Advanced Graph Theory Notes

Sam Spiro

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Contents

0 Overview and Basic Definitions

These notes are intended for a graduate course in graph theory which assumes the reader is already familiar with basic graph theory terms and definitions (see also Section 0.1 for a recap of these definitions). **You should expect many typos and missing references**.

The first half of these notes centers on two of the main areas of modern graph theory: extremal graph theory and structural graph theory. Broadly speaking, extremal graph theory ask questions of the form: how “large” can a graph be if it satisfies a certain property? Structural graph theory, on the other hand, broadly speaking aims to characterize families of graphs which satisfy a certain property. It is worth noting that the exact line between these two areas is rather vague, so some topics may have crossover between each other. It should also be said that I am an extremal graph theorist, so there will certainly be a bias these topics.

The second half of the text centers around “bonus” material which delves into specific methods for solving graph theory problems, as well as auxiliary topics which could be entire courses on their own.

0.1 Very Basic Graph Theory Definitions

Here we briefly recall the basic definitions and notations for graphs that we use throughout the text.

The Essentials:

- A *graph* G is a pair of sets (V, E) with E a set of 2-element subsets of V, i.e. $E \subseteq \{\{x, y\} : x, y \in V,\ x \neq y\}$. The set V is called the *vertex set* of G and its elements are called *vertices*, while the set E is called the *edge set* of G and its elements are called *edges*. We will typically denote edges $\{x, y\}$ by the simpler notation xy.

  Eg $(\{1, 2, 3, 4\}, \{12, 23, 13, 14\})$ is a graph. Often it's easier to depict graphs by pictures (and how exactly we draw the picture doesn't matter).

- Throughout this text we will only consider finite graphs, ie graphs with $|V| < \infty$, but interesting things can be said regarding infinite graphs.

- Throughout this text we will almost always work with graphs without repeated edges (ie E is a set rather than a multiset) and graphs without oriented edges (ie each edge is an *unordered* pair of vertices, meaning $xy = yx$).

- We will often write $V(G)$ and $E(G)$ to denote the vertex and edge sets of a graph G, and we write $v(G) = |V(G)|$ and $e(G) = |E(G)|$.

- We say two vertices x, y are *adjacent* or *neighbors* if $xy \in E(G)$, and we sometimes denote this be writing $x \sim y$.

- Given a vertex x we define the *neighborhood* of x by $N(x) = \{$vertices that are adjacent to x in $G\}$. We define the *degree* of x by $\deg(x) = |N(x)|$. Whenever the graph G is not clear from context we will write $N_G(x)$ and $\deg_G(x)$.

- We say that a graph $G' = (V', E')$ is a *subgraph* of another graph $G - (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In this case we write $G' \subseteq G$.

- We say two graphs G, H are isomorphic if there exists a bijection $\phi : V(G) \to V(H)$ such that $x, y \in V(G)$ are adjacent in G if and only if $\phi(x), \phi(y) \in V(H)$ are adjacent in H for all x, y.

Paths and Connectivity:

- A *path* in a graph G is sequence of distinct adjacent vertices $(x_1, x_2, \ldots, x_t)$, and we say such a path is a path from x_1 to x_t and that it has *length* $t-1$ (i.e. the length of the path is the number of edges it has).

- A graph is *connected* if for any two pair of vertices there exists a path from x to y.

- The *distance* between two vertices x, y, denoted $\mathrm{dist}(x, y)$, is the length of the shortest path from x to y (with $\mathrm{dist}(x, y) = \infty$ if no such path exists).

Graph Operations and Subgraphs

- Given a set S and an integer k, we let $\binom{S}{k}$ denote the set of all subsets of S of size k. For example, our definition of a graph is equivalent to saying that $E \subseteq \binom{V}{2}$.

- Given a graph G we define its *complement* $\overline{G}$ to be the graph obtained by replacing all edges with non-edges and vice versa. That is, $\overline{G}$ is the graph with vertex set $V(G)$ and edge set $\binom{V(G)}{2} \setminus E(G)$.

- Given a graph G and a set of vertices $S \subseteq V(G)$, we define $G - S$ to be the graph obtained by deleting S and all edges incident to it. That is, $V(G - S) = V(G) \setminus S$ and $E(G - S) = E(G) \setminus \{e : e \cap S \neq \emptyset\}$. If $S = \{x\}$ then we will denote this simply by $G - x$. Similarly if xy is an edge of G we define $G - xy$ to be the graph obtained by deleting the edge xy.

- A subgraph $G' \subseteq G$ is said to be *induced* if it is of the form $G - S$ for some set of vertices S. Given a set of vertices V we will sometimes write $G[V]$ to be the induced subgraph with vertex set V, i.e. $G[V] = G - V(G) \setminus V$.

- A subgraph $G' \subseteq G$ is called *spanning* if $V(G') = V(G)$.

Independent Sets and Colorings

- A set of vertices I is *independent* if no two vertices $x, y \in I$ are adjacent to each other.
- A graph is bipartite if there exists a partition of $V(G)$ into two independent sets.
- Given a graph G and an integer k, a *proper k-coloring* is a map $\phi : V(G) \to [k]$ with the property that adjacent vertices $x, y \in V(G)$ have $\phi(x) \neq \phi(y)$. The smallest k for which G has a proper k-coloring is called the *chromatic number* of G and is denoted $\chi(G)$.

Forests and Trees

- A graph is a *forest* if it contains no cycles (i.e. no subgraph isomorphic to a cycle graph C_ℓ). A *tree* is a forest which is connected.
- A vertex of degree 0 is called an *isolated vertex*. A vertex of degree 1 (especially in the context of trees and forests) is called a *leaf*.

0.2 Common Graph Families and Parameters

We record notation for various graphs that will appear throughout the text.

- K_n denotes the n-vertex complete graph, i.e. the unique n-vertex graph with all $\binom{n}{2}$ edges.
- $K_{s,t}$ denotes the complete bipartite graph which has s vertices in one part and t vertices in the other.
- C_ℓ denotes the cycle graph of length ℓ.
- P_r denotes the path graph with r vertices (NOTE: some authors would denote this by P_{r-1}).

We record notation for graph parameters that will appear throughout the text, where here G denotes an arbitrary graph.

- $\delta(G)$ is the minimum degree of G, i.e. $\delta(G) = \min_{x \in V(G)} \deg(x)$.
- $\Delta(G)$ is the maximum degree of G, i.e. $\Delta(G) = \max_{x \in V(G)} \deg(x)$.
- $\alpha(G)$ is the independence number of G, which is the largest size of an independent set of G.
- $\chi(G)$ is the chromatic number of G, which is the smallest integer k such that G has a proper k-coloring.

0.3 Asymptotic Notation

Eventually in the text it will be convenient for us to make use of the following asymptotic notation which we record here for ease of reference. We emphasize that this notation will be redefined when it first appears in the text, so there is no need to memorize this right now.

Let $f(n), g(n)$ be two functions.

- We write $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(n)$ for all n.
- We write $f(n) = \Omega(g(n))$ if there exists a constant $c > 0$ such that $f(n) \leq cg(n)$ for all n.
- We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. In this case we say that f, g have the same *order of magnitude*.
- We write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. In this case we say that f, g are *asymptotic* to each other.
- We write $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. In particular, writing $f(n) = o(1)$ means $\lim_{n\to\infty} f(n) = 0$.

0.4 Inequalities

Many proofs in extremal combinatorics rely on basic inequalities from analysis. Here we record the most important of these that we will use.

Theorem (Cauchy-Schwarz Inequality). *If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are real numbers, then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$$

Personally, we like to remember the statement of Cauchy-Schwarz by noting that it follows from the vector equality $\langle \mathbf{x}, \mathbf{y} \rangle = \cos\theta \, \|\mathbf{x}\| \, \|\mathbf{y}\|$ where θ is the angle between the vectors $\mathbf{x}, \mathbf{y}$.

For the next inequality, recall that a function $\phi : \mathbb{R} \to \mathbb{R}$ is *convex* if for all $0 \leq t \leq 1$ and $x, y \in \mathbb{R}$ we have $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$.

Theorem (Jensen's Inequality). *If ϕ is a convex function and $x_1, \ldots, x_n \in \mathbb{R}$, then*

$$\sum_{i=1}^{n} \phi(x_i) \geq n\phi\left(n^{-1} \sum_{i=1}^{n} x_i\right).$$

That is, this sum is minimized when each x_i is equal to their average $n^{-1} \sum x_{i'}$. We note for later that for any integer $t \geq 1$, the function $\binom{x}{t} := \frac{x(x-1)\cdots(x-t+1)}{t!}$ is convex.

0.5 Exercises

Each chapter will end with a set of exercises. Following the notation of Stanley, we will add numbers after each exercise to indicate the problem's rough level of difficulty as follows:

- [1] problems are elementary and routine requiring little to no thought,
- [2] problems have simple solutions (though that does not necessarily mean it is easy to find such a solution!),
- [3] problems tend to have involved solutions,
- [4] problems have extremely difficult solutions (to the extent that such questions should never be used in a classroom setting),
- [5] problems are unsolved open problems.

Additionally, plus and minus symbols may be used to indicate higher or lower levels of difficulty for the problem. For example, a [2+] problem might have a simple solution that's pretty challenging to find, while a [3-] problem might have an involved solution that's actually not too hard to work out. Ultimately, all of the ratings that I give are only rough estimates and the reader may find a given [3] problem easier to solve than a [2-] depending on the circumstances.

With that preamble out of the way, we begin with some "elementary" (though not necessarily easy) graph theory problems.

1. (Handshaking Lemma) Prove that every graph G has $\sum_{x\in V(G)} \deg(x) = 2e(G)$ [2-].
2. Prove that every graph G with $v(G) \geq 2$ contains two vertices with the same degree [2-].
3. Prove that for every graph G, either G or its complement $\overline{G}$ is connected [2-].
4. Prove that a graph is bipartite if and only if it contains no odd cycles [2-].
5. Prove that for every graph G, the set of edges $E(G)$ can be partitioned into cycles if and only if every very vertex of G has even degree [2+].

* * *

6. Recall that a graph is d-regular if $\deg(u) = d$ for every vertex u. Prove for all integers $0 \leq d < n$ that there exists an n-vertex d-regular graph if and only if at least one of d or n is even [2].
7. A graph is said to have girth g if it contains a cycle of length g and no cycles of shorter length.

(a) Prove that for all integers $d, g \geq 2$, there exists a d-regular graph of girth g [2+].

(b) Prove that if G is a d-regular graph of girth g, then

$$v(G) \leq ???.$$

[2]

(c) Show that the bound above is tight for $d = 3, g = 5$ [1+].

* * *

8. Prove that $\chi(G)\alpha(G) \leq v(G)$ for all graphs G [2-].

9. Prove that $\alpha(G) \geq \frac{v(G)}{\Delta(G)+1}$ for all graphs G [2].

10. Prove that if a graph G is triangle-free (i.e. if G contains no subgraph isomorphic to K_3) then $\alpha(G) \geq \sqrt{v(G)}$ [2-].

* * *

11. Prove that every tree T with $v(T) \geq 2$ has at least two leaves.

12. Prove that for every tree T, there exists an ordering of its vertices $v_1, \ldots, v_n$ such that for all $2 \leq i \leq n$, there exists an integer j_i such that $N(v_i) \cap \{v_1, \ldots, v_{i-1}\} = \{v_{j_i}\}$ [1+].

13. Prove various characterizations of trees

14. (Helly Theorem for Trees) Let T be a tree and $\mathcal{T}$ a set of subtrees of T (i.e. a set of subgraphs of T which are themselves trees). Prove that if $V(T') \cap V(T'') \neq \emptyset$ for all $T', T'' \in \mathcal{T}$, then there exists a vertex $v \in \bigcap_{T' \in \mathcal{T}} V(T')$ [2+].

Part I

Extremal Graph Theory

As mentioned in the introduction, extremal graph theory broadly speaking asks questions of the form: how "large" can a graph be if it satisfies a certain property?

What exactly "large" means depends on the type of problem one is considering, with some popular choices being the number of edges, the number of vertices, and the minimum degree of the graph in question. Each of these choices (together with an appropriate choice of "property") gives rise to three of the main topics of extremal graph theory: Turán problems, Ramsey problems, and Dirac problems; see the table below for a brief outline. Each of these types of problems will be the main topic of focus for the forthcoming chapters.

Measurement		Property		Type of Problem
Number of edges	+	Triangle-free	=	Turán Problems: Section 1
Number of vertices	+	G and $\overline{G}$ are triangle-free	=	Ramsey Problems: Section 3
Minimum degree	+	non-Hamiltonian	=	Dirac Problems: Section 2

Figure 1: A table of measures of "largeness", properties that one can consider, and the problems that these produce. Note that in each case, the given property is harder to fulfill the "larger" G is with respect to its measurement, which is a hallmark of a good extremal problem.

