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Chapter · January 2009

DOI: 10.1007/978-1-4020-8678-6_32

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Intelligent Engineering Systems and Computational Cybernetics

Springer

On the Fractional Order Control of Heat Systems

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Summary. The differentiation of non-integer order has its origin in the seventeenth century, but only in the last two decades appeared the first applications in the area of control theory. In this paper we consider the study of a heat diffusion system based on the application of the fractional calculus concepts. In this perspective, several control methodologies are investigated namely the fractional PID and the Smith predictor. Extensive simulations are presented assessing the performance of the proposed fractional-order algorithms.

1 Introduction

Fractional calculus (FC) is a generalization of integration and differentiation to a complex order α, being the fundamental operator ${}_aD_t^{\alpha}$, where a and t are the limits of the operation [1,2]. The FC concepts constitute a useful tool to describe several physical phenomena, such as heat, flow, electricity, magnetism, mechanics or fluid dynamics and presently it is applied in almost all areas of science and engineering.

In the last years, FC has been used increasingly to model the constitutive behavior of materials and physical systems exhibiting hereditary and memory properties. This is the main advantage of fractional-order derivatives in comparison with classical integer-order models, where these effects are neglected.

It is well-known that the fractional-order operator $s^{0.5}$ appears in several types of problems [3]. The transmission lines, the heat flow or the diffusion of neutrons in a nuclear reactor are examples where the half-operator is the fundamental element. On the other hand, diffusion is one of the three fundamental partial differential equations of mathematical physics [4].

In this paper we investigate the heat diffusion system in the perspective of applying the FC theory. Several control strategies based on fractional-order algorithms are presented and compared with the classical schemes. The adoption of fractional-order controllers has been justified by its superior performance, particularly when used with fractional-order dynamical systems, such as the case of the heat system under study. The fractional-order PID ($\mathrm{PI}^{\alpha}\mathrm{D}^{\beta}$ controller) involves an integrator of order $\alpha \in \Re+$ and a differentiator of order $\beta \in \Re+$. It

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was demonstrated the good performance of this type of controller, in comparison with the conventional PID algorithms.

Bearing these ideas in mind, the paper is organized as follows. Section 2 gives the fundamentals of fractional-order control systems. Section 3 introduces the heat diffusion system. Section 4 points out several control strategies for the heat system and discusses their results. Finally, Section 5 draws the main conclusions and addresses perspectives towards future developments.

2 Fractional-Order Control Systems

Fractional-order control systems are characterized by differential equations that have, in the dynamical system and/or in the control algorithm, an integral and/or a derivative of fractional-order. Due to the fact that these operators are defined by irrational continuous transfer functions, in the Laplace domain, or infinite dimensional discrete transfer functions, in the Z domain, we often encounter evaluation problems in the simulations. Therefore, when analyzing fractional-order systems, we usually adopt continuous or discrete integer-order approximations of fractional-order operators [5–7].

The mathematical definition of a fractional-order derivative and integral has been the subject of several different approaches [1, 2]. One commonly used definition for the fractional-order derivative is given by the Riemann–Liouville definition ($\alpha > 0$):

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \;\; n-1 < \alpha < n, \tag{1} $$

where $f(t)$ is the applied function and $\Gamma(x)$ is the Gamma function of x. Another widely used definition is given by the Grünwald–Letnikov approach ($\alpha \in \Re$):

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh) \tag{2a} $$

$$ \binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \tag{2b} $$

where h is the time increment and $[x]$ means the integer part of x.

The "memory" effect of these operators is demonstrated by (1) and (2), where the convolution integral in (1) and the infinite series in (2), reveal the unlimited memory of these operators, ideal for modeling hereditary and memory properties in physical systems and materials.

