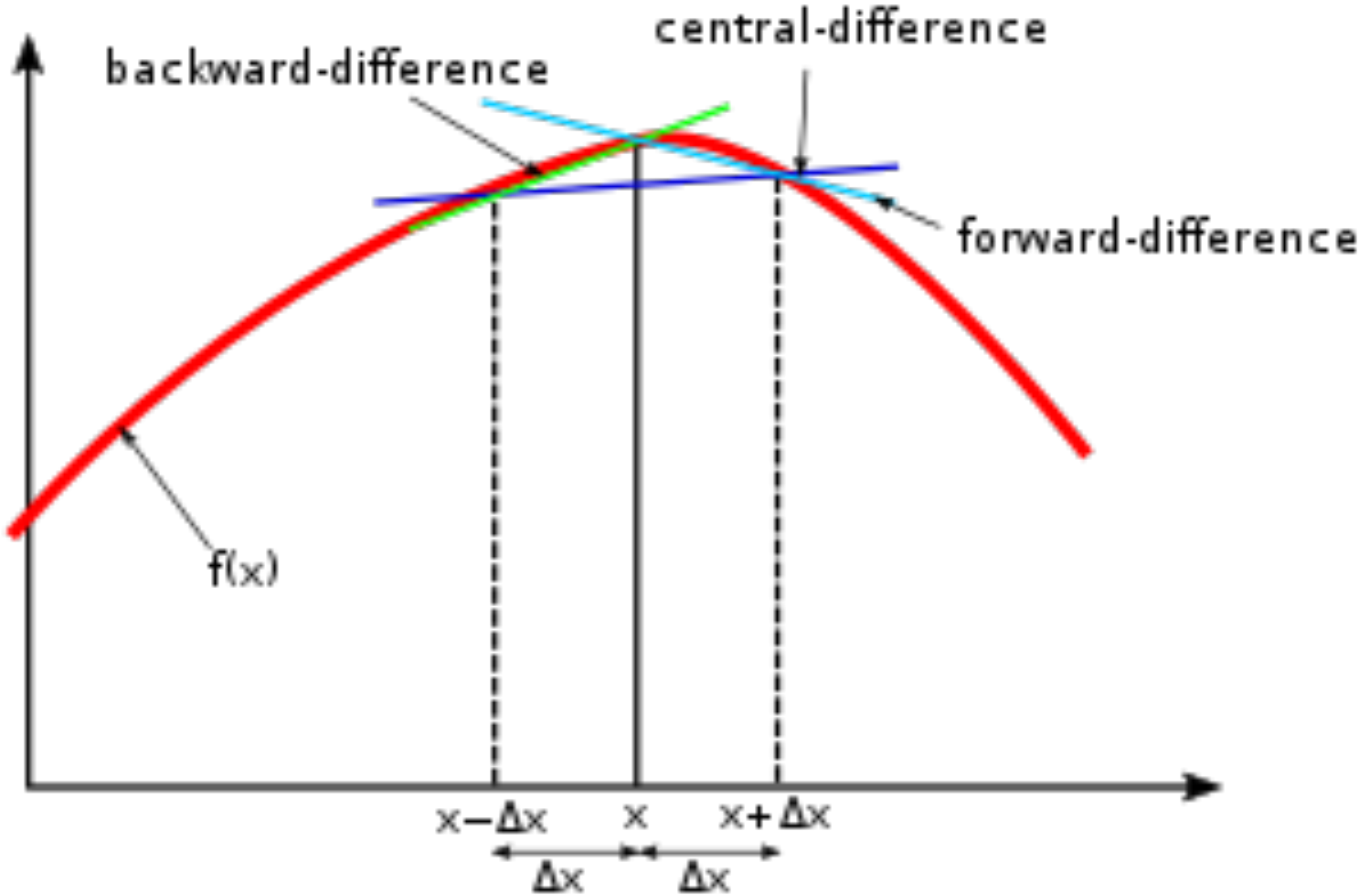
UCLA Math151A Fall 2021
Lecture 16
20211101

Richardson Extrapolation

Last time


central-difference
backward-difference
forward-difference
f(x)
x−Δx
x
x+Δx
Δx
Δx

$$\boxed{\frac{f(x_0+h)-f(x_0)}{h}} = f'(x_0) + h\frac{f''(\xi)}{2}$$

Forward Difference Formula $O(h)$

$$\boxed{\frac{f(x_0)-f(x_0-h)}{h}} = f'(x_0) + \frac{h}{2}f''(\xi).$$

Backward Difference Formula $O(h)$

$$\boxed{\frac{f(x_0+h)-f(x_0-h)}{2h}} = f'(x_0) + (f'''(\xi_1) + f'''(\xi_2))\frac{h^2}{12}$$

Centered Difference Formula $O(h^2)$

Richardson Extrapolation (R.E.)

Basic idea: generate high accuracy results using low order formulas.

Recall, for $f \in C^2([a,b])$,

$$\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) + \frac{h}{2} f''(\xi),$$

i.e., Forward Difference formula gives $O(h)$ error.

If $f \in C^3([a,b])$, then

$$\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) + \frac{h}{2} f''(x_0) + \frac{h^2}{3!} f'''(\xi),$$

error is still $O(h)$, but we kept one more term in Taylor expansion.

Want next: derive an approximation to f'(x_0), solely based on these O(h) formulas, but with error O(h^2) (a more accurate approximation).

$$\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) + \frac{h}{2}f''(x_0) + \frac{h^2}{3!}f'''(\xi),$$

first define a notation for forward difference

$$D_h^+ f(x_0) := \frac{f(x_0+h)-f(x_0)}{h}$$

$$D_{h/2}^+ f(x_0) = \frac{f(x_0+\frac{h}{2})-f(x_0)}{h/2}$$

$\Rightarrow$

$$2D_{h/2}^+ f(x_0) - D_h^+ f(x_0)$$

$$= \left(2f'(x_0) + h\frac{1}{2}f''(x_0) + 2\frac{h^2}{4}\frac{1}{3!}f''(\xi_1)\right) - \left(f'(x_0) + h\frac{1}{2}f''(x_0) + h^2\frac{1}{3!}f''(\xi_2)\right)$$

$$= f'(x_0) + O(h^2)$$

In summary, we combined two first order formula to get a second order method.

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This is a powerful idea, not restricted to numerical differentiation.

let M be the true quantity that we want to compute,
$N(h)$ be the approximation $\longrightarrow$ "N" for "Numerical".

E.g., $M = f'(x_0)$, $N = D_h^+ f(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$

Further, assume M can be written as

$$M = N(h) + k_1 h + k_2 h^2 + k_3 h^3 + \dots, \qquad (*)$$

where k_1, k_2, k_3 are constants independent of h. Then,

$$M = N(\frac{h}{2}) + k_1 \frac{h}{2} + k_2 (\frac{h}{2})^2 + k_3 (\frac{h}{2})^3 + \dots, \qquad (**)$$

$2(**) - (*)$

$$M = 2N(\frac{h}{2}) - N(h) \underbrace{- \frac{1}{2} k_2 h^2 - \frac{3}{4} k_3 h^3 + \dots}_{O(h^2)}$$

What if we have higher order? For instance, suppose

$$M = N(h) + k_1 h^2 + k_2 h^4 + k_3 h^6 + \ldots,$$

E.g.

$$\frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + (f'''(\xi_1) + f'''(\xi_2))\frac{h^2}{12}$$

We can repeat the process, cancel out h^2 terms, and get:

$$M = \frac{1}{3}\left(4N(\frac{h}{2}) - N(h)\right) + O(h^4).$$

Proof: will be in HW6