Generation as Computation

Generalised Computation in the Set-theoretic Universe and Beyond
(Thesis Proposal)

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Outline

1. Introduction
   - Background: Infinitary Computation
   - Background: Formulas as Programs
   - Overview
2. Generalised Computation Within V
   - Generalised Sequential Algorithms
   - Comparisons with Other Notions
3. Generalised Computation Beyond V
   - Degrees of Small Extensions
   - Local Method Definitions

Outline

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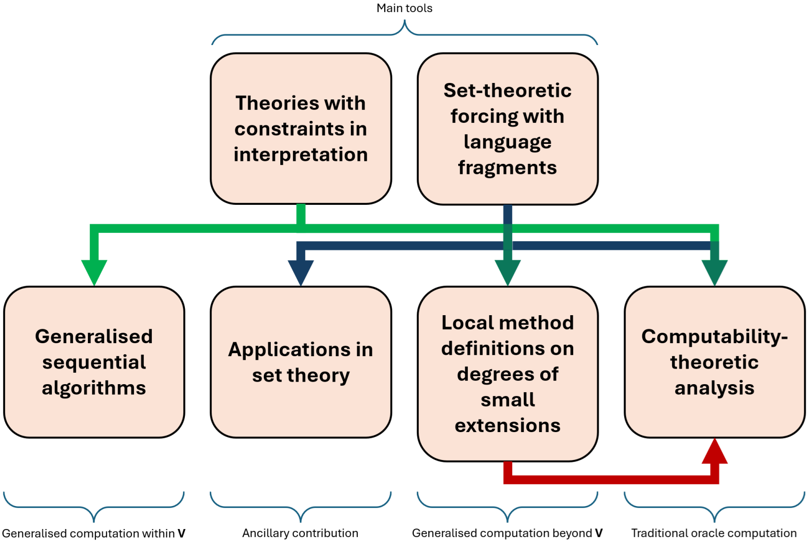
Overview Chart



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Property \ Relation	$\leq_\alpha$	$\preceq_\alpha$	$\leq_\alpha^P$	$\leq_\alpha^{P,s}$
Oracle-analogue?	✓	✓	✗	✗
Transitive?	✓	✗	✓	✓
Upward consistent?	✗	✓	✓	✓
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The Meta-theory

- What do we mean by "sets outside V"?
- Step out of V and treat V as a countable transitive model of ZFC (henceforth denoted CTM).
- Look at those sets that can be adjoined to V to give a nice CTM.
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Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation “being an outer model of” is transitive.

Definition

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The next proposition is easy to see.

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Let V be a CTM. Then

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- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

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- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
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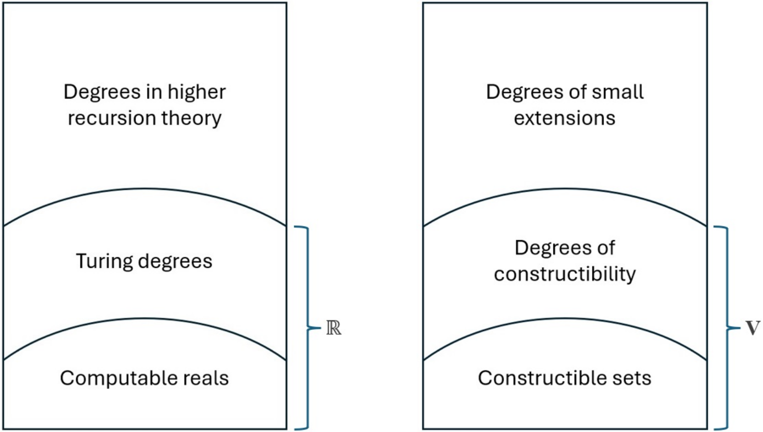

Figure: comparison between conventional notions of relative computability (left) and our generalised notions (right).

Multiverse of Generic Extensions

Definition

Let V be a CTM. The *outward generic multiverse centred at V* is the set

$$\mathbf{M}_F(V) := \{W : W \text{ is a forcing extension of } V\}.$$

The following fact is basic.

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$\mathbf{M}_F(V)$ is downward-closed in $\mathbf{M}(V)$, and thus also in $\mathbf{M}_S(V)$.

On the other hand, the next fact can be derived from arguments using class forcing.