An alternative definition to (1) and (2), which reveals useful for the analysis of fractional-order control systems, is given by the Laplace transform method. Considering vanishing initial conditions, the fractional *differintegration* is defined in the Laplace domain, $F(s) = L\{f(t)\}$, as:

$$ L\{{}_aD_t^{\alpha} f(t)\} = s^{\alpha} F(s), \quad \alpha \in \Re \tag{3} $$

An important aspect of fractional-order algorithms can be illustrated through the elemental control system with open-loop transfer function $G(s) = Ks^{-\alpha}$ ($1 < \alpha < 2$) in the forward path and unit feedback. The open-loop Bode diagrams of amplitude and phase have a slope of -20α dB/dec and a constant phase of $-\alpha\pi/2$ rad over the entire frequency domain. Therefore, the closed-loop system has a constant phase margin of PM $= \pi(1-\alpha/2)$ rad that is independent of the system gain K. Therefore, the closed-loop system will be robust against gain variations exhibiting step responses with an iso-damping property [8].

In this paper we adopt discrete integer-order approximations to the fundamental element s^{α} ($\alpha \in \Re$) of a fractional-order control (FOC) strategy. The usual approach for obtaining discrete equivalents of continuous operators of type s^{α} adopts the Euler, Tustin and Al-Alaoui generating functions [5].

It is well known that rational-type approximations frequently converge faster than polynomial-type approximations and have a wider domain of convergence in the complex domain. Thus, by using the Euler operator $w(z^{-1}) = (1-z^{-1})/T$, and performing a power series expansion of $\left[w(z^{-1})\right]^{\alpha} = \left[(1-z^{-1})/T\right]^{\alpha}$ gives the discretization formula corresponding to the Grünwald–Letnikov definition (2):

$$D^{\alpha}\left(z^{-1}\right) = \left(\frac{1-z^{-1}}{T}\right)^{\alpha} = \sum_{k=0}^{\infty} h^{\alpha}(k)\, z^{-k} \tag{4a}$$

$$h^{\alpha}(k) = \left(\frac{1}{T}\right)^{\alpha} \binom{k-\alpha-1}{k} \tag{4b}$$

where the impulse response sequence $h^{\alpha}(k)$ is given by the expression (4b) ($k \geq 0$).

A rational-type approximation can be obtained by applying the Padé approximation method to the impulse response sequence (4b) $h^{\alpha}(k)$, yielding the discrete transfer function:

$$H\left(z^{-1}\right) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} = \sum_{k=0}^{\infty} h(k)\, z^{-k}, \tag{5}$$

where $m \leq n$ and the coefficients a_k and b_k are determined by fitting the first $m+n+1$ values of $h^{\alpha}(k)$ into the impulse response $h(k)$ of the desired approximation $H(z^{-1})$. Thus, we obtain an approximation that has a perfect match to the desired impulse response $h^{\alpha}(k)$ for the first $m+n+1$ values of k [8]. Note that the above Padé approximation is obtained by considering the Euler operator but the determination process will be exactly the same for other types of discretization schemes.

3 Heat Diffusion

The heat diffusion is governed by a linear partial differential equation (PDE) of the form:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{6}$$

where k is the diffusivity, t is the time, u is the temperature and (x,y,z) are the space cartesian coordinates. The system (6) involves the solution of a PDE of parabolic type for which the standard theory guarantees the existence of a unique solution [9].

For the case of a planar perfectly isolated surface we usually apply a constant temperature U_0 at $x = 0$ and we analyze the heat diffusion along the horizontal coordinate x. Under these conditions, the heat diffusion phenomenon is described by a non-integer order model, yielding:

$$U(x,s) = \frac{U_0}{s} G(s),\; G(s) = e^{-x\sqrt{\frac{s}{k}}}, \tag{7}$$

where x is the space coordinate, U_0 is the boundary condition and $G(s)$ is the system transfer function.

In our study, the simulation of the heat diffusion is performed by adopting the Crank–Nicholson implicit numerical integration based on the discrete approximation to differentiation [10–13].

4 Control Strategies for Heat Diffusion Systems

In this section we consider three strategies for the control of the heat diffusion system. In the first two subsections, we analyze the system of Fig. 1 by adopting classical PID and fractional PID$^\beta$ controllers, respectively. In the third subsection we use a Smith predictor (SP) structure with a fractional PID$^\beta$ controller (Fig. 5).

