ORD & PRTL DIFF EQUATIONS-MATH 353-FALL 2020-EXAM 1

Name Solutions

Section 1

Thursday, September 24, 12:00-1:15 pm.

Please sign the "no assistance" pledge:

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	

NOTICE: y' stands for dy/dx.

1. Check all appropriate boxes. Leave the other boxes blank (points taken off for incorrect checks). DO NOT SOLVE THE ODEs.

   Hint: An ODE that is separable is automatically exact. An ODE that is homogeneous is automatically separable.

ODE	linear [H]	linear [NH]	separable	exact	homogeneous
$y' \sin x + x \cos x = -y \cos x$		✓		✓	
$y' = \frac{3x^2y^2+y^2 \sin x}{-2x^3y+2y \cos x}$	✓		✓*	✓	
$y' = \frac{x^2-y^2}{x^2+y^2}$			✓*	✓**	✓
$y' = \frac{(x^2+1)\cos y}{y^2+1}$			✓	✓*	
$y' + y^2 \cos x = \sin x$					

✓* secondary characterization

✓** tertiary -"-

✓ 2pts if correct, −1 pt if incorrect

✓*, ✓** collectively 2 pts, 1pt or no pts

2. Find the general solution of the following differential equations. If you cannot solve for x or y it is OK to leave your answer in implicit form. CIRCLE YOUR ANSWERS.

(a) $xy' = y + 2x^3, \qquad x > 0.$

$y' - \frac{1}{x} y = 2x^2$

Integrating factor: $e^{-\ln x} = \frac{1}{x}$

$\frac{y'}{x} - \frac{y}{x^2} = 2x$, $\left(\frac{y}{x}\right)' = 2x$, $\frac{y}{x} = x^2 + c$

$y = x^3 + cx$

(b) $y''' + y'' - 2y' = 1$

Integrate both sides

$y'' + y' - 2y = x + \text{const.}$

[H]: Let $y = e^{\lambda x}$

[H]: Gen sol. $y = c_1 e^{x} + c_2 e^{-2x}$

Particular solution: $y_{part} = Ax + B$

$A = -\frac{1}{2}$ $B =$ free

$y = -\frac{1}{2}x + B + c_1 e^{x} + c_2 e^{-2x}$

Alternatively: [H]: $\lambda^3 + \lambda^2 - 2\lambda = 0$, $\lambda(\lambda^2 + \lambda - 2) = 0$ etc

3. (a) What does it mean for an operator $\mathbb{L}$ to be linear?

$\mathbb{L}$ operates distributively, that is

$$\mathbb{L}(c_1 f_1 + c_2 f_2) = c_1 \mathbb{L} f_1 + c_2 \mathbb{L} f_2$$

for arbitrary inputs f_1, f_2 and arbitrary constants c_1, c_2

(b) Prove that the second order differential operator $\mathbb{L} = \frac{d^2}{dx^2} + q(x)$, where $q(x)$ is a function, is linear.

$$\left(\frac{d^2}{dx^2} + q\right)(c_1 f_1 + c_2 f_2)$$

$$= c_1 f_1'' + c_2 f_2'' + c_1 q f_1 + c_2 q f_2$$

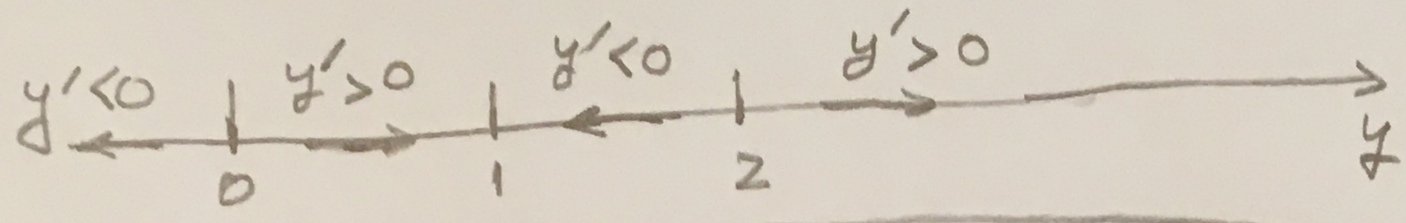
$$= c_1 (f_1'' + q f_1) + c_2 (f_2'' + q f_2)$$

$$= c_1 \mathbb{L} f_1 + c_2 \mathbb{L} f_2$$

4. (a) Consider the initial value problem

$$\frac{dy}{dx} = y(y^2 - 3y + 2), \qquad y(0) = 1.5 \qquad y' = y(y-1)(y-2)$$

The solution $y(x)$ approaches what value as $x \to +\infty$? Justify your answer.



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$y(x) \to 1$ as $x \to +\infty$

For what initial values of y does the solution display a vertical asymptote? CIRCLE YOUR ANSWER

$y<0$ and $y>2$. If no vertical asymptote, as x grows, y will reach values at which $y' > y^2$. We know however that $y' = y^2$ has a solution that blows-up in finite x.

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(b) The solution of a first order ODE is given by

$$x^3y + y^2x^2 - 4x^5 = c$$

where c is an arbitrary constant. What is the ODE? CIRCLE YOUR ANSWER.

We have $F(x,y) = c$

thus $dF = 0$, $\underbrace{F_x\,dx + F_y\,dy = 0}_{\text{ODE in differential form}}$

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or $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ (standard form)

Thus $\dfrac{dy}{dx} = -\dfrac{3x^2y + 2xy^2 - 20x^4}{x^3 + 2yx^2}$

5. (a) Find the value of b for which the ODE

$$\frac{dy}{dx} = -\frac{xy^2 + bx^2y}{x^3 + x^2y + y^3}$$

is exact. CIRCLE YOUR ANSWER. cross multiply

$$\underbrace{(xy^2 + bx^2y)}_{M = F_x}dx + \underbrace{(x^3 + x^2y + y^3)}_{N = F_y}dy = 0$$

$M_y = 2xy + 6x^2 \qquad N_x = 3x^2 + 2xy \qquad b = 3$

(b) Calculate the general solution of the ODE for this value of b. CIRCLE YOUR ANSWER.

$$F = \int xy^2 + 3x^2y \, dx = \frac{x^2y^2}{2} + x^3y + A(y)$$

$$F_y = x^2y + x^3 + A' = N = x^3 + x^2y + y^3$$

$$A' = y^3 \implies A = \frac{y^4}{4}$$

solution $\quad \frac{x^2y^2}{2} + x^3y + \frac{y^4}{4} = c$

6. Find the general solution of the ODE

$$y'' - 4y = e^{-x}$$

CIRCLE YOUR ANSWER.

[H]: $y = c_1 e^{2x} + c_2 e^{-2x}$

particular solution:

We notice that e^{-x} is not a solution of [H].

So seek $y = Ae^{-x}$

$$Ae^{-x} - 4Ae^{-x} = e^{-x}$$

$$-3A = 1 \qquad A = -\frac{1}{3}$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^{-x}$$