# Generation as Computation

## Generalised Computation in the Set-theoretic Universe and Beyond
(Thesis Proposal)

Desmond Lau

National University of Singapore

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# Outline

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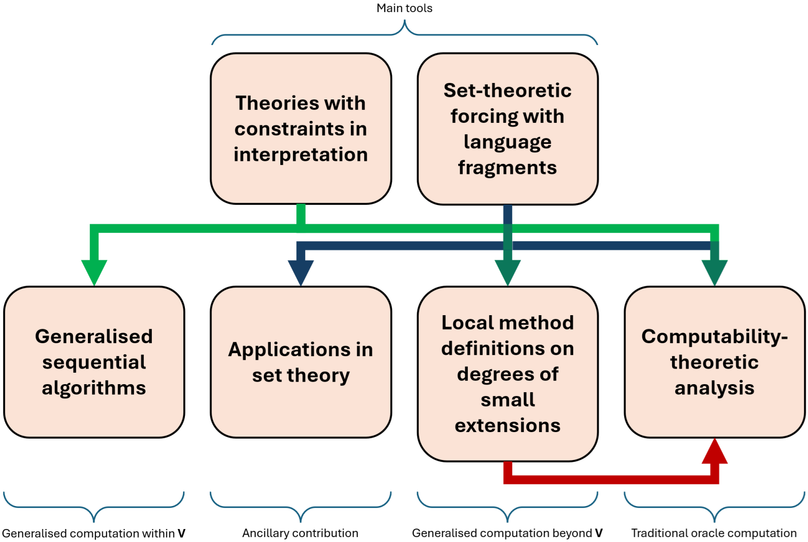
# Overview Chart



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- All reasonable transfinite abstract algorithms are expressible as GSeqAPs.
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In the following table we compare various relative computability relations when they are restricted to subsets of admissible ordinals $\alpha$.

| Property \ Relation | $\leq_\alpha$ | $\preceq_\alpha$ | $\leq_\alpha^P$ | $\leq_\alpha^{P,s}$ |
|---|---|---|---|---|
| Oracle-analogue? | ✓ | ✓ | ✗ | ✗ |
| Transitive? | ✓ | ✗ | ✓ | ✓ |
| Upward consistent? | ✗ | ✓ | ✓ | ✓ |
| Appears in... | $\alpha$-recursion | $\alpha$-computability | | |

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# Generators over $V$

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- In essence, we want to group sets outside $V$ based on their power as generators over $V$.

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# The Meta-theory

- What do we mean by "sets outside $V$"?
- Step out of $V$ and treat $V$ as a countable transitive model of $\mathrm{ZFC}$ (henceforth denoted CTM).
- Look at those sets that can be adjoined to $V$ to give a nice CTM.
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# Outer Models

- Let $U_1$ and $U_2$ be CTMs. $U_2$ is an outer model of $U_1$ iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

**Definition**

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Due to Jensen's result on "coding the universe", we have the following theorem.

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The next proposition is easy to see.

**Proposition**

*Let $V$ be a CTM. Then*

$$\mathbf{M}_S(V) = \{V[x] : x \in \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}\}.$$

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- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of* $V$ with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

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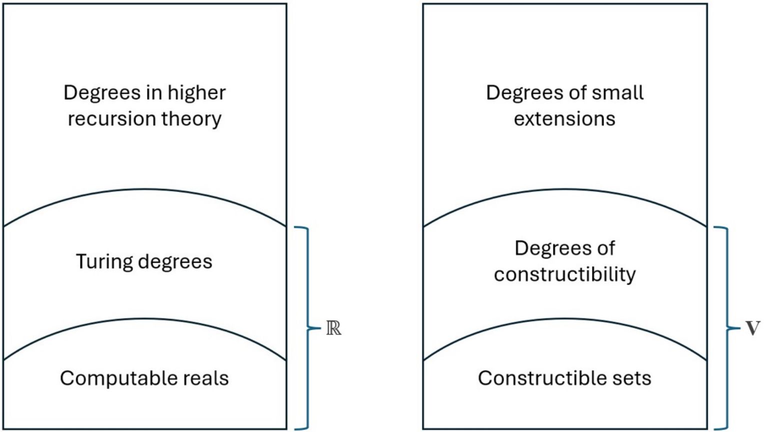


Figure: comparison between conventional notions of relative computability (left) and our generalised notions (right).

