UCLA Math151A Fall 2021
Lecture 6
20211006

Newton's Method
and
Quick Into to Secant Method

$\boxed{\text{Optional reading: book 2.3.}}$

Newton's Method and Secant Method are root finding methods

The goal is to solve $f(x) = 0$ for x.

Newton's Method

Newton's Method (N.M.) is a classic technique.

It's used in science and engineering, research and industry all the time.

Many different ways for deriving it.. We will cover 3 of them.

Analytic Derivation of Newton's Method

an analytic derivation based on Taylor polynomials

Let $f \in C^2([a,b])$ $\quad$ p is a root $(f(p)=0)$

suppose p_n is "close to" p, i.e., $|p_n - p|$ is "small".

$$\Rightarrow \qquad 0 = f(p) = f(p_n) + f'(p_n)(p-p_n) + f''(\xi)\frac{(p-p_n)^2}{2},$$

ξ is between p and p_n

If $|p-p_n|$ is "small", then $|p-p_n|^2$ is "really small".

$\Rightarrow$ Up to an error of size $\approx (p-p_n)^2$,

$$0 = f(p) = f(p_n) + f'(p_n)(p-p_n)$$

$$\boxed{\Rightarrow p = p_n - \frac{f(p_n)}{f'(p_n)}}$$

Theorem 6.1 (Taylor's theorem)**.** Let $f \in C^n([a,b])$, let $x_0 \in [a,b]$, and let $f^{(n+1)}$ exists on (a,b). Then $\forall x \in [a,b]$, $\exists$ some $\xi(x) \in \mathbb{R}$ s.t. $x_0 < \xi < x$ and

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)(x-x_0)^2/2! + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

(Taylor's polynomial)

$$R_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)^{n+1} \qquad \text{(The remainder term)}$$

Let $f \in C^2([a,b])$ $\quad$ p is a root $(f(p) = 0)$

suppose p_n is "close to" p, i.e., $|p_n - p|$ is "small".

$$\Rightarrow p = p_n - \frac{f(p_n)}{f'(p_n)}$$

This can be used to "invent" Newton's Method (N.M.):

Definition 6.1 (Newton's Method).

Start with p_0 close to p, then do $\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.}$

Remark 6.1. The initial guess p_0 must be close to p, otherwise the analytic derivation breaks down.

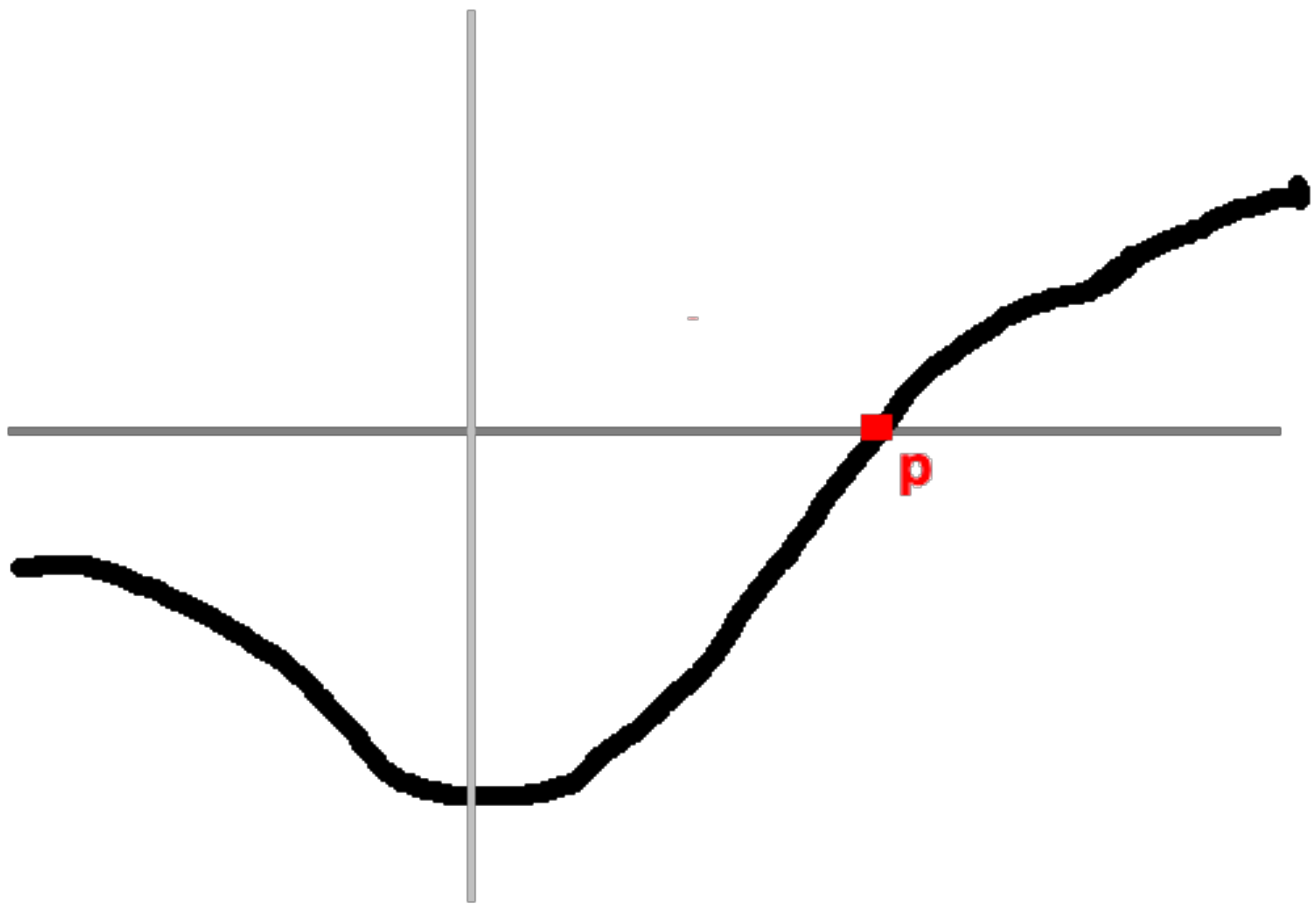
Graphical Derivation of Newton's Method

pretty smooth.

