Unpolarized Cross-Section of $e^+e^- \longrightarrow \mu^+\mu^-$

Quantum Field Theory

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1 Introduction

In this report, we derive equation (5.11) from Peskin and Schroeder's book on Quantum Field Theory, which calculates the unpolarized cross section for a QED process. We start by introducing the necessary concepts, including Dirac spinors and Feynman rules. Following that, we will compute the matrix element and perform a detailed evaluation of the trace. Finally, we will derive the cross section, averaging over spins, and simplifying the expression step by step.

The QED process under consideration involves the scattering of an electron and positron into a pair of heavier leptons. The matrix element involves spinor contractions and gamma matrix identities, which we will evaluate explicitly.

2 Feynman Rules in QED

To calculate scattering amplitudes in quantum electrodynamics (QED), we use Feynman rules. These rules dictate how to write down an expression for the matrix element $\mathcal{M}$. For an electron-muon scattering process, the matrix element is given by

$$\mathcal{M} = \bar{v}^{s'}(p')(-ie\gamma^\mu)\mu^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k') \tag{1}$$

This expression is readily apparent by drawing the Feynman diagram for the process:

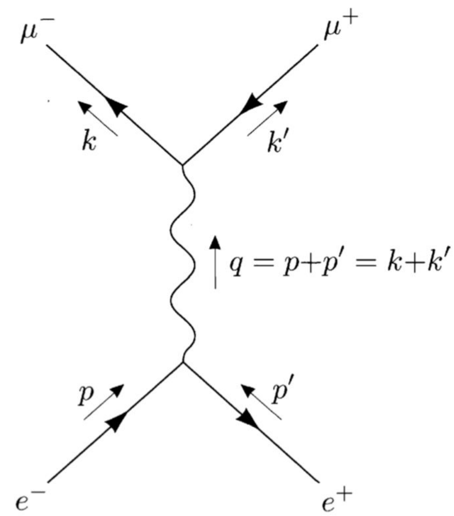


Figure 1: Electron-Positron Scattering Feynman Diagram

3 Simplifying Amplitude Expression

Rearranging this equation and using $g^{\mu\nu}[\gamma^\nu] = \gamma_\mu$:

$$i\mathcal{M} = \frac{ie^2}{q^2}\left(\bar{v}(p^{'})\gamma^\mu u\,(p)\right)\left(\bar{u}\,(k)\,\gamma_\mu v(k^{'})\right) \tag{2}$$

Next we calculate $|M|^2$ to compute the differential cross section. Generally the complex conjugation follows the follow rules:

$$(\bar{v}\gamma^\mu u)^* = u^\dagger\,(\gamma^\mu)^\dagger\left(\gamma^0\right)^\dagger v = \bar{u}\gamma^\mu v \tag{3}$$

Where $(\gamma^0)^\dagger$ keeps the equation Lorentz invariant. Thus the square matrix is:

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left(\bar{v}(p^{'})\gamma^\mu u\,(p)\,\bar{u}(p)\gamma^\nu v(p^{'})\right)\left(\bar{u}\,(k)\,\gamma_\mu v(k^{'})\bar{v}(k^{'})\gamma_\nu u(k)\right) \tag{4}$$

4 Clifford Algebra

We define Clifford Algebra by:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I \tag{5}$$

$$(\gamma^0)^2 = I \qquad (\gamma^i)^2 = -I$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Where σ^i are the Pauli matrices ; {i:1,3}
Then for all $\mu = 0, 1, 2, 3$ we have

$$\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger$$

We now define the Dirac adjoint:

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0 \tag{6}$$

We also define:

$$A_\mu \gamma^\mu = \not{A} \tag{7}$$

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^i \partial_i \tag{8}$$

For space-like part $\mu = i$ for $i = 1, 2, 3$

$$\gamma^i \partial_i = \begin{pmatrix} 0 & \sigma^i \partial_i \\ -\sigma^i \partial_i & 0 \end{pmatrix} \tag{9}$$

Adding with the time-like part we have:

$$\begin{pmatrix} 0 & I\partial_0 + \sigma^i \partial_i \\ I\partial_0 - \sigma^i \partial_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix}$$

with $\sigma^\mu = \left(I, \sigma^i\right)$ and $\bar{\sigma}^\mu = \left(I, -\sigma^i\right)$
Next we introduce:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

which has the following properties:

$$\left(\gamma^5\right)^\dagger = \gamma^5$$
$$\left(\gamma^5\right)^2 = 1$$
$$\left\{\gamma^5, \gamma^\mu\right\} = 0$$

