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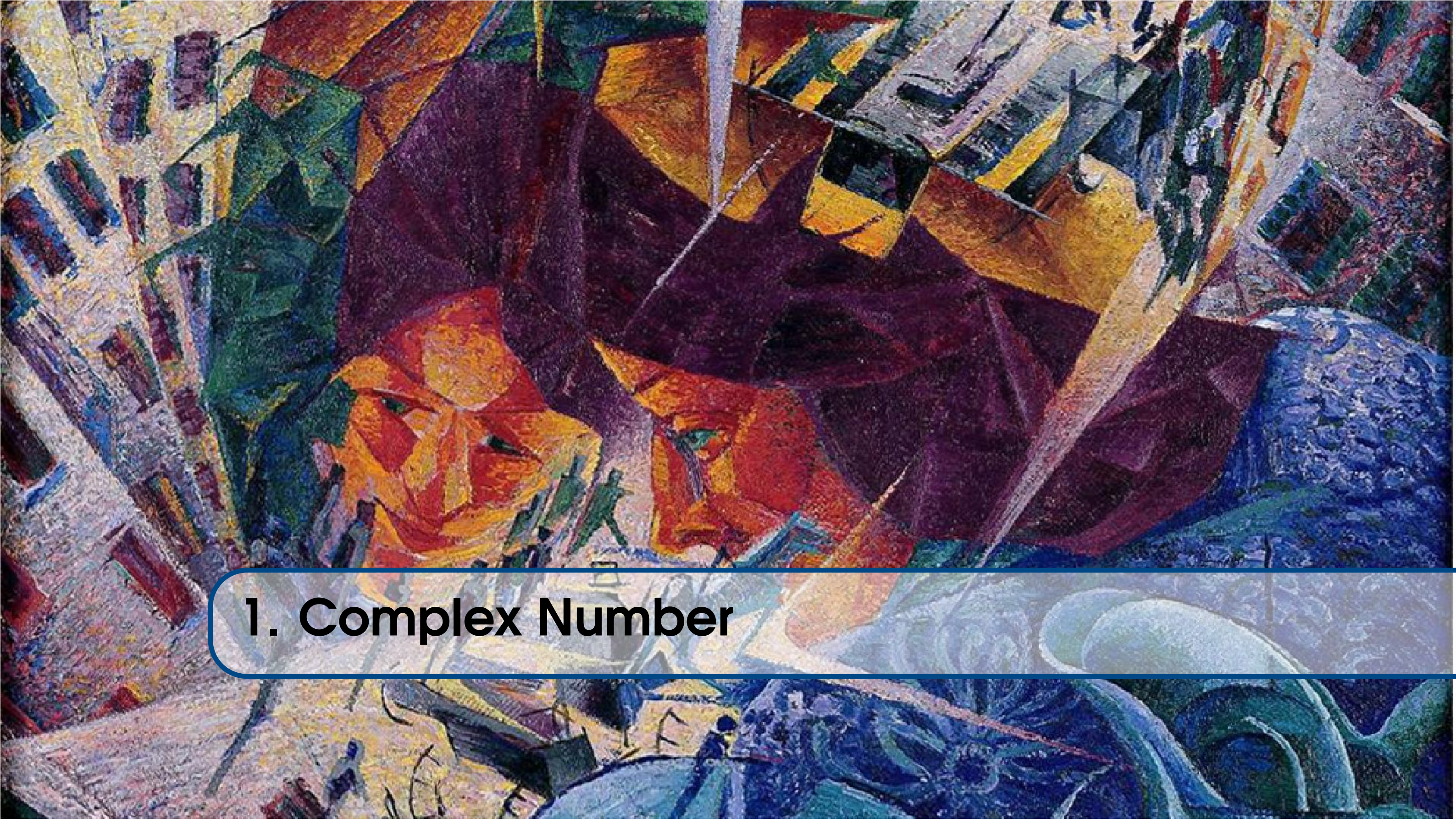
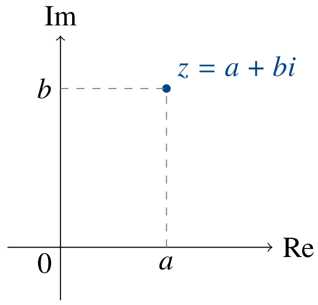


1. Complex Number

1.1 Definition of Complex Number

Mathematicians define $i := \sqrt{-1}$ which is completely different from real number so that i add one more dimension on $\mathbb{R}$. Note that $\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}$ which $\mathbb{R} \subset \mathbb{C}$ and we can consider $a + bi = (a, b)$ where $a + bi$ is a coordinate in $\mathbb{R}^2$ plane because a and i are mutually independent. Also, suppose we change i with variable $y \in \mathbb{R}$, we can also consider $a + by$ a coordinate in a $\mathbb{R}^2$ plane.



Sometime the variable is not that important, the corresponding coefficient which shows the coordinate is more important, however, the definition of i allows us to carry on intuitive multiplication.

Definition 1.1.1 — Complex Numbers under Addition and Multiplication. For some $a_j, b_j \in \mathbb{R}$, $\mathbf{v_j} \in \mathbb{C}$ and $\mathbf{v_j} = a_j + b_j i$, the addition of complex numbers is defined as:

$$\mathbf{v_n} + \mathbf{v_m} := (a_n + a_m) + (b_n + b_m)i$$

Multiplication is defined as:

$$\mathbf{v_n} \cdot \mathbf{v_m} = (a_n a_m - b_n b_m) + (a_n b_m + a_m b_n)i$$

The multiplicative identity element, i.e. $z \oplus e = z$, $\forall z, e \in E$, where e is the $\oplus$ identity element in E, is defined as:

$$\frac{z}{z} = 1 \in \mathbb{R}, \quad \forall z \in \mathbb{C}^{\times}$$

Note that the imaginary part is a real number, and complex numbers are closed under addition and multiplication. For example: $\sum_j \mathbf{v_j} = \overbrace{\sum_j a_j}^{\text{Real Part}} + \overbrace{(\sum_j b_j)}^{\text{Imaginary Part}} \; i \in \mathbb{C}$

Example 1.1 — Sets that are closed under addition. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Example 1.2 — Sets that are not closed under addition. $\mathbb{R} - \mathbb{Q}$

Exercise 1.1 Verify that $\mathbb{Q}$ is closed under addition and multiplication but $\mathbb{R} - \mathbb{Q}$ is not for both of them.

1.2 Fundamental Tools for Complex number

In DSE, we are required to show our ability on solving some arithmetic problems instead of showing understanding on complex number, So, its queit enough to just know the operations below with respect to solving DSE type questions.

Theorem 1.2.1 — Useful Arithmetic. For any $a \neq 0$ or $b \neq 0$, we have the followings:
Square difference:

$$(a + bi)(a - bi) = a^2 + b^2 \in \mathbb{R}$$

Rationalization (expressed fraction in the form of $x + yi$):

$$\frac{1}{a + bi} = \frac{1}{a + bi}\frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \in \mathbb{C}$$

Theorem 1.2.2 — Periodicity. $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i\ldots$

Exercise 1.2 Solve the followings:

(a) Is 123 divisible by 4? If not, what is the remainder?

(b) Use (a), calculate i^{123}.

1.3 Past Paper Region

Exercise 1.3 Let α be a real number. Define $u = w + \frac{1}{w}$, $v = w - \frac{1}{w}$, where $w = \frac{\alpha + i}{\alpha - i}$. Which of the following must be true?

I u is a real number.

II The real part of v is equal to 0.

III The imaginary part of w is equal to the imaginary part of $2w$.

Exercise 1.4 Define $z_1 = \dfrac{2+ki}{1+i}$ and $z_2 = \dfrac{k+5i}{2-i}$, where $k \in \mathbb{R}$. If the imaginary part of z_1 is equal to the imaginary part of z_2, prove $z_1 - z_2 = 3$.

Exercise 1.5 If $a \in \mathbb{R}$, find the real part of $\dfrac{4+i^5}{a+i} - i^6$.

Exercise 1.6 Find the real part of $\dfrac{2i^{12} + 3i^{13} + 4i^{14} + 5i^{15} + 6i^{16}}{1 - i}$.

Exercise 1.7 If k and $\dfrac{5}{2 - i} + ki$ are real numbers, find k.

