14. Series

In this section, we define the notion of infinite series and study its convergence.

14.1. Notation and Definitions

- *Finite Summation.* We define the (finite) summation notation by

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n.$$

More formally, the finite summation can be defined recursively by

$$\sum_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \sum_{k=m}^{n+1} a_k = \left(\sum_{k=m}^{n} a_k \right) + a_{n+1}.$$

- *Infinite Series.* We assign meaning to the symbol $\sum_{n=m}^{\infty} a_n$ by the following way: Consider a sequence $(a_n)_{n=m}^{\infty}$ and define its **partial sums** by

$$s_n = \sum_{k=m}^{n} a_k.$$

Then the **(infinite) series** $\sum_{k=m}^{\infty} a_k$ is said to **converge** provided the sequence of partial sums (s_n) converge to a real number, in which case, we define

$$\sum_{k=m}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=m}^{n} a_k.$$

A series that do not converge is said to **diverge**. Two particular scenarios deserves special attentions:

  - We say that $\sum_{k=m}^{\infty} a_k$ **diverges to** $+\infty$ and write $\sum_{k=m}^{\infty} a_k = +\infty$ provided $\lim s_n = +\infty$.
  - We say that $\sum_{k=m}^{\infty} a_k$ **diverges to** $-\infty$ and write $\sum_{k=m}^{\infty} a_k = -\infty$ provided $\lim s_n = -\infty$.

Finally, the symbol $\sum_{n=m}^{\infty} a_n$ is left undefined unless the series converges or diverges to $+\infty$ or $-\infty$.

- *Conventions.*
  - Sometimes we write $\sum_n a_n$ for $\sum_{n=m}^{\infty} a_n$ when the start index is not important.
  - Sometimes the infinite series $\sum_{n=m}^{\infty} a_n$ is denoted using the symbolic expression

$$a_m + a_{m+1} + a_{m+2} + \cdots .$$

- $\sum_n a_n$ is said to **converge absolutely** if $\sum_n |a_n|$ converges.
- **A Fair Warining.** Infinite series does not inherit all the nice properties of finite summation. So the reader should not attempt to manipulate an infinite series just like what we have done for finite summations. Any manipulation on infinite series needs to be justified based on the definition of infinite series as the **limit of partial sums**.

Example 14.1. (Geometric Series) A series of the form

$$\sum_{n=0}^{\infty} ar^n$$

for constants a and r is called a **geometric series**. To study the convergence of a geometric series, it is useful to know:

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1} = \begin{cases} \dfrac{1-r^n}{1-r}, & \text{if } r \neq 1; \\ n, & \text{if } r = 1; \end{cases}$$

For $|r| < 1$, taking limit to both sides of this identity and utilizing $\lim_{n\to\infty} r^n = 0$ proves that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \qquad \text{if} \quad |r| < 1. \tag{1}$$

When $|r| \geq 1$, the sequence $(r^n)_{n=0}^{\infty}$ does not converge to 0, so the series $\sum_n r^n$ diverges by the General Term Test.

The next result collects some immediate consequences of the definition of series.

Theorem 14.1.

(a) If both $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ converge, then $\sum_{n=m}^{\infty}(a_n + b_n)$ also converges and

$$\sum_{n=m}^{\infty}(a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n$$

(b) If $\sum_{n=m}^{\infty} a_n$ converges and k is a real number, then $\sum_{n=m}^{\infty} ka_n$ also converges and

$$\sum_{n=m}^{\infty} ka_n = k \sum_{n=m}^{\infty} a_n.$$

(c) If both $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ exist in $\overline{\mathbb{R}}$, and if $a_n \leq b_n$ for all $n \geq m$, then

$$\sum_{n=m}^{\infty} a_n \leq \sum_{n=m}^{\infty} b_n.$$

Moreover, the inequality becomes strict if $a_n < b_n$ holds for some $n \geq m$.

(d) $\sum_{n=m}^{\infty} a_n$ converges if and only if $\sum_{n=M}^{\infty} a_n$ converges for some $M > m$. If this is the case, then the two series are related by

$$\sum_{n=m}^{\infty} a_n = \left(\sum_{n=m}^{M-1} a_n\right) + \left(\sum_{n=M}^{\infty} a_n\right).$$

14.2. Basic Convergence Tests

Theorem 14.2. If $a_n \geq 0$ for all n, then $\sum_n a_n$ always exists in $[0, +\infty]$.

Proof. By the assumption, its sequence (s_n) of partial sums is non-negative and increasing. Therefore $\lim s_n$ always exists in $[0, +\infty]$. □

The next result is an immediate consequence of Theorem 10.10 of Note 7, and it highlights a useful idea for investigating the convergence of a series.

Theorem 14.3. A series $\sum_n a_n$ converges if and only if the following condition is satisfied:

$$\begin{gathered} \text{for each } \varepsilon > 0, \quad \text{there exists a number } N \quad \text{such that} \\ n \geq m > N \quad \text{implies} \quad \left| \sum_{k=m}^{n} a_k \right| < \varepsilon. \end{gathered} \tag{2}$$

The condition (2) is often called the **Cauchy criterion** for series.

