17. Continuous Functions

In this section, we introduce the notion of continuity of functions from a subset of $\mathbb{R}$ to $\mathbb{R}$ and study its properties.

Definition 17.1. Let f be a real-valued function whose domain $\mathrm{dom}(f)$ lies in $\mathbb{R}$.

(a) f is **continuous at** a in $\mathrm{dom}(f)$ if, for every sequence (x_n) in $\mathrm{dom}(f)$ such that $x_n \to a$, we have $f(x_n) \to f(a)$.

(b) Let $S \subseteq \mathrm{dom}(f)$. Then f is **continuous on** S if f is continuous at each point of S.

(c) f is **continuous** if it is continuous on $\mathrm{dom}(f)$.

The next result provides an alternative characterization of continuity, called the ϵ–δ definition. Despite its abstract formulation, it is often adopted as the definition of continuity in the literature for its versatility and generalizability.

Theorem 17.2. Let f be a real-valued function defined on a subset of $\mathbb{R}$. Then f is continuous at a point $a \in \mathrm{dom}(f)$ if and only if

$$\begin{gathered}\text{For each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ x \in \mathrm{dom}(f) \text{ and } |x - a| < \delta \text{ implies } |f(x) - f(a)| < \varepsilon.\end{gathered} \tag{1}$$

Example 17.1. Let $f(x) = x^2 + 1$ for $x \in \mathbb{R}$. Prove f is continuous on $\mathbb{R}$ by

(a) Using the definition,

(b) Using the ϵ–δ property of Theorem 17.2.

Example 17.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0; \end{cases}$$

(a) Prove f is not continuous at 0.

(b) Prove that $g(x) = xf(x)$ for $x \in \mathbb{R}$ is continuous at 0.

17.1. Continuity and Operations on Functions

If f and g are real-valued functions, then we can combine f and g to obtain new functions:

Function	Domain	Codomain	Formula
$f+g$	$\mathrm{dom}(f) \cap \mathrm{dom}(g)$	$\mathbb{R}$	$(f+g)(x) = f(x) + g(x)$
fg	$\mathrm{dom}(f) \cap \mathrm{dom}(g)$	$\mathbb{R}$	$(fg)(x) = f(x)g(x)$
f/g	$\mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) : g(x) \neq 0\}$	$\mathbb{R}$	$(fg)(x) = f(x)/g(x)$
$g \circ f$	$\{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}$	$\mathbb{R}$	$(g \circ f)(x) = g(f(x))$
$\max\{f,g\}$	$\mathrm{dom}(f) \cap \mathrm{dom}(g)$	$\mathbb{R}$	$\max\{f,g\}(x) = \max\{f(x), g(x)\}$
$\min\{f,g\}$	$\mathrm{dom}(f) \cap \mathrm{dom}(g)$	$\mathbb{R}$	$\min\{f,g\}(x) = \min\{f(x), g(x)\}$

These new functions are continuous if f and g are continuous.

Theorem 17.3. Let f and g be real-valued functions defined on subsets of $\mathbb{R}$ that are continuous at $a \in \mathbb{R}$. Then

(a) $f+g$ is continuous at a;

(b) fg is continuous at a;

(c) f/g is continuous at a provided $g(a) \neq 0$.

Theorem 17.4. If f is continuous at a and g is continuous at $f(a)$, then the composite function $g \circ f$ is continuous at a.

Example 17.3.

(a) Prove $x \mapsto kx$ for $x \in \mathbb{R}$ is continuous for any constant $k \in \mathbb{R}$.

(b) Prove $x \mapsto |x|$ for $x \in \mathbb{R}$ is continuous.

(c) Use the identity

$$\max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2} \quad \text{for any} \quad x, y \in \mathbb{R}$$

to show that if f and g are continuous at a, then $\max\{f, g\}$ is continuous at a.

Example 17.4. (Vacuously Continuous) Let D be the subset of $\mathbb{R}$ given by

$$D = [1,2] \cup \{3\}.$$

(a) Let (x_n) be a sequence in D that converges to 3. Prove $x_n = 3$ for all sufficiently large n.

(b) Prove that any function $f : D \to \mathbb{R}$ is continuous at 3.

18. Properties of Continuous Functions

18.1. Continuity and Extreme Values

- For a function $f : A \to B$ and a subset $S \subseteq A$, the **image of S under f** is defined as:

$$f(S) = \{f(x) : x \in S\}$$

- A function $f : A \to \mathbb{R}$ is said to be **bounded** if its range $f(A)$ is a bounded subset of $\mathbb{R}$.

Theorem 18.1. (Extreme Value Theorem)[1] Let f be a continuous real-valued function on a closed interval $[a, b]$. Then f is bounded. Moreover, f assumes its maximum and minimum values on $[a, b]$; that is, there exists $x_{\min}, x_{\max} \in [a, b]$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \text{for all} \quad x \in [a, b].$$

Example 18.1. In each of the following problems, find an example of a real-valued function on an open interval (a, b) satisfying the given condition.

