MATH 353-ORD & PRTL DIFF EQUATIONS-FALL 2018-EXAM 1

Name Key

Section 1

Thursday, October 4, 2018.

Closed book and notes. Use of calculators and cellphones is not allowed.
Course formula sheet allowed.
You may use the back of the pages.

Please sign "no assistance" pledge____________________

ALL ANSWERS SHOULD BE CIRCLED

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
TOTAL	

1. Identify the type(s) of each ODE by checking the appropriate boxes. Leave the other boxes blank (points taken off for incorrect checks). DO NOT SOLVE THE ODEs.

   An ODE that is separable is automatically exact. Nevertheless, if you check the "separable" box, please leave the "exact" box unchecked.

   Notation: y' is the derivative of y with respect to x.

ODE	linear homog.	linear nonhom.	separable	exact
$y' = -\frac{3x^2-y}{3y^2-x}$				✓
$x(y^2+1)y' = y\tan x$			✓	
$xy' - x = (\sin x)y + \cos x$		✓		
$(x+1)y' = y(x + \ln x + 1)$	✓		✓	
$y' = y^2 + x$				

2. True or false? In each of the following statements, circle the appropriate word (points taken off for wrong guess):

   (a) The following operation is linear.

      i. $f(x) \quad \mapsto \quad f''(x) + 2x^2 f(x)$. **True** **False**

      ii. $f(x) \quad \mapsto \quad 1 + \int_0^x f(t)dt$. **True** **False**

      iii. $f(x) \quad \mapsto \quad \int_0^x f(t)dt + 2f(x)$. **True** **False**

      iv. $x \quad \mapsto \quad \ln x, \quad$ where $x > 0$. **True** **False**

      v. $x \quad \mapsto \quad |x|$. **True** **False**

   (b) Convergence of power series

      i. The power series $\sum_{n=0}^{\infty} n!x^n$ has zero radius of convergence. **True** **False**

      ii. The power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has infinite radius of convergence **True** **False**

   (c) If $y_1(x)$ and $y_2(x)$ are solutions of a linear nonhomogeneous ODE, their difference $y(x) = y_1(x) - y_2(x)$ is a solution of the corresponding homogeneous ODE.
   Hint. Think of the ODE as $\mathbb{L}y = f$ and check. **True** **False**

   (d) The ODE $y'' + y = \cos x$ has at least one bounded solution *i.e.* there is a positive M such that $|y(x)| < M$ for all x. **True** **False**

   (e) The general solution of the ODE $y''' + xy'' - 3y' - x^2y = 0$ has the form $y = c_1y_1(x) + c_2y_2(x) + c_3y_3(x)$, where $y_n(x), \quad n = 1, 2, 3$ are three linearly independent solutions of the ODE. **True** **False**

3. You are given the ODE

$$\frac{dy}{dx} = -\frac{12xy^2 - 3x^2 + y}{12x^2y - 2y + x}$$

(a) Find the general solution. It does not have to be in a form that is solved for y or for x. Circle your answer.

$$\underbrace{(12xy^2 - 3x^2 + y)}_{M \overset{?}{=} F_x} dx + \underbrace{(12x^2y - 2y + x)}_{N \overset{?}{=} F_y} = 0$$

$F_{xy} = F_{yx}$ implies the necessary condition:

$$M_y = N_x$$

$$M_y = 24xy + 1 \qquad N_x = 24xy + 1$$

$$F = \int (12xy^2 - 3x^2 + y)\, dx = 6x^2y^2 - x^3 + xy + A(y)$$

$$F_y = 12x^2y + x + A'(y).$$

Necessarily, $A' = -2y \qquad A = -y^2$

Thus, $\quad F = 6x^2y^2 - x^3 + xy - y^2$

ODE: $\quad F_x dx + F_y dy = 0 \quad , \quad dF = 0 \qquad F = \text{const.}$

general solution $6x^2y^2 - x^3 + xy - y^2 = c$
$c = \text{constant}$

(b) Find the solution that satisfies the initial condition $y(1) = 2$.

Insert $x = 1$, $y = 2$ in the general solution

$$6 \cdot 4 - 1 + 2 - 4 = c \qquad c = 21$$

Solution: $6x^2y^2 - x^3 + xy - y^2 = 1$

4. Consider the ODE

$$\frac{dy}{dx} = f(y).$$

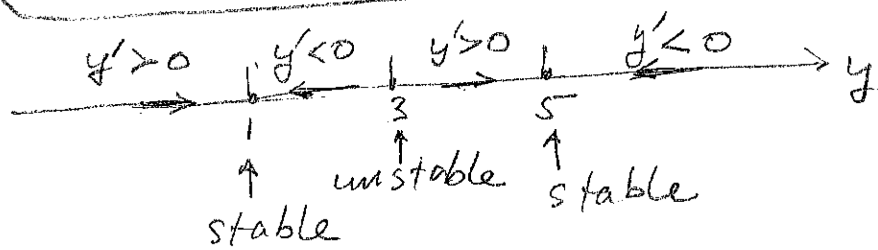
(a) Find the function $f(y)$, given that

i. f is a polynomial of y of the third degree and $f(0) = 30$,

ii. the ODE has three equilibrium points $y = 1$, $y = 3$, $y = 5$

iii. the points $y = 1$ and $y = 5$ are stable, the point $y = 3$ is unstable. Circle your answer.

