Let the convergents R_n of $x = \sqrt{N}$ for $n = 1, 2, ...$

$$R_n = \frac{p_n}{q_n} = \frac{p_{n-1}a_n + p_{n-2}}{q_{n-1}a_n + q_{n-2}}$$

$$\sqrt{N} = x = \frac{p_{n-1}x_n + p_{n-2}}{q_{n-1}x_n + q_{n-2}}$$

Where :

$$x = [a_0; a_1, a_2, a_3, ...] = \lim_{n\to\infty} R_n$$

$$x_n = [a_n; a_{n+1}, a_{n+2}, a_{n+3}, ...]$$

We have the relations :

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1} \ (1)$$

$$\left| x - \frac{p_n}{q_n} \right| = \left| \frac{(-1)^{n+1}}{q_n(q_n x_{n+1} + q_{n-1})} \right| = \frac{1}{q_n(q_n x_{n+1} + q_{n-1})} \ (2)$$

Let the semi-convergents for $k = 1, 2, ..., a_{n+1} - 1$

$$s_k = \frac{p_n k + p_{n-1}}{q_n k + q_{n-1}}$$

We have :

$$|x - s_k| = \left| \frac{p_{n-1}x_n + p_{n-2}}{q_{n-1}x_n + q_{n-2}} - \frac{p_n k + p_{n-1}}{q_n k + q_{n-1}} \right|$$

$$= \left| \frac{(x_{n+1} - k)(p_n q_{n-1} - p_{n-1} q_n)}{(q_{n-1}x_n + q_{n-2})(q_n k + q_{n-1})} \right|$$

$$= \frac{(x_{n+1} - k)}{(q_{n-1}x_n + q_{n-2})(q_n k + q_{n-1})} \ (3) \ ; \ \textit{by using (1)}$$

Remark : $s_{a_{n+1}} = R_{n+1}$

By (3) $|x - s_k|$ decrease as k increase

so we must choose the biggest value for k : $K = \dfrac{10^{12} - q_{n-1}}{q_n}$

By (2)(3) $|x - s_K| < \left| x - \dfrac{p_n}{q_n} \right| \Leftrightarrow \dfrac{(x_{n+1} - K)}{(q_n K + q_{n-1})} < \dfrac{1}{q_n}$

$\Leftrightarrow x_{n+1} - 2\ K < \dfrac{q_{n-1}}{q_n}$ (4)

as $\lfloor x_{n+1} \rfloor = a_{n+1}$ and $\dfrac{q_{n-1}}{q_n} < 1$ we have

- for $2\ k < a_{n+1}$: $\dfrac{p_n}{q_n}$ better than s_K

- for $2\ k > a_{n+1}$: s_K better than $\dfrac{p_n}{q_n}$

For $a_{n+1} == 2\ K$ two solutions:

Solution 1 : direct comparison between $\dfrac{p_n}{q_n}$ and s_K

a) If $\dfrac{p_n}{q_n}, s_K$ are on the same side of $x = \sqrt{N}$, s_K is better $\Leftrightarrow$

$s_K - \dfrac{p_n}{q_n}$ and $\sqrt{N} - s_K$ have the same sign

for example if $\dfrac{p_n}{q_n} < \sqrt{N}$ *and* $s_K < \sqrt{N}$, s_K *is better if* $s_K > \dfrac{p_n}{q_n}$

b) If $\dfrac{p_n}{q_n}$ are on opposite side of $x = \sqrt{N}$

we choose by comparison between $\dfrac{(\frac{p_n}{q_n} + s_K)}{2}$ and $\sqrt{N}$

The comparison is done by evaluation of $num^2 - N \times den^2$

the only part of code which needs more than 64 bits precision

Solution 2 : $x_{n+1} - 2\ K = x_{n+1} - a_{n+1} = \dfrac{1}{x_{n+2}}$

Using (4) : computation of x_{n+2} and comparison to $\dfrac{q_n}{q_{n-1}}$

Comparison to $\dfrac{q_n}{q_{n-1}}$ using the alternate convergence of $x_{n+2} = [a_{n+2}; a_{n+3}, a_{n+4}, a_{n+4}, ...]$