UCLA Math151A Fall 2021
Lecture 5
20211004

continued..
Fixed Point Iteration

Optional reading: book 2.2.

Recall last time

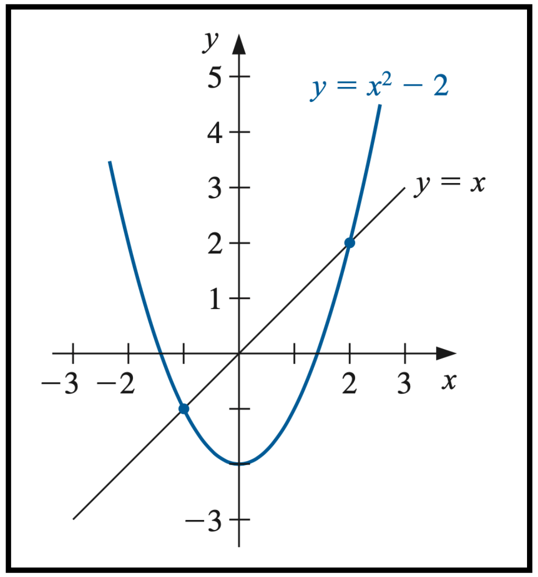
A *fixed point problem* is a problem where one seeks solutions to

$$x = g(x)$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is a function.

A solution x^* such that $x^* = g(x^*)$ is called a *fixed point* of g.

A fixed point for g $\longrightarrow$ $y = g(x)$ intersects $y = x$



A *fixed point iteration* is an iteration of the form $x_{k+1} = g(x_k)$, $k = 0, 1, 2, \ldots$.

Existence of a Fixed Point

From last time, we listed the existance theorem for fixed points,

Theorem 5.1 (Existence).

Let $g \in \mathbf{C}\,[a,b])$ with $a \leq g(x) \leq b\ \forall x \in [a,b]$,

then $\exists$ at least one fixed point p s.t. $g(p) = p$.

let's prove it.

Theorem 5.1 (Existence).

Let $g \in \mathrm{C}\,[a,b])$ with $a \leq g(x) \leq b \ \forall x \in [a,b]$,

then $\exists$ at least one fixed point p s.t. $g(p) = p$.

Proof. First check if an end point is a fixed point.

If $g(a) = a$ or $g(b) = b$, we are done.

Otherwise, let's define $G(x) := g(x) - x$.

Goal: Use IVT to prove that G has a root.