(i) $f = -2(y-1)(y-3)(y-5)$



(b) For the above ODE, let $L = \lim_{x \to +\infty} y(x)$. In each of the following cases, fill in the blanks:

$y(0) = 0$, $L =$ 0

$y(0) = 2$, $L =$ 0

$y(0) = 4$, $L =$ 5

$y(0) = 7$, $L =$ 5

5. We are seeking the power series solution

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=-\infty}^{\infty} a_n x^n, \qquad a_n = 0 \text{ if } n < 0.$$

of the ODE

$$y'' - 2y' + x^2 y = 0.$$

that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

(a) Use the given initial values to calculate the coefficients a_0 and a_1.

$y = a_0 + a_1 x + a_2 x^2 + \cdots$

Circle your answer: $a_0 = 0$, $a_1 = 1$

$y' = a_1 + 2a_2 x + \cdots$

$y(0) = a_0 = 0, \quad y'(0) = a_1 = 1$

(b) Derive the equations that the coefficients a_n satisfy. Circle your answer. Write one short sentence justifying your derivation. Circle the sentence.

$y = \sum_{n=-\infty}^{\infty} a_n x^n, \quad a_n = 0 \text{ if } n<0$

$\sum_{n=-\infty}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=-\infty}^{\infty} 2n a_n x^{n-1} + \sum_{n=-\infty}^{\infty} a_n x^{n+2}$

shifts: $n \to n-1$ $\quad n \to n-4$

$\sum_{n=-\infty}^{\infty} \left(n(n-1)a_n - 2(n-1)a_{n-1} + a_{n-4}\right) x^{n-1} = 0$ for all x

Eq A. $n(n-1)a_n - 2(n-1)a_{n-1} + a_{n-4} = 0$ for all n

(c) Calculate the recurrence relation and point out the set of indices n for which it does not apply. Circle your answer.

Eq. B $\quad a_n = \dfrac{2a_{n-1}}{n} - \dfrac{a_{n-4}}{n(n-1)}, \quad n \neq 0, \; n \neq 1$

(d) What are the first three nonzero terms of the series solution? Circle your answer.

$a_0 = 0, \; a_1 = 1$ from above. IC

Eq. B: $a_2 = \dfrac{2a_1}{2} = 1$ $\qquad y = x + 4x^2 + \frac{2}{3}x^3 + \cdots$

$a_3 = \dfrac{2a_2}{3} = \dfrac{2}{3}$

(e) Without calculation. What is the radius of convergence of the series? Justify and circle your answer.

$R = \infty$, $p = -2$ $q = x^2$ have no singularities

(f) How would you construct a second linearly independent solution. Answer in a line and circle your answer. Let $a_0 = 1$ and $a_1 = 0$

Note

From Eq A: $n=0$ $\; 0a_0 = 0$ $\; a_0$ is free; $n=1$ $\; 0a_1 = 0$ $\; a_1$ is free

The free variables are given by the IC in this case

6. For each (no exception) positive value of the frequency w, calculate a particular solution of the following ODE that consists of only one term.

$$y'' + 4y = \cos wx$$

If A is the amplitude (maximum of the absolute value) of the solution you found, draw the graph of A versus the frequency w. *Hint.* A single solution formula is not sufficient for all frequencies.

Case 1 $\quad \omega^2 \neq 4 \quad$ Try $y = a \cos \omega x$

$$y'' = -a\omega^2 \cos \omega x$$

$$-a\omega^2 \cos \omega x + 4a \cos \omega x = \cos \omega x$$

$$a(4-\omega^2) = 1, \quad y = \frac{1}{4-\omega^2} \cos \omega x$$

$$A = \frac{1}{|4-\omega^2|}$$

Case 2: $\quad \omega^2 = 4 \quad$ Try $y = cx \sin 2x$

$$y'' = -4cx \sin 2x + 4c \cos 2x$$

ODE $\quad -4cx\cancel{\sin 2x} + 4c\cos 2x + 4cx\cancel{\sin 2x} = \cos 2x$

$$4c = 1$$

$$y = \frac{x}{4} \sin 2x$$