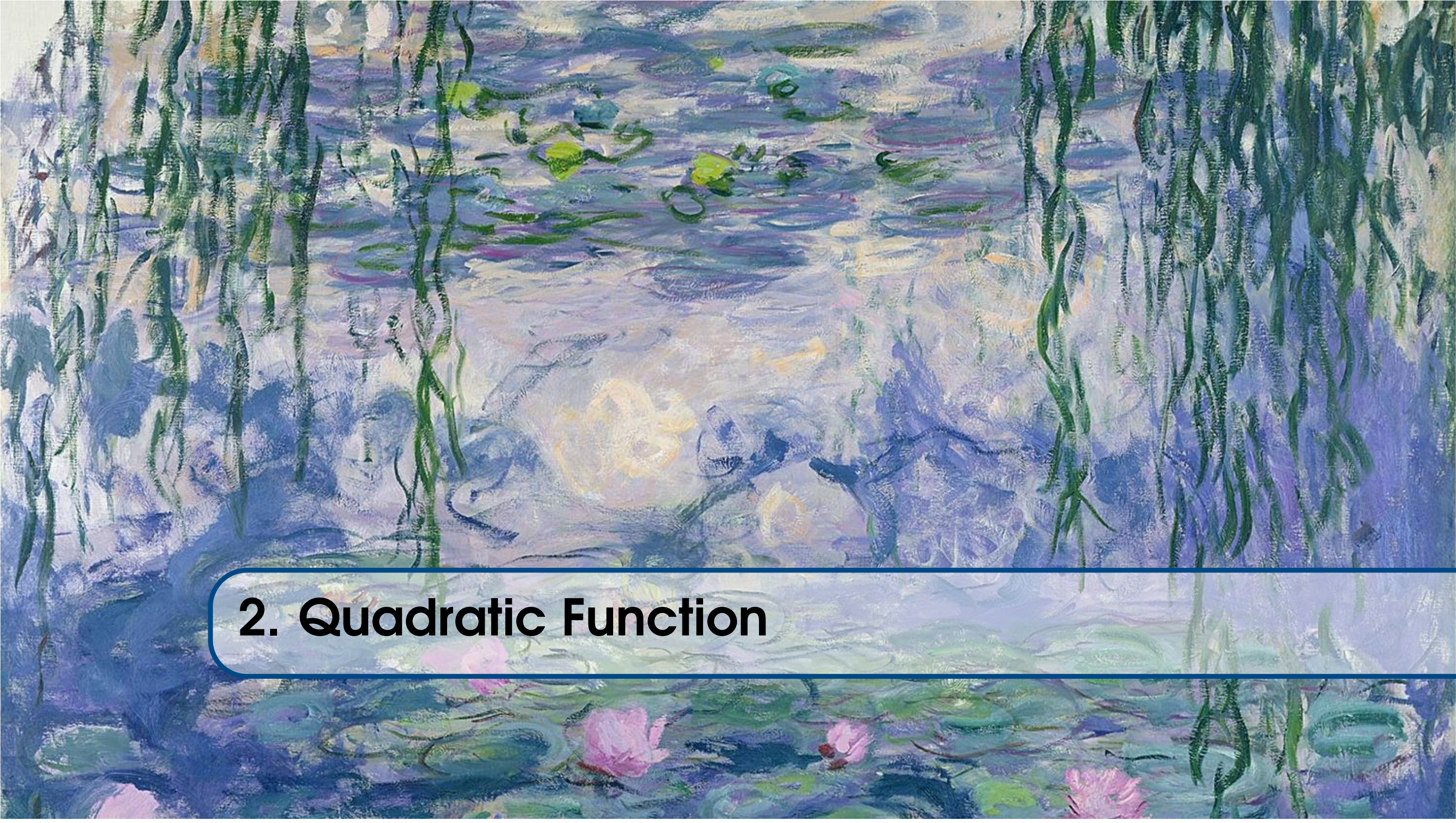
Exercise 1.8 Let $z = (a + 5)i^6 + (a - 3)i^7$, where $a, z \in \mathbb{R}$. Find a.

Exercise 1.9 Let $u = \dfrac{7}{a+i}$ and $v = \dfrac{7}{a-i}$, where a is a real number. Which of the following must be true?

I $u \cdot v$ is a rational number.

II The real part of u is equal to the real part of v.

III The imaginary part of $\dfrac{1}{u}$ is equal to the imaginary part of $\dfrac{1}{v}$.



2. Quadratic Function

2.1 Vertex

Definition 2.1.1 — Quadratic Function. For some real constant $a \neq 0, b, c$, a quadratic function f is defined as follows:

$$f(x) = ax^2 + bx + c$$

Theorem 2.1.1 — Vertex Form. For any quadratic function $f(x) = ax^2 + bx + c$ with real constant $a \neq 0, b, c$, there exists $h, k \in \mathbb{R}$ so that we have the following vertex form:

$$f(x) = a(x - h)^2 + k$$

where a is the coefficient of x^2 and (h, k) are the extrema.

Proof. We can derive the vertex form by completing the square:

$$\begin{aligned} f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \end{aligned}$$

Since $(x + \frac{b}{2a})^2 \geq 0$, we have either $f(x) \geq -a(\frac{b}{2a})^2 + c$ or $f(x) \leq -a(\frac{b}{2a})^2 + c$ which depends on the sign of a and f attain its extreme value when $x = -\frac{b}{2a}$. ■

You may find the expression so complicated, so instead of strictly remembering the form, you should use the key idea-completing square to develop it.

Exercise 2.1 Express $f(x) = 100x^2 + 2x + 1$ in vertex form.

Exercise 2.2 Solve the following questions:

(a) Solve $x^4 + 3x^2 + 1 = 0$.

(b) Find the minimum of $x^4 + 3x^2 + 1 = y$.

Exercise 2.3 Make use of $a^2 - b^2 = (a - b)(a + b)$, prove that for any x in $\mathbb{R}$,

$$x^2 > (x + 1)(x - 1) > (x + 2)(x - 2) > \dots$$

2.2 Sketch the Quadratic Curve

2.2.1 Increasing/Decreasing Function

Definition 2.2.1 — Increasing/Decreasing Function. For any $x_1, x_2 \in I$,

- Increasing Function on I : If $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- Decreasing Function on I : If $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Corollary 2.2.1 For $a > 0$, a quadratic function increase on $[h, \infty)$ where (h, k) is its extrema and it decrease on $(-\infty, h]$.

Proof.
- Express a quadratic function f in vertex form where $a > 0$: $f(x) = a(x-h)^2 + k$.
- If $x_2 > x_1 > h$, then $(x_2 - h) > (x_1 - h)^2$.
- We have $f(x_2) - f(x_1) = a\underbrace{\left[(x_2-h)^2 - (x_1-h)^2\right]}_{>0}$, thus the sign of f depends on a.

■

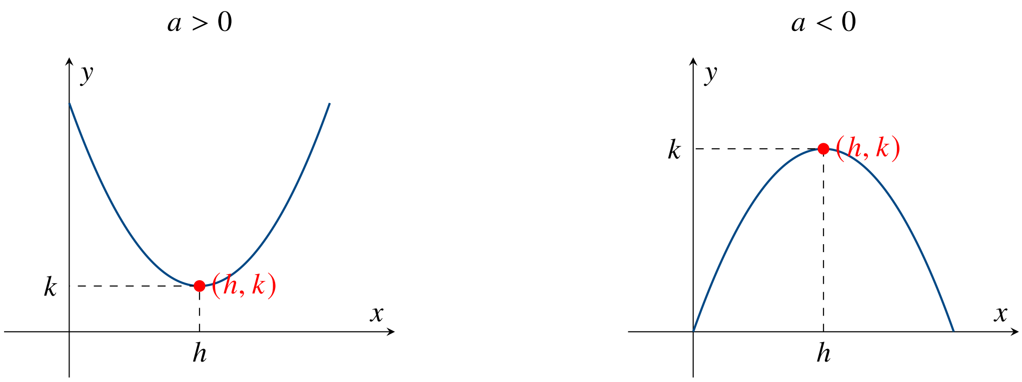
Note that we can draw a similar conclusion for $a < 0$. Note that $a(x-h)^2$ is the dominant term in a quadratic function f, so when x goes farther away from h, the speed of increasing/decreasing accelerates for both $a > 0$ and $a < 0$.

2.2.2 Axis of Symmetry

When you draw a graph of a quadratic function f, it is intuitive that when $x = h$ (axis of symmetry). For the rigorous proof of it, we prove the corollary:

Corollary 2.2.2 If (h, k) is the extrema of a quadratic function $f(x)$, we have $f(h+x) = f(h-x)$ for any $x \in \mathbb{R}$.

Proof. Consider $f(x) = a(x-h)^2 + k$, for any $x \in \mathbb{R}$, $f(h+x) = ax^2 + k = f(h-x)$. ■



2.3 Roots

2.3.1 Determinant Δ and Root

Definition 2.3.1 — Root. A value p is a root/zero for function $f(x) = y$ if $f(p) = 0$.

