ORD & PRTL DIFF EQUATIONS-MATH 353-753 FALL 2024-EXAM 1

Name Key

Section 02

Wednesday, October 9, 2024, 3:05-4:20 PM, Physics 235.

Calculators, cellphones, computers not allowed. Closed book and notes. Allowed: Formula sheet given to you plus one standard page sheet written by you on both sides

PLACE ALL ANSWERS IN BOXES

Student "no assistance" pledge. Please sign:

MATH 353, ORD & PRTL DIFF EQUATIONS-EXAM 1

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	

1. Identify the type(s) of each ODE by checking the appropriate boxes. Leave the other boxes blank (points taken off for incorrect checks). Remember: "separable" implies "exact". y' stands for dy/dt.

	linear homog.	linear nonhom.	separable	exact
$ty' = (y + \sin y)(t^2 + 1)$			✓	✓
$y' = -\frac{2t + y\cos t + \sin y}{\sin t + t\cos y}$			✓	
$y' = \frac{f(t)}{g(y) + 2y}$			✓	✓
$y' + 5t^2 + 5y\cos t = \sin^2 t$		✓		
$y'^2 = y^2 - t^2$				

2. (a) What does it mean to solve the eigenvalue problem of a given square matrix A?

Find the pairs $\lambda, \vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}, \quad \vec{x} \neq 0$$

(b) Write down a third order linear nonhomogeneous ODE with coefficients that are not all constants. Do not attempt to solve it.

$$y''' + 5ty' - 6t^2 y = t^2 + 1$$

(c) Write down a third order linear homogeneous ODE with constant coefficients. Do not solve it, but explain the solution process in a few words.

$$y''' + 5y'' - 6y' - 2y = 0$$

Let $y = e^{rt}$ then $r^3 + 5r^2 - 6r - 2 = 0$. Roots r_1, r_2, r_3

If all distinct: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t}$ (general solution)

If nonunique te^{rt} and $t^2 e^{rt}$ may appear.

(d) Write down the first order ODE that has general solution $t^2+3t^3y+y^3 = \text{constant}$.

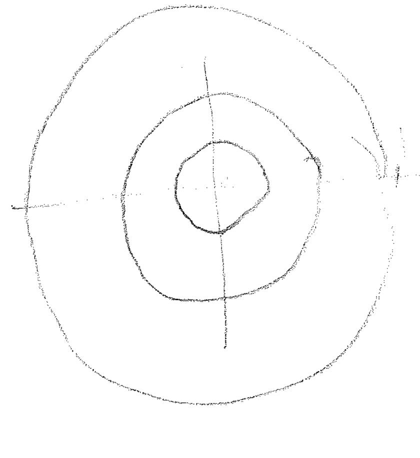
Take the derivative on both sides

$$2t + 9t^2 y + 3t^3 y' + 3y^2 y' = 0$$

$$y' = -\frac{2t + 9t^2 y}{3(t^3 + y^2)}$$

(e) Plot the solutions of the ODE $xdx + ydy = 0$ in the x, y plane.

Solution $x^2 + y^2 = c$



3. a) Find a basis for the vector space of all real-valued solutions of the ODE

$$y'' - 2y' + 5y = 0$$

$y = e^{rt}$

$r^2 - 2r + 5 = 0 \qquad r = 1 \pm \sqrt{-4} = 1 \pm 2i$

one complex solution $\quad y = e^t(\cos 2t + i \sin 2t)$

$= e^t \cos 2t + i e^t \sin 2t$

Each of the two terms is a solution

Basis: $y_1 = e^t \cos 2t, \; y_2 = e^t \sin 2t$

b) Solve the the ODE

$$y' + 4ty - 5t = 0$$

[H]: $y' + 4ty = 0 \qquad \frac{dy}{dt} = -4ty, \qquad y_{[H]} = c e^{-2t^2}$

Particular solution:

Check $y = a$ = constant

$4ta = 5t \qquad a = \frac{5}{4}$

$y = c e^{-2t^2} + \frac{5}{4}$

4. Consider the ODE

$$y'' + 4y = \cos at \quad a > 0$$

Provide formulae for the solution that would cover ALL the values of a

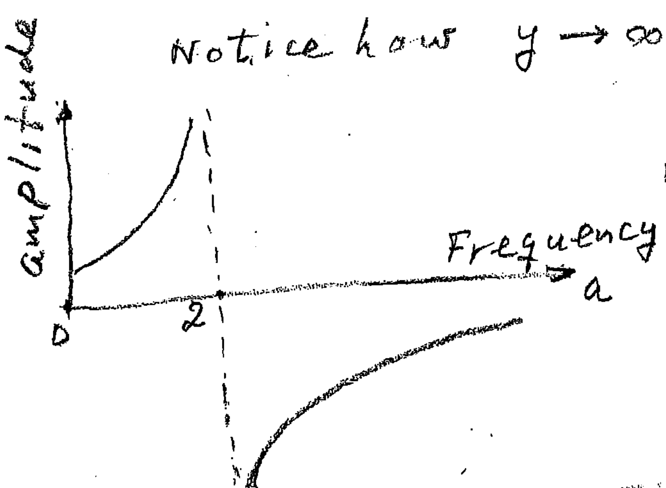
- Solution of [H] $y = c_1 \cos 2t + c_2 \sin 2t$
- Particular solution:

