3. The Set $\mathbb{R}$ of Real Numbers

In this section, we will discuss the properties that characterize the set $\mathbb{R}$ of real numbers.

3.1. Ordered Fields

- We first discuss the algebraic and order structure of $\mathbb{R}$. This will be accomplished by introducing a class of mathematical objects called ordered fields, which will encompass both $\mathbb{Q}$ and $\mathbb{R}$.
- We begin by defining a **field**, an algebraic structure in which all the basic operations (addition, subtraction, multiplication, division) can be fully studied.

Definition 3.1. Consider a set F which is endowed with two binary operations $+$ and $\cdot$. Then F is called a **field** if all of the following properties hold:[1]

A1.

A2.

A3.

A4.

M1.

M2.

M3.

M4.

DF.

[1]When defining a term in mathematics, it is customary to use the phrase ‘if’ to actually refer to ‘if and only if’.

- Assuming familiarity with the space of $\mathbb{Q}$ rational numbers (and we indeed do so), we know that $\mathbb{Q}$ satisfies all the above properties with the usual addition and multiplication on $\mathbb{Q}$. Hence $\mathbb{Q}$ is an example of a field.
- Many algebraic properties of $\mathbb{Q}$ can be proved on the basis of the field axioms, and hence are valid in any field. Although we will not pursue this direction in full detail, we will demonstrate some examples:

Theorem 3.2. Let F be a field. Then the following properties hold:

(i) $a + c = b + c$ implies $a = b$;

(ii) $a \cdot 0 = 0$;

(iii) $(-a)b = -(ab)$;

(iv) $(-a)(-b) = ab$;

(v) $ac = bc$ and $c \neq 0$ imply $a = b$;

(vi) $ab = 0$ implies either $a = 0$ or $b = 0$;

for any $a, b, c \in F$.

Proof.

- The set $\mathbb{Q}$ also has an order structure, meaning that every pair of elements in $\mathbb{Q}$ can be compared and that comparison is 'compatible' with the operations on $\mathbb{Q}$.

Definition 3.3. Consider a field F which is equipped with a binary relation $\leq$. Then F is called an **ordered field** if all of the following properties hold:

O1.

O2.

O3.

O4.

O5.

- Again, we will not attempt to verify all the familiar properties of $\mathbb{Q}$ regarding the order structure from scratch. Nevertheless, we discuss some consequences of the above definition in order to assure the reader that such a quest is indeed possible.

Theorem 3.4. Let F be an ordered field. Then the following properties hold:

(i) If $a \leq b$, then $-b \leq -a$;

(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$;

(iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ac$;

(iv) $0 \leq a^2$;

(v) $0 < 1$;

(vi) If $0 < a$, then $0 < a^{-1}$;

(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$;

for any $a, b, c \in F$.

Here, $a < b$ means $a \leq b$ and $a \neq b$.

Proof.

- The order axioms on F allows to introduce the idea of a distance between elements in F. To do this, we begin with the following definition:

Definition 3.5. Let F be an ordered field. The **absolute value function** on F is a function $|\cdot| : F \to F$ defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

- The next result collects some important properties of the absolute value function.

Theorem 3.6. Let F be an ordered field. Then the following properties hold:

(i) $|a| \geq 0$, and $|a| = 0$ if and only if $a = 0$;

(ii) $|a| = |-a|$;

(iii) $-|a| \leq a \leq |a|$;

(iv) $|a| \leq b$ if and only if $-b \leq a \leq b$;

(v) $|a + b| \leq |a| + |b|$;

(vi) $|ab| = |a| \cdot |b|$;

for any $a, b \in F$.

Proof.

As a corollary, we can define the notion of a distance on F:

Corollary 3.7. (Triangle Inequality) Let F be an ordered field, and define the distance function on F by $\operatorname{dist}(a, b) = |a - b|$. Then

$$\operatorname{dist}(a, c) \leq \operatorname{dist}(a, b) + \operatorname{dist}(b, c)$$

for all $a, b, c \in F$.

