if $x^2 + 5xy + 3y^2 = z^2$

A) if y is even, it is easy to check that y is multiple of 4 and z is odd

so if $y = 4y_1,\ x^2 + 20xy_1 + 12y_1^2 = (x + 10y_1)^2 - 52y_1^2 = z^2$

if $u = x + 10y_1$ we have: $u^2 - z^2 = 52y_1^2$

$\Leftrightarrow \dfrac{u+z}{2} \times \dfrac{u-z}{2} = 13y_1^2$

$gcd(x, y) = 1 \Rightarrow gcd(x, y_1) = 1 \Rightarrow gcd(u, z) = 1 \Rightarrow gcd(\dfrac{u+z}{2}, \dfrac{u-z}{2}) = 1$

so it exists m,n with gcd(m,n)=1, gcd(n,13)=1, mn even, $y_1 = m \times n$ and

$\{\dfrac{u+z}{2} = 13m^2\ and\ \dfrac{u-z}{2} = n^2\}$ or $\{\dfrac{u+z}{2} = n^2\ and\ \dfrac{u-z}{2} = 13m^2\}$

In the 2 cases: $z = |n^2 - 13m^2|\ ;\ y = 4nm\ ;\ x = 13m^2 - 10mn + n^2$

B) if y is odd we have: $4x^2 + 20xy + 12y^2 = (2x + 5y)^2 - 13y^2 = 4z^2$

if $u = 2x + 5y$ we have: $u^2 - 4z^2 = (u + 2z) \times (u - 2z) = 13y^2$

$gcd(x, y) = 1\ and\ y\ odd \Rightarrow gcd(u, y) = 1\ and\ u\ odd \Rightarrow gcd(u + 2z, u - 2z) = 1$

so it exists m,n with gcd(m,n)=1, gcd(n,13)=1, mn odd, $y = m \times n$ and

$\{u + 2z = 13m^2\ and\ u - 2z = n^2\}$ or $\{u + 2z = n^2\ and\ u - 2z = 13m^2\}$

In the 2 cases: $z = |\dfrac{n^2 - 13m^2}{4}|\ ;\ y = nm\ ;\ x = \dfrac{13m^2 - 10mn + n^2}{4}$

for A) and B) $x > 0 \Leftrightarrow \dfrac{n}{m} < \sqrt{5 - 2\sqrt{3}}\ or\ \dfrac{n}{m} > \sqrt{5 + 2\sqrt{3}}$

In summary the solutions are:

m,n integer gcd(m,n)=1, gcd(n,13)=1, $\dfrac{n}{m} < \sqrt{5 - 2\sqrt{3}}\ or\ \dfrac{n}{m} > \sqrt{5 + 2\sqrt{3}}$

- if mn odd, $z = |\dfrac{n^2 - 13m^2}{4}| \le N$
- if mn even, $z = |n^2 - 13m^2| \le N$

Let D(N) the number of solutions with $gcd(m, n) > 2$ we have:

$$C(N) = D(N) - \sum_{p\ prime}^{p \neq 2,13} D(\frac{N}{p^2}) + \sum_{p_1,p_2\ prime}^{p_i \neq 2,13} D(\frac{N}{p_1^2 p_2^2}) - \sum_{p_1,p_2,p_3\ prime}^{p_i \neq 2,13} D(\frac{N}{p_1^2 p_2^2 p_3^2}) + \ \ldots$$

let $C_{13}(N)\ resp.\ D_{13}(N)$ the functions without the condition gcd(n,13)=1

we have the same relation between $C_{13}(N)\ and\ D_{13}(N)$

and as if $n = 13n_1,\ |13m^2 - n^2| = 13|m^2 - 13n_1^2| \Rightarrow C(N) = C_{13}(N) - C_{13}(N/13)$

$$A_e = \frac{1}{2}(\sqrt{\frac{N}{5\sqrt{3}-6}}, \sqrt{\frac{13 \times N}{5\sqrt{3}+6}}) \ A_o = (\sqrt{\frac{N}{5\sqrt{3}-6}}, \sqrt{\frac{13 \times N}{5\sqrt{3}+6}})$$

$$B_e = \frac{1}{2}(\sqrt{\frac{N}{5\sqrt{3}+6}}, \sqrt{\frac{13 \times N}{5\sqrt{3}-6}}) \ B_o = (\sqrt{\frac{N}{5\sqrt{3}+6}}, \sqrt{\frac{13 \times N}{5\sqrt{3}-6}})$$