Generation as Computation

Generalised Computation in the Set-theoretic Universe and Beyond
(Thesis Proposal)

Desmond Lau

National University of Singapore

October 31, 2025

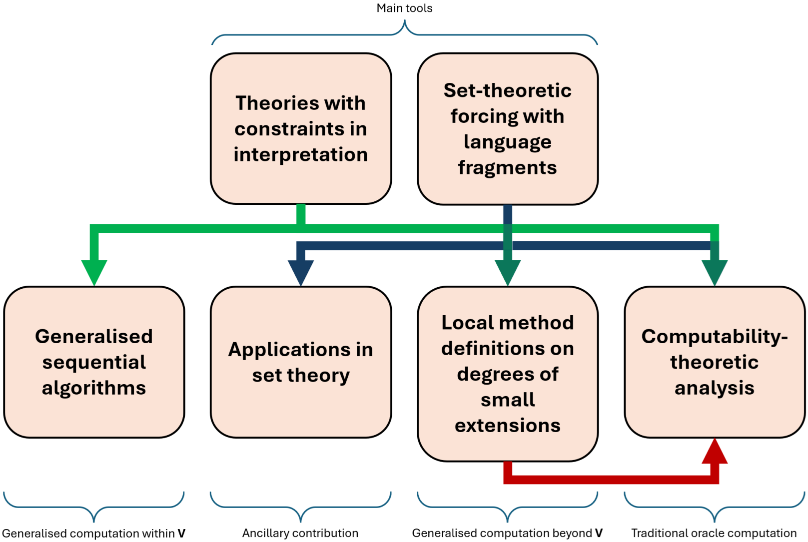
Outline

Outline

Outline

Outline

Overview Chart



Outline

Upshot

- All reasonable transfinite abstract algorithms are expressible as GSeqAPs.
- Here *reasonable* means satisfying the boundedness and locality conditions.

Upshot

- All reasonable transfinite abstract algorithms are expressible as GSeqAPs.
- Here *reasonable* means satisfying the boundedness and locality conditions.

Upshot

- All reasonable transfinite abstract algorithms are expressible as GSeqAPs.
- Here *reasonable* means satisfying the boundedness and locality conditions.

Outline

Summary Table

In the following table we compare various relative computability relations when they are restricted to subsets of admissible ordinals α.

Property \ Relation	$\leq_\alpha$	$\preceq_\alpha$	$\leq_\alpha^P$	$\leq_\alpha^{P,s}$
Oracle-analogue?	✓	✓	✗	✗
Transitive?	✓	✗	✓	✓
Upward persistent?	✗	✓	✓	✓
Appears in...	α-recursion	α-computability		

Summary Table

In the following table we compare various relative computability relations when they are restricted to subsets of admissible ordinals α.

Relation / Property	$\leq_\alpha$	$\preceq_\alpha$	$\leq_\alpha^P$	$\leq_\alpha^{P,s}$
Oracle-analogue?	✓	✓	✗	✗
Transitive?	✓	✗	✓	✓
Upward persistent?	✗	✓	✓	✓
Appears in...	α-recursion	α-computability		

Summary Table

In the following table we compare various relative computability relations when they are restricted to subsets of admissible ordinals α.

Relation / Property	$\leq_\alpha$	$\preceq_\alpha$	$\leq_\alpha^P$	$\leq_\alpha^{P,s}$
Oracle-analogue?	✓	✓	✗	✗
Transitive?	✓	✗	✓	✓
Upward persistent?	✗	✓	✓	✓
Appears in...	α-recursion	α-computability		

Outline

Generators over V

- Going further in the direction of generalised computation, can we maintain the definition of constructibility degrees, but swop L for V?
- In essence, we want to group sets outside V based on their power as generators over V.

Generators over V

- Going further in the direction of generalised computation, can we maintain the definition of constructibility degrees, but swop L for V?
- In essence, we want to group sets outside V based on their power as generators over V.

Generators over V

- Going further in the direction of generalised computation, can we maintain the definition of constructibility degrees, but swop L for V?
- In essence, we want to group sets outside V based on their power as generators over V.

The Meta-theory

- What do we mean by "sets outside V"?
- Step out of V and treat V as a countable transitive model of ZFC (henceforth denoted CTM).
- Look at those sets that can be adjoined to V to give a nice CTM.
- What do we mean by "a nice CTM"?

The Meta-theory

- What do we mean by "sets outside V"?
- Step out of V and treat V as a countable transitive model of ZFC (henceforth denoted CTM).
- Look at those sets that can be adjoined to V to give a nice CTM.
- What do we mean by "a nice CTM"?

The Meta-theory

- What do we mean by "sets outside V"?
- Step out of V and treat V as a countable transitive model of ZFC (henceforth denoted CTM).
- Look at those sets that can be adjoined to V to give a nice CTM.
- What do we mean by "a nice CTM"?

The Meta-theory

- What do we mean by “sets outside V”?
- Step out of V and treat V as a countable transitive model of ZFC (henceforth denoted CTM).
- Look at those sets that can be adjoined to V to give a nice CTM.
- What do we mean by “a nice CTM”?

The Meta-theory

- What do we mean by "sets outside V"?
- Step out of V and treat V as a countable transitive model of ZFC (henceforth denoted CTM).
- Look at those sets that can be adjoined to V to give a nice CTM.
- What do we mean by "a nice CTM"?

Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

Definition

Let V be a CTM. The *outward multiverse centred at V* is the set

$$\mathbf{M}(V) := \{W : W \text{ is an outer model of } V\}.$$

Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

Definition

Let V be a CTM. The *outward multiverse centred at V* is the set

$$\mathbf{M}(V) := \{W : W \text{ is an outer model of } V\}.$$

Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

Definition

Let V be a CTM. The *outward multiverse centred at V* is the set

$$\mathbf{M}(V) := \{W : W \text{ is an outer model of } V\}.$$

Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

Definition

Let V be a CTM. The *outward multiverse centred at V* is the set

$$\mathbf{M}(V) := \{W : W \text{ is an outer model of } V\}.$$

Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

Definition

Let V be a CTM. The *outward multiverse centred at V* is the set

$$\mathbf{M}(V) := \{W : W \text{ is an outer model of } V\}.$$

Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

Definition

Let V be a CTM. The *outward multiverse centred at V* is the set

$$\mathbf{M}(V) := \{W : W \text{ is an outer model of } V\}.$$

Outer Models

- Let U_1 and U_2 be CTMs. U_2 is an outer model of U_1 iff
  - $U_1 \subset U_2$, and
  - $ORD^{U_1} = ORD^{U_2}$.
- The binary relation "being an outer model of" is transitive.

Definition

Let V be a CTM. The *outward multiverse centred at* V is the set

$$\mathbf{M}(V) := \{W : W \text{ is an outer model of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of U_1* iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of U_2 over U_1*, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at V* is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of* U_1 iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of* U_2 *over* U_1, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at* V is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of* U_1 iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of* U_2 *over* U_1, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at* V is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of U_1* iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of U_2 over U_1*, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at V* is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of* U_1 iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of* U_2 *over* U_1, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at* V is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of* U_1 iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of* U_2 *over* U_1, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at* V is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of* U_1 iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of* U_2 *over* U_1, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at* V is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Small Extensions

Definition (L.)

Let U_1 and U_2 be CTMs. U_2 is a *small extension of U_1* iff

- U_2 is an outer model of U_1, and
- there is $x \in U_2$ such that U_2 is the smallest outer model of U_1 containing x.

In this case we call x a *generator of U_2 over U_1*, and denote U_2 as $U_1[x]$.

The binary relation "being a small extension of" is transitive.

Definition

Let V be a CTM. The *small outward multiverse centred at V* is the set

$$\mathbf{M}_S(V) := \{W : W \text{ is a small extension of } V\}.$$

Assorted Multiversal Properties

Due to Jensen's result on "coding the universe", we have the following theorem.

Theorem (Jensen)

Given a CTM V, $(\mathbf{M}_S(V), \subset)$ is a cofinal subposet of $(\mathbf{M}(V), \subset)$.

Assorted Multiversal Properties

Due to Jensen's result on "coding the universe", we have the following theorem.

Theorem (Jensen)

Given a CTM V, $(\mathbf{M}_S(V), \subset)$ is a cofinal subposet of $(\mathbf{M}(V), \subset)$.

Assorted Multiversal Properties

Due to Jensen's result on "coding the universe", we have the following theorem.

Theorem (Jensen)

Given a CTM V, $(\mathbf{M}_S(V), \subset)$ is a cofinal subposet of $(\mathbf{M}(V), \subset)$.

Assorted Multiversal Properties

The next proposition is easy to see.

Proposition

Let V be a CTM. Then

$$\mathbf{M}_S(V) = \{V[x] : x \in \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}\}.$$

Assorted Multiversal Properties

The next proposition is easy to see.

Proposition

Let V be a CTM. Then

$$\mathbf{M}_S(V) = \{V[x] : x \in \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}\}.$$

Assorted Multiversal Properties

The next proposition is easy to see.

Proposition

Let V be a CTM. Then

$$\mathbf{M}_S(V) = \{V[x] : x \in \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}\}.$$

Assorted Multiversal Properties

The next proposition is easy to see.

Proposition

Let V be a CTM. Then

$$\mathbf{M}_S(V) = \{V[x] : x \in \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}\}.$$

Assorted Multiversal Properties

- Given a CTM V, we want to define a degree structure on generators of small extensions of V, analogous to the constructibility degrees on sets in V.
- By the previous proposition, it suffices to consider equivalence classes arising from the natural reducibility relation on

$$\mathscr{G}(V) := \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}.$$

Assorted Multiversal Properties

- Given a CTM V, we want to define a degree structure on generators of small extensions of V, analogous to the constructibility degrees on sets in V.
- By the previous proposition, it suffices to consider equivalence classes arising from the natural reducibility relation on

$$\mathscr{G}(V) := \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}.$$

Assorted Multiversal Properties

- Given a CTM V, we want to define a degree structure on generators of small extensions of V, analogous to the constructibility degrees on sets in V.
- By the previous proposition, it suffices to consider equivalence classes arising from the natural reducibility relation on

$$\mathscr{G}(V) := \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}.$$

Assorted Multiversal Properties

- Given a CTM V, we want to define a degree structure on generators of small extensions of V, analogous to the constructibility degrees on sets in V.
- By the previous proposition, it suffices to consider equivalence classes arising from the natural reducibility relation on

$$\mathscr{G}(V) := \bigcup \mathbf{M}(V) \cap \bigcup \{\mathscr{P}(\alpha) : \alpha \in ORD^V\}.$$

Degrees over V

- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of V* with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

Degrees over V

- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of V* with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

Degrees over V

- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of V* with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

Degrees over V

- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of V* with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

Degrees over V

- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of V* with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

Degrees over V

- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of V* with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

Degrees over V

- Specifically, define the binary relation $\leq^S$ on $\mathscr{G}(V)$ by

$$x \leq^S y \iff V[x] \subset V[y].$$

- $\leq^S$ partially orders $\mathscr{G}(V)$, so we can define the equivalence relation $\equiv^S$ the usual way.
- Call $(\mathscr{G}/\equiv^S, \leq^S/\equiv^S)$ the *degrees of small extensions of* V with its standard partial ordering.
- Observe that

$$(\{\text{small extensions of } V\}, \subset) \cong (\mathscr{G}/\equiv^S, \leq^S/\equiv^S),$$

 so "small extension(s)" and "degree(s) of small extensions" are often used interchangeably.

