# UCLA Math151A Fall 2021
# Lecture 7
# 20211008

# Secant Method,
# more importantly,
# [Newton Convergence Theorem]

Optional reading: book 2.3.

$\boxed{\textbf{Secant Method}}$

Recall Newton's Method (N.M.) is defined as:

$$\text{Given } p_0, \qquad p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

This requires evaluation of $f'$.

In general, this could be expensive or unknown.

E.g., if in higher dimensions
or if $f(x)$ comes from experimental data
(no analytical expression for $f$).

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.}$$

Recall the definiton of the derivative $f'(x)$.

$$f'(x) := \lim_{h \to 0} \frac{f(x) - f(x-h)}{x - (x-h)} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}.$$

So when $h$ is small,

the derivative can be approximaated by "fintie difference":

$$f'(x) \approx \frac{f(x) - f(x-h)}{x - (x-h)}$$

So if we let $x = p_n$, and $x - h = p_{n-1}$, then this becomes

$$\boxed{f'(p_n) \approx \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}},}$$

which holds true when $p_n - p_{n-1}$ is small.

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

$$f'(p_n) \approx \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}},$$

**Definition 7.1** (Secant Method)**.** Given $p_0, p_1$, Secant Method does

$$\boxed{p_{n+1} = p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})},}$$

where the fraction is approximating $(f'(p_n))^{-1}$.

# How to get p1?

running one iteration of Bisection Method (B.M.), e.g.

# Local Convergence of Newton's Method

**Theorem 7.1** (Newton Converges for Sufficently Close Initial Guess).

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$,

(ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

I.e., $p_n \to p$ as $n \to \infty$ $\square$

Unforutnately there is no guideline to find the exact $\delta$.

(Therefore you don't know what close-enough means in practice unfortunately.)

**Newton Convergence Theorem**

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

The idea of the proof is to apply the Fixed Point Iteration (F.P.I.) theorem from our previous lectures to some to-be-defined function g.

To do this, need to show that g satisfies some assumptions. (But what is g?)

**Newton Convergence Theorem**

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

# Recall the F.P.I. convergence theorem from lecture 5:

**Fixed Point Iteration Convergence Theorem**

$(g \in C([a,b]), g \in [a,b] \quad (*))$

$\exists k \in (0,1)$ s.t. $|g(x) - g(y)| \leq k|x-y|, \ \forall x, y \in [a,b]$ $\quad (**)$

Or $g \in C^1[a,b]$ and that $\forall x \in [a,b]$, $\exists k \in (0,1)$ s.t. $|g'(x)| \leq k$, $\quad (**)$

1. $\exists$ unique $p$ s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to $p$.
3. Error estimate: $|p_n - p| \leq k^n \max\{b - p_0, p_0 - a\}$.

**Key conditions to satisfy:** $(I)[\hat{a}, \hat{b}] \to [\hat{a}, \hat{b}]$ (II) $g$ is $C^1$; (III) $g$ has bounded derivative with bound in $(0,1)$.

$[\hat{a}, \hat{b}] \in [a, b]$

## Newton Convergence Theorem

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p)=0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

Key conditions to satisfy: $(I)[\hat{a},\hat{b}] \to [\hat{a},\hat{b}]$ (II) $g$ is $C^1$;

(III) $g$ has bounded derivative with bound in $(0,1)$.

Define $g(x) := x - \dfrac{f(x)}{f'(x)}$.

N.M. on $f(x)$ is the same as F.P.I. on $g(x)$:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad \Leftrightarrow \quad g(p_n) = p_{n+1}.$$

Therefore we just need to show the three postulates about $g$.

## Newton Convergence Theorem

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

Key conditions to satisfy: $(I)[\hat{a}, \hat{b}] \to [\hat{a}, \hat{b}]$ (II) $g$ is $C^1$;

(III) $g$ has bounded derivative with bound in $(0,1)$.

Define $g(x) := x - \dfrac{f(x)}{f'(x)}$.

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**Show (II):** $f \in C^2([a,b])$,

so $f \in C([a,b])$ and $f' \in C([a,b])$ and $f'' \in C([a,b])$.

Let's compute $g'(x)$

$$g'(x) = 1 - \left( \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} \right) = \frac{f(x)f''(x)}{(f'(x))^2}$$

there exists a region $[p-\delta_1, p+\delta_1]$ in $[a,b]$ such that $f'(x) \neq 0$.

$\Rightarrow$ $g'$ is continous in $[p-\delta_1, p+\delta_1]$ $\boxed{\text{This proves (II).}}$

Newton Convergence Theorem

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

Key conditions to satisfy: $(I)[\hat{a}, \hat{b}] \to [\hat{a}, \hat{b}]$ (II) $g$ is $C^1$;

(III) $g$ has bounded derivative with bound in $(0,1)$.