Then since $g \in [a,b]$, we know

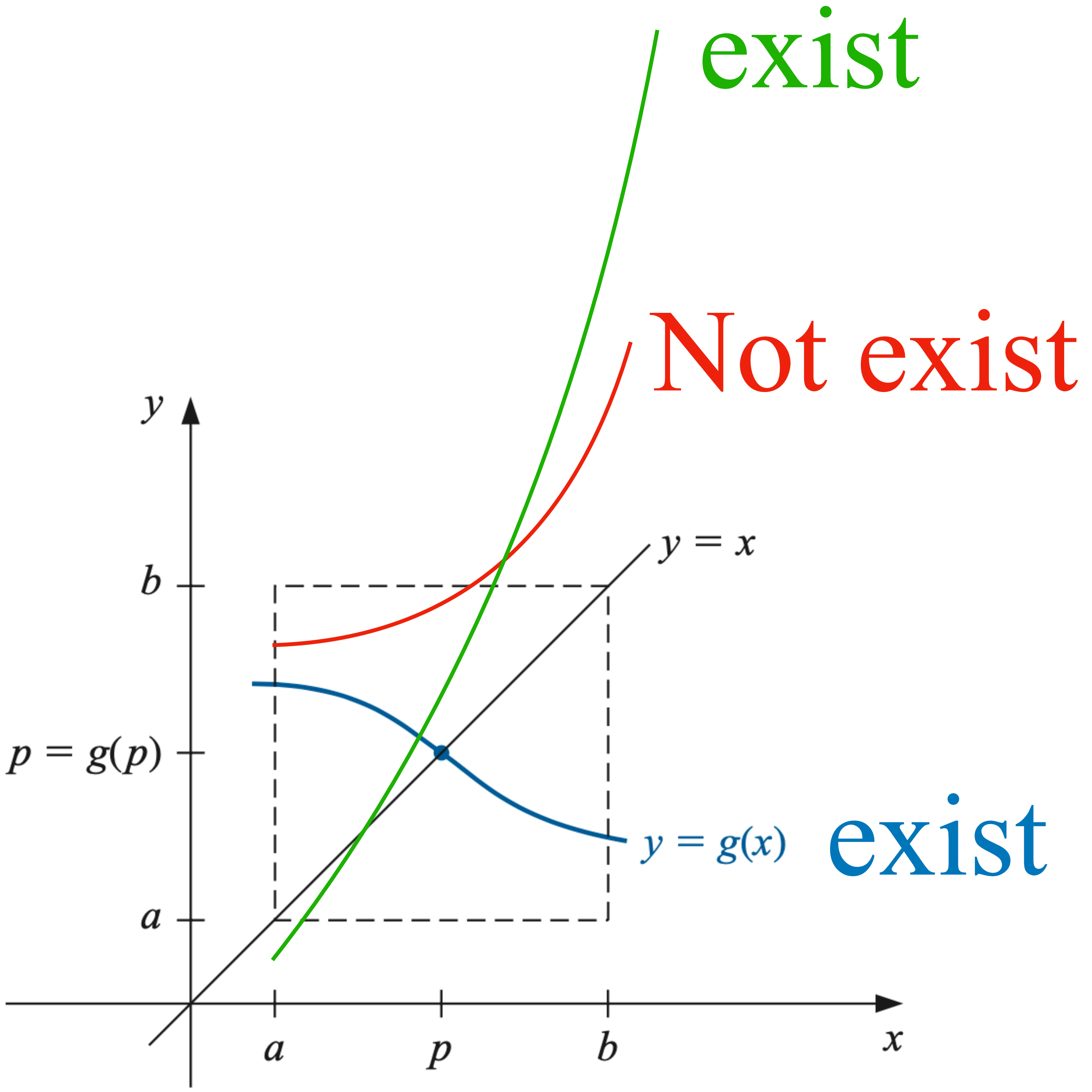
$$G(a) = g(a) - a > 0, \quad G(b) = g(b) - b < 0, \quad \Rightarrow G(a)G(b) < 0.$$

strict inequality
because we already checked the end points.

$G \in C([a,b])$ (because sum of continous functions is continuous).

Therefore by I.V.T., $\exists p$ s.t. $G(p) = 0$. I.e., $\exists p \in [a,b]$ s.t. $g(p) = p$.

Example



Remark: the theorem is just a sufficient condition for existence.

Uniqueness and F.P.I Convergence

Theorem 5.2 (FPI convergence with Lipschitz continuity).

First, everything from the existance theorem

$$(g \in C([a,b]), g \in [a,b] \quad (*))$$

Further assume

$$\exists k \in (0,1) \text{ s.t. } \underline{|g(x) - g(y)| \le k|x-y|, \ \forall x, y \in [a,b]} \qquad (**)$$

This is called a Lipschitz condition
In general $k \ge 0$ is called the Lipschitz constant.

Here in this theorem we are saying g has Lipschitz constant $\in (0,1)$.

Then

1. $\exists$ unique p s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to p.
3. Error estimate: $|p_n - p| \le k^n \max\{b - p_0, p_0 - a\}$.

$\square$

$(g \in C([a,b]),\ g \in [a,b] \quad (*))$

$\exists k \in (0,1)$ s.t. $|g(x) - g(y)| \le k|x-y|,\ \forall x, y \in [a,b] \qquad (**)$

1. $\exists$ unique p s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to p.
3. Error estimate: $|p_n - p| \le k^n \max\{b - p_0, p_0 - a\}$.