Drawing Parallels

Drawing Parallels

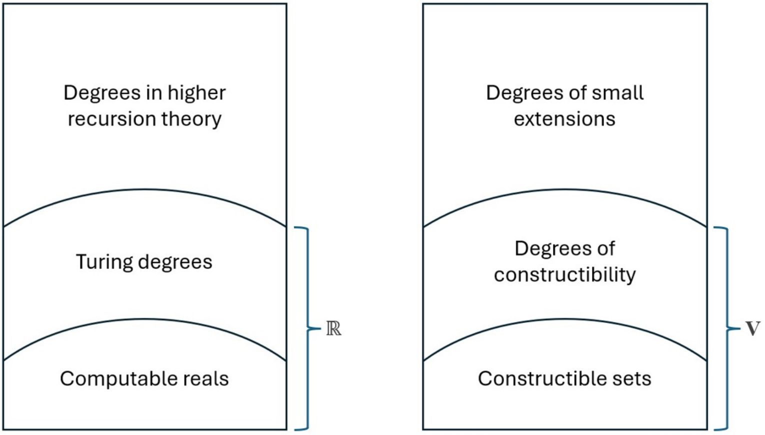


Figure: comparison between conventional notions of relative computability (left) and our generalised notions (right).

Multiverse of Generic Extensions

Definition

Let V be a CTM. The *outward generic multiverse centred at V* is the set

$$\mathbf{M}_F(V) := \{W : W \text{ is a forcing extension of } V\}.$$

The following fact is basic.

Fact

Every generic extension is a small extension. Equivalently,

$$\mathbf{M}_F(V) \subset \mathbf{M}_S(V).$$

Multiverse of Generic Extensions

Definition

Let V be a CTM. The *outward generic multiverse centred at V* is the set

$$\mathbf{M}_F(V) := \{W : W \text{ is a forcing extension of } V\}.$$

The following fact is basic.

Fact

Every generic extension is a small extension. Equivalently,

$$\mathbf{M}_F(V) \subset \mathbf{M}_S(V).$$

Multiverse of Generic Extensions

Definition

Let V be a CTM. The *outward generic multiverse centred at V* is the set

$$\mathbf{M}_F(V) := \{W : W \text{ is a forcing extension of } V\}.$$

The following fact is basic.

Fact

Every generic extension is a small extension. Equivalently,

$$\mathbf{M}_F(V) \subset \mathbf{M}_S(V).$$

Multiverse of Generic Extensions

Definition

Let V be a CTM. The *outward generic multiverse centred at V* is the set

$$\mathbf{M}_F(V) := \{W : W \text{ is a forcing extension of } V\}.$$

The following fact is basic.

Fact

Every generic extension is a small extension. Equivalently,

$$\mathbf{M}_F(V) \subset \mathbf{M}_S(V).$$

Multiverse of Generic Extensions

Definition

Let V be a CTM. The *outward generic multiverse centred at* V is the set

$$\mathbf{M}_F(V) := \{W : W \text{ is a forcing extension of } V\}.$$

The following fact is basic.

Fact

Every generic extension is a small extension. Equivalently,

$$\mathbf{M}_F(V) \subset \mathbf{M}_S(V).$$

Multiverse of Generic Extensions

The following is a rephrasing of a well-known result on intermediate models of forcing extensions.

Fact

$\mathbf{M}_F(V)$ is downward-closed in $\mathbf{M}(V)$, and thus also in $\mathbf{M}_S(V)$.

On the other hand, the next fact can be derived from arguments using class forcing.

Fact

Given a CTM V, $(\mathbf{M}_S(V) \setminus \mathbf{M}_F(V), \subset)$ is a cofinal subposet of $(\mathbf{M}_S(V), \subset)$.

Multiverse of Generic Extensions

The following is a rephrasing of a well-known result on intermediate models of forcing extensions.

Fact

$\mathbf{M}_F(V)$ is downward-closed in $\mathbf{M}(V)$, and thus also in $\mathbf{M}_S(V)$.

On the other hand, the next fact can be derived from arguments using class forcing.

Fact

Given a CTM V, $(\mathbf{M}_S(V) \setminus \mathbf{M}_F(V), \subset)$ is a cofinal subposet of $(\mathbf{M}_S(V), \subset)$.

Multiverse of Generic Extensions

The following is a rephrasing of a well-known result on intermediate models of forcing extensions.

Fact

$\mathbf{M}_F(V)$ is downward-closed in $\mathbf{M}(V)$, and thus also in $\mathbf{M}_S(V)$.

On the other hand, the next fact can be derived from arguments using class forcing.

Fact

Given a CTM V, $(\mathbf{M}_S(V) \setminus \mathbf{M}_F(V), \subset)$ is a cofinal subposet of $(\mathbf{M}_S(V), \subset)$.

Multiverse of Generic Extensions

The following is a rephrasing of a well-known result on intermediate models of forcing extensions.

Fact

$\mathbf{M}_F(V)$ is downward-closed in $\mathbf{M}(V)$, and thus also in $\mathbf{M}_S(V)$.

On the other hand, the next fact can be derived from arguments using class forcing.

Fact

Given a CTM V, $(\mathbf{M}_S(V) \setminus \mathbf{M}_F(V), \subset)$ is a cofinal subposet of $(\mathbf{M}_S(V), \subset)$.

Multiverse of Generic Extensions

The following is a rephrasing of a well-known result on intermediate models of forcing extensions.

Fact

$\mathbf{M}_F(V)$ is downward-closed in $\mathbf{M}(V)$, and thus also in $\mathbf{M}_S(V)$.

On the other hand, the next fact can be derived from arguments using class forcing.

Fact

Given a CTM V, $(\mathbf{M}_S(V) \setminus \mathbf{M}_F(V), \subset)$ is a cofinal subposet of $(\mathbf{M}_S(V), \subset)$.

Accessibility of Forcing

- The previous two facts tell us there are many objects inaccessible by forcing.
- Do these objects have "local first-order properties" not shared by any set in any forcing extension?

Accessibility of Forcing

- The previous two facts tell us there are many objects inaccessible by forcing.
- Do these objects have "local first-order properties" not shared by any set in any forcing extension?

