# UCLA Math151A Fall 2021
# Lecture 11
# 20211020

# Divided Differences, Runge’s Phenomenon

Optional reading: book 3.3

# LAST TIME

## Neville's Method

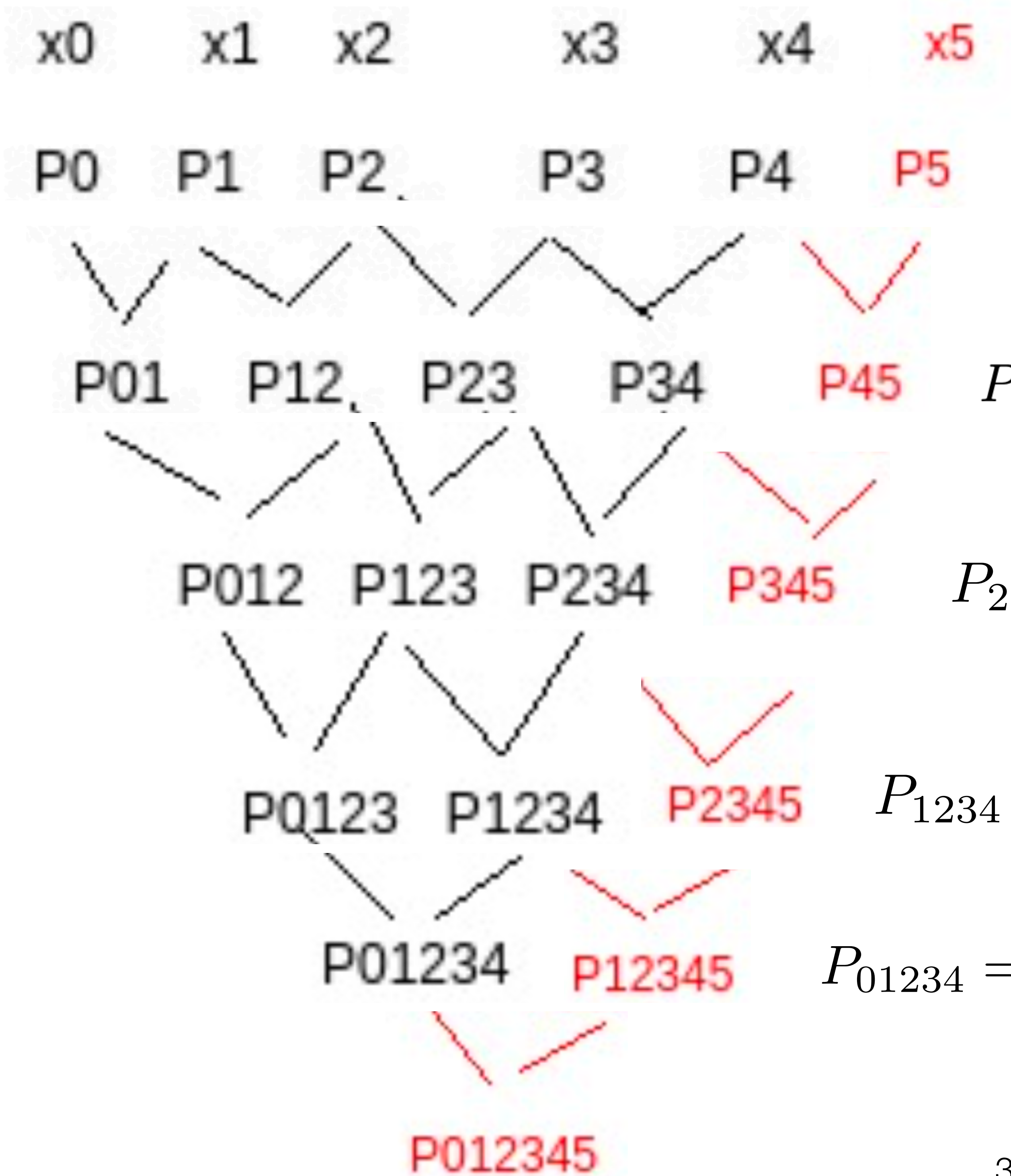
**Neville's Method** lets us re-use our previous work to get higher degree polynomial approximation to f(x) for a specific x.