Proof. Write (s_n) for the sequence of partial sums of (a_n). Then we have:

$$\begin{aligned} (s_n) \text{ converges} \quad &\iff \quad (s_n) \text{ is a Cauchy sequence} \\ &\iff \quad \forall \varepsilon > 0,\ \exists N \text{ s.t. } [m, n > N \implies |s_m - s_n| < \varepsilon] \\ &\iff \quad \forall \varepsilon > 0,\ \exists N \text{ s.t. } [n > m > N \implies |s_n - s_m| < \varepsilon] \\ &\iff \quad \forall \varepsilon > 0,\ \exists N \text{ s.t. } [n \geq m > N \implies |s_n - s_{m-1}| < \varepsilon]. \end{aligned}$$

Indeed,

- The first step is a consequence of Theorem 10.10 of Note 7.
- The third step follows by the symmetry in the roles of m and n in the defining condition for the Cauchy sequence.
- The last step follows by replacing m and N by $m-1$ and $N-1$, respectively.

Therefore we conclude by noting that the last condition is precisely (2). □

As an immediate corollary, we obtain the next result. It is sometimes called the General Term Test or the N-th Term Test.

Theorem 14.4. If a series $\sum_n a_n$ converges, then $\lim a_n = 0$. Equivalently, if the sequence (a_n) does not converge to 0, then the series $\sum_n a_n$ diverges.

Remark. The converse of the General Term Test does not hold, as we see from the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

The next result is extremely useful throughout.

Theorem 14.5. (Comparison Test)

(a) If $|b_n| \le a_n$ for all n and $\sum_n a_n$ converges, then $\sum_n b_n$ also converges.

(b) If $b_n \ge a_n \ge 0$ for all n and $\sum_n a_n = +\infty$, then $\sum b_n = +\infty$.

Exercise 14.2. (Limit Comparison Test)

Consider a series $\sum_n a_n$ where $a_n > 0$ for all n. Prove the following statements:

(a) If $\limsup |b_n|/a_n < +\infty$ and $\sum_n a_n$ converges, then $\sum_n b_n$ also converges.

(b) If $\liminf b_n/a_n > 0$ and $\sum_n a_n = +\infty$, then $\sum b_n = +\infty$.

The following corollary is sometimes called the Absolute Convergence Test.

Theorem 14.6. Absolutely convergent series are convergent.

14.3. p-Series

The next result provides an elementary tool for inspecting the convergence of certain types of series.

Theorem 14.7. (Cauchy Condensation Test) Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$$

converges.

Example 14.3. (p-Series) A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for a constant p is called a p-**series**. Let us study the convergence of the p-series.

- For $p \leq 0$, we have $n^{-p} \geq 1$ for all n, and so, n^{-p} cannot converge to 0. Hence $\sum_{n=1}^{\infty} n^{-p}$ diverges by the General Term Test.
- For $p > 0$, the sequence $a_n = n^{-p}$ is non-negative and decreasing. Moreover,

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^{(1-p)k}$$

is a geometric series, hence it converges precisely when $2^{1-p} < 1$, or equivalently, $p > 1$. So by the Cauchy Condensation Test, $\sum_{n=1}^{\infty} n^{-p}$ converges if and only if $p > 1$.

14.4. Root and Ratio Test

Theorem 14.8. (Root Test) Let $\sum_n a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$.

(a) If $\alpha < 1$, then the series $\sum_n a_n$ converges absolutely.

(b) If $\alpha > 1$, then the series $\sum_n a_n$ diverges.

(c) If $\alpha = 1$, then the test gives no information.

Theorem 14.9. (Ratio Test) Let $\sum_n a_n$ be a series of non-zero terms.

(a) If $\limsup |a_{n+1}/a_n| < 1$, then the series $\sum_n a_n$ converges absolutely.

(b) If $\liminf |a_{n+1}/a_n| > 1$, then the series $\sum_n a_n$ diverges.

(c) Otherwise, then the test gives no information.

Example 14.4. Study the convergence of the following series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ **(c)** $\displaystyle\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots.$

15. Conditional Convergence

All the tests that we have covered in the previous section only covers absolutely convergent series. However, it turns out that some convergent series do not converge absolutely.

- A series $\sum_n a_n$ is said to **converge conditionally** if $\sum_n a_n$ converges but $\sum_n |a_n| = +\infty$.
- For a conditionally convergent series, the convergence hinges on a delicate cancellation between positive terms and negative terms as the partial sum progresses. As such, this cancellation behavior is easily disrupted by rearranging the order of terms. We will briefly mention a related result in this direction, which shows a stark difference between finite summation and infinite series.

15.1. Alternating Series Test

Lemma 15.1. (Summation by Parts) Consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, and write

$$A_0 = 0, \qquad A_n = \sum_{k=1}^{n} a_k$$

for the sequence of partial sums of (a_n). Then for any integers $1 \leq m \leq m$, we have

$$\sum_{k=m}^{n} a_k b_k = (A_n b_n - A_{m-1} b_m) + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1})$$

This is a summation analogue of the integration by parts formula.

