MATH 353.02-ORD & PRTL DIFF EQUATIONS-Fall 2022-EXAM 1

Name *Solutions*

Section ______________________

Monday, October 3, 2022.

Closed book, notes, internet, cell phones.
Allowed: Class formula sheet
Please sign "no assistance" pledge______________________

ALL ANSWERS SHOULD BE CIRCLED

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
TOTAL	

1. READ CAREFULLY: Check the boxes in the table, in which the ODE has the corresponding property. Points will be taken out for wrong checks.

   READ CAREFULLY: It is possible that more than one boxes should be checked for the same ODE. It is also possible that no boxes at all should be checked. You may have to do simple algebra to bring an ODE to one of the standard solavable forms. DO NOT SOLVE THE ODEs.

   Notation: y' is the derivative of y with respect to x.

ODE	lin. [H]	lin. [NH]	separable	exact	homogeneous	autonomous
$y' = xy + y + x$		✓				
$y' + 3y^2 - 5 = 0$			✓	✓		✓
$(x^2+2)y' + xy^3 + xy = 0$			✓	✓		
$(x+2y)y' = 5x - y$			✓	✓	✓	
$y' + \frac{2xy^2+y}{2x^2y+x} = 0$	✓		✓	✓	✓	
$(x^2+1)y' = 3x + \sin y$						

Note:

- An *autonomous* ODE is necessarily *separable*.
- A *linear [H]* ODE is necessarily *separable*.
- A *homogeneous* ODE reduces to *separable*.
- A *separable* ODE is or reduces to *exact*.

2. (a) Solve the initial value problem of the ODE

$$y' = 2y^2 + xy^2, \quad y(0) = -1$$

and determine the subset of the x axis over which the solution exists as a continuous function. Circle your answers.

$$\frac{dy}{y^2} = (2+x)dx, \qquad -\frac{1}{y} = \frac{x^2}{2} + 2x + c,$$

Inserting the initial condition obtains: $c = 1$. Thus,

$$y = -\frac{2}{x^2 + 4x + 2}$$

The denominator is zero at the two roots, $-2 - \sqrt{2} < -2 + \sqrt{2}$, both negative. The solution of the initial value problem is initiated at $x = 0$. From there, the solution proceeds continuously to the left up to the larger root $x = -2 + \sqrt{2}$, where continuity is lost; it proceeds from $x = 0$ to the right, all the way.

$$\text{Answer:} \quad -2 + \sqrt{2} < x < \infty.$$

(b) Find the general solution of the ODE

$$(3y + 1)dx + (3x + 6y)dy = 0,$$

The ODE is exact, it can thus be written as $F_x dx + F_y dy = 0$, where $F_x = 3y + 1$ and $F_y = 3x + 6y$. Then,

$$F = \int F_x dx = \int (3y + 1)dx = 3xy + x + A(y).$$

$$F_y = 3x + A'(y)$$

Comparing this with the second parenthesis of the ODE obtains

$$A' = 6y, \qquad \text{thus} \quad A = 3y^2, \qquad F = 3xy + x + 3y^2.$$

$$\text{Answer:} \qquad 3xy + x + 3y^2 = \text{constant}.$$

3. Use the method of undetermined coefficients to find the solution of the ODE initial value problem (only one initial condition is given).

$$y'' + y = \cos x, \quad y(0) = 0.$$

The forcing is resonant since $y = \cos x$ is a solution of [H]. Thus, try

$$y = ax \sin x + bx \cos x$$

and insert into the ODE.

$$-\cancel{ax \sin x} + 2a \cos x - \cancel{bx \cos x} - 2b \sin x + \cancel{ax \sin x} + \cancel{bx \cos x} = \cos x$$

Necessarily, $b = 0, \quad a = \frac{1}{2}$, thus,

$$\text{particular solution:} \qquad y = \frac{x \sin x}{2}.$$

Need to add a solution of [H], such that the initial condition os satisfied.

$$\text{Answer:} \qquad y = \frac{x \sin x}{2} + c \sin x.$$

Make a rough graph of a solution. The x positions at which $y(x) = 0$ should be placed correctly in the graph.

4. Given the ODE

$$\frac{dy}{dt} = \frac{(y-1)(y-3)}{y^2+1},$$

(a) Determine the equilibrium solutions and characterize their stability.

$$\text{Equilibrium solutions:}\quad y = 1 \text{ (stable)}, \qquad y = 3 \text{ (unstable)}$$

(b) Consider the initial value problem of the above ODE with initial condition $y(0) = 2$. What does the solution $y(t)$ do, as $t \to +\infty$. What does it do, as $t \to -\infty$.

$$\lim_{t\to\infty} y(t) = 1; \qquad \lim_{t\to-\infty} y(t) = 3$$

(c) Given $y(0) = 4$, what are the limits of y and $\frac{dy}{dt}$ as t tends to $+\infty$?

$$\lim_{t\to\infty} y(t) = +\infty; \qquad \lim_{t\to\infty} \frac{dy}{dt} = 1;$$

Notice that the fraction $\frac{(y-1)(y-3)}{y^2+1}$ tends to 1, when y tends to infinity.

5. a) Given the ODE (known as the Airy equation)

$$y'' - xy = 0$$

calculate the recurrence relation for the coefficients of power series solutions of the ODE centered at $x = 0$.

$$y = \sum_{n=-\infty}^{\infty} a_n x^n, \qquad a_n = 0 \text{ when } n < 0.$$

$$\sum_{n=-\infty}^{\infty} a_n n(n-1)x^{n-2} - \underbrace{\sum_{n=-\infty}^{\infty} a_n x^{n+1}}_{\text{shift } n \mapsto n-3}.$$

$$\sum_{n=-\infty}^{\infty} [n(n-1)a_n - a_{n-3}] = 0$$

$$n(n-1)a_n - a_{n-3} = 0, \qquad \text{for all } n.$$

- True when $n < 0$ ($a_n = 0$)
- True when $n = 0, 1$ (a_0 and a_1 are free variables.)
- Recurrence relation:

$$a_n = \frac{a_{n-3}}{n(n-1)} \qquad n \geq 2$$

b) Calculate the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

The radius of convergence is infinite (ratio test: $\lim_{n\to\infty} \frac{|x|}{n+1} = 0$, for any fixed x).

6. Use the Laplace transform to solve the initial value problem

$$y' + y = 1 - u_1(x); \qquad y(0) = 0.$$

$$\mathcal{L}(y'+y) = \mathcal{L}(1-u_1); \qquad \mathcal{L}y' + \mathcal{L}y = \mathcal{L}1 - \mathcal{L}u_1; \qquad sY + Y = \frac{1}{s} - \frac{e^{-s}}{s};$$

$$Y = \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}, \qquad \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$Y = \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+1}$$

$$y = 1 - e^{-t} - u_1(t) + u_1(t)e^{-(t-1)} = 1 - e^{-t} - u_1(t)(1 - e^{-t+1})$$

$$\begin{cases} y = 1 - e^{-t}, & t < 1 \\ y = (e-1)e^{-t}, & t > 1 \end{cases}$$