1 Forbidden Subgraphs and Turán Problems

Turán Problems broadly ask: how many edges can an n-vertex graph have if it does not contain a copy of a given graph F? Specifically, the we will work with the following throughout this chapter.

Definition 1. Given two graphs F, G, we say that G is *F-free* if G does not contain a subgraph which is isomorphic to F. Given an integer $n \geq 1$, we define the *Turán number* $\mathrm{ex}(n, F)$ to be the maximum number of edges that an n-vertex F-free graph can have.

The name of the game now is to try and either determine or bound $\mathrm{ex}(n, F)$ for various choices of F.

1.1 Forbidding C_4 and Complete Bipartite Graphs

Perhaps the first question we need to answer is: why should we care about Turán problems in the first place? There are many possible answers to this question, here are a few of my own personal reasons:

- They are natural extremal problem to consider.
- They have applications to various areas of mathematics.
- Solutions to Turán problems often use cool and deep results from other areas of mathematics in interesting ways.
- They're fun!

To try and illustrate these points above, we will begin by studying the Turán problem for $F = C_4$. Historically, this is the second Turán problem to be considered (we will look at the first problem in the following section) and was largely solved by Erdős in year due to its connection to a certain problem in number theory.

The Upper Bound. We begin by establishing an upper bound for this Turán number.

Theorem 1.1. *We have*
$$\mathrm{ex}(n, C_4) \leq \frac{n\sqrt{4n-3}+n}{4}.$$
That is, every n-vertex C_4-free graph has at most this many edges.

We emphasize that this is not a very pretty looking upper bound; we will address this further shortly after the proof.

Proof. In order to prove any upper bound for this problem, we need to get some understanding of what it means for a graph to be C_4-free graph. After thinking about it for long enough, one might come up with the following observation: a graph is C_4-free if and only if every pair of distinct vertices u, v has at most one common neighbor, i.e. there is at most one vertex in $N(u) \cap N(v)$. Indeed, the existence of two vertices in this set together with u, v would exactly define a C_4 in our graph.

Now, a priori, it is not immediate how to use the fact that pairs of vertices have at most one common neighbor to bound the number of edges in our graph. However, one can use it to bound the number of some other object which is "almost" an edge. Namely, let

$$\mathcal{P} = \{(u, x, v) \in V(G)^3 : u \sim x \sim v,\ u \neq v\},$$

which is just the set of P_3's in G. Note that each element of $\mathcal{P}$ can be uniquely identified by picking two distinct vertices to play the roles of u, v together with a common neighbor of these vertices to play the role of x. As such, we have

$$|\mathcal{P}| = \sum_{u \neq v} |N(u) \cap N(v)| \leq \sum_{u,v} 1 = \binom{n}{2},$$

with the inequality using that our graph is C_4-free. Now, we got the first equality above by identifying each element of $\mathcal{P}$ by its first and last vertices u, v and then picking some common neighbor x. Alternatively, we could identify each element of $\mathcal{P}$ by specifying its middle vertex x together with two distinct neighbors u, v of x. As such, we also have

$$|\mathcal{P}| = \sum_{x \in V(G)} \binom{\deg(x)}{2} \geq n \binom{n^{-1} \sum_x \deg(x)}{2} = n \binom{n^{-1} \cdot 2e(G)}{2},$$

where this inequality used Jensen's inequality together with the fact that $\binom{a}{2}$ is a convex function, and the last equality used that $\sum_x \deg(x) = 2e(G)$. Comparing this to the upper bound for $|\mathcal{P}|$ we found above gives

$$n \binom{n^{-1} \cdot 2e(G)}{2} \leq \binom{n}{2}, \tag{1}$$

or equivalently

$$(2e(G))(2n^{-1}e(G) - 1) \leq n(n-1).$$

This in turn is equivalent to having

$$4e(G)^2 - 2ne(G) - n^2(n-1) \leq 0,$$

and solving this exactly gives the desired bound on $e(G)$.

Somewhere in the text I should call this a double counting argument and maybe mention the word cherries/P_2.

□

While the bound of Theorem 1.1 is truly the best we can do using our approach, it is often not a good idea in extremal combinatorics to do things so precisely.

Mantra 1. It is often better to use (slightly) "wastefull" bounds in extremal combinatorics in order to have cleaner proofs and theorem statements.

Knowing when exactly and how to derive such “crude” bounds is an important skill to have in extremal combinatorics, since in practice we do not know a priori if the approach we are currently playing around with is going to give something useful in the end, and until that point it is a bad idea to harp over minute details in the argument.

For example, let us consider the point in the proof where we reached (1). Here an expert might simplify their lives by observing that simple inequalities for binomial coefficients yield

$$n \cdot \frac{1}{2}(n^{-1}2e(G) - 1)^2 \leq n\binom{n^{-1} \cdot 2e(G)}{2} \leq \binom{n}{2} \leq \frac{1}{2}n^2,$$

and rearranging this gives

$$n^{-1}2e(G) - 1 \leq n^{1/2},$$

and hence

$$e(G) \leq \frac{1}{2}n^{3/2} + n.$$

Note that this is extremely close to the optimal bound we get in Theorem 1.1. In particular, one can show that both bounds are ultimately of the form $\mathrm{ex}(n, C_4) \leq \frac{1}{2}n^{3/2} + Cn$ for some sufficiently large constant C. This means that our weakening above captures the “main part” of the bound from Theorem 1.1, in the sense that for n very large the two numbers are very close to each other.

It will be useful going forward to develop notation to measure more precisely what exactly we mean by “very close to each other”.

Definition 2. Let $f(n), g(n)$ be two functions.

- We write $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(n)$ for all n. In particular, our remark in the paragraph above is equivalent to saying that our two bounds give[1] $\mathrm{ex}(n, C_4) \leq \frac{1}{2}n^{3/2} + O(n)$.

- We write $f(n) = \Omega(g(n))$ if there exists a constant $c > 0$ such that $f(n) \leq cg(n)$ for all n.

- We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. In this case we say that f, g have the same *order of magnitude*.

- We write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. In this case we say that f, g are *asymptotic* to each other.

- We write $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. In particular, writing $f(n) = o(1)$ means $\lim_{n\to\infty} f(n) = 0$.

[1] A very persnickety reader might object that actually this doesn't exactly agree with the definition given: the real thing that should be written is $\mathrm{ex}(n, C_4) - \frac{1}{2}n^{3/2} = O(n)$ and the “algebra” of moving $\frac{1}{2}n^{3/2}$ to the other side is not actually valid. It is, however, common practice in the field to use these somewhat imprecise notational implementations in order to make statements easier to read and write, which is the ultimate goal of introducing this in the first place.

Applications. Theorem 1.1 has a number of applications to other areas of mathematics. We will consider one quick example from discrete geometry.

Let $\mathcal{P}$ be a set of points of $\mathbb{R}^2$ and let $\mathcal{L}$ be a set of lines in $\mathbb{R}^2$. We say that a point $p \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$ are *incident* if p lies on the line ℓ. We let $I(\mathcal{P}, \mathcal{L})$ to denote the number of pairs $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ with p and ℓ incident. A natural extremal question to ask is: what is the maximum number of incidences that a given number of points and line can obtain? Trivially one can do no better than n^2, but it is not so immediate how to improve this. We will be able to achieve such an improvement using our Turán result Theorem 1.1.

Corollary 1.2. *If $\mathcal{P}$ is a set of n points in $\mathbb{R}^2$ and if $\mathcal{L}$ is a set of n lines in $\mathbb{R}^2$, then*

$$I(\mathcal{P}, \mathcal{L}) = O(n^{3/2}).$$

Proof. As is often the case for applications, we begin by defining an auxiliary graph related to our problem at hand. To this end, define a bipartite graph G whose vertex set is $\mathcal{P} \cup \mathcal{L}$ where we have $p \sim \ell$ if and only if p and ℓ are incident. Observe that $I(\mathcal{P}, \mathcal{L}) = e(G)$, so bounding the number of incidences is exactly the same thing as bounding the number of edges of G.

Now, for arbitrary bipartite graphs G we could of course have $e(G)$ as large as n^2, but we have some additional structure to work with because G is coming from a set of points and lines. In particular, because every pair of lines intersect in at most one point, G can not contain a C_4 (since such a subgraph would consist of vertices p_1, p_2, ℓ_1, ℓ_2 with p_1, p_2 points common to both ℓ_1 and ℓ_2). This together with the fact that $v(G) = |\mathcal{P}| + |\mathcal{L}| = 2n$ immediately implies that

$$I(\mathcal{P}, \mathcal{L}) = e(G) \le \mathrm{ex}(2n, C_4) = O((2n)^{3/2}) = O(n^{3/2}),$$

with this last step using that this "big oh" notation is not affected by multiplying by a fixed constant. $\square$

While it is neat that we could obtain this purely geometric result using graph theory, we should note that the bound of Corollary 1.2 is not tight, and in fact the true bound is $I(\mathcal{P}, \mathcal{L}) = O(n^{4/3})$. The fact that we obtained a subpar bound should perhaps not come as a surprise, as we used almost no information about the geometry of the Euclidean plane $\mathbb{R}^2$ in our argument. It is, however, possible to derive this optimal bound of $O(n^{4/3})$ if one uses Theorem 1.1 together with some appropriate geometric tools (such as real polynomial partitiontionings). We will not go into this here, but see eg the book by Sheffer for a lot more on this problem and more.

The Lower Bound. Theorem 1.1 shows that $\mathrm{ex}(n, C_4) = O(n^{3/2})$. The immediate question is: is this tight? This is an important question for us to figure out, since e.g. any improvement to Theorem 1.1 would give an improvement to our bound in Corollary 1.2 as well as to any other application we can come up with for $\mathrm{ex}(n, C_4)$.

To see whether our bound is tight, we need to lower bound $\mathrm{ex}(n, C_4)$, i.e. to construct n-vertex graphs with many edges and no C_4's. This, as the reader is welcome to try for themselves, is not so easy to do. To make some headway on this, we usethe following mantra.

Mantra 2. To find a lower bound construction for extremal problems, we should ask ourselves what would need to happen for our extremal upper bound to be (exactly) sharp.

In our case we ask: what would need to happen for us to have $\text{ex}(n, C_4) = \frac{n\sqrt{4n-3}+n}{4}$? Well, this would happen precisely if every inequality throughout our proof of Theorem 1.1 were in fact an *equality*. In particular, our very first inequality $\sum_{u \neq v} |N(u) \cap N(v)| \leq \binom{n}{2}$ must be an equality, and this would imply that *every* pair of distinct vertices in G has exactly 1 common neighbor. Now we have to ask...is this ever possible?

Well, if you think about it for long enough, you might have the wild idea that "every two vertices has exactly 1 common neighbor" is kind of analogous to the statement "every two points in $\mathbb{R}^2$ lie on exactly one common line." Riffing off of this as well as what we did for our application in Corollary 1.2, what if we defined a bipartite graph G by taking a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ and making a point p adjacent to a line ℓ if and only if they are incident? Such a graph will automatically be C_4-free due to the geometry of the situation, so we will win if we can find some points and lines with many incidences.

As hinted at just after Corollary 1.2, it is possible to find n points and lines in $\mathbb{R}^2$ such that $I(\mathcal{P}, \mathcal{L}) = \Omega(n^{4/3})$, giving a corresponding lower bound to $\text{ex}(n, C_4)$, but this is as good as we can hope to do in Euclidean space. However, another wild thought based on what we said around Corollary 1.2 is that our idea of using points and lines does not fundamentally rely on the full geometry of Euclidean space: we only needed the very basic property that two points line on at most one line, and such a property holds for many different types of geometries. In particular, since we're working with finite graphs...why not try and do something with geometries over finite fields?

