# Random Turán Problems

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**Theorem (Erdős-Stone 1946)**

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}.$$

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The lower bound is tight when $p = 1$. The upper bound is tight if $p$ is "small."

$$\frac{1}{2}p\binom{n}{2} \lesssim \text{ex}(G_{n,p}, K_3) \lesssim p\binom{n}{2},$$

with the lower bound tight for $p = 1$ and the upper bound tight for $p \ll n^{-1/2}$.

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Theorem (Frankl-Rödl 1986)

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Theorem (Conlon-Gowers, Schacht 2010)

Whp,

$$\mathrm{ex}(G_{n,p}, F) = p \cdot \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2} \qquad p \gg n^{-1/m_2(F)},$$

where $m_2(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subseteq F\}$.

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Conjecture

*If $F$ is a bipartite graph which is not a forest, then whp*

$$\text{ex}(G_{n,p}, F) = \begin{cases} \Theta(p \cdot \text{ex}(n, F)) & p \gg n^{-1/m_2(F)}, \\ (1 + o(1))p\binom{n}{2} & p \ll n^{-1/m_2(F)}. \end{cases}$$
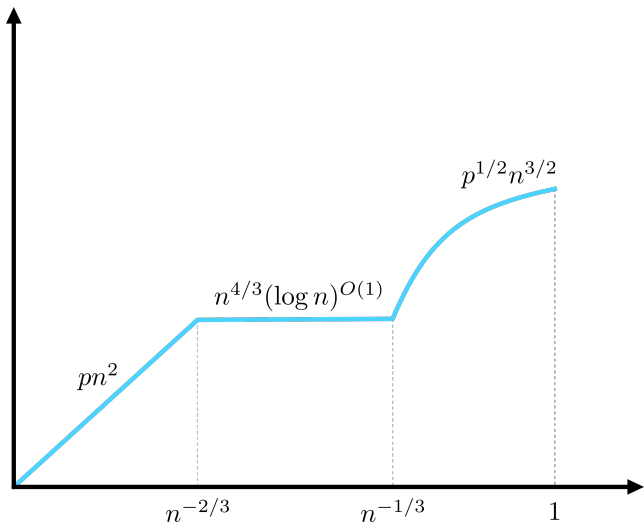
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This conjecture turns out to be completely false!

Plot of $\mathrm{ex}(G_{n,p}, C_4)$ (Füredi 1991)

## Conjecture (McKinley-S.)

If $F$ is a graph with $\mathrm{ex}(n, F) = \Theta(n^\alpha)$ for some $\alpha \in (1, 2]$, then whp

$$\mathrm{ex}(G_{n,p}, F) = \max\{\Theta(p^{\alpha-1}n^\alpha), n^{2-1/m_2(F)}(\log n)^{O(1)}\},$$

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The heart of this problem is in showing $\mathrm{ex}(G_{n,p}, F) = \Theta(p^{\alpha-1}n^{\alpha})$ when $p$ is large.

**Theorem (Kővari-Sós-Turán 1954)**

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

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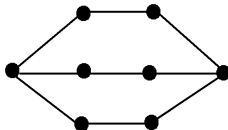
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## Theorem (Morris-Saxton 2013)

$$\text{ex}(G_{n,p}, K_{s,t}) = O(p^{1-1/s} n^{2-1/s}) \text{ for } p \text{ large}.$$

Moreover, this bound is tight whenever $\text{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$.

### Theorem (Bondy-Simonovits 1974)

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$$\mathrm{ex}(G_{n,p}, C_{2b}) = O(p^{1/b}n^{1+1/b}) \text{ for } p \text{ large.}$$

*Moreover, this is tight whenever* $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2b}\}) = \Theta(n^{1+1/b})$.

$$\text{ex}(n, \theta_{a,b}) = O(n^{1+1/b}).$$

**Theorem (Faudree-Simonovits 1974)**

$$\text{ex}(n, \theta_{a,b}) = O(n^{1+1/b}).$$
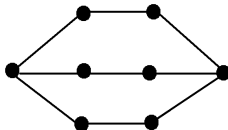
**Theorem (McKinley-S. 2023)**

$$\text{ex}(G_{n,p}, \theta_{a,b}) = O(p^{1/b} n^{1+1/b}) \text{ for } p \text{ large}.$$
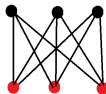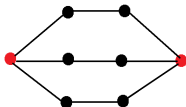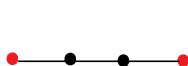
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$$\text{ex}(G_{n,p}, \theta_{a,b}) = O(p^{1/b} n^{1+1/b}) \text{ for } p \text{ large.}$$

