Solutions for Advanced Graph Theory

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Due to limited time the following are only **sketches** of full solutions, and in particular these solutions alone wouldn't necessarily constitute a solution worth full marks. I also emphasize that there may exist other (and possibly simpler) solutions these problems.

1 HW1

0.1 *(Handshaking Lemma) Prove that every graph G has $\sum_{x\in V(G)} \deg(x) = 2e(G)$ [2-].*

Consider the set of pairs $\mathcal{P} = \{(v, e) : v \in V(G),\ e \in E(G),\ v \in e\}$ and for each vertex v let $\mathcal{P}_v = \{(v, e) : e \in E(G),\ v \in e\}$. Then the $\mathcal{P}_v$ sets partition $\mathcal{P}$, giving

$$2e(G) = |\mathcal{P}| = \sum |\mathcal{P}_v| = \sum \deg(v).$$

Alternatively you could consider the set of pairs $\{(v, w) : vw \in E(G)\}$ which gives a similar argument.

0.4 *Prove that a graph is bipartite if and only if it contains no odd cycles [2-].*

If G has an odd cycle $(v_1, \ldots, v_{2\ell+1})$ and a bipartition $V_1 \cup V_2$ with, say, $v_1 \in V_1$, then one can prove inductively that $v_i \in V_j$ iff $i \equiv j \mod 2$. But this implies $v_1 v_{2\ell+1} \in E(G)$ has both vertices in V_1, a contradiction.

For the other direction, one should prove (or at least cite) (i) a graph has an odd cycle if and only if it has a closed walk of odd length, and (ii) a graph is bipartite if and only if each of its connected components is bipartite. With this, if we assume our graph has no odd cycles then you can define a bipartition on each connected component by picking an arbitrary vertex v and defining $V_1 \cup V_2$ by having $u \in V_1$ iff $\mathrm{dist}(u, v)$ is odd. One can check that this can not have, say, $uu' \in E(G)$ with $u, u' \in V_1$ as otherwise this plus the paths from u, u' to v would create an odd closed walk.

1.3a *Prove that*

$$z(m, n; s, t) \le (t-1)^{1/s} m n^{1-1/s} + (s-1)n.$$

(Hint: if you're struggling with this, try solving the previous problem first) [2].

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Consider the set $\mathcal{P}$ of pairs (S, v) where $v \in V$ and $S \subseteq N(v)$ is a set of size s. If G avoids a $K_{s,t}$ then each of the $\binom{m}{s}$ sets S can belong to at most $t-1$ pairs, proving $|\mathcal{P}| \leq (t-1)\binom{m}{s}$. On the other hand a convexity argument shows $|\mathcal{P}| \geq n\binom{n^{-1}e(G)}{s}$, and now some algebra gives the result.

1.3b *Prove that if G is an n-vertex bipartite C_4-free graph then $e(G) \leq 2^{-3/2}n^{3/2} + o(n^{3/2})$, i.e. the lower bound we got for $\mathrm{ex}(n, C_4)$ using G_q was best possible in the setting of bipartite graphs [2-].*

Let G be such a graph with parts of sizes m and $n-m$, and (importantly) without loss of generality we may assume $m \leq n/2$. By the first part with $s = t = 2$ we have $e(G) \leq m(n-m)^{1/2} + (n-m) = m(n-m)^{1/2} + o(n^{3/2})$. One can check (using calculus, for example), that this expression is maximized in the range $0 \leq m \leq n/2$ at the value $m = n/2$, giving the bound. Note crucially that if we did not assume $m \leq n/2$ then the maximum would be at $m = 2n/3$, which would give a suboptimal bound.