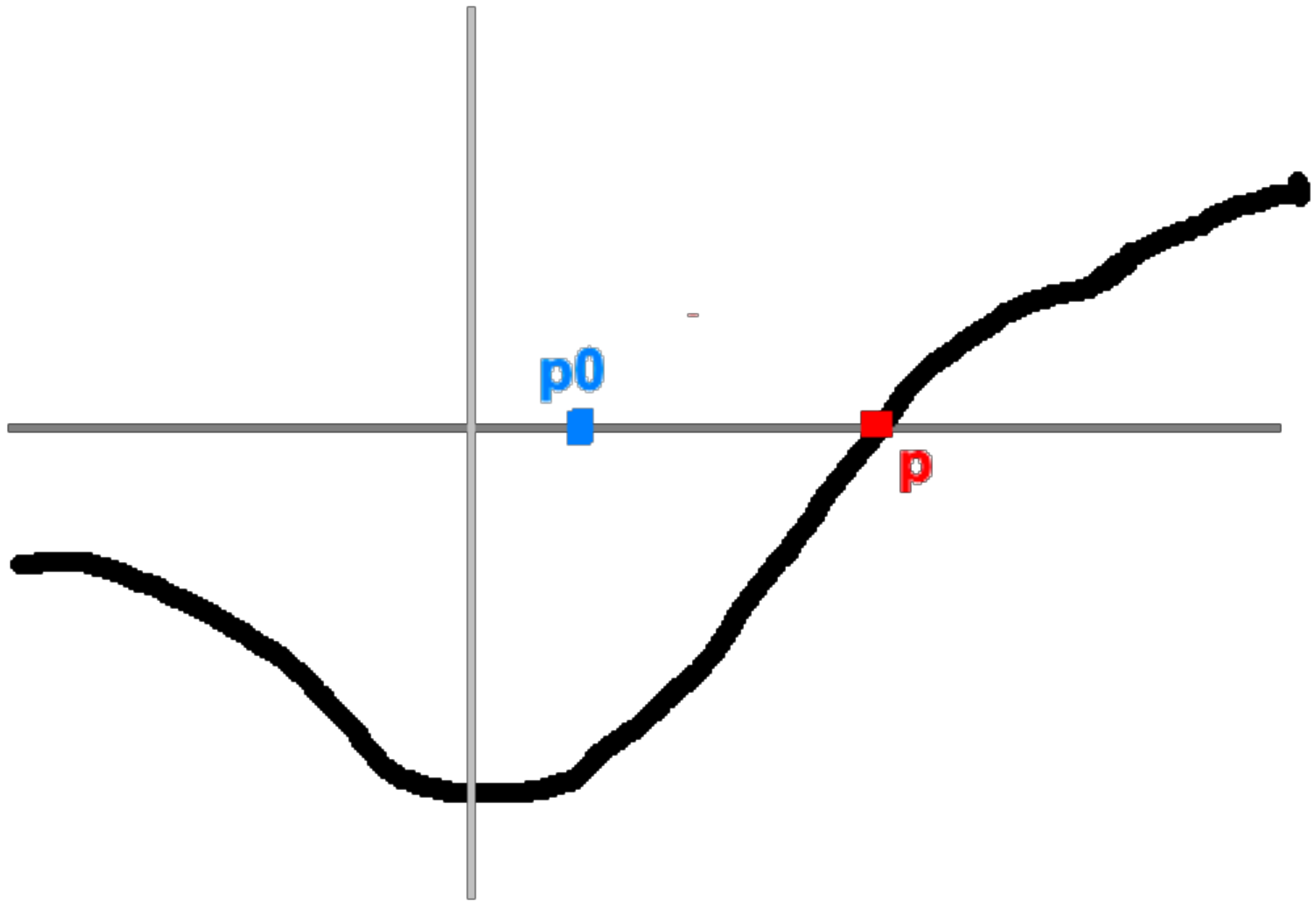
Graphical Derivation of Newton's Method

pretty smooth.

The true root is p.



Graphical Derivation of Newton's Method



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The true root is p.

We pick p_0 close by

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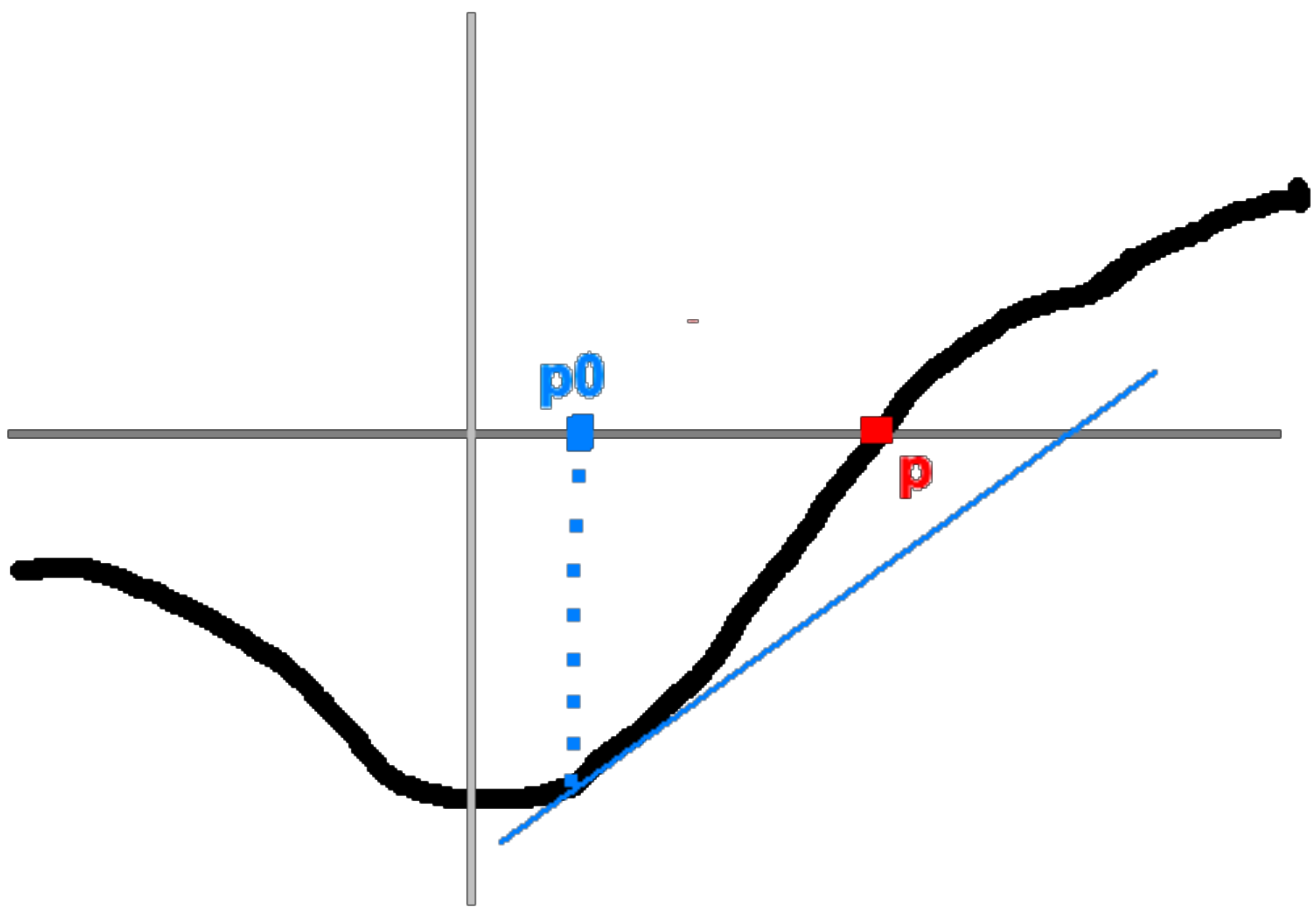

