Math 151A - Homework 6

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Question 1.

Backwards difference approximations for derivative of $\sin x$ at $x = \pi/3$. Note $(\sin x)'|_{x=\pi/3} = \cos(\pi/3) = 0.5$. The error decreases by about a factor of 10 each time.

$$f'(\pi/3) \approx \frac{\sin(\pi/3) - \sin(\pi/3 - 0.1)}{0.1} = 0.54243 \qquad e_{abs} = 0.04243$$
$$f'(\pi/3) \approx \frac{\sin(\pi/3) - \sin(\pi/3 - 0.01)}{0.01} = 0.50432 \qquad e_{abs} = 0.00432$$
$$f'(\pi/3) \approx \frac{\sin(\pi/3) - \sin(\pi/3 - 0.001)}{0.001} = 0.50043 \qquad e_{abs} = 0.00043$$

Question 2.

Formula with $h/2$.

$$f'(x_0) = \frac{f(x_0 + \frac{h}{2}) - f(x_0 - \frac{h}{2})}{h} - \left(\frac{h^2}{24} f'''(x_0) + \frac{h^4}{1920} f^{(5)}(\xi)\right)$$

New approximation.

$$\begin{aligned}
4f'(x_0) &= 4\frac{f(x_0 + \frac{h}{2}) - f(x_0 - \frac{h}{2})}{h} - \left(\frac{h^2}{6} f'''(x_0) + \frac{h^4}{480} f^{(5)}(\xi)\right) \\
3f'(x_0) &= 4\frac{f(x_0 + \frac{h}{2}) - f(x_0 - \frac{h}{2})}{h} - \left(\frac{h^2}{6} f'''(x_0) + \frac{h^4}{480} f^{(5)}(\xi)\right) - \frac{f(x_0 + h) - f(x_0 - h)}{4h} + \left(\frac{h^2}{6} f'''(x_0) + \frac{h^4}{120} f^{(5)}(\xi)\right) \\
&= 4\frac{f(x_0 + \frac{h}{2}) - f(x_0 - \frac{h}{2})}{h} - \frac{f(x_0 + h) - f(x_0 - h)}{4h} - \frac{3h^4}{480} f^{(5)}(\xi) \\
&= \frac{16f(x_0 + \frac{h}{2}) - 16f(x_0 - \frac{h}{2}) - f(x_0 + h) - f(x_0 - h)}{4h} - \frac{3h^4}{480} f^{(5)}(\xi) \\
f'(x_0) &= \frac{16f(x_0 + \frac{h}{2}) - 16f(x_0 - \frac{h}{2}) - f(x_0 + h) - f(x_0 - h)}{12h} - \boxed{\frac{h^4}{480} f^{(5)}(\xi)}
\end{aligned}$$

Question 3.

Taylor expansion at x_1.

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2!}(x - x_1)^2 + \frac{f^{(3)}(x_1)}{3!}(x - x_1)^3 + \frac{f^{(4)}(\xi)}{4!}(x - x_1)^4$$

Integral of Taylor expansion at x_1 on x_0, x_2.

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,\mathrm{d}x &= \int_{x_0}^{x_2} f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2!}(x - x_1)^2 + \frac{f^{(3)}(x_1)}{3!}(x - x_1)^3 + \frac{f^{(4)}(\xi)}{4!}(x - x_1)^4\,\mathrm{d}x \\
&= f(x_1)x\Big|_{x_0}^{x_2} + f'(x_1)\frac{(x - x_1)^2}{2}\Big|_{x_0}^{x_2} + \frac{f''(x_1)}{2!}\frac{(x - x_1)^3}{3}\Big|_{x_0}^{x_2} + \frac{f^{(3)}(x_1)}{3!}\frac{(x - x_1)^4}{4}\Big|_{x_0}^{x_2} + \frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(\xi)(x - x_1)^4\,\mathrm{d}x \\
&= f(x_1)(x_2 - x_0) + \frac{f''(x_1)}{2!}\frac{(x_2 - x_1)^3 - (x_0 - x_1)^3}{3} + \frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(\xi)(x - x_1)^4\,\mathrm{d}x \\
&= 2hf(x_1) + \frac{f''(x_1)}{2!}\frac{h^3 + h^3}{3} + \frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(\xi)(x - x_1)^4\,\mathrm{d}x \\
&= 2hf(x_1) + \frac{h^3}{3}f''(x_1) + \frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(\xi)(x - x_1)^4\,\mathrm{d}x
\end{aligned}$$

Question 4.