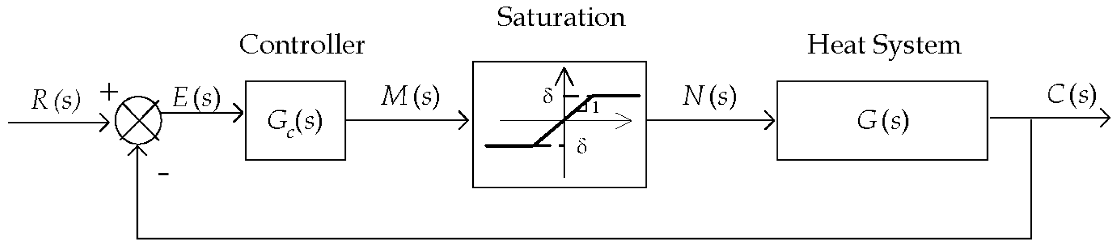


Fig. 1: Closed-loop system with PID or PID$^\beta$ controller $G_c(s)$

The generalized PID controller $G_c(s)$ has a transfer function of the form:

$$G_c(s) = K\left[1 + \frac{1}{T_i s^\alpha} + T_d s^\beta\right], \tag{8}$$

where α and β are the orders of the fractional integrator and differentiator, respectively. The constants K, T_i and T_d are correspondingly the proportional gain, the integral time constant and the derivative time constant.

Clearly, taking $(\alpha, \beta) = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$ we get the classical {PID, PI, PD, P} controllers, respectively. Other PID controllers are possible, namely: PD$^\beta$ controller, PI$^\alpha$ controller, PID$^\beta$ controller, and so on. The PI$^\alpha$D$^\beta$ controller is more flexible and gives the possibility of adjusting more carefully the closed-loop system characteristics.

4.1 The PID Controller

In this subsection we analyze the closed-loop system with a conventional PID controller given by the transfer function (8) with $\alpha = \beta = 1$. Often, the PID parameters (K, T_i, T_d) are tuned by using the so-called Ziegler–Nichols open loop (ZNOL) method. The ZNOL heuristics are based on the approximate first-order plus dead-time model:

$$\hat{G}(s) = \frac{K_p}{\tau s + 1} e^{-sT} \tag{9}$$

A step input is applied at $x = 0.0$ m and the output $c(t)$ analyzed for $x = 3.0$ m, without the saturation. The resulting parameters are $\{K_p, \tau, T\} = \{0.52,\ 162, 28\}$ leading to the PID constants $\{K, T_i, T_d\} = \{18.07, 34.0, 8.5\}$. We verify that the system with a PID controller, tuned according with the ZNOL heuristics, does not produce satisfactory results giving a significant overshoot, and a large settling time. Moreover, the step response reveals a considerable time delay. The poor results obtained indicate that the method of tuning as well the structure of the system may not be the most adequate for the control of the heat system under consideration. In fact, the inherent fractional-order dynamics of the system lead us to consider other configurations. In this perspective, we propose the use of fractional-order controllers and the SP to achieve a superior control performance.

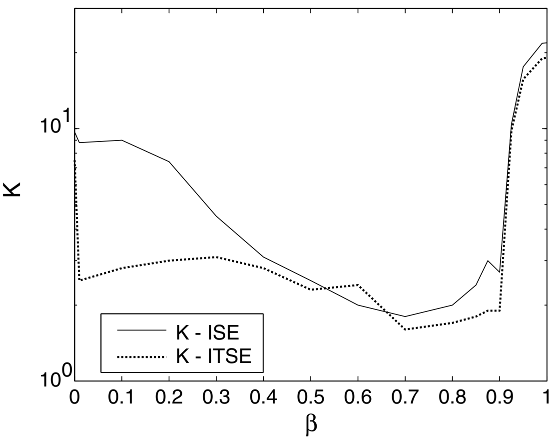
4.2 The PID$^{\beta}$ Controller

In this subsection we analyze the closed-loop system with a PID$^{\beta}$ controller given by the transfer function (8) with $\alpha = 1$. The fractional-order derivative term $T_d s^{\beta}$ in (8) is implemented by using a fourth-order Padé discrete rational transfer function of type (5). It is used a sampling period of $T = 0.1$ s.