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We pick p_0 close by

following the tangent lines at each point $(p_n, f(p_n))$

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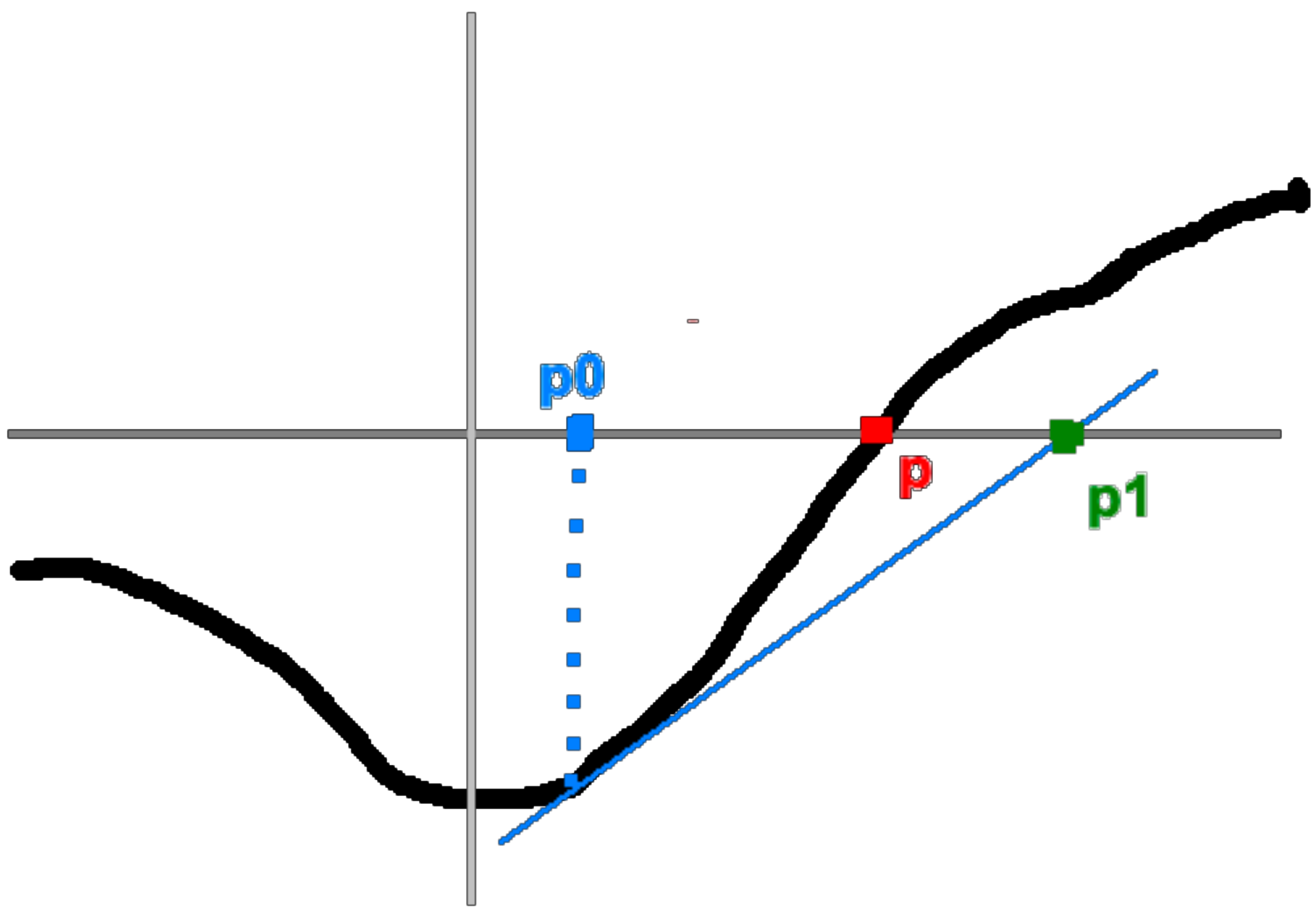