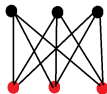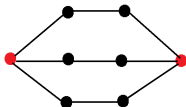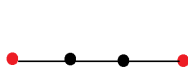*Moreover, this bound is tight whenever a is sufficiently large in terms of b.*

**Theorem (Bukh-Conlon 2015)**

If $T^\ell$ is the "$\ell$th power of a balanced tree with density $b/a$", then $\mathrm{ex}(n, T^\ell) = \Omega(n^{2-a/b})$ if $\ell$ is sufficiently large.

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**Theorem (S. 2022)**

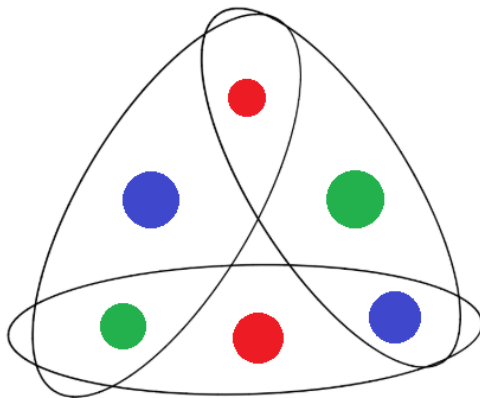$$\mathrm{ex}(G_{n,p}, T^\ell) = \Omega(p^{1-a/b}n^{2-a/b}).$$

## Theorem (Jiang-Longbrake 2022)

*If $F$ satisfies "mild conditions", then*

$$\text{ex}(G_{n,p}, F) = O(p^{1-m_2^*(F)(2-\alpha)} n^\alpha) \ \text{for } p \ \text{large},$$

*where $m_2^*(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subsetneq F, \ e(F') \geq 2\}$.*

# Hypergraphs

## Theorem (S.-Verstraëte 2021)

Let $K^r_{s_1,\ldots,s_r}$ denote the complete $r$-partite $r$-graph with parts of sizes $s_1,\ldots,s_r$. There exist constants $\beta_1,\beta_2,\beta_3,\gamma$ depending on $s_1,\ldots,s_r$ such that, for $s_r$ sufficiently large in terms of $s_1,\ldots,s_{r-1}$, we have whp

$$\mathrm{ex}(G^r_{n,p}, K^r_{s_1,\ldots,s_r}) = \begin{cases} \Theta\left(pn^r\right) & n^{-r} \ll p \leq n^{-\beta_1}, \\ n^{r-\beta_1+o(1)} & n^{-\beta_1} \leq p \leq n^{-\beta_2}(\log n)^\gamma, \\ \Theta(p^{1-\beta_3}n^{r-\beta_3}) & n^{-\beta_2}(\log n)^\gamma \leq p \leq 1. \end{cases}$$

We define the *loose cycle* $C_\ell^r$ to be the $r$-uniform hypergraph obtained by inserting $r - 2$ distinct vertices into each edge of the graph cycle $C_\ell$.
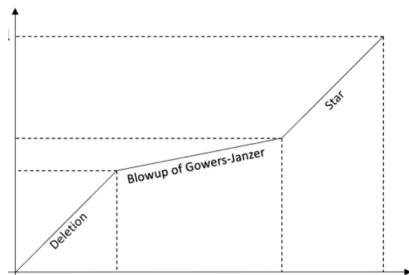
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Theorem (Nie-S.-Verstaëte 2020; Nie 2023)

*For $r \geq 3$, if $p \gg n^{-r+3/2}$ then whp*

$$\mathrm{ex}(G_{n,p}^r, C_3^r) = \max\{p^{\frac{1}{2r-3}} n^{2+o(1)}, pn^{r-1+o(1)}\}.$$



Picture due to Jiaxi Nie.

