Spectral decomp example

July 28, 2022

Outer product: if $|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, then

$$|v\rangle\langle v| = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \end{pmatrix} = \begin{pmatrix} a\overline{a} & a\overline{b} \\ b\overline{a} & b\overline{b} \end{pmatrix}.$$

This matrix will always be Hermitian (entries across the diagonal are complex conjugates of each other) no mater what $|v\rangle$ is. Can we get all Hermitian matrices this way? No but by taking the sums of matrices like this, we can.

It turns out that if A is Hermitian, we can write it as a sum

$$A = \sum_i \lambda_i |v_i\rangle\langle v_i|,$$

where the eigenvalues are λ_i and the vectors $|v_i\rangle$ are a set of orthonormal (orthogonal with each other and norm 1) eigenvectors. That is

$$A|v_i\rangle = \lambda_i |v_i\rangle.$$

This way of rewriting A is called the spectral decomposition.

As an example, let's use the

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Then $\lambda_1 = 2$ and $\lambda_2 = -2$ and

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Checking that this works

$$A = 2|v_1\rangle\rangle v_1| + (-2)|v_2\rangle\rangle v_2| = 2\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - 2\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Now you can find A^{-1} by inverting eigenvalues in the decomposition:

$$A^{-1} = \sum_i \frac{1}{\lambda_i} |v_i\rangle\langle v_i| = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

You can also do things like square the matrix this way:

$$A^2 = \sum_i \lambda_i^2 |v_i\rangle\langle v_i| = 2^2\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + (-2)^2\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

and find the matrix exponential of A

$$e^{A} = \sum_i e^{\lambda_i} \left|v_i\right\rangle \left\langle v_i\right| = e^{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{-2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{2}+e^{-2} & e^{2}-e^{-2} \\ e^{2}-e^{-2} & e^{2}+e^{-2} \end{pmatrix}.$$

It turns out that if A is a Hermitian matrix, then e^{iAt} is a unitary. allowing it to be implemented as quantum gate. The explanation for why this is the case is because

$$e^{iAt} = \sum_s e^{i\lambda_s t} \left|v_s\right\rangle \left\langle v_s\right|$$

has eigenvalues, $e^{i\lambda_s t}$, which are phases. Thus, is just a phase gate applying phases, $e^{i\lambda_s t}$, to the set of quantum states $\left|v_s\right\rangle$.