Recall from algebra[2] that for every prime power q there exists a field $\mathbb{F}_q$ of order q. Again going off what we did in Euclidean space, we want to consider a set of points and lines from the plane $\mathbb{F}_q^2 = \{(x, y) : x, y \in \mathbb{F}_q\}$. There might be some particularly clever choices of points and lines that we could make here, but since we are just playing around, why don't we go ahead and just take all of them. That is, we will take $\mathcal{P} = \mathbb{F}_q^2$ and $\mathcal{L}$ all of the lines in $\mathbb{F}_q^2$. To be clear, lines in $\mathbb{F}_q^2$ are just sets of points in $\mathbb{F}_q^2$ taking on one of two forms: for $a, b \in \mathbb{F}_q$ we define the line $\ell_{a,b} = \{(x, ax + b) : x \in \mathbb{F}_q\}$, and for $c \in \mathbb{F}_q$ we define the vertical lines $\ell_c = \{(c, y) : y \in \mathbb{F}_q\}$. Now define a bipartite graph G_q on $\mathcal{P} \cup \mathcal{L}$ where $p \sim \ell$ if and only if $p \in \ell$. We leave it as an exercise to the reader to verify that G_q is indeed C_4-free. To count $e(G_q)$, we observe that the total number of lines is $q^2 + q$ and that each line is incident to exactly q points, and as such $e(G_q) = q^3 + q^2$. Because the total number of vertices in G_q is exactly $2q^2 + q$, we in total conclude for any prime power q that

$$\text{ex}(2q^2 + q, C_4) \geq q^3 + q^2.$$

By considering $n = 2q^2 + q \approx 2q^2$ or equivalent $q \approx (n/2)^{1/2}$, we find that for infinitely many integers n that $\text{ex}(n, C_4)$ is at least $q^3 \approx (n/2)^{3/2} = 2^{-3/2} n^{3/2}$. As such, the upper bound of $\text{ex}(n, C_4) = O(n^{3/2})$ really is the best we can do for general n! In fact, some basic number theory facts let us prove the following.

Theorem 1.3. *We have* $\text{ex}(n, C_4) = \Theta(n^{3/2})$.

Proof. By Theorem 1.1 we have for n large enough that, say, $\text{ex}(n, C_4) \leq n^{3/2}$, proving $\text{ex}(n, C_4) = O(n^{3/2})$.

[2]The reader who is scared of algebra should be reassured that this is the only fact you need to recall from algebra.

Now consider any integer n. By Bertrend's postulate, there exists a prime number p with $\frac{1}{2}\sqrt{n/3} \le p \le \sqrt{n/3}$. This in particular implies $n \ge 3p^2 \ge 2p^2 + p$, which together with our discussion above implies

$$\mathrm{ex}(n, C_4) \ge \mathrm{ex}(2p^2 + p, C_4) \ge p^3 \ge (12)^{-3/2} n^{3/2},$$

proving that $\mathrm{ex}(n, C_4) = \Omega(n^{3/2})$ and hence that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$ as desired. $\square$

We personally find it fascinating that one can use ideas from algebra and geometry to solve the purely combinatorial problem of determining $\mathrm{ex}(n, C_4)$. This is in fact a very common phenomenon.

Mantra 3. To solve a combinatorics problem, one often needs ideas and tools from other areas of math. As such, any extra knowledge you have outside of combinatorics is always useful to keep in the back of your mind!

This mantra is intended to be inspirational rather than intimidating. In particular, even if you don't have hardly any knowledge in areas outside of combinatorics (such as myself), you can still make it very far, its just that some problems in particular may elude your grasps until you figure out the right tool needed to crack it.

Even Better Lower Bounds. We've done pretty good so far with our lower bounds for $\mathrm{ex}(n, C_4)$, but we can go even farther.

Mantra 4. Once you prove something, see if you can prove something even better.

In particular, given that we have determined the order of magnitude $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$, we should next ask ourselves if we can prove that $\mathrm{ex}(n, C_4) \sim cn^{3/2}$ for some constant c. We emphasize that doing this will require a bit more algebra/geometry than before, and as such the reader may wish to skip over this part of the text if they're already overwhelmed.

Returning back to the problem at hand, we know

$$2^{-3/2} n^{3/2} + o(n^{3/2}) \le \mathrm{ex}(n, C_4) \le 2^{-1} n^{3/2} + o(n^{3/2}),$$

and we need to figure out if we can sharpen either of these bounds. For this, it is useful to analyze "why" our lower bound proof does not match the bound we got in the upper bound. After all, in our construction every pair of points really does have exactly one common neighbor. However, if we look back at what motivated our construction in the first place, we recall that for the upper bound for Theorem 1.1 to be exactly sharp that we need every pair of *vertices* to have a common neighbor, and there is no hope of that happening for our current graph because G_q is bipartite (meaning a given point and a given line will never have any common neighbors in G_q).

It is not so immediate how to fix this problem, as the underlying motivation for our construction relied on working with both points and lines which intrinsically are different objects from each other. But, if we stare at things long enough, we might realize that our lines $\ell_{a,b}$ are indexed by points in $\mathbb{F}_q^2$, and as such, one might possibly have the idea where we could consider a graph G where its vertex set is just $\mathbb{F}_q^2$ but where a point (x, y) corresponds to both the point itself and the line $\ell_{x,y}$. That is, we want to define a graph on $\mathbb{F}_q^2$ where $(x, y) \sim (a, b)$ if and only if $(x, y) \sim \ell_{a,b}$. While this is a noble idea, an immediate issue in this definition is that this edge relation is not symmetric. That is, having $(x, y) \in \ell_{a,b}$ does not imply $(a, b) \in \ell_{x,y}$ (i.e. $y = ax + b$ does not mean $b = xa + y$). At a very high level the issue here with the idea of identifying points with a corresponding line is that points and lines are not truly "dual" to each other in $\mathbb{F}_q^2$. However, this can be fixed by going to yet another type of geometry, namely projective geometry.

Insert better intuition on projective geometries at some point.

To define things, consider the set of triples $T = \{(x, y, z) : x, y, z \in \mathbb{F}_q^3\} \setminus \{(0, 0, 0)\}$ and define an equivalence relation (not to be confused with an edge relation) by having $(x, y, z) \equiv (\alpha x, \alpha y, \alpha z)$ for all $\alpha \in \mathbb{F}_q \setminus \{0\}$. Let $[x, y, z]$ denote the equivalence class containing (x, y, z), and define our set of "points" $\mathcal{P}$ to be the set of all such equivalence classes. For each $[a, b, c] \in \mathcal{P}$ we define the line $\ell_{[a,b,c]} = \{[x, y, z] : ax + by + cz = 0\}$. Note that this definition is well-defined (i.e. it does not matter whether we write $[x, y, z]$ or $[\alpha x, \alpha y, \alpha z]$) since having $ax + by + cz = 0$ implies $\alpha ax + \alpha by + \alpha cz = 0$ for all $\alpha \neq 0$. Also note that this definition is truly "dual" in points and lines, in that $[x, y, z] \in \ell_{[a,b,c]}$ if and only if $[a, b, c] \in \ell_{[x,y,z]}$. Motivated by this and our ideas from above, we define a graph G_q^* on $\mathcal{P}$ where $[x, y, z] \sim [a, b, c]$ if and only if $[x, y, z] \in \ell_{[a,b,c]}$. We leave it as an exercise to verify that G_q^* is C_4-free, that $v(G_q^*) = q^2 + q + 1$, and that $e(G_q^*) = \frac{1}{2}(q + 1)(q^2 + q + 1)$.

Similar to before, if we take $n = q^2 + q + 1 \approx q^2$, then we see that this shows $\mathrm{ex}(n, C_4)$ is at least $\frac{1}{2}q^3 \approx \frac{1}{2}n^{3/2}$, exactly matching the asymptotic bound from Theorem 1.1! Actually, even more is true: one can check that the upper bound $\frac{n\sqrt{4n-3}+n}{4}$ is actually *exactly* tight in this case. That is, for all prime powers q, we have

$$\mathrm{ex}(q^2 + q + 1, C_4) = \frac{1}{2}(q + 1)(q^2 + q + 1).$$

Generalizations. Given our success with studying the Turán problem for C_4, we should go on and ask to what extent can the ideas here be used to prove bounds for other graphs F. Naively one might first consider the problem for other cycles C_ℓ, but this turns out to be pretty difficult. Instead, the "correct" generalization of the ideas we have here are for complete bipartite graphs $K_{s,t}$ in general beyond just that of $K_{2,2} = C_4$. For example, we leave it as an exercise to generalize the upper bound in Theorem 1.1 to prove the following general upper bound.

Theorem 1.4 (Kővári-Sós-Turán Theorem)**.** *For all integers $s, t \geq 1$, we have*

$$O_{s,t}(n^{2-1/s}).$$

Here we add the s, t subscript to the big-oh notation to emphasize that the implicit constant depends on s, t. This is not entirely necessary since we fix s, t at the start of the theorem, but it is sometimes nice to emphasize this for clarity.

This gives an upper bound, what about a corresponding lower bound? Our lower bound $\mathrm{ex}(n, C_4) = \Omega(n^{3/2})$ immediately implies $\mathrm{ex}(n, K_{2,t}) = \Omega(n^{3/2})$ for all $t \geq 2$, giving the correct order of magnitude. In fact, Füredi**REF**improved the lower bound for $\mathrm{ex}(n, K_{2,t})$ even further, giving a tight asymptotic bound. With some effort, one can generalize the geometric intuition we had for C_4 to prove $\mathrm{ex}(n, K_{3,t}) = \Theta(n^{5/3})$ for all $t \geq 3$. Despite this success, the next case of this problem remains open.

Open Problem 1.5. *Determine the order of magnitude of* $\mathrm{ex}(n, K_{4,4})$.

Similarly $\mathrm{ex}(n, K_{s,s})$ remains open for all $s \geq 4$. However, it turns out that we can solve this problem for $K_{s,t}$ whenever t is sufficiently large in terms of s.

Theorem 1.6. *For all $s \geq 2$, there exists an integer t_0 such that $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$ for all $t \geq t_0$.*

The first result of this form was proven by Authors who showed one can take $t_0 =$Something by using an explicit algebraic construction like we had for G_q^*. The best current bound is due to Bukh who recently showed one can take $t_0 =$Something by using a *random* algebraic construction.

1.2 Forbidding Cliques

Now that we've all been convinced that studying $\mathrm{ex}(n, F)$ is an interesting problem, we need to figure out some graphs F for which we can effectively bound (or even determine) $\mathrm{ex}(n, F)$. As a starting step, we can think about this problem for small graphs F. A moment's thought shows that it is quite easy to determine $\mathrm{ex}(n, F)$ for every graph F with $v(F) \leq 3$ *except* for the graph $F = K_3$, which is the smallest non-trivial instance of this problem. The full solution to this problem is a classical result of Mantel from 1907, which is perhaps the first ever theorem in extremal graph theory.

Theorem 1.7 (Mantel's Theorem)**.** *We have $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ for all $n \geq 1$. Moreover, the only n-vertex K_3-free graphs with $\lfloor n^2/4 \rfloor$ edges are those which are isomorphic to the balanced complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

There are many proofs for Mantel's Theorem (the textbook "Proofs from the Book" contains a full 7 proofs, and there are many more than just these!). We will content ourselves with only a single proof here, though we sketch out a few more in the exercises.

Proof. Perhaps one reasonable approach when given a problem like this is to try and prove things by induction on n, which is indeed what we shall try and do, though we will have to be a little careful.

Indeed, consider the following naive approach of using induction: let G be an n-vertex K_3-free graph and v an arbitrary vertex of G. Inductively we know that $e(G - v) \leq \lfloor (n-1)^2/4 \rfloor$, and hence $e(G) \leq \lfloor (n-1)^2/4 \rfloor + \deg(v)$. Unfortunately though this bound is not good enough: if, say $G = K_{1,n-1}$ and v were the center of the star then this would give a bound of $\lfloor (n-1)^2/4 \rfloor + n - 1$, which is too large. One can try and be smarter by picking v to be a vertex of minimum degree, but as far as we are aware this still is not enough to prove the result. To deal with this, we will prove the result by removing *two* vertices at a time rather than just one.

To this end, observe that the result is true for $n = 1, 2$. Assume we have proven the result up to some value $n \geq 3$ and let G be an n-vertex triangle-free graph. If $e(G) = 0$ then we are done, so we can assume G has an edge xy. By induction, we know that $e(G-x-y) \leq \lfloor (n-2)^2/4 \rfloor = \lfloor n^2/4 \rfloor - n + 1$, and hence that

$$e(G) = e(G-x-y) + \deg(x) + \deg(y) - 1 \leq \lfloor n^2/4 \rfloor + \deg(x) + \deg(y) - n.$$

Finally, because G is triangle-free (which is a fact we must use somewhere in our argument), we must have $N(x) \cap N(y) = \emptyset$, as any common neighbor z would form a triangle with the edge xy. We conclude then that

$$\deg(x) + \deg(y) = |N(x)| + |N(y)| = |N(x) \cup N(y)| \leq n,$$

which combined with the bound above give the desired bound.

