Physics 112 - Homework 5

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Question 1.

Number of photons in cavity of volume $V = L^3$.

$$\begin{aligned}
N &= \sum_n \langle s_n \rangle \\
&= \sum_n \frac{1}{e^{E_n/\tau} - 1} \\
&= \sum_n \frac{1}{e^{\hbar\omega_n/\tau} - 1} \\
&= \sum_{n_x, n_y, n_z} \frac{1}{e^{\hbar\omega_n/\tau} - 1} \\
&= \frac{3}{8} \int_0^\infty \frac{4\pi n^2}{e^{\hbar\omega_n/\tau} - 1} \, \mathrm{d}n \\
&= \frac{3\pi}{2} \int_0^\infty \frac{n^2}{e^{\hbar n\pi c/\tau L} - 1} \, \mathrm{d}n \\
&= \frac{3\pi}{2} \left(\frac{\tau L}{\hbar \pi c} \right)^3 \int_0^\infty \frac{x^2}{e^x - 1} \, \mathrm{d}x \\
&= \frac{3\pi}{2} \left(\frac{\tau L}{\hbar \pi c} \right)^3 \left[-x^2 e^{-x} \big|_0^\infty + \int_0^\infty 2xe^{-2x} \, \mathrm{d}x \right] \\
&= \frac{3\pi}{2} \left(\frac{\tau L}{\hbar \pi c} \right)^3 \left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \right]_0^\infty \\
&= \frac{3\pi}{2} \left(\frac{\tau L}{\hbar \pi c} \right)^3 \left[-e^{-x}(x^2 + 2x + 2) \right]_0^\infty \\
\lim_{x \to 0} -\frac{x^2 + 2x + 2}{e^x} &= \lim_{x \to 0} -\frac{2x + 2}{e^x} \\
&= \lim_{x \to 0} -\frac{2}{e^x} \\
&= -2 \\
\lim_{x \to \infty} -\frac{x^2 + 2x + 2}{e^x} &= \lim_{x \to \infty} -\frac{2}{e^x} \\
&= 0 \\
N &= \frac{3\pi}{2} \left(\frac{\tau L}{\hbar \pi c} \right)^3 (2) \\
&= \frac{3L^3}{\pi^2} \left(\frac{\tau}{\hbar c} \right)^3 \\
&= \boxed{\frac{3V}{\pi^2} \left(\frac{\tau}{\hbar c} \right)^3}
\end{aligned}$$

I'm off by a little bit of the coefficient... maybe it had to do with the number of polarisations.

Question 2.

Pressure of a photon gas and so on.

$$U = \sum_j s_j \hbar \omega_j$$

$$p = -\frac{\partial U}{\partial V} = -\sum_j \frac{\partial s_j \hbar \omega_j}{\partial V} = \boxed{-\sum_j s_j \hbar \frac{\mathrm{d}\omega_j}{\mathrm{d}V}}$$

$$\frac{\mathrm{d}\omega_j}{\mathrm{d}V} = \frac{\mathrm{d}}{\mathrm{d}V}\frac{n\pi c}{L} = \frac{\mathrm{d}}{\mathrm{d}V}\frac{j\pi c}{L} = \frac{\mathrm{d}}{\mathrm{d}V}\frac{j\pi c}{V^{1/3}} = -\frac{j\pi c}{3V^{4/3}} = -\frac{j\pi c}{L}\frac{1}{3V} = \boxed{-\frac{\omega_j}{3V}}$$

$$p = -\sum_j s_j \hbar \frac{\mathrm{d}\omega_j}{\mathrm{d}V} = \sum_j s_j \hbar \frac{\omega_j}{3V} = \frac{1}{3V}\sum_j s_j \hbar \omega_j = \boxed{\frac{U}{3V}}$$

Kinetic pressure p_k and thermal radiation pressure p of gas of H atoms.

$$p_k = \frac{N}{V}\tau = (1\,\mathrm{mol/cm^3})(6.022 \times 10^{23}\,/\mathrm{mol})(1 \times 10^{6}\,\mathrm{cm^3/m^3})\tau$$
$$= \boxed{1.66 \times 10^{14}\,\mathrm{N/m^2}}$$
$$p = \frac{U}{3V} = \frac{1}{3}\frac{8\pi^2(\tau)^4}{15h^3c^3} = \boxed{4.03 \times 10^{13}\,\mathrm{N/m^2}}$$

Temperature for which the pressures are equal.

$$(6.022 \times 10^{29}\,/\mathrm{m^3})\tau = \frac{8\pi^2\tau^4}{45h^3c^3}$$
$$\tau = hc\left(\frac{45}{8\pi^2}(6.022 \times 10^{29}\,/\mathrm{m^3})\right)^{1/3}$$
$$= 1.39 \times 10^{-15}\,\mathrm{J}$$
$$T = \boxed{1.01 \times 10^{8}\,\mathrm{K}}$$

Question 3.

Heat capacity of one-dimensional EM wave/photons. Note the solution to the one-dimensional wavefunction is $E = E_0 \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$.

$$
\begin{aligned}
C_v &= \frac{\partial U}{\partial \tau} \\
U &= \sum_j \frac{\hbar\omega_j}{e^{\hbar\omega_j/\tau} - 1} \\
&= \sum_n \frac{\hbar n\pi v/L}{e^{\hbar n\pi v/L\tau} - 1} \\
&= \int_0^\infty \frac{\hbar n\pi v/L}{e^{\hbar n\pi v/L\tau} - 1}\,\mathrm{d}n \\
&= \frac{L\tau^2}{\hbar\pi v}\int_0^\infty \frac{x}{e^x - 1}\,\mathrm{d}x \\
&= \frac{L\tau^2}{\hbar\pi v}\left[-e^{-x}(x+1)\right]_0^\infty \\
\lim_{x\to 0} -\frac{x+1}{e^x} &= -1 \\
\lim_{x\to\infty} -\frac{x+1}{e^x} &= \lim_{x\to\infty} -\frac{1}{e^x} \\
&= 0 \\
U &= \frac{L\tau^2}{\hbar\pi v} \\
C_v &= \frac{\partial}{\partial\tau}\frac{L\tau^2}{\hbar\pi v} \\
&= \boxed{\frac{2L\tau}{\hbar\pi v}}
\end{aligned}
$$

im too tired to finish up this problem set :(