The Random Turán Problem

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Based on various joint works with

Gwen McKinley, Jiaxi Nie, and Jacques Verstraëte

Extremal Combinatorics

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Question (Erdős-Turán 1936)

How large can a subset $S_n \subseteq \{1, 2, \ldots, n\}$ be if S_n does not contain a k-term arithmetic progression?

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Theorem (Roth 1953; Szemerédi 1975)

The largest subset of $\{1, 2, \ldots, n\}$ which does not contain a k-term arithmetic progression has size $o(n)$.

That is,

$$\lim_{n\to\infty} \frac{|S_n|}{n} = 0.$$

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Theorem (Conlon-Gowers, Schacht 2010)

$$\mathbb{E}[|S_n|] = \begin{cases} pn + o(pn) & p \ll n^{-1/(k-1)}, \\ o(pn) & p \gg n^{-1/(k-1)}. \end{cases}$$

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Theorem (Green-Tao 2008)

If P is a "pseudo-random" set of primes, then the largest subset $S \subseteq P$ which contains no k-term arithmetic progression has size $o(|P|)$.

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Theorem (Mantel 1907)

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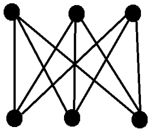

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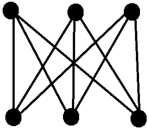

Theorem (Turán 1941)

$$\mathrm{ex}(n, K_t) = \left\lfloor \binom{t-1}{2} \frac{n^2}{(t-1)^2} \right\rfloor$$

Extremal Graph Theory

Theorem (Erdős-Stone, Simonovits 1946)

For any graph F, we have

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2}.$$

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If F is bipartite this only says $\mathrm{ex}(n, F) = o(n^2)$.

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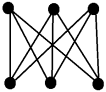
- Complete bipartite graphs: $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$ for $t \gg s$.



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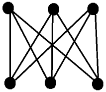
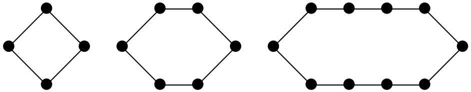


- Even cycles: $\mathrm{ex}(n, C_{2b}) = \Theta(n^{1+1/b})$ for $2b \in \{4, 6, 10\}$.



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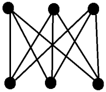

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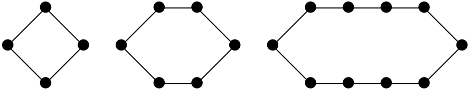
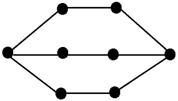


- Theta graphs: $\mathrm{ex}(n, \theta_{a,b}) = \Theta(n^{1+1/b})$ for $a \gg b$.



$\theta_{a,b}$: a internally disjoint paths of length b.

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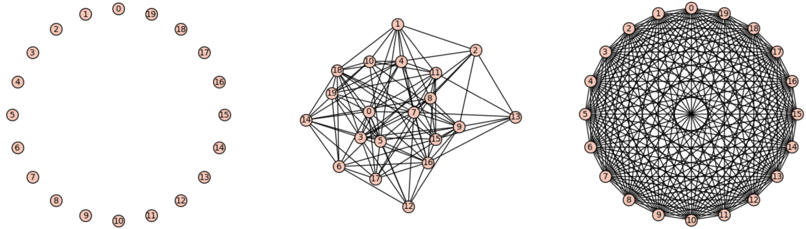

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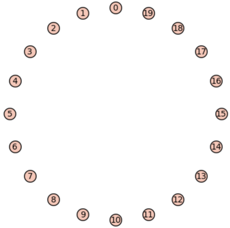