To prove the equality case, again one can show this holds for $n = 1, 2$. Inductively then, the only way for the bound $e(G-x-y) \leq \lfloor (n-2)^2/4 \rfloor$ to be tight is if $G-x-y = K_{\lfloor n/2 \rfloor - 1, \lceil n/2 \rceil - 1}$, and similarly the only way the bound $|N(x) \cup N(y)| \leq n$ can be tight is if every vertex of $G-x-y$ is adjacent to exactly one of x, y, which is only possible if G is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. $\square$

Similar to how the "correct" way to generalize our bound for C_4 in Theorem 1.1 was to consider complete bipartite graphs, it turns out that the "correct" way to generalize Mantel's Theorem is to consider larger cliques K_r. And indeed, just like the case of triangles, the Turán number for cliques in general can be solved exactly and has a unique extremal construction which is defined as follows.

Definition 3. Given integers $r, n \geq 1$, we define the *Turán graph* $T_{r-1}(n)$ to be the $(r-1)$-partite graph whose part sizes are as equal as possible, i.e. such that each part either has size $\lfloor n/(r-1) \rfloor$ or size $\lceil n/(r-1) \rceil$

Theorem 1.8 (Turán's Theorem)**.** *For all integers $r \geq 2$ and $n \geq 1$, we have $\mathrm{ex}(n, K_r) = e(T_{r-1}(n))$. Moreover, the only n-vertex K_r-free graph with $e(T_{r-1}(n))$ edges are those which are isomorphic to $T_{r-1}(n)$.*

Again there are many different proofs of Turán's Theorem, and again we limit ourselves to just a single one here based on the following idea.

Mantra 5. If you think an extremal problem has a unique optimal construction, then try and prove this by "shifting" an arbitrary construction to look like the optimal construction.

For example, in the setting of Turán's Theorem we might want to somehow shift an arbitrary K_r-free graph into a graph that, like the Turán graph $T_{r-1}(n)$, is complete $(r-1)$-partite. And indeed this is always possible to do.

Lemma 1.9 (Zykov Symmeterization)**.** *For every K_r-free graph G, there exists a graph G' satisfying the following:*

- $V(G') = V(G)$,

- $\deg_{G'}(x) \geq \deg_G(x)$ *for all* $x \in V(G)$*, and*
- G' *is complete* $(r-1)$*-partite.*

Proof. We prove the result by induction on r, the case $r = 2$ being trivial. Let $x \in V(G)$ be a vertex of maximum degree. Observe that $H := G[N(x)]$ must be K_{r-1}-free, as any K_{r-1} in H together with x would form a K_r. By induction we can find a complete $(r-2)$-partite graph H' satisfying the conditions of the lemma for H. Now define G' to be the graph formed by starting with H' and then adding every edge from $V(H') = N_G(x)$ to the remaining vertices $x \cup (V(G) \setminus N_G(x))$.

Observe that $V(G') = V(G)$ and that G' is complete $(r-1)$-partite (namely by considering the $r-2$ parts from H' together with the part $x \cup (V(G) \setminus N_G(x))$, so it remains to check the degree condition. If $y \in V(H') = N_G(x)$ then

$$\deg_{G'}(y) = \deg_{H'}(y) + (1 + v(G) - \deg_G(x)) \geq \deg_H(y) + |N_G(y) \setminus N_G(x)| = \deg_G(y).$$

On the other hand, for every $y \notin N_G(x)$ we have

$$\deg_{G'}(y) = v(H') = \deg_G(x) \geq \deg_G(y),$$

with this last inequality using that x was chosen to be a vertex of maximum degree. □

We now use this result to prove Turán's Theorem, though for simplicity we will omit the proof of uniqueness.

Proof of Turán's Theorem. Let G be an n-vertex K_r-free graph. By Zykov symmeterization, we know that there exists an n-vertex complete $(r-1)$-partite graph G' with at least as many edges as G, and it is a simple exercise to show that any such graph has at most as many edges as $T_{r-1}(n)$, proving the result. □

As a historical aside, Turán proved this result without being aware of Mantel's Theorem, and in this paper he went on to introduce the general problem of determining $\mathrm{ex}(n, F)$ for various graphs F, which is why the "Turán number" bears his name.

1.3 Forbidding Trees

We have now solved the Turán problem for the "densest" graphs K_r. We now turn to solving the problem for the "sparsest" graphs, namely that of forests and trees. The simplest case of this problem is that of stars, which is easy to solve exactly.

Proposition 1.10. *For all* $r \geq 2$*, we have* $\mathrm{ex}(n, K_{1,r-1}) \leq \frac{r-2}{2}n$ *with equality if and only if at least one of* r *or* n *is even.*

Proof. A graph G being $K_{1,r-1}$-free is the same as saying that G has maximum degree at most $r-2$. Thus, any n-vertex $K_{1,r-1}$-free graph satisfies

$$e(G) = \frac{1}{2} \sum \deg(x) \leq \frac{1}{2} \sum r - 2 = \frac{r-2}{2} n,$$

proving the upper bound. This upper bound is tight whenever there exists an n-vertex $(r-2)$-regular graph, which holds precisely if at least one of r or n is even. $\square$

Note that in this example there are infinitely many extremal constructions, which is a significantly different phenomenon compared to what we saw when forbidding cliques.

We next turn to the problem of avoiding an arbitrary tree T, for which we might ideally like to generalize our argument for stars. Unfortunately unlike in this case we can not say that an arbitrary T-free graph has small maximum degree, but we can prove the slightly weaker statement that such a graph has small minimum degree.

Lemma 1.11. *If T is a tree with r vertices and if G is a graph with minimum degree at least $r-1$, then G contains a copy of T.*

Note that the bound of $r-1$ is best possible, as can be seen by considering graphs G which are disjoint unions of copies of K_{r-1}. We present two essentially equivalent proofs of this result, the first of which is a little vaguer but requires less knowledge of trees while the second is a bit more explicit/algorithmic.

First Proof. We prove the result by induction on r, the case $r=2$ being trivial. Assume we have the proven the result up to some $r \geq 3$ and let T be an arbitrary r-vertex tree.

Because T is a tree, there exists some leaf x with some vertex y its unique neighbor. Because G has minimum degree at least $r-1 \geq r-2$, we inductively can assume that G has a copy of $T' = T - x$. Now the vertex playing the role of y in this copy of T' has at least $r-1$ neighbors, of which at most $r-2$ of them lie in this copy of T'. In particular, there exists at least one neighbor which is not in T', and taking this together with the copy of T' gives a copy of T gives the desired result. $\square$

Second Proof. We build up our copy of T algorithmically "vertex by vertex." To do this we require the fact that for every r-vertex tree, there exists an ordering of the vertices $v_1, \ldots, v_r$ such that for all $2 \leq i \leq r$ there exists an integer $j_i < i$ such that $N_T(v_i) \cap \{v_1, \ldots, v_{i-1}\} = \{v_{j_i}\}$.

Let y_1 be an arbitrary vertex of G. Iteratively given that we have chosen vertices $y_1, \ldots, y_{i-1}$ in G for some $i \leq r$, we choose y_i to be an arbitrary vertex in $N_G(y_{j_i})$ which is not in the set $\{y_1, \ldots, y_{i-1}\} \setminus \{y_{j_i}\}$. Note that the number of such vertices is at least $r-1-(i-2) \geq 1$, so there does indeed exist a valid choice for y_i, and as such this algorithm will successfully terminate. With this, it is not difficult to see that the y_i vertices form a copy of T, giving the result. $\square$

The result above gives a tight bound on the minimum degree needed to contain a copy of T, but we ultimately want a bound on $\mathrm{ex}(n, T)$, i.e. on the *average* degree needed to find a copy of T. Fortunately, there is a general result which allows us to translate between the concept of minimum degrees and average degrees.

Proposition 1.12. *If G is a graph of average degree at least d, then there exists a non-empty subgraph $G' \subseteq G$ with minimum degree at least $d/2$ and average degree at least d.*

For most applications of this result we will only need the conclusion that G' has large minimum degree, but sometimes it is useful to also have this additional average degree condition (see for example Theorem 2.8). Again we offer two essentially equivalent proofs of this result, both of which implicitly use that the average degree by definition is

$$v(G)^{-1} \sum \deg(x) = \frac{2e(G)}{v(G)}.$$

First Proof. Assume the result is false for a given d and graph G, and choose such a counterexample with $v(G)$ as small as possible. If $\delta(G) \geq d/2$ then taking $G' = G$ gives the desired subgraph, a contradiction. As such, we can assume that G contains a vertex x with $\deg(x) < d/2$. In this case, the graph $G - x$ has a smaller number of vertices and average degree

$$\frac{2e(G-x)}{v(G-x)} = \frac{2e(G) - 2\deg(x)}{v(G)-1} \geq \frac{2e(G) - d}{v(G)-1} \geq d,$$

with this last step using that $2e(G) \geq dv(G)$ by hypothesis. Since $G - x$ is a graph with fewer vertices than G and with average degree d, our choice of G having $v(G)$ as small as possible implies that there exists $G' \subseteq G - x \subseteq G$ satisfying the properties of the statement, giving another contradiction. $\square$

Second Proof. The key idea of the argument is to start with $G' = G$ and then iteratively remove vertices of low degree, i.e. as long as G' contains a vertex of degree less than $d/2$ then we remove this vertex and we repeat this until no such vertices exist. Note that the total number of edges that we remove in this process is certainly less than

$$(d/2) \cdot v(G) \leq e(G),$$

with this inequality being equivalent to saying that G has average degree at least $d/2$. As such, the resulting graph G' has at least one edge and has minimum degree at least $d/2$ by construction. One can similarly check that it has average degree at least d, proving the result. $\square$

This in total lets us prove the following.

Theorem 1.13. *For any r-vertex tree T, we have*

$$\frac{r-2}{2}n - O_r(1) \leq \mathrm{ex}(n, T) \leq (r-2)n$$

Proof. For the lower bound we take the disjoint union of copies of K_{r-1}, which is certainly T-free and which has the stated number of edges.

For the lower bound, assume that there exists an n-vertex T-free graph G with $e(G) > (r-2)n$, i.e. with average degree more than $2(r-2)$. By Proposition 1.12 there exists a subgraph G' of G with minimum degree more than $r-2$, i.e. with $\delta(G') \geq r-1$. By Lemma 1.11 $G' \subseteq G$ contains a copy of T, a contradiction. $\square$

While Theorem 1.13 solves the Turán problem for trees up to a factor of 2, one can ask if one can give an even more precise answer. In particular, given that the lower bound of Theorem 1.13 is the truth for the case of stars, it is natural to believe this should be the answer in general.

Conjecture 1.14 (Erdős-Sós)**.** *Every r-vertex tree T satisfies* $\mathrm{ex}(n,T) \le \frac{r-2}{2}n$.

There are a number of special cases for which the Erdős-Sós Conjecture is known to be true (such as for paths; see Theorem 2.8), but overall the problem of improving the small gap from Theorem 1.13 for all T seems difficult to do

1.4 An Aside: Degenerate vs Non-Degenerate Turán Problems

At this point we've studied $\mathrm{ex}(n,F)$ for a lot of classes of graphs F, but still we've said almost nothing about graphs in general. Part of the issue is that the Turán number can behave in very different ways depending on the structure of the graph F, in the following sense.

Proposition 1.15. *Let F be a graph.*

- *If F is non-bipartite, then* $\mathrm{ex}(n,F) = \Theta(n^2)$.
- *If F is bipartite, then* $\mathrm{ex}(n,F) = O(n^{2-1/v(F)})$.

Proof. For any graph we have $\mathrm{ex}(n,F) \le e(K_n) = \binom{n}{2} = O(n^2)$. If F is further non-bipartite, then the balanced complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is F-free and shows that $\mathrm{ex}(n,F) \ge \lfloor n^2/4\rfloor = \Omega(n^2)$, proving the first part. For the second part, because F is bipartite, we have $F \subseteq K_{v(F),v(F)}$, and hence by Kővári-Sós-Turán,

$$\mathrm{ex}(n,F) \le \mathrm{ex}(n, K_{v(F),v(F)}) = O(n^{2-1/v(T)}).$$

□

This observation divides the study of Turán number into two distinct cases: the *non-degenerate* case which studies non-bipartite F (i.e. those graphs with $\mathrm{ex}(n,F) = \Theta(n^2)$), and the *degenerate* case which studies bipartite F (i.e. those graphs with $\mathrm{ex}(n,F) = o(n^2)$). In what follows we very briefly survey some each of these cases. Some of these results require some real machinery to prove, and as such will be deferred until much later in the text.

The Non-Degenerate Case. For non-bipartite graphs F, the most important theorem is undoubtedly the following.

