This document contains solution for selected problems from "Set Theory: A First Course" by Daniel W. Cunningham.
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Exercises 1.1

1. By definition of $a \notin A/B$ we have $\neg(a \in A/B)$, it means that $a \notin A$ or $a \in B$ and by supposition we know $a \in A$. thus it can not be the case that $a \notin A$, so $a \in B$.

2. By definition of $A \subseteq B$ we know that for all x, $x \in A \Rightarrow x \in B(*)$. Suppose $x \in C \wedge x \notin B$. by $x \notin B$ and contraposition of (*) we have $x \notin A$. by supposition we can make claim that $x \in C \wedge x \notin A$, so we can write for all x ($x \in C \wedge x \notin B \Longrightarrow x \in C \wedge x \notin A$) which is definition of $C/B \subseteq C/A$.

3. For all x we have, $x \in A \wedge x \notin B \Rightarrow x \in C(*)$. Suppose an x that $x \in A \wedge x \notin C$, by $x \notin C$ and (*) using modus tollens we have $\neg(x \in A \wedge x \notin B)$ which is equal to say that $x \notin A \vee x \in B$. by the last sentence and $x \in A$ from supposition, we have $x \in B$. thus we can say $x \in A \wedge x \notin C \Rightarrow x \in B$ which is definition of $A/C \subseteq B$.

4. Suppose an x such that $x \in A$, by $A \subseteq B$ we know $x \in B$ and by $x \subseteq C$, $x \in C$. so we can say that for all x, $x \in A \Rightarrow x \in B \wedge x \in C$ which is definition of $A \subseteq B \cap C$.

5. We prove this by contradiction. Suppose there exist an a such that $a \in A$ but $a \notin B/C$ wich is equal to say that $a \notin B \vee a \in C$. by $a \in A$ from our supposition and $A \subseteq B$ from problem's supposition, we get $a \in B(*)$, thus it can not be the case that $a \notin B$, so it must be the case that $a \in C$ which together with (*) contradict problem's supposition $B \cap C = \emptyset$.

6. Suppose an x such that $x \in A/(B/C)$ which is equal to say that $x \in A \wedge x \notin B/C$. The second conjunct is equal to $x \notin B \vee x \in C$. At least one of the disjuncts must be true, if $x \notin B$, by supposition we have $x \in A$, so we can write $x \in A/B$. we can also say $x \in A/B \cup C$. if $x \in C$ then $x \in C \cup A/B$. We can conclude that $x \in A/(B/C) \Rightarrow x \in A/B \cup C$ which is definition of $A/(B/C) \subseteq A/B \cup C$.

7. $A \not\subseteq C$ means that there exist an a such that $a \in A$ and $a \notin C$. by $a \notin C$ and $A/B \subseteq C$ we know that $a \notin A/B$ which means that $a \notin A$ or $a \in B$. because of $a \in A$ it is only possible $a \in B$. so we have $a \in A$ and $a \in B$ which means $A \cap B \neq \emptyset$.

Exercises 1.5

1. By paring axiom we get the set $\{\{u\}, \{v, w\}\}$. Now by union axiom there exist a set that contains member of member of this set, i.e. $\{u, v, w\}$.

2. By the pairing axiom for every two set there is a set that contains them. take both set A, then we get $\{A\}$.

3. Axiom of regularity says that every non-empty set S contains at least one set x such that $x \cap S = \emptyset$. because the set $\{A\}$ contains just one set A, it must be the case that $A \cap \{A\} = \emptyset(*)$. Now suppose that $A \in A$, together with the fact that $A \in \{A\}$, there must be a common object in the two sets which contradict our first result (*).

4. By the axiom of regularity the set $\{A, B\}$ must contain a set which has nothing in common with that (i.e. $\exists(S \in \{A, B\}) S \cap \{A, B\})$. Because the set $\{A, B\}$ just contains two set, it must be A or B. it could not be be B because $A \in B$ and $A \in \{A, B\}$. it just remains A, so $A \cap \{A, B\} = \emptyset$. Clearly, $B \notin A$ because it contradicts former claim.

5. According to the regluarity axiom the set $\{A, B, C\}$ must contains a member x which $A \notin x$ and $B \notin x$ and $C \notin x$. x could not be B, because by problem supposition we know that $A \in B$. By the same justification x is not C. it just remains A, therefore the third conjucnt implies that $C \notin A$.

6. By power set axiom we have $\mathcal{P}(A)$. Now by subset axiom we can define $\{x \in \mathcal{P}(A) : x \in B\}$ which is equal to $\mathcal{P}(A) \cap B$.

9. To prove $A = \emptyset$ we must show that for all x $x \in A \Leftrightarrow x \in \emptyset$. the $\Rightarrow$ side is vacuously true because we supposed A to have no member. the $\Leftarrow$ side is true because empty set doesn't have any member.

10. Suppose that for an x $\phi(x, y_0)$ and $\phi(x, y_1)$ are both true, we prove that $y_0 = y_1$. Since $\forall z(z \in y_0 \leftrightarrow z = x)$ and $\forall z(z \in y_1 \leftrightarrow z = x)$ we have $\forall z(z \in y_0 \leftrightarrow z \in y_1)$, thus $y_0 = y_1$ and ϕ describe uniquely such a y. So by $\phi(x, y)$ and replacement axiom for every set A we have a set $\{\{x\} : x \in A\}$.

Exercises 2.1

1. Let $x \in A$, then it is also true to say $x \in A \vee x \in B$, so $A \subseteq A \cup B$. if $x \in A \cap B$ it is in both A and B, so it is in A, thus $A \cap B \in A$.

5. i.e. $A \notin A$.

8. $x \in A \cap (B \cup C)$ iff $x \in A \wedge x \in B \cup C$
iff $x \in A \wedge (x \in B \vee x \in C)$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$
iff $x \in (A \cap B) \cup (A \cap C)$.

12. $x \in C/(A \cup B)$ iff $x \in C \wedge x \notin A \cup B$
iff $x \in C \wedge (x \notin A \vee x \notin B)$ iff $(x \in A \wedge x \in B \vee x \in A \wedge x \in C)$
iff $x \in (A \cap B) \cup (A \cap C)$.

16. $x \in (A \cup B)/(A \cap B)$ iff $(x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B)$. Assume $x \in A$, one disjunct of $(x \notin A \vee x \notin B)$ must be true, the first leads to contradiction, so it must be $x \notin B$. we have $x \in A \wedge x \notin B$ which is equal to $x \in A/B$, then it is also true to say that $x \in A/B \cup B/A$. Suppose $x \in B$ then we can prove the former sentence just by previous reasoning.

20. Let $x \in \mathcal{P}(A)$ then $x \subseteq A$. Since $A \subseteq B$ then $x \subseteq B$ (by transitivty of $\subseteq$), so $x \in \mathcal{P}(B)$.

21. Let $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$. if $x \in \mathcal{P}(A)$ we can say that $x \subseteq A$, we can say for every t, $t \in x \to t \in A$, and also $t \in x \to (t \in A \vee t \in B)$ which is equal to say $x \subseteq A \cup B$, so by definition of power set, we have $x \in \mathcal{P}(A \cup B)$.

24. let $A = \{\emptyset\}$ and $B = \emptyset$, then we have $P(\{\emptyset\}/\emptyset) = \{\emptyset, \{\emptyset\}\} \neq P(\{\emptyset\})/P(\emptyset) = \{\emptyset, \{\emptyset\}\}/\{\emptyset\} = \{\{\emptyset\}\}$.