(1) Prove by contradiction.

Assume $\exists$ two different fixed points p, q, then

$$|g(p) - g(q) = |p - q|.$$

But by $(**)$ we know

$$|g(p) - g(q)| \le k|p - q|.$$

This implies that

$$|p - q| \le k|p - q|,$$

which cannot be true since $p \neq q$ and $k \in (0,1)$. Contradiction!

$(g \in C([a,b]),\ g \in [a,b] \quad (*))$

$\exists k \in (0,1)$ s.t. $|g(x) - g(y)| \le k|x-y|,\ \forall x, y \in [a,b] \qquad (**)$

1. $\exists$ unique p s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to p.
3. Error estimate: $|p_n - p| \le k^n \max\{b - p_0, p_0 - a\}$.

(2,3) OK, let's directly look at $|p_n - p|$

By (**) we know that differences of g values are bounded.

So let's try to convert this to something with g values.

We know F.P.I is $g(p_n) = p_{n+1}$, also let p the solution $g(p) = p$.

$$\text{So } |p_n - p| = |g(p_{n-1}) - g(p)|.$$

By $(**)$, know: $|p_n - p| = |g(p_{n-1}) - g(p)| \le k|p_{n-1} - p|$.

Further, $k|p_{n-1} - p| = k|g(p_{n-2}) - g(p)| \le k^2|p_{n-2} - p|$.

Recursing downwards we get $|p_n - p| \le k^n|p_0 - p|$.

$(g \in C([a,b]),\ g \in [a,b] \quad (*))$

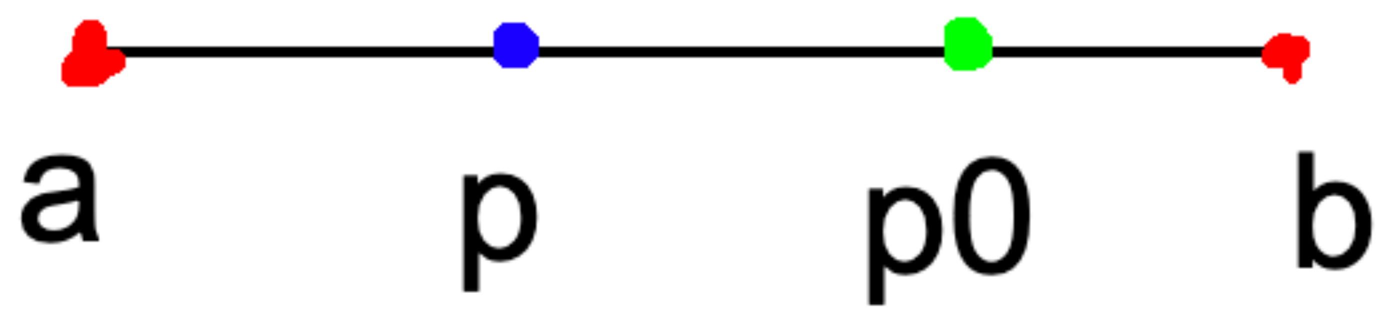
$\exists k \in (0,1)$ s.t. $|g(x) - g(y)| \leq k|x-y|,\ \forall x, y \in [a,b] \qquad (**)$

$\Rightarrow$

1. $\exists$ unique p s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to p.
3. Error estimate: $|p_n - p| \leq k^n \max\{b - p_0, p_0 - a\}$.