**Theorem (Mubayi-Yepremyan 2020; Nie 2023)**

For $r \geq 4$, if $p \gg n^{-r+1+\frac{1}{2\ell-1}}$ then whp

$$\text{ex}(G_{n,p}^r, C_{2\ell}^r) = \max\{n^{1+\frac{1}{2\ell-1}}, pn^{r-1}\}.$$
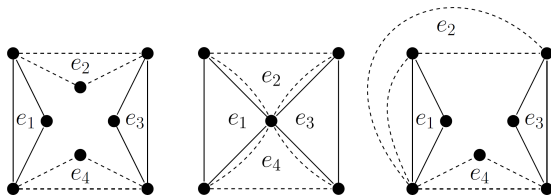
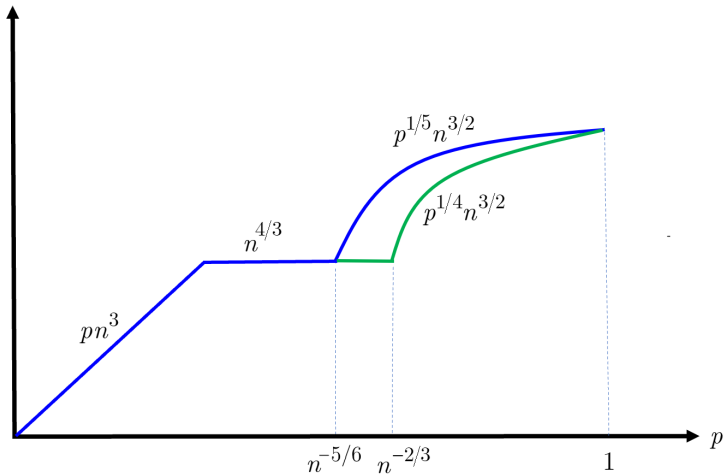For $r \geq 4$, if $p \gg n^{-r+1+\frac{1}{2\ell-1}}$ then whp

$$\mathrm{ex}(G_{n,p}^r, C_{2\ell}^r) = \max\{n^{1+\frac{1}{2\ell-1}}, pn^{r-1}\}.$$

It's suspected that this continues to hold for $r = 3$, but there is a gap for medium values of $p$.

We say that $F$ is a Berge $C_\ell$ if it has edges $e_1, \ldots, e_\ell$ and distinct vertices $v_1, \ldots, v_\ell$ with $v_i \in e_i \cap e_{i+1}$ for all $i$.

Plot of $\text{ex}(G^3_{n,p}, \mathcal{B}^3_4)$



$p^{1/5}n^{3/2}$

$n^{4/3}$

$p^{1/4}n^{3/2}$
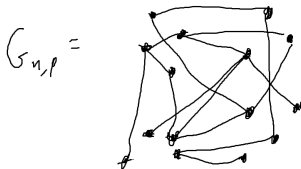
$pn^3$

$n^{-5/6}$ $\quad n^{-2/3}$ $\qquad\qquad 1 \qquad p$

S.-Verstraëte 2021; Nie 2023

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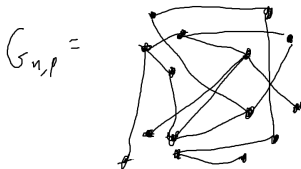


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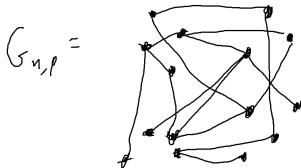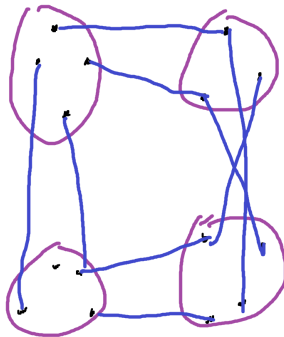
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Theorem (Füredi, 1991; Morris-Saxton, 2013)

If $m \geq n^{4/3}(\log n)^2$, then

$$N_m(n, C_4) \leq e^{cm}(\log n)^m \left( \frac{n^{3/2}}{m} \right)^{2m}$$

# Upper Bound Techniques

**Corollary**

If $p \geq n^{-1/3}(\log n)^3$, then a.a.s.

$$\text{ex}(G_{n,p}, C_4) \leq O\left(p^{1/2}n^{3/2}\log n\right).$$

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$$\Pr[\mathrm{ex}(G_{n,p}, C_4) \geq m] = \Pr[X \geq 1] \leq \mathbb{E}[X] = p^m \cdot N_m(n, C_4) \approx p^m \cdot \left(\frac{n^{3/2}}{m}\right)^{2m}$$

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- Plenty of problems left to solve!