25. if $C \in \mathcal{F}$, we can write $\bigcup \mathcal{F} = C \cup X_1 U X_2 ...$ for every $X_n \in \mathcal{F}$. So it is obviously true to say $C \subseteq C \cup X_1 U X_2 ... = \bigcup \mathcal{F}$.
Second Proof: if $C \in \mathcal{F}$ we can say for every member of C there exist some set (namely, C) that belongs to $\mathcal{F}$, so by union axiom it also belongs to $\bigcup \mathcal{F}$.

26. Suppose $\bigcap \mathcal{F} \not\subset C$ which means that there is some a, $a \in \bigcap \mathcal{F}$ $a \notin C$, on the other hand $a \in \bigcap \mathcal{F}(*)$ means that a belongs to every member of $\mathcal{F}$ which C is among them (by problems supposition), so it must be true to say $a \in C$. by (*) it leads to contradiction.

28. $A \subseteq C$ for all $C \in \mathcal{F}$, means that every thing in A belongs to all member of $\mathcal{F}$ which means $A \subseteq \bigcap \mathcal{F}$.

29. Suppose $\bigcup \mathcal{F} \not\subset A$, that is, there is some a, $a \in \bigcup \mathcal{F}(*)$ but $a \notin A$. Since $C \subseteq A$ for all $C \in \mathcal{F}$, there is no $C \in \mathcal{F}$ such that $a \in C$, which contradicts (*), so $\bigcup \mathcal{F} \subseteq A$.

30. Let $x \in \bigcup \mathcal{P}(A)$, there is some X, $X \in \mathcal{P}(A)(*)$ such that $x \in X(**)$. Because of (*) we have $X \subseteq A(***)$, then by (**) and (***) we have $x \in A$. so $\bigcup \mathcal{P}(A) \subseteq A$.
Now we prove $A \subseteq \bigcup \mathcal{P}(A)$. Suppose $x \in A$, then there is some set X such that $x \in X$ and $X \subseteq A$ which means $X \in \mathcal{P}(A)$ (for example $\{x\}$). it is equal to say $A \subseteq \bigcup \mathcal{P}(A)$.

31. Let $X \in A$, then $X \subseteq X \cup X_1 \cup X_2 .... = \bigcup A$, so $X \in \mathcal{P}(\bigcup A)$.

32. if $C \in \mathcal{F}$ then $C \subseteq \bigcup \mathcal{F}$ (by Theorem 25[1]), and also $\mathcal{P}(C) \subseteq \mathcal{P}(\bigcup \mathcal{F})$(by Theorem 20), so we have $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\bigcup \mathcal{F}))$

33. It is about the collection of sets that contained by something. Suppose there exist such a set, call it A, i.e. $A = \{x : \exists y(x \in y)\}$. By axiom of regularity we know that $A \notin A$, which means it does not belong to any set ($\neg \exists y(A \in y)$), but by pairing axiom we have $\{A\}$ such that $A \in \{A\}$. so supposition of the existence of A leads to contradiction.

[1]Exercise Number 25

34. It is contraposition of Theorem 2.1.3. $\{x : \phi(x)\}$ is a set(not a proper class) $\Leftrightarrow \exists A \forall x(\phi(x) \rightarrow x \in A)$, so $\{x : \phi(x)\}$ is a proper class (not a set) $\Leftrightarrow$. $\forall A \exists x(\phi(x) \wedge x \notin A)$.

Exercises 2.2

6. Since $Y = A \cap B \in \mathcal{P}(A)$ and $\mathcal{P}(A) \in \mathcal{P}(\mathcal{P}(\bigcup \mathcal{F}))$ by Theorem 32 in Exercises 2.1. we can conclude that $\mathcal{P}(A) \subseteq \bigcup \mathcal{P}(\mathcal{P}(\bigcup \mathcal{F}))$, then $Y \in \bigcup \mathcal{P}(\mathcal{P}(\bigcup \mathcal{F}))$. so there exists a set satisfying condition $Y = A \cap B$ for some $A \in \mathcal{F}$ and some $B \in \mathcal{G}$ by Theorem 2.1.3.

7. We prove it by contradiction. Suppose there is an element x in $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$ but not in $\bigcup\{A \cap B : A \in \mathcal{F} \wedge B \in \mathcal{G}\}$. then there is an x such that $x \in \bigcup \mathcal{F} \wedge x \in \bigcup \mathcal{G}$. it means that there is some $C \in \mathcal{F}$ which $x \in C$ and some $D \in \mathcal{G}$ which $x \in D$(*). but $x \notin \bigcup\{A \cap B : A \in \mathcal{F} \wedge B \in \mathcal{G}\}$ means that there is no $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $x \in A$ and $x \in B$, it is in contradiction with (*).(proof of right to left side is similar).

8. Because $Y = A \cup B \subseteq \bigcup \mathcal{F} \cup \bigcup \mathcal{G}$ we have $Y \in \mathcal{P}(\bigcup \mathcal{F} \cup \bigcup \mathcal{G})$, by Theorem 2.1.3 there is a set satisfying this property.

9. Let $x \in (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$ but $x \notin \bigcap\{A \cup B : A \in \mathcal{F} \wedge B \in \mathcal{G}\}$, $x \in (\bigcap \mathcal{F})$, so it means that $x \in f$ for every $f \in \mathcal{F}$(*) or $x \in \bigcap \mathcal{G}$ which means $x \in g$ for every $g \in \bigcap \mathcal{G}$(**). the second part means that there are some $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $x \notin A \cup B$ which means that $x \notin A \wedge x \notin B$ for some A and B. it is in contradiction with both (*) and (**) (proof of right to left side is similar).

11. Let $x \in \bigcup(\mathcal{F} \cup \mathcal{G})$, it means there is some $X \in (\mathcal{F} \cup \mathcal{G})$ and $x \in X$. if $X \in \mathcal{F}$ we can say there is some $X \in \mathcal{F}$ and $x \in X$ which is equal to say $x \in \bigcup \mathcal{F}$, then $x \in (\bigcup \mathcal{F} \cup \bigcup \mathcal{G})$ is also true. But if $X \in \mathcal{G}$, then $x \in \bigcup \mathcal{G}$ and also $x \in (\bigcup \mathcal{F} \cup \bigcup \mathcal{G})$.

12. $x \in \bigcap(\mathcal{F} \cup \mathcal{G})$ iff x belongs to every member of $(\mathcal{F} \cup \mathcal{G})$, i.e. it must belong to every $f \in \mathcal{F}$ and every $g \in \mathcal{G}$, which is equal to say $x \in (\bigcap \mathcal{F} \cap \bigcap \mathcal{G})$.

13. Let $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$, it means that there is some $X \in (\mathcal{F} \cap \mathcal{G})$ which $x \in X$. it means there is some X, $X \in \mathcal{F} \wedge X \in \mathcal{G}(*)$. from (*) we can conclude two propositions, there is some $X \in \mathcal{F}$ and there is some $Y \in \mathcal{G}$ which $x \in X$ and $x \in Y$, it is equal to $x \in (\bigcup \mathcal{F} \cap \bigcup \mathcal{G})$, so $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\bigcup \mathcal{F} \cap \bigcup \mathcal{G})$ (the converse is not hold because of existential quantification rules).

14. $x \in \mathcal{P}(\bigcap \mathcal{F})$ iff $x \subseteq \bigcap \mathcal{F}$ iff every y, $y \in x$ then $y \in \bigcap \mathcal{F}$. $y \in \bigcap \mathcal{F}$ iff $y \in C$ for every $C \in \mathcal{F}$, it means $x \subseteq C$, so $x \in \mathcal{P}(C)$ for every $C \in \mathcal{F}$. it is equal to say $x \in \bigcap\{\mathcal{P}(C) : C \in \mathcal{F}\}$.