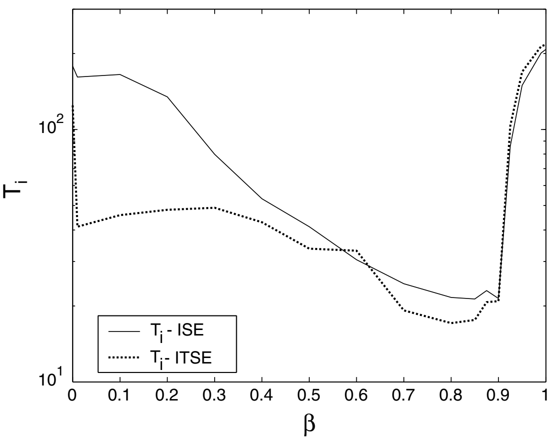
The PID$^{\beta}$ controller is tuned by minimization of an integral performance index. For that, we adopt the integral square error (ISE) and the integral time square error (ITSE) criteria defined as:

$$\mathrm{ISE} = \int_{0}^{\infty} [r(t) - c(t)]^2 \, dt \tag{10}$$

$$\mathrm{ITSE} = \int_{0}^{\infty} t\,[r(t) - c(t)]^2 \, dt \tag{11}$$



(a)



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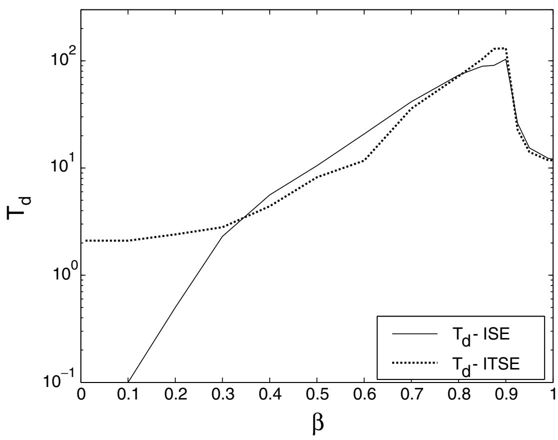


(c)

Fig. 2: The PID$^{\beta}$ parameters (K, T_i, T_d) versus β for the ISE and the ITSE criteria

We can use other integral performance criteria such as the integral absolute error (IAE) or the integral time absolute error (ITAE) but the ISE and the ITSE criteria have produced the best results. Furthermore, the ITSE criterion enable us to study the influence of time in the error generated by the system.

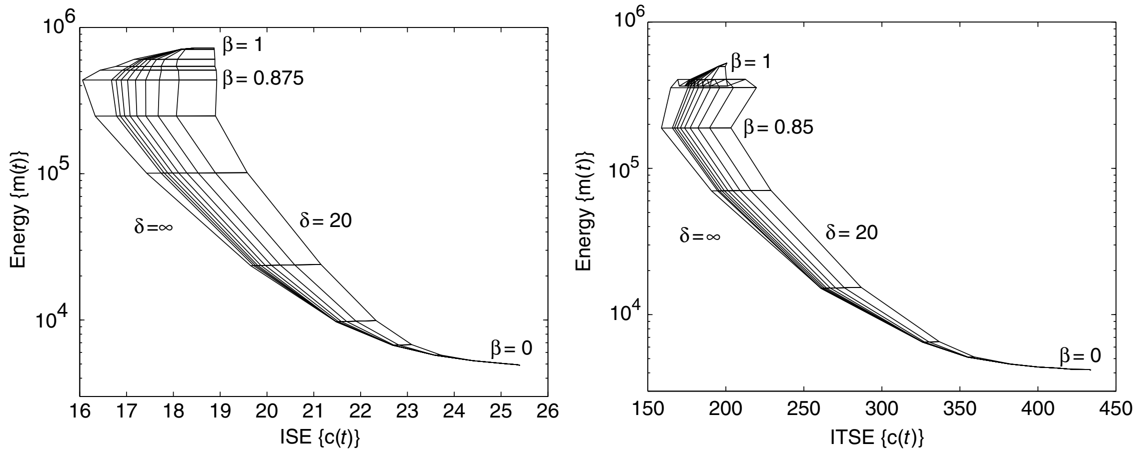
A step reference input $R(s) = 1/s$ is applied at $x = 0.0$ m and the output $c(t)$ is analyzed for $x = 3.0$ m, without considering saturation. The heat system is simulated for 3,000 s. Figure 2 illustrates the variation of the fractional PID parameters (K, T_i, T_d) as function of the order's derivative β, tuned according with the ISE and the ITSE criteria.

