MATH 353-ORD & PRTL DIFF EQUATIONS-Spring 2018-EXAM 1

Name KEY

Section 1 (Venakides)

Thursday, February 22, 2018.

Closed book and notes. Use of calculators and cellphones is not allowed.
Formula sheet in sakai general site allowed.
You may use the back of the pages.

Please sign "no assistance" pledge____________________

ALL ANSWERS SHOULD BE CIRCLED

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
TOTAL	

1. Identify the type(s) of each ODE by checking the appropriate boxes. Leave the other boxes blank (points taken off for incorrect checks). DO NOT SOLVE THE ODEs.

   An ODE that is separable is automatically exact. Nevertheless, if you check the "separable" box, please leave the "exact" box unchecked.

   Notation: y' is the derivative of y with respect to x.

ODE	linear homog.	linear nonhom.	separable	exact
$xy' = x(1+y^2)e^{-x}$			✓	
$y' = xy + e^x$		✓		
$(y^3 - x^2y)y' = xy^2 + 1$				✓
$y' = y(\tan^2 x + 1)$	✓		✓	
$xy' - \sin x = y + 1$		✓		

2. True or false? In each of the following statements, circle the appropriate word (points taken off for wrong guess):

(a) The following operation is linear.

i. $f(x) \mapsto f'(x)$. **True** **False**

ii. $f(x) \mapsto \int_0^x f(t)dt$. **True** **False**

iii. $f(x) \mapsto \dfrac{\int_0^x f(t)dt}{1+\int_0^x f(t)dt}$. **True** **False**

iv. $x \mapsto \sqrt{x}$, where $x > 0$. **True** **False**

v. $x \mapsto 3|x|$. **True** **False**

(b) If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges when $z = 3.1 + 4i$, where $i = \sqrt{-1}$, it necessarily converges when $z = 5i$. **True** **False**

(c) The following ODE has at least one solution that tends to plus or minus infinity, as the independent variable tends to $+\infty$. The independent variable is x.

i. $y' = y + 1$ **True** **False**

ii. $y' = -y + 1$ **True** **False**

iii. $y'' + 2y = \cos x$ **True** **False**

iv. $y'' + y = \sin x$ **True** **False**

3. Calculate the function $y(x)$ that solves the initial value problem

$$y'' + 2(xy)' = 0, \quad y(0) = 0,\ y'(0) = 1$$

HINT: This is really a first (not second) order ODE.

$y' + 2xy = c$ from initial condition

Insert $x = 0$: $1 + 0 = c$ $c = 1$

Thus $y' + 2xy = 1$

This a linear homogeneous ODE

Integrating factor. e^{x^2}

$(e^{x^2} y)' = e^{x^2}$

$e^{x^2} y = \int_0^x e^{s^2} ds + K$ K = constant

Insert $x = 0$ $y(0) = K = 0$ from initial condition

$$y = e^{-x^2} \int_0^x e^{s^2} ds$$

4. Given the ODE

$$y' = (y-1)(y-3)(y-5)(y-7)$$

where $y = y(x)$.

(a) Write down all the equilibrium solutions of the ODE, *i.e.*the solutions that are independent of the variable x (circle your answer).

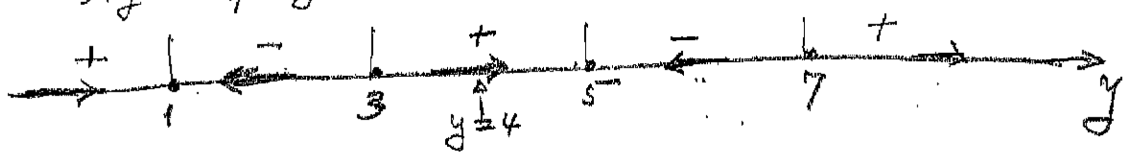
(b) Characterize all the equilibrium points as stable or unstable (circle your answer).