Accessibility of Forcing

- The previous two facts tell us there are many objects inaccessible by forcing.
- Do these objects have "local first-order properties" not shared by any set in any forcing extension?

Outline

Referring to Small Extensions in V

- Often it is useful to refer to small extensions of V within V.
- Such references in general cannot isolate any non-trivial small extension.
- Think of them as descriptions in V picking out sets of small extensions of V, when evaluated outside V.
- We want the evaluations of these descriptions to be reasonably absolute.
- A straightforward way to ensure absoluteness is to make evaluations local to the parameters given in the descriptions.

Referring to Small Extensions in V

- Often it is useful to refer to small extensions of V within V.
- Such references in general cannot isolate any non-trivial small extension.
- Think of them as descriptions in V picking out sets of small extensions of V, when evaluated outside V.
- We want the evaluations of these descriptions to be reasonably absolute.
- A straightforward way to ensure absoluteness is to make evaluations local to the parameters given in the descriptions.

Referring to Small Extensions in V

- Often it is useful to refer to small extensions of V within V.
- Such references in general cannot isolate any non-trivial small extension.
- Think of them as descriptions in V picking out sets of small extensions of V, when evaluated outside V.
- We want the evaluations of these descriptions to be reasonably absolute.
- A straightforward way to ensure absoluteness is to make evaluations local to the parameters given in the descriptions.

Referring to Small Extensions in V

- Often it is useful to refer to small extensions of V within V.
- Such references in general cannot isolate any non-trivial small extension.
- Think of them as descriptions in V picking out sets of small extensions of V, when evaluated outside V.
- We want the evaluations of these descriptions to be reasonably absolute.
- A straightforward way to ensure absoluteness is to make evaluations local to the parameters given in the descriptions.

Referring to Small Extensions in V

- Often it is useful to refer to small extensions of V within V.
- Such references in general cannot isolate any non-trivial small extension.
- Think of them as descriptions in V picking out sets of small extensions of V, when evaluated outside V.
- We want the evaluations of these descriptions to be reasonably absolute.
- A straightforward way to ensure absoluteness is to make evaluations local to the parameters given in the descriptions.

Referring to Small Extensions in V

- Often it is useful to refer to small extensions of V within V.
- Such references in general cannot isolate any non-trivial small extension.
- Think of them as descriptions in V picking out sets of small extensions of V, when evaluated outside V.
- We want the evaluations of these descriptions to be reasonably absolute.
- A straightforward way to ensure absoluteness is to make evaluations local to the parameters given in the descriptions.

Referring to Small Extensions in V

- *Theories with constraints in Interpretation* (*TCIs*), and their models, are a formalisation of this idea.
- TCIs were used as a convenient means of defining state spaces of restricted abstract state machines.
- They also naturally capture the intuition behind forcing.
- A TCI in V describes potential generators of small extensions of V.
- Models of this TCI in $\mathbf{M}_S(V)$ carve out the set of small extensions it describes.

Referring to Small Extensions in V

- *Theories with constraints in Interpretation* (*TCIs*), and their models, are a formalisation of this idea.
- TCIs were used as a convenient means of defining state spaces of restricted abstract state machines.
- They also naturally capture the intuition behind forcing.
- A TCI in V describes potential generators of small extensions of V.
- Models of this TCI in $\mathbf{M}_S(V)$ carve out the set of small extensions it describes.

Referring to Small Extensions in V

- *Theories with constraints in Interpretation* (*TCIs*), and their models, are a formalisation of this idea.
- TCIs were used as a convenient means of defining state spaces of restricted abstract state machines.
- They also naturally capture the intuition behind forcing.
- A TCI in V describes potential generators of small extensions of V.
- Models of this TCI in $\mathbf{M}_S(V)$ carve out the set of small extensions it describes.

Referring to Small Extensions in V

- *Theories with constraints in Interpretation* (*TCIs*), and their models, are a formalisation of this idea.
- TCIs were used as a convenient means of defining state spaces of restricted abstract state machines.
- They also naturally capture the intuition behind forcing.
- A TCI in V describes potential generators of small extensions of V.
- Models of this TCI in $\mathbf{M}_S(V)$ carve out the set of small extensions it describes.

Referring to Small Extensions in V

- *Theories with constraints in Interpretation* (*TCIs*), and their models, are a formalisation of this idea.
- TCIs were used as a convenient means of defining state spaces of restricted abstract state machines.
- They also naturally capture the intuition behind forcing.
- A TCI in V describes potential generators of small extensions of V.
- Models of this TCI in $\mathbf{M}_S(V)$ carve out the set of small extensions it describes.

Referring to Small Extensions in V

- *Theories with constraints in Interpretation* (*TCIs*), and their models, are a formalisation of this idea.
- TCIs were used as a convenient means of defining state spaces of restricted abstract state machines.
- They also naturally capture the intuition behind forcing.
- A TCI in V describes potential generators of small extensions of V.
- Models of this TCI in $\mathbf{M}_S(V)$ carve out the set of small extensions it describes.

Theories with Constraints in Interpretation

- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
- These requirements include
  - a set upper bound (under $\subset$) to the base set M of $\mathfrak{M}$,
  - a set upper bound to $\dot{X}^{\mathfrak{M}}$, for each $\dot{X} \in \sigma$, and
  - whether each $\dot{X}^{\mathfrak{M}}$ depends entirely on its upper bound and M.

Theories with Constraints in Interpretation

- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
- These requirements include
  - a set upper bound (under $\subset$) to the base set M of $\mathfrak{M}$,
  - a set upper bound to $\dot{X}^{\mathfrak{M}}$, for each $\dot{X} \in \sigma$, and
  - whether each $\dot{X}^{\mathfrak{M}}$ depends entirely on its upper bound and M.

Theories with Constraints in Interpretation

- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
- These requirements include
  - a set upper bound (under $\subset$) to the base set M of $\mathfrak{M}$,
  - a set upper bound to $\dot{X}^{\mathfrak{M}}$, for each $\dot{X} \in \sigma$, and
  - whether each $\dot{X}^{\mathfrak{M}}$ depends entirely on its upper bound and M.