The curves reveal that for $\beta < 0.4$, the parameters (K, T_i, T_d) are slightly different, for the two ISE and ITSE criteria, but for $\beta \geq 0.4$ they almost lead to similar values. This fact indicates a large influence of a weak order derivative on system's dynamics.

To further illustrate the performance of the fractional-order controllers a saturation nonlinearity is included in the closed-loop system of Fig. 1 and inserted in series with the output of the PID controller $G_c(s)$. The saturation element is defined as:

$$n(m) = \begin{cases} +\delta, & m > \delta \\ m, & |m| \leq \delta \\ -\delta, & m < -\delta \end{cases}$$

The controller performance is evaluated for $\delta = \{20, \ldots, 100, \infty\}$, where the last case corresponds to a system without saturation. We use the same fractional-PID parameters obtained without considering the saturation nonlinearity.



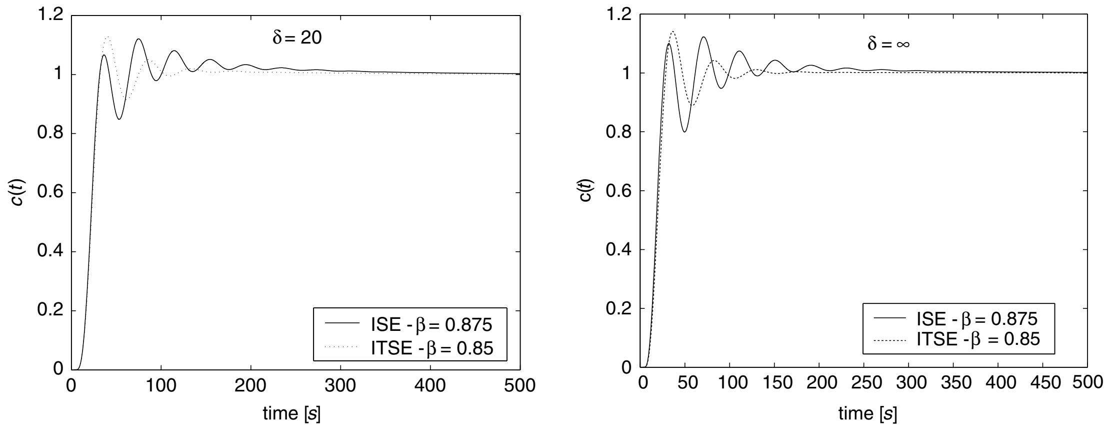
(a) (b)

Fig. 3: Energy E_m *versus* the ISE and the ITSE for $\delta = \{20, \ldots, 100, \infty\}$, $0 \leq \beta \leq 1$

The energy E_m at the output $m(t)$ of the fractional PID controller is also analyzed through the expression:

$$E_m = \int_0^{T_s} m^2(t)dt, \tag{12}$$

where T_s is the time corresponding to the 5% settling time criterion of the system output $c(t)$.



(a) (b)

Fig. 4: Step responses of the closed-loop system for the ISE and the ITSE indices, with a PID$^\beta$ controller, for $\delta = \{20, \infty\}$ and $x = 3.0$ m