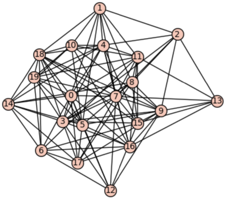

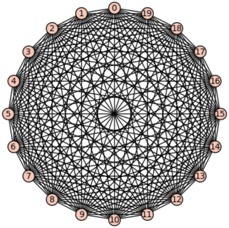

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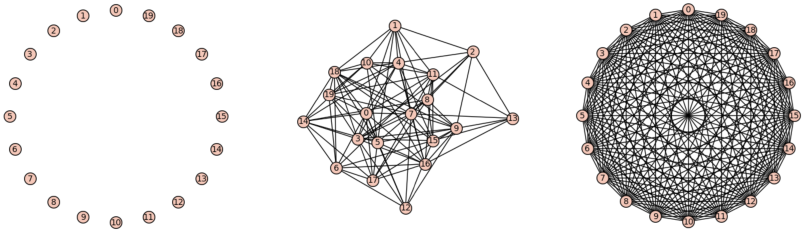

Let $\mathrm{ex}(G_{n,p}, F)$ be the maximum number of edges that an F-free subgraph of $G_{n,p}$ can have. For example,

$$\mathrm{ex}(G_{n,1}, F) = \mathrm{ex}(n, F).$$

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With high probability (Whp):

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For $F = K_3$,

$$\frac{1}{2}p\binom{n}{2} \lesssim \mathrm{ex}(G_{n,p}, K_3) \lesssim p\binom{n}{2},$$

with the lower bound tight for $p = 1$ and the upper bound tight for $p \ll n^{-1/2}$.

Theorem (Frankl-Rödl 1986)

Whp,

$$\mathrm{ex}(G_{n,p}, K_3) \sim \frac{1}{2} p \binom{n}{2} \qquad\qquad p \gg n^{-1/2}.$$

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$$\mathrm{ex}(G_{n,p}, F) = p \cdot \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2} \qquad p \gg n^{-1/m_2(F)},$$

where $m_2(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subseteq F\}$.

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Natural Guess

$$\mathrm{ex}(G_{n,p}, F) = \begin{cases} \Theta(p \cdot \mathrm{ex}(n, F)) & p \gg n^{-1/m_2(F)}, \\ (1 + o(1))p\binom{n}{2} & p \ll n^{-1/m_2(F)}. \end{cases}$$

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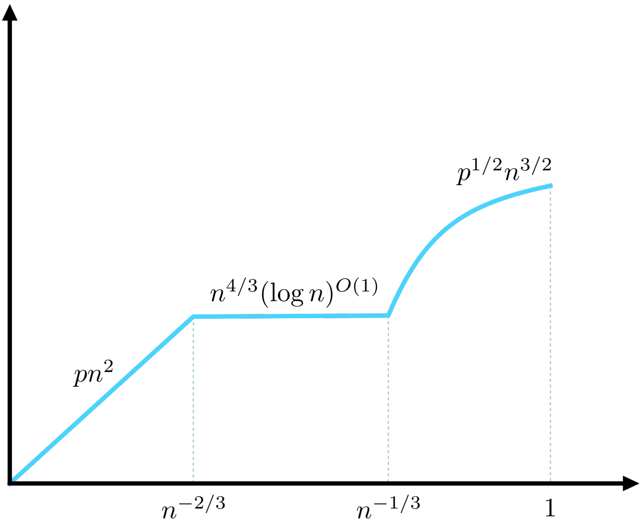
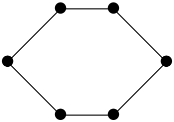
This guess turns out to be completely false!



Plot of $\mathsf{ex}(G_{n,p}, C_4)$ (Füredi 1991)

Theorem (Haxell-Kohayakawa-Łuczak 1995)

Whp $\mathrm{ex}(G_{n,p}, C_{2b}) = o(pn^2)$ *for* $p \gg n^{-1+1/(2b-1)}$.



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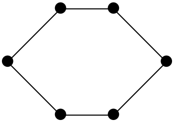

Theorem (Kohayakawa-Kreuter-Steger 1998)

For $n^{-1+1/(2b-1)} \ll p \ll n^{-1+1/(2b-1)+1/(2b-1)^2}$, *we have whp*
$\mathrm{ex}(G_{n,p}, C_{2b}) = n^{1+1/(2b-1)} \log^{O(1)}(n)$

Theorem (Morris-Saxton 2013)

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This was all that was known for *specific* F, but more can be said about *general* F.

