MATH 353-ORD & PRTL DIFF EQUATIONS-Spring 2017-EXAM 1

Name KEY

Section Venakides

Thursday, February 23, 2017.

Closed book and notes. Use of calculators and cellphones is not allowed.
Formula sheet in sakai general site allowed.
You may use the back of the pages.

Please sign "no assistance" pledge____________________

ALL ANSWERS SHOULD BE CIRCLED

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
TOTAL	

1. Identify the type(s) of each ODE by checking the appropriate boxes. Leave the other boxes blank (points taken off for incorrect checks). DO NOT SOLVE THE ODEs.

   An ODE that is separable is automatically exact. Nevertheless, if you check the "separable" box, please leave the "exact" box unchecked.

   Notation: y' is the derivative of y with respect to x.

ODE	linear homog.	linear nonhom.	separable	exact	homogeneous
$xy' = y\sin x$	✓		✓		
$y' = xy^2 + \sin x$					
$y' = \frac{3x^2+y^2}{2y-2xy}$				✓	
$y' = \frac{xy^2-y^3}{x^3+x^2y}$					✓
$xy' - \sin x = y + 1$		✓			

15pts

2. True or false? In each of the following statements, circle the appropriate word (points taken off for wrong guess):

(a) Consider the ODE
$$y' = y(y-2)(y-4)$$

i. The ODE has exactly two stable equilibrium solutions. **True** **False**

ii. If $y(0) = 4.5$, then $\lim_{x \to +\infty} y(x) = +\infty$. **True** **False**

(b) The following operation is linear.

i. $f(x) \longrightarrow \int_0^x f(t)dt$. **True** **False**

ii. $x \longrightarrow 1 + \frac{1}{x}$, where $x > 0$. **True** **False**

(c) At least one solution of any second-order linear homogeneous ODE, can be obtained by the method of reduction of order. **True** **False**

16 pts

(d) If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges when $z = 3 + 4i$, where $i = \sqrt{-1}$, it necessarily converges when $z = 5i$. **True** **False**

(e) The Laplace transform is a linear transformation. **True** **False**

(f) Consider the ODE
$$y'' + 4y = \sin(2x) + \cos(3x)$$

i. The general solution of the ODE equals the general solution of the homogeneous ODE plus two particular solutions obtained when the two forcings are considered separately. **True** **False**

ii. Every solution remains bounded when x ranges over the real numbers. **True** **False**

3. Given the ODE

$$\frac{dy}{dx} = -\frac{3x^2 + y}{x + 2y},$$

(a) Find the general solution of the ODE. (circle your answer)

(b) Calculate the real-valued function $y(x)$ that satisfies the ODE with initial condition $y(0) = 1$. Remember that the solution of the IVP is unique. (circle your answer)

(c) What integer and half-integer values of x (positive, zero and negative) lie in the domain of existence of the calculated $y(x)$ of question (b)? (circle your answer)

(a) $\underbrace{(3x^2+y)}_{M}dx + \underbrace{(x+2y)}_{N}dy = 0 \qquad (1)$

seek a function $f(x,y)$ such that $M = f_x$, $N = f_y$

Necessary condition $M_y = N_x$ $(f_{xy} = f_{yx})$

$M_y = 1 \quad N_x = 1$, condition is satisfied.

15pts

$f = \int M(x,y)\,dx = x^3 + xy + g(y)$

$f_y = x + g'(y) = \underbrace{x+2y}_{N}$, $g'(y) = 2y$, $g = y^2$

constant of integration taken $= 0$

Thus $f = x^3 + xy + y^2$

ODE (1) becomes $f_x dx + f_y dy = 0$, $df = 0$, $f = \text{const.}$

Answer to (a): $x^3 + xy + y^2 = c \qquad c = \text{const.}$

(b) Inserting $x=0$ $y=1$ obtains $c=1$

solve for y: $y = \frac{1}{2}\left(-x + \sqrt{x^2 - 4x^3 + 4}\right)$

The minus sign in front of the root fails to satisfy the initial condition.

(c) All negative ones as well as $0, \frac{1}{2}, 1$ produce quantity > 0 under the radical

4. (a) The rate of change of a variable y with respect to a variable x is a polynomial of y whose roots are the numbers $0, 1, 2, 4$. The roots are simple (as opposed to multiple).

   i. Express the given information as an ODE. (circle your answer)

   ii. Write down all the solutions of the ODE that are independent of the variable x (circle your answer).

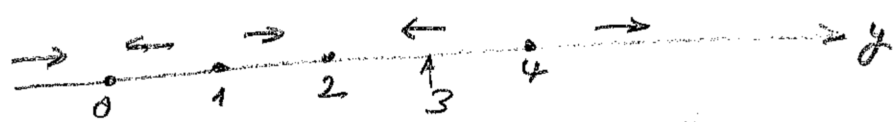
   iii. Given the initial condition $y(0) = 3$, evaluate the limits

$$\lim_{x \to -\infty} y(x) = \qquad , \qquad \lim_{x \to +\infty} y(x) =$$

(i) $\frac{dy}{dx} = y(y-1)(y-2)(y-4)$

(ii) $y(x) = 0$, $y(x) = 1$, $y(x) = 2$, $y(x) = 4$

(iii)



$\lim_{x \to -\infty} y(x) = 4$ $\quad \lim_{x \to +\infty} y(x) = 2$

14 pts

(b) Fill in the blanks using the most appropriate of the following four choices: vector space, matrix, vector, system of equations

eigenvalue of a ....matrix....

eigenvector of a ....matrix....

nullspace of a ....matrix....

dimension of a ....vector space....

free variables of a ....system of equations....

magnitude (norm) of a ....vector....

