The Random Turán Problem

Sam Spiro, Rutgers University

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Theorem (Erdős-Stone 1946)

$$\mathrm{ex}(n, F) \sim \left(1 - \frac{1}{\chi(F) - 1}\right) \binom{n}{2}.$$

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The lower bound is tight when $p = 1$. The upper bound is tight if p is small enough so that $G_{n,p}$ contains no triangles (i.e. $p \ll n^{-1}$), or if almost no edges are contained in triangles (i.e. if $p \ll n^{-1/2}$).

Theorem (Frankl-Rödl 1986)

Whp,

$$\mathrm{ex}(G_{n,p}, K_3) \sim \frac{1}{2}p\binom{n}{2} \qquad p \gg n^{-1/2}.$$

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Theorem (Conlon-Gowers, Schacht 2010)

Whp,

$$\mathrm{ex}(G_{n,p}, F) \sim \left(1 - \frac{1}{\chi(F) - 1}\right) p \binom{n}{2} \qquad p \gg n^{-1/m_2(F)},$$

where $m_2(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subseteq F\}$.

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Conjecture

If F is a bipartite graph which is not a forest and $\mathrm{ex}(n, F) = \Theta(n^{\alpha})$, then whp

$$\mathrm{ex}(G_{n,p}, F) = \begin{cases} \Theta(pn^{\alpha}) & p \gg n^{-1/m_2(F)}, \\ (1 + o(1))p\binom{n}{2} & p \ll n^{-1/m_2(F)}. \end{cases}$$

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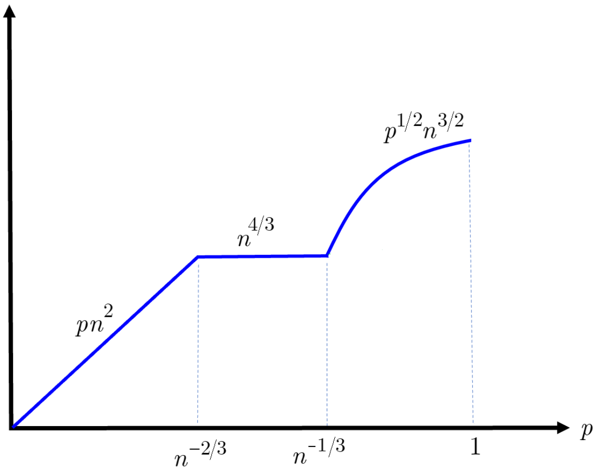
This conjecture turns out to be completely false!



Plot of $\mathrm{ex}(G_{n,p}, C_4)$ (Füredi 1991)

Conjecture (McKinley-S.)

If F is a 2-balanced graph with $\mathrm{ex}(n,F)=\Theta(n^{\alpha})$ for some $\alpha\in(1,2]$, then whp

$$\mathrm{ex}(G_{n,p},F)=\max\{\Theta(p^{\alpha-1}n^{\alpha}),\,n^{2-\frac{v(F)-2}{e(F)-1}}(\log n)^{O(1)}\},$$

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The heart of this problem is in showing $\mathrm{ex}(G_{n,p}, F) = \Theta(p^{\alpha-1} n^{\alpha})$ when p is large.

Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

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Moreover, this bound is tight whenever $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$.

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Moreover, this is tight whenever $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2b}\}) = \Theta(n^{1+1/b})$.

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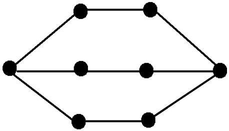

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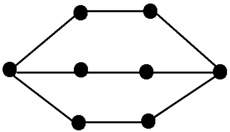

Theorem (McKinley-S. 2023++)

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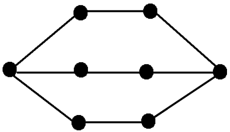

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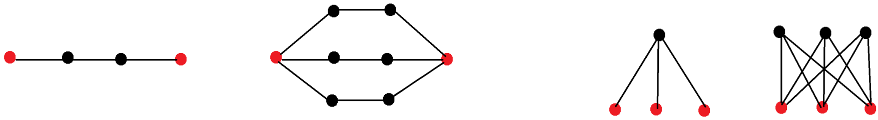
Moreover, this bound is tight whenever a is sufficiently large in terms of b.