16. Let $x \in \mathcal{P}(\bigcup \mathcal{F})$ then $x \subseteq \bigcup \mathcal{F}$ which means that for every t, $t \in x$ then $t \in \bigcup \mathcal{F}$, but we kow $t \in \bigcup \mathcal{F}$ iff $t \in X$ for some X, $X \in \mathcal{F}$. but from supposition we know that for every $C \in \mathcal{F}$, $C \subseteq A$, so $X \subseteq A$, thus $t \in A$. we can conclude that $x \subseteq A$ and $x \in \mathcal{P}(A)$. also it is true to say that $x \in \mathcal{P}(C)$ for some C, therfore $x \in \bigcup\{\mathcal{P}(C) : C \in \mathcal{F}\}$.

Exercises 3.1

2. let $x \in (A \cup B) \times C$, it means that $x = (u, v)$ which $u \in A \cup B$ and $v \in C$. $u \in A \cup B$ is true iff $u \in A$ or $u \in B$. if $u \in A$ then $(u, v) \in A \times C$. it is also true to say $(u, v) \in (A \times C) \cup (B \times C)$. if $u \in B$ then $(u, v) \in B \times C$, similarly $(u, v) \in (A \times C) \cup (B \times C)$.

3. $x \in (A/B) \times C$ iff $x = (u, v) \in A/B \times C$ iff $u \in A$ and $u \notin B$ and $v \in C$. from $u \in A$ and $v \in C$ we know $(u, v) \in A \times C$. from $u \notin B$ we can say $(u, v) \notin B \times C$, so $x \in A \times C \wedge x \notin B \times C$ which is definition of $x \in A \times C/B \times C$.

Exercises 3.2

1. Suppose that empty set is not a relation, so for some s, $s \in \emptyset$ such that $s \neq (u, v)$ for any u, v. but empty set has no member and it contradicts with supposition.

2.1. $x \in dom(R^{-1})$ iff $(x, y) \in R^{-1}$ for some y, iff $(x, y) \in R^{-1}$ then $(y, x) \in R$ for some y, so $x \in ran(R)$.

2.2. Let $y \in ran(R^{-1})$, so there is some x, such that $(x, y) \in R^{-1}$. it is also true to say that $(y, x) \in R$ for some x, so $y \in dom(R)$.

2.3 $(u, v) \in (R^{-1})^{-1}$, iff $(v, u) \in (R^{-1})$ iff $(u, v) \in R$.

3. let $y \in R[A]$, so there is some $x \in A$ such that $(x, y) \in R$. because $A \subseteq B$, we can conclude that $x \in B$. so we can say there is some $x \in B$ such that $(x, y) \in R$, so $y \in R[B]$.

5. let $(u, v) \in R|(A \cup B)$, it means that $(u, v) \in R$ and $u \in A \cup B$. if $u \in A$, we can conclude that $(u, v) \in R|A$ and also $(u, v) \in (R|A) \cup (R|B)$. if $u \in B$, with similar argument we can conclude that $(u, v) \in (R|A) \cup (R|B)$.

6. let $x \in fld(R)$, so $x \in dom(R) \cup ran(R)$. which means that there is some $y \in ran(R)$ such that $(x, y) \in R$ or $(y, x) \in R$ for some $y \in dom(R)$. for both situation, we know that $(x, y) = \{\{x\}, \{x, y\}\} \in R$, so $\{x\}, \{x, y\} \in \bigcup R$, thus $x, y \in \bigcup\bigcup R$. (it is true similarly for (y, x))

7. $R|A$ is a set because $R|A \subseteq R$. and $R^{-1}[B]$,$R[C] \subseteq \bigcup\bigcup R$. and $RoS \subseteq dom(S) \times ran(R)$. all of them are set.

8. Define for every $x \in \mathcal{G}$, $\phi(x, y) := y = R[x]$, We prove uniqness of this definition: Assume $x = x'(1)$ we prove $R[x] = R[x']$. let $u \in R[x]$ then $(v, u) \in R$ for some $v \in x$, so by (1) $v \in x'$, so there is some $v \in x'$ such that $(v, u) \in R$ which is means that $v \in R[x']$, so $R[x] = R[x']$. then by axiom of replacement there exist a set U that containns $R[x]$ for every $x \in \mathcal{G}$. Assume that $\mathcal{G}$ is not empty, so there is some $C \in \mathcal{G}$, then $R[C]$, then $R[C]$ is empty or non-empty, in both case U is not empty(in the first case U is equal to $\{\emptyset\}$).

9. let $x \in A$ then $(x, y) \in R$ for some y, we know for every $(x, y) \in R$ iff $(y, x) \in R^{-1}$. thus we have $(x, y) \in R$ and $(y, x) \in R^{-1}$, it means that $(x, x) \in R^{-1}oR$. by $R^{-1}oR \subseteq R$ we can conclude $(x, x) \in R$, so R is reflexive.

10. let $(x, y) \in R$, then $(y, x) \in R^{-1}$, because $R^{-1} \subseteq R$ then $(y, x) \in R$.

11. Let $(x, y) \in R$ and $(y, z) \in R$, we can concludeq that $(x, z) \in RoR$, by assumption $(x, z) \in R$. thus R is transitive.

12. By Theorem 3.2.7 we have $R^{-1} = (R^{-1}oR)^{-1} = R^{-1}o(R^{-1})^{-1} = R^{-1}oR = R$. thus $R^{-1}oR = RoR \subseteq R$ by Exercise 11 R is transitive.
Because of $R = R^{-1}$ we can say $R^{-1} \subseteq R$, by Exercise 10, R is symmetric.

14. Let $(x, y), (y, z) \in \bigcap \mathcal{G}$ then both of them are in every $C \in \mathcal{G}$. because C is transitive and $(x, y), (y, z) \in C$ we can conclude that $(x, z) \in C$ for every $C \in \mathcal{G}$, thus $(x, z) \in \bigcap \mathcal{G}$ and $\bigcap \mathcal{G}$ is transitive.

15. Let $x \in A$, because R is reflexive $(x, x) \in R$ and also $(x, x) \in R^{-1}$, thus R^{-1} is reflexive. let $(x, y) \in R^{-1}$, then $(y, x) \in R$, because R is symmetric $(x, y) \in R$, thus $(y, x) \in R^{-1}$ and R^{-1} is symmetric.

Let $(x, y), (y, z) \in R^{-1}$, then $(y, x), (z, y) \in R$, because of transitivity of R we have $(z, x) \in R$, then $(x, z) \in R^{-1}$, so R^{-1} is transitive.

16. Let $(x, y) \in RoR$, then $(x, t) \in R$ and $(t, y) \in R$ for some t. because R is transitive $(x, y) \in R$. to prove converse, let $(x, y) \in R$, because R is symmetric, $(y, y) \in R$. so we can conclude that $(x, y) \in RoR$.

17. Let $(x, y) \in S$, because $y \in A$ and R is reflexive on A, $(y, y) \in R$. so we can say $(x, y) \in RoS$. again, we can say that $(x, x) \in R$, then $(x, y) \in SoR$.

18. Suppose $S \subseteq R(*)$ and $(x, y) \in SoR$. then $(x, t) \in R$ and $(t, y) \in S$ for some t. because of (*) $(t, y) \in R$, on the other hand R is transitive, so $(x, y) \in R$.
Let $(x, y) \in R$, S is reflexive on A so we have $(y, y) \in S$, therefore $(x, y) \in SoR$.
To prove the converse, assume $SoR = R$ and $(x, y) \in S$, because R is reflexive $(x, x) \in R$, so we have $(x, y) \in SoR = R$, therefore $S \subseteq R$.

21. Let $x \in R[A]/R[B]$, it means that $(a, x) \in R$ for some $a \in A$ and there is no $b \in B$ such that $(b, x) \in R$, the latter means for every $(y, x) \in R$ then $y \notin B(*)$. so by first, we can say there is some $a \in A$ such that $(a, x) \in R$ and by (*) $a \notin B$. therefore $x \in R[A/B]$.