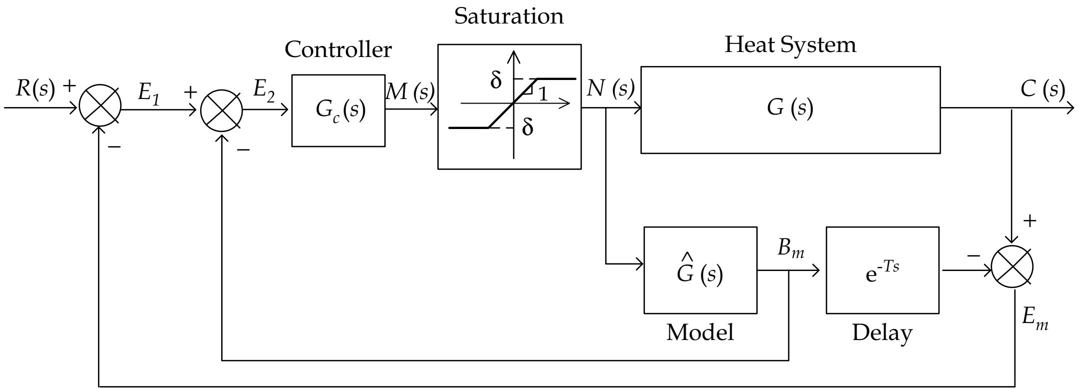


Fig. 5: Closed-loop system with a Smith predictor and a PID$^\beta$ controller $G_c(s)$

Figure 3 depicts the energy E_m as function of the ISE and the ITSE indices for $0 \le \beta \le 1$. As can be seen, the energy changes smoothly for different values of δ when considering a given order β. However, fixing the value δ, we verify that the energy increases significantly with β.

On the other hand, we observe that the ISE decreases with δ for $\beta \le 0.875$, while for $\beta > 0.875$ the ISE increases very quickly. The same conclusions can be outlined relatively to the ITSE criterion. The results confirm the good performance of the system for low values of the fractional-order derivative term.

When comparing the two indices, we also note that the value of the ITSE is generally larger than the corresponding for the ISE case. This occurs because of the large simulation time needed to stabilize the system, which is $T_s \sim 700$ s.

Figure 4 shows the step responses of the closed-loop system for the PID$^\beta$ tuned in the perspective of ISE and ITSE.

The controller parameters $\{K,\ T_i,\ T_d,\ \beta\}$ correspond to the minimization of those indices leading to the values ISE: $\{K, T_i, T_d, \beta\} \equiv \{3, 23, 90.6, 0.875\}$ and ITSE: $\{K, T_i, T_d, \beta\} \equiv \{1.8, 17.6, 103.6, 0.85\}$.

The step responses reveal a large diminishing of the overshoot and the rise time when compared with the integer PID, showing a good transient response and a zero steady-state error. As should be expected, the PID$^\beta$ leads to better results than the PID controller tuned