(a) f is unbounded on (a, b);

(b) f is bounded but does not attain its maximum on (a, b).

[1]In topology, this statement generalizes as follows: A continuous function maps compact sets to compact sets.

18.2. Continuity and Intermediate Values

Theorem 18.2. (Intermediate Value Theorem)[2] Let f be a continuous real-valued function on an interval I. Then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$, and y lies between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ such that $f(c) = y$.

Corollary 18.3. Let f be a continuous real-valued function on an interval I. Then the image $f(I)$ is an interval or a singleton.

The above result easily follows from the Intermediate Value Theorem by the characterization of intervals in $\mathbb{R}$: a non-empty subset J of $\mathbb{R}$ is either an interval or a singleton if and only if

$$x, y \in J \text{ and } x < y \quad \text{implies} \quad [x, y] \subseteq J.$$

See the proof of Corollary 18.3 of the textbook for more details.

Example 18.2.[3] Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a **fixed point** in $[0, 1]$, i.e., a point c in $[0, 1]$ such that $f(c) = c$.

[2] In topology, this statement generalizes as follows: A continuous function maps connected sets to connected sets.

[3] This is the one-dimensional version of the Brouwer Fixed-Point Theorem.

18.3. Continuity and Inverse Function

The next theorem characterizes continuous one-to-one functions on an interval.

Theorem 18.4. Let f be a continuous function on an interval I. Then f is one-to-one if and only if it is strictly monotone [i.e., either strictly increasing or strictly decreasing].

Proof. The direction $(\Leftarrow)$ is obvious. So we will only prove $(\Rightarrow)$.

Suppose f is a continuous, one-to-one function on an interval I. We first show that f is strictly monotone on each finite subset of I:

Claim. For any $x_0 < x_1 < \cdots < x_n$ in I, either

$$f(x_0) < f(x_1) < \cdots < f(x_n) \qquad \text{or} \qquad f(x_0) > f(x_1) > \cdots > f(x_n).$$

Since the claim is trivial if $n = 0$ or $n = 1$, we will only prove the claim for $n \geq 2$.

- *Base Case.* Suppose the claim is not true for $n = 2$. Then either

  $$f(x_1) > \max\{f(x_0), f(x_2)\} \qquad \text{or} \qquad f(x_1) < \min\{f(x_0), f(x_2)\}$$

  In the first case, pick y so that $f(x_1) > y > \max\{f(x_0), f(x_2)\}$. Then by the Intermediate Value Theorem, we can find $c_1 \in (x_0, x_1)$ and $c_2 \in (x_1, x_2)$ such that $f(c_1) = y = f(c_2)$, contradicting the assumption that f is one-to-one. Arguing similarly, we can derive a contradiction in the second case as well. This contradiction shows that the claim must be true for $n = 2$.

- *Inductive Step.* Suppose the claim is true for a given $n \geq 2$, and let $x_0 < x_1 < \cdots < x_{n+1}$.

  We first consider the case $f(x_1) < f(x_n)$. By applying the induction hypothesis to $x_0 < x_1 < \cdots < x_n$, we have $f(x_0) < f(x_1) < \cdots < f(x_n)$. Similarly, by the induction hypothesis applied to $x_1 < x_2 < \cdots < x_{n+1}$, we have $f(x_1) < f(x_2) < \cdots < f(x_{n+1})$. Combining these two inequalities proves that $f(x_0) < f(x_1) < \cdots < f(x_{n+1})$.

  A similar argument shows that if $f(x_1) > f(x_n)$, then $f(x_0) > f(x_1) > \cdots > f(x_n)$.

Therefore the induction step is established and hence the claim is true for any n.

Now fix $a < b$ in I and suppose we have $f(a) < f(b)$. Then for any $x < y$ in I, the above claim shows that f must be strictly monotone on the set $\{a, b, x, y\}$:

$$f(s) < f(t) \quad \text{whenever} \quad s < t \quad \text{and} \quad s, t \in \{a, b, x, y\}.$$

In particular, $f(x) < f(y)$ holds and hence f is strictly increasing. Likewise, if $f(a) > f(b)$ then a similar reasoning shows that f is strictly decreasing.

Theorem 18.5. Let g be a function on an interval J such that

(i) g is **strictly increasing** on J, that is, $g(x) < g(y)$ whenever $x, y \in J$ and $x < y$;

(ii) $g(J)$ is also in interval.

Then g is continuous on J.

The next result is immediately obtained by combining the two theorems above.

Corollary 18.6. Let f be a continuous one-to-one function on an interval I. Then the inverse f^{-1} defines a continuous one-to-one function on the interval $f(I)$.

Proof. By the Intermediate Value Property, $f(I)$ is an interval. Also, by Theorem 18.4, f is either strictly increasing or strictly decreasing. Now suppose f is strictly increasing; the other case is similar. Then it is easy to verify that f^{-1} is a strictly increasing function from $f(I)$ to I, hence f^{-1} is continuous by Theorem 18.5.