Theorem (Jiang-Longbrake 2022)

If F satisfies "mild conditions" and $\mathrm{ex}(n, F) = \Theta(n^{\alpha})$, *then whp*

$$\mathrm{ex}(G_{n,p}, F) = O(p^{1-m_2^*(F)(2-\alpha)} n^{\alpha}) \textit{ for } p \textit{ large},$$

where $m_2^*(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subsetneq F,\ e(F') \geq 2\}$.

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Theorem (S.-Verstraëte 2020)

If F satisfies "moderate conditions" and $\mathrm{ex}(n,F)=\Theta(n^\alpha)$, then whp

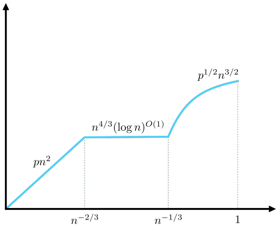
$$\mathrm{ex}(G_{n,p},F)=\Omega(p^{\alpha-1}n^\alpha) \text{ for } p \text{ large}.$$

Conjecture (McKinley-S. 2023)

If F is a graph with $\mathrm{ex}(n, F) = \Theta(n^\alpha)$ *for some* $\alpha \in (1, 2]$, *then whp*

$$\mathrm{ex}(G_{n,p}, F) = \max\{\Theta(p^{\alpha-1} n^\alpha), n^{2-1/m_2(F)} (\log n)^{O(1)}\},$$

provided $p \gg n^{-1/m_2(F)}$.



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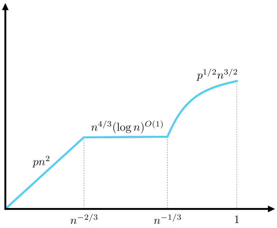


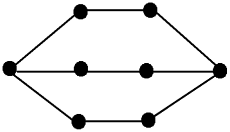
Theorem (Nie-S. 2023 (Informal))

If a graph F satisfies this conjecture, then it also satisfies Sidorenko's conjecture.

Theorem (Faudree-Simonovits 1974; Conlon 2014)

$$\mathrm{ex}(n, \theta_{a,b}) = \Theta(n^{1+1/b}) \text{ for } a \gg b.$$

$\theta_{a,b}$: a internally disjoint paths of length b.



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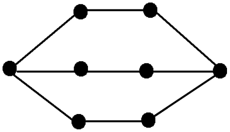

Theorem (Corsten-Tran 2021; Jiang-Longbrake 2022)

$$\mathrm{ex}(G_{n,p}, \theta_{a,b}) = O(p^{\frac{2}{ab}} n^{1+1/b}) \text{ for } p \text{ large.}$$

Note: our conjecture predicts $p^{\frac{1}{b}} n^{1+1/b}$.

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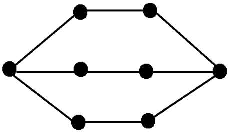

Theorem (McKinley-S. 2023)

For $a \geq 100$,

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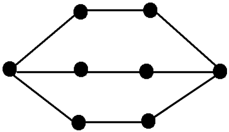

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Theorem (S. 2022)

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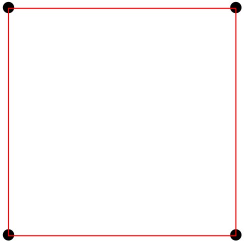
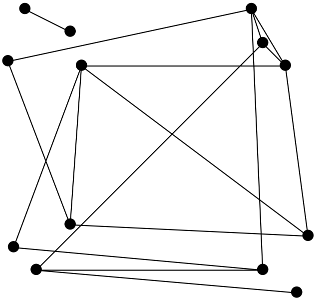