1.3c *Prove that for all s, t there exists a constant $C > 0$ such that if G is an n-vertex $K_{s,t}$-free graph, then the number of edges $xy \in E(G)$ with $\deg(x) \geq Cn^{1-1/s}$ is at most $O(n)$. Find an example of a graph which has $\Theta(n)$ edges of this form (Hint: the intended proof I have in mind works with $C \approx (s+t-1)^{1/s}$) [2].*

Define an auxilliary bipartite graph B where one part U consists of all vertices of G with $\deg_G(x) \geq Cn^{1-1/s}$ and the other part V consists of a disjoint copy of $V(G)$ where we have $xy \in E(B)$ if $xy \in E(G)$ and $x \in U$. Observe that B can not contain a $K_{s,t+s}$ with the part of size s in U, since if it did then removing the at most s vertices that appear in both parts of the $K_{s,t+s}$ from the part of size $t+s$ would give a $K_{s,t}$ in G. It follows from (a) that $e(B) \leq (s+t-1)^{1/s}|U|n^{1-1/s} + (s-1)n$. On the other hand, by construction we have $e(B) \geq Cn^{1-1/s}|U|$, so if say $C = 2(s+t-1)^{1/s}|U|n^{1-1/s}$ then this implies $(s-1)n \geq (s+t-1)^{1/s}|U|n^{1-1/s}$, and hence $e(B) \leq (s+t-1)^{1/s}|U|n^{1-1/s} + (s-1)n \leq 2(s-1)n$. But $e(B)$ is exactly the number of edges of the form that we wish to bound, proving the result.

An example of a graph which this is tight is an n-vertex star.

1.4a *Prove that if G is an n-vertex graph then G contains at least $e(G) - \mathrm{ex}(n, F)$ copies of F for any graph F with at least one edge [1].*

One can prove this by induction on $\Delta := e(G) - \mathrm{ex}(n, F)$, for example, the case $\Delta = 0$ being trivial. Inductively for $\Delta > 0$, we have by definition of $\mathrm{ex}(n, F)$ that G contains some copy of F, let e be any such edge. Then inductively $G - e$ contains at least $\Delta - 1$ copies of F and these copies must be distinct from the copy of F containing e (since these copies live in $G - e$), giving the desired result.

1.4b *Prove that if G is an n-vertex graph with $e(G) \geq 100n^{3/2}$ then G contains at least $\Omega(n^{-4}e(G)^4)$ copies of C_4 (the number 100 does not matter in case you'd rather prove this result with a different constant) [2].*

For notational convenience let $d(u, v) = |N(u) \cap N(v)|$. It is not too difficult to see that the total number of C_4's in a graph G is $\sum_{u\neq v}\binom{d(u,v)}{2}$. By convexity this is at least

$$\binom{n}{2}\binom{\binom{n}{2}^{-1}\sum d(u,v)}{2}\approx n^{-2}(\sum d(u,v))^2.$$

Note that $\sum d(u,v)=|\mathcal{P}|$ the set of pairs that we defined in class, and this in turn is lower bounded by roughly $n^{-1}e(G)^2$ by the convexity argument we did in class, and plugging this in gives the desired result.

2 HW2

1.8 *Verify that if G' is an n-vertex complete $(r-1)$-partite graph then $e(G')\le e(T_{r-1}(n))$ [1+].*

Let $n_1,\ldots,n_{r-1}$ be the sizes of a complete $(r-1)$-partite graph on n vertices. Then its number of edges equals

$$\sum_{i<j} n_i n_j.$$

If there exists some i,j with say $n_i\ge 2+n_j$, then one can consider a new sequence of integers defined by replacing n_i,n_j with n_i-1,n_j+1 and one can easily check that this strictly increases the sum above. As such, the sum is maximized when all of these integers are within 1 of each other, and this in turn is only possible if each value is equal to the floor or ceiling of $n/(r-1)$ (since in particular, some value must be at least the floor and some value must be at least the ceiling simply by averaging).

1.9a *Observe that if G is a triangle-free graph, then $\deg(x)+\deg(y)\le v(G)$ for all $xy\in E(G)$. Use this to prove Mantel's Theorem (which is in fact the original way Mantel proved his result) [2].*

Let G be an n-vertex triangle-free graph. Then our observation above implies

$$\sum_{xy\in E(G)}\deg(x)+\deg(y)\le ne(G).$$

On the other hand, each term $\deg(x)$ in this sum appears exactly $\deg(x)$ times, meaning the sum equals $\sum_x \deg(x)^2$. Note that Cauchy-Schwarz implies that for any sequence of n real numbers x_i that $n\sum_i x_i^2=\sum_i 1^2\cdot\sum_i x_i^2\ge(\sum_i x_i)^2$. Applying this here gives $\sum\deg(x)^2\ge n(2e(G))^2$ which exactly gives the bound that we want.

