.M.). Given p_0, define

$$\mu(x) := \frac{f(x)}{f'(x)},$$

$$p_{n+1} = p_n - \frac{\mu(p_n)}{\mu'(p_n)}$$

$$\Rightarrow \quad p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{(f'(p_n))^2 - f(p_n)f''(p_n)}.$$

This allows us to find p without worrying about division by zero.

Drawback: more computations, second derivative evaluation.

New Topic of the Course: Interpolation

Goal: Given n discrete points,

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)),$$

want to find polynomial P(x)

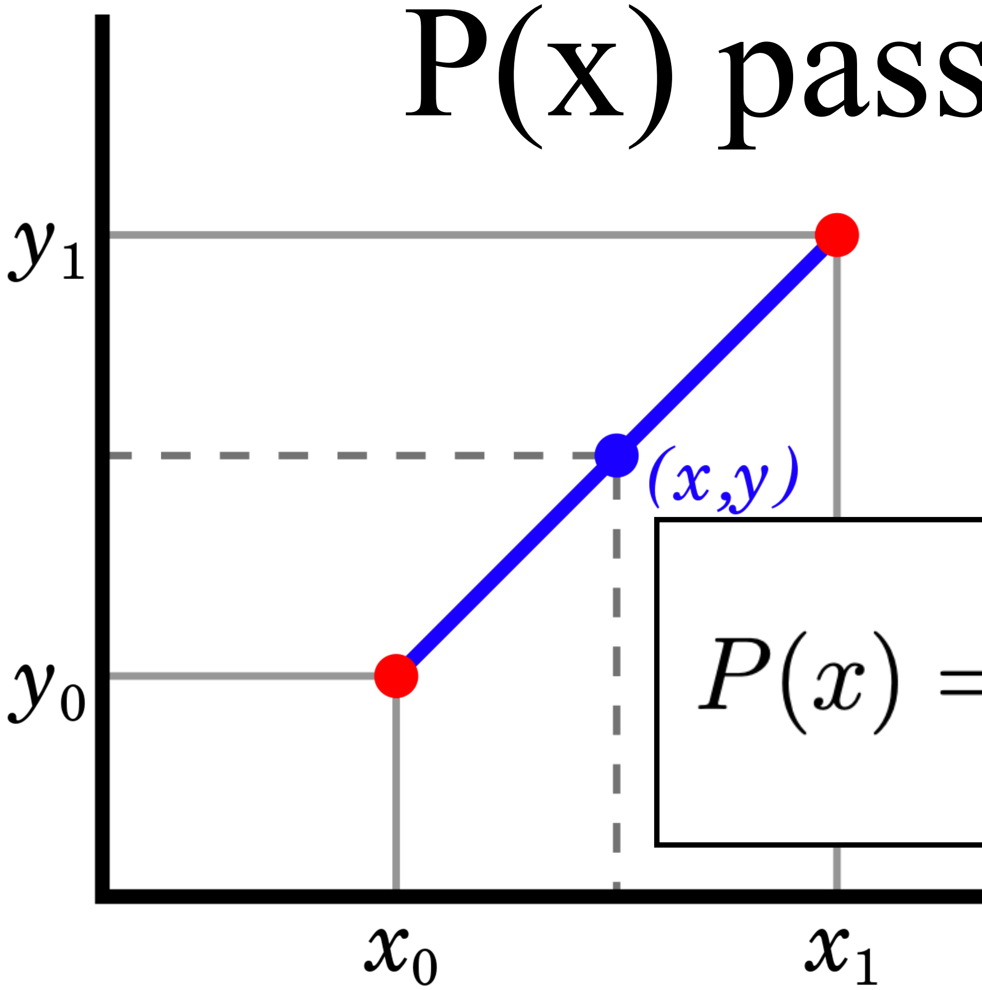
$$P(x) \;\; = f(x), \qquad \text{at } x = x_i, \; \forall 0 \leq i \leq n.$$

How? Lagrangian polynomials. Given n + 1 data points, these will produce a polynomial of degree n.

E.g., 1 data point gives a constant function, 2 gives a line, etc.

Example 9.1 (Linear interpolation).

P(x) passes through the two points



$$P(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0},$$

clearly $P(x_0) = f(x_0)$, $P(x_1) = f(x_1)$.

Example 9.1 (Linear interpolation).

$$\boxed{P(x) = f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0},}$$

Strategy:

Sum up polynomials so that each piece vanishes at other data points.

$$L_0(x_i) = \delta_{i0}$$
$$L_1(x_i) = \delta_{i1}.$$

$$L_0(x) := \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) := \frac{x - x_0}{x_1 - x_0}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then $P(x) = f(x_0)L_0(x) + f(x_1)L_1(x)$.

With n points

suppose we have $n+1$ distinct points, $i = 0, 1, 2, \ldots, n$.

Then we define

$$L_i(x) := \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)},$$

or more compactly,

$$L_i(x) = \Pi_{j=0,j\neq i}^{n} \frac{x-x_j}{x_i-x_j} \qquad 0 \leq i \leq n. \qquad L_i(x_j) = \delta_{ij}.$$

Definition 9.1. A Lagrangian polynomial of degree n of $f(x)$ is:

$$P(x) = \sum_{i=0}^{n} f(x_i) L_i(x).$$

Example

https://www.wolframalpha.com/input/?
i=interpolating+polynomial+calculator&a
ssumption=%7B%22F%22%2C+
%22InterpolatingPolynomialCalculator%
22%2C+%22data%22%7D+-
%3E%22%7B1%2C+3%2C+7%2C+2%2
C9%7D%22