

# Averages of static electric and magnetic fields over a spherical region: a derivation based on the mean-value theorem

Ben Yu-Kuang Hu

*Department of Physics, 250 Buchtel Commons, University of Akron, Akron, OH 44325-4001.*

(February 2, 2008)

The electromagnetic theory of dielectric and magnetic media deals with macroscopic electric and magnetic fields, because the microscopic details of these fields are usually experimentally irrelevant (with certain notable exceptions such as scanning tunneling microscopy). The macroscopic fields are the average of the microscopic fields over a microscopically large but macroscopically small region.<sup>1,2</sup> This averaging region is often chosen to be spherical, denoted here as

$$\bar{\mathbf{E}} \equiv \frac{3}{4\pi R^3} \int_{\mathcal{V}} d\mathbf{r}' \mathbf{E}(\mathbf{r}'), \quad (1)$$

where  $\mathcal{V}$  is a sphere of radius  $R$  centered at  $\mathbf{r}$  (with a similar definition for  $\bar{\mathbf{B}}$ ).  $R$  is a distance which is macroscopically small but nonetheless large enough to enclose many atoms. The macroscopic  $\bar{\mathbf{E}}$  and  $\bar{\mathbf{B}}$  fields obtained by averaging over a sphere exhibit properties which prove useful in certain arguments and derivations. These properties are as follows:<sup>3,4</sup>

1. if all the sources of the  $\mathbf{E}$ -field are outside the sphere, then  $\bar{\mathbf{E}}$  is equal to the electric field at the center of the sphere,
2. if all the sources of the  $\mathbf{E}$ -field are inside the sphere,  $\bar{\mathbf{E}} = -\mathbf{p}/(4\pi\epsilon_0 R^3)$ , where  $\mathbf{p}$  is the dipole moment of the sources with respect to the center of the sphere,
3. if all the sources of the  $\mathbf{B}$ -field are outside the sphere, then  $\bar{\mathbf{B}}$  is equal to the magnetic field at the center of the sphere,
4. if all the sources of the  $\mathbf{B}$ -field are inside the sphere,  $\bar{\mathbf{B}} = \mu_0 \mathbf{m}/(2\pi R^3)$ , where  $\mathbf{m}$  is the magnetic dipole moment of the sources.

These results can be derived in a variety of ways. For example, Griffiths<sup>3</sup> derives properties 1 and 2 using a combination of results from Coulomb's law and Gauss' law, and properties 3 and 4 by writing down the  $\mathbf{B}$ -field in terms of the vector potential in the Coulomb gauge, and explicitly evaluating angular integrals.<sup>5</sup> The purpose of this note is to describe a relatively simple derivation of all four results, based on the well-known mean-value theorem (described in most textbooks on electromagnetic theory): if a scalar potential  $\Phi(\mathbf{r})$  satisfies Laplace's equation in a sphere, then the average of  $\Phi$  over the surface of the sphere is equal to  $\Phi$  at the center of the sphere;<sup>6,7</sup> that is, if  $\nabla^2 \Phi = 0$  in a spherical region of radius  $r'$  centered at  $\mathbf{r}$ , then

$$\Phi(\mathbf{r}) = \frac{1}{4\pi} \oint d\Omega \Phi(\mathbf{r} + r' \hat{\mathbf{n}}_\Omega), \quad (2)$$

where  $\Omega$  is the solid angle relative to  $\mathbf{r}$  and  $\hat{\mathbf{n}}_\Omega$  is the unit vector pointing in the direction of  $\Omega$ .

Taking the gradient  $\nabla$  of both sides of Eq. (2) with respect to  $\mathbf{r}$ , we immediately obtain

$$\mathbf{E}(\mathbf{r}) = -\nabla \Phi(\mathbf{r}) = \frac{1}{4\pi} \oint d\Omega \left[ -\nabla \Phi(\mathbf{r} + r' \hat{\mathbf{n}}_\Omega) \right] = \frac{1}{4\pi} \oint d\Omega \mathbf{E}(\mathbf{r} + r' \hat{\mathbf{n}}_\Omega); \quad (3)$$

that is, if  $\nabla \cdot \mathbf{E} = 0$  inside a sphere, the average of  $\mathbf{E}$  over the surface of the sphere is equal to the  $\mathbf{E}$  at the center of sphere. Eq. (3) is the basis of the derivation of the four properties listed above. To simplify the notation, henceforth in this note it is assumed that  $\mathcal{V}$  is a sphere of radius  $R$  and is centered at the origin  $\mathbf{0}$ .

### 1. Electric field with sources outside sphere

When all the static charge sources are outside  $\mathcal{V}$ ,  $\mathbf{E} = -\nabla \Phi$  and  $\nabla \cdot \mathbf{E} = 0$  inside  $\mathcal{V}$  and hence Eq. (3) is valid. We shall now see that property 1 follows quite trivially as a special case of Eq. (3) [and hence of Eq. (2)].