1.9b *Generalize our inductive proof of Mantel's Theorem to give an alternative proof of Turán's Theorem (which is in fact the original way that Turán proved his result).*

Instead of deleting a pair of vertices in an edge like we did for Mantel, we now delete a set of vertices forming a K_{r-1}. In this case no vertex outside the K_{r-1} can be adjacent to every vertex of this K_{r-1}, and the same sort of analysis as we did before proves the result.

1.10 *Let F denote the unique 4-vertex graph with 5 edges (i.e. the graph consisting of two triangles sharing an edge). Prove (without using* **??***) that* $\mathrm{ex}(n, F) = \lfloor n^2/4 \rfloor$ *for all* $n \geq 4$ *[2].*

We prove this by induction on n, the base cases being straightforward. If we have a graph with more than $n^2/4$ edges then we know there exists some triangle xyz. Look at $G - x - y - z$ and observe now that every vertex here has at most one neighbor in x, y, z since otherwise we create our forbidden graph. Then the same analysis we did for our previous inductive proof gives the result.

3 HW3

1.13 *Prove that for every integer $s \geq 1$ and real $\varepsilon > 0$, there exists a graph with average degree at least $2s - \varepsilon$ which contains no non-empty subgraph with minimum degree greater than $s + 1$; that is, the $d/2$ in* **??** *is essentially best possible [2-].*

$K_{s,t}$ with t large has average degree $\frac{2st}{s+t} = 2s - \varepsilon$, and we can guarantee min degree at least s but nothing more.

1.14 *Prove (without using* **??***) that for all $\ell \geq 2$ we have* $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2\ell}\}) = O(n^{1+1/\ell})$ *(Hint: first prove the result under the additional assumption that every vertex of G has degree at least $n^{1/\ell} + 1$) [2].*

We claim that if G is a graph which has no cycles of length at most 2ℓ and minimum degree δ, then for all $0 \leq i \leq \ell$ and every vertex u there exist at least δ^i vertices v such that there exists a directed walk of length i from u to v. We prove this by induction on i, the base case $i = 0$ being trivial. Note that for each vertex v' which can be reached by a walk of length $i - 1$ there exist at least δ vertices which can be reached by a walk of length i, namely by appending any neighbor of v' to its walk. On the other hand, each vertex can be reached by at most one walk of length i as otherwise there would exist a short cycle in G. This implies that the number of vertices we can reach increases by a factor of δ, proving this claim.

For the actual problem, it suffices to prove the result under this min degree condition due to our lemma allowing us to translate from average degree d to min degree $d/2$. By our claim, any given vertex of our graph can reach more than n vertices, a contradiction to our graph having only n vertices.

2.1a *Prove that* $\mathrm{ex}(n, C_n) = \binom{n-1}{2} + 1$ *[2].*

Lower bound comes from K_{n-1} plus a leaf. For the upper bound, take an n-vertex C_n-free graph G with at least $\binom{n-1}{2} + 2$ edges. If we look at two non-adjacent vertices x, y then trivially $\deg(x) + \deg(y) \geq e(G) - \binom{n-2}{2} \geq n$ with the first inequlity using that every edge of G is either between two vertices of $G - x - y$ (of which there can be at most $\binom{n-2}{2}$) or is incident to exactly one of x, y. This together with Ore gives the result.

2.4 *Prove Pósa's Theorem [2].*

Go through our same proof of Dirac's Theorem except this time pick x_1, x_n to be non-adjacent vertices with $\deg(x_1) + \deg(x_n)$ as large as possible. If $\deg(x_1) + \deg(x_n) \geq n$ then we are done as before. Otherwise we must have, say, $\deg(x_1) := k < n/2$. Now for each of the k neighbors x_i of x_1, we must have that $x_{i-1} \not\sim x_n$, which by our choice of non-edge x_1, x_n means that $\deg(x_{i-1}) \leq k$ for all k of these vertices. This contradicts our hypothesis on Ore's Theorem.