22. Let $(x,y) \in RoS$ and $(t,y) \in RoS$. then $(x,z) \in S$ and $(z,y) \in R$ for some z. on the other hand, $(t,u) \in S$ and $(u,y) \in R$ but because R is single root relation we can say $z = u$, thus $(x,u) \in S$ and $(t,u) \in S$. but because S is single root we have $t = x$.

23. Let $(u,y) \in S$ and $(v,y) \in S$. because $dom(R) = ran(S)$ and $y \in ran(S)$ there is some t such that $(y,t) \in R$ thus $(u,t) \in RoS$ and $(v,t) \in RoS$ fo some t. but because RoS is single root $u = v$.

24. $x \sim x$ because for all $x \in A$ there is some $C \in P$ such that $x \in C$ and $x \in C$. let $x \sim y$, so there is some $C \in P$ such that $x \in C$ and $y \in C$, because "and" is comutative it is true that $y \sim x$. Let $x \sim y$ and $y \sim z$, so there is some $C \in P$, $x \in C$ and $y \in C$. and there is some $D \in P$ such that $y \in D$ and $z \in D$. but because P is partition $C \cap D = \emptyset$ or $C = D$, because y is in both of them the first case can not be true, so $C = D$, so $x \sim z$.

Exercises 3.3

1.(lemma 3.3.5) Let $y \in F$, then there is some $x \in dom(F)$ such that $y = F(x)$, but from supposition, we have $y = F(x) = G(x)$ for all x in their ommon domain, it means that $(x,y) \in G$.

1.(lemma 3.3.13) Suppose F is a one-to-one function, let $(x,y),(z,y) \in F$ it means that $F(x) = y = F(z)$, but because F is one-to-one, we can say $z = x$, thus for every $(x,y),(z,y) \in F$ we have $x = z$.

2. $y \in F[A]$ iff $y = F(x)$ for some $x \in A$, but because $A \subseteq B \subseteq dom(F)$, we have $y = F(x)$ for some $x \in B$, thus $y \in F[B]$.

3. Let $x \in A \subseteq dom(F)$, then there is unique $y = F(x) \in F[A]$, but because $F(x) \in B$ iff $x \in F^{-1}[B]$ (by definition), we can say $F(x) \in F[A]$ iff $x \in F^{-1}[F[A]]$.

4. Let $f(x) \in f[A]$, it means there is some $u \in A$ such that $f(x) = f(u)$, but because f is one-to-one, $x = u$, thus $x \in A$.

5. Assume $g[A] \cap g[B] \neq \emptyset$, so there is some $a \in g[A], g[B]$. it means that there is some $u \in A$ such that $a = g(u)$ and some $v \in B$ such that $a = g(v)$, so $g(u) = g(v)$ for some $v \in B$ and $u \in A$, but g is one-to-one and we have $u = v$, it means $u \in A \cap B$ which contradicts with assumption $A \cap B = \emptyset$.

6. $A \subseteq F^{-1}[F[A]]$ by Exercise 3. we prove other side. let $x \in F^{-1}[F[A]]$ then $F(x) \in F[A]$ which means that there is some $z \in A$ such that $F(x) = F(z)$, but because F is one-to-one, we can say $x = z$ and $x \in A$.

7. $x \in F^{-1}[C]$, iff $F(x) \in C$, because $C \subseteq D$, thus we have $F(x) \in D$, so $x \in F^{-1}[D]$.

8. Let $y \in C$, because F is onto there is some $x \in X$ such that $y = F(x) \in C$, then $x \in F^{-1}[C]$, becuase of $F^{-1}[C] \subseteq F^{-1}[D]$, we have $x \in F^{-1}[D]$, the last sentence is true iff $F(x) \in D$, thus $C \subseteq D$.

9. $x \in F^{-1}[C \cap D]$ iff $F(x) \in C \cap D$, because $F(x) \in C$ we have $x \in F^{-1}[C]$ and because of $F(x) \in D$, we can say $x \in F^{-1}[D]$ so $x \in F^{-1}[C] \cap F^{-1}[D]$. to prove converse, let $x \in F^{-1}[C] \cap F^{-1}[D]$, we have $F(x) \in C$ and $F(x) \in D$ wich means $F(x) \in C \cap D$, so $x \in F^{-1}[C \cap D]$.

10. We have $F \subseteq G$ by assumption so we just prove $G \subseteq F$. let $(x, y) \in G$, because $x \in A$ and F is a function from A to B, there must be some $(x, z) \in F$, because both $(x, z), (x, y) \in F$ by definition of function we have $z = y$, so $(x, y) \in F$.

11(a). Let $(x, y), (x, z) \in \bigcup \mathcal{C}$, then there are some $f, g \in \mathcal{C}$ such that $(x, y) \in f$ and $(x, z) \in g$, by assumption either $(x, y), (x, z) \in f$ or $(x, y), (x, z) \in g$, in both case by definition of function we have $y = z$, so $\bigcup \mathcal{C}$ is a function.

11(b). Let $(x, y), (z, y) \in \bigcup \mathcal{C}$ then there are some $f, g \in \mathcal{C}$ such that $(x, y) \in f$ and $(z, y) \in g$, by assumption either $(x, y), (z, y) \in f$ or $(x, y), (z, y) \in g$, both of them are one-to-one, so in either case we have $x = z$.

12. Let $y \in f[f^{-1}[C]]$, then there is some $x \in f^{-1}[C]$ such that $y = f(x)$, but $x \in f^{-1}[C]$ means that $f(x) \in C$, so $y \in C$.

13. One side is provided by last exercise. we prove other side. let $y \in C$, because f is onto, there must be some $x \in A$ such that $y = f(x)$, so $f(x) \in C$ iff $x \in f^{-1}[C]$ iff $f(x) \in f[f^{-1}[C]]$.

14. Suppose $y \in (FoG)[A]$, it means that there is some $x \in A$ such that $z = G(x)$ and $y = F(z)(*)$, so $z \in G[A]$ and because of (*) $y \in F[G[A]]$.

15. Let $G[X] = G[Y]$ for some $X, Y \in \mathcal{P}(A)$, it means that $f[X] = f[Y]$ i.e. $y \in f[X]$ iff $y \in f[Y]$. let $x \in X$ then $f(x) \in f[Y]$ which means there is some $y \in Y$ such that $f(x) = f(y)$, because f is one-to-one $y = x$. the converse is similar.

16. $x \sim x$ because it is always true that $F(x) = F(x)$, so $\sim$ is reflexive. let $x \sim y$ which means $F(x) = F(y)$, it is also true to say $F(y) = F(x)$, so $y \sim x$. let $x \sim y$ and $y \sim z$, it means that $F(x) = F(y)$ and $F(y) = F(x)$, because of transitivity of $=$ we have $F(x) = F(z)$ which means that $x \sim z$ holds, so $\sim$ is transitive.

18. Let $(x, y) \in F$ which means $(x, f(x)) \in F$, so $y = f(x)$ by problems assumption we have $f(y) = x$, it means that $(y, x) \in F$. to prove the other side, let $x \in A$, there is some $(x, y) \in F$ such that $F(x) = y$, because F is symmetric we also have $(y, x) \in F$, so $F(y) = x$, by substition of later we have $F(F(x)) = x$.

19. Suppose that $F(F(x)) = F(x)$ for all $x \in A$, let $(x, y) \in F$ and $(y, z) \in F$, it means that $F(x) = y$ and $F(y) = z$, from supposition we have $F(F(x)) = F(y) = F(x) = y$, so we have $F(x) = z$ which means that $(x, z) \in F$, so F is transitive.