Theorem 1.16 (Erdős-Stone-Simonovits)**.** *For any graph F with at least one edge, we have*

$$\mathrm{ex}(n,F) = \left(\frac{\chi(F)-2}{\chi(F)-1} + o(1)\right)\binom{n}{2}.$$

In particular, this result determines the asymptotic value of $\mathrm{ex}(n,F)$ for *any* non-bipartite[3] graph F. The lower bound for this is rather easy: the Turán graph $T_{\chi(F)-1}(n)$ has chromatic

[3]If F is bipartite the theorem simply says $\mathrm{ex}(n,F) = o(n^2)$, which follows from Kővári-Sós-Turán.

number $\chi(F) - 1$ and hence is F-free and has about $\frac{\chi(F)-2}{\chi(F)-1}\binom{n}{2}$ edges. The upper bound is somewhat difficult to prove, and we will defer this until we have the powerful machinery of the regularity lemma at our disposal.

Because of the power of the Erdős-Stone-Simonovits theorem, the non-degenerate case of the Turán problem is often considered to be largely solved. That being said, one can ask for your favorite non-bipartite graph F if one can prove sharper (possibly even exact) bounds on $\mathrm{ex}(n, F)$ or to determine the full set of optimal extremal constructions. There are a number of results in this direction, with perhaps the most useful being the following result of Simonovits.

Theorem 1.17. *Let F be a graph which is "edge-critical", meaning it contains an edge e with $\chi(F - e) < \chi(F)$. Then $\mathrm{ex}(n, F) = e(T_{\chi(F)-1}(n))$ for all n sufficiently large, and moreover the unique extremal construction for n sufficiently large is $T_{\chi(F)-1}(n)$.*

We emphasize that the assumption of n being sufficiently large is necessary in general. Indeed, we always have $\mathrm{ex}(n, F) = \binom{n}{2}$ whenever $n < v(F)$, and for small n this will typically be better than the bound given in Theorem 1.17.

Finally, we note that while the Erdős-Stone-Simonovits Theorem largely solves the case of non-degenerate Turán problems for graphs, the analogous problem for *hypergrpahs* remains very wide open. We'll touch on this a bit more somewhere later.

The Degenerate Case. While non-degenerate Turán problems for graphs are largely solved, nothing could be farther from the case for degenerate Turán problems. Indeed, even determining the order of magnitude of relatively simple bipartite graphs remain open despite decades of study. We already mentioned that for complete bipartite graphs that $\mathrm{ex}(n, K_{s,s})$ remains open for all $s \geq 4$. Similarly for even cycles (which is perhaps the next most natural class of bipartite graphs to study) our knowledge of what's going on is largely summarized as follows.

Theorem 1.18. *For all $\ell \geq 2$, we have $\mathrm{ex}(n, C_{2\ell}) = O(n^{1+1/\ell})$. Moreover, this is known to be best possible whenever $\ell = 2, 3,$ or 5.*

That is, we know the Turán number for C_4, C_6, and C_{10}, but frustratingly not for C_8! This is roughly because there happen to exists a class of very particular algebraic objects which just so happen to solve these cases and no others. Another frustrating problem is that of the 3-dimensional hypercube graph Q_3, which can be viewed as the "skeleton" of a usual cube. Determining $\mathrm{ex}(n, Q_3)$ was one of the original problems that Turán raised back in his 1941 paper on the topic, but to date only the following bounds are known.

Theorem 1.19. *We have $\mathrm{ex}(n, Q_3) = O(n^{8/3})$ and $\mathrm{ex}(n, Q_8) = \Omega(n^{3/2})$.*

The lower bound comes simply by considering an extremal C_4-free graph. The upper bound is based on a "supersaturation" argument of Erdős and Simonovits from 1969.

Much more can be said about what we do not know about Turán numbers of bipartite graphs, see Survey.

1.5 Exercises

1. Verify that the graphs G_q, G_q^* defined in the first subsection are C_4-free and that $v(G_q^*) = q^2+q+1$ and $e(G_q^*) = \frac{1}{2}(q+1)(q^2+q+1)$ [1+].

2. Prove the Kővári-Sós-Turán Theorem, Theorem 1.4 [1+].

3. Given integers $m, n, s, t \geq 1$, define the *Zarankiewicz number*[4] $z(m, n; s, t)$ to be the maximum number of edges in a bipartite graph G with parts U, V satisfying $|U| = m, |V| = n$, and that G no copy of $K_{s,t}$ with the part of size s in U and the part of size t in V.

   (a) Prove that
   $$z(m, n; s, t) \leq (t-1)^{1/s} mn^{1-1/s} + (s-1)n.$$
   (Hint: if you're struggling with this, try solving the previous problem first) [2].

   (b) Prove that if G is an n-vertex bipartite C_4-free graph then $e(G) \leq 2^{-3/2} n^{3/2} + o(n^{3/2})$, i.e. the lower bound we got for $\mathrm{ex}(n, C_4)$ using G_q was best possible in the setting of bipartite graphs [2-].

   (c) Prove that for all s, t there exists a constant $C > 0$ such that if G is an n-vertex $K_{s,t}$-free graph, then the number of edges $xy \in E(G)$ with $\deg(x) \geq Cn^{1/s}$ is at most $O(n)$. Find an example of a graph which has $\Theta(n)$ edges of this form (Hint: the intended proof I have in mind works with $C \approx (s+t-1)^{1/s}$) [2].

   (d) Use (a) with $s = t = 2$ to give a generalization of Corollary 1.2 [1].

4. The Turán problem involves graphs with 0 copies of a given graph F (where here by a *copy* we mean a subgraph isomorphic to F). What about graphs with more copies?

   (a) Prove that if G is an n-vertex graph then G contains at least $e(G) - \mathrm{ex}(n, F)$ copies of F for any graph F [1].

   (b) Prove that if G is an n-vertex graph with $e(G) \geq 100n^{3/2}$ then G contains at least $\Omega(n^{-6} e(G)^4)$ copies of C_4 (The number 100 does not matter here in case you'd rather prove this result with a different constant) [2].

   Note that the number of copies guaranteed in (b) is far more than the naive bound given by (a). This sort of phenomenon of graphs with $e(G)$ just above $\mathrm{ex}(n, F)$ having a surprisingly large jump in the number of copies of F is known as *supersaturation*.

   (c) Prove that for all m with $100n^{3/2} \leq m \leq \binom{n}{2}$ that there exists an n-vertex graph G with $e(G) = \Theta(m)$ and with $\Theta(n^{-6} m^4)$ copies of C_4 (Hint: consider something random) [2+].

   Insert something about triangles at some point.

[4] Some texts define $z(m, n; s, t)$ with respect to G which are $K_{s,t}$-free rather than simply avoiding things on one side like we have here.

5. Prove that $\mathrm{ex}(n, K_{3,3}) = \Omega(n^{5/3})$ [3].

6. Prove that $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-1/s})$ for all t sufficiently large in terms of s [3+].

* * *

7. Determine $\mathrm{ex}(n, F)$ for all graphs F with $2 \le v(F) \le 3$ other than $F = K_3$. Why did I leave out the case $v(F) = 1$? [1].

8. Verify that if G' is an n-vertex complete $(r-1)$-partite graph then $e(G') \le e(T_{r-1}(n))$ [1+].

9. Here we sketch a few alternative proofs of Mantel's Theorem and Turán's Theorem.

   (a) Observe that in a triangle-free graph G, we have $\deg(x) + \deg(y) \le v(G)$ for all $xy \in E(G)$. Use this to prove Mantel's Theorem (which is in fact the original way that Mantel proved his result) [2].

   (b) Generalize our inductive proof of Mantel's Theorem to give an alternative proof of Turán's Theorem (which is in fact the original way that Turán proved his result) [2].

10. Let F denote the unique 4-vertex graph with 5 edges (i.e. the graph consisting of two triangles sharing an edge). Prove (without using Theorem 1.17) that $\mathrm{ex}(n, F) = \lfloor n^2/r \rfloor$ for all $n \ge 4$ [2].

11. If F denotes the "bowtie" graph consisting of two triangles sharing a vertex, show that $\mathrm{ex}(n, F) = \lfloor n^2/r \rfloor + 1$ for all $n \ge 6$ Double check this [3-].

* * *

12. Determine $\mathrm{ex}(n, P_4)$ exactly for all n (Hint: characterize all connected P_4-free graphs) [2].

13. Prove that for every integer $s \ge 1$ and real $\varepsilon > 0$, there exists a graph with average degree at least $2s - \varepsilon$ which contains no non-empty subgraph with minimum degree greater than $s + 1$; that is, the $d/2$ in Proposition 1.12 is essentially best possible [2].

* * *

14. One can consider Turán problems which avoids more than just a single graph at a time. To this end, given a set of graphs $\mathcal{F}$, we say that a graph G is $\mathcal{F}$-free if G is F-free for all $F \in \mathcal{F}$.

    Prove that for all $\ell \ge 2$ we have $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2\ell}\}) = O(n^{1+1/\ell})$ (Hint: first prove the result under the additional assumption that every vertex of G has degree at least $n^{1/\ell}$) [2].

15. Prove that if F is a graph with $\mathrm{ex}(n, F) = \Omega(n)$ and if F' is a graph obtained from F by adding a new vertex x and making it adjacent to a vertex $y \in V(F)$, then $\mathrm{ex}(n, F') = \Theta(\mathrm{ex}(n, F))$. In other words, to determine the order of magnitude of $\mathrm{ex}(n, F)$ for all graphs F, it suffices to do so for all graphs with minimum degree at least 2 [2].

16. (Füredi) Prove that if F is a bipartite graph where every vertex on one side of the bipartition has degree at most r, then $\mathrm{ex}(n, F) = O(n^{2-1/r})$. Show that this bound is best possible for all r [3+].

2 Spanning Subgraphs and Dirac Problems

Up to this point we have considered the Turán number $\mathrm{ex}(n, F)$ where we think of F as a fixed graph and n as tending towards infinity, but this is not the only regime that could be considered. For example, $\mathrm{ex}(n, C_n)$ asks for the maximum number of edges that an n-vertex graph can have without containing a Hamiltonian cycle. More generally, we might consider $\mathrm{ex}(n, F_n)$ where F_n is some sequence of spanning subgraphs of K_n.

Unfortunately the Turán problem for spanning subgraph tends not to be very interesting. For example, one can show $\mathrm{ex}(n, C_n) \geq \binom{n-1}{2} + 1$ by taking G to be a clique on $n-1$ vertices together with a single vertex of degree 1, and it is not too difficult to show that this somewhat silly construction is best possible. More generally, $\mathrm{ex}(n, F_n)$ tends to be ludicrously large for a number of natural choices of F_n simply by considering graphs G which have a single vertex of small degree. This leads us to another mantra.

Mantra 6. If an extremal problem has a known or boring optimal construction, try modifying or adding extra restrictions to the problem in such a way that any solution to this new problem must be "far" from the known/boring construction.

In particular, our current construction for $\mathrm{ex}(n, C_n)$ is boring because we can trivially make constructions by using vertices of very small degrees. So what if we instead forced our constructions to all have large minimum degree? This leads to the following broad type of problem.

Definition 4. Given a graph F, we define[5] the *Dirac number* $\delta^*(F)$ to be the smallest number δ^* such that any $v(F)$-vertex graph G with $\delta(G) \geq \delta^*$ has a copy of F as a spanning subgraph.

Note that we have already seen some problems somewhat similar to δ^* when we were working on Turán numbers for trees via Lemma 1.11. We will see another application of min degree results to Turán problems with Theorem 2.8.

2.1 Hamiltonian Cycles

Recall that a graph G is Hamiltonian if it contains a cycle passing through all of its vertices. Historically, the first study of Dirac numbers came from Dirac who determined $\delta^*(C_n)$, i.e. the smallest minimum degree of an n-vertex graph G which guarantees that G is Hamiltonian.

To start our investigation, let us try to think of some graphs with large minimum degree which do not have a Hamiltonian cycle. One immediate way to tell that a graph does not have a Hamilotonian cycle is if the graph is disconnected. In particular, if we consider G to be the n-vertex graph which is the disjoint union of $K_{\lceil n/2 \rceil}$ and $K_{\lfloor n/2 \rfloor}$, then this is a graph with no Hamiltonian cycle and with minimum degree $\lfloor n/2 \rfloor - 1$, showing that we must have $\delta^*(C_n) \geq \lfloor n/2 \rfloor$. While perhaps not as obvious, there exists another construction that gives a very similar bound which one might discover by looking at the cases of small n, for example. Specifically, any graph of the form $K_{m,n-m}$ with $m < \lceil n/2 \rceil$ will fail to be Hamiltonian. Indeed,

[5]This name is completely made up since we are not aware of any standard name for this parameter in the literature.

if n is odd this is immediate because $K_{m,n-m}$ is bipartite and hence can not contain C_n. If n is even then any Hamilton cycle in such a graph must have exactly $n/2 = \lceil n/2 \rceil$ of its vertices lying in each part of $K_{m,n-m}$, which is impossible to do under the condition $m < \lceil n/2 \rceil$. This construction thus implies that $\delta^*(C_n) \geq \lceil n/2 \rceil$, which matches the bound in the previous construction if n is even and does a little better if n is odd. In total it turns out that this bound is indeed the correct one.

