Card Guessing with Partial Feedback

Sam Spiro, UC San Diego.

Joint work with Persi Diaconis, Ron Graham, and Xiaoyu He

Feedback Models

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To begin the game, shuffle the deck uniformly at random. Each round a Guesser tries to guess what the next card in the deck is, and then the card is revealed and discarded, and we continue this way until the deck is depleted. The score at the end of the game is the total number of correct guesses made during the mn rounds.

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The strategy $\mathrm{G}^{\pm}$ *of guessing a most/least likely card at each stage achieves* $\mathcal{C}^{\pm}_{m,n}$.

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The strategy $\mathrm{G}^{\pm}$ *of guessing a most/least likely card at each stage achieves* $\mathcal{C}^{\pm}_{m,n}$*. Moreover, for* n *fixed*

$$\mathcal{C}^{\pm}_{m,n} = m \pm c_n\sqrt{m} + o_n(\sqrt{m}).$$

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Define the expected score under the partial feedback model by $P_{m,n}(\mathbf{G})$ and its optimal scores by $\mathcal{P}^{+}_{m,n} = \max_{\mathbf{G}} P(\mathbf{G})$ and $\mathcal{P}^{-}_{m,n} = \min_{\mathbf{G}} P(\mathbf{G})$.

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so for n fixed we have $\mathcal{P}^+_{m,n} \sim m$, and similarly $\mathcal{P}^-_{m,n} \sim m$.

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so for n fixed we have $\mathcal{P}^+_{m,n} \sim m$, and similarly $\mathcal{P}^-_{m,n} \sim m$. What happens when n is large?

Feedback Models

Theorem (Diaconis-Graham-He-S., 2020)

For m fixed,

$$\mathcal{C}^{+}_{m,n} \sim H_m \log(n),$$
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where H_m is the mth harmonic number.

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With this we have the trivial bounds

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In particular, with partial information and fixed m, you can't get many points compared to having no feedback at all. This is somewhat intuitive since saying "this card is not type i" becomes less useful as the number of card types grow.

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That is, our upper bound is strongest when g_i and S is small. These conditions are necessary: if i has been guessed incorrectly $g_i = mn - m$ times, then we know the card must be an i, and similarly if the first $S = mn - m$ cards were guessed correctly and not equal to i, then the next card must be i.

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Open Problems

Theorem (Diaconis-Graham-He-S., 2020)

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Conjecture (Diaconis-Graham-He-S., 2020)

$$\mathcal{P}^{+}_{m,n} = m + m^{1/2+o(1)}.$$

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Question

Is there a reasonable choice of feedback so that the maximum expected score is, say, $\Theta_m(\log\log n)$?

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This lower bound is simply the probability of guessing at least one card correctly (the best strategy for this is to just guess each card type exactly m times).

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Increasing Subsequences

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Consider the following strategy in the partial feedback model: guess 1 until you guess one correct, then 2 until you guess one correct, then 3, and so on. After guessing n play arbitrarily.

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Consider the following strategy in the partial feedback model: guess 1 until you guess one correct, then 2 until you guess one correct, then 3, and so on. After guessing n play arbitrarily. If the deck is shuffled according to π, then the Guesser's score is at least $L(\pi)$.

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$$\mathbb{E}[L(\pi)] \le \mathcal{P}^+_{m,n}.$$

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Theorem (Clifton-Deb-Huang-S.-Yoo)

We have

$$\left| \lim_{n\to\infty} \mathcal{L}_{m,n} - \left(m + 1 - \frac{1}{m+2} \right) \right| \leq O(e^{-\beta m})$$

for some $\beta > 0$.

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In particular, the strategy "guess 1 until you get one correct, then 2,..." does not give much more than the trivial bound.

Increasing Subsequences

More precisely: if $\alpha_1, \ldots, \alpha_m$ are the zeroes of $\sum_{k=0}^{m} \frac{x^k}{k!}$, then

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This implies $\mathcal{L}_{1,n} \to e - 1$ and that

$$\mathcal{L}_{2,n} \to e(\cos(1) + \sin(1)) - 1.$$

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Mathematically it's equivalent to have Shuffler iteratively choose each subsequent card in an online fashion (but your friends may disagree if you try this in real life).

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In any case, we let $\mathcal{C}_{m,n}(\mathbf{G}, \mathbf{S})$ be the expected number of points Guesser scores when the two players follow strategies $\mathbf{G}, \mathbf{S}$.

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Theorem (S., 2021+)

If Shuffler wants to minimize the number of correct guesses and Guesser wants to maximize this, then under their optimal strategies G', S' *we have*

$$\mathcal{C}_{m,n}(\mathrm{G}', \mathrm{S}') = \log n + o_m(\log n).$$

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This theorem is a first for me, since normally I prove a result, then makes jokes about it during my talk.

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A strategy that gives this is the "greedy strategy", which is such that if there are r types of cards remaining in the deck, then Shuffler draws each of these card types with probability r^{-1} (regardless of how many copies are left in the deck of each type).

Online Card Guessing

Theorem

There exists a strategy S' *for Shuffler so that* $\mathcal{C}_{m,n}(\mathsf{G},\mathsf{S}') \leq \log n + o_m(\log n)$.

A strategy that gives this is the "greedy strategy", which is such that if there are r types of cards remaining in the deck, then Shuffler draws each of these card types with probability r^{-1} (regardless of how many copies are left in the deck of each type). E.g. if the deck has a hundred 1's and one 2, we draw a 1 or 2 with probability $\frac{1}{2}$.

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Interestingly, the greedy strategy is also the "unique" strategy which maximizes the number of correct guesses if Guesser tries to minimize their score.

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Question

Consider a "semi-restricted" version of Rock Paper Scissors: $3m$ rounds of the game is played but one of the players must use each move exactly m times. What are the optimal strategies/scores in this game?

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Question

What happens if Guesser can guess each card type at most k times?

The End

Thank You!