Learning Objectives

- ☐ Review some basic about logical operations.
- ☐ Learn the fundamental properties of the set of natural numbers.

0. Statements and Logical Operations

(Disclaimer: A large portion of this section is borrowed from Michael Hutchings's note.[1])

In mathematics, we study **statements**. Loosely speaking, a statement is a sentence that are either true or false, but not both. For example,

'6 is an even integer'

is a statement that is true, whereas

'12 is a prime number'

is a statement that is false. In this course, generic statements are often denoted by letters such as 'P' and 'Q'.

0.1. Logical Operations

If we have statements, then we can perform operations on statements to obtain another statement. Examples include 'not', 'and', 'or', and 'if ... then ...'. Although these logical operations share many similarities with their English counterparts, they have precise mathematical meanings that sometimes slightly differ from common usage.

- **Negation (Not).** If P is a statement, then

  'not P' (or $\neg P$, in symbols)

  is defined to be

  - true, if P is false;
  - false, if P is true.

- **Conjunction (And).** If P and Q are statements, then

  'P and Q' (or $P \wedge Q$, in symbols)

  is defined to be

  - true, when P and Q are both true;
  - false, when at least one of P or Q is false.

[1] Michael Hutchings, *Introduction to mathematical arguments*, [link].

- **Disjunction (Or).** If P and Q are statements, then

$$\text{‘}P \text{ or } Q\text{’} \quad (\text{or } P \vee Q, \text{ in symbols})$$

  is defined to be

  - true, when at least one of P and Q is true;
  - false, when both P and Q are false.

  In English, sometimes "P or Q" means that P is true of Q is true, but not both. However, this is *never* the case in mathematics.

- **Conditional (If ... then ...).** If P and Q are statements, then

$$\text{‘if } P \text{ then } Q\text{’} \quad (\text{or } P \Rightarrow Q, \text{ in symbols})$$

  is defined to be

  - true, when either both P and Q are true or P is false;
  - false, when P is true and Q is false.

  Again, this slightly deviates from the usage of the conditional in English especially when the assumption is false. Indeed, if P is false, then we *always* interpret $P \Rightarrow Q$ to be true.[2]

- **Equivalence (If and only if).** If P and Q are statements, then

$$\text{‘}P \text{ if and only if } Q\text{’} \quad (\text{or } P \Leftrightarrow Q, \text{ in symbols})$$

  is defined to be

  - true, when P or Q are both true or both false;
  - false, when one of P, Q is true and the other is false.

0.2. Predicates and Quantifiers

- Consider the sentence

$$P(x) \quad : \quad \text{‘}x \text{ is an even integer.’}$$

  This is not a statement in the sense we learned, because we can't determine whether it is true or false without knowing what x is. In general, a sentence whose truth value depends on some variables is called a **predicate**. So, how can we turn a predicate to a statement?

- A first and obvious way is to specify the value of x. For instance, if we substitute x with 32 in the above example of predicate, then

$$P(32) \quad : \quad \text{‘}32 \text{ is an even integer.’}$$

[2] This type of truth is often called ‘vacuously true’ since the resulting statement does not carry any valuable information regarding the conclusion Q.

is now a statement that is true.

- Instead of talking about the truth of $P(x)$ for an individual value of x, we may also ask whether $P(x)$ is simultaneously true for many values of x or whether $P(x)$ is true for some values of x. Such logical operations are called **quantifiers**. The two most important examples of quantifiers are as follows:

- **Universal quantifier (For all).** If $P(x)$ is a predicate in the variable x and S is a set, then we may consider the statement

  'For every x in S, $P(x)$ is true' (or $(\forall x \in S)\, P(x)$, in symbols).

  The phrase 'for every' (or 'for all', 'for any', ...) is called a universal quantifier.

- **Existential quantifier (There exists).** If $P(x)$ is a predicate in the variable x and S is a set, then we may consider the statement

  'There exists x in S such that $P(x)$ is true' (or $(\exists x \in S)\, P(x)$, in symbols).

  The phrase 'there exists' (or 'there is at least one', 'for some', ...) is called an existential quantifier.

- The order of quantifiers is very important, since changing the order of quantifiers may alter the meaning of a statement. For example,

$$(\forall x \in \mathbb{Z})\, (\exists y \in \mathbb{Z})\, x < y$$

  is true, whereas

$$(\exists y \in \mathbb{Z})\, (\forall x \in \mathbb{Z})\, x < y$$

  is false.