Theories with Constraints in Interpretation

- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
- These requirements include
  - a set upper bound (under $\subset$) to the base set M of $\mathfrak{M}$,
  - a set upper bound to $\dot{X}^{\mathfrak{M}}$, for each $\dot{X} \in \sigma$, and
  - whether each $\dot{X}^{\mathfrak{M}}$ depends entirely on its upper bound and M.

Theories with Constraints in Interpretation

- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
- These requirements include
  - a set upper bound (under $\subset$) to the base set M of $\mathfrak{M}$,
  - a set upper bound to $\dot{X}^{\mathfrak{M}}$, for each $\dot{X} \in \sigma$, and
  - whether each $\dot{X}^{\mathfrak{M}}$ depends entirely on its upper bound and M.

Theories with Constraints in Interpretation

- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
- These requirements include
  - a set upper bound (under $\subset$) to the base set M of $\mathfrak{M}$,
  - a set upper bound to $\dot{X}^{\mathfrak{M}}$, for each $\dot{X} \in \sigma$, and
  - whether each $\dot{X}^{\mathfrak{M}}$ depends entirely on its upper bound and M.

Theories with Constraints in Interpretation

- TCIs provide bounds to the interpretation of a theory T over a signature σ.
- A TCI $\mathscr{T}$ expands on T by specifying additional requirements on a possible model $\mathfrak{M}$ of T.
- These requirements include
  - a set upper bound (under $\subset$) to the base set M of $\mathfrak{M}$,
  - a set upper bound to $\dot{X}^{\mathfrak{M}}$, for each $\dot{X} \in \sigma$, and
  - whether each $\dot{X}^{\mathfrak{M}}$ depends entirely on its upper bound and M.

Consistency of TCIs

Definition (L.)

A TCI is *consistent* iff it has a model in some outer model of V.

By examining a suitable forcing notion in V, we can decide if a TCI in V is consistent.

Proposition

Let $\mathscr{T}$ be a TCI. Then for all sufficiently large cardinals λ,

$$\Vdash_{Col(\omega,\lambda)} \exists\mathfrak{M}\,(\text{“}\mathfrak{M} \text{ is a model of } \mathscr{T}\text{”}) \iff \mathscr{T} \text{ is consistent.}$$

Consistency of TCIs

Definition (L.)

A TCI is *consistent* iff it has a model in some outer model of V.

By examining a suitable forcing notion in V, we can decide if a TCI in V is consistent.

Proposition

Let $\mathscr{T}$ be a TCI. Then for all sufficiently large cardinals λ,

$$\Vdash_{Col(\omega,\lambda)} \exists\mathfrak{M}\ (\text{“}\mathfrak{M} \textit{ is a model of } \mathscr{T}\text{”}) \iff \mathscr{T} \textit{ is consistent.}$$

Consistency of TCIs

Definition (L.)

A TCI is *consistent* iff it has a model in some outer model of V.

By examining a suitable forcing notion in V, we can decide if a TCI in V is consistent.

Proposition

Let $\mathscr{T}$ be a TCI. Then for all sufficiently large cardinals λ,

$$\Vdash_{Col(\omega,\lambda)} \exists \mathfrak{M} \, (\text{“}\mathfrak{M} \textit{ is a model of } \mathscr{T}\text{”}) \iff \mathscr{T} \textit{ is consistent.}$$

Consistency of TCIs

Definition (L.)

A TCI is *consistent* iff it has a model in some outer model of V.

By examining a suitable forcing notion in V, we can decide if a TCI in V is consistent.

Proposition

Let $\mathscr{T}$ be a TCI. Then for all sufficiently large cardinals λ,

$$\Vdash_{Col(\omega,\lambda)} \exists \mathfrak{M}\ (\text{“}\mathfrak{M} \textit{ is a model of } \mathscr{T}\text{”}) \iff \mathscr{T} \textit{ is consistent.}$$

Consistency of TCIs

Definition (L.)

A TCI is *consistent* iff it has a model in some outer model of V.

By examining a suitable forcing notion in V, we can decide if a TCI in V is consistent.

Proposition

Let $\mathscr{T}$ be a TCI. Then for all sufficiently large cardinals λ,

$$\Vdash_{Col(\omega,\lambda)} \exists \mathfrak{M} \, (\text{“}\mathfrak{M} \textit{ is a model of } \mathscr{T}\text{”}) \iff \mathscr{T} \textit{ is consistent.}$$

Local Method Definitions

Definition (L.)

A *local method definition of V* is a non-empty class of TCIs definable in V.

- In the meta-theory, define a function Eval^V from the set of TCIs $\mathscr{T} \in V$ into the set of small extensions of V, such that

$$\mathrm{Eval}^V(\mathscr{T}) = \{V[\mathfrak{M}] \in \mathbf{M}_S(V) : \mathfrak{M} \text{ is a model of } \mathscr{T}\}.$$

- Each local method definition then picks out multiple sets of small extensions of V.

Local Method Definitions

Definition (L.)

A *local method definition of* V is a non-empty class of TCIs definable in V.

- In the meta-theory, define a function Eval^V from the set of TCIs $\mathscr{T} \in V$ into the set of small extensions of V, such that

$$\mathrm{Eval}^V(\mathscr{T}) = \{V[\mathfrak{M}] \in \mathbf{M}_S(V) : \mathfrak{M} \text{ is a model of } \mathscr{T}\}.$$

- Each local method definition then picks out multiple sets of small extensions of V.

Local Method Definitions

Definition (L.)

A *local method definition of* V is a non-empty class of TCIs definable in V.

- In the meta-theory, define a function Eval^V from the set of TCIs $\mathscr{T} \in V$ into the set of small extensions of V, such that

$$\mathrm{Eval}^V(\mathscr{T}) = \{V[\mathfrak{M}] \in \mathbf{M}_S(V) : \mathfrak{M} \text{ is a model of } \mathscr{T}\}.$$

- Each local method definition then picks out multiple sets of small extensions of V.