Using the vertex form, when $f(x) = 0$, we can find the discriminant Δ and the roots of f (This is left as an exercise). Some questions may tell you to find the root of the function

$$f(x) = ax^2 + bx + c$$

or solve the quadratic equation

$$ax^2 + bx + c = 0$$

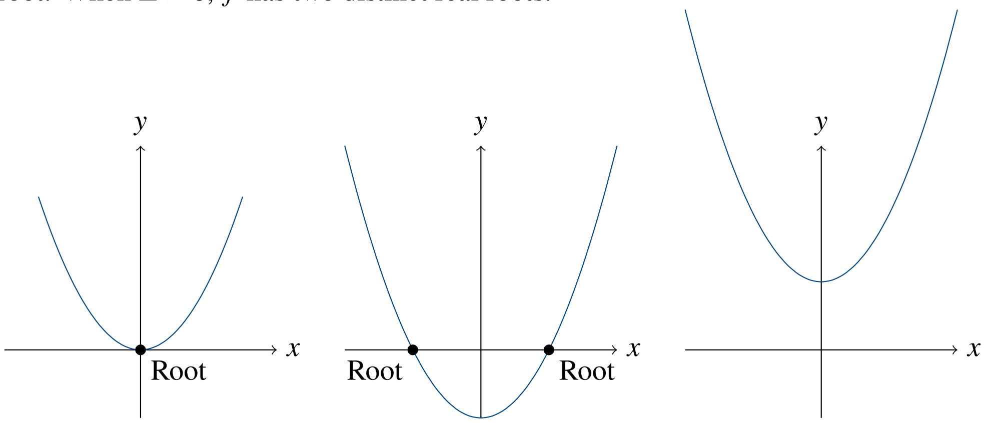
These two questions are in fact equivalent.

Theorem 2.3.1 For a quadratic function $f(x) = ax^2 + bx + c$ with some real constants a, b, c, suppose α, β are the roots of f, we have

$$\alpha = \frac{-b + \sqrt{\Delta}}{2a}, \quad \beta = \frac{-b - \sqrt{\Delta}}{2a}$$

where $\Delta = b^2 - 4ac$.

When $\Delta < 0$, $\sqrt{\Delta}$ would generate two complex number which brings f two complex roots (no intersection with real x axis). When $\Delta = 0$, $\sqrt{\Delta} = 0$ and so $\alpha = \beta$ which means that f has only one real root. When $\Delta > 0$, f has two distinct real roots.



Case 1: $\Delta = 0$
One real root (repeated)

Case 2: $\Delta > 0$
Two distinct real roots

Case 3: $\Delta < 0$
No real roots (complex)

Exercise 2.4 Prove the above expression of the root of a general quadratic function.

Corollary 2.3.2 — Root Form. Let α, β bethe roots of a quadratic function $f(x) = ax^2 + bx + c$, we have

$$f(x) = a(x - \alpha)(x - \beta)$$

Proof. Note that,

$$\begin{aligned} f(x) &= a\left(x - \frac{b^2 - \sqrt{\Delta}}{2a}\right)\left(x - \frac{b^2 - \sqrt{\Delta}}{2a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2 - (b^2 - 4ac)}{4a^2}\right) \\ &= ax^2 + bx + c \end{aligned}$$

■

Exercise 2.5 Construct a quadratic function that has $x = 1, 2$ as its roots and passes through $(3, -4)$.

2.3.2 Intersections of Quadratic Function and Constant Function

The meaning of $ax^2 + bx + c = k$ could be written as the following:

$$\begin{cases} ax^2 + bx + c = y \\ k = y \end{cases}$$

So, it's basically the intersections of a quadratic function and any constant k in $\mathbb{R}$. To actually find the intersections, you may need to have a curve sketching in your mind.

Exercise 2.6 Sketch the curve of $3x^2 + 5x + 2 = y$ and $y = 0$ and find the intersection of them.

Exercise 2.7 Sketch the curve of the function

$$g(x) = \begin{cases} 3x^2 + 5x + 2 = y \\ 17 = y \end{cases}$$

and find the intersection of them.

2.3.3 Intersections of Two Functions

When finding the intersections of two functions, we rely heavily on curve sketching. Consider any function $f(x)$ and $g(x)$ and the following:

$$\begin{cases} f(x) = y \\ g(x) = y \end{cases}$$

It is hard to find the required intersections using this simultaneous equation. However, by considering $f(x) = g(x)$, we have $f(x) - g(x) = 0$, we write:

$$\begin{cases} f(x) - g(x) = y \\ 0 = y \end{cases}$$

where $f(x) - g(x)$ is the vertical difference of $f(x)$ and $g(x)$. (Again, by sketching, you can obtain the geometric meaning of vertical difference)

Exercise 2.8 When the vertical difference of $f(x)$, $g(x)$ equals to zero, what kind of point do we get?

Exercise 2.9 Solve the following question:

(a) Find the intersecting points of the two functions $f(x) = 5x^2 + 3x + 3$, $l(x) = 3x + 10$.

(b) Find the intersecting points of the two functions $f(x) = kx^2 + 3x + 3$, $l(x) = 3x + 10$.

2.3.4 Use Set to Understand the Simultaneous Equation of Two Functions

In elementary school, we know any curve or line is a set of infinitely many points. For any function f on the $\mathbb{R}$-axis, f is the set of all possible points $(x, f(x))$ where $(x, f(x)) \neq (f(x), x)$. So we can consider:

$$\begin{cases} f(x) = y \\ g(x) = y \end{cases} \sim \begin{cases} \text{All possible points of } (x, f(x)) \\ \text{All possible points of } (x, g(x)) \end{cases}$$
$$\sim \{\text{All possible points of } (x, f(x))\} \bigcap \{\text{All possible points of } (x, g(x))\}$$

Exercise 2.10 For any $x \in \mathbb{R}$, is the set of intersections of $\sin x = y$ and $1 = y$ finite?

2.4 Vieta's Formula

Using the formula for finding roots, we can easily prove Vieta's formula.

Definition 2.4.1 — Vieta's Formula. Let α, β be two roots (intersections of f and the x-axis) of a quadratic function f,

$$f(x) = a(x^2 - (\alpha + \beta)x + \alpha\beta)$$

where a is the coefficient of x^2.

Exercise 2.11 Prove Vieta's formula.

Exercise 2.12 Let α, β be two roots of f. Find $\alpha + \beta$ and $\alpha\beta$ of $f(x) = 7x^2 + 5x + 3$.

Exercise 2.13 Let $f(x) = x^2 - (\alpha + \beta)x + \alpha\beta$. Define $\alpha := m + ni, \beta := p + qi$. When $f(x)$ has zeros on real axis,

(a) Is α, β the root of $f(x)$?

(b) Find n and q.

Exercise 2.14 The equation of parabola Γ is $y = 2x^2 - 2kx + 2x - 3k + 8$, where k is a real constant. Denote the straight line $y = 19$ by L.

(a) Prove that L and Γ intersect at two distinct points.

(b) The points of intersection of L and Γ are A and B.

(i) Let a and b be the x-coordinates of A and B respectively. Prove that

$$(a - b)^2 = k^2 + 4k + 23$$

(ii) Is it possible that the distance between A and B is less than 4? Explain.

Exercise 2.15 Let $g(x) = 3x^2 + 12kx + 16k^2 + 8$, where k is a non-zero real constant.

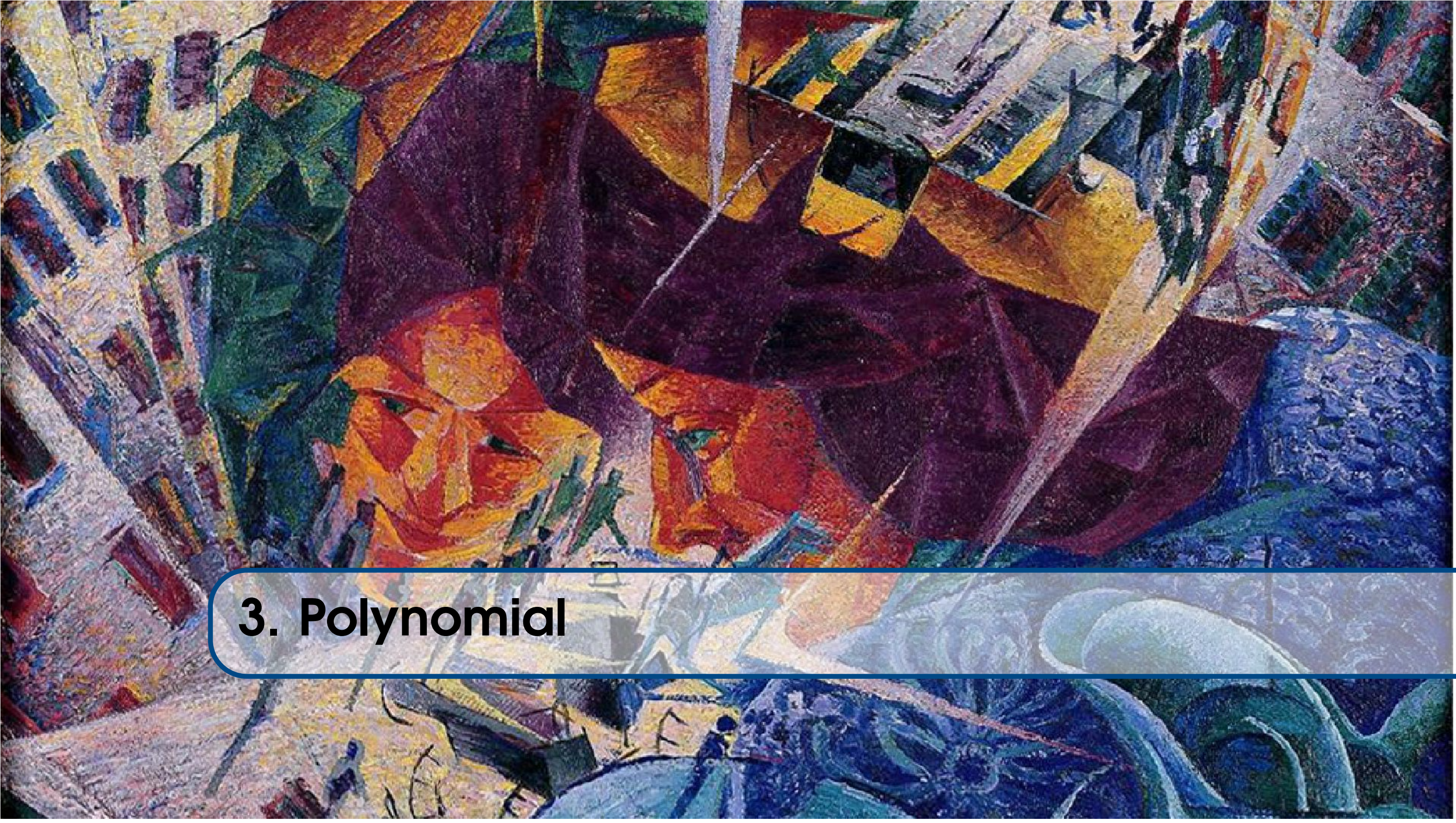
(a) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y = g(x)$.