## Neville's Method

Let $f$ be defined at points $x_0, x_1, x_2, \ldots, x_k$

$$P_0 = f(x_0), P_1 = f(x_1), P_2 = f(x_2)$$

$$P(x) = \left[(x - x_j)P_{012\ldots(j-1)(j+1)\ldots k}(x) - (x - x_i)P_{012\ldots(i-1)(i+1)\ldots k}(x)\right] \frac{1}{x_i - x_j}$$




EXAMPLE

$$P_{01}(x) := \frac{1}{x_1 - x_0}((x - x_0)P_1 - (x - x_1)P_0)$$

$$P_{234} = \frac{1}{x_4 - x_2}((x - x_2)P_{34} - (x - x_4)P_{23})$$

$$P_{1234} = \frac{1}{x_4 - x_1}((x - x_1)P_{234} - (x - x_4)P_{123})$$

$$P_{01234} = \frac{1}{x_4 - x_0}((x - x_0)P_{1234} - (x - x_4)P_{0123})$$

# EXAMPLE

Values of various interpolating polynomials at x = 1.5

$x_0 = 1.0, x_1 = 1.3, x_2 = 1.6,$

| | | | | |
|---|---|---|---|---|
| 1.0 | 0.7651977 P0 | | | |
| 1.3 | 0.6200860 P1 | 0.5233449 P01(1.5) | | |
| 1.6 | 0.4554022 P2 | 0.5102968 P12(1.5) | 0.5124715 P012(1.5) | |
| $\boxed{1.9}$ | $\boxed{0.2818186}$ P3 | $\boxed{0.5132634}$ P23(1.5) | $\boxed{0.5112857}$ P123(1.5) | $\boxed{0.5118127}$ |

P0123(1.5)

$x_3 = 1.9$

# Divided Differences

**Neville's Method** lets us re-use our previous work to get higher degree polynomial approximation to f(x) for a specific x.

**Divided difference method** is useful for successively generating higher degree polynomial expressions (as a function of x).

# Divided Difference Method

$$f[x_i] = f(x_i)$$

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$P_n(x) = P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1}).$$

$f[x_0]$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$f[x_1]$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$\boxed{f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}}$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

$$\boxed{f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}}$$

$f[x_2]$

$$\boxed{f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}}$$

$\boxed{f[x_3]}$

| i | $x_i$ | $f[x_i]$ | 1st order differences | 2nd order differences | 3rd order differences | 4th order differences |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | | | | |
| | | | $\frac{1-0}{1-0}=1$ | | | |
| 1 | 1 | 1 | | $\frac{7-1}{2-0}=3$ | | |
| | | | $\frac{8-1}{2-1}=7$ | | $\frac{6-3}{3-0}=1$ | |
| 2 | 2 | 8 | | $\frac{19-7}{3-1}=6$ | | $\frac{1-1}{4-0}=0$ |
| | | | $\frac{27-8}{3-2}=19$ | | $\frac{9-6}{4-1}=1$ | |
| 3 | 3 | 27 | | $\frac{37-19}{4-2}=9$ | | |
| | | | $\frac{64-27}{4-3}=37$ | | | |
| 4 | 4 | 64 | | | | |

$$P_n(x) = P(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots$$
$$+ f[x_0,x_1,\dots,x_n](x-x_0)(x-x_1)\dots(x-x_{n-1}).$$

$$P(x) = 0 + 1(x-0) + 3(x-0)(x-1) + 1(x-0)(x-1)(x-2) + 0(x-0)(x-1)(x-2)(x-3)$$
$$= x + 3x(x-1) + x(x-1)(x-2)$$

Recall MVT:

**Theorem 1.2.** If $f \in C[a,b]$ and $f$ is differentiable on $(a,b)$, then $c \in (a,b)$ exists with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1st order divided difference is highly related to the first derivative.

How about the high order ones?

How does the k-th order divided difference relate to the k-th derivative?

**Theorem 12.1.**

Suppose $f \in C^n([a,b])$ with $\{x_i\}_{i=0}^n \in [a,b]$ distinct,

then $\exists \xi \in (a,b)$ s.t. $f[x_0, x_1, \ldots, x_n] = \dfrac{f^{(n)}(\xi)}{n!}$

Proof. The proof uses generalized Rolle's theorem and Derivative of Multiplied Monomials (Recall lecture 10)

**Theorem 10.1** ((I) Generalized Rolle's Theorem)**.**

Let $f \in C^n([a,b])$. Suppose $\exists\ n+1$ distinct roots of $f$ on $[a,b]$.