Theorem (Bukh-Conlon 2015)

If T^ℓ is the "ℓth power of a balanced tree with density b/a", then $\mathrm{ex}(n, T^\ell) = \Omega(n^{2-a/b})$ *if ℓ is sufficiently large.*



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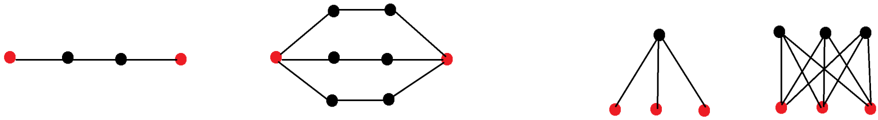


Theorem (S. 2022)

$$\mathrm{ex}(G_{n,p}, T^\ell) = \Omega(p^{1-a/b} n^{2-a/b}).$$

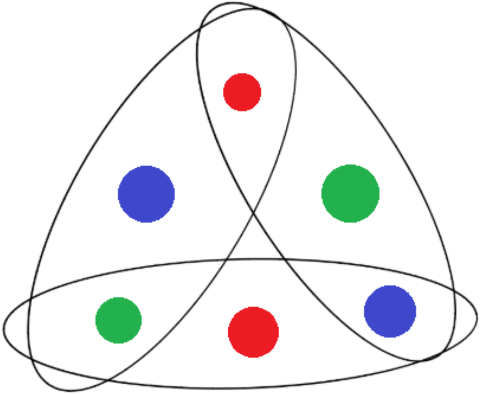
Theorem (Jiang-Longbreak 2022)

If F satisfies "mild conditions", then

$$\mathrm{ex}(G_{n,p}, F) = O(p^{1-m_2^*(F)(2-\alpha)} n^{\alpha}) \text{ for } p \text{ large,}$$

where $m_2^*(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subsetneq F,\ e(F') \geq 2\}$.

Hypergraphs



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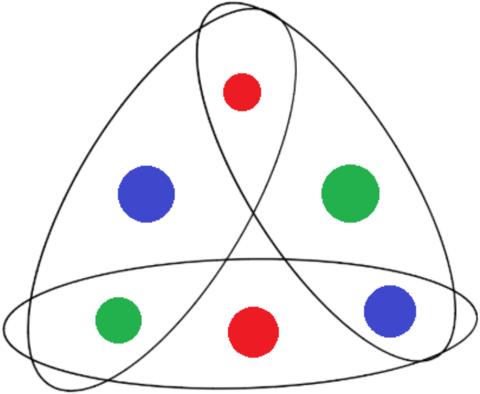
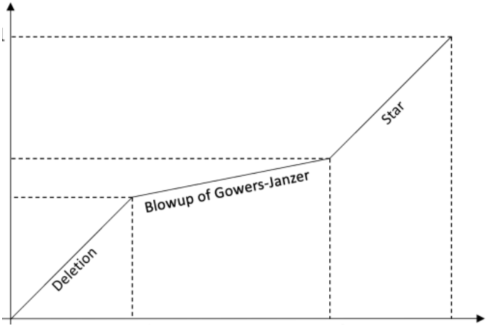


We define the *loose cycle* C_ℓ^r to be the r-uniform hypergraph obtained by inserting $r-2$ distinct vertices into each edge of the graph cycle C_ℓ.

Theorem (Nie-S.-Verstaëte 2020; Nie 2023)

For $r \geq 3$, if $p \gg n^{-r+3/2}$ then whp

$$\mathrm{ex}(G^r_{n,p}, C^r_3) = \max\{p^{\frac{1}{2r-3}} n^{2+o(1)}, pn^{r-1+o(1)}\}.$$



Picture due to Jiaxi Nie.

Theorem (Mubayi-Yepremyan 2020; Nie 2023)

For $r \geq 4$, if $p \gg n^{-r+1+\frac{1}{2\ell-1}}$ then whp

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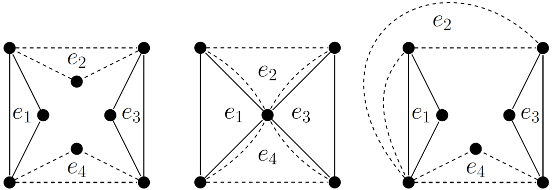
It's suspected that this continues to hold for $r = 3$, but there is a gap for medium values of p.

We say that F is a Berge C_ℓ if it has edges $e_1, \ldots, e_\ell$ and distinct vertices $v_1, \ldots, v_\ell$ with $v_i \in e_i \cap e_{i+1}$ for all i.