Theorem 2.1 (Dirac's Theorem)**.** *Every n-vertex graph G with $\delta(G) \geq n/2$ contains a Hamiltonian cycle.*

The reader should double check that this, together with our constructions from above, is equivalent to saying that $\delta^*(C_n) = \lceil n/2 \rceil$. Before we get on with the proof, let us make the meta-observation that for n even there are two extremal constructions for Dirac's Theorem (the disjoint union of two equally sized cliques, and a slightly unbalanced complete bipartite graph). This is non-ideal due to the following

Mantra 7. Extremal problems tend to be harder if they have more than one extremal constructions, especially if these constructions look very different from each other.

Indeed, part of the ease of proving Turán's Theorem is that there is only one possible extremal construction, which means we can hope to do arguments like Zykov symmeterization which move us closer to this unique extremal example. However, this approach as well as many others fail when there are multiple different looking extremal examples because whatever argument we make must simultaneously be optimal for all of our possible constructions.

To partially deal with this issue, we will utilize another mantra.

Mantra 8. If during a proof you assume that there exists some counterexample to your statement, it is sometimes useful to assume this counterexample is "extremal" in some sense.

We will see a concrete example of this in our following proof of Dirac's Theorem, which is originally due to Posa maybe.

Proof of Dirac's Theorem. Assume for some integer n that there exists a counterexample G and, crucially, choose such a counterexample with as many edges as possible. Intuitively by choosing a graph with more edges should make it easier for us to construct a Hamiltonian cycle, giving the desired contradiction. In particular, this assumption gives us the following key fact.

Claim 2.2. *The graph G contains a Hamiltonian path $x_1 \cdots x_n$.*

Proof. This is trivial if $G = K_n$, so assume this is not the case, i.e. that there exists some non-edge $xy \notin E(G)$. Because $G + xy$ is an n-vertex graph with $\delta(G + xy) \geq \delta(G) \geq n/2$ and with strictly more edges than G, it must be that $G + xy$ contains a Hamiltonian cycle C by assumption of G being a counterexample with the maximum number of edges. The subgraph $C - xy$ then must be a Hamiltonian path. □

The other key observation we will need is the following.

Claim 2.3. *If there exists an integer $2 \leq i \leq n$ such that $x_i \sim x_1$ and $x_{i-1} \sim x_n$, then G is Hamiltonian.*

Proof. Consider the following sequence of vertices:

$$P = (x_1, x_i, x_{i+1}, \ldots, x_{n-1}, x_n, x_{i-1}, x_{i-2}, \ldots, x_2).$$

It is not difficult to see that P is a Hamiltonian path (i.e. every vertex appears exactly once and consecutive vertices are adjacent) with its first and last vertices being adjacent to each other. Therefore this defines a Hamiltonian cycle in G, proving the claim. $\square$

As an aside, the idea in this claim of "rotating" the Hamiltonian path we started with into a new one P is a common idea known as a Pósa rotation.

Back to our problem at hand, we want to show that an index i as in the claim exists. To this end, define

$$X_1 = \{i : x_i \sim x_1\},$$
$$X_n = \{i : x_{i-1} \sim x_1\}.$$

By the claim above and our assumption that G is Hamiltonian, we can assume that X_1, X_n are disjoint subsets of $\{2, \ldots, n\}$. This implies that

$$n - 1 \geq |X_1 \cup X_n| = |X_1| + |X_n| = \deg(x_1) + \deg(x_n) \geq n,$$

a contradiction. $\square$

Even though Dirac's Theorem is tight, it is still possible to ask for strengthenings of this result as follows.

Mantra 9. After proving a theorem, check to see where you use the hypothesis of your theorem and if these can be relaxed in any way.

For example, the only place where we really used $\delta(G) \geq n/2$ in our proof of Dirac's Theorem was to show that $\deg(x_1)+\deg(x_n) \geq n$. A moments thought then shows that our proof actually implies the following stronger result.

Theorem 2.4 (Ore's Theorem)**.** *If G is an n-vertex graph such that for every non-edge $xy \notin E(G)$ we have $\deg(x) + \deg(y) \geq n$, then G is Hamiltonian.*

In fact, our proof has much more flexibility that can be exploited to prove a number of other extensions. We state another one here and leave its proof as an exercise to the reader.

Theorem 2.5 (Pósa's Theorem)**.** *If G is an n-vertex graph such that for all integers $k < n/2$,*

$$|\{x \in V(G) : \deg(x) \leq k\}| < k,$$

then G is Hamiltonian.

These extensions of Dirac's Theorem, in addition to being nice on their own, also have various applications to them, such as the following.

Theorem 2.6. *If G is an n-vertex graph with $\delta(G) \geq \frac{n+1}{2}$, then for every edge $xy \in E(G)$ there exists a Hamiltonian cycle in G which uses the edge xy.*

Proof. Let $xy \in E(G)$ be an arbitrary edge, and consider a new graph G' obtained by adding a new vertex v which is adjacent to only x, y. This $(n+1)$-vertex graph G' satisfies the conditions of Pósa's Theorem (it has only 1 vertex of degree at most 2, and every other vertex has degree at least $v(G')/2$), so G' contains a Hamiltonian cycle C. Note that this Hamiltonian cycle must contain the edges xv, vy since these are the only two neighbors of v. As such, the graph $C - v + xy$ is a Hamiltonian cycle in G using the edge xy, proving the result. □

2.2 Applications to Paths

Having just determined the optimal minimum degree needed to guarantee a graph contains a Hamiltonian cycle, it is natural to ask what conditions guarantee a Hamiltonian path. In fact, this turns out to be a consequence of Dirac's Theorem.

Theorem 2.7. *We have $\delta(P_n) = \lfloor n/2 \rfloor$. Equivalently, any n-vertex graph G with $\delta(G) \geq \frac{n-1}{2}$ contains a Hamiltonian path and this bound is best possible.*

Proof. The fact that this bound is best possible follows by considering G to be the disjoint union of two cliques of sizes $\lfloor n/2 \rfloor, \lceil n/2 \rceil$.

Now let G be an n-vertex graph with $\delta(G) \geq \frac{n-1}{2}$ and consider a new graph G' obtained by adding a vertex v which is adjacent to every vertex of G. Then $\delta(G') \geq (n+1)/2 = v(G')/2$, so by Dirac's Theorem G' contains a Hamiltonian cycle C, and thus $C - v$ is a Hamiltonian path in G. □

The trick we used in the proof above lets us easily translate many of the results that we have for Hamiltonian cycles to that of Hamiltonian paths; see the exercises for more.

We can also use Dirac's Theorem to prove good bounds for Turán numbers of paths.

Theorem 2.8 (Erdős-Gallai)**.** *For all $r \geq 2$, we have $\mathrm{ex}(n, P_r) \leq \frac{r-2}{2} n$.*

Note that this bound is tight whenever $r - 1 | n$, as can be seen by considering G to be the disjoint union of K_{r-1}'s.

Proof. By prove the result by double induction on r and n. The result for all n is trivial when $r = 2$, so assume we have proven the result for all n up to some value r. This result in turn is trivial if $n \leq r - 1$, so we assume we have the proven the result up to some value $n \geq r$. With this in mind, let G be an extremal n-vertex P_r-free graph and assume for contradiction that $e(G) > \frac{r-2}{2} n$.

Because our extremal example looks like a disjoint union of K_{r-1}'s, a perhaps reasonable thing to try and prove is the following.

Claim 2.9. *The graph G contains a cycle C with $r - 1$ vertices.*

Proof. By Proposition 1.12, there exists a subgraph $G' \subseteq G$ with minimum degree at least $\frac{r-1}{2}$ (i.e. strictly more than $\frac{r-2}{2}$) and average degree strictly more than $r-2$. By induction on r and the fact that G' has average degree more than $r-2$, we conclude that G' must contain a path $x_1 \cdots x_{r-1}$.

Now all of the neighbors for x_1, x_{r-1} must lie within $\{x_1, \ldots, x_{r-1}\}$, as otherwise $G' \subseteq G$ would contain a path on r vertices. Because $\deg_{G'}(x_1), \deg_{G'}(x_{r-1}) \geq \frac{r-1}{2}$, the exact same argument that we used in the proof of Dirac's Theorem implies that there exists a cycle C using all of the vertices in $\{x_1, \ldots, x_{r-1}\}$. □

Observe that every vertex in C can only be adjacent to other vertices of C, as one could use any additional neighbor together with C to construct a P_r in G. As such, the number of edges incident to the vertices of C is at most $\binom{r-1}{2}$, and as such the graph $G - V(C)$ is a smaller order graph which has

$$e(G - V(C)) > \frac{r-2}{2}n - \binom{r-1}{2} = \frac{r-2}{2}(n-r+1),$$

and since $G - V(C)$ has $n-r+1$ vertices, we conclude by induction on n that $G - V(C)$ has a P_r, giving the result. □

2.3 Clique Factors

Perhaps after Hamiltonian cycles and paths, the next most natural spanning structure to consider is that of a perfect matching, i.e. a disjoint union of K_2's which cover every vertex of the graph exactly once. Note that perfect matchings can only exist if the number of vertices in our graph is even.

While a natural problem to consider, perfect matchings will turn out to not be very interesting to study for two reasons. First, any graph with an even number of vertices and a Hamiltonian cycle (or path) contains a perfect matching, so by Dirac's Theorem we know that $\delta(G) \geq n/2$ is enough to guarantee a perfect matching, and this is best possible by considering $K_{n/2-1,n/2+1}$. Second, one can in fact characterize *exactly* when a given graph has a perfect matching as we shall see in later section, so just proving a sufficient condition is not so interesting.

While the exact problem of determining minimum degree conditions for perfect matchings is not so exciting, there are generalizations of perfect matchings for which this is very interesting. To this end, we say that a K_r*-matching* in a graph G is a subgraph of G which is the disjoint union of copies of K_r, and we say that G has a K_r*-factor* if G has a K_r-matching which contains every vertex of G exactly once. Note that G can only hope to have a K_r-factor if $r|n$.

Theorem 2.10 (Hajnal-Szemerédi Theorem Version I)**.** *If G is an n-vertex graph with $r|n$ and $\delta(G) \geq (r-1)n/r$, then G contains a K_r-factor.*

The Hajnal-Szemerédi Theorem is a deep result with a number of applications, see for example coloring chapter. The original proof of this result was quite difficult. There does exist a quite short proof due to Kierstead and Kostochka, but it is a little too dense to present here I think that's the case; double check. Rather than spending time on proving this in full, we will instead sketch out how to prove a somewhat weaker result.

Proposition 2.11. *If G is an n-vertex graph with $r|n$ and $\delta(G) \geq (r-1)n/r$, then G has a K_r-matching which contains all but at most $(r-1)^2 r$ vertices of G.*

Sketch of Proof. The rough idea is to consider a largest K_r-matching in G and argue that it has at least this size. However, to make the argument work we need to assume something slightly stronger about our matching.

To this end, let $S_1, \ldots, S_{n/r}$ be a partition of $V(G)$ into sets of size r such that $G[S_i]$ contains K_r for as many i as possible, and conditional on this, we choose this partition so that $G[S_i]$ contains a K_{r-1} for as many i as possible, and so on. Let $C_i \subseteq S_i$ denote a largest clique in $G[S_i]$ and assume for contradiction that $G[C_i] \neq K_r$ for at least $(r-1)^2+1$ values of i. By the Pigeonhole principle, this implies there is some $\ell \in [r-1]$ such that $|C_i| = \ell$ for at least r values of i, say for all $i \in [r]$ without loss of generality. Let $N(C_i)$ denote the set of common neighbors of C_i, i.e. the vertices adjacent to every vertex of C_i.

