1. For each ODE, place a check in one or more boxes, where the ODE has the indicated property. Leave the other boxes blank (points taken off for incorrect checks). Remember that a separable ODE is necessarily exact. Also a homogeneous ODE (not the [NH] type) can be made separable. DO NOT SOLVE THE ODEs.

   Notation: y' is the derivative of y with respect to x.

ODE	linear homog.	linear nonhom.	separable	exact
$(x+y)y' = x - y$			✓	✓
$y' = xy^2 + 1$				
$xy' - (\sin x)y - x\cos x = 0$		✓		
$(x^2 + \ln x)y' = x^2(y^2+1)$			✓	✓
$y' + \frac{y\cos x - xy\sin x - 2x\cos y}{x\cos x + x^2\sin y} = 0$				✓
$(\sin x)y' = y\cos x$	✓		✓	✓

ODE

$$y' = \frac{x-y}{x+y}$$

Let $y = ux$

$y' = u'x + u$

$$\frac{x-y}{x+y} = \frac{1-u}{1+u}$$

$$u'x + u = \frac{1-u}{1+u}$$

separable

2. True or false? In each of the following statements, circle the appropriate word (points taken off for wrong guesses):

(a) Consider the ODE

$$y' = (y-1)(y-2)(y-3)(y-5)$$

i. The ODE has exactly two stable equilibrium solutions. True False

ii. If $y(0) = 6$, then $\lim_{x\to+\infty} y(x) = +\infty$. True False

iii. The limit $\lim_{x\to+\infty} y(x) = -\infty$ does not occur for any initial data $y(0)$. True False

(b) The following operation is linear.

i. The Lapalce transform. True False

ii. Taking the square root of a positive function. True False

iii. Taking the second derivative of a function True False

(c) The initial value problem $y' = y^{\frac{1}{3}}, \quad y(0) = 0$ has a unnique solution. True False

(d) If one solution of a second order linear homogeneous ODE is known, a formula for a second,linearly independent solution of the ODE can be derived. True False

(e) If the complex power series $\sum_{n=0}^{\infty} a_n z^n$ converges when $z = 3 + 4i$, where $i = \sqrt{-1}$, it necessarily converges when $z = 5i$. True False

(f) The set of solutions of a linear homogeneous third order ODE is a three dimensional vector space True False

(g) If the coefficient functions of a linear ODE and the forcing term are bounded, then all its solutions are also bounded. True False

(h) The general solution of a linear nonhomogeneous ODE equals the sum of the general solution of the homogeneous ODE plus a particular solution. True False

3. (a) Use the methodof undetermined coefficients to derive the solution of the ODE

$$y'' + 4y = \sin(2t)$$

with zero initial conditions, $y(0) = 0, \quad y'(0) = 0$. Circle your answer.

The forcing is resonant

Let $y = At\cos(2t)$

Insert into the ODE to obtain

$A = -\frac{1}{2}$

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{1}{4}t\cos(2t)$$

$y(0) = 0 \Rightarrow c_1 = 0 \quad y = c_2\sin(2t) - \frac{1}{4}t\cos(2t)$

$y' = 2c_2\cos(2t) - \frac{1}{4}\cos 2t + \frac{1}{2}\sin(2t)$

$y'(0) = 0 \Rightarrow 2c_2 - \frac{1}{4} = 0, \quad c_2 = \frac{1}{8}, \quad y = \frac{1}{8}\sin(2t) - \frac{1}{4}t\cos(2t)$

(b) Without using series, find a basis of the space of solutions of the ODE

$$x^2y'' - 2y = 0.$$

HINT. Look for solutions of the form $y = x^n$, where n is an integer.
Circle your answer.

$y = x^n \quad y' = nx^{n-1} \quad y'' = n(n-1)x^{n-2}$

Insert into the ODE

$n(n-1)x^n - 2x^n = 0$

$n(n-1) - 2 = 0 \quad n^2 - n - 2 = 0$

$n_1 = 2 \quad n_2 = -1$

Basis: $y_1 = x^2, \quad y_2 = x^{-1}$

4. Find the general solution of the ODE

$$\underbrace{(2x + 3x^2 + 6xy)}_{M}dx + \underbrace{(3x^2 + 1)}_{N}dy = 0$$

Circle your answer.