Then $\exists\ \xi \in (a,b)$ s.t. $f^{(n)}(\xi) = 0$. $\square$



**Lemma 10.1** $\dfrac{d^{n+1}}{dt^{n+1}}(t-t_0)(t-t_1)\ldots(t-t_n) = (n+1)!$ $\square$

**Theorem 12.1.**

Suppose $f \in C^n([a,b])$ with $\{x_i\}_{i=0}^n \in [a,b]$ distinct,
then $\exists \xi \in (a,b)$ s.t. $f[x_0, x_1, \ldots, x_n] = \dfrac{f^{(n)}(\xi)}{n!}$

Let $\quad g(x) := f(x) - P_n(x)$

$$P_n(x) = P(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \ldots$$
$$+ f[x_0,x_1,\ldots,x_n](x-x_0)(x-x_1)\ldots(x-x_{n-1}).$$

Then $\quad g(x_i) = f(x_i) - P_n(x_i) = 0 \quad$ for $0 \le i \le n$.

By (I) Generalized Rolle's Theorem, $\exists \xi \in [a,b]$ s.t. $g^{(n)}(\xi) = 0$.

$$g^{(n)}(x) = f^{(n)}(x) - P_n^{(n)}(x) = f^{(n)}(x) - f[x_0,x_1,\ldots,x_n]n!$$

(all terms vanish except the final one, and use the lemma)

$$g^{(n)}(\xi) = f^{(n)}(\xi) - f[x_0,x_1,\ldots,x_n]n! = 0$$

$$\boxed{\Rightarrow \quad \frac{f^{(n)}(\xi)}{n!} = f[x_0,x_1,\ldots,x_n].}$$

■

## Runge's Phenomenon

Now let's look into potential challenges with Lagrangian polynomial.

Recall the theorem from lecture 10.

We want to do high degree polynomial interpolation (large n).

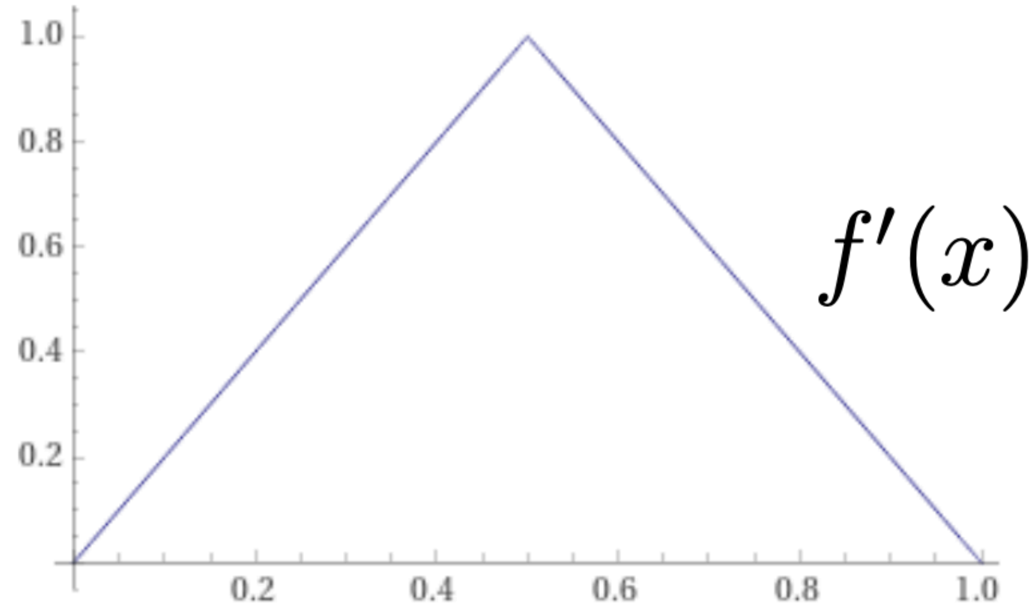
**Theorem 10.2** (Error of Lagrangian Polynomial Interpolation).

Let $\{x_0, x_1, \ldots, x_n\} \in [a, b]$ be distinct. Let $f \in C^{n+1}([a, b])$, $P(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$, then $\forall x \in [a, b], \exists \xi(x) \in (a, b)$ s.t.