$a \neq 2$ $\quad y = A\cos at$ (including $B\sin at$ is wasteful. Only cosines come up)

$$-Aa^2 \cos at + 4A\cos at = \cos at$$

$$-Aa^2 + 4A = 1 \qquad A = \frac{1}{4-a^2}, \qquad y = \frac{\cos at}{4-a^2}, \; a \neq 2$$

Notice how $y \to \infty$ as $a \to 2$



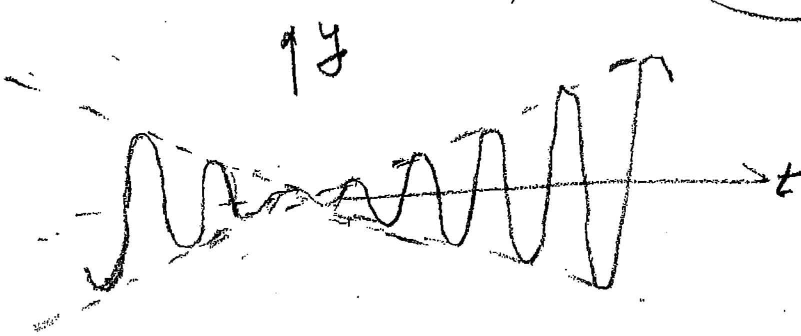
Resonant forcing $a = 2$

Try $y = At\sin 2t$

$$4A\cos 2t - \cancel{4At\sin 2t} + \cancel{4At\sin 2t} = \cos 2t$$

$$A = \frac{1}{4}$$

$$y = \frac{t\sin 2t}{4}; \quad a = 2$$

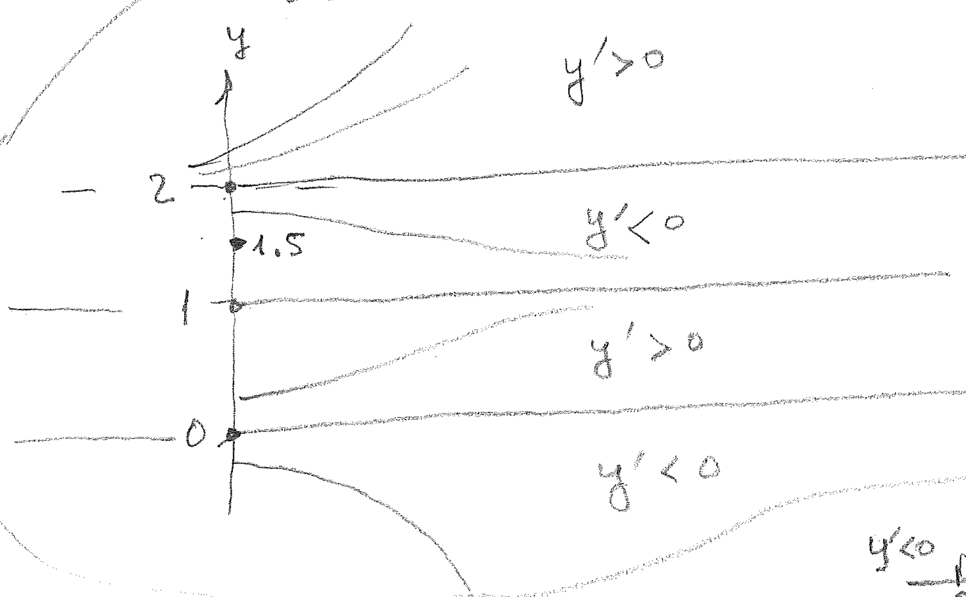


5. Consider the initial value problem

$$\frac{dy}{dt} = y(y^2 - 3y + 2), \qquad y(0) = 1.5$$

a) Find all equilibrium solutions and classify them as stable or unstable.

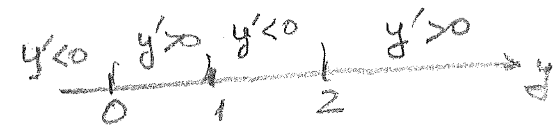
$$\frac{dy}{dt} = y(y-2)(y-1)$$



equil sol. $y = 2$ unstable

equil. sol $y = 1$ stable

equil. sol $y = 0$ unstable



b) The solution $y(t)$ approaches what value as $t \to +\infty$? Justify your answer.

$y(t) \longrightarrow 1$ as $t \to +\infty$

6. Use the Laplace transform to solve the initial value problem,

$$y'' + 4y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 1.$$

$\mathcal{L}y'' = s^2Y - 1, \quad \mathcal{L}y = Y, \quad \mathcal{L}\,\delta(t-1) = \int_0^\infty e^{-st}\delta(t-1)\,dt = e^{-s}.$

$s^2Y - 1 + 4Y = e^{-s}, \quad (s^2+4)Y = 1 + e^{-s}$

$Y = \frac{1}{s^2+4} + e^{-s}\frac{1}{s^2+4}$

$\mathcal{L}^{-1}\frac{1}{s^2+4} = \frac{\sin 2t}{2}$

$\mathcal{L}^{-1}e^{-s}\frac{1}{s^2+4} = u_1(t)\frac{\sin 2(t-1)}{2}$ acc to Table #13

Thus, $y = \frac{\sin 2t}{2} + u_1(t)\frac{\sin 2(t-1)}{2}$

The graph of the second term is