Local Method Definitions

Definition (L.)

A *local method definition of* V is a non-empty class of TCIs definable in V.

- In the meta-theory, define a function Eval^V from the set of TCIs $\mathscr{T} \in V$ into the set of small extensions of V, such that

$$\mathrm{Eval}^V(\mathscr{T}) = \{V[\mathfrak{M}] \in \mathbf{M}_S(V) : \mathfrak{M} \text{ is a model of } \mathscr{T}\}.$$

- Each local method definition then picks out multiple sets of small extensions of V.

Local Method Definitions

Definition (L.)

A *local method definition of* V is a non-empty class of TCIs definable in V.

- In the meta-theory, define a function Eval^V from the set of TCIs $\mathscr{T} \in V$ into the set of small extensions of V, such that

$$\mathrm{Eval}^V(\mathscr{T}) = \{V[\mathfrak{M}] \in \mathbf{M}_S(V) : \mathfrak{M} \text{ is a model of } \mathscr{T}\}.$$

- Each local method definition then picks out multiple sets of small extensions of V.

Comparing Local Methods

Definition (L.)

If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

> "there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \mathrm{Eval}^V(F(\mathscr{T})) \subset \mathrm{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all consistent descriptions in X.
- $\leq^M$ partially orders local method definitions, so we can define the equivalence relation $\equiv^M$ the usual way.

Observation

Let X, Y be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

Comparing Local Methods

Definition (L.)

If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

"there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \mathrm{Eval}^V(F(\mathscr{T})) \subset \mathrm{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all consistent descriptions in X.
- $\leq^M$ partially orders local method definitions, so we can define the equivalence relation $\equiv^M$ the usual way.

Observation

Let X, Y be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

Comparing Local Methods

Definition (L.)

If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

"there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \mathrm{Eval}^V(F(\mathscr{T})) \subset \mathrm{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all consistent descriptions in X.
- $\leq^M$ partially orders local method definitions, so we can define the equivalence relation $\equiv^M$ the usual way.

Observation

Let X, Y be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

Comparing Local Methods

Definition (L.)

If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

> "there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \text{Eval}^V(F(\mathscr{T})) \subset \text{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all consistent descriptions in X.
- $\leq^M$ partially orders local method definitions, so we can define the equivalence relation $\equiv^M$ the usual way.

Observation

Let X, Y be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

Comparing Local Methods

Definition (L.)

If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

"there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \mathrm{Eval}^V(F(\mathscr{T})) \subset \mathrm{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all consistent descriptions in X.
- $\leq^M$ partially orders local method definitions, so we can define the equivalence relation $\equiv^M$ the usual way.

Observation

Let X, Y be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

Comparing Local Methods

Definition (L.)

If X and Y are local method definitions of V, $X \leq^M Y$ denotes the statement

"there is a function $F : X \longrightarrow Y$ definable in V such that $\emptyset \neq \mathrm{Eval}^V(F(\mathscr{T})) \subset \mathrm{Eval}^V(\mathscr{T})$ for all consistent $\mathscr{T} \in X$".

- Intuitively, $X \leq^M Y$ if V can see that Y provides non-trivial refinements to all consistent descriptions in X.
- $\leq^M$ partially orders local method definitions, so we can define the equivalence relation $\equiv^M$ the usual way.

Observation

Let X, Y be local method definitions. If $X \subset Y$, then $X \leq^M Y$.

Comparing Local Methods

Comparing Local Methods

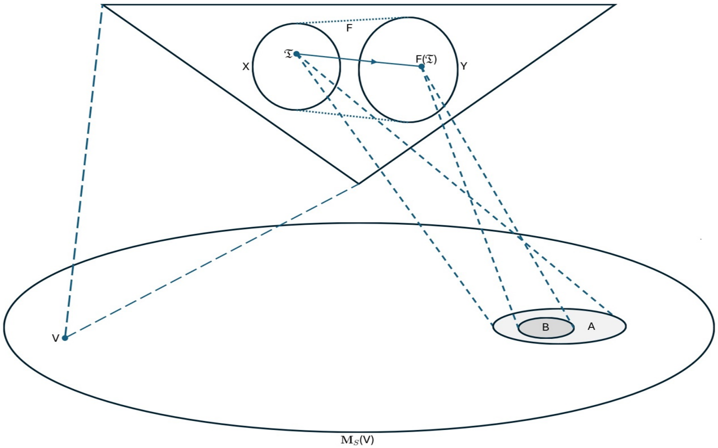


Figure: Visual representation of a function F witnessing $X \leq^M Y$, where X and Y are local method definitions.

The Local Method Hierarchy

- We can also define the complexity of a TCI by just looking at its first-order theory.
- For example, a Σ_n TCI has a first-order theory containing only Σ_n sentences.
- The classes of Σ_n and Π_n TCIs — denoted Σ_n^{M} and Π_n^{M} respectively — then form a hierarchy under the relation $\leq^M$, called the local method hierarchy.

The Local Method Hierarchy

- We can also define the complexity of a TCI by just looking at its first-order theory.
- For example, a Σ_n TCI has a first-order theory containing only Σ_n sentences.
- The classes of Σ_n and Π_n TCIs — denoted $\Sigma^{\mathrm{M}}_{\mathrm{n}}$ and $\Pi^{\mathrm{M}}_{\mathrm{n}}$ respectively — then form a hierarchy under the relation $\leq^M$, called the local method hierarchy.

The Local Method Hierarchy

- We can also define the complexity of a TCI by just looking at its first-order theory.
- For example, a Σ_n TCI has a first-order theory containing only Σ_n sentences.
- The classes of Σ_n and Π_n TCIs — denoted $\Sigma^{\mathrm{M}}_{\mathrm{n}}$ and $\Pi^{\mathrm{M}}_{\mathrm{n}}$ respectively — then form a hierarchy under the relation $\leq^M$, called the local method hierarchy.