Define $g(x) := x - \dfrac{f(x)}{f'(x)}$.

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# Show (III):

$$g'(x) = 1 - \left(\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}\right) = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(p) = 0.$$

Due to continuity of $g'$ in $[p-\delta_1, p+\delta_1]$, there exists a region (with $0 < \delta < \delta_1$) s.t. $|g'(x)| \leq k$ in $[p-\delta, p+\delta]$ for any $k \in (0,1)$.

$\boxed{\text{This proves (III).}}$

**Newton Convergence Theorem**

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

Key conditions to satisfy: $(I)[\hat{a}, \hat{b}] \to [\hat{a}, \hat{b}]$ (II) $g$ is $C^1$;

(III) $g$ has bounded derivative with bound in $(0,1)$.

Define $g(x) := x - \dfrac{f(x)}{f'(x)}$.

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# Show (I):

need to prove $g$ maps $[p-\delta, p+\delta]$ to $[p-\delta, p+\delta]$

$$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)||x - p| \leq k|x-p|$$

M.V.T. $\xi$ between $x$ and $p$ $\qquad < |x-p|$.

Thus when $x \in [p-\delta, p+\delta]$,
$g$ must also be in $[p-\delta, p+\delta]$.

$\boxed{\text{This proves (I).}}$

## Newton Convergence Theorem

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p) = 0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

Key conditions to satisfy: $(I)[\hat{a},\hat{b}] \to [\hat{a},\hat{b}]$ (II) $g$ is $C^1$;

(III) $g$ has bounded derivative with bound in $(0,1)$.

Define $g(x) := x - \dfrac{f(x)}{f'(x)}$.

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N.M. on $f(x)$ is the same as F.P.I. on $g(x)$:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad \Leftrightarrow \quad g(p_n) = p_{n+1}.$$

Now we proved that F.P.I. converges to $p$ for any $p_0 \in [p-\delta, p+\delta]$.

Equivalently, N.M. converges for $f$ at $p$. ■

**Newton Convergence Theorem**

Let $f \in C^2([a,b])$, and $p \in (a,b)$ s.t. (i) $f(p)=0$, (ii) $f'(p) \neq 0$.

Then $\exists \delta > 0$ s.t. N.M. will converge for $\forall p_0 \in [p-\delta, p+\delta]$.

**Remark** $\delta$ cannot be a priori measured.

So in practice, we can:

- Begin with some $p_0 \in [a,b]$
- Run several iterations of B.M. (a global method)
- Switch to N.M.

**Example** $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. $\boxed{1.365230013}$

**(a)** $x = g_1(x) = x - x^3 - 4x^2 + 10$

**(b)** $x = g_2(x) = \left(\dfrac{10}{x} - 4x\right)^{1/2}$

**(c)** $x = g_3(x) = \dfrac{1}{2}(10 - x^3)^{1/2}$

**(d)** $x = g_4(x) = \left(\dfrac{10}{4 + x}\right)^{1/2}$

**(e)** $x = g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

| $n$ | $(a)$ | $(b)$ | $(c)$ | $(d)$ | $(e)$ |
|---|---|---|---|---|---|
| 0 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 |
| 1 | −0.875 | 0.8165 | 1.286953768 | 1.348399725 | 1.373333333 |
| 2 | 6.732 | 2.9969 | 1.402540804 | 1.367376372 | 1.365262015 |
| 3 | [illegible] | [illegible] | [illegible] | [illegible]57015 | 1.365230014 |
| 4 | [illegible] | [illegible] | [illegible] | [illegible]64748 | 1.365230013 |
| 5 | [illegible] | [illegible] | [illegible] | [illegible]25594 | $\boxed{\text{excellent}}$ |
| 6 | [illegible] | [illegible] | [illegible] | [illegible]30576 | |
| 7 | [illegible] | [illegible] | [illegible] | [illegible]29942 | |
| 8 | [illegible] | [illegible] | [illegible] | [illegible]30022 | |
| 9 | | | 1.364878217 | 1.365230012 | |
| 10 | | | 1.365410062 | 1.365230014 | |
| 15 | | | 1.365223680 | 1.365230013 | |
| 20 | | | 1.365230236 | $\boxed{\text{excellent}}$ | |
| 25 | | | 1.365230006 | | |
| 30 | | | 1.365230013 | $\boxed{\text{excellent}}$ | |

Define $g(x) := x - \frac{f(x)}{f'(x)}$.

N.M. on $f(x)$ is the same as F.P.I. on $g(x)$:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad \Leftrightarrow \quad g(p_n) = p_{n+1}.$$

Next time:

Convergence Order Theorem for F.P.I. / N.M.