Claim 2.12. *We have $|N(C_i)| \geq (r-\ell)n/r$ and $N(C_i) \cap C_j = \emptyset$ for all $i, j \in [r]$.*

Proof. The lower bound $|N(C_i)| \geq (r-\ell)n/r$ follows from the fact that each of the ℓ vertices of C_i have minimum degree at least $(r-1)n/r$, i.e. are non-adjacent to at most n/r vertices. For the second part, assume for contradiction that there exists some $v \in N(C_i) \cap C_j$ and let $w \in S_j \setminus C_j$ be arbitrary (which exists since $|C_j| < r = |S_j|$). In this case, we could change our partition by replacing S_i, S_j with $S_i \cup \{v\} \setminus \{w\}$ and $S_j \setminus \{v\} \cup \{w\}$, which would increase the number of sets in the partition which contain a $K_{\ell+1}$ while not decreasing the number of sets containing any larger clique, contradicting how we chose our partition. We conclude that no such v exists. $\square$

In total this claim implies $\sum_{i=1}^{r} |N(C_i) \cap \bigcup_{j>r} C_j| \geq (r-\ell)n$, which by the Pigeonhole principle implies there is some $j > r$ such that

$$\sum_{i=1}^{r} |N(C_i) \cap C_j| \geq \left\lceil \frac{(r-\ell)n}{n/r - r} \right\rceil \geq r(r-\ell)+1.$$

Claim 2.13. *There exists some distinct $i', i'' \in [r]$ and disjoint $C_j', C_j'' \subseteq C_j$ of sizes 1 and $r-\ell$ such that $C_j' \subseteq N(C_{i'}) \cap C_j$ and $C_j'' \subseteq N(C_{i''}) \cap C_j$.*

Proof. By the inequality above and the Pigeonhole principle, there exists $i' \in [r]$ such that $|N(C_{i'}) \cap C_j| \geq r-\ell+1$, and since $|N(C_{i'}) \cap C_j| \leq r$ we have

$$\sum_{i \in [r] \setminus \{i'\}} |N(C_i) \cap C_j| \geq r(r-\ell-1)+1,$$

so again by the Pigeonhole principle there exists $i'' \neq i'$ such that $|N(C_{i''}) \cap C_j| \geq r-\ell$. Let $C_j'' \subseteq N(C_{i''}) \cap C_j$ be an arbitrary subset of size $r-\ell$ and let $C_j' \subseteq N(C_{i'}) \cap C_j$ be an arbitrary vertex disjoint from C_j', giving the result. $\square$

Let $w \in S_{i'} \setminus C_{i'}$ be arbitrary. If we consider modifying the partition by replacing $S_{i'}, S_{i''}, S_j$ (whose largest cliques have sizes ℓ, ℓ, r) with the r-sets $S_{i'} \cup C_j' \setminus \{w\}$, $C_{i''} \cup C_j''$, and $S_j \cup \{w\} \setminus$

$(C_j' \cup C_j'')$ (whose largest cliques have sizes at least $\ell+1, r, 1$), we see that this strictly increases the number of sets in our partition containing a $K_{\ell+1}$ while maintaining the sizes of all larger cliques, a contradiction to how we chose our partition. $\square$

We emphasize that for many Dirac-type problems it is relatively easy to find an "almost spanning" subgraph like we did here, but finding a genuinely spanning structure is often difficult. One general tool for doing this is the absorption method which we probably won't talk about, but we'll see what happens.

2.4 Exercises

1. Prove that $\mathrm{ex}(n, C_n) = \binom{n}{2} + 1$ [2+].

2. The original proof of Dirac's theorem went as follows:

   (a) Prove that if G is a (finite) graph with minimum degree $d \geq 2$, then G contains a cycle on at least $d+1$ vertices. [1+]

   (b) Define a *lollipop* to be a graph which consists of a cycle on vertices $v_1, \ldots, v_\ell$ together with a path on vertices $u_1, \ldots, u_t$ with $u_1 = v_1$. Given a graph G, consider its "largest" lollipop, i.e. the one which has ℓ as large as possible and conditional on this has t as large as possible

   Prove that if such a largest lollipop has $\ell \geq 3$ and $t \geq 2$, then u_t is not adjacent to any two consecutive vertices in $v_1, \ldots, v_\ell$. Similarly prove that if $\ell \geq 3$ then u_t is not adjacent to any v_i vertex which is "close" to v_1 (in particular, prove this is true for v_ℓ, v_2, then generalize this as much as you can) [2].

   (c) Conclude Dirac's Theorem [2]. .

3. Prove Pósa's Theorem [2].

4. Prove that if G is an n-vertex graph with $\delta(G) \geq (n+k)/2$ for some integer $k \geq 0$, then for any path $P \subseteq G$ on k edges there exists a Hamiltonian cycle of G which contains P as a subgraph (Hint: the trick we did before for $k = 1$ using Pósa's Theorem no longer works here, so you'll have to go back and modify the proof of Dirac's Theorem we presented instead) [2].

5. Prove that if G is an n-vertex graph and $\delta(G) \geq n/2$, then for every edge of G there exists a Hamiltonian path of G containing this edge [1+].

3 Ramsey Theory

Turán's original motivation for the Turán problem came from another area of extremal combinatorics known as Ramsey theory. In a very abstract sense, Ramsey theory (which extends far beyond just that of graphs) aims to prove that every sufficiently large structure contains relatively simple and orderly substructures. The original problem, as well as the namesake of the theory, comes from the following foundational result of Ramsey[6] from **REF**.

Definition 5. A *red-blue edge coloring* of a graph G is a map $\chi : E(G) \to \{\text{red, blue}\}$. We say that such a coloring has a *monochromatic* K_n if there exists a subgraph of G isomorphic to K_n such that either every edge of the subgraph is colored red or if every edge of the subgraph is colored blue.

Theorem 3.1 (Ramsey's Theorem)**.** *For all integers $n \geq 1$, there exists a (finite) N such that every red-blue edge coloring of K_N contains a monochromatic K_n.*

Equivalently, this says that for all integers $n \geq 1$, there exists some (finite) N such that every N-vertex graph G either contains a clique of size n or an independent set of size n (as can be seen by coloring the edges of K_N red if they belong to G and blue otherwise). That is, large graphs can not simultaneously have arbitrarily large clique and independent numbers.

The original proof of Ramsey's Theorem does not give explicit bounds on the size of N, and the central problem ins Ramsey Theory is to get better bounds on this quantity.

Definition 6. We define the *(diagonal) Ramsey number* $R(n)$ to be the smallest integer N such that every red-blue edge coloring of K_N contains a monochromatic K_n.

There are many variants of this classical Ramsey number $R(n)$, several of which we will discuss below.

3.1 Classical Bounds

Let us start by working some small examples to give a little intuition for the problem in general. It is immediate that $R(1) = 1$ and $R(2) = 2$, so the first non-trivial case of the problem is to determine[7] $R(3)$.

Proposition 3.2. *We have $R(3) = 6$.*

Proof. The lower bound comes from giving a coloring of the edges of K_5 which does not contain a triangle. The unique way to do this is to take a $C_5 \subseteq K_5$ and color its edges red with the remaining edges (which also form a C_5) being colored blue. It is easy to check that such a coloring has no monochromatic triangle.

[6] Funnily enough Ramsey was not a combinatorialist but rather a logician, and to this day there is still a lot of work on Ramsey theoretic problems from the perspectives of both logic and combinatorics.

[7] Colloquially this result is known as the "party problem" due to the following interpretation of its statement: if there are 6 people at a party, then there exist 3 people there who either all know each other or who all do not know each other.

For the upper bound, consider an arbitrary red-blue coloring of the edges of K_6 and assume for contradiction that this did not contain a monochromatic triangle. Let u be an arbitrary vertex, and observe that u has 5 total edges incident to it each of which is given one of 2 colors, so by the pigeonhole principle at least 3 of the edges of u all have the same color, say without loss of generality that the edges uv_1, uv_2, uv_3 are all colored red. Now if any edge v_iv_j is colored red then u, v_i, v_j would form a red triangle, so we can assume that all of the edges v_iv_j are colored blue. But in this case v_1, v_2, v_3 forms a blue triangle, again yielding a contradiction. $\square$

At its core, the reason that the upper bound proof worked is that if a red-blue coloring does not contain a monochromatic K_3, then the "red neighborhood" of any vertex u can not contain either a red K_2 nor a blue K_3. Building on this idea leads to the following definition.

Definition 7. Given integers m, n, we define $R(m, n)$ to be the smallest integer N such that if every edge of K_N is colored either red or blue, then there either exists a red K_m or a blue K_n.

For example, one can check that $R(2, 3) = 3$ which is implicitly what we used in our upper bound proof for $R(3)$. Generalizing this idea gives the following observation of Erdős and Szekeres.

Lemma 3.3 (Erdős-Szekeres)**.** *For all $m, n \geq 2$, we have*

$$R(m, n) \leq R(m-1, n) + R(m, n-1).$$

Proof. Let $N = R(m-1, n) + R(m, n-1)$ and assume for contradiction that there exists a red-blue edge coloring of K_N which does not contain a red K_m nor a blue K_n. Let u be an arbitrary vertex and let V_R denote the set of vertices v such that uv is colored red, and similarly define V_B. Note that $|V_R| + |V_B| = N - 1 = R(m-1, n) + R(m, n-1) - 1$, and that we must either have $|V_R| \geq R(m-1, n)$ or $|V_B| \geq R(m, n-1)$ (since otherwise $|V_R| + |V_B| \leq R(m-1, n) + R(m, n-1) - 2$).

First consider the case that $V_R \geq R(m-1, n)$. By definition of $R(m-1, n)$, the coloring on $K_N[V_R]$ must contain either a red K_{m-1} or a blue K_n. The latter case can not happen by assumption of our coloring, and if the former happens then this K_{m-1} together with u would form a red K_m, again giving a contradiction. A similar conclusion holds if $|V_B| \geq R(m, n-1)$, proving the result. $\square$

Using this recurrence relation together with the boundary condition $R(1, n) = R(n, 1) = 1$ gives the following.

Theorem 3.4. *For all $m, n \geq 1$, we have*

$$R(m, n) \leq \binom{m+n-2}{m-1}.$$

Indeed, by induction on $m + n$ we have that

$$R(m, n) \leq R(m-1, n) + R(m, n-1) \leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1},$$

with the last step being Pascal's identity. Finally, taking $m = n$ in this bound gives bounds for diagonal Ramsey numbers.

Corollary 3.5. *For all $n \geq 1$, we have*

$$R(n) \leq \binom{2n-2}{n-1} \leq 4^n.$$

Let us turn now to lower bounds, starting with an elementary bound.

Lemma 3.6. *We have $R(n) \geq (n-1)^2 + 1$.*

Proof. Color the edges of $R_{(n-1)^2}$ via breaking up the vertex sets in to $n-1$ parts $V_1, \ldots, V_{n-1}$ each of size $n-1$ and coloring all the edges within each part red and all the edges between two parts blue. It is easy to see that this avoids monochromatic copies of K_n. $\square$

Note that in this coloring that the blue edges form a copy of the Turán graph $T_{n-1}(n-1)$ and I think there's some connection here but I forget the details. It was believed for some time that $R(n)$ should grow polynomialy like in this lemma here, but Erdős disproved this in a very strong form.

Theorem 3.7. *We have*

$$R(n) \geq (1 + o(1)) \frac{n}{e\sqrt{2}} 2^{n/2}.$$

This is a strange bound and not one should necessarily expect to understand how to prove even if you work out the right proof idea. Indeed, our proof will utilize the following.

Mantra 10. First figure out how your proof works using an abstract set of parameters, then go back and choose whatever parameters you need in order for the arithmetic to go through

Let us see this in action.

Proof. To partially motivate the idea of the argument, we observe that it is very easy to show $R(n) \geq n + 1$ for $n \geq 3$. Indeed, there are only two colorings of K_n which contain a monochromatic K_n, and as long as $n \geq 3$ we can find a coloring which avoids one of these two bad ones. To get our stated lower bound, we will similarly use an elementary counting argument to bound the number of "bad" colorings of K_N and then argue that if N is not too large then there are more total colorings than bad colorings, proving that there exists some coloring which is not bad.

From now on we fix an integer N which we will determine later once we see how the numbers work out. For each subset $S \subseteq [N]$ of size n, let B_S denote the set of edge colorings of K_N which have a monochromatic K_n on S. Because the total number of edge-colorings of K_N is $2^{\binom{N}{2}}$ and because a coloring avoids monochromatic K_n's if and only if it does not lie in any B_S set, we see that there exists an edge-coloring of K_N avoiding monochromatic K_n's if and only if

$$2^{\binom{N}{2}} > |\bigcup_{S \in \binom{[N]}{n}} B_S|.$$

It thus remains to show that this latter set is small. Using elementary arguments we have

$$|\bigcup_{S\in\binom{[N]}{n}} B_S| \le \sum_{S\in\binom{[N]}{n}} |B_S| = \binom{N}{n} 2^{1+\binom{N}{2}-\binom{n}{2}}$$

where this last step used that every coloring in B_S has 2 choices for how it can act on the edges of S (either all red or all blue) together with $2^{\binom{N}{2}-\binom{n}{2}}$ choices for the remaining edges. As such, we will succeed if

$$\binom{N}{n} 2^{1-\binom{n}{2}} < 1.$$

To get a handle on this, we use the well-known binominal inequality $\binom{m}{k} \le (em/k)^k$ to conclude that it suffices to have N such that

$$2\left(\frac{eN2^{(n-1)/2}}{n}\right) < 1,$$

and in particular the result holds provided $N < 2^{1/n} \cdot \frac{n}{e\sqrt{2}} 2^{-n/2}$, and picking such an N gives the desired bound. □

This counting argument is all well and good, but we can give a more modern perspective by rewriting our proof in the language of probability.