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n)$$

$$= P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^{n}(x - x_k) \qquad \square$$

$f(x) = 1 - 2|x - \frac{1}{2}|$




$f'(x)$ is not continuous.

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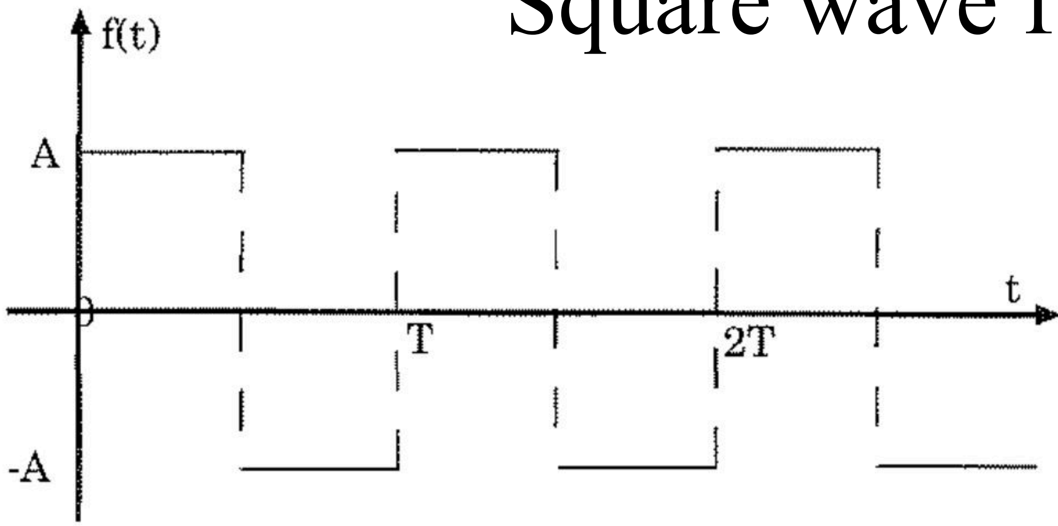
Square wave function.

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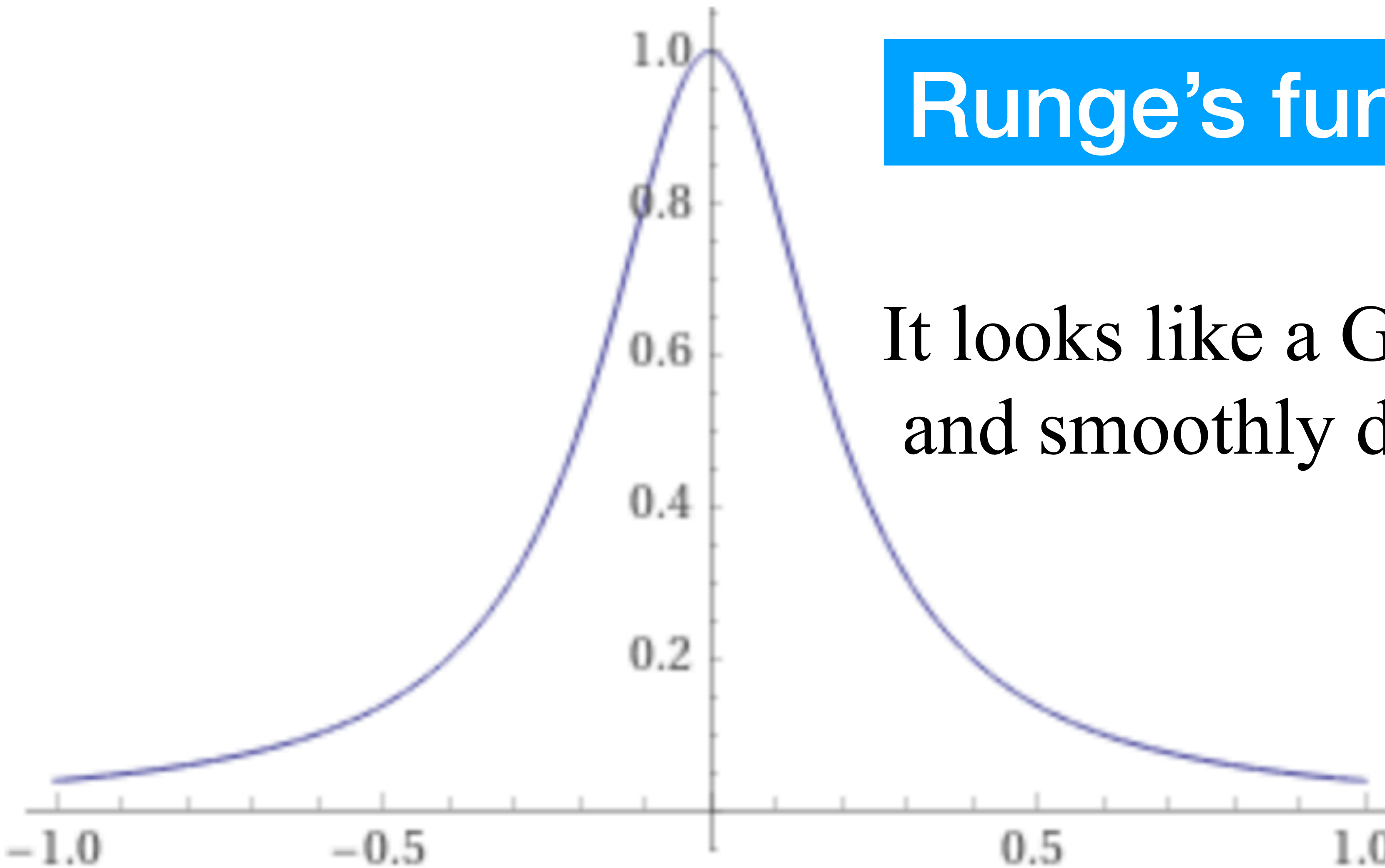
$$\begin{aligned} f(x) &= P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n) \\ &= P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{k=0}^n (x - x_k) \end{aligned}$$

