Card Guessing with Feedback

Sam Spiro, Rutgers University.

Feedback Models

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Theorem (Diaconis-Graham, 1981)

For n fixed,

$$\mathcal{C}^{\pm}_{m,n} = m \pm c_n\sqrt{m} + o_n(\sqrt{m}).$$

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What happens when n is large?

Feedback Models

Theorem (Diaconis-Graham-He-S., 2020)

For m fixed,

$$\mathcal{C}^{+}_{m,n} \sim H_m \log(n),$$
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where H_m is the mth harmonic number.

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With this we have the trivial bounds

$$m \leq \mathcal{P}^{+}_{m,n} \leq \mathcal{C}^{+}_{m,n} = O_m(\log n).$$

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There exist $c, C > 0$ such that if n is sufficiently large in terms of m, we have

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There exist $c, C > 0$ such that if n is sufficiently large in terms of m, we have

$$m + c\sqrt{m} \leq \mathcal{P}^{+}_{m,n} \leq m + Cm^{3/4} \log m.$$

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That is, our upper bound is strongest when g_i and S is small. These conditions are necessary: if i has been guessed incorrectly $g_i = mn - m$ times, then we know the card must be an i.

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Corollary

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Open Problems

Theorem (Diaconis-Graham-He-S., 2020)

$$m + \Omega(\sqrt{m}) \leq \mathcal{P}^{+}_{m,n} \leq m + O(m^{3/4} \log m).$$

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(1) If you made less than $m/2$ correct guesses, guess 1 the rest of the game.

(2) Else guess 2 the rest of the game.

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Practical Strategies

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for some $\beta > 0$.

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Another simple strategy is the *shifting strategy*, which guesses 1 until a correct guess is made, then 2 until a correct guess is made, and so on.

Practical Strategies

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Corollary

If n is sufficiently large in terms of m, then

$$\mathcal{L}_{m,n} := \mathbb{E}[L(\pi)] \leq m + O(m^{3/4} \log m).$$

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Theorem (Clifton-Deb-Huang-S.-Yoo, 2021)

We have

$$\left| \lim_{n \to \infty} \mathcal{L}_{m,n} - \left(m + 1 - \frac{1}{m+2} \right) \right| \leq O(e^{-\beta m})$$

for some $\beta > 0$.

Practical Strategies

More precisely: if $\alpha_1, \ldots, \alpha_m$ are the zeroes of $\sum_{k=0}^{m} \frac{x^k}{k!}$, then

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This implies $\mathcal{L}_{1,n} \to e - 1$ and that

$$\mathcal{L}_{2,n} \to e(\cos(1) + \sin(1)) - 1.$$

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More precisely, we are now considering a two player game played by Shuffler and Guesser. Let $\mathcal{C}_{m,n}(G, S)$ be the expected number of points Guesser scores when the two players follow strategies G, S.

Adversarial Card Guessing

$$\Theta_m(n^{-1/m}) \le \mathcal{C}_{m,n}(G, \mathrm{Uniform}) \le H_m \log n + o_m(\log n).$$

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There exists a strategy S' for Shuffler so that

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This theorem is a first for me, since normally I prove a result, then makes jokes about it during my talk.

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Interestingly, the greedy strategy is also the "unique" strategy which maximizes the number of correct guesses if Guesser tries to minimize their score.

Semi-restricted Games

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The adversarial card guessing game can be viewed as a "semi-restricted" version of this game where mn rounds of Matching Pennies is played and player B must use each number exactly m times.

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In semi-restricted Rock, Paper, Scissors the "greedy strategy" is the unique optimal strategy for the restricted player.

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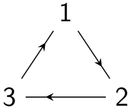
Theorem (S.-Surya-Zeng, 2022)

““Almost every”” semi-restricted game fails to have an optimal strategy which is greedy.

Semi-restricted Games

Given a digraph D, we define its skew adjacency matrix A by $A_{u,v} = +1$ if $u \to v$, $A_{u,v} = -1$ if $v \to u$, and $A_{u,v} = 0$ otherwise.



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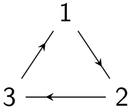

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Question

Which digraphs D are such that their skew-adjacency matrix A satisfies $Null(A) = span(\vec{1})$?