MATH 353-753 ORD & PRTL DIFF EQUATIONS-SPRING 2023   EXAM 1

Instructor: *Venakides* *Date* *February* 20, 2023.

Student Name: *Solutions*

Closed book, notes, internet, cell phones.
Allowed: Class formula sheet

Please sign "no assistance" pledge______________________________

ALL ANSWERS SHOULD BE CIRCLED

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
TOTAL	

1. READ CAREFULLY: Check the boxes in the table, in which the ODE has the corresponding property. Points will be taken out for wrong checks. Leave the other boxes blank.

   READ CAREFULLY: It is possible that more than one boxes should be checked for the same ODE. It is also possible that no boxes at all should be checked. You may have to do simple algebra to bring an ODE to one of the standard solavable forms. DO NOT SOLVE THE ODEs.

   REMEMBER: A separable equation is necessarily also exact.

   Notation: y' is the derivative of y with respect to x.

ODE	lin. [H]	lin. [NH]	separable	exact	autonomous
$y' = xy + x$		✓	✓	✓	
$y' + y\cos y = 0$			✓	✓	✓
$(x^2 + y^2)y' = x$					
$xy' = 5x^2 - y$		✓		✓	
$y' + 2x^2y = x^2$		✓	✓	✓	
$(2xy + y^2)dx + (2xy + x^2)dy = 0$				✓	

2. Find the solution of the ODE

$$y' = -2xy + 2x,$$

in two ways.

(a) Using an integrating factor. Circle your answer.

Integrating factor: $e^{\int 2x dx} = e^{x^2}$.

$$y'e^{x^2} + 2xe^{x^2}y = 2xe^{x^2}, \qquad \left(ye^{x^2}\right)' = 2xe^{x^2}, \qquad ye^{x^2} = e^{x^2} + c$$

$$Answer: \quad y = 1 + ce^{-x^2},$$

where c is a constant.

(b) Using the general solution of the homogeneous equation and an easy to find particular solution of the nonhomogeneous equation. Circle your answer.

General solution of the homogeneous ([H]) equation (separable).

$$y_{[H]} = ce^{-x^2}$$

Particular solution of the nonhomogeneous ([NH]) equation.

$$y_{[NH]} = 1$$

$$Answer: \quad y = 1 + ce^{-x^2}$$

3. (a) Find the equilibrium solutions of the ODE

$$y' = y^2 - 4y + 3,$$

and determine whether they are stable or unstable.

Circle your answer.

$$Answer: y = 1, \text{ stable}, \quad y = 3, \text{ unstable}$$

(b) Determine the limit of the solution $y(x)$ as $x \to +\infty$, given the initial value $y(0) = 2$. Circle your answer.

$$Answer: \quad \lim_{x \to \infty} y = 1,$$

(c) Consider the solution of the ODE with initial value $y(0) = 4$. Name two different scenarios for this solution, as x grows from the value zero.

Answer:

1) Scenario that does **not** happen in this case: $x \to \infty$, $y \to \infty$.

2) Scenario that actually **does** happen. The solution $y(x)$ has a vertical asymptote at some point $x = x_1 > 4$. Thus, $x \to x_1$ and $y \to \infty$; in particular, $y \to +\infty$, since $y' > 0$.

Please let me know, if you can prove that scenario 2 is the correct one.

4. You are given the ODE

$$y'' + 2py' + y = 0, \quad y = y(x).$$

a)Assuming that $0 < p < 1$, calculate the distance between two successive points of the x-axis, at which a solution $y(x)$ equals zero.

b) Assuming that $0 < p < 1$, make a rough graph of a nonzero solution of the ODE. Circle your answer

c) Calculate the smallest value of $p \in \mathbb{R}$ for which the solutions of the ODE exhibit no oscillations.

a) SOLUTION

Looking for exponential solutions $y = e^{rx}$, where r is a constant, we obtain the characteristic polynomial

$$r^2 + 2pr + 1 = 0,$$

which has roots $-p + ia$ and $-p - ia$, where $a = \sqrt{1 - p^2}$.

The general solution of the ODE is

$$y = c_1 e^{-p} \cos ax + c_2 e^{-p} \sin ax.$$

It can also be written as

$$y = ce^{-p} \cos(ax + b),$$

where c and b arbitrary constants. The distance between successive zeroes of the cosine and the sine functions is π. Thus, the distance d between two successive zeroes of $y(x)$ is given by

$$ad = \pi, \quad \mathit{Answer}: \quad d = \frac{\pi}{a}.$$

b) SOLUTION

The graph should exhibit an oscillatory profile $y(x)$ with decaying amplitude and with equally spaced zeroes.

c) SOLUTION

The solutions exhibit oscillations, when the roots of the characteristic polynomial have nonzero imaginary parts, leading to the emergence of sines and cosines in the solution. In our case, this occurs when $-1 < p < 1$. The smallest *positive* value of p for which there are no oscillations is clearly $p = 1$. There are also no oscillations when $p \leq -1$. Clearly, there is no smallest value of p in this region.

5. Use the Laplace transform to solve the initial value problem

$$y' + y = 1 - u_1(t); \qquad y(0) = 0.$$

Circle your answer.

SOLUTION

Apply the Laplace Transform $\mathcal{L}$ to both sides of the equation. Use the table to invert the transform.

$$sY + Y = \frac{1}{s} - \frac{e^{-s}}{s}, \qquad Y = \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}.$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right) = 1 - e^{-t} \qquad (\#\ 1,\ 2 \text{ in table})$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s+1)}\right) = u_1(t)(1 - e^{-t+1}) \qquad (\#\ 13 \text{ in table})$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}\right) = 1 - e^{-t} - u_1(t)(1 - e^{-t+1})$$

$$\boxed{\textit{Answer}: \quad y = \begin{cases} 1 - e^{-t}, & 0 \le t < 1, \\ e^{-t+1} - e^{-t}, & t \ge 1 \end{cases}}$$

Notice that $y(t)$ is continuous at $t = 1$. This should be expected. A jump discontinuity of $y(t)$ at $t = 1$ would produce a delta function in the derivative y'. This delta function would have to be balanced in the ODE by another delta function, which is clearly not there.

6. Use the method of undetermined coefficients to find the general solution of the ODE

$$y'' + y = \sin 2t + e^{-t}$$

Circle your answer.

Hint: Linearity allows you to solve the problem separately for each of the terms on the right.

SOLUTION. The general solution of the [H] equation is

$$y_{[H]} = c_1 \cos 2t + c_2 \sin 2t$$

Following the hint, we find particular solutions for each forcing term separately.

a) $\sin 2t$: The forcing is not resonant. The natural choice would be a linear combination of a sine and a cosine. We notice though that if we use only a sine, no cosines would be generated. So we take

$$y = a \sin 2t.$$

Inserting this in the ODE obtains after the cancellation of the sines

$$-4a + a = 1, \qquad a = -\frac{1}{3}.$$

b) The forcing e^{-t}, also does not satisfy the [H] equation, thus, it does not exhibit resonant behavior. We try $y = be^{-t}$. Inserting this into the ODE, we obtain

$$b + b = 1, \qquad b = \frac{1}{2}.$$

$$\textit{Answer}: \quad y = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{3} \sin 2t + \frac{1}{2} e^{-t}$$