$\square$

Let's look at a super smooth function instead..

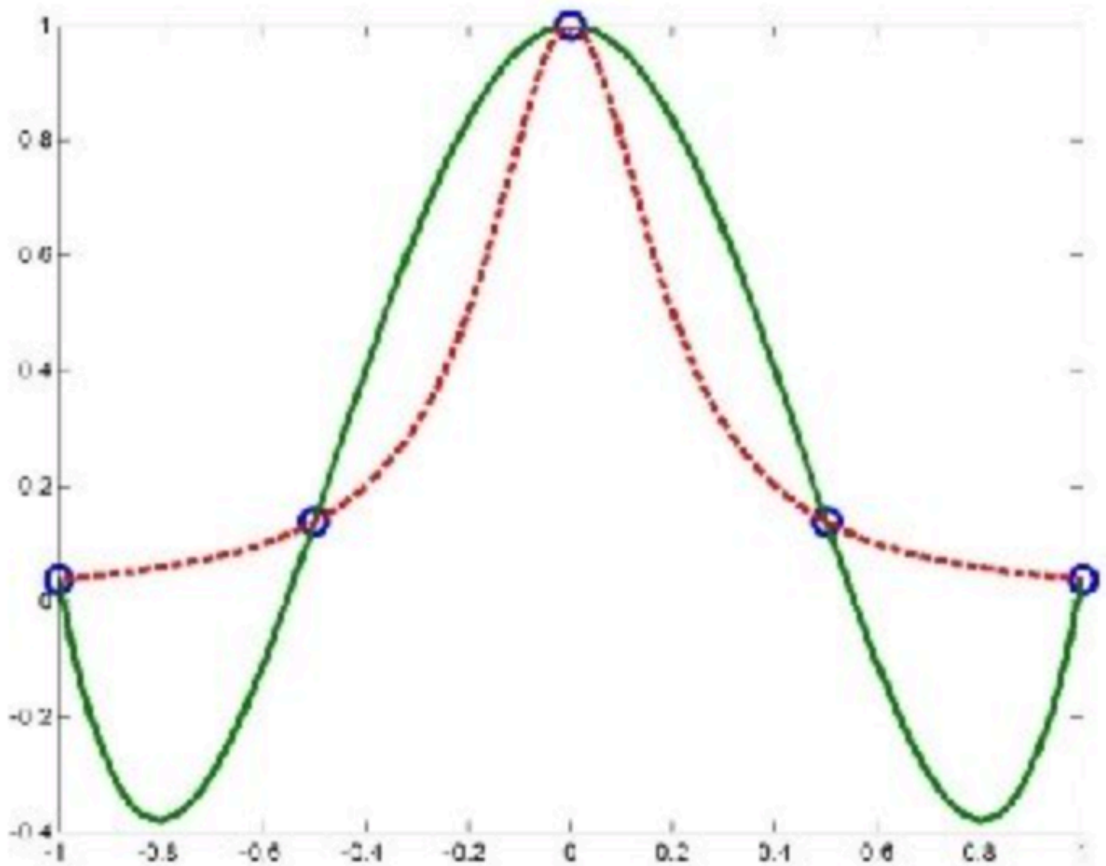
$$f(x) = \frac{1}{1+25x^2} \quad \in C^\infty[-1,1]$$