5. (a) Use the Laplace transform to solve the following initial value problem.

$$\begin{cases} y'' + 4y = \delta(t-2) + e^{-t} \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

where $\delta(t)$ is the Dirac delta function. Circle your answer.

$$\mathcal{L}(y''+4y) = \mathcal{L}(\delta(t-2) + e^{-t}), \quad \mathcal{L}(y'') + 4\mathcal{L}y = \mathcal{L}\,\delta(t-2) + \mathcal{L}e^{-t}$$

$$s^2Y + 4Y = e^{-2s} + \frac{1}{s+1}, \quad Y = \frac{e^{-2s}}{s^2+4} + \frac{1}{(s+1)(s^2+4)}$$

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}. \quad A = \frac{1}{5}, \quad Bs+C = \frac{1}{s+1}\Big|_{s=\pm 2i}$$

$$\left.\begin{aligned} 2Bi + C &= \frac{1}{1+2i} \\ -2Bi + C &= \frac{1}{1-2i} \end{aligned}\right\} \quad C = \frac{1}{5}, \; B = -\frac{1}{5} \qquad Y = \frac{1}{2} e^{-2s} \frac{2}{s^2+4} + \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s}{s^2+4} + \frac{1}{10}\frac{2}{s^2+4}$$

$$y(t) = \frac{1}{2} u_2(t) \sin[2(t-2)] + \frac{1}{5}e^{-t} - \frac{1}{5}\cos(2t) + \frac{1}{10}\sin(2t)$$

(b) Find a basis of the space of solutions of the ODE

$$y''' + xy'' = 0$$

16 pts

Circle your answer. Hint: Do NOT use series.

Let $y'' = z$ $\qquad z' + xz = 0$ separable

$$z = c_1 e^{-\frac{x^2}{2}}$$

$$y' = c_1 \int_0^x e^{-\frac{s^2}{2}} ds + c_2$$

$$y = c_1 \int_0^x \int_0^t e^{-\frac{s^2}{2}} dt\, ds + c_2 x + c_3$$

Basis of solution

$$y_1 = 1, \quad y_2 = x, \quad y_3 = \int_0^x \int_0^t e^{-\frac{s^2}{2}} dt\, ds$$

6. Consider the ODE

$$y'' + xy' + y = 0.$$

We are seeking a basis of the space of solutions of the ODE in the form of power series that are centered at $x = 0$. It is advisable to write the series as

$$y = \sum_{-\infty}^{\infty} a_n x^n, \qquad a_n = 0 \text{ when } n < 0.$$

(a) Derive the free variables (show your work) and choose their values for producing a most convenient basis. Circle the free variables, the recursion relation, and each of the basis series together with the numerical values of the free variables that produce the series. You can write each basis series in your answer by giving its first three or four terms +... if you so prefer.

Inserting the series into the ODE

$$\sum_{n=-\infty}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=-\infty}^{\infty} n a_n x^n + \sum_{n=-\infty}^{\infty} a_n x^n$$

$n \to n-2$ $\qquad$ $n \to n-2$

16 pts

$$\sum_{n=-\infty}^{\infty} \left(n(n-1)a_n + (n-2)a_{n-2} + a_{n-2} \right) x^{n-2} = 0 \quad \text{"} \forall x \text{"}$$

$$n(n-1)a_n + (n-1)a_{n-2} = 0 \quad \forall \text{ integers } n$$

$n < 0$ $\quad 0 = 0$

$n = 0$ $\quad 0 = 0$

$n = 1$ $\quad 0 = 0$

$n > 1$ $\quad a_n = -\dfrac{a_{n-2}}{n}$ $\qquad$ a_0, a_1 : free variables

$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $y_1 = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{2 \cdot 4} - \dfrac{x^6}{2 \cdot 4 \cdot 6} + \dots$

$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad y_2 = x - \dfrac{x^3}{3} + \dfrac{x^5}{3 \cdot 5} - \dfrac{x^7}{3 \cdot 5 \cdot 7} + \dots$

(b) Calculate the radius of convergence of the series using the recursion relation and the ratio test.

For each series the x's that lie within the circle of convergence can be found from

y_1: $\displaystyle \lim_{n \to \infty} \left| \frac{(n+1)^{th} \text{ term}}{n^{th} \text{ term}} \right| < 1, \quad |x^2| \underbrace{\lim_{n \to \infty} \frac{1}{2(n-1)}}_{=0} < 1$ True for all x

y_2: similar $\qquad R = \infty$