MATH 353.01-ORD & PRTL DIFF EQUATIONS-Fall 2021-EXAM 1

Name *Solutions* ______________________

Section ______________________

Thursday, September 23, 2021.

Open book and notes. Please sign "no assistance" pledge______________________

ALL ANSWERS SHOULD BE CIRCLED

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
TOTAL	

1. Check the boxes in the table, in which the ODE has the corresponding property. It is possible that more than one boxes have to be checked, or no boxes at all have to be checked for one ODE. Points will be taken out for wrong checks. You may have to do simple algebra to bring an ODE to one of the standard solavable forms. DO NOT SOLVE THE ODEs.

   Notation: y' is the derivative of y with respect to x.

ODE	lin. [H]	lin. [NH]	separable	exact	homogeneous	autonomous
$y' = xy + y + x + 1$		v	v	v		
$(x^2+1)y' = \cos x - y$		v				
$(x^2+1)y' + xy^2 + xy = 0$			v	v		
$(x+y)y' = x - y$				v	v	
$y' + \frac{2xy^2+y+1}{2x^2y+x+1} = 0$				v		
$y' - 2y^2 - 3 = 0$			v	v		v

- boxes not checked do not earn or lose points.
- checks earn 1 point if correct, lose a half-point if incorrect. Exceptions: (a) checks for the ODE being separable and exact as a result of it being homogeneous earn no points. (b) only two of the checks for ODEs being exact as a result of being separable earn a point.
- A total score, that is not an integer, is rounded of upwards to an integer.

2. (a) Find the general solution of the ODE

$$xy' + y = x^2.$$

Circle your answer.

The left hand side of the equation is an exact derivative, namely $(xy)'$.
The equation becomes $(xy)' = x^2$,

Integrate both sides $xy = \frac{x^3}{3} + c,$ divide by x,

Answer: $\boxed{y = \frac{x^2}{3} + \frac{c}{x}}$

(b) Find the general solution of the ODE

$$(2x + y + 1)dx + (2y + x)dy = 0,$$

Circle your answer.

Exactness test: $\frac{\partial(2x+y+1)}{\partial y} = 1,$ $\frac{\partial(2y+x+1)}{\partial x} = 1,$ the ODE is exact

The ODE is now $F_x dx + F_y dy = 0,$ where $F_x = 2x + y + 1,$ $F_y = 2y + x.$

$F_x dx + F_y dy = 0$ integrates to $dF = 0$, which means $F(x, y) =$ constant.

Use F_x and F_y to calculate F: $F(x, y) = \int F_x dx = \int (2x+y+1)dx = x^2+xy+x+a(y),$
Taking the y-derivative, $F_y = x + a'(y)$. But also $F_y = 2y + x$. Thus, $\frac{da}{dy} = 2y,$ $a = y^2,$ $F = x^2 + xy + x + y^2$ (no need for a constantof integration, we need just one F).

Answer: $\boxed{F = x^2 + xy + x + y^2 = \text{constant}}$

3. Use the method of undetermined coefficients to find a particular solution of the ODE

$$2y'' - y' - y = 6e^x, \quad y(0) = 1.$$

Circle your answer.

A first guess for a particular solution of the ODE (without considering the initial condition) is $y = Ae^x$. We find that it does not work, because $y = e^x$ satisfies the [H] equation. One then tries $y = Axe^x$.

$$y' = Axe^x + Ae^x, \qquad y'' = Axe^x + 2Ae^x.$$

Inserting into the ODE

$\cancel{2Axe^x} + 4Ae^x - \cancel{Axe^x} - Ae^x - \cancel{Axe^x} = 6e^x, \qquad A = 2; \qquad y = 2xe^x.$

The particular solution obtained does not satisfy the initial condition (I.C.) $\quad y(0) = 1.$

We satisfy the I.C. by adding a solution of the $[H]$ equation.

$y = -3xe^x + Be^x, \quad y(0) = B. \quad$ Thus, $B = 1$

$\boxed{\text{Answer: } y = 2xe^x + e^x}$

Remark: The answer is clearly not unique. The $[H]$ equation has the basis solutions e^x (that is used above to satisfy the I.C.) and $e^{-\frac{1}{2}x}$. The I.C. is satisfied by using appropriate linear combinations of the two solutions.

4. Given the ODE

$$\frac{dy}{dt} = -(y-1)(y-3)(y-5),$$

(a) Determine the equilibrium solutions and characterize their stability. Beware of the minus sine up front.

Circle your answer.