The average of  $\mathbf{E}$  over  $\mathcal{V}$  can be written as weighted integral of the average of  $\mathbf{E}$  over surfaces of spheres with radius  $r' < R$  centered at  $\mathbf{0}$ ,

$$\bar{\mathbf{E}} \equiv \frac{3}{4\pi R^3} \int_{\mathcal{V}} d\mathbf{r}' \mathbf{E}(\mathbf{r}') = \frac{3}{R^3} \int_0^R r'^2 dr' \left[ \frac{1}{4\pi} \int d\Omega \mathbf{E}(r' \hat{\mathbf{n}}_\Omega) \right]. \quad (4)$$

Using Eq. (3) in Eq. (4) immediately yields property 1,

$$\bar{\mathbf{E}} = \frac{3}{R^3} \int_0^R r'^2 dr' \mathbf{E}(\mathbf{0}) = \mathbf{E}(\mathbf{0}). \quad (5)$$

## 2. Electric field with sources inside sphere

For clarity we first prove property 2 for a single point charge inside the sphere. The general result can be inferred from the single point charge result using the superposition principle, but for completeness we generalize the proof for a continuous charge distribution.

Utilizing the vector identity<sup>2,8</sup>

$$\int_{\mathcal{V}} \nabla \Phi \, d\mathbf{r} = \oint_S \Phi \, d\mathbf{a}, \quad (6)$$

where  $S$  is the surface of  $\mathcal{V}$ , the average of the electric field over  $\mathcal{V}$  centered at  $\mathbf{0}$  can be written as<sup>4</sup>

$$\bar{\mathbf{E}} = -\frac{3}{4\pi R^3} \int_{\mathcal{V}} \nabla \Phi(\mathbf{r}) \, d\mathbf{r} = -\frac{3}{4\pi R^3} \oint_S \Phi \, d\mathbf{a}. \quad (7)$$

Eq. (7) implies the average  $\bar{\mathbf{E}}$  is determined completely by the potential on the surface  $S$ .

We now use a well-known result from the method of images solution for the potential of a point charge next to a conducting sphere: the potential on the surface  $S$  for a charge  $q$  at  $\mathbf{r}$  inside the sphere is reproduced exactly by an image charge  $q' = Rq/d$  at  $\mathbf{r}' = (R^2/r)\hat{\mathbf{r}}$  outside the sphere ( $\hat{\mathbf{r}}$  is a unit vector). Eq. (7) therefore implies that the  $\bar{\mathbf{E}}$  for a point charge  $q$  at  $\mathbf{r}$  is exactly equal to that of an image point charge  $q'$  at  $(R^2/r)\hat{\mathbf{r}}$ . But since the image charge is *outside* the sphere, we can use property 1 to determine  $\bar{\mathbf{E}}$ ,

$$\bar{\mathbf{E}} = \mathbf{E}_{\text{image}}(\mathbf{0}) = -\frac{q'\hat{\mathbf{r}}}{4\pi\epsilon_0|\mathbf{r}'|^2} = -\frac{q\mathbf{r}}{4\pi\epsilon_0 R^3} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}, \quad (8)$$

where  $\mathbf{p} = q\mathbf{r}$  is the dipole moment for a single point charge.

*Generalization to continuous charge distributions* – Assume the charge distribution inside the spherical volume  $\mathcal{V}$  is  $\rho(\mathbf{r})$ . A volume element  $dV$  at  $\mathbf{r}$  inside  $\mathcal{V}$  contains charge  $dq = \rho(\mathbf{r}) dV$ . The image charge element outside the sphere which gives the same average electric field as  $dq$  is  $dq' = \rho'(\mathbf{r}') dV' = dq R/r$  at  $\mathbf{r}' = (R^2/r)\hat{\mathbf{r}}$ . As in the discrete case, the contribution of  $dq$  to the average  $\mathbf{E}$ -field in  $\mathcal{V}$  is equal to the electric field at the origin due to  $dq'$ ,

$$d\bar{\mathbf{E}} = d\mathbf{E}_{\text{image}}(\mathbf{0}) = -\frac{\rho'(\mathbf{r}') dV'}{4\pi\epsilon_0 r'^2} \hat{\mathbf{r}} = -\frac{\mathbf{r} \rho(\mathbf{r}) dV}{4\pi\epsilon_0 R^3}. \quad (9)$$

The averaged electric field due to all charges in the sphere  $\mathcal{V}$  is therefore

$$\bar{\mathbf{E}} = \int_{\mathcal{V}} d\bar{\mathbf{E}} = -\frac{1}{4\pi\epsilon_0 R^3} \int_{\mathcal{V}} dV \, \mathbf{r} \rho(\mathbf{r}) = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}, \quad (10)$$

where  $\mathbf{p} = \int_{\mathcal{V}} dV \, \mathbf{r} \rho(\mathbf{r})$  is the dipole moment with respect to the origin of all the charges in  $\mathcal{V}$ .

## 3. Magnetic field with sources outside the sphere

When magnetic field sources are absent in  $\mathcal{V}$ , both  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{B} = 0$ , and hence  $\mathbf{B} = -\nabla\phi_M$  where  $\nabla^2\phi_M = 0$  inside the sphere. Therefore, the same derivation for property 1 holds here; that is, the average of the  $\mathbf{B}$ -field over a sphere is equal to its value at the center of the sphere.