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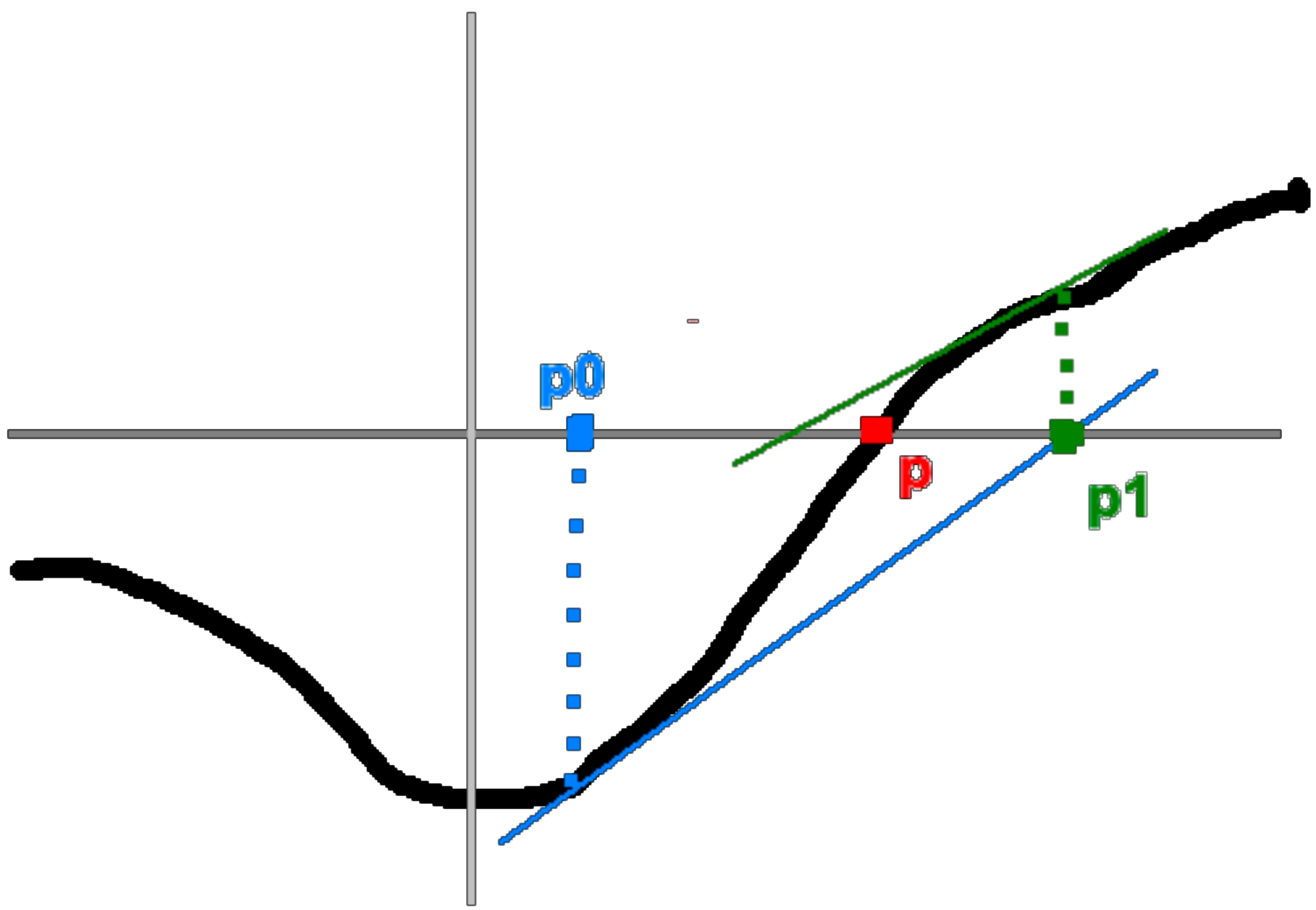

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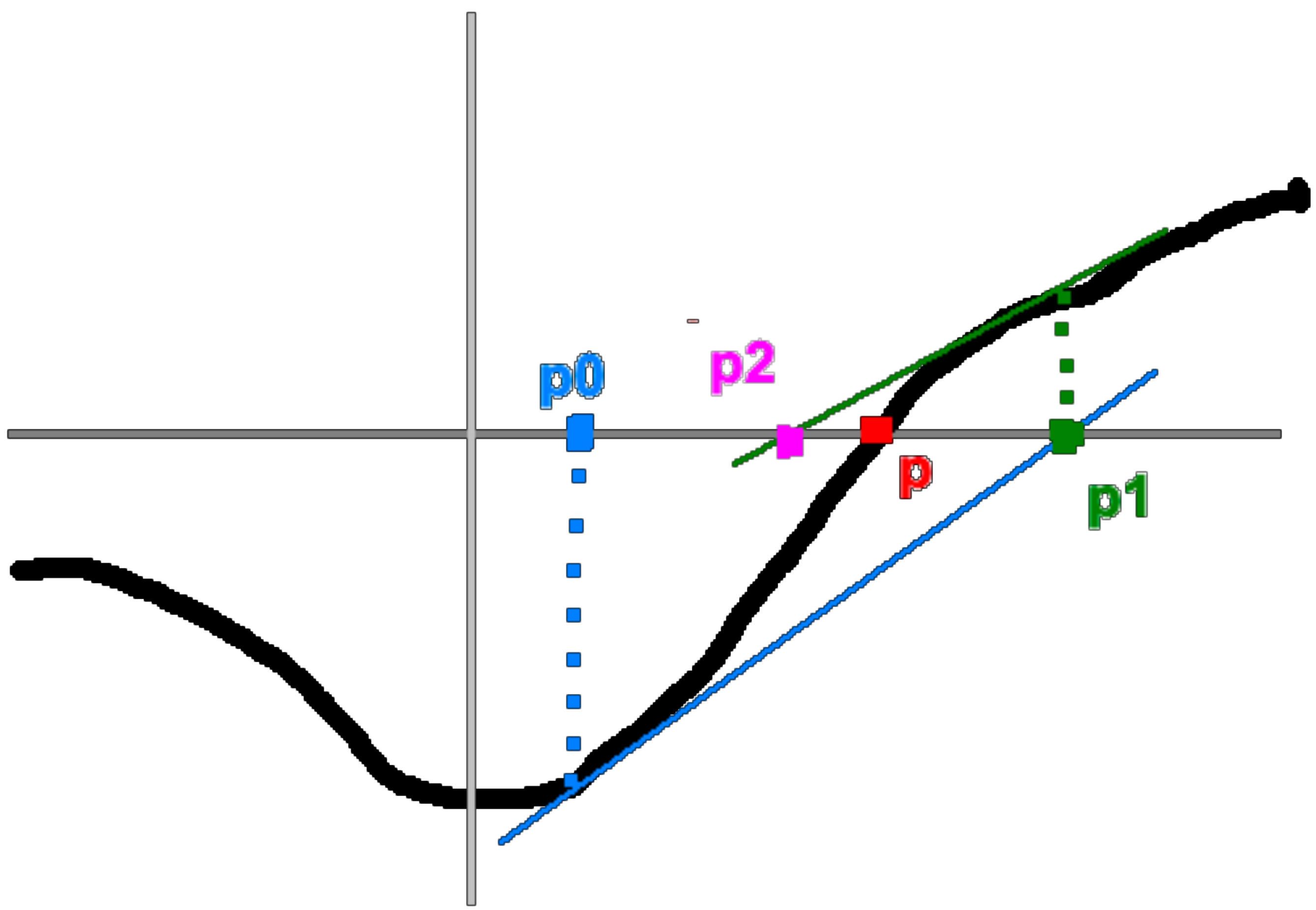