Proof.

3.2. The Completeness Axiom

- In this section, we will define $\mathbb{R}$ as an ordered field that satisfies the completeness axiom. This is the axiom that ensures $\mathbb{R}$ to have no "gaps", and almost every significant result in this course will critically depend on it.

Definition 3.8. (Boundedness) Consider a subset S of an ordered field F.

(a) If $M \in F$ satisfies $s \leq M$ for all $s \in S$, then M is called an **upper bound** of S and S is said to be **bounded above**.

(b) If $m \in F$ satisfies $s \geq m$ for all $s \in S$, then m is called a **lower bound** of S and S is said to be **bounded below**.

(c) S is said to be **bounded** if it is bounded above and bounded below.

Definition 3.9. (Supremum/Infimum) Consider a subset S of an ordered field F. An element $\beta \in F$ is called a **supremum** (or **least upper bound**), if

(a) β is an upper bound of S, and

(b) $\beta \leq M$ holds for any upper bound M of S.

Likewise, an element $\alpha \in F$ is called an **infimum** (or **greatest lower bound**), if

(a) α is a lower bound of S, and

(b) $\alpha \geq m$ holds for any lower bound m of S.

Example 3.1. Consider the set

$$S = \{r \in \mathbb{Q} : 3 < r \leq 42\}$$

in $\mathbb{Q}$. Find the supremum and infimum of S in $\mathbb{Q}$.

Solution.

Example 3.2. A subset of an ordered field need not have supremum or infimum.

(a) Consider the set $\mathbb{N}$ in $\mathbb{Q}$. We have $\inf \mathbb{N} = 1$ because 1 is the minimum of $\mathbb{N}$. However, $\mathbb{N}$ is not bounded above. (Can you prove this?) So $\mathbb{N}$ does not have a supremum in $\mathbb{Q}$.

(b) Consider the set $S = \{r \in \mathbb{Q} : 0 \leq r \text{ and } r^2 \leq 2\}$ in $\mathbb{Q}$. Intuitively, the supremum of S should be the positive number whose square is 2, that is, $\sqrt{2}$. However, no such element exists in $\mathbb{Q}$. Therefore S does not have a supremum in $\mathbb{Q}$.

- Part (b) of the above example is particularly interesting, because it reveals another source of "gaps" in $\mathbb{Q}$. This motivates the following definition:

Definition 3.10. (Completeness) Consider an ordered field F. Then F is said to be **complete** if the following property holds:

CA.

The property CA in the above definition is often called the **completeness axiom** or **least-upper-bound property**. The above example shows that $\mathbb{Q}$ does not satisfy the completeness axiom, hence is not complete. Now the following result is the core of this chapter:

Theorem 3.11. (Existence of $\mathbb{R}$) There exists an "essentially unique" complete ordered field, called the **set $\mathbb{R}$ of real numbers**.

By "essentially unique", we mean that any two completely ordered fields have exactly the same structure, hence it is meaningless to distinguish them as complete ordered fields. This legitimates the use of the phrase "the set of real numbers".

We will not prove this result in our course, though, because any construction of $\mathbb{R}$ requires a substantial amount of development.

- The completeness axiom for sets bounded below comes free.

Corollary 3.12. Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Proof.

- Another important consequence of the completeness axiom is the **Archimedean property**.

Theorem 3.13. (Archimedean Property) If $a \in \mathbb{R}$, then there exists a positive integer n such that $n > a$.

This result is not as obvious as it may appear. Intuitively, this tells that $\mathbb{R}$ does not contain "infinitely large numbers" and "infinitely small numbers". As a comparison, there are examples of ordered fields that violate the Archimedean property.

Proof.

Corollary 3.14. Let $a, b \in \mathbb{R}$ with $a > 0$.

(i) There is a positive integer n such that $an > b$.

(ii) There is a positive integer n such that $0 < \frac{1}{n} < a$.