(b) On the same rectangular coordinate system, denote the vertex of the graph of $y = g(x)$ and the vertex of the graph of $y = 2g(-x)$ by A and B respectively. Let M be a point lying on AB such that the area of ΔOBM is three times the area of ΔOAM, where O is the origin. Express, in terms of k, the coordinates of M.

Exercise 2.16 Define

$$|x| := \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

solve $x^2 + |x| + 3=0$.



3. Polynomial

The DSE curriculum has recently emphasized the importance of dealing with abstract polynomials. However, most textbooks do not provide the original definition of a polynomial and a relevant exercise for it. This chapter would give readers a clearer flow of doing polynomials.

3.1 Integer

We always talk about divisibility in polynomials which gives the concept of divisor and remainder, recalling that in elementary school, we also have these terms for integer division, the proof for many theorems and algorithm in integer and polynomial are actually identical. For example, remainder theorem , long (short) division and Euclidean algorithm (this is out of the syllabus).

Theorem 3.1.1 — Remainder Theorem for Integer. Let $a, b \in \mathbb{Z}$ such that $b \neq 0$. Then, there are unique integers q and r such that

$$a = bq + r \qquad 0 \leq r < |b|$$

Proof. For uniqueness, suppose there is $q', r' \in \mathbb{Z}$ so that $qb+r = q'b+r'$, we have $(q'-q)b = r'-r$ which implies that $|q'-q||b| = |r'-r|$. Suppose $q' \neq q$, we have $|q'-q| \geq 1$ and so we get

$$\max\{|r|, |r'|\} \leq |b| < |r-r|$$

which leads to contradiction. Then, we have $q = q'$ and so $r = r'$.

For existence, if $b > 0$, there exists $q \in \mathbb{Z}$ so that $q \leq \frac{a}{b} < q+1$ which implies $qb \leq a < qb+b$. Set $r = a - qb$, we have $a = qb + r$ with $0 \leq r < |b|$ by considering $a - qb < a < b$. If $b < 0$, we consider a integer q so that $q - 1 \leq \frac{a}{b} < q$ which implies that $qb \leq a < qb - b$. Take $r = a - qb$, we have $a = r + qb$ with $0 < r = a - qb < -b = |b|$. ■

The proof may sound abstract but the key idea of the theorem is intuitive. Consider a classical clock which the hour hand point the hour $\{1, 2, ..., 12\}$, when the hour exceed "12", it ends a cycle and will reset to the begining. This explain why $0 \leq r < |b|$ (reset operation) and there is a remainder r (hour).

3.2 Foundation of Polynomial

Definition 3.2.1 — Polynomial. For some complex constant a_n and any x in $\mathbb{C}$, we say

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

is a polynomial with degree n where a_k is the coefficient of x^k and $a_n \neq 0$.

Note that we consider a complex polynomial here to introduce the theorem below, in most cases, you just need to consider a real polynomial.

Theorem 3.2.1 — Fundamental Theorem of Algebra. Any polynomial in $\mathbb{C}$ with degree n has exactly n complex roots. This means for $\deg P \geq 1$, we have

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = a_n(x - b_1)(x - b_2)\cdots(x - b_n)$$

where $a_i \in \mathbb{C}$ is the coefficient of x^i and $P(b_1) = P(b_2) = \ldots = P(b_n) = 0$.

Note that when $b_m = b_p$ where m, p are some natural numbers, we say they are repeated roots where we still count them as two roots.

Exercise 3.1 — Zeros. How do we call the points $(b_1, P(b_1)), (b_2, P(b_2)), \ldots, (b_n, P(b_n))$ in the above theorem?

R For a polynomial f with degree n, if we have its n roots and one point which is distinct from the roots, then we can construct f a polynomial expression without solving the coefficient for each x_i

Exercise 3.2 — Constructing Polynomial. Solve the following questions:

(a) For real constant a, b, c, d, e, f, suppose $a + bi$ and $c + di$ are the zeros and (e, f) is a different passing point of a quadratic function p, write the analytic expression of p.

(b) For any $j \in \{1, ..., n\}$ and real constant a_j, b_j, e, f, suppose $a_j + b_j i$ are the zeros and (e, f) is a different passing point of a polynomial p where $\deg p = n$, write the analytic expression of p.

3.3 Generalized Remainder Theorem

Theorem 3.3.1 — Division Algorithm. For any polynomial p, $d \neq 0$, there exists two unqiue polynomials p and q such that

$$p = qd + r \quad \text{where } \deg r < \deg d$$

When $r = 0$, we say p is divisible by d.

Proof. For uniqueness, suppose there exists polynomial q, r, q', r' such that $(q - q')d = r' - r$. If $q - q' \neq 0$, we have $\deg(q - q') \geq$ and so

$$\deg d \leq \deg d + \deg(q - q') = \deg d(q - q') = \deg(r' - r) \leq \max\{\deg r' . \deg r\}$$

This leads to contradiction and we have $q - q' = r - r' = 0$.

For existence, if $p = 0$, we just take $q = r = 0$. If $\deg p \geq 0$, by the remainder theorem on $\mathbb{Z}$, it still hold. Then, assume that the required statement is true for some $n \in \mathbb{N}$, let $p(x) = a_0 + a_1x + \ldots + a_nx^{n+1}$ and $d(x) = b_0 + b_1x + \ldots + b_mx^m$ where $b_m, a_m \neq 0$ and $n + 1 \geq m$ (If $n + 1 < m$, we just take $q = 0, r = p$). Then, by letting $h(x) = p(x) - \frac{a_n}{b_m}x^{n+1-m}d(x)$ which allows us to vanish the leading term in p and our assumption,there exists polynomial q_1, r_1 so that

$$p(x) - \frac{a_n}{b_m}x^{n+1-m}d(x) = h(x) = q_1(x)d(x) + r_1(x)$$

Rearranging

$$p(x) = q_1(x)d(x) + r_1(x) + \frac{a_n}{b_m}x^{n+1-m}d(x)$$

By applying our assumption to $\frac{a_n}{b_m}x^{n+1-m}d(x)$, there exists polynomial q_2, r_2 so that

$$p = q_1d + r + q_2d + r_2 = (q_1 + q_2)d + (r_1 + r_2)$$

By induction, the required statement is true for any $n \in \mathbb{N}$. ■

Here, recall the metaphor in the section of integer, you may consider the polynomial d is a cycle and is $\deg r$ try to exceed $\deg d$, r will be written as $r = q_2d + r_2$ (reset to the begining). Also, theorem 3.1.1 is in fact a special case of theorem 3.3.1 which the degree of polynomial is zero.

Corollary 3.3.2 — Remainder Theorem. For polynomial p, q, d, r, suppose we that

$$p(x) = q(x)d(x) + r(x) \quad \text{where } \deg r < \deg d$$

If $x = a$ is the zero of $d(x)$, then we have $p(a) = r(a)$.

Exercise 3.3 — Concept. For some polynomial $p(x)$, if $p(x)$ is divisible by $x^2 - 3x + 2$, find two roots of $p(x)$.

Exercise 3.4 — Lagrange Interpolation. There is a typical question in kindergarten: $a_1 = 1, a_2 = 2, a_3 = 3, a_4 =?$. It's obvious that $a_4 = 100$ as considering

$$P(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 2\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}$$

$$+3\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 100\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

We have for $k = 1, 2, 3, 4$, $a_k = P(k)$.

(a) Verify that for $k = 1, 2, 3, 4$, $a_k = P(k)$.

(b) Is $P(x)$ a polynomial?

(c) How many roots does $P(x)$ have?