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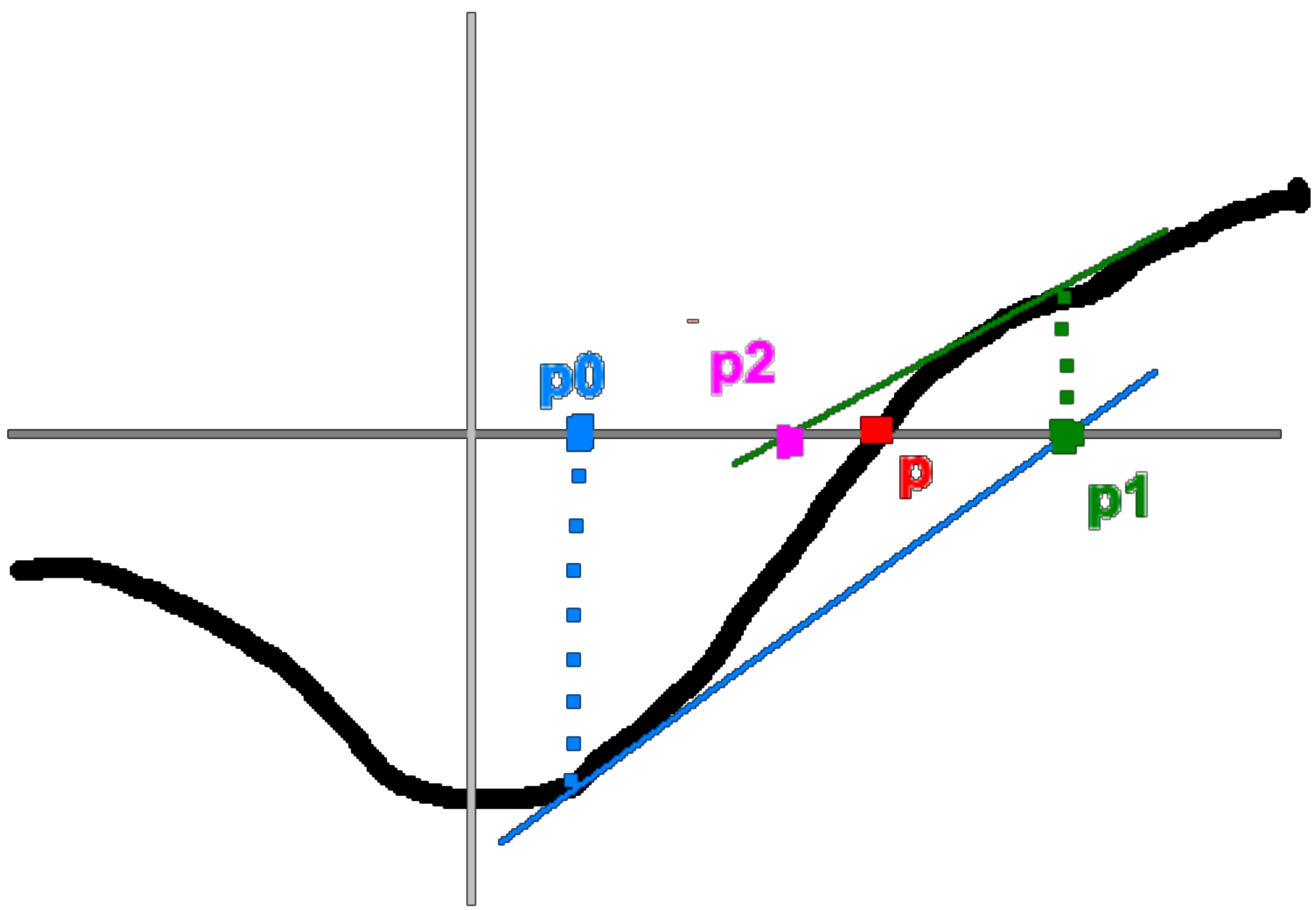