20(a). let $x \in \bigcup_{i \in I} A_i$, it means that there is some $i \in I$ such that $x \in A_i$, because $A_i \subseteq B_i$ for all $i \in I$, we have $x \in B_i$, so we have last statement for some $i \in I$, it means that $x \in \bigcup_{i \in I} B_i$.

21. Let $C = \{A_i, i \in I\}$ be the range of indexed function. According to Theorem 3.3.24 there is a function $H : C \to \bigcup_{i \in I} A_i$ such that for all $A_i \in C$, $H(A_i) \in A_i$. so let $x : I \to \bigcup_{i \in I} A_i$ such that for each $i \in I$, $x(i) = H(A_i)$.

22. Let $(A_n, n \in N^+)$ be an indexed function such that $A_n = \{x \in R \mid 0 < x < \frac{1}{n}\}$, by axiom of choice we have a indexed function $(x_n, n \in N^+)$ such that $x_n \in A_n$ which means that $0 < x_n < \frac{1}{n}$.

Exercises 3.4

1. We want to prove that $\preceq$ is partial order. for every x we have $x \leq x$ and $x + x = 2x$ and is even, so we have $x \preceq x$ for all x and relation is reflexive. let $x \preceq y$ and $y \preceq x$, it means that $x \leq y$ and $x + y = 2k$. also we have $y \leq x$ and $y + x = 2k$, because $\leq$ is a anti-symmetric relation, we have $x = y$. so $\preceq$ is also anti-symmetric. let $x \preceq y$ and $y \preceq z$, we have $x \leq y$ and $y \leq z$, and also $x + y = 2k$ and $y + z = 2k'$, by transitivity of $\leq$ we have $x \leq z$ and also sum of them is even, because $x + z = 2k - y + 2k' - y = 2(k + k' - y)$.
1(a). No, because some members are not comparable, for example 2 and 3. sum of them is an odd number.
1(b). Yes, because the usual order is total and sum of two odd number is always even.
1(c). It has no lower and upper bound.
1(d). maximal:4,5. minimal : 1,2.

2.1 Assume that $x \prec x$, it means that $x \leq x$ and $x \neq x$ but it is imposible.
2.2 Let $x \prec y$, then $x \leq y$ (*) and $x \neq y$. because $\leq$ is antisymmetric we have either $x \not\leq y$ or $y \not\leq x$. because of (*) $y \not\leq x$ so we have $y \not\prec x$.
2.3 Suppose $x \prec y$ and $y \prec z$, by transivity of $\leq$ we have $x \leq z$, it remains to prove $x \neq z$, assume it's negation $x = z$, so we have $x \prec y$ and $y \prec x$ which is in contradiction with asymmetric property of $\prec$.
2.4 for every x and y we have $x = y$ or $x \neq y$, if first conclusion follows, if $x \neq y$, because $\leq$ is total we have $x \leq y$ or $y \leq x$, in conjunct with $x \neq y$ follows that $x \prec y$ or $u \prec x$ or $x = y$.

3. greatesst lower bound is 5 and least upper bound is 60.

4. Because $x \preceq b$ for every $x \in S$ it is also among upper bounds of S. Assume that it is not least upper bounsd, so it is not the case that $b \preceq x$ for every x among upper bounds. so there is some l among upper bounds such that $bnot \preceq l$, but because l is in upper bounds we have $b \preceq l$ which contradicts latter.

5. because $l \preceq g$ for every l among lower bounds, we have also $g' \preceq g$. also from similar argument we kno what $g \preceq g'$, because of antisymmetry we have $g = g'$.

6. (a) upper bound= $\{\{a, b\}, \{a, b, c\}\}\}$, lub=$\{a, b\}$. lower bound=$\{a, \emptyset\}$, glb=$\{a\}$.
(b) upper bound= $\{\{a, b\}, \{a, b, c\}\}\}$, lub=$\{a, b\}$. lower bound=$\{\emptyset\}$.
(c)

7. An argument similar to Lemma 3.4.10.

8. Upper Bound =$\{N, Q^+, Q, R^+, R\}$. least upper bound is N. (Does it have a maximal element? think of Zorn lemma)

9. No.

10. Let $h(x) = h(y)$, so we have $h(x) \leq h(y)$ and $h(y) \leq h(x)$. also $x \leq y$ and $y \leq x$. so because $\leq$ is antisymmetric we have $x = y$.

11. Reflexive: let $x7inA$, we have $h(x) \preceq' h(x)$ because order is reflexive, thus by definition of $\preceq$ we have $x \preceq x$.
Antisymmetry: let $x \preceq y$ and $y \preceq x$, so we have $h(x) \preceq' h(y)$ and $h(y) \preceq' h(x)$ we have $h(x) = h(y)$, because h is one-to-one $x = y$.
Transitive: let $x \preceq y$ and $y \preceq z$, it means that $h(x) \preceq' h(y)$ and $h(y) \preceq' h(z)$, because of transitivity we have $h(x) \preceq' h(z)$, thus $x \preceq z$.

12. From last Exercise we know that $\preceq$ is poset. we need to prove that it is total. let $x, y \in A$, by totallity of $\preceq'$ we have either $h(x) \preceq' h(y)$ or $h(y) \preceq' h(x)$, from this we can conlude that either $x \preceq y$ or $y \preceq x$.

13. Becuase $a \preceq x$ for every $x \in S$ it is also among lower bound of S. assume that it is not greatest lower bound of S. so it is not the case that $x \preceq a$ for all x in lower bound. so there is some l among lower bound such that $l \not\preceq a$, but because l is lower bound of S and $a \in S$ we have $l \preceq a$ which is in contradiction with latter.

14. $x \preceq_C y$ iff $x \preceq y$ and $x, y \in C$. Reflexive: for every $x \in C$ we have $x \preceq x$, thus $x \preceq_C x$.
Antisymmetry: let $x \preceq_C y$ and $y \preceq_C x$, it means that $x, y \in C$ and $x \preceq y$ and $y \preceq x$, thus $x = y$.
Transitive: let let $x \preceq_C y$ and $y \preceq_C z$, it means $x, y, z \in C$ and $x \preceq y$ and $y \preceq z$, thus by $x \preceq z$ and $x, z \in C$ we have $x \preceq_C z$.
if $\preceq$ is total, for every $x, y \in A$ we have either $x \preceq y$ or $y \preceq x$, because $C \subseteq A$ it is also true to say that for every $x, y \in A$ we have $x \preceq y$ or $y \preceq x$ which means that $x \preceq_C y$ or $y \preceq_C x$.

15. Let $x \preceq y$ and $a \in P_x$, then we have $a \preceq x$, by transitivity we can say $a \preceq y$ so $x \in P_y$. to prove converse, let $P_x \subseteq P_y$, because $\preceq$ is reflexive we have $x \preceq x$, so $x \in P_x$, thus $x \in P_y$ and also $x \preceq y$. define $f(x) = P_x$ for every x. f is a isomorphic function.

16(a). Because C is a chain, we have $x \preceq y$ or $y \preceq x$ for all $x, y \in C$, thus we have $h(x) \preceq' h(y)$ or $h(y) \preceq' h(x)$ which both belong to $h[C]$.

16(b). if $h[C]$ is a chain, then for all $u, v \in h[C]$ we have $u = h(x)$ and $v = h(y)$ for some $x, y \in C$ such that $h(x) \preceq' h(y)$ or $h(y) \preceq' h(x)$, then also we have $x \preceq y$ or $y \preceq x$.by Exercise 10 we know that h is one-to-one, thus this is true for all distinct element of C.