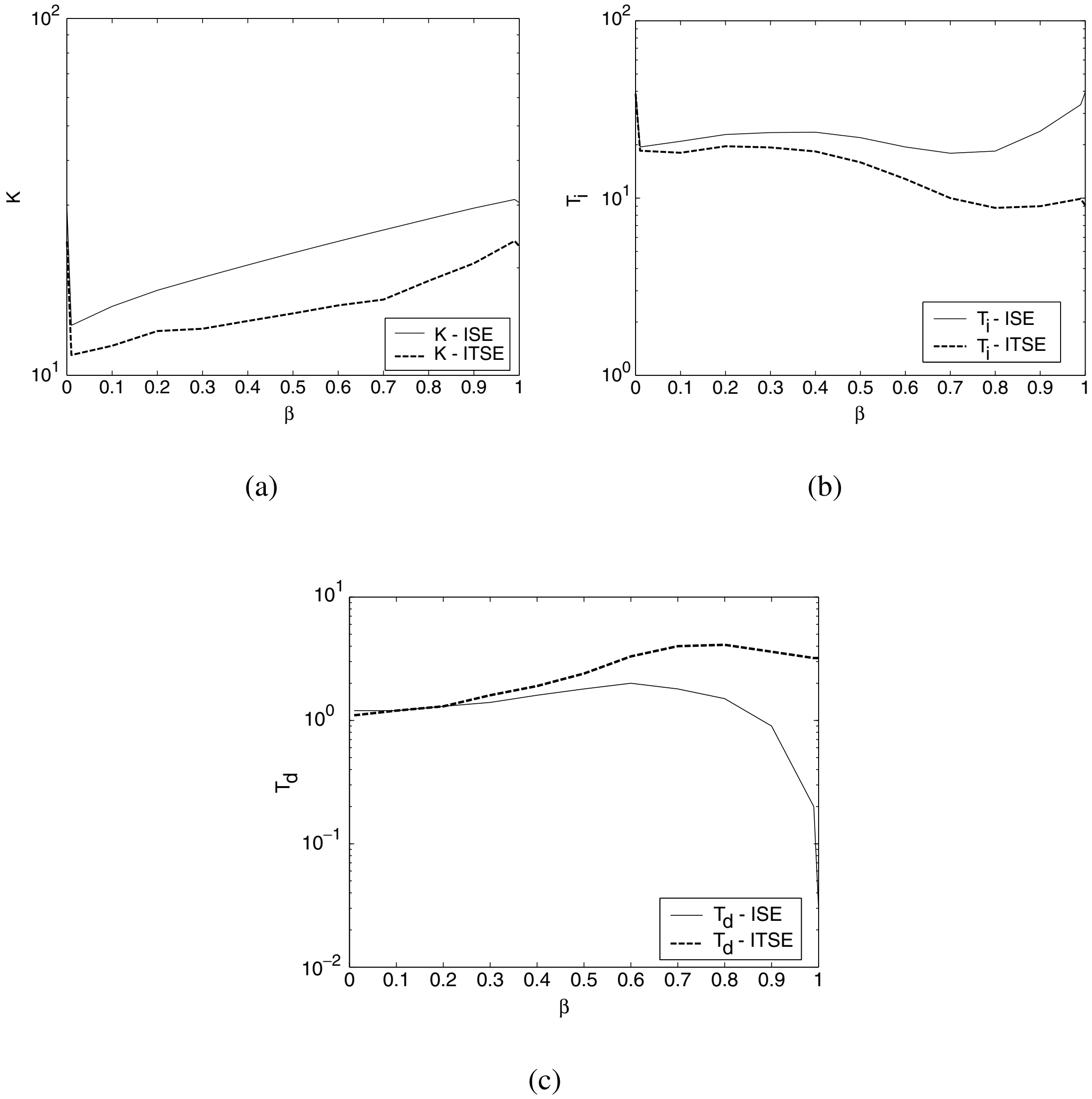


Fig. 6: The PID$^\beta$ parameters (K, T_i, T_d) versus β for the ISE and the ITSE criteria

through the ZNOL method. These results demonstrate the effectiveness of the fractional-order algorithms when used for the control of fractional-order systems.

4.3 The Smith Predictor with a PID$^\beta$ Controller

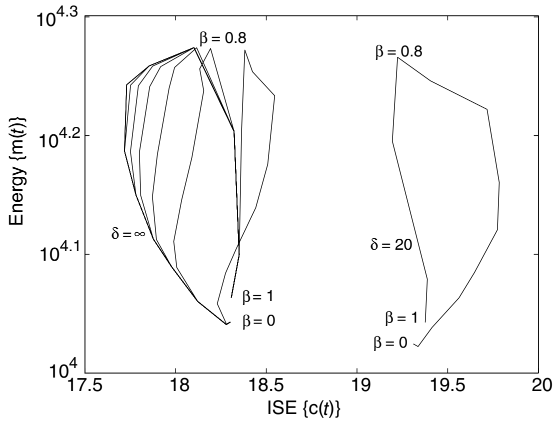
In this subsection we adopt a fractional PID$^\beta$ controller inserted in a SP structure, as represented in Fig. 5. This algorithm, developed by O. J. M. Smith in 1957, is a dead-time compensation technique, very effective in improving the control of processes having time delays [14, 15].

The transfer function $\hat{G}(s)$, inserted in the second branch of the SP, is described by the following first-order plus dead-time model:

$$\hat{G}(s) = \frac{0.52}{162s+1} e^{-28s} \tag{13}$$

where the parameters $(K_p, \tau, T) = (0.52,\ 162,\ 28)$ were estimated by a least-squares fit between the frequency responses of the numerical model of the heat system and the SP model $\hat{G}(s)$.