Let's check:

$$\left\{\gamma^5, \gamma^\mu\right\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$$

CASE 1: for $\mu = 0$

$$\gamma^5\gamma^0 + \gamma^0\gamma^5$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3$$

since $\left(\gamma^0\right)^2 = 1$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^1\gamma^2\gamma^3$$

Now let's check the anti-commutator relation of γ^0 with γ^1

$$\left\{\gamma^0, \gamma^i\right\} = 2g^{0i}$$

$$\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$$

$$\gamma^0\gamma^i = -\gamma^i\gamma^0$$

$$\boldsymbol{\gamma}^i\boldsymbol{\gamma}^0 = -\boldsymbol{\gamma}^0\boldsymbol{\gamma}^i$$

Flipping the order of matrices gives us a negative sign:

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^1\gamma^2\gamma^3$$

$$= -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3$$

$$= -i\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3$$

$$= 0$$

CASE 2: for $\mu = i$

$$\left\{\gamma^i, \gamma^j\right\} = 2g^{ij}$$

when $i = j$

$$\left\{\gamma^i, \gamma^i\right\} = -2$$

$$2\gamma^i\gamma^i = -2$$

$$\left(\gamma^i\right)^2 = -1$$

Next, when $i \neq j$

$$\left\{\gamma^i, \gamma^j\right\} = 0$$

$$\gamma^i\gamma^j = -\gamma^j\gamma^i$$

Now let's check:

$$\left\{\gamma^5, \gamma^i\right\} = \gamma^5\gamma^i + \gamma^i\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^i + i\gamma^i\gamma^0\gamma^1\gamma^2\gamma^3$$

When $i = j$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^i + i\gamma^i\gamma^0\gamma^1\gamma^2\gamma^3$$

Let's check for $i = 3$

$$\left(\gamma^i\right)^2 = -1$$

$$\begin{aligned} &= -i\gamma^0\gamma^1\gamma^2 + i\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -i\gamma^0\gamma^1\gamma^2 - i\gamma^3\gamma^3\gamma^0\gamma^1\gamma^2 \\ &= -i\gamma^0\gamma^1\gamma^2 + i\gamma^0\gamma^1\gamma^2 = 0 \end{aligned}$$

When $i \neq j$, we have either $i = 1$ or $i = 2$. For $i = 1$:

$$i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 + i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 - i\gamma^0\gamma^2\gamma^3 + i\gamma^0\gamma^2\gamma^3 = 0$$

Similarly, $i = 2$ gives 0. Hence:

$$\left\{\gamma^5, \gamma^\mu\right\} = 0$$

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

5 Dirac Spinors and Gamma Matrices

We now begin by introducing some necessary notation and mathematics to make our calculations easier. We begin by recalling that Dirac spinors describe fermions in relativistic quantum field theory. A Dirac spinor $\psi(x)$ satisfies the Dirac equation:

$$(i\partial\!\!\!/ - m)\psi(x) = 0. \tag{10}$$

In momentum space, the solutions to this equation can be written as

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad \bar{\psi}(x) = \bar{u}(p)e^{ip\cdot x}. \tag{11}$$

where $u(p)$ are the spinors for particles and $v(p)$ for antiparticles.

The spinor outer products, which are essential for computing scattering amplitudes, are governed by the completeness relations:

$$\sum_s u^s(p)\bar{u}^s(p) = (p\!\!\!/ + m), \quad \sum_s v^s(p)\bar{v}^s(p) = (p\!\!\!/ - m). \tag{12}$$

Where

$$u^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\bar{\sigma}}\xi^s \end{pmatrix} \tag{13}$$

is the positive frequency plane wave solution to the Dirac equation.
And

$$v^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\eta^s \\ -\sqrt{p.\bar{\sigma}}\eta^s \end{pmatrix} \tag{14}$$

is the negative frequency plane wave solution to the Dirac equation.