(c) Given the initial condition $y(0) = 4$, evaluate the limits

$$\lim_{x\to-\infty} y(x) = 3 \quad , \quad \lim_{x\to+\infty} y(x) = 5$$

(d) What are the initial conditions $y(0)$ for which the solution $y(x)$ tends to infinity? (circle your answer)

sign of y'



(a) $y(x)=1$, $y(x)=3$, $y(x)=5$, $y(x)=7$

(b) $y=1$ and $y=5$ are stable
$y=3$ and $y=7$ are unstable

(c) see above

(d) $y(0)>7$

5. We are seeking the power series solution

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

of the ODE

$$y'' + 2xy' + 2y = 0.$$

that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

(a) Use the given initial values to calculate the coefficients a_0 and a_1.

Circle your answer: $a_0 = 0$, $a_1 = 1$

$\left.\begin{array}{l} y(0) = a_0 = 0 \\ y'(0) = a_1 = 1 \end{array}\right\}$ from series

(b) Recommended: $y = \sum_{-\infty}^{\infty} a_n x^n$, $a_n = 0$ when $n < 0$.

(c) Derive the infinite set of equations that the coefficients a_n satisfy. Circle your answer.

$\left\{\begin{array}{l} y = \sum_{n=-\infty}^{\infty} a_n x^n, \\ a_n = 0 \text{ if } n < 0 \end{array}\right.$ $y' = \sum_{n=-\infty}^{\infty} n a_n x^{n-1}$, $2xy' = \sum_{n=-\infty}^{\infty} 2n a_n x^n$

Insert into ODE $\qquad y'' = \sum_{n=-\infty}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=-\infty}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=-\infty}^{\infty} 2n a_n x^n + \sum_{n=-\infty}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=-\infty}^{\infty} n(n-1) a_n x^{n-2} + \underbrace{\sum_{n=-\infty}^{\infty} 2 a_n (n+1) x^n}_{n \mapsto n-2} = 0 \longrightarrow \text{becomes } \sum_{n=-\infty}^{\infty} 2 a_{n-2} (n-1) x^{n-2}$$

$$\sum_{n=-\infty}^{\infty} \left(n(n-1) a_n + 2(n-1) a_{n-2}\right) x^{n-2} = 0 \quad \forall n$$

Equations: $n(n-1) a_n + 2(n-1) a_{n-2} = 0$

(d) Calculate the recurrence relation and the set of indices n for which it applies. Circle your answer. for all n

Recurrence relation $a_n = \dfrac{-2 a_{n-2}}{n}$, $n \neq 0, 1$

(e) Calculate the first three nonzero terms of the series. Circle your answer.

$a_0 = 0$ & recurrence relation $\Rightarrow a_n = 0$ when n is even

$a_1 = 1 \quad a_3 = -\dfrac{2a_1}{3} = -\dfrac{2}{3}$, $a_5 = -\dfrac{2\left(-\frac{2}{3}\right)}{5} = \dfrac{4}{15}$

$a_1 = 1$, $a_3 = -\dfrac{2}{3}$, $a_5 = \dfrac{4}{15}$ $\qquad y = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots$

(f) What is the radius of convergence of the series? Justify and circle your answer.

Ratio test counting only the nonzero terms:

For n odd: $\dfrac{a_n}{a_{n-2}} = -\dfrac{2}{n} \longrightarrow 0$ $\quad\therefore$ Radius of convergence $R = \infty$

6. Find a particular solution of the ODE

$$y'' + y = \sin x$$

The forcing is resonant ($y = \sin x$ satisfies the homogeneous equation $y'' + y = 0$

Undetermined coefficients:

$y = Ax\cos x + Bx\sin x$ Insert into ODE

$$-\cancel{Ax\cos x} - 2A\sin x - \cancel{Bx\sin x} + 2B\cos x$$
$$+\cancel{Ax\cos x} + \cancel{Bx\sin x} = \sin x$$

$$-2A\sin x + B\cos x = \sin x \quad \forall x$$

Balancing coefficients:

$B = 0, \ 2A = -1 \qquad B = 0, \ A = -\frac{1}{2}$

Answer $y = -\frac{1}{2}x\sin x$