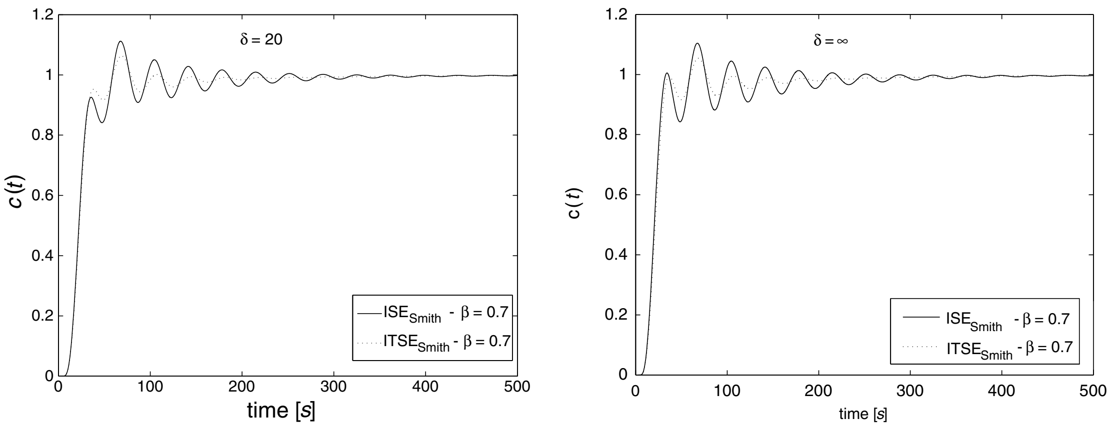
The PID$^\beta$ controller is tuned by applying the ISE and the ITSE integral performance criteria, as described in the previous subsection.



(a) (b)

Fig. 7: Energy E_m versus the ISE and the ITSE indices for $\delta = \{20, ..., 100, \infty\}$, $0 \leq \beta \leq 1$

Figure 6 illustrates the variation of the PID$^\beta$ parameters (K, T_i, T_d) as function of the order's derivative β, for the ISE and the ITSE indices, without the occurrence of saturation.



(a) (b)

Fig. 8: Step responses of the closed-loop system for the PID$^\beta$ and the Smith predictor with PID$^\beta$, for the ISE and the ITSE indices, $\delta = \{20, \infty\}$, $x = 3.0$ m

Figure 7 shows the relation between the energy E_m and the values of the ISE and ITSE indices. We verify that the best case is achieved when $\beta = 0.7$, revealing that, with a SP structure, the effectiveness of the fractional-order controller is more evident.

Since the effect of the system time delay is diminished by the use of the SP, the PID$^{\beta}$ controller and, more specifically, the fractional-order derivative D^{β} will be more effective and, consequently, produce better results.

Figure 8 illustrates the step responses for $x = 3.0$ m when applying a step input $R(s) = 1/\mathrm{s}$ at $x = 0.0$ m, for the SP and the PID$^{\beta}$ controller with $\beta = 0.7$, both for the ISE and the ITSE indices. The graphs show a better transient response for the SP with a PID$^{\beta}$, namely smaller values of the overshot, rise time and time delay. The settling time is approximately the same for both configurations.

However, for small values of β, the step response of the SP is significantly better than the corresponding one for the PID$^{\beta}$ controller. In the SP step response we still observe a time delay due to an insufficient match between the system model $G(s)$ and the first-order approximation $\hat{G}(s)$. Therefore, for a superior use of this method a better approximation of the heat system should be envisaged. This subject, namely a fractional order SP model will be addressed in future research.

5 Conclusions

This paper presented the fundamental aspects of the FC theory. We demonstrated that FC is a tool for modelling physical phenomena having superior capabilities than those of traditional methodologies. In this perspective, we studied a heat diffusion system, which is described through the fractional-order operator $s^{0.5}$. The dynamics of the system was analyzed in the perspective of FC, and some of its implications upon control algorithms and systems with time delay were also investigated.

Acknowledgments

The authors thank GECAD – *Grupo de Investigação em Engenharia do Conhecimento e Apoio à Decisão*, for the financial support to this work.

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