9. Limit Theorems for Sequences

- A sequence (s_n) is said to be **bounded** if there exists $M \in \mathbb{R}$ such that $|s_n| \leq M$ for all n.

Theorem 9.1. Convergent sequences are bounded.

Theorem 9.2. Let $(s_n)_{n=m}^{\infty}$ be a sequence of real numbers such that $s_n \neq 0$ for all n. If $(s_n)_{n=m}^{\infty}$ converges to a real number $\alpha \neq 0$, then

$$\inf\{|s_n| : n \geq m\} > 0.$$

9.1. Basic limit laws

Theorem 9.3. Consider sequences (s_n) and (t_n) and real numbers α and β. Suppose (s_n) converges to α and (t_n) converges to β. Then the followings hold:

(a) For any $k \in \mathbb{R}$, (ks_n) converges to $k\alpha$. That is,

$$\lim_{n\to\infty} ks_n = k\Big(\lim_{n\to\infty} s_n\Big).$$

(b) $(s_n + t_n)$ converges to $\alpha + \beta$. That is,

$$\lim_{n\to\infty} (s_n + t_n) = \Big(\lim_{n\to\infty} s_n\Big) + \Big(\lim_{n\to\infty} t_n\Big).$$

(c) $(s_n t_n)$ converges to $\alpha\beta$. That is,

$$\lim_{n\to\infty} s_n t_n = \Big(\lim_{n\to\infty} s_n\Big)\Big(\lim_{n\to\infty} t_n\Big).$$

(d) If $s_n \neq 0$ for all n, and if $\alpha \neq 0$, then (s_n^{-1}) converges to α^{-1}. That is,

$$\lim_{n\to\infty} s_n^{-1} = \Big(\lim_{n\to\infty} s_n\Big)^{-1}.$$

(e) If $s_n \neq 0$ for all n, and if $\alpha \neq 0$, then (t_n/s_n) converges to β/α. That is,

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}.$$

(f) If $s_n \leq t_n$ eventually holds in n, then $\alpha \leq \beta$.

Proof of (a).

Proof of (b).

Proof of (c).

Proof of (d).

Example 9.1. Prove $\lim s_n = \frac{1}{5}$, whree

$$s_n = \frac{n^3 + 3n^2 - 7}{5n^3 - 9n + 11}.$$

Solution.

9.2. Basic examples of limits

- Recall the **Binomial Theorem**: For any numbers x, y and for any non-negative integer $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient and the convention $x^0 = y^0 = 1$ is adopted. (See Exercise 1.12 of the textbook.)

Theorem 9.4.

(a) $\lim\limits_{n\to\infty} \dfrac{1}{n^p}$ for $p > 0$.[1]

(b) $\lim\limits_{n\to\infty} a^n = 0$ for $|a| < 1$.

(c) $\lim\limits_{n\to\infty} n^{1/n} = 1$.

(d) $\lim\limits_{n\to\infty} a^{1/n} = 1$ for $a > 0$.

Proof of (a).

Proof of (b).

[1]Here, we assume familiarity with exponentiation to a real power.

Proof of (c).

Proof of (d).

9.3. Infinite limits

Definition 9.5. Consider a sequence (s_n) of real numbers.

(a) We say that (s_n) **diverges to** $+\infty$ provided

$$\text{for each } M \in \mathbb{R}, \quad \text{there exists a number } N \text{ such that,} \quad n > N \text{ implies } s_n > M.$$

In this case, we write $\lim_{n\to\infty} s_n = +\infty$.

(b) Similarly, we say that (s_n) **diverges to** $-\infty$ provided

$$\text{for each } M \in \mathbb{R}, \quad \text{there exists a number } N \text{ such that,} \quad n > N \text{ implies } s_n < M.$$

In this case, we write $\lim_{n\to\infty} s_n = -\infty$.

Example 9.2. Give a formal proof that

$$\lim_{n\to\infty} \frac{n^2+3}{n+1} = +\infty.$$

Solution.

- **Fair Warning:** Sequences with infinite limits are special cases of divergent sequences. For this reason, you should not attempt to apply the limit laws (Theorem 9.3) to infinite limits.
- Instead, a version of limit laws for infinite limits hold:

Theorem 9.6. Consider sequences (s_n) and (t_n) and real numbers. Then the followings hold:

(a) If $\lim s_n = +\infty$ and $\lim t_n \in (-\infty, +\infty]$, then $\lim(s_n + t_n) = +\infty$.

(b) If $\lim s_n = -\infty$ and $\lim t_n \in [-\infty, +\infty)$, then $\lim(s_n + t_n) = -\infty$.

(c) If $\lim s_n = +\infty$ and $\lim t_n \in (0, +\infty]$, then $\lim s_n t_n = +\infty$.

(d) If $\lim s_n = +\infty$ and $\lim t_n \in [-\infty, 0)$, then $\lim s_n t_n = -\infty$.

(e) If $\lim s_n = -\infty$ and $\lim t_n \in (0, +\infty]$, then $\lim s_n t_n = -\infty$.

(f) If $\lim s_n = -\infty$ and $\lim t_n \in [-\infty, 0)$, then $\lim s_n t_n = +\infty$.

(g) If $\lim s_n = +\infty$ or $-\infty$, then $\lim s_n^{-1} = 0$.

(h) If $\lim s_n = 0$ and $s_n > 0$ for all n, then $\lim s_n^{-1} = +\infty$.

(i) If $\lim s_n = 0$ and $s_n < 0$ for all n, then $\lim s_n^{-1} = -\infty$.

Proof of (c).

Proof of (h).

- The infinite limit laws (Theorem 9.6) motivates us to add new algebraic laws to $\overline{\mathbb{R}}$ given by:

$$
\begin{aligned}
a+(+\infty) &= (+\infty)+a = +\infty && \text{for } a \in (-\infty, +\infty]; \\
a+(-\infty) &= (-\infty)+a = -\infty && \text{for } a \in [-\infty, +\infty); \\
a \cdot (\pm\infty) &= (\pm\infty) \cdot a = \pm\infty && \text{for } a \in (0, +\infty]; \\
a \cdot (\pm\infty) &= (\pm\infty) \cdot a = \mp\infty && \text{for } a \in [-\infty, 0); \\
1/(\pm\infty) &= 0.
\end{aligned}
$$

Example 9.3. Find $\lim s_n$, where

$$s_n = \frac{n^2+3n+5}{n+1}.$$

Solution.