Alternative Proof. Let N be an integer to be determined later and consider a uniform random red-blue edge coloring of K_N. Let X be the random variable which is equal to the number of monochromatic K_n's that are in the random coloring of K_N. Crucially, we observe that if $\mathbb{E}[X] < 1$, then $R(n) > N$. Indeed, because X is integer valued, the only way $\mathbb{E}[X] < 1$ is possible is if there exists some coloring of K_N such that $X = 0$, i.e. a coloring without any monochromatic copies of K_N.

To get a handle on $\mathbb{E}[X]$, for each $S \in \binom{[N]}{n}$ we let $\mathbb{1}_S$ denote the indicator random variable for $K_N[S]$ being monochromatic. That is, $\mathbb{1}_S$ is the random variable defined by having $\mathbb{1}_S = 1$ if $K_N[S]$ is monochromatic and $\mathbb{1}_S = 0$ otherwise. With this $X = \sum \mathbb{1}_S$, so by linearity of expectation we have

$$\mathbb{E}[X] = \sum \mathbb{E}[\mathbb{1}_S] = \sum \Pr[\mathbb{1}_S = 1] = \binom{N}{n} 2^{1-\binom{n}{2}},$$

as can be checked by a simple counting argument. Thus in total, we conclude $R(n) > N$ provided $\binom{N}{n}2^{1-\binom{n}{2}} < 1$, which as we showed in the previous version of the proof happens for $N = (1 + o(1))\frac{n}{e\sqrt{2}} 2^{-n/2}$. □

While both the counting argument and the probabilistic argument for [theorem] are effectively equivalent to each other, the prespective of "thinking probabilistically" has proven to be the more useful in general. Indeed, it is hard at this point not to find an important result in Ramsey theory where the lower bound (and sometimes even the upper bound) does not use some amount of ideas or techniques motivated by probability theory. Since we are not assuming the reader has any knowledge of probability we will not dwell on this point any further at this point, though the interested reader is invited to go to [probabilistic methods section] for much more on this perspective.

We note that in both cases of our argument, the lower bound for $R(n)$ we gave was non-constructive, i.e. we did not explicitly construct a coloring of K_N which avoids monochromatic K_n's, we only showed that such a coloring must exist. It is a major open problem to find a constructive argument which gives anywhere close to these bounds here.

Open Problem 3.8. *For some $c > 1$, find "explicit" red-blue edge colorings of K_{c^n} which avoid monochromatic K_n's.*

Observe that our proof not only shows that constructions should exist for $c = \sqrt{2}$, but in fact a more careful inspection shows that for any $c < \sqrt{2}$ that *almost every* coloring should work. Nevertheless, how to explicitly find such a coloring problem remains quite elusive.

The results we have mentioned in this sections are all classical, and the reader might wonder what is the current state of the art. For the lower bound, the only improvement over [result] is an argument due to Lovász using a slightly more involved probabilistic approach that gives a lower bound of [whatever], improving the bound of [result] by a multiplicative factor of [whatever].

For the upper bound, modest results showing bounds of the form $4^{n-o(n)}$ for an increasing series of $o(n)$ functions were obtained over the years until a recent major breakthrough by [authors in year] who proved that $R(n) \leq$???, and since then some further optimizations of their argument has yielded a bound of $R(n) \leq$???. At present this is all that is known for diagonal Ramsey numbers despite decades of hard work from an armada of talented mathematicians.

In addition to the diagonal Ramsey numbers $R(n)$, a lot of work has been put into studying the assymetric case $R(m, n)$. In particular, the study of these numbers when m is fixed and n tends towards infinity is referred to as "off-diagonal" Ramsey numbers. These problems are essentially equivalent to asking: how large can $\alpha(G)$ be if G is K_m-free and contains a given number of vertices? Indeed, more exposition, also comment on how $m = 3, 4$ are reasonably well understood due to complex probabilistic arguments.

3.2 More Colors and Arithmetic Ramsey Theory

There are a ton of variants for Ramsey numbers that one can consider. One of the immediate ones to consider is using more than just two colors. To this end, we define the *multi-color Ramsey number* $R_r(n)$ to be the smallest number N such that every r-coloring of the edges of K_N contains a monochromatic copy of K_n. Similar to [before] one can show that these numbers exist. In particular, we leave it as an exercise to prove the following bounds for the first non-trivial case of $n = 3$.

Maybe use q instead of r to avoid potential confusion.

Theorem 3.9. *We have*

$$2^q < R_q(3) \leq 3 \cdot q!$$

Another direction is to consider coloring combinatorial objects other than graphs. One natural choice would be the integers $[N]$, from which we can ask if there exists a monochromatic subset satisfying some sort of arithmetic condition. One classical result due to Schur is as follows.

Theorem 3.10 (Schur). *For all $q \geq 1$, there exists a finite number N_q such that any q-coloring of $[N]$ contains a monochromatic solution to the equation $x+y=z$, i.e. there exist three integers x, y, z with $x+y=z$ which are all assigned the same color.*

Proof. We will in fact prove that

$$N_q \leq R_q(3),$$

following a common theme in Ramsey theory of upper bounding one Ramsey problem by a function of another. To prove this, we will start with some coloring $\chi : [N] \to [q]$ and then use this to construct an auxiliary coloring $\chi' : E(K_N) \to [q]$ in such a way that monochromatic triangles under χ' correspond to monochromatic solutions to $x+y=z$ under χ. There are a couple of plausible ways one might try and define χ'. For example, given the edge $xy \in E(K_N)$ it is perhaps natural try coloring this edge to be the same color as either $\min(x,y)$ or $\max(x,y)$, but neither of these are really "compatible" with the goal of finding a solution to $x+y=z$.

With a bit more thought, one might come up with the (correct) idea of defining $\chi'(xy) = \chi(|x-y|)$. To see why this does what we want, assume that χ' has a monochromatic triangle on $u < v < w$. This implies that $\chi(v-u), \chi(w-v), \chi(w-u)$ all have the same color. Moreover, we have $(v-u)+(w-v)=(w-u)$, so taking $x = v-u,\ y = w-v$, and $z = w-u$ gives a monochromatic solution under χ. In total this implies that if $N \geq R_q(3)$ and χ is an arbitrary coloring then, because χ' must contain a monochromatic triangle since $N \geq R_q(3)$, χ contains a monochromatic solution to $x+y=z$. This proves $N_r \leq R_q(3)$, and in particular that this number is finite. ☐

A lot more can be said about this area known as arithmetic Ramsey theory. Perhaps the most famous result in this direction is Van der Waerden's Theorem.

Theorem 3.11 (Van der Waerden's Theorem). *For all k, q, there exists a finite number $N_{k,q}$ such that any r-coloring of $[N_{k,q}]$ contains a monochromatic k-term arithmetic progression. That is, there exist integers $a, d \geq 1$ such that $a, a+d, \ldots, a+(k-1)d$ are all given the same color.*

Proving this is not so easy, and the bounds for $N_{k,q}$ are horrendous even in the case of $q=2$. In fact, an even stronger statement than Van der Waerden's Theorem is known to be true.

Theorem 3.12 (Szemerédi's Theorem). *Every subset of $[N]$ which does not contain a k-term arithmetic progression has size $o(N)$.*

To see this implication, observe that every r-coloring of $[N]$ contains a subset of size at least $r^{-1}N$ which, by Szermedi's Theorem, must contain a k-term arithmetic progression whenever N is sufficiently large. This is an example of a general phenomenon where Turán results (which bound how dense a structure can be before it contains a given substructure) often upper bound Ramsey results (which bound how large a structure can be with the property that it can be partitioned into r substructures avoiding a given substructure) simply because one of the partition elements in a Ramsey result must have relatively large density.

3.3 Ramsey Without Colors

We will omit this for time unless requested by popular by demand. Broadly speaking it will be around the theme that Ramsey isn't just about saying that colored objects contain things. Some examples include monotone sequences and convex sets.

3.4 Exercises

1. Let's look at some small Ramsey numbers:

   - Prove that $R(3, 4) = 9$???
   - Prove that $R(4) \leq 18$ [1].
   - Prove that $R(4) = 18$ [3].
   - Determine[8] $R(5)$ [5].

2. Prove that every n-vertex graph has a clique or independent set on at least $\frac{1}{2}\log_2(n)$ vertices [1+].

3. Recall that a tournament is a digraph obtained by giving an orientation to each edge of a complete graph, and that a tournament is transitive if one can order its vertices $v_1, \ldots, v_n$ in such a way that $v_i \to v_j$ if and only if $i < j$. Prove that every tournament on n vertices contains a transitive tournament of size at least $\lfloor \log_2(n) \rfloor + 1$ [2-].

* * *

4. Prove for all $n, q \geq 2$ that $R_q(n) \leq q^{qn}$ [2-].

5. Let us look at the multi-color Ramsey number $R_q(3)$.

   (a) Prove that $R_q(3) > 2^q$ [2-].

   (b) Prove that $R_q(3) \leq 3 \cdot q!$, noting that this is best possible for $q = 2, 3$ [2].

   (c) Improve this upper bound to $R_q(3) \leq \lfloor e \cdot q! \rfloor + 1$, which as far as we know is still the best known upper bound [3].

6. One of the most important results in general Ramsey theory is the Hales-Jewett Theorem which is a sort of "high-dimensional tic-tac-toe" theorem that goes as follows: Insert statement, exercise is to derive Van der Waerden

[8] Currently the best known bounds are $43 \leq R(5) \leq 46$. The fact that this is still open should demonstrate how hard determining $R(n)$ exactly is. Indeed, Erdős once said something to the effect of: if aliens came to Earth and demanded we tell them what $R(5)$ was in the next 10 years or they would destroy us, then we should dedicate all our resources to this problem. If instead they ask for $R(6)$, then we should instead dedicate all our resources to fighting the aliens because we have no hope of doing what they ask.

$* * *$

7. We say that a graph G is K_n-Ramsey if any red-edge edge coloring of G contains a monochromatic copy of K_n, and we define the *size Ramsey number* $\hat{R}(n)$ to be the smallest number of edges in a graph which is K_n-Ramsey.

   (a) Observe that $R(n)$ can be defined to be the smallest number of *vertices* in a graph which is K_n-Ramsey, motivating this definition [1].

   (b) Prove that $\hat{R}(n) \leq \binom{R(n)}{2}$ [1+].

   (c) Prove that $\hat{R}(n) = \binom{R(n)}{2}$; noting crucially that this equality holds despite us largely not understanding what $R(n)$ is (Hint: prove that any K_n-Ramsey graph must have chromatic number at least $R(n)$) [2+].

Part II

Structural Graph Theory

4 Colorings

TODO. Likely topics: Brooks Theorem and Hajnal-Szemerédi, List Colorings, Edge Colorings and Vizing's Theorem, triangle-free graphs

5 Matchings and Factors

TODO. Likely topics: Konig's Theorem, Hall's Theorem, stable matchings, Tutte's Theorem, Factors

6 Connectivity and Flows

TODO. Likely topics: k-connectivity, blocks, Menger's Theorem, min cut max flow, ear decompositions

Part III

Methods

7 Probabilistic Methods

TODO. Likely topics: high girth and chromatic, deletion method for Turán problems, random polynomial graphs

8 Regularity and Removal Lemmas

TODO. Likely topics: Erdős-Stone-Simonovits, Removal Lemma, Roth's Theorem, Property Testing, Ramsey-Turán Problems

9 Linear Algebra Methods

TODO. Likely topics: Huang's Theorem, girth 5 regular graphs

Part IV

Bonus Topics

10 Hypergraphs

TODO. Likely topics: generalized KST, codegree arguments and loose cycle Turán problems, Turán densities exist and supersaturation, Fisher's inequality, hypergraph ramsey

11 Random Graphs

TODO. Likely topics: thresholds, connectivity, spreadness theorems

12 Planar Graphs

TODO. Likely topics: Euler's formula, Wagner's Theorem characterizing planar graphs, 5-color theorem.

13 Spectral Graph Theory

TODO. Likely topics: adjacency matrix, Laplacian matrix, matrix-tree theorem, Cheeger inequality, expanders

14 Advanced Methods

TODO. Likely topics: entropy, hypergraph containers, spreadness, absorption, homomorphism counting

Note: many of these topics would be covered in exactly the same way as in my notes here.