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Given a CTM V, $(\mathbf{M}_S(V) \setminus \mathbf{M}_F(V), \subset)$ is a cofinal subposet of $(\mathbf{M}_S(V), \subset)$.

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- Such references in general cannot isolate any non-trivial small extension.
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- We want the evaluations of these descriptions to be reasonably absolute.
- A straightforward way to ensure absoluteness is to make evaluations local to the parameters given in the descriptions.

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Referring to Small Extensions in V

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- Such references in general cannot isolate any non-trivial small extension.
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- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
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Consistency of TCIs

Definition (L.)

A TCI is *consistent* iff it has a model in some outer model of V.

By examining a suitable forcing notion in V, we can decide if a TCI in V is consistent.

Proposition

Let $\mathscr{T}$ be a TCI. Then for all sufficiently large cardinals λ,

$$\Vdash_{Col(\omega,\lambda)} \exists \mathfrak{M} \text{ (“}\mathfrak{M} \textit{ is a model of } \mathscr{T}\text{”)} \iff \mathscr{T} \textit{ is consistent.}$$

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Local Method Definitions

Definition (L.)

A *local method definition of* V is a non-empty class of TCIs definable in V.

- In the meta-theory, define a function Eval^V from the set of TCIs $\mathscr{T} \in V$ into the set of small extensions of V, such that

$$\mathrm{Eval}^V(\mathscr{T}) = \{V[\mathfrak{M}] \in \mathbf{M}_S(V) : \mathfrak{M} \text{ is a model of } \mathscr{T}\}.$$

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Comparing Local Methods

Definition (L.)

If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

> "there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \mathrm{Eval}^V(F(\mathscr{T})) \subset \mathrm{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all consistent descriptions in X.
- $\leq^M$ partially orders local method definitions, so we can define the equivalence relation $\equiv^M$ the usual way.

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Let X, Y be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

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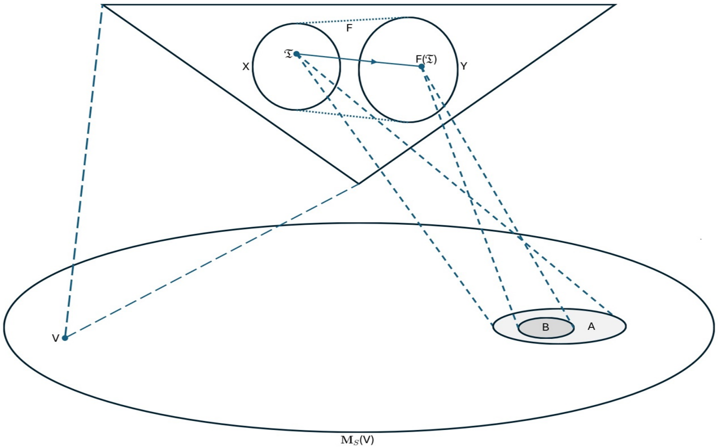

Figure: Visual representation of a function F witnessing $X \leq^M Y$, where X and Y are local method definitions.

The Local Method Hierarchy

- We can also define the complexity of a TCI by just looking at its first-order theory.
- For example, a Σ_n TCI has a first-order theory containing only Σ_n sentences.
- The classes of Σ_n and Π_n TCIs — denoted $\Sigma^{\mathrm{M}}_{\mathrm{n}}$ and $\Pi^{\mathrm{M}}_{\mathrm{n}}$ respectively — then form a hierarchy under the relation $\leq^M$, called the local method hierarchy.

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Let $1 \leq n < \omega$. Then for every $\mathscr{T} \in \Pi^{\mathsf{M}}_{\mathsf{n}+1}$ there is $\mathscr{T}' \in \Sigma^{\mathsf{M}}_{\mathsf{n}}$ such that

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- There is an obvious way of representing any forcing notion as a Π_2 TCI.
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By adapting our framework for constructing forcing notions to the context of TCIs, we associate with each Π_2 TCI $\mathscr{T}$ a forcing notion $\mathbb{P}(\mathscr{T})$, such that a unique model of $\mathscr{T}$ can be read off every $\mathbb{P}(\mathscr{T})$-generic filter over V.

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Further Results

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Some Open Questions

(Q1) What structural properties of $(\mathbf{M}_F(L), \subset)$ also hold for $(\mathbf{M}_S(L), \subset)$?

(Q2) For example, is $(\mathbf{M}_S(L), \subset)$ downward directed?

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