$\mathrm{ex}(pn, F)$



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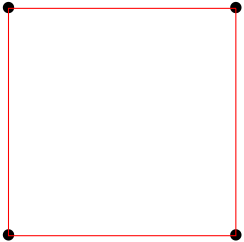

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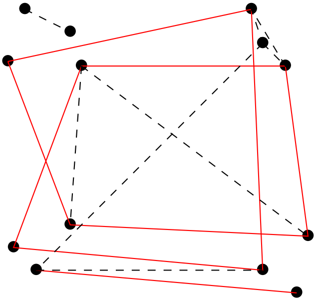

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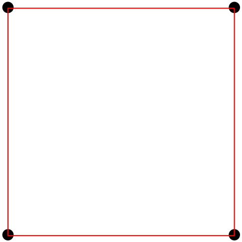

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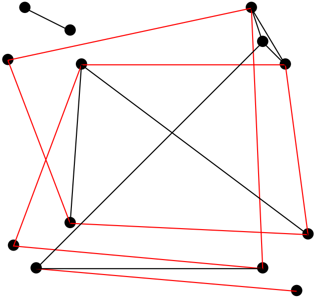

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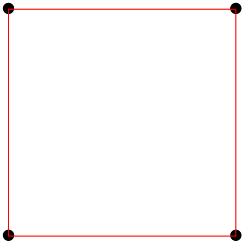

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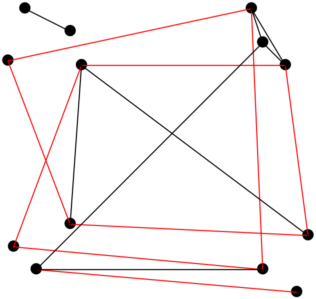

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Theorem (S.-Verstraëte 2020)

To lower bound $\mathrm{ex}(G_{n,p}, F)$, it suffices to lower bound $\mathrm{ex}_{\mathrm{Fold}}(n, F)$.

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$$\mathbb{E}[e(G)] \approx pn^2 - p^e n^v,$$

assuming F has v vertices and e edges. Taking $p \approx n^{-\frac{v-2}{e-1}}$ gives

$$\mathrm{ex}(n, F) \geq n^{2-\frac{v-2}{e-1}}.$$

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Crucial idea: do a deletion argument not for the random graph $G_{n,p}$, but for a random **algebraic** graph G.

Given polynomials $f_i : \mathbb{F}_q^b \times \mathbb{F}_q^b \to \mathbb{F}_q$ of degree at most d, define the bipartite graph $G_{f_1,\ldots,f_r}$ with vertex set $\mathbb{F}_q^b \sqcup \mathbb{F}_q^b$ where $x \sim y$ if and only if $f_i(x, y) = 0$ for all i.

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If $f_1, \dots, f_r$ are random polynomials of degree d, then $G_{f_1,\dots,f_r}$ behaves "locally" like $G_{n,p}$ with $n = q^b$ and $p = q^{-r/b}$.

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If $a \gg b$*, then* $\mathrm{ex}(n, \theta_{a,b}) = \Omega(n^{1+1/b})$.

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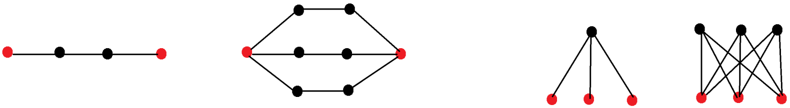
By the previous lemma, G' will be $\theta_{a,b}$-free for $a \geq C$. We know that G contains about $p^e n^v$ copies of $\theta_{a,b}$, so we can not delete "too many" vertices in going from G to G', implying that $e(G') \approx e(G)$ has many edges. $\square$

Theorem (Bukh-Conlon 2017)

We can effectively lower bound $\mathrm{ex}(n, F)$ *whenever* F *is a large power of a rooted tree.*



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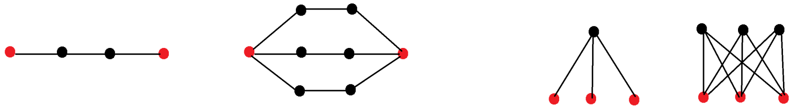

Theorem (S. 2022)

This result holds more generally when:

- *We replace* F *with any rooted graph,*
- *We forbid multiple rooted graphs* $F_1, \ldots, F_t$*,*
- *We replace* $\mathrm{ex}(n, F)$ *with* $\mathrm{ex}_{\mathrm{Fold}}(n, F)$ *or* $\mathrm{ex}(G_{n,p}, F)$*.*

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To do this, we use a powerful technique known as **hypergraph containers**, which is a general tool for counting combinatorial objects.