16(c) let $u \in A$ such that $x \preceq u$ for all $x \in C$, it is also true that $h(x) \preceq' h(a)$ for all $x \in C$, in other word, for all $h(x) \in C$ there is some $h(a)$ such that $h(x) \preceq' h(a)$.

16(d). There is some $u \in B$ such that $y \preceq' u$ for all $y \in h[C]$, it means that $u = h(x)$ for some $x \in A$ (because h is onto) and $y \preceq' h(x)$. but for all $y \in h[C]$ there is a uniqe $t \in C$ such that $y = h(t)$ because h is one-to-one, thus we have $h(t) \preceq' h(x)$ for all $t \in C$ and some $x \in A$, so we can conclude that $t \preceq' x$ for some $x \in A$ and all $t \in C$.

17. Let $x \in A$ then $(x, x) \in \preceq$ or equivalently $\{\{x\}, \{x, x\}\} = \{\{x\}\} \in \preceq$, it means that $\{x\} \in \bigcup \preceq$, and also $x \in \bigcup\bigcup \preceq$, so $A \subseteq \bigcup\bigcup \preceq$. To prove converse, let $x \in \bigcup\bigcup \preceq$, it means that there is some $C \in \bigcup \preceq$ such that $x \in C$. but $C \in \bigcup \preceq$ means that there is some $D \in \preceq$ and $C \in D$, but every member of $\preceq$ are in the form of (u, v), it means that $C = \{u\}$ or $C = \{u, v\}$ which means that $x = u$ or $x = v$ but because $\preceq \subseteq A \times A$, we

have $u, v \in A$ so $x \in A$, thus $A = \bigcup\bigcup \preceq$ but by Exercise number 6 of 3.2 we have $\bigcup\bigcup \preceq = fld(\preceq)$ thus $fld(\preceq) = A$.

18(a), Let $\preceq \subseteq \preceq'$, we have $\bigcup \preceq \subseteq \bigcup \preceq'$, also $\bigcup\bigcup \preceq \subseteq \bigcup\bigcup \preceq'$, so by Exercise 6 of 3.2 we have $fld(\preceq) \subseteq fld(\preceq')$.

Exercises 3.5

1. Assume $f(x) = f(y)$, it means that $[x] = [y]$. because $A/$ is singleton we have $x = y$ which means $x = y$, thus f is one-to-one. Assume some $y \in [x]$ such that $x \neq y$, then we have $[x] = [y]$ so $f(x) = f(y)$ but f is one-to-one, thus $x = y$ which is contradiction. 2.

Exercises 4.1

1. Because I and K both are inductive set, $\emptyset \in I \cap J$. let $x \in I \cap J$, it means $x \in I$ and $x \in J$, so $x^+ \in I$ and $x^+ \in J$, thus we have $x^+ \in I \cap J$.

2. We know that $A \subseteq A^+$, by Theorem 4.1.10 for every transitive set A we have $\bigcup A^+ = A$, thus we have $\bigcup A^+ \subseteq A^+$.

3. IF $A \subseteq \mathcal{P}(A)$, then $\bigcup A \subseteq \bigcup \mathcal{P}(A)$, by Exercise 2.1, 30 we have $\bigcup A \subseteq A$. to prove converse let $\bigcup A \subseteq A$, then $\mathcal{P}(\bigcup A) \subseteq \mathcal{P}(A)$ by Exercise 2.1,31 $A \subseteq \mathcal{P}(\bigcup A)$, so we have $A \subseteq \mathcal{P}(A)$.

4.if $\bigcup A \subseteq A$, then $\bigcup\bigcup A \subseteq \bigcup A$.

5. Let $x \in \bigcup\bigcap \mathcal{A}$, it means that there is some $C \in \bigcap \mathcal{A}$ such that $x \in C$. but we have $C \in B$ for all $B \in \mathcal{A}$, thus $x \in B$ for all $B \in \mathcal{A}$ because all member of $\mathcal{A}$ are transitive, thus we have $x \in \bigcap \mathcal{A}$.

Let $x \in \bigcup\bigcup \mathcal{A}$, it means that there is some $C \in \bigcup \mathcal{A}$ such that $x \in C$, if $C \in \bigcup \mathcal{A}$ then there is some $B \in \mathcal{A}$ such that $C \in B$, but because B is transitive we have $x \in B$, it means that $x \in \bigcup \mathcal{A}$ so $x \in \bigcup\bigcup \mathcal{A} \subseteq \bigcup \mathcal{A}$.

7. Let $I = \{n \in \omega : n \neq n^+\}$, obviously $0 \in I$, let $n \in I$, so $n \neq n^+$ then by Thorem 4.1.12 we have $n^+ \neq (n^+)^+$, so $I = \omega$.

8. $n \neq n^+$ means that $n \not\subseteq n^+$ or $n^+ \not\subseteq n$, obviously $n \subseteq n^+$, thus we just have $n^+ \not\subseteq n$.

9. Let $n \subseteq m$, if $m \in n$ then $m \subseteq n$(by Theorem 4.1.11), so we have $n = m$, thus $n \in n$ which is contradiction.

Proof 2. Let $m \in \omega$ and $I = \{n \in \omega : (m \in n \to n \not\subseteq m)\}$, we have $0 \in I$ vacuously. let $n \in I$, it means that $m \in n \to n \not\subseteq m$. let $m \in n^+$ which means that $m = n$ or $m \in n$. if first, by Exercise 8 we have $n^+ \not\subseteq m$. if $m \in n$, then $n \not\subseteq m$, then $n^+ \not\subseteq m$.

10. Let $m = n$ in Exercise 9, then if $n \in n$ we have $n \not\subseteq n$ which is contradiction.

11. if $\bigcup A = A$ then $\bigcup A \subseteq A$.

Let $x \in A$, then $x \in \bigcup A$, which means that there is some $y \in A$ such that $x \in y$.

Exercises 4.2

1. 2. Let $j, k \in A$ and $f(j) = f(k)$ and $j \neq k$. because h is onto, we have some $x, y \in \omega$ such that $j = h(x), k = h(y)$, thus we have $f(h(x)) = f(h(y))$, but by (2) we have $h(x^+) = f(h(x)) = f(h(y)) = h(y^+)$ it means that $x^+ = y^+$, thus $x = y$ and it means that $j = h(x) = h(y) = k$ which is contradiction.

3. Let $I = \{n \in \omega : h(n) \in h(n^+)\}$, $0 \in I$ because $h(0) = a \in f(a)$ it means $h(0) \in f(h(0))$ or equavalently $h(0) \in h(0^+)$.

Assume $n \in I$, it means that $h(n) \in h(n^+)$, because $h(n), h(n^+) \in A$ by (b) we have $f(h(n)) \in f(h(n^+))$ which mean that $h(n^+) \in h((n^+)^+)$, so $n^+ \in I$, thus $I = \omega$.

4.(a) is a set, because $\mathcal{S} \subseteq \mathcal{P}(A)$.

(b) is non-empty, because A itself satisfies condition.

(c) $y \in C = \bigcap \mathcal{S}$ because $y \in B$ for all $B \in \mathcal{S}$. let $x \in F[C]$ it means that there is some $t \in C$ such that $x = f(t)$, but $t \in C$ means that $t \in B$ for all

$B \in C$ for which we have $F[B] \subseteq B$, thus $x = f(t) \in B$ for all $B \in C$, so $x \in C$.

(d) for every $B \subseteq A$, if $y \in B$ and $F[B] \subseteq B$ then $B \in \mathcal{S}$ and because $C = \bigcap \mathcal{S}$ we have $C \subseteq B$.

(e) By (c) we kno that $F[C] \subseteq C$, because $F[C], C \subseteq A = dom(F)$ by Exercise 3.3, 2 we have $F[F[C]] \subseteq F[C]$ and because $y \in F[C]$ we have $F[C] \in \mathcal{S}$, thus $F[C] \subseteq C$. The other side hold by (c) again, thus we have $F[C] = C$.