## 4. Magnetic field for sources inside the sphere –

Using the vector potential description of the magnetic field,  $\mathbf{B} = \nabla \times \mathbf{A}$ , the average over the sphere  $\mathcal{V}$  can be written as<sup>4</sup>

$$\bar{\mathbf{B}} = \frac{3}{4\pi R^3} \int_{\mathcal{V}} \nabla \times \mathbf{A} = -\frac{3}{4\pi R^3} \oint_S \mathbf{A} \times d\mathbf{a}. \quad (11)$$

The second equality in the above equation is a vector identity.<sup>8</sup> This equation shows that, as in the case of the  $\mathbf{E}$ -field and the scalar potential, the average  $\mathbf{B}$ -field over any volume  $\mathcal{V}$  can be computed from  $\mathbf{A}$  on the surface of  $\mathcal{V}$ .

We now consider the contribution to  $\bar{\mathbf{B}}$  of current element  $\mathbf{J}(\mathbf{r}) dV$  inside  $\mathcal{V}$ . We can do this by determining the image current element outside the sphere that exactly reproduces the vector potential due to  $\mathbf{J}(\mathbf{r}) dV$  on the surface of the sphere. We choose the Coulomb gauge

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J_i(\mathbf{x})}{|\mathbf{r} - \mathbf{x}|} d\mathbf{x}, \quad (12)$$

where  $i$  denotes spatial component. Since  $A_i$  is related to  $J_i$  in the same way as the electric potential  $\Phi$  is to the charge density  $\rho$ , the method of electrostatic images can also be used here to determine the image current element.

The proof of property 4 proceeds analogously to that of property 2. For  $\mathbf{J}(\mathbf{r}) dV$  inside the sphere  $\mathcal{V}$ ,  $\mathbf{A}$  on the surface  $\mathcal{S}$  is reproduced by an image current element  $\mathbf{J}'(\mathbf{r}') dV' = (R/r)\mathbf{J}(\mathbf{r}) dV$ , where  $\mathbf{r}' = (R^2/r)\hat{\mathbf{r}}$ . Since the image current is outside the sphere, we can use property 3. Thus, the  $\mathbf{J}(\mathbf{r}) dV$  contribution to the average  $\mathbf{B}$ -field in  $\mathcal{V}$  is equal to the  $\mathbf{B}$ -field at the center due to  $\mathbf{J}'(\mathbf{r}') dV'$ ,<sup>9</sup>

$$d\overline{\mathbf{B}} = d\mathbf{B}_{\text{image}}(\mathbf{0}) = -\frac{\mu_0}{4\pi} \frac{\mathbf{J}'(\mathbf{r}') \times \hat{\mathbf{r}}}{r'^2} dV' = \frac{\mu_0}{4\pi R^3} \mathbf{r} \times \mathbf{J}(\mathbf{r}) dV. \quad (13)$$

The contribution for the entire current distribution in  $\mathcal{V}$  is therefore

$$\overline{\mathbf{B}} = \frac{\mu_0}{4\pi R^3} \int_{\mathcal{V}} dV \mathbf{r} \times \mathbf{J}(\mathbf{r}) = \frac{\mu_0 \mathbf{m}}{2\pi R^3}, \quad (14)$$

where  $\mathbf{m} = \frac{1}{2} \int_{\mathcal{V}} dV \mathbf{r} \times \mathbf{J}(\mathbf{r})$  is magnetic moment of a current distribution in  $\mathcal{V}$ .

Finally, note that similar arguments hold for charge distributions which are constant along the  $z$ -direction, since the potentials for these satisfy Laplace's equation in two dimensions, and the method of images is also applicable for line charges.

### Acknowledgement

Useful correspondence with Prof. David Griffiths is gratefully acknowledged.

<sup>1</sup> David J. Griffiths, Introduction to Electrodynamics, 3rd Ed. (Prentice-Hall, New Jersey, 1999).

<sup>2</sup> John David Jackson, Classical Electrodynamics, 3rd Ed. (Wiley, New York, 1999).

<sup>3</sup> Ref. 1, pgs. 156–157 and pg. 253.

<sup>4</sup> Ref. 2, pgs. 148–149 and pgs. 187–188.

<sup>5</sup> David J. Griffiths, Solutions manual to Introduction to Electrodynamics, 3rd Ed. (Prentice-Hall, New Jersey, 1999), pg. 108–109.

<sup>6</sup> Ref. 1, pg. 114.

<sup>7</sup> Ref. 2, Problem 1.10, pg. 52.

<sup>8</sup> Ref. 1, pg. 56.

<sup>9</sup> The fact that  $\nabla' \cdot \mathbf{J}'(\mathbf{r}')$  may be non-zero is irrelevant, because we know from Jefimenko's equations (see Ref. 1, pg. 427–429; Ref. 2, pg. 246–248) that the  $\mathbf{B}$ -field is dependent only on  $\mathbf{J}$  and  $\dot{\mathbf{J}}$  and not  $\dot{\rho}$ . Retardation effects in Jefimenko's equations can be ignored by assuming that the currents were turned on infinitely long ago. See also Ref. 1, Problem 7.55, pg. 339.