Upper Bound Techniques

Theorem (Morris-Saxton; Balogh-Morris-Samotij; Saxton-Thomasson)

To prove upper bounds on $\mathrm{ex}(G_{n,p}, F)$*, it suffices to prove a "balanced supersaturation", i.e. that every dense graph* G *contains a large collection of copies of* F *such that no set of edges* σ *of* G *is contained in many copies of* F*.*

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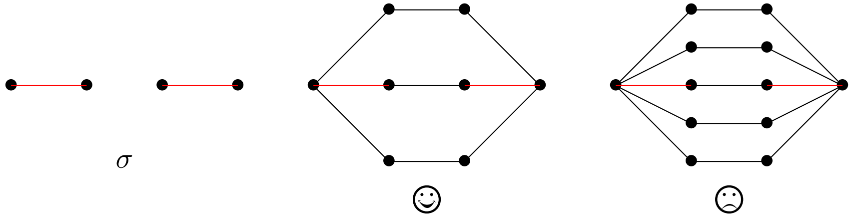

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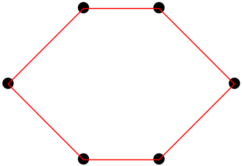
Theorem (Corsten-Tran; Jiang-Longbrake)

If G is a dense graph, then one can find a large collection of copies of $\theta_{a,b}$ such that any set of edges σ is not contained in many copies <u>except for edges forming a cycle</u>.



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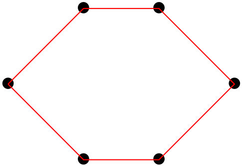

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Morally speaking, the difficulty is that we algorithmically build each copy of $\theta_{a,b}$ vertex by vertex, but for balanced supersaturation we need to control "edges" not "vertices."

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3) We impose *asymmetric* codegree conditions for our vertices (eg we may demand that every pair of vertices $\{u, v\}$ is in at most 1000 copies of $\theta_{a,b}$ overall, and that at most 10 copies contain these as the two high-degree vertices of $\theta_{a,b}$).

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Lower bounding $\mathrm{ex}(G_{n,p}, F)$:

- Use a "template graph" to reduce to lower bounding $\mathrm{ex}_{\mathrm{Fold}}(n, F)$. (S.-Verstraëte 2020).
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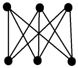
Upper bounding $\mathrm{ex}(G_{n,p}, F)$:

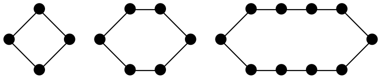
- Use edge-balanced supersaturation to upper bound $\mathrm{ex}(G_{n,p}, F)$ (Morris-Saxton 2013).
- Use *vertex*-balanced supersaturation for $\theta_{a,b}$ (McKinley-S. 2023).

Summary of Bipartite Random Turán Results

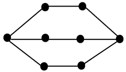
- Complete bipartite graphs $K_{s,t}$ with $t \gg s$ (Morris-Saxton 2013).



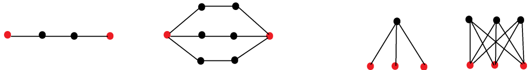
- Even cycles C_{2b} with $2b \in \{4, 6, 10\}$ (Morris-Saxton 2013).



- Theta graphs $\theta_{a,b}$ with $a \gg b$ (McKinley-S. 2023).



- **Lower bounds** for large powers of rooted trees (S. 2022).



Thanks!