# Multiverse of Generic Extensions

**Definition**

Let $V$ be a CTM. The *outward generic multiverse centred at* $V$ is the set

$$\mathbf{M}_F(V) := \{W : W \text{ is a forcing extension of } V\}.$$

The following fact is basic.

**Fact**

*Every generic extension is a small extension. Equivalently,*

$$\mathbf{M}_F(V) \subset \mathbf{M}_S(V).$$

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The following is a rephrasing of a well-known result on intermediate models of forcing extensions.

**Fact**

*$\mathbf{M}_F(V)$ is downward-closed in $\mathbf{M}(V)$, and thus also in $\mathbf{M}_S(V)$.*

On the other hand, the next fact can be derived from arguments using class forcing.

**Fact**

*Given a CTM $V$, $(\mathbf{M}_S(V) \setminus \mathbf{M}_F(V), \subset)$ is a cofinal subposet of $(\mathbf{M}_S(V), \subset)$.*

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- The previous two facts tell us there are many objects inaccessible by forcing.
- Do these objects have "local first-order properties" not shared by any set in any forcing extension?

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# Outline

# Referring to Small Extensions in $V$

- Often it is useful to refer to small extensions of $V$ within $V$.
- Such references in general cannot isolate any non-trivial small extension.
- Think of them as descriptions in $V$ picking out sets of small extensions of $V$, when evaluated outside $V$.
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# Referring to Small Extensions in $V$

- *Theories with constraints in Interpretation* (*TCIs*), and their models, are a formalisation of this idea.
- TCIs were used as a convenient means of defining state spaces of restricted abstract state machines.
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- TCIs provide bounds to the interpretation of a theory $T$ over a signature $\sigma$.
- A TCI $\mathscr{T}$ expands on $T$ by specifying additional requirements on a possible model $\mathfrak{M}$ of $T$.
- These requirements include
  - a set upper bound (under $\subset$) to the base set $M$ of $\mathfrak{M}$,
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# Consistency of TCIs

**Definition (L.)**

A TCI is *consistent* iff it has a model in some outer model of $V$.

By examining a suitable forcing notion in $V$, we can decide if a TCI in $V$ is consistent.

**Proposition**

*Let $\mathscr{T}$ be a TCI. Then for all sufficiently large cardinals $\lambda$,*

$$\Vdash_{Col(\omega,\lambda)} \exists \mathfrak{M} \,(\text{“}\mathfrak{M} \textit{ is a model of } \mathscr{T}\text{”}) \iff \mathscr{T} \textit{ is consistent.}$$

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# Local Method Definitions

**Definition (L.)**

A *local method definition of* $V$ is a non-empty class of TCIs definable in $V$.

- In the meta-theory, define a function $\mathrm{Eval}^V$ from the set of TCIs $\mathscr{T} \in V$ into the set of small extensions of $V$, such that

$$\mathrm{Eval}^V(\mathscr{T}) = \{V[\mathfrak{M}] \in \mathbf{M}_S(V) : \mathfrak{M} \text{ is a model of } \mathscr{T}\}.$$

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# Comparing Local Methods

**Definition (L.)**

If $X$ and $Y$ are local method definitions of $V$, $X \leq^M Y$ denotes the statement

> "there is a function $F : X \longrightarrow Y$ definable in $V$ such that $\emptyset \neq \mathrm{Eval}^V(F(\mathscr{T})) \subset \mathrm{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if $V$ can see that $Y$ provides non-trivial refinements to all consistent descriptions in $X$.
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Let $X, Y$ be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