pretty smooth.
The true root is p.
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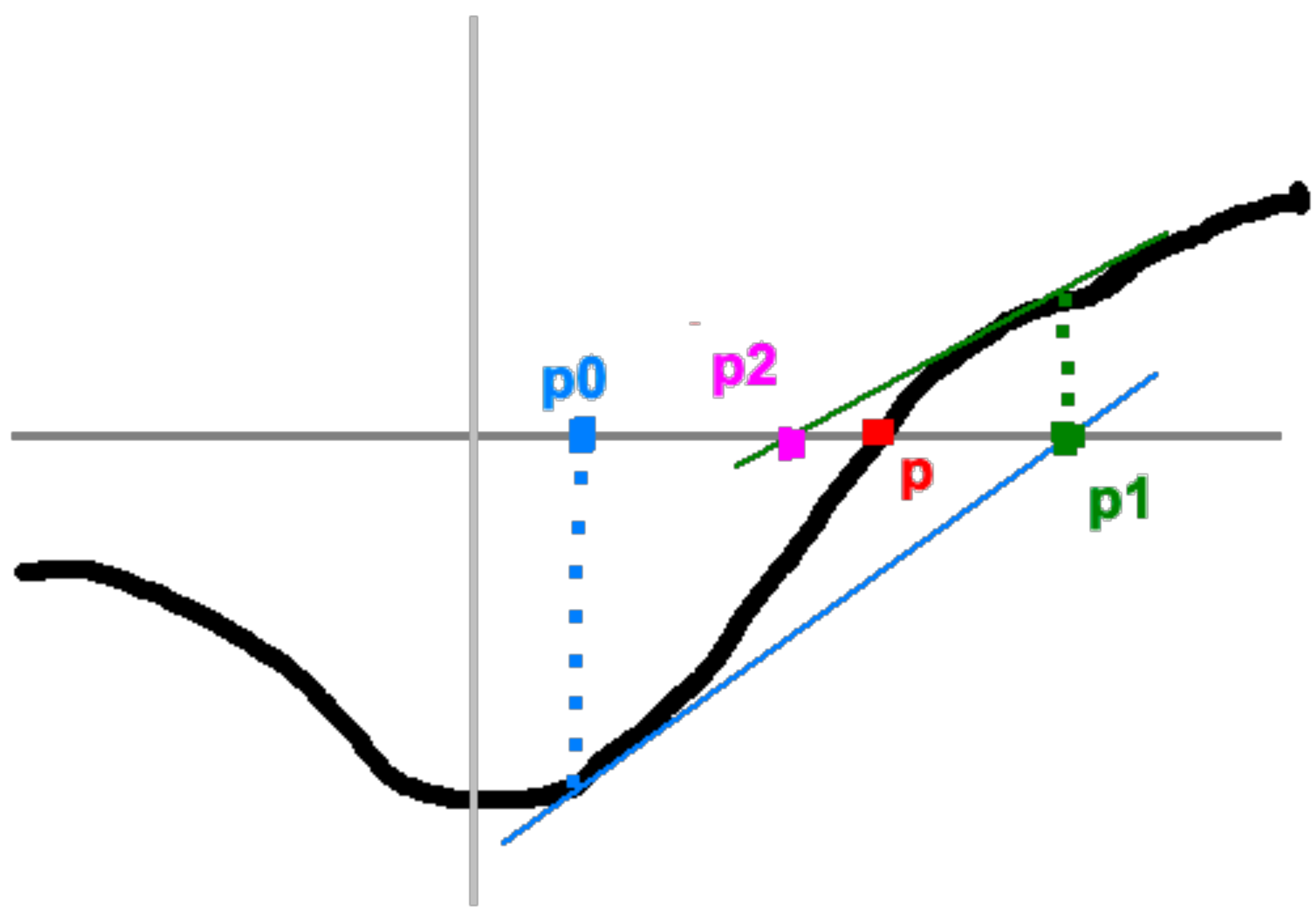
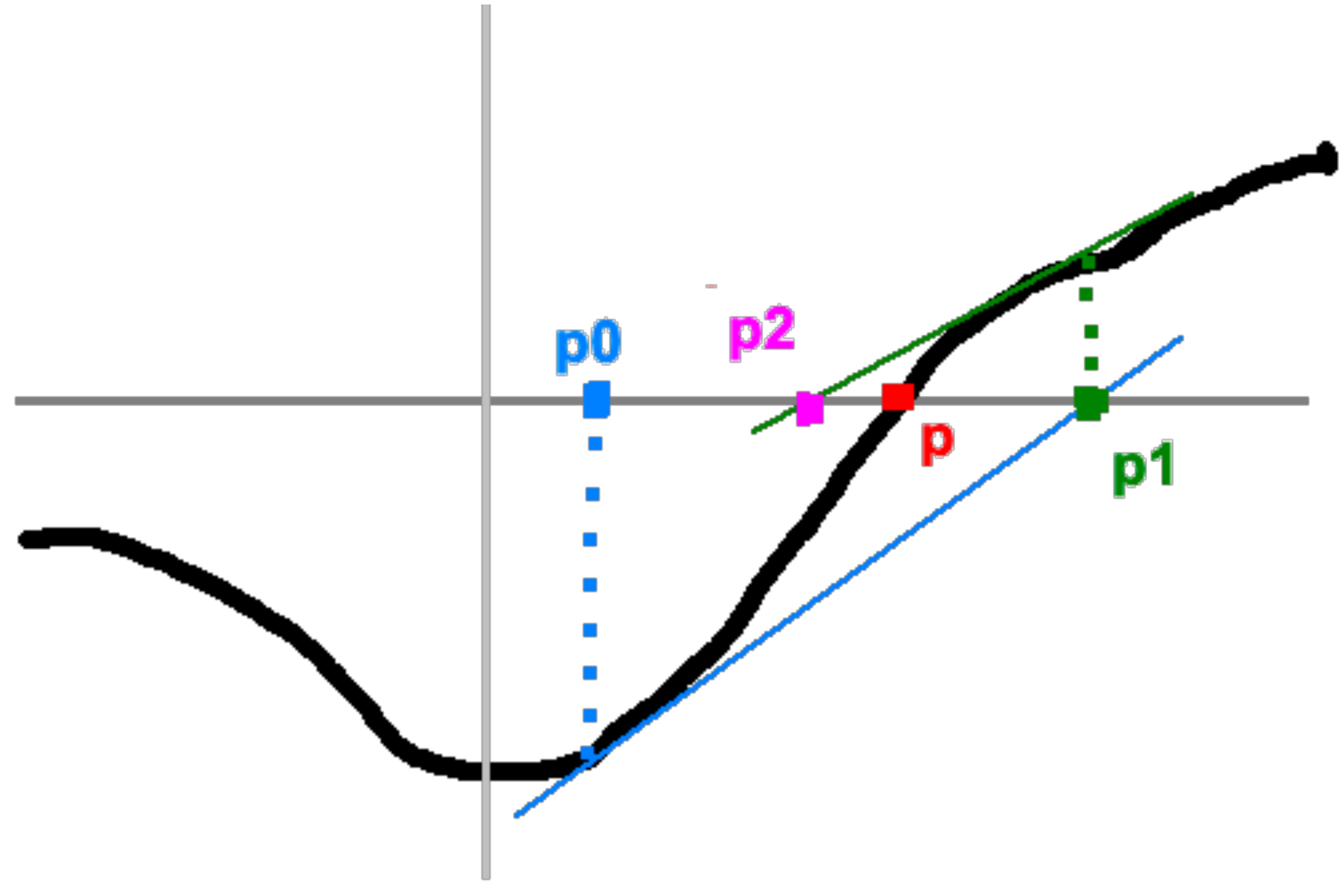
following the tangent lines at each point $(p_n, f(p_n))$ we can see that we get closer and closer to p.