Exercises 4.3

1. Assume that $n \neq 0$, by Theorem 4.1.6 we have $n = k^+$ for some $k \in \omega$. then we have $m + k^+ = 0$, so by (A2) we have $(m + k)^+ = 0$ which means that $m + k \in 0$, but it is contradiction because $0 = \emptyset$. similar argument can be given for $m = 0$.

2. Assume $m \neq 0$ and $n \neq 0$, so we have $m.k^+ = 0$ for some k, then by (M2) we have $m.k + m = 0$. by previous exercise we have $m = 0$ and it is contrdiction.

3. Let $I = \{p \in \omega : m + p = n + p \to m = n\}$ we prove it is an inductive set. $0 \in I$ trivially, let $p \in I$ and assume $m + p^+ = n + p^+$, by (A2) we have $(m + p)^+ = (n + p)^+$ so we have $m + p = n + p$, but $p \in I$ thus $m = n$. so we have $p^+ \in I$ and $I = \omega$.

4. Let $I = \{n \in \omega : 0.n = 0\}$, $0 \in I$ trivially. let $n \in I$, we have $0.n^+ = 0.n + 0$ but because $0.n = 0$ we have $0.n^+ = 0$ thus $n^+ \in I$.

5. for a $m \in \omega$ let $I = \{n \in \omega : m^+.n = m.n + n\}$. $0 \in I$ because $m^+.0 = m.0 + 0 = 0$. Let $n \in I$, we have $m^+.n^+ = m^+.n + m^+$(by (A2)) but because $n \in I$ we have

$$
\begin{aligned}
m^+.n^+ &= (m.n + n) + m^+ \\
&= m.n + (n + m^+) && \text{by Associative prop} \\
&= m.n + (n + m)^+ && \text{by (A2)} \\
&= m.n + (m + n)^+ && \text{by Commutative prop} \\
&= m.n + (m + n^+) && \text{by (A2)} \\
&= (m.n + m) + n^+ && \text{by Associative prop} \\
&= m.n^+ + n^+ && \text{by (M2)}
\end{aligned}
$$

thus $n^+ \in I$ and $I = \omega$.

6. Let $n \in \omega$ and $I = \{m \in \omega : m.n = n.m\}$, $0 \in I$ because $0.n = 0$ by Exercise 4, and $0 = n.0$ by (M1), thus $0.n = n.0$. Assume $m \in I$, then

$$\begin{aligned} m^+.n &= m.n + n && \text{by previous exercise} \\ &= n.m + n && \text{because } m \in I \\ &= n.m^+ && \text{by (M2)} \end{aligned}$$

so we have $m^+ \in I$ and $I = \omega$.

7. Let $I = \{n \in \omega : (\exists k \in \omega) n = 2.k \vee (\exists i \in \omega) n = 2.i + 1\}$. $0 \in I$ because $0 = 2.0$ for $k = 0$. Let $n \in I$, then either there is some $k \in \omega$ such that $n = 2.k$ or there is some $i \in \omega$ such that $n = 2.i + 1$. Assume that the first sentence is true. then let $n^+ = n + 1$ but because the first hold, we have $n^+ = 2.k + 1$ thus n^+ is odd and $n^+ \in I$. Now assume that the second, then $n^+ = n + 1 =$, by replace n with hypothesis, for some $i \in \omega$ we have

$$\begin{aligned} n^+ &= (2.i + 1) + 1 \\ &= 2.i + (1 + 1) && \text{by Associative} \\ &= 2.i + 2 && \text{because } 1 + 1 = 2 \\ &= 2.i^+ && \text{by (M2)} \end{aligned}$$

so n^+ is even, thus $n^+ \in I$ and $I = \omega$.

8. Let $I = \{n \in \omega : \neg(n \text{ is even and } n \text{ is odd})\}$, assume that 0 is even and is odd, if it is odd, then $0 = 2.i + 1$ for some $i \in \omega$. but then $0 = (2.i)^+$ so zero is succesor of a number which is contradiction. thus it is not the case that n is both even and odd, thus $0 \in I$.

Let $n \in I$, then by n is not even or is not odd. if first, there is no $k \in \omega$ such that $n = 2.k$ and so $n^+ = 2.k + 1$, thus n^+ is not odd and is in I. if second, then there is no $i \in \omega$ such that $n = 2.i + 1$ and also $n^+ = 2.i^+$(from previous exercise), thus n^+ is not even, so $n^+ \in I$.

9. For $m, n \in \omega$ let $I = \{k \in \omega : m^{n+k} = m^n.m^k\}$. because $m^{n+0} = m^n = m^n.1 = m^n.m^0$, so $0 \in I$. Let $k \in I$, we have

$$\begin{aligned} m^{n+k^+} &= m^{(n+k)^+} && \text{by (A2)} \\ &= m^{n+k}.m && \text{by (E2)} \\ &= (m^n.m^k).m && \text{because } k \in I \\ &= m^n.(m^k.m) && \text{by Associativ-} \end{aligned}$$

ity

$$= m^n.m^{k^+} \qquad \text{by (E2)}$$

so $k^+ \in I$.

10. For $m, n \in \omega$ let $I = \{k \in \omega : (m.n)^k = m^k.n^k\}$. because $(m.n)^0 = 1 = 1.1 = m^0.n^0$ then we have $0 \in I$. let $k \in I$, we have $(m.n)^{k^+}$

$$\begin{aligned}
&= (m.n)^k.(m.n) && \text{by (E2)} \\
&= m^k.n^k.(m.n) && \text{because } k \in I \\
&= m^k.n^k.(n.m) && \text{by Theorem 4.1.13} \\
&= m^k.(n^k.n).m && \text{by Theorem 4.1.12} \\
&= m^k.n^{k^+}.m && \text{by (E2)} \\
&= m^k.m.n^{k^+} && \text{by Theorem 1.4.13} \\
&= m^{k^+}.n^{k^+} && \text{by (E2)}
\end{aligned}$$

Thus, $k^+ \in I$ and $I = \omega$.

11. For $m, n \in \omega$ let $I = \{k \in \omega : (m^n)^k = m^{n.k}\}$. because $(m^n)^0 = 1 = m^{n.0}$, 0 belongs to I. let $k \in I$, then $(m^n)^{k^+}$

$$\begin{aligned}
&= (m^n)^k.(m^n) && \text{by (E2)} \\
&= m^{n.k}.m^n && \text{because } k \in I \\
&= m^{n.k+n} && \text{by Exercise 9} \\
&= m^{n.k^+} && \text{by (M2) 9}
\end{aligned}$$

so $k^+ \in I$, thus $I = \omega$.

Exercises 4.4

1. let $I = \{n \in \omega : 1 \underline{\in} n^+\}$. because $1 = 0^+$ we have $0 \in I$. assume $n \in I$, then $1 \underline{\in} n^+$ iff $1 = n^+$ or $1 \in n^+$. if first, we know that $n^+ \in (n^+)+$, thus $1 \in (n^+)+$. if second, $1 \in (n^+)^+$ by transitivity. in either case $1 \underline{\in} (n^+)^+$, thus $(n^+)^+ \in I$.

Proof two: by Lemma 4.4.6 we have $0 \underline{\in} n$ for every $n \in \omega$. by Lemma 4.4.7 we have $0^+ \underline{\in} n^+$, which means $1 \underline{\in} n^+$ for all $n \in \omega$.

2. if $m \in n^+$ then $m \in n$ or $m = n$, in both case we have $m \underline{\in} n$.