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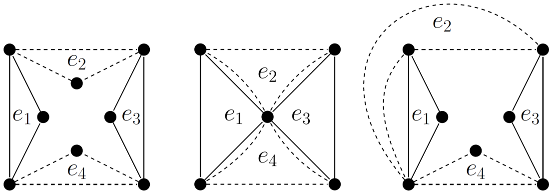
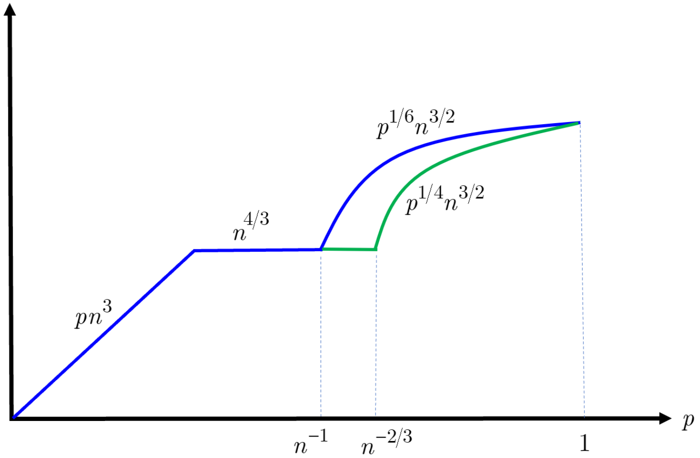


Let $\mathcal{B}^r_\ell$ denote the set of r-uniform Berge C_ℓ's.



Plot of $\mathrm{ex}(G^3_{n,p}, \mathcal{B}^3_4)$ (S.-Verstraëte 2021)

Theorem (S.-Verstraëte 2021)

Let $K^r_{s_1,\dots,s_r}$ denote the complete r-partite r-graph with parts of sizes $s_1,\dots,s_r$. There exist constants $\beta_1,\beta_2,\beta_3,\gamma$ depending on $s_1,\dots,s_r$ such that, for s_r sufficiently large in terms of $s_1,\dots,s_{r-1}$, we have whp

$$\mathrm{ex}(G^r_{n,p}, K^r_{s_1,\dots,s_r}) = \begin{cases} \Theta\left(pn^r\right) & n^{-r} \ll p \le n^{-\beta_1}, \\ n^{r-\beta_1+o(1)} & n^{-\beta_1} \le p \le n^{-\beta_2}(\log n)^\gamma, \\ \Theta(p^{1-\beta_3} n^{r-\beta_3}) & n^{-\beta_2}(\log n)^\gamma \le p \le 1. \end{cases}$$

Lower Bound Techniques

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For the bipartite case, lower bounds for $\mathrm{ex}(n, F)$ often imply lower bounds for $\mathrm{ex}(G_{n,p}, F)$ through "random homomorphisms."

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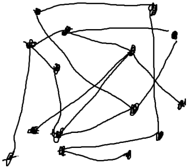

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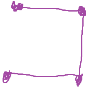

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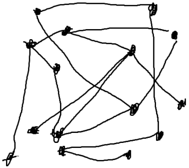

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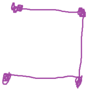

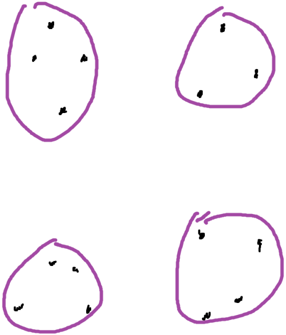

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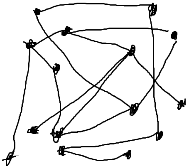

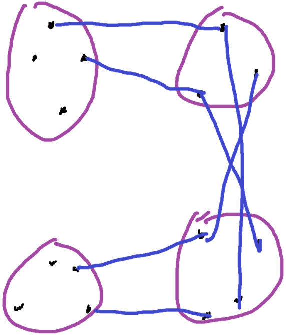

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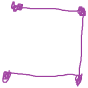

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Theorem (Füredi, 1991; Morris-Saxton, 2013)

If $m \geq n^{4/3}(\log n)^2$, *then*

$$N_m(n, C_4) \leq e^{cm}(\log n)^m \left(\frac{n^{3/2}}{m}\right)^{2m}$$

Upper Bound Techniques

Corollary

If $p \geq n^{-1/3}(\log n)^3$, *then a.a.s.*

$$\mathrm{ex}(G_{n,p}, C_4) \leq O\left(p^{1/2} n^{3/2} \log n\right).$$

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$$\Pr[\mathrm{ex}(G_{n,p}, C_4) \geq m] = \Pr[X \geq 1] \leq \mathbb{E}[X] = p^m \cdot N_m(n, C_4) \approx p^m \cdot \left(\frac{n^{3/2}}{m}\right)^{2m}$$

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- Plenty of problems left to solve!