- You are encouraged to discuss the problems with other students. However, you must write the solutions using your own words. Violation of the honor code may void your submissions.
- The assignments must be submitted through Gradescope. **No late homework will be accepted or graded.** Please allow plenty of time to upload your assignments, especially if you are using Gradescope for the first time.
- You should demonstrate your works that lead to the final answers in order to receive full credit.

Problems

1. Find all rational solutions of the equation

$$x^4 - 2x^3 + 3x^2 + 5x - 1 = 0.$$

2. Prove (iv) of Theorem 3.1 of the textbook (or Theorem 3.2 of the lecture note 3). It is fine to follow the proof in the book, but please clarify the properties that you use.

3. Let F be an ordered field. Prove that $(a^2 + b^2)/2 \geq ab$ for any $a, b \in F$.[1]

4. Prove the **reverse triangle inequality**: Let F be an ordered field. For any $a, b \in F$, we have

$$|a - b| \geq ||a| - |b||.$$

5. Consider real numbers a and b where $a < b$. We define

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}, \qquad (a, b) = \{x \in \mathbb{R} : a < x < b\},$$
$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}, \qquad (a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

Show that $\inf[a, b) = a$ and $\sup[a, b) = b$.

6. Let S and T be non-empty subsets of $\mathbb{R}$. Prove: If T is bounded above and $S \subseteq T$, then S is also bounded above and $\sup S \leq \sup T$.

7. Let A and B be non-empty subsets of $\mathbb{R}$ which are bounded above. Define the sum of A and B by

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Prove that

$$\sup(A + B) = \sup A + \sup B.$$

(*Hint:* To show $\sup A + \sup B \leq \sup(A + B)$, note that $a + b \leq \sup(A + B)$ for any $a \in A$ and $b \in B$, and then take supremum over $a \in A$ and then over $b \in B$.)

8. Show that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for all $a \in \mathbb{R}$.

[1]Here, $x^2 = x \cdot x$ and $x/y = x \cdot y^{-1}$.

More Practice Problems

Here are some more practice problems for the interested student. However, these problems will not be graded and need not be turned in.

9. Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n.

10. This exercise proves that $x^2 = 2$ can be solved in $\mathbb{R}$.

- **(a)** Show that $S := \{x \in \mathbb{R} : x^2 < 2\}$ has a supremum in $\mathbb{R}$. Let us call $\alpha := \sup S$. Show that $0 < \alpha < 2$.
- **(b)** Show that $(\alpha + \frac{1}{n})^2 < 2$ for some $n \in \mathbb{N}$ if $\alpha^2 < 2$.
- **(c)** Show that $(\alpha - \frac{1}{n})^2 > 2$ for some $n \in \mathbb{N}$ if $\alpha^2 > 2$.
- **(d)** Conclude that $\alpha^2 = 2$.

11. Let a and b real numbers satisfying $a < b$.

- **(a)** Show that there are infinitely many rational numbers between a and b.
- **(b)** Show that there are infinitely many irrational numbers between a and b.

12. Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Prove that S has a unique supremum. That is, if β and β' are both least upper bounds of S, then $\beta = \beta'$. (Your proof should be short.)

13. This exercise illustrates another proof of Corollary 3.12 of the lecture note 3. Let S be a non-empty subset of $\mathbb{R}$ that is bounded below. Define a subset T of $\mathbb{R}$ by

$$T = \{m \in \mathbb{R} : m \text{ is a lower bound of } S\}.$$

- **(a)** Show that T is non-empty and bounded above.
- **(b)** In light of the completeness axiom, the supremum of T exists in $\mathbb{R}$. Let us write $\alpha = \sup T$. Show that α is the infimum of S.