Now let's derive the expression of finding the intersection of the tangent line with the x axis



tangent line be $y = ax + b$,

intersection with the x axis be p_{n+1}



we know

$$f(p_n) = ap_n + b,$$

$$0 = ap_{n+1} + b,$$

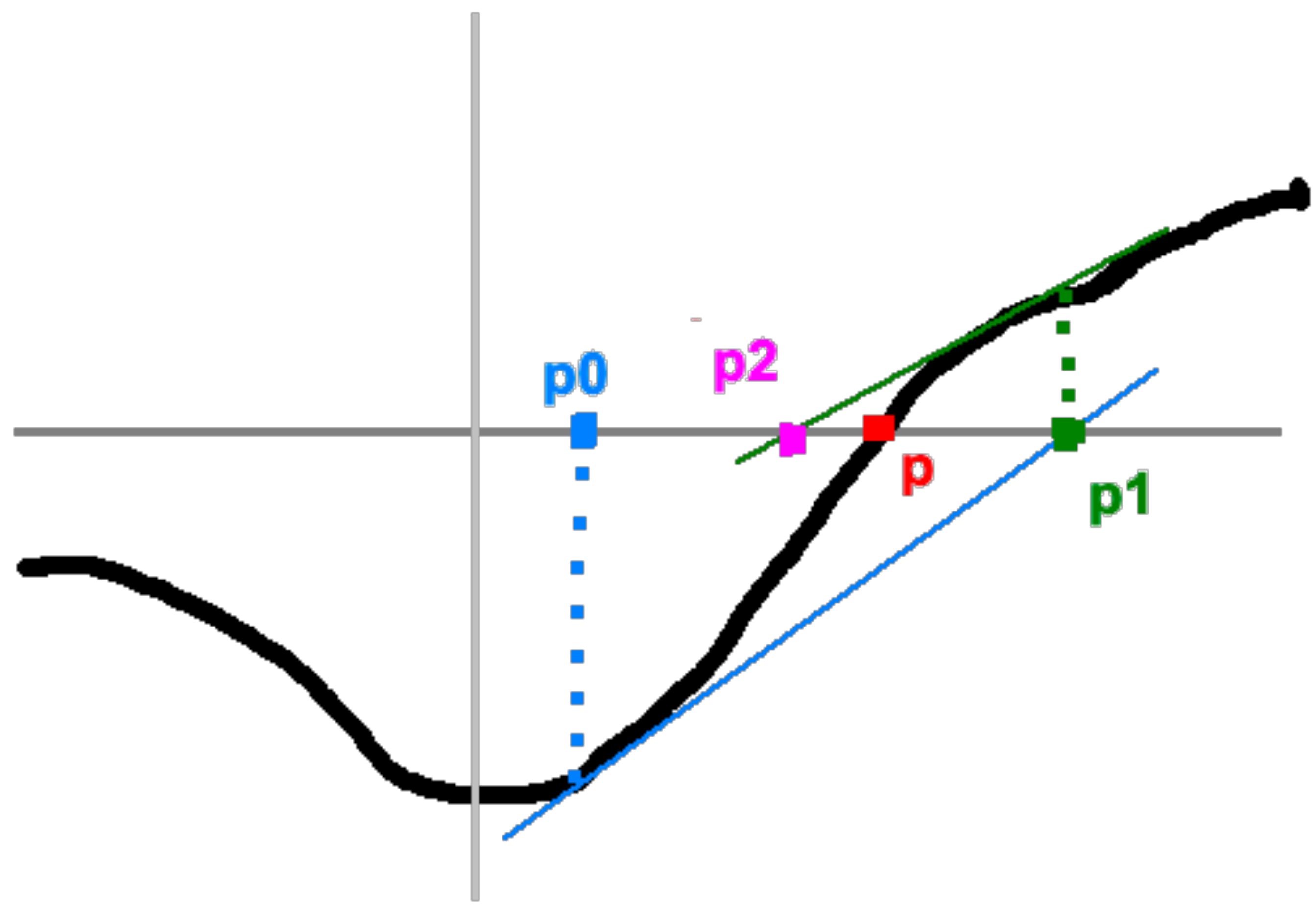
$$a = f'(p_n),$$

The unknowns are a, b, p_{n+1}.

Solving them we get

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.}$$

what happens if we picked a p_0 that is far away from p?