Proof. Note that $a_n = A_n - A_{n-1}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} (A_k - A_{k-1}) b_k &= \sum_{k=m}^{n} A_k b_k - \sum_{k=m}^{n} A_{k-1} b_k \\ &= \left(A_n b_n + \sum_{k=m}^{n-1} A_k b_k \right) - \left(A_{m-1} b_m + \sum_{k=m}^{n-1} A_k b_{k+1} \right).\end{aligned}$$

Simplifying the last line proves the summation by parts formula.

The summation by parts formula allows us to prove the next result.

Theorem 15.2. (Dirichlet's Test) Consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. Suppose

- **(i)** The sequence $A_n = \sum_{k=1}^{n} a_k$ of partial sums of (a_n) is bounded;
- **(ii)** $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$;
- **(iii)** $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

As a corollary, we obtain:

Theorem 15.3. (Alternating Series Test) Consider a sequence $(b_n)_{n\in\mathbb{N}}$. Suppose

(i) $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$;

(ii) $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

Proof. Apply the Dirichlet's Test with $a_n = (-1)^{n-1}$. □

Example 15.1. Study the convergence of the following series

$$\textbf{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad\qquad \textbf{(b)} \sum_{n=1}^{\infty} \frac{\cos(2n\pi/3)}{\sqrt{n}}$$

15.2. (Optional) Rearrangements

- In finite summation, it is quite clear that rearranging the order of summation does not alter the value of the sum, owing to the commutative law of addition. Does this observation persist for infinite series? Surprisingly, the answer depends on the mode of convergence of the series.
- Let us begin by formalizing the meaning of rearranging a series. Consider a sequence $(a_n)_{n\in\mathbb{N}}$. Then any sequence of the form $(a_{\sigma(n)})_{n\in\mathbb{N}}$ for some bijection[1] $\sigma : \mathbb{N} \to \mathbb{N}$ is called a **rearrangement** of $(a_n)_{n\in\mathbb{N}}$.

The next result tells that the answer is affirmative for absolutely convergent series.

Theorem 15.4. If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely to $S \in \mathbb{R}$, then any of its rearrangement $\sum_{n=1}^{\infty} a_{\sigma(n)}$ also converges absolutely to S.

Proof. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection, and write $s_n = \sum_{k=1}^{n} a_k$ and $\tilde{s}_n = \sum_{k=1}^{n} a_{\sigma(k)}$. Given $\varepsilon > 0$, there exists N such that

$$n \geq m > N \quad \implies \quad \sum_{k=m}^{n} |a_k| < \varepsilon.$$

Now choose $\tilde{N} \geq N$ such that all of the integers $1, 2, \ldots, N$ are contained in the list $\sigma(1), \sigma(2), \ldots, \sigma(\tilde{N})$. Then for any $n > \tilde{N}$, the terms $a_1, a_2, \ldots, a_N$ will cancel out in the difference $\tilde{s}_n - s_n$, and so, we get

$$|\tilde{s}_n - s_n| < \epsilon.$$

This shows that $\tilde{s}_n - s_n$ converges to 0 and therefore the conclusion follows. □

The next result, named after the famous mathematiciann Bernhard Riemann, tells that we can exploit the cancellation behavior of conditionally convergent series to rearrange the series so that the new series assumes different behaviors than the original series.

Theorem 15.5. (Riemann Rearrangement Theorem) Suppose $\sum_n a_n$ converges conditionally. Then for any $-\infty \leq \alpha \leq \beta \leq \infty$, we can find a rearrangement $(a_{\sigma(n)})_{n\in\mathbb{N}}$ such that

$$\liminf_{n\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} = \alpha \quad \text{and} \quad \limsup_{n\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} = \beta.$$

Its proof is rather lengthy, but the idea is as follows:[2]

Sketch of Proof. Write

$$a_n^+ = \frac{|a_n| + a_n}{2} \qquad \text{and} \qquad a_n^- = \frac{|a_n| - a_n}{2}$$

for the positive and negative parts of a_n, respectively. Since $\sum_n a_n$ converges conditionally, we

[1] A function $f : A \to B$ is called bijective if it is both one-to-one (injective) and onto (surjective).

[2] For an actual proof, check: Rudin, *"Principles of Mathematical Analysis"*, 3rd Ed., McGraw-Hill, 1976, pp76–77.

have

$$\sum_n a_n^+ = +\infty, \qquad \sum_n a_n^- = +\infty, \qquad \lim_{n\to\infty} a_n^+ = 0, \qquad \lim_{n\to\infty} a_n^- = 0.$$

In other words, both the partial sums $P_n = \sum_{k=1}^n a_k^+$ and $N_n = \sum_{k=1}^n a_k^-$ diverges to $+\infty$ but the rate of increase tends to slow down as the sequence progresses. Using this, we can alternatively splice positive terms and negative terms of (a_n) so that the partial sums of the rearranged series almost alternates between α and β.