(iii) There is a positive integer n such that $n - 1 \leq a < n$.

Proof.

Corollary 3.15. (Densenessf of $\mathbb{Q}$) If $a, b \in \mathbb{R}$ and $a < b$, then there exists a rational $r \in \mathbb{Q}$ such that $a < r < b$.

Proof.

3.3. Some Exercises

Example 3.3. Find the supremum and infimum of each of the following sets, if exist.

(a) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

(b) $S = \{r \in \mathbb{Q} : r^2 < 3\}$

Solution.

Example 3.4. Let S be a subset of $\mathbb{R}$ which is bounded above, and let $k > 0$ be a positive real number. Show that

$$\sup(kS) = k \sup S,$$

where $kS = \{kx : x \in S\}$.

Solution.

3.4. The Symbols $+\infty$ and $-\infty$

It is extremely useful to introduce symbols $+\infty$ (or simply ∞) and $-\infty$, even though they are not real numbers. By adjoining these symbols to $\mathbb{R}$, we obtain the **extended real number line**, which is denoted by $\overline{\mathbb{R}}$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

Here are discussions on the structure of $\overline{\mathbb{R}}$:

- **Ordering on $\overline{\mathbb{R}}$.** We will endow $\overline{\mathbb{R}}$ with the ordering $\leq$ that extends that of $\mathbb{R}$ by declaring that

  $$-\infty \leq a \leq \infty \qquad \text{for any } a \in \overline{\mathbb{R}}.$$

  We can check that the resulting ordering on $\overline{\mathbb{R}}$ satisfies O1–O3.
- **Algebraic Structure on $\overline{\mathbb{R}}$?** We will not attempt to extend the algebraic structure on $\mathbb{R}$ to to $\overline{\mathbb{R}}$, since it is simply impossible to do so unless we give up many of our familiar algebraic rules in $\mathbb{R}$. We remark, however, that *some* algebraic rules on $\mathbb{R}$ do carry over to $\overline{\mathbb{R}}$. We will briefly discuss this later when we define the notion of infinite limit and investigate its properties.

So, how does the extended real number line help?

- **Intervals.** Recall that there are four types of (bounded) intervals in $\mathbb{R}$. More precisely, for any real numbers $a < b$, we define:

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$	open interval
$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$	closed interval
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$	half-open interval

  This interval notation can be extended to $\overline{\mathbb{R}}$ by allowing the endpoints to be either of $\pm\infty$. This results in unbounded intervals. For instance, for any real numbers a and b,

$(a, \infty) = \{x \in \mathbb{R} : a < x\}$ $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$	(unbounded) open interval
$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$ $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$	(unbounded) closed interval
$(-\infty, \infty) = \mathbb{R}$	set of real numbers

- **Supremum/Infimum.** The definition of supremum/infimum also extends to $\overline{\mathbb{R}}$. Moreover, the least-upper-bound property for $\overline{\mathbb{R}}$ takes a simpler form:

> **Theorem 3.16.** Any non-empty subset of $\overline{\mathbb{R}}$ has a supremum in $\overline{\mathbb{R}}$. Moreover, if S is a non-empty subset of $\mathbb{R}$, then
>
> $$[\sup S \text{ in } \overline{\mathbb{R}}] = \begin{cases} [\sup S \text{ in } \mathbb{R}] \in \mathbb{R}, & \text{if } S \text{ is bounded above,} \\ +\infty, & \text{if } S \text{ is not bounded above.} \end{cases}$$

An analogous statement for the infimum in $\overline{\mathbb{R}}$ holds as well. In light of discussion, we will extend the definition of supremum/infimum in $\mathbb{R}$ as follows: for any non-empty subset S of $\mathbb{R}$ we define

$$\sup S = +\infty \qquad \text{if } S \text{ is not bounded above,}$$

and

$$\inf S = -\infty \qquad \text{if } S \text{ is not bounded below.}$$