Additionally, the gamma matrices γ^μ satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I. \tag{15}$$

6 Initial Amplitude Calculation

We now proceed with calculating the squared amplitude. For an unpolarized positron beam, we take the average of spins s and s', while for muons we sum over all their spins r and r' as muon detectors are blind to polarization. The squared matrix element is then given by,

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2. \tag{16}$$

At this point, we use the spin summation identities for the spinors to reduce the products of spinors to traces over gamma matrices. Using them for the first half of equation (4):

$$\begin{aligned} \sum_{s,s'} \bar{v}_a^{s'}(p^{'})\gamma^\mu_{ab} u^s_b(p)\, \bar{u}^s_c(p)\gamma^\nu_{cd} v^{s'}_d(p^{'}) &= (\not{p}' - m)_{da}\gamma^\mu_{ab}(\not{p} + m)_{bc}\gamma^\nu_{cd} \\ &= \operatorname{tr}[(\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu] \end{aligned} \tag{17}$$

Repeating this for the second part. the matrix element becomes a product of traces:

$$\frac{1}{4}|\mathcal{M}|^2 = \frac{e^4}{4q^4} \operatorname{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] \operatorname{tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right]. \tag{18}$$

7 Trace Technology

Having been equipped with the necessary Clifford algebra, we can begin to check different trace relations

$$\operatorname{tr} \gamma^\mu = \operatorname{tr} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} = 0$$

An abstract proof for this is as follows

$$\begin{aligned}
\operatorname{tr}\gamma^\mu &= \operatorname{tr}\gamma^5\gamma^5\gamma^\mu && \text{since } \left(\gamma^5\right)^2 = 1 \\
&= -\operatorname{tr}\gamma^5\gamma^\mu\gamma^5 && \text{since } \left\{\gamma^\mu,\gamma^5\right\} = 0 \\
&= -\operatorname{tr}\gamma^5\gamma^5\gamma^\mu && \text{using cyclic property of trace; tr[AB]=tr[BA]} \\
&= -\operatorname{tr}\gamma^\mu .
\end{aligned}$$

This tells us that when there are an odd number of γ matrices, the trace must vanish

Next we have

$$\begin{aligned}
\operatorname{tr}\gamma^\mu\gamma^\nu &= \operatorname{tr}(2g^{\mu\nu}\cdot 1 - \gamma^\nu\gamma^\mu) \\
&= 8g^{\mu\nu} - \operatorname{tr}\gamma^\mu\gamma^\nu \\
2\operatorname{tr}\gamma^\mu\gamma^\nu &= 8g^{\mu\nu} \\
\operatorname{tr}\gamma^\mu\gamma^\nu &= 4g^{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= \operatorname{tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\right) \\
&= \operatorname{tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2g^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\right) \\
&= \operatorname{tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2g^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho 2g^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right) \\
&= g^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma - g^{\mu\rho}\operatorname{tr}\gamma^\nu\gamma^\sigma + g^{\mu\sigma}\operatorname{tr}\gamma^\nu\gamma^\rho \\
&= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}\gamma^5 &= \operatorname{tr}\left(\gamma^0\gamma^0\gamma^5\right) \\
&= -\operatorname{tr}\left(\gamma^0\gamma^5\gamma^0\right) \quad \text{since } \left\{\gamma^\mu,\gamma^5\right\} = 0 \\
&= -\operatorname{tr}\left(\gamma^0\gamma^0\gamma^5\right) \quad \text{tr[AB]=tr[BA]} \\
&= -\operatorname{tr}\gamma^5
\end{aligned}$$

A summary of trace theorems is as follows:

$$\begin{aligned}
\operatorname{tr}(\mathbf{1}) &= 4 \\
\operatorname{tr}\left(\text{ any odd \# of } \gamma' \text{ s }\right) &= 0 \\
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\right) &= 4g^{\mu\nu} \\
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \\
\operatorname{tr}\left(\gamma^5\right) &= 0 \\
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) &= 0 \\
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) &= -4i\epsilon^{\mu\nu\rho\sigma}
\end{aligned}$$

Another useful identity allows us to reverse the order of all the γ matrices in a trace:

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma ...) = \mathrm{tr}(...\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu)$$

To prove this relation consider the matrix $C = \gamma^0\gamma^2$ where $C^2 = I$

$$\begin{aligned} C\gamma^\mu C &= \gamma^0\gamma^2\gamma^\mu\gamma^0\gamma^2 \\ \text{since } \{\gamma^0\gamma^i\} &= 0 \\ &= -\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2 \\ &= \gamma^2\gamma^2\gamma^\mu\gamma^0\gamma^0 \\ \text{since } \left(\gamma^i\right)^2 &= -1 \\ &= -\gamma^0\gamma^\mu\gamma^0 = -\left(\gamma^\mu\right)^\dagger \end{aligned}$$

Thus if there are $n\gamma$-matrices inside the trace,

$$\begin{aligned} \mathrm{tr}\left(\gamma^\mu\gamma^\nu\cdots\right) &= \mathrm{tr}\left(C\gamma^\mu CC\gamma^\nu C\cdots\right) \\ &= (-1)^n\,\mathrm{tr}\left[\left(\gamma^\mu\right)^T\left(\gamma^\nu\right)^T\cdots\right] \\ &= \mathrm{tr}\left(\cdots\gamma^\nu\gamma^\mu\right) \end{aligned}$$