The Local Method Hierarchy

- We can also define the complexity of a TCI by just looking at its first-order theory.
- For example, a Σ_n TCI has a first-order theory containing only Σ_n sentences.
- The classes of Σ_n and Π_n TCIs — denoted $\mathsf{\Sigma^M_n}$ and $\mathsf{\Pi^M_n}$ respectively — then form a hierarchy under the relation $\leq^M$, called the local method hierarchy.

The Local Method Hierarchy

Theorem (L.)

Let $1 \leq n < \omega$. Then for every $\mathscr{T} \in \Pi^{\mathsf{M}}_{\mathsf{n+1}}$ there is $\mathscr{T}' \in \Sigma^{\mathsf{M}}_{\mathsf{n}}$ such that

$$\mathrm{Eval}^V(\mathscr{T}) = \mathrm{Eval}^V(\mathscr{T}').$$

Corollary

$\Pi^{\mathsf{M}}_{\mathsf{n+1}} \leq^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$ *for all* $1 \leq n < \omega$.

The Local Method Hierarchy

Theorem (L.)

Let $1 \leq n < \omega$. Then for every $\mathscr{T} \in \Pi^{\mathsf{M}}_{\mathsf{n+1}}$ there is $\mathscr{T}' \in \Sigma^{\mathsf{M}}_{\mathsf{n}}$ such that

$$\mathrm{Eval}^V(\mathscr{T}) = \mathrm{Eval}^V(\mathscr{T}').$$

Corollary

$\Pi^{\mathsf{M}}_{\mathsf{n+1}} \leq^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$ *for all* $1 \leq n < \omega$.

The Local Method Hierarchy

Theorem (L.)

Let $1 \leq n < \omega$. Then for every $\mathscr{T} \in \Pi^{\mathsf{M}}_{\mathsf{n+1}}$ there is $\mathscr{T}' \in \Sigma^{\mathsf{M}}_{\mathsf{n}}$ such that

$$\mathrm{Eval}^V(\mathscr{T}) = \mathrm{Eval}^V(\mathscr{T}').$$

Corollary

$\Pi^{\mathsf{M}}_{\mathsf{n+1}} \leq^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$ *for all* $1 \leq n < \omega$.

The Local Method Hierarchy

Theorem (L.)

Let $1 \leq n < \omega$. Then for every $\mathscr{T} \in \Pi^{\mathsf{M}}_{\mathsf{n+1}}$ there is $\mathscr{T}' \in \Sigma^{\mathsf{M}}_{\mathsf{n}}$ such that

$$\mathrm{Eval}^V(\mathscr{T}) = \mathrm{Eval}^V(\mathscr{T}').$$

Corollary

$\Pi^{\mathsf{M}}_{\mathsf{n+1}} \leq^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$ *for all* $1 \leq n < \omega$.

Set Forcing

- There is an obvious way of representing any forcing notion as a Π_2 TCI.
- Thus set forcing, as a technique of accessing small extensions of V, can be represented by a local method definition, denoted Fg.
- We want to see if Fg fits nicely in the local method hierarchy.
- From what we know so far, $\mathrm{Fg} \leq^M \Sigma_1^{\mathrm{M}}$.

Set Forcing

- There is an obvious way of representing any forcing notion as a Π_2 TCI.
- Thus set forcing, as a technique of accessing small extensions of V, can be represented by a local method definition, denoted Fg.
- We want to see if Fg fits nicely in the local method hierarchy.
- From what we know so far, $\mathrm{Fg} \leq^M \Sigma_1^{\mathrm{M}}$.

Set Forcing

- There is an obvious way of representing any forcing notion as a Π_2 TCI.
- Thus set forcing, as a technique of accessing small extensions of V, can be represented by a local method definition, denoted Fg.
- We want to see if Fg fits nicely in the local method hierarchy.
- From what we know so far, Fg $\leq^M \Sigma_1^{\mathrm{M}}$.

Set Forcing

- There is an obvious way of representing any forcing notion as a Π_2 TCI.
- Thus set forcing, as a technique of accessing small extensions of V, can be represented by a local method definition, denoted Fg.
- We want to see if Fg fits nicely in the local method hierarchy.
- From what we know so far, $\mathrm{Fg} \leq^M \Sigma_1^{\mathrm{M}}$.

Set Forcing

- There is an obvious way of representing any forcing notion as a Π_2 TCI.
- Thus set forcing, as a technique of accessing small extensions of V, can be represented by a local method definition, denoted Fg.
- We want to see if Fg fits nicely in the local method hierarchy.
- From what we know so far, Fg $\leq^M \Sigma_1^{\mathrm{M}}$.

Set Forcing is Σ_1 (is Π_2)

Theorem (L.)

$\mathrm{Fg} \equiv^M \Sigma_1^{\mathrm{M}}$.

Proof Idea.

By adapting our framework for constructing forcing notions to the context of TCIs, we associate with each Π_2 TCI $\mathscr{T}$ a forcing notion $\mathbb{P}(\mathscr{T})$, such that a unique model of $\mathscr{T}$ can be read off every $\mathbb{P}(\mathscr{T})$-generic filter over V.

This allows us to define a class function witnessing $\Pi_2^{\mathrm{M}} \leq^M \mathrm{Fg}$. $\square$

Set Forcing is Σ_1 (is Π_2)

Theorem (L.)

$\mathsf{Fg} \equiv^M \Sigma_1^{\mathsf{M}}$.

Proof Idea.

By adapting our framework for constructing forcing notions to the context of TCIs, we associate with each Π_2 TCI $\mathscr{T}$ a forcing notion $\mathbb{P}(\mathscr{T})$, such that a unique model of $\mathscr{T}$ can be read off every $\mathbb{P}(\mathscr{T})$-generic filter over V.

This allows us to define a class function witnessing $\Pi_2^{\mathsf{M}} \leq^M \mathsf{Fg}$. □

Set Forcing is Σ_1 (is Π_2)

Theorem (L.)