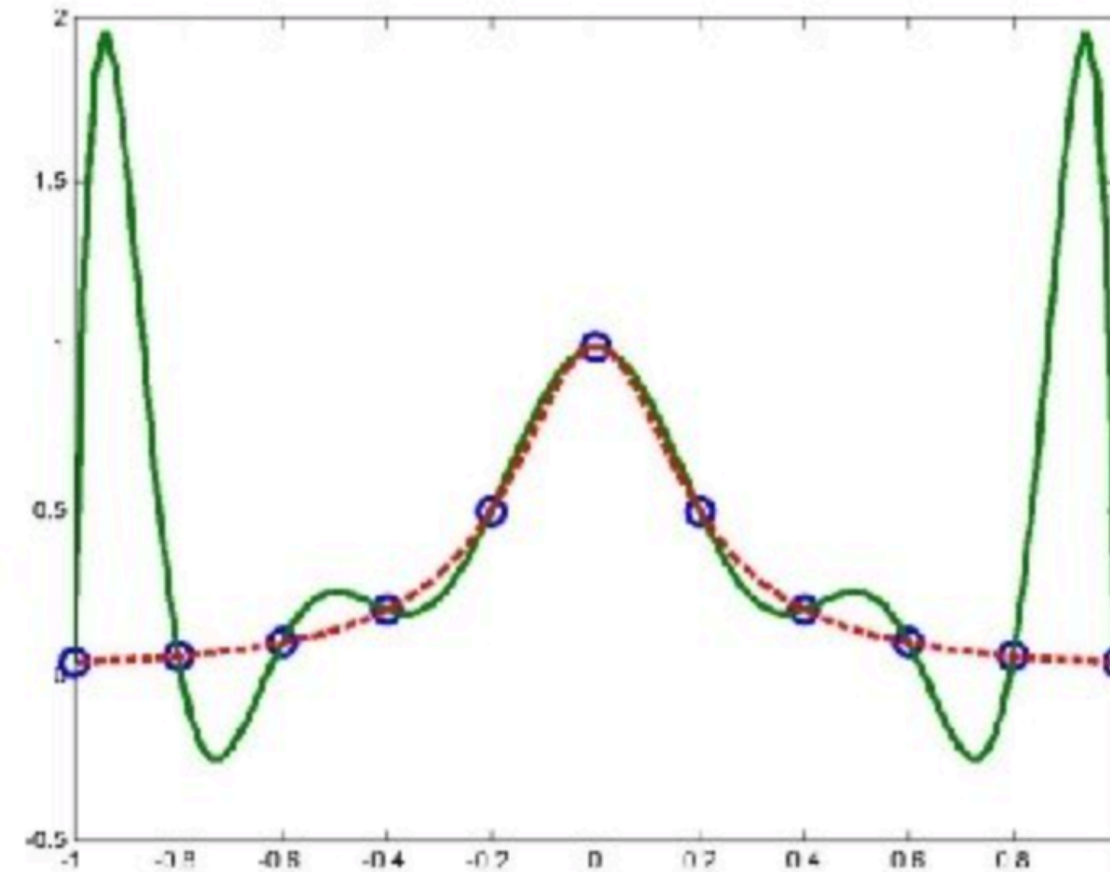

**Runge's function**

It looks like a Gaussian with max at 1 and smoothly decreases to 0.