R By Lagrange interpolation, if we have (a_i, b_i) for $i \in \{1, ..., n\}$, we can construct a polynomial that passes through all the points (a_i, b_i).

3.4 Past Paper Region

Theorem 3.4.1 — Identity in Polynomial. Consider $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $Q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$, we have for $x \in \mathbb{R}$,

$$a_k = b_k \text{ for } k = 0, 1, \ldots, n \iff P(x) = Q(x)$$

There is a trick in doing comparing coefficients: you can find the corresponding coefficients by summation. For example, for finding $(x + 3)^4 + (x + 3)^2$, we first go to find the coefficients of $x^4, x^3, \ldots$ respectively.

Exercise 3.5 Let $P(x) = (3x + 3)^{100} + (x + 3)^{50} + x^{30}$.

(a) What is the degree of $P(x)$?

(b) Find the coefficient of the term with the largest degree.

(c) Suppose $Q(x) = k_{100}x^{100} + k_{99}x^{99} + \ldots + k_0$ and $Q(x) = P(x)$ for all x in $\mathbb{R}$. Find k_{100} and k_0.

Exercise 3.6 — Comparing Coefficient. Let $P(x) = 6x^4 + 7x^3 + ax^2 + bx + c$, where a, b, c are constants. When $P(x)$ is divided by $x + 2$ and $x - 2$, the remainders are equal. It is given that $P(x) = (lx^2 + 5x + 8)(2x^2 + mx + n)$ where l, m, n are constants.

(a) Find l, m, n.

(b) How many real roots does $P(x) = 0$ have?

Exercise 3.7 — Abstract Polynomial. The polynomial $p(x)$ is divisible by $x-5$. When $p(x)$ is divided by x^2+x+1, the quotient and the remainder are $2x^2-37$ and $cx+c-1$ respectively, where c is a constant.

(a) Find c.

(b) Prove that $x+3$ is a factor of $p(x)$.

(c) Someone claims that all the roots of the equation $p(x)=0$ are real numbers. Is the claim correct? Explain your answer.

Exercise 3.8 — Curve Sketching. Answer the following questions:

(a) Find the value of k such that $x - 2$ is a factor of $kx^3 - 21x^2 + 24x - 4$.

(b) Let $f(x) = 15x^2 - 63x + 72$. Q is a variable point on the curve f in the first quadrant. P and R are the feets of perpendiculars from Q to the x-axis and the y-axis respectively.

(i) Let $(m, 0)$ be the coordinate of P. Express the area of the rectangle $OPQR$ in terms of m.

(ii) Are there three different positions of Q such that the area of the rectangle $OPQR$ is 12? Explain your answer.

Exercise 3.9 Let $f(x) = 6x^3 - 13x^2 - 46x + 34$. When $f(x)$ is divided by $2x^2 + ax + 4$, the quotient and the remainder are $3x + 7$ and $bx + c$ respectively, where a, b are constants.

(a) Find a.

(b) Let $g(x)$ be a quadratic polynomial such that $g(x)$ is divided by $2x^2 + ax + 4$, the remainder is $bx + c$.

(i) Prove that $f(x) - g(x)$ is divisible by $2x^2 + ax + 4$.

(ii) Someone claims that all the roots of the equation $f(x) - g(x) = 0$ are integers. Do you agree? Explain your answer.

Exercise 3.10 — Cubic Polynomial. Let $p(x)$ be a cubic polynomial. When $p(x)$ is divided by $x - 1$, the remainder is 50. When $p(x)$ is divided by $x + 2$, the remainder is -52. It is given that $p(x)$ is divisible by $2x^2 + 9x + 14$.

(a) Find the quotient when $p(x)$ is divided by $2x^2 + 9x + 14$.

(b) How many rational roots does the equation $p(x) = 0$ have? Explain your answer.

Exercise 3.11 — Trap. Let $f(x) = 4x(x+1)^2 + ax + b$, where a, b are constants. It is given that $x - 3$ is a factor of $f(x)$. When $f(x)$ is divided by $x + 2$, the remainder is $2b + 165$.

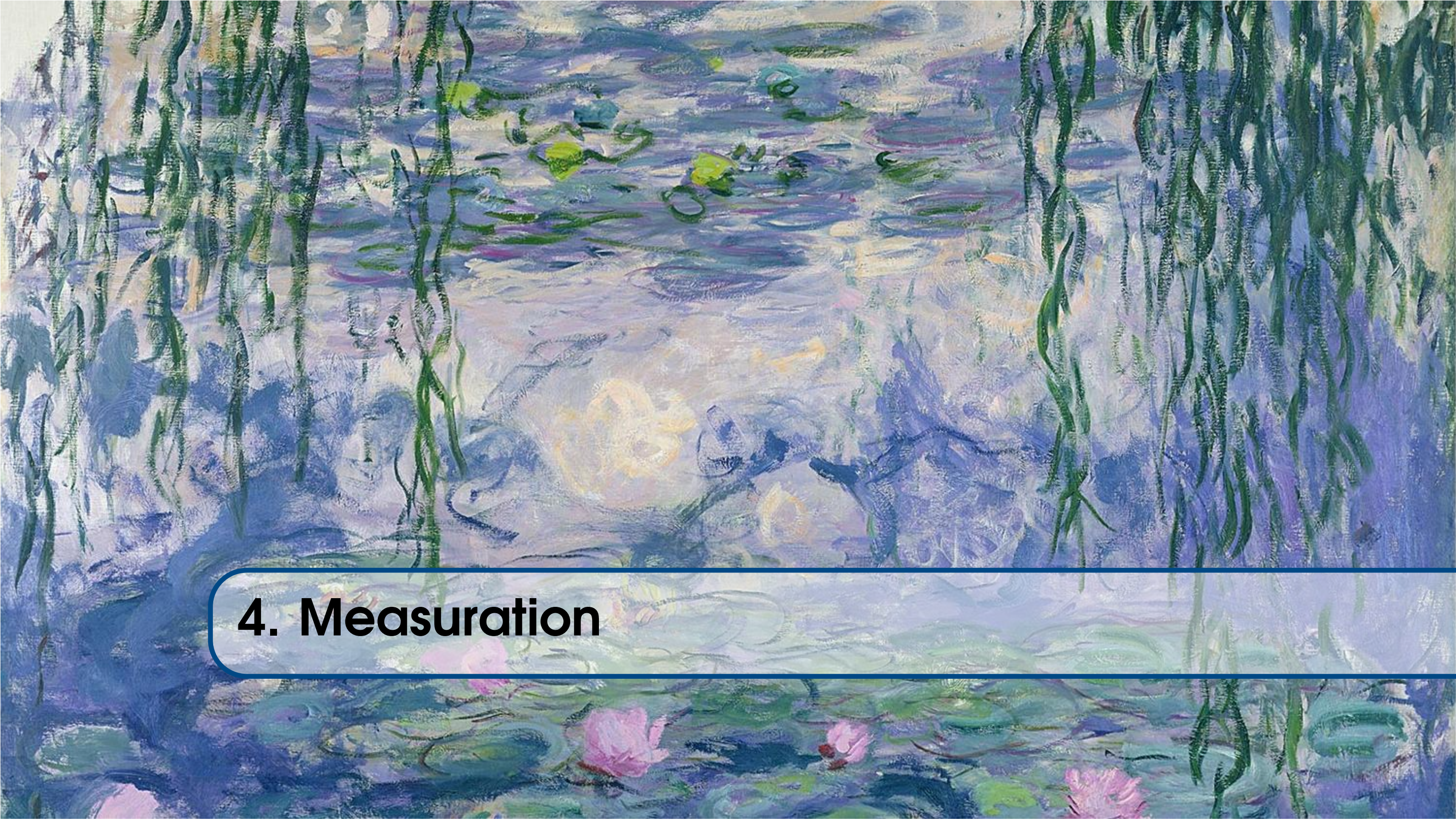
(a) Find a and b.

(b) Someone claims that the equation $f(x) = 0$ has at least one irrational root. Do you agree? Explain your answer.

Exercise 3.12 — Multiple Choice. Let $p(x)$ be a polynomial. When $p(x)$ is divided by $x + 1$, the remainder is -2. If $p(x)$ is divided by $x - 1$, find the linear remainder when $p(x)$ is divided by $x^2 - 1$.

Exercise 3.13 — Multiple Choice. Let $g(x) = ax^3 + 4ax^2 - 24$, where a is a constant. If $x + 2$ is a factor of $g(x)$, find $g(2)$.

Exercise 3.14 — Multiple Choice. Let k be a constant such that $2x^4 + kx^3 - 4x - 16$ is divisible by $2x + k$. Find k.



4. Measuration

4.1 Area and Volume

4.1.1 Measure

This proof of the area and volume of many objects are beyond syllabus, even for the area of circle which is taught in elementary school, it is just told to be πr^2. For area, we can make use of the definition that

Definition 4.1.1 — Area. We define the area:

$$A(E) := \sum_{k=1}^{n} (a_k - b_k)(c_k - d_k)$$

for $E := \coprod_{k=1}^{n} \langle a_k, b_k \rangle \times \langle c_k, d_k \rangle$ where E is a set of all points bounded in a disjointed union of rectangles and $A(E)$ is an area of the required rectangles.