$\mathsf{Fg} \equiv^M \Sigma_1^{\mathsf{M}}$.

Proof Idea.

By adapting our framework for constructing forcing notions to the context of TCIs, we associate with each Π_2 TCI $\mathscr{T}$ a forcing notion $\mathbb{P}(\mathscr{T})$, such that a unique model of $\mathscr{T}$ can be read off every $\mathbb{P}(\mathscr{T})$-generic filter over V.

This allows us to define a class function witnessing $\Pi_2^{\mathsf{M}} \leq^M \mathsf{Fg}$. $\square$

Set Forcing is Σ_1 (is Π_2)

Theorem (L.)

$\mathsf{Fg} \equiv^M \Sigma_1^{\mathsf{M}}$.

Proof Idea.

By adapting our framework for constructing forcing notions to the context of TCIs, we associate with each Π_2 TCI $\mathscr{T}$ a forcing notion $\mathbb{P}(\mathscr{T})$, such that a unique model of $\mathscr{T}$ can be read off every $\mathbb{P}(\mathscr{T})$-generic filter over V.

This allows us to define a class function witnessing $\Pi_2^{\mathsf{M}} \leq^M \mathsf{Fg}$. □

Further Results

- We can refine the function witnessing $\Pi_2^{\mathrm{M}} \leq^M \mathrm{Fg}$ by iteratively pruning the $\mathbb{P}(\mathscr{T})$s of atoms.
- There are analogues of the previous theorem in V pertaining to certain countable Π_2 TCIs.
    - Essentially, we can define functions F of low complexity taking these TCIs to reals such that every $F(\mathscr{T})$-1-generic real codes a model of $\mathscr{T}$.

Further Results

- We can refine the function witnessing $\Pi_2^{\mathsf{M}} \leq^M \mathsf{Fg}$ by iteratively pruning the $\mathbb{P}(\mathscr{T})$s of atoms.
- There are analogues of the previous theorem in V pertaining to certain countable Π_2 TCIs.
  - Essentially, we can define functions F of low complexity taking these TCIs to reals such that every $F(\mathscr{T})$-1-generic real codes a model of $\mathscr{T}$.

Further Results

- We can refine the function witnessing $\Pi_2^{\mathsf{M}} \leq^M \mathsf{Fg}$ by iteratively pruning the $\mathbb{P}(\mathscr{T})$s of atoms.
- There are analogues of the previous theorem in V pertaining to certain countable Π_2 TCIs.
    - Essentially, we can define functions F of low complexity taking these TCIs to reals such that every $F(\mathscr{T})$-1-generic real codes a model of $\mathscr{T}$.

Further Results

- We can refine the function witnessing $\Pi_2^{\mathsf{M}} \leq^M \mathsf{Fg}$ by iteratively pruning the $\mathbb{P}(\mathscr{T})$s of atoms.
- There are analogues of the previous theorem in V pertaining to certain countable Π_2 TCIs.
    - Essentially, we can define functions F of low complexity taking these TCIs to reals such that every $F(\mathscr{T})$-1-generic real codes a model of $\mathscr{T}$.

Some Open Questions

(Q1) What structural properties of $(\mathbf{M}_F(L), \subset)$ also hold for $(\mathbf{M}_S(L), \subset)$?

(Q2) For example, is $(\mathbf{M}_S(L), \subset)$ downward directed?

(Q3) Are there $m, n < \omega$ for which $\Sigma^{\mathsf{M}}_{\mathsf{m}} \not\equiv^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$?

(Q4) Is there a TCI $\mathscr{T}$ such that $\{\mathscr{T}\} \not\leq^M \mathsf{Fg}$?

Some Open Questions

(Q1) What structural properties of $(\mathbf{M}_F(L), \subset)$ also hold for $(\mathbf{M}_S(L), \subset)$?

(Q2) For example, is $(\mathbf{M}_S(L), \subset)$ downward directed?

(Q3) Are there $m, n < \omega$ for which $\Sigma^{\mathsf{M}}_{\mathsf{m}} \not\equiv^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$?

(Q4) Is there a TCI $\mathscr{T}$ such that $\{\mathscr{T}\} \not\leq^M \mathrm{Fg}$?

Some Open Questions

(Q1) What structural properties of $(\mathbf{M}_F(L), \subset)$ also hold for $(\mathbf{M}_S(L), \subset)$?

(Q2) For example, is $(\mathbf{M}_S(L), \subset)$ downward directed?

(Q3) Are there $m, n < \omega$ for which $\Sigma^{\mathsf{M}}_{\mathsf{m}} \not\equiv^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$?

(Q4) Is there a TCI $\mathscr{T}$ such that $\{\mathscr{T}\} \not\leq^M \mathsf{Fg}$?

Some Open Questions

(Q1) What structural properties of $(\mathbf{M}_F(L), \subset)$ also hold for $(\mathbf{M}_S(L), \subset)$?

(Q2) For example, is $(\mathbf{M}_S(L), \subset)$ downward directed?

(Q3) Are there $m, n < \omega$ for which $\Sigma^{\mathsf{M}}_{\mathsf{m}} \not\equiv^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$?

(Q4) Is there a TCI $\mathscr{T}$ such that $\{\mathscr{T}\} \not\leq^M \mathsf{Fg}$?

Some Open Questions

(Q1) What structural properties of $(\mathbf{M}_F(L), \subset)$ also hold for $(\mathbf{M}_S(L), \subset)$?

(Q2) For example, is $(\mathbf{M}_S(L), \subset)$ downward directed?

(Q3) Are there $m, n < \omega$ for which $\Sigma^{\mathsf{M}}_{\mathsf{m}} \not\equiv^M \Sigma^{\mathsf{M}}_{\mathsf{n}}$?

(Q4) Is there a TCI $\mathscr{T}$ such that $\{\mathscr{T}\} \not\leq^M \mathsf{Fg}$?

References

Thank You!