If L.P. with equilspaced nodes is used: $x_i = x_0 + ih, 0 \le i \le n, h = \frac{b-a}{n}$,

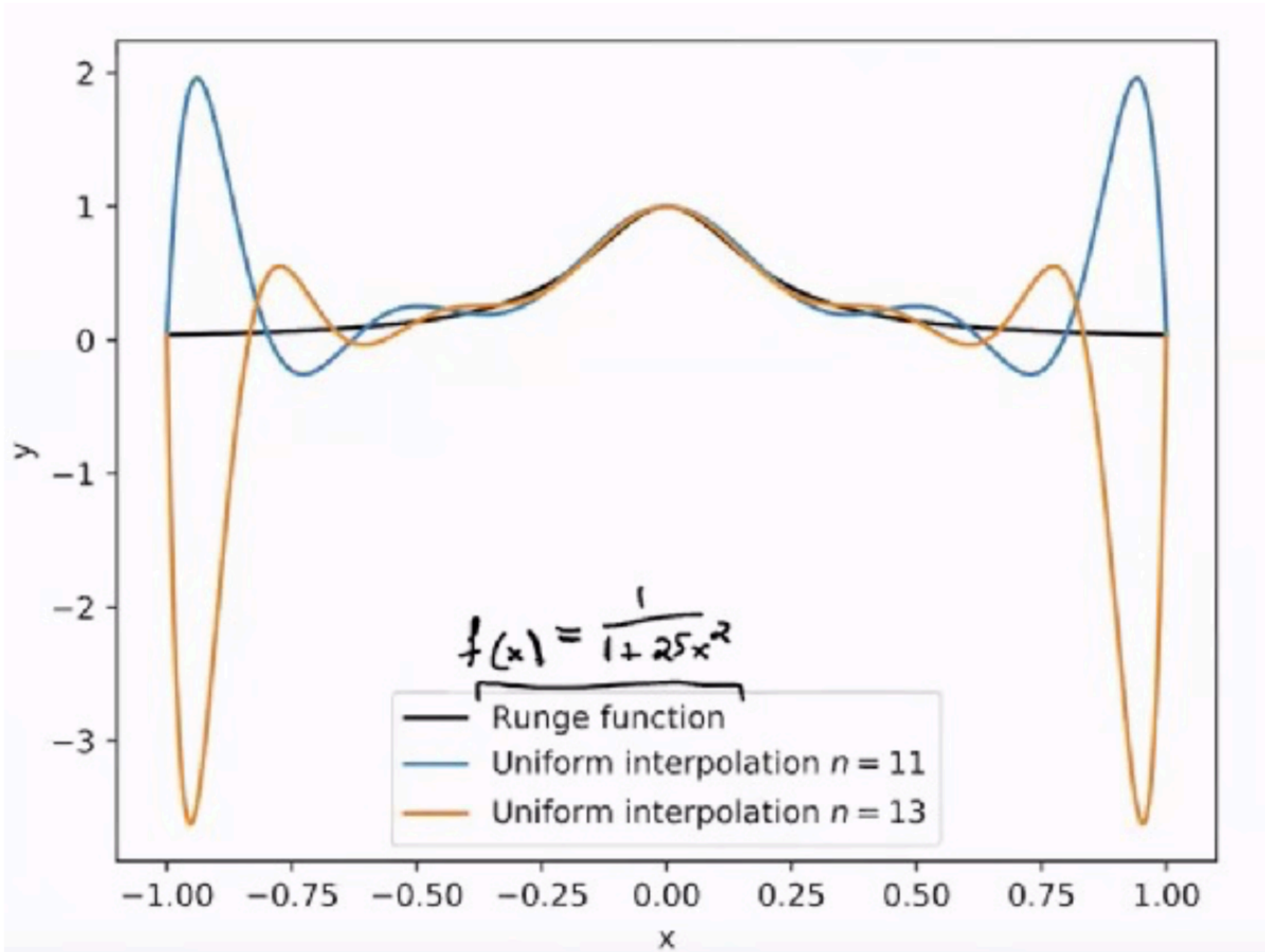


**4th-order**



**10th-order**

If we increase n further, oscillation will have higher magnitude.
This is called Runge's phenomenon.




It was discovered by Carl David Tolmé Runge (1901) when exploring the behavior of errors when using polynomial interpolation to approximate certain functions.

The discovery was important because it shows that going to higher degrees does not always improve accuracy.