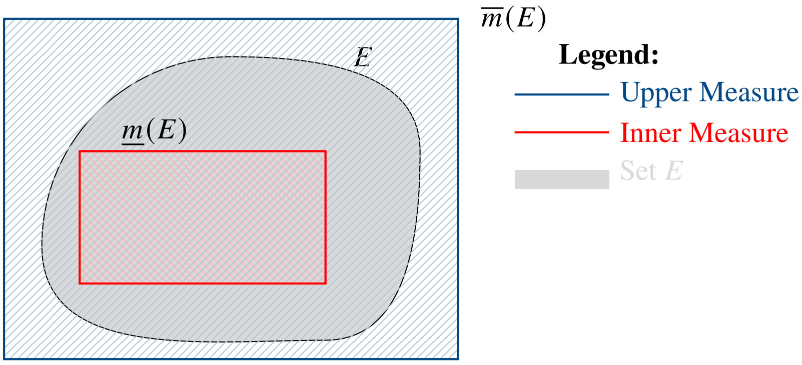
Then, we use the union of rectangles E to enclose an arbitrary shape S and use S to enclose E^* so as to approximate the required area $A(S)$. We define a set of all unions of rectangles to be simple region.

Definition 4.1.2 — Jordan Measure in $\mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be a non-empty bounded set. We define its inner Jordan measure $\mu_*(\Omega)$ and outer Jordan measure $\mu^*(\Omega)$ to be:

$$\mu_*(\Omega) := \sup\{A(S) : S \subset \Omega \text{ and } S \text{ is a simple region in } \mathbb{R}^2\}$$
$$\mu^*(\Omega) := \inf\{A(S) : T \supset \Omega \text{ and } T \text{ is a simple region in } \mathbb{R}^2\}$$

If $\mu_*(\Omega) = \mu^*(\Omega)$, then we say Ω is Jordan measurable and define its Area $\mu(\Omega)$.

Therefore, we can use many small rectangles to approximate an area by taking limit if the error of approximation eventually goes to zero when the number of rectangles goes to infinity (This is a foundation for calculus). The case for $\mathbb{R}^3$ can define volume by a similar argument.



4.1.2 Surface Area and Volume of Objects

This part focuses on the surface area and volume of objects since area for $\mathbb{R}^2$ objects are taught.

Theorem 4.1.1 — Surface area of Objects. We have the followings:
- Sphere : $4\pi r^2$ where r is the radius
- Cone : $\pi r l + \pi r^2$ where r is the radius of base and l is the slant height

Theorem 4.1.2 — Volume of Objects. We have the followings
- Sphere : $\frac{4}{3}\pi r^2$ where r is the radius
- Cone and Pyramid : $\frac{1}{3}bh$ where b is the base area and h is the height

4.2 Similarity

4.2.1 Angle and Similar Triangle

There are some propositions about finding ratio of heights if two triangle or other objects are similar. However, those are non-necessary to be memorised because we can notice those propositions by drawing graph.

Note that if two similar cones A and B have volume $\frac{1}{3}\pi r_1^2 h_1$ and $\frac{1}{3}\pi r_2^2 h_2$, we have $\tan\theta_1 = \tan\theta_2 = \frac{h_1}{r_1} = \frac{h_2}{h_2}$ where θ_1 is the angle between the slant height and the radius of A and similar for B. Similar argument can be drawed for pyramids and We can use this argument to solve the question beneath.

Theorem 4.2.1 — Ratio. Given A, B as two similar cones or pyramids. Use the common notation above and $S(A)$ and $V(A)$ be the required slant surface area respectively and similar for B. We have

$$h_A^2 : h_B^2 = S(A) : S(B) \quad \text{and} \quad h_A^3 : h_B^3 = V(A) : V(B)$$

Replace h by r, same conclusion can be drawn for sphere.

Exercise 4.1 Prove the above theorem.

4.3 Past Paper Region

Exercise 4.2 The base radius of the solid right circular cylinder X and the base radius of the solid right circular cone Y are equal. The heights of X and Y are 20 cm and 24 cm respectively. The volume of the solid right circular cone Z is equal to the sum of the volume of X and the volume of Y. The base radius of Z is equal to the base diameter of X. A craftsman finds that the volume of Y is 800π cm^3.

(a) Find the base radius of Y.

(b) Are Y and Z similar? Explain your answer.

(c) The craftsman claims that the sum of the curved surface area of X and the curved surface area of Y is greater than the curved surface area of Z. Do you agree? Explain your answer.

Exercise 4.3 The height and the base radius of a solid right circular cone are 36 cm and 15 cm respectively. The circular cone is divided into three parts by two planes which are parallel to its base. The heights of the three parts are equal.

(a) Find the volume of the middle part of the circular cone.

(b) Find he curved surface area of the middle part of the circular cone.

Exercise 4.4 The sum of the volumes of two spheres is 324π cm^3. The radius of the larger sphere is equal to the diameter of the smaller sphere.

(a) Find the volume of the larger sphere.

(b) Find the sum of the surface areas of the two spheres.

Exercise 4.5 A right circular cylindrical container of base radius 8 cm and height 64 cm and an inverted right circular conical vessel of base radius 20 cm and height 60 cm are held vertically. The container is fully filled with water. The water in the container is now poured into the vessel.

(a) Find the volume of water in the vessel in terms of π.

(b) Find the depth of water in the vessel.

(c) If a solid metal sphere of radius 14 cm is then put into the vessel and the sphere is totally immersed in the water, will the water overflow? Explain your answer.

Exercise 4.6 A solid metal right prism of base area 84 cm^2 and height 20 cm is melted and recast into two similar solid right pyramids. The bases of the two pyramids are squares. The ratio of the base area of the smaller pyramid to the base area of the larger pyramid is 4:9.

(a) Find the volume of the larger pyramid.

(b) If the height of the larger pyramid is 12 cm, find the total surface area of the smaller pyramid.

Exercise 4.7 An inverted right circular conical vessel contains some milk. The vessel is held vertically. The depth of milk in the vessel is 12 cm. Peter then pours 444π cm^3 of milk into the vessel without overflowing. He now finds that the depth of milk in the vessel is 16 cm.

(a) Express the final volume of milk in the vessel in terms of π.

(b) Peter claims that the final area of the wet curved surface of the vessel is at least 800 cm^2. Do you agree? Explain your answer.

Exercise 4.8 The radius and the area of a sector are 12 cm and 30π cm^2 respectively.

(a) Find the angle of the sector.

(b) Express the perimeter of the sector in terms of π.

Exercise 4.9 Figure 3 shows a vessel in the form of a frustum which is made by cutting off the lower part of an inverted right circular cone of base radius 72 cm and height 96 cm. The height of the vessel is 60 cm. The vessel is placed on a horizontal table. Some water is now poured into the vessel. John finds that the depth of water in the vessel is 28 cm.

(a) Find the area of the wet curved surface of the vessel in terms of π.

(b) John claims that the volume of water in the vessel is greater than 0.1 m^3. Do you agree? Explain your answer.

Exercise 4.10 In a workshop, 2 identical solid metal right circular cylinders of base radius R cm are melted and recast into 27 smaller identical solid right circular cylinders of base radius r cm and height 10 cm. It is given that the base area of a larger circular cylinder is 9 times that of a smaller one.

(a) Find

(i) $r : R$,

(ii) the height of a larger circular cylinder.

(b) A craftsman claims that a smaller circular cylinder and a larger circular cylinder are similar. Do you agree? Explain your answer.

Exercise 4.11 In Figure 2, the volume of the solid right prism $ABCDEFGH$ is 1020 cm^3. The base $ABCD$ of the prism is a trapezium, where AD is parallel to BC. It is given that $\angle BAD = 90°$, $AB = 12$cm, $BC = 6$cm and $DE = 10$cm.

(a) Find the length of AD,

(b) Find the total surface area of the prism $ABCDEFGH$.