When two matrices inside a trace are dotted together, it is easiest to eliminate them before evaluating the trace. Let's see this:

$$\begin{aligned} \gamma^\mu\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu \\ &= \frac{2}{2}g_{\mu\nu}\gamma^\mu\gamma^\nu \end{aligned}$$

since $g_{\mu\nu} = g_{\nu\mu}$

$$= \frac{1}{2}[g_{\mu\nu}\gamma^\mu\gamma^\nu + g_{\nu\mu}\gamma^\mu\gamma^\nu]$$

The second part of the equation in the brackets can be treated to have dummy indices so we can take $\mu = \nu$

$$= \frac{1}{2}[g_{\mu\nu}\gamma^\mu\gamma^\nu + g_{\mu\nu}\gamma^\nu\gamma^\mu]$$

$$= \frac{1}{2}g_{\mu\nu}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu]$$

since $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$

$$\gamma^\mu\gamma_\mu = \frac{1}{2}g_{\mu\nu}2g^{\mu\nu} = g_{\mu\nu}g^{\mu\nu} = 4$$

Let's prove the following set of contraction identities:

1) $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$

Proof:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$$

$$\gamma^\mu\gamma^\nu = 2g^{\mu\nu}I - \gamma^\nu\gamma^\mu$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = (2g^{\mu\nu}I - \gamma^\nu\gamma^\mu)\gamma_\mu = 2g^{\mu\nu}I\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu = 2g^{\mu\nu}I\gamma_\mu - 4\gamma^\nu = 2\gamma^\nu - 4\gamma^\nu$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$$

2) $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$

Proof:

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = (2g^{\mu\nu}I - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma_\mu = 2g^{\mu\nu}\gamma^\rho\gamma_\mu - \gamma^\nu\gamma^\mu\gamma^\rho\gamma_\mu$$

We can plug in the result from 1 in the second part after the negative sign

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 2\gamma^\rho\gamma^\nu + 2\gamma^\nu\gamma^\rho = 2\{\gamma^\rho, \gamma^\nu\} = 4g^{\nu\rho}$$

The same procedure can be repeated to prove the identity:

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$$

8 Evaluation of the Trace

We now evaluate the traces with the help of the properties of γ matrices. For the electron, the trace is:

$$\mathrm{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p \cdot p' + m_e^2)\right]. \quad (19)$$

Similarly, for the muon, we have:

$$\mathrm{tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right] = 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k \cdot k' + m_\mu^2)\right]. \quad (20)$$

The product of the traces gives us:

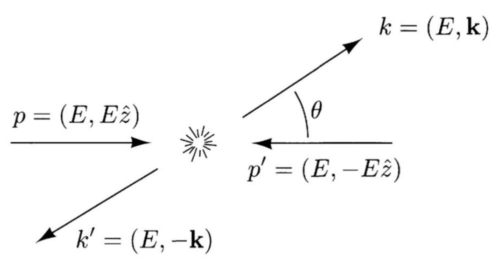
$$|\mathcal{M}|^2 = \frac{16e^4}{q^4}\left[(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) + m_\mu^2(p \cdot p')\right]. \quad (21)$$

9 Unpolarized Cross Section

Finally, the unpolarized cross section is obtained by averaging over the initial spins and summing over final spins. Taking $m_e = 0$ leads to the following result for the squared matrix element:

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2 (p\cdot p')\right]. \qquad (22)$$

Specializing to the center of mass frame and simplifying the above equation in terms of the energies and angles we get



$$|\mathbf{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\mathbf{k}\cdot\hat{z} = |\mathbf{k}|\cos\theta$$

$$\begin{aligned}
p\cdot k = p'\cdot k' &= E^2 - E|\mathbf{k}|\cos\theta \\
p\cdot k' = p'\cdot k &= E^2 + E|\mathbf{k}|\cos\theta \\
p\cdot p' &= 2E^2 \\
q^2 = (p+p')^2 &= 4E^2
\end{aligned}$$

The squared matrix element may then be written as:

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{8e^4}{16E^4}\left[E^2(E - |\mathbf{k}|\cos\theta)^2 + E^2(E + |\mathbf{k}|\cos\theta)^2 + m_\mu^2 E^2\right]. \\
&= e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos\theta\right]
\end{aligned}$$