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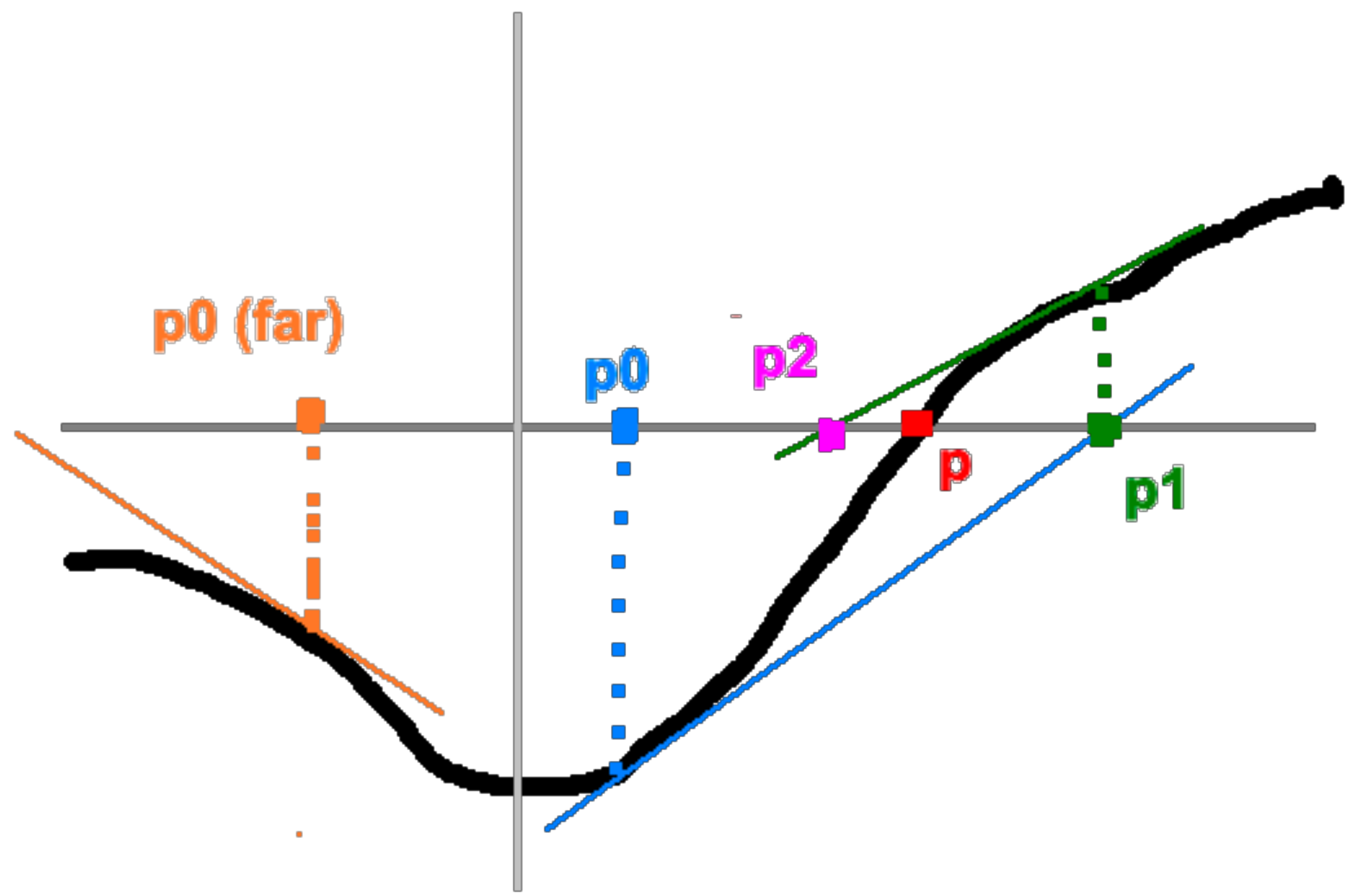


Probably lead us further away.

A Third Derivation: Fixed Point Derivation

Theorem

Let $g(x) := x - \frac{f(x)}{f'(x)}$ for some $f \in C^1([a,b])$ where also $f'(x) \neq 0$ for $x \in [a,b]$.

Then $\quad g(p) = p \quad$ if and only if $f(p) = 0$. $\square$

Proof. Basic algebra. $\blacksquare$

So define a fixed point iteration from g:

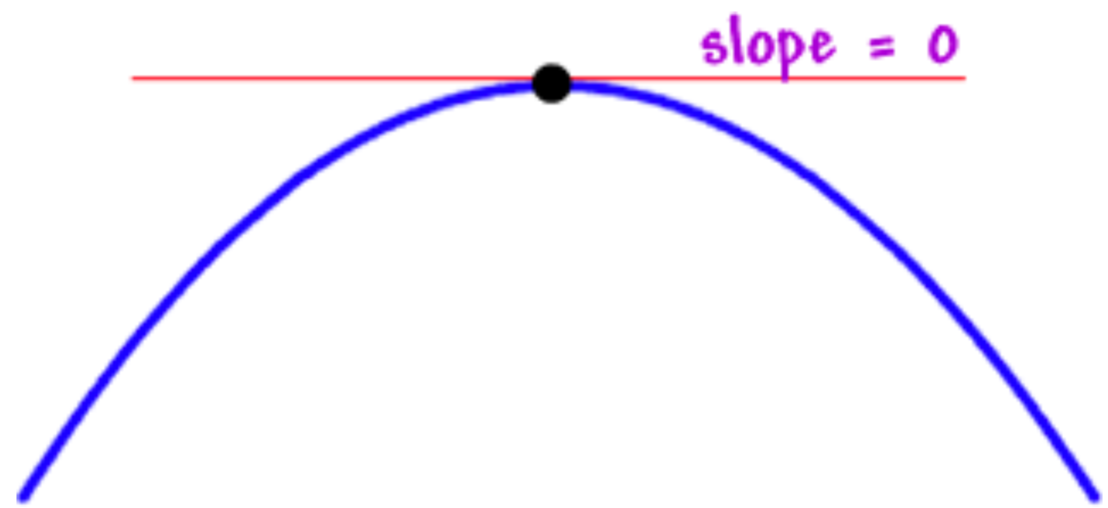
$$\boxed{p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)}.}$$

Remarks about Newton's Method

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

Must have $f'(p_n) \neq 0, \forall n.$

Otherwise N.M. will fail.



Pro of N.M.:

- It will converge faster than the B.M. to the root p of function $f(x)$ (when it does converge).

Con of N.M.:

- Unlike the B.M., N.M. is a local method, not global.

  That means p_0 must be sufficiently close to p for success.

- N.M. requires knolwedge of $f'(x)$ and evaluation of $f'(x)$

  (could be costly, especially when f is $\mathbb{R}^n \to \mathbb{R}^m$).

In higher dimensions, if f is $\mathbb{R}^n \to \mathbb{R}^n$, then N.M. is:

$$\boxed{\mathbf{x}_{n+1} = \mathbf{x}_n - \left(\mathbf{J}(\mathbf{x}_n)\right)^{-1}\mathbf{f}(\mathbf{x}_n)}$$

where $\mathbf{J}(\mathbf{x})$ is the Jacobian matrix and

$$J_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}).$$

Secant Method Quick Intro

- N.M. requires knolwedge of $f'(x)$ and evaluation of $f'(x)$

$f'(p_n) \approx \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}$, This defines the **Secant Method**.

Definition 6.2. Secant Method Given some p_0 and p_1, define

$$p_{n+1} = p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})},$$

where $\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})} \approx (f'(p_n))^{-1}$.

Secant method is useful when you don't have access to $f'(x)$,

E.g., when f comes from experimental data!

Next time:

Newton Convergence Theorem:
Newton converges for
sufficiently close initial guess!