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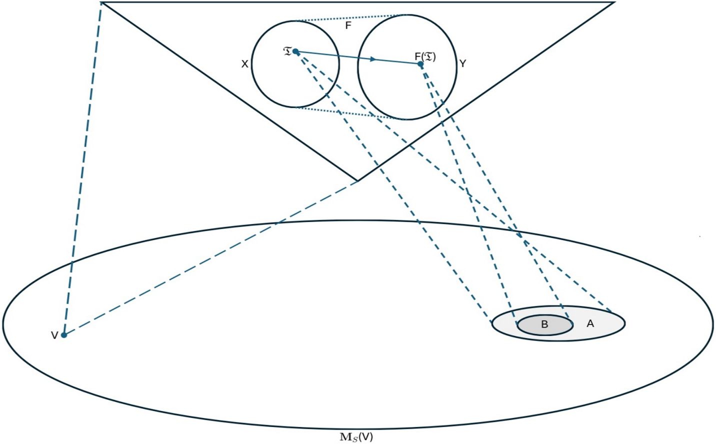


Figure: Visual representation of a function $F$ witnessing $X \leq^M Y$, where $X$ and $Y$ are local method definitions.

# The Local Method Hierarchy

- We can also define the complexity of a TCI by just looking at its first-order theory.
- For example, a $\Sigma_n$ TCI has a first-order theory containing only $\Sigma_n$ sentences.
- The classes of $\Sigma_n$ and $\Pi_n$ TCIs — denoted $\Sigma^{\mathrm{M}}_{\mathrm{n}}$ and $\Pi^{\mathrm{M}}_{\mathrm{n}}$ respectively — then form a hierarchy under the relation $\leq^M$, called the local method hierarchy.

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## Theorem (L.)

*Let $1 \leq n < \omega$. Then for every $\mathscr{T} \in \Pi^{\mathsf{M}}_{\mathsf{n+1}}$ there is $\mathscr{T}' \in \Sigma^{\mathsf{M}}_{\mathsf{n}}$ such that*

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## Corollary

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- Thus set forcing, as a technique of accessing small extensions of $V$, can be represented by a local method definition, denoted Fg.
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# Set Forcing is $\Sigma_1$ (is $\Pi_2$)

**Theorem (L.)**

$\mathrm{Fg} \equiv^M \Sigma_1^{\mathrm{M}}$.

**Proof Idea.**

By adapting our framework for constructing forcing notions to the context of TCIs, we associate with each $\Pi_2$ TCI $\mathscr{T}$ a forcing notion $\mathbb{P}(\mathscr{T})$, such that a unique model of $\mathscr{T}$ can be read off every $\mathbb{P}(\mathscr{T})$-generic filter over $V$.

This allows us to define a class function witnessing $\Pi_2^{\mathrm{M}} \leq^M \mathrm{Fg}$. $\square$

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# Further Results

- We can refine the function witnessing $\Pi_2^{\mathrm{M}} \leq^M \mathrm{Fg}$ by iteratively pruning the $\mathbb{P}(\mathscr{T})$s of atoms.
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- There are analogues of the previous theorem in $V$ pertaining to certain countable $\Pi_2$ TCIs.
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# Some Open Questions

(Q1) What structural properties of $(\mathbf{M}_F(L), \subset)$ also hold for $(\mathbf{M}_S(L), \subset)$?

(Q2) For example, is $(\mathbf{M}_S(L), \subset)$ downward directed?

(Q3) Are there $m, n < \omega$ for which $\Sigma^{\mathsf{M}}_{\mathsf{m}} \neq^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$?

(Q4) Is there a TCI $\mathscr{T}$ such that $\{\mathscr{T}\} \not\leq^M \mathsf{Fg}$?

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# References

Thank You!