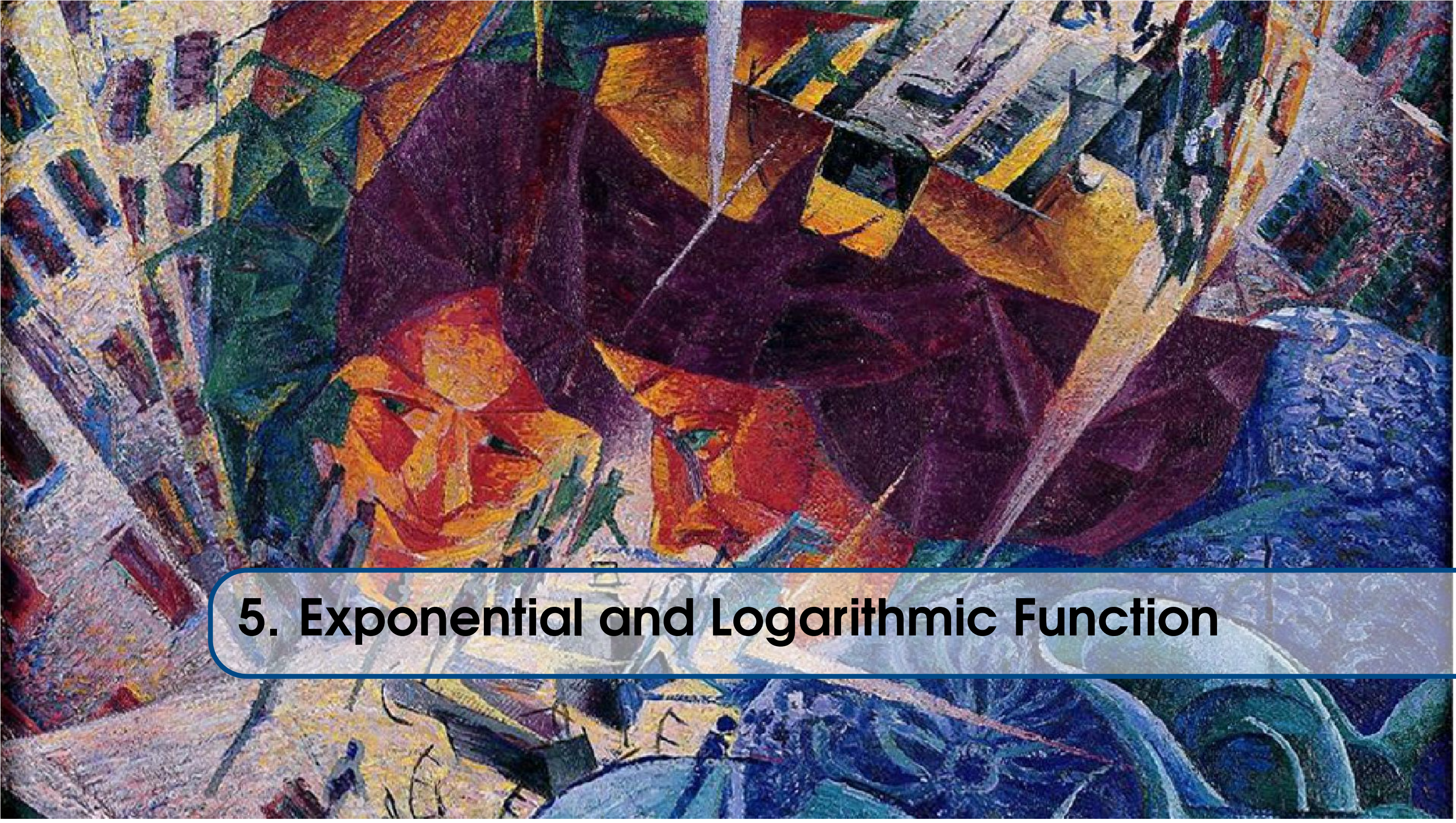
Exercise 4.12 Figure 3(a) shows a solid metal right circular cone of base radius 48 cm and height 96 cm.

(a) Find the volume of the circular cone in terms of π.

(b) A hemispherical vessel of radius 60 cm is held vertically on a horizontal surface. The vessel is fully filled with milk.

(i) Find the volume of the milk in the vessel in terms of π.

(ii) The circular cone is now held vertically in the vessel as shown in Figure 3(b). A craftsman claims that the volume of the milk remaining in the vessel is greater than 0.3 m^3. Do you agree? Explain your answer.



5. Exponential and Logarithmic Function

5.1 Invertibility

Definition 5.1.1 — Function. A map $f|_D$ is said to be a function if for any $x^* \in D$, there exists a unique $y^* \in f(D)$ such that $f(x^*) = y^*$.

Definition 5.1.2 — Invertibility. A function $f|_D$ is said to be invertible if for any $y^* \in f(D)$, there exists a unique $x^* \in D$ such that $f(x^*) = y^*$. Also, the above definition means that if such "f^{-1}" is not a function, then f is not invertible.

The logarithmic function is the inverse function of exponential function i.e. set $f(x) = y$ where f is invertible, we denote the inverse function of f as f^{-1} such that $f^{-1}(y) = x$.

■ **Example 5.1**
- $y = \pm\sqrt{x}$ is not a function.
- $y = x^2$ is a function.
- $y = x^2$ is not invertible.
- The circle equation : $(x-a)^2 + (x-b)^2 = r^2$ is not a function.
- $\sin\theta, \cos\theta, \tan\theta$ is invertible on $(0°, 90°)$ but not on $(0°, 180°)$ (we will learn it later).
- a^x is a function and is invertible.

■

5.2 Arithmetic Rule of Logarithmic Function

Definition 5.2.1 For any $a \in \mathbb{R}_{>0} \backslash \{1\}$, if $0 < y = a^x$ for any $x \in \mathbb{R}$, we define the logarithmic function as:

$$\log_a y = x$$

where a is called the base of the log function.

Exercise 5.1 Solve the followings:

(a) Prove that for any $a \in \mathbb{R}_{>0} \backslash \{1\}$, $\log_a 1 = 0$.

(b) Prove that for any $a \in \mathbb{R}_{>0}\backslash\{1\}$, $\log_a a = 1$.

(c) Explain why base $a \in \mathbb{R}_{>0}\backslash\{1\}$.

Then, we go to prove some arithmetic rules of log function.

Theorem 5.2.1 For any $y_1, y_2 \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}_{>0} \backslash \{1\}$, we have

$$\log_a y_1 y_2 = \log_a y_1 + \log_a y_2$$

Proof. There exists $x_1, x_2 \in \mathbb{R}$ so that we have

$$a^{x_1} = y_1 \quad \text{and} \quad a^{x_2} = y_2$$

This implies that $a^{x_1+x_2} = y_1 y_2$. Hence, by considering $\log_a y_1 = x_1$ and $\log_a y_2 = x_2$, we have

$$\log_a y_1 + \log_a y_2 = x_1 + x_2 = \log_a y_1 y_2$$

■

Theorem 5.2.2 For any $y \in \mathbb{R}_{>0}$, $a \in \mathbb{R}_{>0} \backslash \{1\}$ and $b \in \mathbb{R}$,

$$\log_a y^b = b \log_a y$$

Proof. There exists some $x \in \mathbb{R}$ such that $a^x = y$, this implies that

$$a^{bx} = y^b$$

and so we get $bx = \log_a y^b$. Hence,

$$b \log_a y = bx = \log_a y^b$$

■

Exercise 5.2 For real constant a, b, c, d, expand

$$\log_a \frac{b}{c^d}$$

where $\frac{b}{c^d} > 0$.

Exercise 5.3 For some function $f > 0$ defined on $\mathbb{R}$, prove that we have

$$\log_a f(1)f(2)\cdots f(n) = \log_a f(1) + \log_a f(2) + \ldots + \log_a f(n)$$

where $a \in \mathbb{R}_{>0}\backslash\{1\}$.

Exercise 5.4 Solve the followings:

(a) Solve $(\log_2 x)^2 - 2\log_2 x + 1 = 0$

(b) Solve $(\log_2 x)^3 - 6(\log_2 x)^2 + 11\log_2 x - 6 = 0$

5.3 Curve Sketching of Logarithmic Function

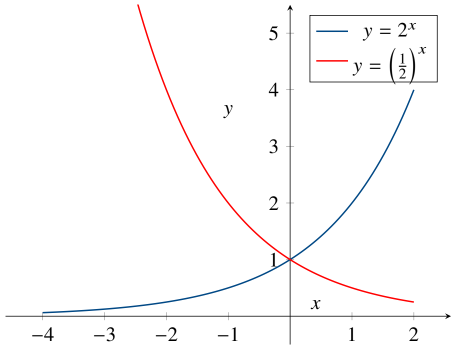
Recall the definition of increasing/decreasing function, we have the following:

Example 5.2
- x increase on $(-\infty, \infty)$.
- x^2 decrease on $(-\infty, 0)$ while increase on $(0, \infty)$.
- $\sqrt{x}$ increase on $(0, \infty)$.
- a^x increase on $(-\infty, \infty)$.
- $\log_a x$ decrease on $(0, \infty)$ for $a \in (0, 1)$.
- $\log_a x$ increase on $(0, \infty)$ for $a \in (1, \infty)$.

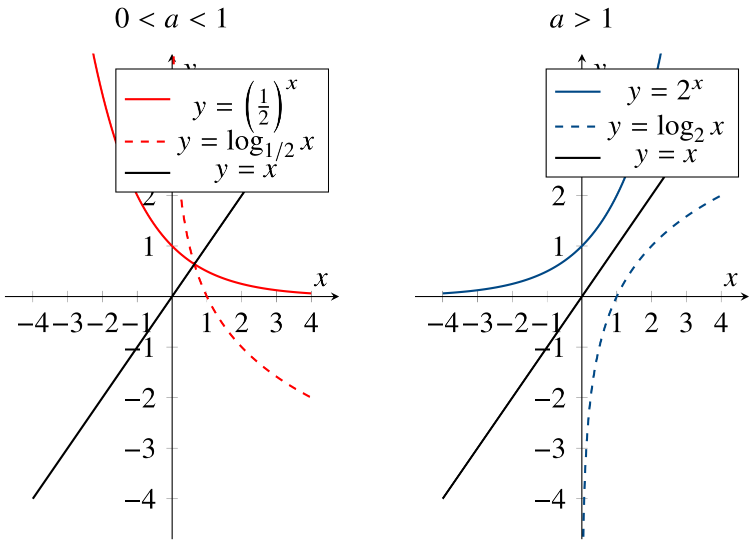
■

Corollary 5.3.1 Any increasing or decreasing function is invertible.

