# Physics 112 - Homework 7

Zooey Nguyen

zooeyn@ucla.edu

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**Question 1.**

Density of orbitals of free electron in one dimension. Recall electrons have two spin states.

$$
\begin{aligned}
\Psi &= A \sin \frac{n\pi x}{L} \\
\epsilon_n &= \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2 \\
N &= 2\int_0^{n_f} \mathrm{d}n_x = 2n_F \\
n_F &= \frac{N}{2} \\
\epsilon_F &= \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \left(\frac{N}{2}\right)^2 \\
N &= \frac{L}{\pi\hbar}\sqrt{8m\epsilon_F} \\
d(\epsilon) &= \frac{\mathrm{d}N}{\mathrm{d}\epsilon} = \frac{2\sqrt{2m}L}{\pi\hbar}\frac{1}{2\sqrt{\epsilon}} = \boxed{\frac{L}{\pi}\left(\frac{2m}{\hbar^2\epsilon}\right)^{1/2}}
\end{aligned}
$$

Density of orbitals of free electron in two dimensions. Recall electrons have two spin states.

$$
\begin{aligned}
\Psi &= A \sin \frac{n_x\pi x}{L} \sin \frac{n_y\pi y}{L} \\
\epsilon_n &= \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2) \\
n^2 &= n_x^2 + n_y^2 \\
N &= 2\int_0^{n_f} \mathrm{d}n_x\,\mathrm{d}n_y = 2\frac{1}{4}\int_0^{n_f} (2\pi n)\,\mathrm{d}n = \frac{\pi n_F^2}{2} \\
n_F &= \sqrt{\frac{2N}{\pi}} \\
\epsilon_F &= \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \frac{2N}{\pi} \\
N &= \frac{mL^2\epsilon_F}{\pi\hbar^2} \\
d(\epsilon) &= \frac{\mathrm{d}N}{\mathrm{d}\epsilon} = \frac{mL^2}{\pi\hbar^2} = \boxed{\frac{mA}{\pi\hbar^2}}
\end{aligned}
$$

**Question 2.**

Fermi energy in limit of $\epsilon \gg mc^2$.

$$\begin{aligned}
\epsilon_n &= pc \\
\epsilon_n &= \frac{\hbar n \pi c}{L} \\
N &= 2\int_0^{n_f} \mathrm{d}n_x \, \mathrm{d}n_y \, \mathrm{d}n_z = 2\frac{1}{8}\int_0^{n_f} 4\pi n^2 \, \mathrm{d}n = \frac{\pi n^3}{3} \\
n_F &= \left(\frac{3N}{\pi}\right)^{1/3} \\
\epsilon_F &= \frac{\hbar \pi c}{L}\left(\frac{3N}{\pi}\right)^{1/3} = \hbar\pi c\left(\frac{3N}{\pi L^3}\right)^{1/3} = \hbar\pi c\left(\frac{3N}{\pi V}\right)^{1/3} = \boxed{\hbar\pi c\left(\frac{3n}{\pi}\right)^{1/3}}
\end{aligned}$$

Total energy of ground state.

$$U_0 = 2\sum_{n=1}^{n_F}\epsilon_n = 2\frac{1}{8}\int_0^{n_F} 4\pi n^2 \frac{\hbar n \pi c}{L}\,\mathrm{d}n = \int_0^{n_F}\frac{\hbar\pi^2 c}{L}n^3\,\mathrm{d}n = \frac{\hbar\pi^2 c}{4L}n_F^4 = \frac{\hbar\pi^2 c}{4L}\left(\frac{\epsilon_F L}{\hbar\pi c}\right)^4 = \boxed{\frac{3}{4}N\epsilon_F}$$

**Question 3.**

Pressure of Fermi gas in ground state.

$$U_0 = \frac{3}{5}N\frac{\hbar^2(3\pi^2)^{2/3}}{2m}\left(\frac{N}{V}\right)^{2/3}$$

$$p = -\frac{\mathrm{d}U_0}{\mathrm{d}V} = -\frac{3}{5}N\frac{\hbar^2(3\pi^2)^{2/3}}{2m}\frac{\mathrm{d}}{\mathrm{d}V}\left(\frac{N}{V}\right)^{2/3} = -\frac{3}{5}\frac{\hbar^2(3\pi^2)^{2/3}}{2m}N^{5/3}\frac{-2}{3}V^{-5/3} = \boxed{\frac{(3\pi^2)^{2/3}\hbar^2}{5m}\left(\frac{N}{V}\right)^{5/3}}$$

**Question 4.**

Fermi sphere parameters for $^3He$.

$$\begin{aligned}
\epsilon_F &= \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = \frac{\hbar^2}{2(3u)}\left(3\pi^2\left(\frac{0.081\,\mathrm{g/cm^3}}{3u}\right)\right)^{2/3} = \boxed{4.2\times10^{-4}\,\mathrm{eV}} \\
v_F &= \frac{2\epsilon_F}{m} = \frac{24.2\times10^{-4}\,\mathrm{eV}}{3u} = \boxed{16\,\mathrm{m/s}} \\
T_F &= \frac{\epsilon_F}{k_B} = \boxed{4.9\,\mathrm{K}}
\end{aligned}$$

Heat capacity at lower temperature, use electron gas equation.

$$\begin{aligned}
C_{el} &= \frac{\pi^2}{3}\left(\frac{3}{2\epsilon_F}\right)Nk_BT \\
&= \frac{\pi^2}{8.4\times10^{-4}\,\mathrm{eV}}Nk_BT
\end{aligned}$$

## Question 5.

Order of magnitude of gravitational energy for white dwarf.

$$U = -\int_0^R \frac{GMm}{r^2}\,\mathrm{d}r = -\int_0^R \frac{G(\frac{4}{3}\pi r^3\rho)(4\pi r^2\rho)}{r^2}\,\mathrm{d}r = -\int_0^R \frac{G(\frac{4}{3}\pi r^3\rho)}{r^2}4\pi r^2\,\mathrm{d}r$$
$$= -\frac{16G\pi\rho^2 R^5}{15} = -\frac{16G\pi(\frac{M}{\frac{4}{3}\pi R^3})^2 R^5}{15} = \boxed{-\frac{3GM^2}{5R}}$$

Same order of magnitude for kinetic and potential energy.

$$U = E$$
$$-\frac{3GM^2}{5R} = \frac{\hbar^2 M^{5/3}}{mM_H^{5/3}R^2}$$
$$M^{1/3}R \propto \frac{\hbar^2}{Gm_e M_H^{5/3}} = \frac{1.82\times 10^{-22}\,\mathrm{mg}^2}{(1.67\times 10^{-24}\,\mathrm{g})^{5/3}} = 7.7\times 10^{20}\,\mathrm{cmg}^{1/3}$$
$$\approx 10^{20}\,\mathrm{cmg}^{1/3}$$

Density if mass is $M_\odot$.

$$\rho = \frac{M_\odot}{4\pi R^3/3} = \frac{3M_\odot}{4\pi(10^{20}\,\mathrm{cmg}^{1/3}/M_\odot^{1/3})^3} = \frac{3(2\times 10^{33}\,\mathrm{g})}{4\pi(10^{60}\,\mathrm{cm}^3\mathrm{g})/(2\times 10^{33}\,\mathrm{g})} = \boxed{10^6\,\mathrm{g/cm}^3}$$