(2,3) $\quad \ldots\ldots \qquad |p_n - p| \leq k^n |p_0 - p|.$

Notice $\quad |p - p_0| \leq \max\{b - p_0, p_0 - a\}$



Thus $\quad |p_n - p| \leq k^n \max\{b - p_0, p_0 - a\}.$

Since $k \in (0,1)$ $\quad$ Apparently this goes to 0. $\blacksquare$

$(g \in C([a,b]),\, g \in [a,b] \quad (*))$

$\exists k \in (0,1)$ s.t. $|g(x) - g(y)| \le k|x-y|, \ \forall x, y \in [a,b] \qquad (**)$

1. $\exists$ unique p s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to p.
3. Error estimate: $|p_n - p| \le k^n \max\{b - p_0, p_0 - a\}$.

Remark

Speed of convergence depends on k.

The closer to 0 k is, the faster it converges.

The theorem gives us a way to guarantee F.P.I. convergence by looking at the Lipschitz condition.

In practice, it is sometimes more convenient to look at the derivative (since it is easy to compute).

Theorem 5.3 (FPI convergence with bounded derivative).
First, everything from the existance theorem

$$(g \in C([a,b]), g \in [a,b] \quad (*))$$

Further assume $g \in C^1[a,b]$
and that $\forall x \in [a,b], \exists k \in (0,1)$ s.t. $|g'(x)| \leq k$,

then $g(x)$ is Lipschitz with constant k.

Therefore, following Theorem 5.2,

F.P.I. converges to the unique solution.

Proof. Here we need to prove that bounded derivative gives Lipschitz.

use the Mean Value Theorem (M.V.T.),

$$\exists c \in (a,b) \text{ s.t. } \forall x, y \in [a,b], \quad g'(c) = \frac{g(x) - g(y)}{x - y}.$$

Thus, $|g(x) - g(y)| = |g'(c)||x - y| \leq k|x - y|.$ ∎

$\boxed{\text{Example of applying the theorem}}$

Example $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. $\boxed{1.365230013}$

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\frac{10}{4 + x}\right)^{1/2}$

(e) $x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5		$\boxed{\text{undefined}}$	1.360094193	1.365225594	$\boxed{\text{excellent}}$
6	$\boxed{\text{divergent}}$		1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236	$\boxed{\text{excellent}}$	
25			1.365230006		
30			1.365230013 $\boxed{\text{excellent}}$		

$x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. $\boxed{1.365230013}$

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

$(g \in C([a,b]),\ g \in [a,b] \quad (*))$

$\exists k \in (0,1)$ s.t. $|g(x) - g(y)| \le k|x - y|,\ \forall x, y \in [a,b] \qquad (**)$

Or $g \in C^1[a,b]$ and that $\forall x \in [a,b], \exists k \in (0,1)$ s.t. $|g'(x)| \le k, \quad (**)$

1. $\exists$ unique p s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to p.
3. Error estimate: $|p_n - p| \le k^n \max\{b - p_0, p_0 - a\}$.

$g_1(1) = 6$ and $g_1(2) = -12$, **Bad!**

$g_1'(x) = 1 - 3x^2 - 8x, \quad |g_1'(x)| > 1$ for all x in $[1, 2]$. **Bad!**

The Theorem doesn't say whether it fails. But there is no reason to expect convergence.

$x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. $\boxed{1.365230013}$

$$\textbf{(d)} \quad x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$$

$(g \in C([a,b]),\ g \in [a,b] \quad (*))$

$\exists k \in (0,1)$ s.t. $|g(x) - g(y)| \le k|x-y|,\ \forall x, y \in [a,b] \qquad (**)$

Or $g \in C^1[a,b]$ and that $\forall x \in [a,b], \exists k \in (0,1)$ s.t. $|g'(x)| \le k, \quad (**)$

1. $\exists$ unique p s.t. $g(p) = p$.
2. The F.P.I. $(p_{n+1} = g(p_n))$ will converge to p.
3. Error estimate: $|p_n - p| \le k^n \max\{b - p_0, p_0 - a\}$.

for all $x \in [1, 2]$.

$$|g_4'(x)| = \left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right| \le \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15,$$

Good!

Next time:

Newton's method (multiple ways of deriving it, including using the F.P.I.)