For the curve of $a^x = y$ where $a > 1$, we notice that when $x_2 > x_1$, we always have $a^{x_2} > a^{x_1}$ and so a^x is an increasing function. For the curve of $a^x = y$ where $0 < a < 1$, we notice that when $x_2 > x_1$, we always have $a^{x_1} > a^{x_2}$ and so a^x is a decreasing function. Also, both curve have a y value that is always positive. In addition, their speed of increasing and decreasing become faster and faster when x is increasing. Combing with these facts, we can sketch the curve:



Note that considering for $a^y = x$, we have $\log_a x = y$, this shows that $\log_a$ is the inverse function of a^x. So, the curve of $\log_a x = y$ is obtained from interchanging the x and y value of $a^x = y$.



Theorem 5.3.2 For any $a > 0$, $a^x = 1$ if and only if $x = 0$.

Proof. Let $x = x_1 + x_2$

$$a^x = 1$$
$$a^{x_1} = a^{-x_2}$$

Since a^x is a increasing function, we have $x = x_1 + x_2 = 0$. The other side is trivial. ■

Definition 5.3.1 We have the following definitions:
- Global Extrema: Extrema on $\mathbb{R}$
- Local Extrema: Extreme in the neighborhoodof some x

Theorem 5.3.3 $h(x) = a^x - x$ has an unique global/local minimum.

Proof. Suppose $x_2 > x_1 > 0$ where $u := x_2 - x_1$, we have

$$h(x_2) - h(x_1) = a^{x_2} - a^{x_1} + x_1 - x_2 = a^{x_1}a^u - u - a^{x_1}$$

We solve that $a^{x_1}a^u - u - a^{x_1} > 0 \iff x_1 > \frac{1}{\ln a} \ln \frac{u}{a^u - 1}$. Note that its similar for $a^{x_1}a^u - u - a^{x_1} > 0$. Since u can be independent of x_1, there exists some ξ such that h decrease on $(-\infty, \xi)$ and increase on (ξ, ∞). Thus, there exists a unique global/local minimum $\min_{x \in \mathbb{R}} h = h(\xi)$. ■

Theorem 5.3.4 For any $a > 1$, we have $a^x \geq x$. Particularly, $a^x = x$ for some $a > 1, x \in \mathbb{R}$.

Proof.
- For any $x \leq 0$, it is trivial that $a^x > 0 \geq x$. For any $x > 0$, since $a > 1$, take $a = 1 + \varepsilon$ for some $\varepsilon > 0$. By Binomial Theorem (we will prove it later),

$$a^x = (1+\varepsilon)^x \geq \sum_{k=0}^{[x]} C_k^{[x]} \varepsilon^k \geq \frac{[x]!\varepsilon^j}{([x]-j)!} \cdot \frac{1}{j!}$$

where given constant $j \in \mathbb{Z}$ and $[x]$ is the integral part of x.
- For $\varepsilon \geq 1$, we take $j = 1$ and so $a^x \geq x$ for any $x \in \mathbb{Z}$. For any $\varepsilon < 1$, when x is sufficiently large, $[x]$ can be sufficiently large such that

$$\frac{[x]!\varepsilon^j}{([x]-j)!} = \prod_{i=0}^{j-1}([x]-i)\varepsilon \geq \underbrace{\left(\prod_{i=j-1}^{k}([x]-i)\varepsilon\right)}_{:=C<1}[x][x-1]\varepsilon \geq [x]j!, \quad ([x]-(k-1))\varepsilon \geq 1$$

where $k \in [0, j-1]$ are constants. So we get $a^x \geq x$ for any $x \in \mathbb{Z}, a > 1$.
- Suppose there exists $x^* \in \mathbb{R}$ such that $a^* < x^*$. Since a^x is continuous, there must exists $\lambda \in ([x^*], x^*)$ and $\mu \in (x^*, [x^*]+1)$ so that

$$a^\lambda = \lambda, \quad a^\mu = \mu$$

But from Theorem 5.2.2, there exists ξ so that h decrease on $(-\infty, \xi)$ and increase on (ξ, ∞) which leads to a contradiction. Hence, for any $a > 1$, we have $a^x \geq x$. Note that if $a > 1$ can be written as $a = k^{k^{-1}}$ for some constant k, $a^x = x$ for $x = k$.

■

R This may explain that why DSE 2025 MC contains a controversial question.

5.4 Logarithmic Inequality

DSE sometimes require us to solve some inequality by using log. Notice that the increasing function, for example $\log x$ and 10^x preserve the inequality.

Corollary 5.4.1 For any increasing function f, suppose we have $u > v$, we also have

$$f(u) > f(v)$$

Proof. Follows from definition. ■

Exercise 5.5 Find the interval of x for $39^x > 100$.

Exercise 5.6 Using the functions from Example 5.2, apply them to the inequalities $2 < 5$ and $5 < 7$ to generate new, valid inequalities.

5.5 Past Paper Region

Exercise 5.7 15. Let a and b be constants. Denote the graph of $y = a + \log_b x$ by G. The x-intercept of G is 9 and G passes through the point $(243, 3)$. Express x in terms of y.

Exercise 5.8 For $\log_4 x$ (horizontal) and $\log_8 y$ (vertical) axis where $(\log_4 x, \log_4 y) = (u, v)$, consider a linear function $f(u) = v$ where the slope and the intercept on the horizontal axis of the graph are $-\frac{1}{3}$ and 3 respectively. Express the relation between x and y in the form $y = Ax^k$, where A and k are constants.

Exercise 5.9 For $\log_5 x$ (horizontal) and $\log_5 y$ (vertical) axis where $(\log_5 x, \log_5 y) = (u, v)$, consider a linear function $f(u) = v$ where passes through $(0, -4)$ and $(0, 2)$ for $(\log_5 x, \log_5 y)$, which of the following must be true?

A. $xy^2 = 625$

B. $x^2 y = 625$

C. $\frac{y^2}{x} = 625$

D. $\frac{y}{x^2} = 625$

Exercise 5.10 Let a, b and c be positive constants. On the same rectangular coordinate system, the graph of $y = a + \log_b x$ and the graph of $y = \log_c x$ cut the x-axis at the points S and T respectively. Denote the origin by O. Find $OT : OS$.

Exercise 5.11 Consider the graph of $y = a^x$ and the graph of $y = b^x$ on the same rectangular coordinate system, where a and b are positive constants. If the graph of $y = a^x$ is the reflection image of the graph of $y = b^x$ with respect to the y-axis where $y = a^x$ is a strictly increasing function, which of the followings are true?

A. $a < 1$

B. $b > 1$

C. $ab = 1$

Exercise 5.12 Consider the graph of $y = ab^x$, where a and b are constants. draw a graph that represent the relation between x and $\log_y y$?

Exercise 5.13 Consider the graph of $y = b^x$ and the graph of $y = c^x$ on the same rectangular coordinate system, where b and c are positive constants. If a horizontal line L cuts the y-axis, the graph of $y = b^x$ and the graph of $y = c^x$ at A, B and C respectively, which of the followings are true?

I. $b < c$

II. $bc > 1$

III. $\frac{AB}{AC} = \log_b c$

Exercise 5.14 Which of the following is the greatest?

A. 12^{241}

B. 24^{214}

C. 412^{142}

D. 42^{124}

Exercise 5.15 The graph $f(u) = v$ shows the linear relation between x and $\log_2 y$. If $y = ab^x$, note that $f(u) = v$ passes through $(0, -2)$ and $(4, 0)$ for $(u, v) = (x, \log_9 y)$, then $b =$

A. -2

B. $\frac{1}{81}$

C. $\frac{1}{2}$

D. 3

Exercise 5.16 In the figure, the straight line L shows the relation between $\log_4 x$ and $\log_4 y$. It is given that L passes through the points $(1, 2)$ and $(9, 6)$ for $(u, v) = (\log_4 x, \log_4 y)$. If $y = kx^a$, then $k =$

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. 2

D. 8

Exercise 5.17 Consider the figure shows the graph of $y = \log_a x$ and the graph of $y = \log_b x$ on $x - y$ plane, where a and b are positive constants and $\log_a x > \log_b x$ for all $x > 1$. If a vertical line cuts the graph of $y = \log_a x$, the graph of $y = \log_b x$ and the x-axis at the points A, B and C respectively, which of the following is/are true?

I. $a > 1$

II. $a > b$

III. $\frac{AB}{BC} = \log_a \frac{b}{a} 123$