Generalised Computability and Complexity in Set-Theoretic Contexts

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Outline

1. New Notions of Generalised (Relative) Computability
   - Computation on Arbitrary Sets
   - Restricting Abstract State Machines
   - The Result
2. Degrees of Small Extensions and Complexity of Local Methods
   - Degrees of Small Extensions
   - Local Method Definitions
   - Results
3. Conclusion

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Conventional Computations

- Turing machines are the de facto abstract models.
- Inputs and outputs of computations are typically finite strings.
- These strings can be coded as natural numbers, and so have hereditarily finite set-theoretic representations.
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Computing with Larger Objects

- What if we want to "compute" things that require more than finite information and/or time-steps?
- For example, one may ask the question

  "Is the Stone-Cech compactification of $\mathbb{R}$ computable from $\mathbb{R}$?"
- This question does not make sense if we interpret "computable" according to convention.
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Abstract State Machines: History

- Gurevich introduced abstract state machines (ASMs) in 2000 as a formalisation of algorithms.
- ASMs are extremely general and flexible, and have no *a priori* size limitations.
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Abstract State Machines: Definition

- An ASM comprises
  - a collection of states, including initial and final states,
  - a transition function mapping one state to another.
- Every state in an ASM is a first-order structure with a finite signature.
- The transition function of an ASM
  - cannot alter the base set of a state,
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Abstract State Machines: Intuition

- States are analogous to Turing machine configurations.
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An Example

- Let A and B be any two sets, and choose an infinite cardinal κ large enough such that $A \cup B \subset V_\kappa$. Define

$$s_1 := (V_\kappa; \in, A, 0)$$
$$s_2 := (V_\kappa; \in, B, 1)$$
$$\text{Set of states} := \{s_1, s_2\}$$
$$\text{Set of initial states} := \{s_1\}$$
$$\text{Transition function} := \{(s_1, s_2), (s_2, s_2)\},$$

  where A, B interpret a unary relation symbol and $0, 1$ interpret a constant symbol.
- This is a valid ASM.
- Morally, it says we can compute B from A.

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The Main Issue

- We can then compute any set from any other set.
- In particular, we can compute any set from $\emptyset$.
- "Everything is computable."
- This is worrying because in mathematics (esp. set theory) there are things that ought to be non-constructive and non-constructible.
- Intuitively, computability is weaker than constructibility, so ASMs are too encompassing a notion for set computation.

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Constraints to the Transition Function

- We want to talk about local features between two states.
- Since they are first-order structures with the same signature and base set, local features are synonymous with properties given through the truth predicate $\models$.
- We modify the predicate into $\models_2$ so that we can talk about any two such structures simultaneously.
- Every ASM has its transitions governed by a transition formula: transition from s_1 to s_2 iff $(s_1, s_2) \models_2 \phi$ for transition formula ϕ.

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Locally Defining Bounded Exploration

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- It suffices but is often not easy to verify.
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Collection of States is not Arbitrary

- To shift the burden of programming to the definition of the transition formula, we streamline the definitions of other components of an ASM.
- We use the language of *theories with contraints in interpretation* (*TCIs*) to define the collection of an ASM's states.
- Basically, TCIs incorporate set bounds that are not first-order definable into first-order theories.
- They generalise a number of constructions in logic.
- Here, they are used to declare the base set, variables and parameters, all of which we require to be uniform across states.

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Ordinal Base Sets

- We specify the base set of any state to be a limit ordinal.
- This simplification is inspired by the fact that every set can be coded as a set of ordinals.
- The base set being an ordinal also fits the basic intuition of sequential memory when one thinks about computation.

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Input/Output Paradigm

- The interpretation of "computing B from A" is implicit in an ASM.
- We strive to make it explicit.
- Every state interprets unary relation symbols In and Out, representing the input and output tapes respectively.
- A machine computes B from A iff it has a run with A interpreting In in an initial state and B interpreting Out in a final state.

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Transfinite Runs

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- This can again be formalised using the first-order truth predicate.
- Under this constraint, we show that taking limits wherever possible (and e.g. resetting to 0 otherwise) point-wise can simulate all other definitions of a limit state.
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Restricted Abstract State Machines with Parameters

- The modifications described hitherto give rise to a type of machine we call *restricted abstract state machines with parameters* (*RASMPs*).
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Relative Computability Relations

- $B \leq_{\kappa}^{P,s} A$ iff some RASMP with base set κ computes B from A in less than κ time-steps.
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- An ordinal κ is *admissible* iff $(L_{\kappa}; \in)$ is a model of Kripke-Platek set theory.
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- We do we mean by sets outside V? Isn't V the entire set-theoretic universe?
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- $\leq^S$ is transitive so $\equiv^S$ is an equivalence relation.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the degrees of small extensions of V with its standard partial ordering.
- Observe that
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so in many contexts we can interchange "small extension(s)" and "degree(s) of small extensions".

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- TCIs and their models are a natural formalisation of this idea.
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- In the meta-theory, define a function Eval^V from the set of TCIs $\mathfrak{T} \in V$ into the set of small extensions of V, such that

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Comparing Local Methods

- A TCI $\mathfrak{T} \in V$ is consistent iff it has a model in some outer model of V.
- If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

  "there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \mathrm{Eval}^V(F(\mathfrak{T})) \subset \mathrm{Eval}^V(\mathfrak{T})$ for all consistent $\mathfrak{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all the descriptions in X.
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The Local Method Hierarchy

- We define the complexity of a TCI to be the maximal complexity of sentences in its first-order theory.
- For example, a Σ_n TCI has a first-order theory containing only Σ_n sentences.
- The classes of Σ_n and Π_n TCIs then form a hierarchy under the relation $\leq^M$, called the (small) local method hierarchy.

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Outline

1. New Notions of Generalised (Relative) Computability
   - Computation on Arbitrary Sets
   - Restricting Abstract State Machines
   - The Result
2. Degrees of Small Extensions and Complexity of Local Methods
   - Degrees of Small Extensions
   - Local Method Definitions
   - Results
3. Conclusion

Set Forcing is Π_2

- We show that $\mathrm{Fg} \equiv^M \Pi_2$, where Π_2 is the local method definition containing precisely the Π_2 TCIs.
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- To do this, we formulate an analogue of the Cantor-Bendixson derivative on certain members of Fg, and leverage on the fact that $\Pi_2 \leq^M \mathrm{Fg}$.

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Computability-theoretic Analogues

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- These results can be proven within V.
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Drawing Parallels

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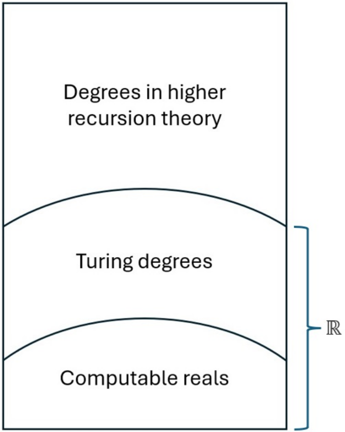

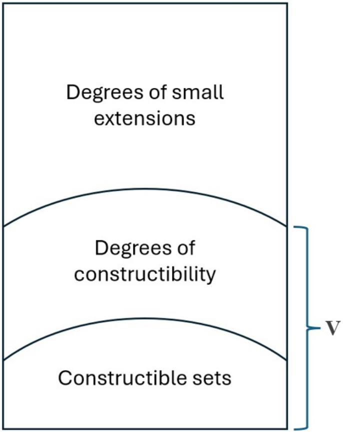

Thank You!