3. Let $n \in a$, by Theorem 4.4.9 we have $a \in n^+$ or $n^+ \in a$ or $n^+ = a$. we prove the first is imposible. if $a \in n^+$ then either $a \in n$ or $a = n$. if first,

then we have both $a \in n$ and $n \in a$ which contradicts the trichotomy law. if $a = n$ then we have $n \in n$ by assumption which is again contradiction. thus it just remain $n^+ \in a$ or $n^+ = a$ which is equal to $n^+ \underline{\in} a$.

4. Let $J = \{n \in I : n \in a \vee a \subseteq n\}$, we know that I is inductive, thus $0 \in I$.because $a \in \omega$ by Lemma 4.4.6 $0 \in a$, thus $0 \in J$. let $n \in J$. then $n \in I$ and $n \in a$ or $a \subseteq n$. if first, because ω is transitive, we have $n \in \omega$, by Lemma 4.4.7 $n^+ \in a^+$, it means that $n^+ \in a$ or $n^+ = a$, in either case $n^+ \in J$. if $a \subseteq n$, then because $n \subseteq n^+$ we have $a \subseteq n^+$, so $n^+ \in J$.

5. By Corollary 4.4.10 if $m \in n$ then $m \subset n$, so $max(m, n) = m \cup n = n$.

6. Let $I = \{p \in \omega : m \in n \to m+p \in n+p\}$. if $m \in n$ then $m+0 \in n+0$, so $0 \in I$. let $p \in I$ and assume that $m \in n$, then we have $m + p \in n + p$, by Lemma 4.4.7 $(m + p)^+ \in (n + p)^+$. by (A2) $m + p^+ \in n + p^+$, thus $p^+ \in I$ and $I = \omega$.

7.Let $I = \{p \in \omega : m + p \in n + p \to m \in n\}$. Clearly $0 \in I$. let $p \in I$, if $m + p^+ \in n + p^+$ then $(m + p)^+ \in (n + p)^+$ by (A2). By Corollary 4.4.8 we have $(m + p) \in (n + p)$ but then because $p \in I$ we have $m \in n$, thus $p^+ \in I$.

8. let $m, n \in \omega$ and $I = \{p \in \omega : m \in n \to m.p^+ \in n.p^+\}$. if $m \in n$ and $p = 0$, then $m.1 \in n.1$ iff $m.0^+ \in n.0^+$, thus $0 \in I$. Assume $m \in n$ and let $p \in I$, then $m.p^+ \in n.p^+$, by Theorem 4.4.11(1) we have $m.p^+ + m \in n.p^+ + m$, also because $m \in n$ again by Theorem 4.4.11(1) we have $n.p^+ + m \in n.p^+ + n$, then by transitivity we have $m.p^+ + m \in n.p^+ + n$ which means that $m.(p^+)^+ \in n.(p^+)^+$, thus $p^+ \in I$.

9. Assume $m.p^+ \in n.p^+$, by Theorem 4.4.9 for every $m, n \in \omega$ we have $m \in n$ or $m = n$ or $n \in m$. if $n \in m$ by previous Exercise we have $n.p^+ \in m.p^+$, by transivity we have $m.^p+ \in m.p^+$ which is contradiciton.

10. (1) Let $m + p = n + p$, by Theorem 4.4.9 we have either $m \in n$ or $n \in m$ or $m = n$. if first then by Theorem 4.4.11(1) we have $m + p \in n + p$ which is contradiction. Suppose the second holds, then $n + p \in m + p$ and again contradiction. the third case just remains.

(2) Assume $p \neg 0$ and $m.p = n.p$, just like above we have either $m \in n$ or $n \in m$ or $m = n$. by Theorem 4.4.11(2) we get that the only possible case is $m = n$.

11. Let $m \in \omega$ and $I = \{p \in \omega : m \in m + p^+\}$. we know that $m \in m^+ = m + 1 = m + 0^+$, thus $0 \in I$. let $p \in I$, then we have $m \in m + p^+$. by Theorem 4.4.11(1) we have $m + 1 \in m + p^+ + 1$. because $m \in m + 1$ and transivity we have $m \in m + p^+ + 1$. by Associavity and (M2), Proposition 4.3.4 we conclude that $m \in m + (p^+)^+$, thus $p^+ \in I$.

12. Let $I = \{n \in \omega : m \in n \to m + p^+ = n\}$ for some $p \in \omega$. $0 \in I$ vacuously. let $n \in I$ and if $m \in n^+$ then either $m \in n$ or $m = n$. if first then because $n \in I$ we have $m + p^+ = n$, then $m + p^+ + 1 = n + 1$, thus we have $m + (p^+)^+ = n^+$ for some $p' = p^+ \in \omega$, thus $n^+ \in I$. if $m = n$ then $m + 1 = n + 1$, which is equal to $m + 1 = n^+$, then we can say $m + p = n^+$ for some $p \in \omega$. thus in either case $n^+ \in I$.

13. One side follows from exercise 12, let's prove the other side. if $m+p^+ = n$ for some $p \in \omega$, by Exercise 11 we have $m \in m + p^+$, then $m \in n$.

14.(a) Let $I = \{n \in \omega : m \in n \to F(m) \in F(n)\}$. Clearly $0 \in I$. let $n \in I$ and suppose that $m \in n^+$, then either $m \in n$ or $m = n$. if $m \in n$ then because $n \in I$ we have $F(m) \in F(n)$. but then because $F(n) \in F(n^+)$ by transitivty we have $F(m) \in F(n^+)$. if $m = n$ then $F(m) = F(n)$, again by replacing $F(n)$ in $F(n) \in F(n^+)$ we get $F(m) \in F(n^+)$, thus $n^+ \in I$ and $I = \omega$.

(b) Assume that $F(m) = F(n)(*)$ for some $m, n \in \omega$ then we have either $m = n$ or $m \in n$ or $n \in m$. if $m \in n$ by previous exercise we have $F(m) \in F(n)$ which contradicts (*), similarly this hold for $n \in m$, then it just remain that $m = n$.

15. Let $S = F[\omega]$, because $S \subseteq \omega$ by Theorem 4.4.13 it has a least element l such that $l = F(k)$ for some $k \in \omega$. but again $k^+ \in \omega$ thus we have $F((k^+)^+) \in F(k^+)$, so $F(k^+) \in S$, it follows then $F(k^+) \in F(k)$ and both $F(k), F(k^+) \in S$ and $F(k)$ is least element which is contradiction.

16. Let $U = \{x \in n^+ : e \subseteq x \text{ for all } e \in E\}$, thus U is upper bound of E. By well-ordering U has a least element m. we prove that $m \in E$. Assume that $m \notin E$. it means that $x \in m$ for all $x \in E$ (*)(and it is not possible $x = m$ for any case). Certainly $m \neq 0$ because $E \neq \emptyset$ (when $E = \emptyset$ we would have $U = n^+$ which contains 0), thus we have $m = k^+$ for some $k \in \omega$. thus we can rewrite (*) like this : $x \in k^+$ for all $x \in E$, by Exerecise 2 we have $x \subseteq k$ for all $x \in E$, thus $k \in U$, but on the other hand we have $k \in m$ (because $k \in k^+ = m$) and m is least element which is a contradiction, thus $m \in E$.

17. Let $I = \{n \in \omega : F[n^+] \text{ has a largest element}\}$. clearly $0 \in I$ because $F[0^+]$ has just one element $F(0)$. Assume $n \in I$ then $F[n^+]$ has a largest element m. $F[(n^+)^] = F[n^+ \cup \{n^+\}] = F[n^+] \cup F[\{n^+\}]$ then $F[n^+]$ just have one element $k = F(n^+)$, by trichotomy law we have just one of either $k \in m$ or $m = k$ or $m \in k$, if first or two then m is largest, if the third then k is largest.