0.3. How to negate a statement?

- Consider a situation where you want to show that a statement P is false. This is often accomplished by showing that its negation (i.e., 'not P') is true. For this reason, it is useful to know how to negate a given statement.

- The following table summarizes the negation of the logical operations discussed above:

Statement	How to Negatie it
P	not P
P and Q	(not P) or (not Q)
P or Q	(not P) and (not Q)
$P \Rightarrow Q$	P and (not Q)
$P \Leftrightarrow Q$	(P and not Q) or (Q and not P)
$(\exists x \in S)\, P(x)$	$(\forall x \in S)$ not $P(x)$
$(\forall x \in S)\, P(x)$	$(\exists x \in S)$ not $P(x)$

Example 0.1. Negate the statement

$$(\exists x \in \mathbb{N})\,(\forall y \in \mathbb{N})\,\Big((y < x) \Rightarrow \big((\forall z \in \mathbb{N})\, x \neq yz\big)\Big)$$

(Note: This reads in common English as 'there exists a prime number.')

1. The set $\mathbb{N}$ of Natural Numbers

- $\mathbb{N}$ denotes the set of all positive integers[3]:

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

- What are the fundamental properties of $\mathbb{N}$? For each $n \in \mathbb{N}$, we write

$$\mathsf{Suc}(n) = n + 1$$

and call it the **successor** of n. Using this, we can describe a set of axioms that characterizes the fundamental properties of $\mathbb{N}$:

Peano Axioms. The set $\mathbb{N}$ satisfies the following properties:

(N1)

(N2)

(N3)

(N4)

(N5)

[3] In the literature, there are two different conventions for defining natural numbers; one includes 0 to the natural numbers, whereas the other excludes 0 from it. In this course, we will reserve the notation $\mathbb{N}$ for the set of positive integers, and will prefer using more clear terms (such as 'positive integers' or 'non-negative integers') to a possibly ambiguous term 'natural numbers'.

- If we represent the statement $\text{Suc}(m) = n$ by the arrow $m \longrightarrow n$, then $\mathbb{N}$ may be visualized as:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

  Moreover, we can visualize each axiom and appreciate how the axioms (N1)–(N5) altogether characterize $\mathbb{N}$ without redundancy:

- Axiom (N5) deserves particular attention, because it serves the basis of one of the most powerful machinery in mathematics, called the **principle of mathematical induction**.[4]

> **Principle of Mathematical Induction (PMI).** Consider a list of statements $P_1, P_2, P_3, \ldots$ so that each P_n may or may not be true, possibly depending on the value of n. Suppose the following properties hold:
>
> **(I1)** P_1 is true, (Base Case)
>
> **(I2)** For each $n \in \mathbb{N}$, if P_n is true, then P_{n+1} is true. (Inductive Step)
>
> Then all the statement $P_1, P_2, P_3, \ldots$ are true.

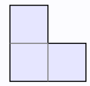
Proof.

> **Example 1.1.** Prove that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ holds for any positive integer n.

Proof.

[4]Despite its name, this is actually a form of deductive reasoning. This should not be confused with inductive reasoning.

Example 1.2. Show that a $2^n \times 2^n$ grid with one square at the corner removed can be tiled with L-triominoes. Here, an L-triomino is a shape consisting of three squares joined in an 'L'-shape:



Proof.

- If we instead have a list of statements $P_m, P_{m+1}, P_{m+2}, \ldots$ that starts at the index m instead of 1, a version of PMI still works for an obvious reason.

Example 1.3. Prove that $2^n > n^2$ holds for any integer $n \geq 5$.

Proof.

- Here is an important remark. In (I2), we are not required to verify the truth of any of P_n and P_{n+1} at all. We are merely required to prove:

  "If we assume that P_n is true, then under that assumption P_{n+1} is also true."

  In particular, it is possible that (I2) holds true even when all of P_n's are false!

Example 1.4. Consider the statement

$$P_n \,:\, n^2 + 5n + 1 \text{ is an even integer.}$$

(a) Verify that the statement (I2), the inductive step, is true for this P_n.

(b) Show that P_n is actually *false* for any $n \in \mathbb{N}$.

Proof.