$M_y = 6x \qquad N_x = 6x \qquad$ exact.

$F_x = M \qquad F_y = N$

$F = \int (2x + 3x^2 + 6xy)\,dx$ (y is treated as a constant)

$F = x^2 + x^3 + 3x^2y + A(y)$

$F_y = 3x^2 + A'(y)$, compare with $N = 3x^2 + 1$

$\cancel{3x^2} + A'(y) = \cancel{3x^2} + 1 \qquad A'(y) = 1 \qquad A = y$

Solution $x^2 + x^3 + 3x^2y + y = c$

$$y = \frac{c - x^2 - x^3}{3x^2 + 1}$$ c : constant

5. (a) Use the Laplace transform to solve the following initial value problem.

$$\begin{cases} y'' + 4y = \delta(t-1) \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

where $\delta(t)$ is the Dirac delta function. Graph your result. Circle your answer.

$$\mathcal{L}(y''+4y) = \mathcal{L}(\delta(t-1))$$

$$s^2 Y + 4Y = e^{-s} \qquad Y = \frac{e^{-s}}{s^2+2^2},$$

$$Y = \frac{1}{2}\,\frac{2}{s^2+2^2}\cdot e^{-s}$$

From table 6.2.1 # 5 and 13

$$y = \frac{1}{2} u_1(t)\sin(2(t-1))$$

(b) Solve the same problem without using the Laplace transform.
HINT. y'' displays a delta function at $t = 1$. What does this mean for y'?

By the existence / uniqueness theorem,

$y(t) = 0$ when $t < 1$

At $t=1$ y' has a unit jump

Thus, we have for $t > 1$ the IVP

$y'' + 4y = 0 \quad y(1) = 0 \quad y'(1) = 1$

Solution for $t > 1$: $y = \frac{1}{2}\sin(2t-2)$

6. Consider the ODE

$$y'' + y' + xy = 0.$$

We are seeking a power series solution centered at $x = 0$,

$$y = \sum_{-\infty}^{\infty} a_n x^n, \qquad a_n = 0 \text{ when } n < 0.$$

(a) Identify the free variables and find the recurrence relation. Circle your answer.

$$\sum n(n-1)a_n x^{n-2} + \underbrace{\sum n a_n x^{n-1}}_{n \mapsto n-1} + \underbrace{\sum a_n x^{n+1}}_{n \to n-3} = 0$$

$$\sum_{n=-\infty}^{\infty} \left(n(n-1)a_n + (n-1)a_{n-1} + a_{n-3}\right) x^{n-2} = 0$$

Necessarily (why?) $n(n-1)a_n + (n-1)a_{n-1} + a_{n-3} = 0 \quad \forall n$

Insert

$n < 0$: $0 = 0$ (true)

$n = 0$: $0 a_0 = 0$ a_0 is free

$n = 1$: $0 a_1 = 0$ a_1 is free

a_0, a_1 free variables

$n > 1$: $a_n = \dfrac{-(n-1)a_{n-1} - a_{n-3}}{n(n-1)}$ recurrence relation

(b) Determine the first four nonzero coefficients of the solution that satisfies $y(0) = 1$ and $y'(0) = 2$. Circle your answer.

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad x = 0 \implies y(0) = a_0 = 1$

$y' = a_1 + 2a_2 x + \cdots \qquad x = 0 \implies y'(0) = a_1 = 2$

Recurrence relation:

$$a_2 = -\frac{a_1 - 0}{2} = -1$$

$$a_3 = \frac{-2a_2 - a_0}{6} = \frac{2-1}{6} = \frac{1}{6}$$

$$y = 1 + 2x - x^2 + \frac{x^3}{6} + \cdots$$