Thinking of $y(t)$ as the position of a particle at time t, we have that the velocity y' equals zero at the values $y = 1, \quad y = 3, \quad y = 5.$

$\boxed{\text{Answer: The equilibrium solutions are the constant solutions } y(t) = 1, \quad y(t) = 3, \quad y(t) = 5.}$

$$\begin{cases} \text{If} \quad 5 < y, & \text{then } y' < 0 \\ \text{If} \quad 3 < y < 5, & \text{then } y' > 0 \\ \text{If} \quad 1 < y < 3, & \text{then } y' < 0 \\ \text{If} \quad y < 1, & \text{then } y' > 0 \end{cases} \qquad \text{(Easier seen in a figure).}$$

Near points $y = 5$ and $y = 1$, the velocity is in the direction towards these points.

Near point $y = 3$ the velocity is in the direction away from this point.

$\boxed{\text{Answer: The solutions } y(t) = 1, \quad y(t) = 5 \text{ are stable; the solution } y(t) = 3 \text{ is unstable.}}$

(b) Consider the initial value problem of the ODE with initial condition $y(0) = 4$. What does the solution $y(t)$ do, as $t \to +\infty$. What does it do, as $t \to -\infty$.

Circle your answer.

$\boxed{\text{Answer: } y \to 5 \text{ as } t \to +\infty \text{ and to } y \to 3 \text{ as } t \to -\infty.}$

5. (a) Solve the initial value problem

$$y'' + a^2 y = \cos x, \qquad a > 0; \qquad y(0) = 0, \quad y'(0) = 0.$$

Circle your answer.

CASE 1 $\quad a \neq 1$ (nonresonant forcing)
Seeking a particular solution of the $[NH]$ equation for each value $0 < a \neq 1$. We try $y = Acosx$. Inserting into the ODE obtains

$$-A\cos x + a^2 A \cos x = \cos x; \qquad A = \frac{1}{a^2-1}, \qquad y_{\text{part}} = \frac{\cos x}{a^2-1}$$

$$\text{General solution :} \qquad y = c_1 \cos\, ax + c_2 \sin\, ax + \frac{\cos x}{a^2-1}$$

$$y(0) = c_1 + \frac{1}{a^2-1} = 0, \quad c_1 = -\frac{1}{a^2-1}; \quad y'(0) = ac_2 = 0; \quad \boxed{y = \frac{-\cos ax + \cos x}{a^2-1}}$$

CASE 2 $\quad a = 1$ (resonant forcing)
Particular solution:

$$y = Ax\cos x + Bx\sin x, \qquad y'' = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x.$$

Inserting into the ODE

$$-2A\sin x - Ax\cos x + 2B\cos x - \cancel{Bx\sin x} + Ax\cos x + \cancel{Bx\sin x} = \cos x$$

$$A = 0, \quad B = \frac{1}{2}; \qquad \boxed{y = \frac{x\sin x}{2}}, \quad \text{satisfies the initial conditions}$$

(b) Draw a rough graph of the maximum of the solution $|y(x)|$ over x, versus the positive values of a.

The maximum of $|y(x, a|$ over all values of x, obtained from the boxed solution of the nonresonant case is the function $\frac{2}{|a^2-1|}$ (why?). The function has a vertical asymptote at $a = 1$.

6. In each of the following true-false statements, circle true or false (points taken off for wrong guess). On the series questions, "explain" basically means stating an appropriate convergence test (points taken off for wrong guess).

   (a) A first order ODE that is exact is necessarily also separable. **True** **<u>False</u>**

   (b) The set of solutions of the ODE $y''' + 5xy'' - x^2y' + y = 0$ is a vector space of dimension 3. **<u>True</u>** **False**

   (c) All solutions $y(x)$ of the ODE $y'' + 4y = 3\sin 2x + 4$ are bounded, *i.e.*, there is a constant M, such that $|y(x)| < M$ for all x. **True** **<u>False</u>**

   (d) The general solution of the ODE $y' = ay, \quad a \neq 0$ is: $y = ce^{ax}$

   (e) The general solution of the ODE $y'' + a^2y = 0, \quad a \neq 0$ is: $y = c_1 \cos ax + c_2 \sin ax$.

   (f) The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ converges. **True** **<u>False</u>**
   Explain: Integral test.

   (g) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ converges. **<u>True</u>** **False**
   Explain: Integral test

   (h) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges. **<u>True</u>** **False**
   Explain: alternating series test

   (i) Express the decimal number 0.77777777..... as a series:
   $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots = 7\left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right)$
   Does the series converge? Yes, the series converges.
   Explain: Geometric series ratio $\frac{1}{10} < 1$.