7. Limits of sequences

In this section, we introduce the limit of a sequence.

7.1. A review of notation on functions

Recall that a function f from a set A to a set B is a relation that associates each element x in A to exactly one element $f(x)$ in B. In this case,

- The set A is called the **domain** of f and denoted by $\operatorname{dom}(f)$.
- The set B is called the **codomain** of f.[1]
- The set of all values of f is called the **range** of f and denoted by $\operatorname{ran}(f) = \{f(x) : x \in \operatorname{dom}(f)\}$.
- The fact of f being a function from A to B is denoted by:

$$f : A \to B$$

- The association rule for a function $f : A \to B$ is often expressed in the arrow notation:

$$x \mapsto f(x)$$

  The arrow notation is also useful for introducing a function anonymously.

7.2. Sequences: definitions and notation

A **sequence** is a function whose domain is a set of the form

$$\{m, m+1, m+2, \dots\} = \{n \in \mathbb{Z} : n \geq m\}$$

for some integer m.

- Here, the first index m is usually 1 or 0.
- For a sequence s, its value $s(n)$ at n is called a **term** and often denoted by s_n.
- It is often convenient to write the sequence as

$$(s_n)_{n=m}^{\infty} \qquad \text{or} \qquad (s_m, s_{m+1}, s_{m+2}, \dots)$$

- If the first index is $m = 1$, then we may also write $(s_n)_{n \in \mathbb{N}}$.
- Sometimes we exploit the notation by writing (s_n) when the first index m is understood or when the value of m is not important.
- The set of values of $s = (s_n)$ is the range of s as a function.

[1]Unlike the domain, a codomain of a function f is not determined solely by the association rule for f. So, this notion becomes useful when we discuss a generic function or a family of functions.

Here is an example:

Example 7.1. For each of the following sequences, list the first five terms and find the set of values.

(a) $(s_n)_{n\in\mathbb{N}}$ where $s_n = \frac{1}{n^2}$. **(b)** $a_n = (-1)^n$ for integers $n \geq 0$.

7.3. The definition of the limit of a sequence

Now we formally define the limit of a sequence. The central idea in the definition is that the limit of (s_n) is an "ideal destination" that the values of s_n become close to for large values of n.

Definition 7.1. A sequence (s_n) of real numbers is said to **converge** to the real number ℓ, provided the following condition holds:

$$\left(\begin{array}{c}\text{for each real } \varepsilon > 0, \quad \text{there exists a number } N, \quad \text{such that} \\ \text{for each integer } n > N, \quad |s_n - \ell| < \varepsilon \text{ holds.}\end{array}\right) \tag{1}$$

- The fact that (s_n) converges to ℓ will be denoted by

$$\lim_{n\to\infty} s_n = \ell \qquad \text{or} \qquad s_n \to \ell.$$

- The number ℓ is called the **limit** of the sequence (s_n).
- A sequence that does not converge to some real number is said to **diverge**.

Here are some comments on this definition.

- In the condition (1), the choice of the number N usually depends on the value of ε. Formally speaking, it is usually the case that N is a function of ε. Sometimes we will write $N = N(\varepsilon)$ when we want to emphasize its dependence on ε.
- In analysis, the symbol ε is often used when the interesting values are the small positive values. For example, the condition (1) is mostly interesting and challenging for small values of ε.

- The condition (1) can be put into a concise form by introducing the following quantifiers: Let (P_n) be a sequence of statements. Then
  - P_n is said to hold **eventually** if P_n holds for all but finitely many values of n.
  - P_n is said to hold **infinitely often** if P_n holds for infinitely values of n.

  Then it is not hard to see that

  $$\text{not } [P_n \text{ is eventually true}] \quad \Leftrightarrow \quad [P_n \text{ do not hold infinitely often}].$$

  Moreover, the above quantifiers are closely related to the notion of limit via the following observation: (1) reduces to the condition that

  $$\text{for each real } \varepsilon > 0, \quad |s_n - \ell| < \varepsilon \text{ eventually holds for } n. \tag{2}$$

The next result shows that limits are unique, if exist. This justifies the use of the term "the limit".

Definition 7.2. If a sequence (s_n) of real numbers converges, then its limits are unique.

7.4. A discussion about proofs

Definition (7.1) provides a rigorous formulation for the notion of limits of sequences. However, it neither tells us how to find the limit nor hints us how to prove the convergence. Here, we will go through some examples to learn some basic techniques for working with Definition (7.1).

Example 7.2. Using the definition, show that the sequence $(a_n)_{n\geq 1}$ with

$$a_n = \frac{1}{n^2}$$

converges.

Example 7.3. Prove
$$\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} = 4.$$

Example 7.4. By negating (1) in Definition 7.1, find a necessary and sufficient condition such that a sequence diverges.

Example 7.5. Prove that the sequence $(a_n)_{n=0}^{\infty}$ with $a_n = (-1)^n$ does not converge.

Example 7.6. Let (s_n) be a sequence of non-negative real numbers, and suppose

$$\lim_{n\to\infty} s_n = \ell$$

for some real number ℓ. Note that $\ell \geq 0$.[2] Prove

$$\lim_{n\to\infty} \sqrt{s_n} = \sqrt{\ell}.$$

[2] This follows from Exercise 8.9(a) of the textbook.

Example 7.7. (Squeeze Lemma) Consider three sequences (a_n), (b_n), and (s_n) of real numbers such that the following conditions are satisfied:

(a) $a_n \leq s_n \leq b_n$ eventually holds for n;

(b) both (a_n) and (b_n) converge to the real number ℓ.

Prove that (s_n) also converges to ℓ.