Intro to Topology

Sam Spiro*

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*Dept. of Mathematics, Rutgers University `sas703@scarletmail.rutgers.edu`. This material is based upon work supported by the National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship under Grant No. DMS-2202730.

Part I

Basic Definitions and Examples

1 What is Topology?

This question is somewhat complicated to answer because people use the word "topology" to refer to two related but distinct areas of study:

- *Modern topology* is roughly speaking the study of geometric objects such as spheres, Möbius strips, Klein bottles, and so on. If a mathematician says they "study topology", this is typically what they're referring to.

- *Point set topology* (also called *general topology*) is a very general framework that includes modern topology, much of calculus, and many other areas of mathematics. **This** is what the present course is all about, and from now on whenever I say the word "topology" I'll be referring to this concept.

The central object studied in topology are mathematical objects called *topologies*. So again we're left with the question: what is a topology?

1.1 What is a Topology: the Short Answer

The definition for a topology is a follows. At this point it should **not** be obvious to you why in the world you would ever consider something like this. Here and throughout, given a set X we let $\mathcal{P}(X)$ denote the power set of X, i.e. the set of all subsets of X.

Definition 1. Given a set X, a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* of X if the following conditions are satisfied:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under (arbitrary) unions, i.e. for any $\mathcal{S} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

(c) $\mathcal{T}$ is closed under *finite* intersections, i.e. for any *finite* subset $\mathcal{S} \subseteq \mathcal{T}$, $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

Again, this is a strange definition that should seem totally bizarre. In the next subsection I'll attempt to provide motivation for why one might possibly come up with this definition. Those that aren't interested/confused by this discussion can completely ignore it without affecting their understanding of the rest of the material in this course.

1.2 What is Topology: the Long Answer

The study of topology originates with the study of calculus/real analysis. When you took courses in these areas, you learned a number of important concepts about the set of real numbers $\mathbb{R}$, as well as about functions f from $\mathbb{R}$ to $\mathbb{R}$. In particular, two very important definitions are:

1. What it means for a sequence of real numbers $(x_n)_{n\geq 1}$ to *converge* to a real number x_0.

2. What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous*.

The central aim of topology is to give a *general framework* which expands these definitions for real numbers to a much broader class of mathematical objects. In particular, it aims to answer the following two questions:

1. What does it mean for a sequence of "objects" $(x_n)_{n\geq 1}$ to *converge* to another object x_0? For example, what does it mean for a sequence of functions $(f_n)_{n\geq 1}$ to converge to another function?

2. What does it mean for a function $f : X \to Y$ between two "nice objects" X, Y to be *continuous*? For example, what does it mean for a map $f : S^2 \to S^2$ from the sphere to itself to be continuous?

We'll postpone the second question and focus on convergence. Of course, any "reasonable" answer should in particular recover the original definition of convergence from real analysis. With this in mind, let's recall this definition and then think about how we might generalize it.

Definition 2. We say that a sequence of real numbers $(x_n)_{n\geq 1}$ *converges* to a real number x_0 if for all $\varepsilon > 0$, there exists an integer N_ε such that $|x_n - x_0| < \varepsilon$ for all $n \geq N_\varepsilon$.

While this is a fine definition, it's a little difficult to generalize. It turns out (for reasons that should not be obvious at this point) that a "better" definition can be made by utilizing the language of *open intervals*, which we recall are sets $I \subseteq \mathbb{R}$ of the form $\{x : a < x < b\}$ for some $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 3. We say that a sequence of real numbers $(x_n)_{n\geq 1}$ *converges* to a real number x_0 if for every open interval I containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

Claim 1.1. *These two definitions are equivalent. That is, a sequence $(x_n)_{n\geq 1}$ and real number x_0 satisfy the conditions of Definition 2 if and only if they satisfy the conditions of Definition 3.*

Definition 3 has several advantages over Definition 2. First, it avoids any mention of the real number ε (which is mathematically nice[1] since we ultimately want to generalize things beyond

[1] This is also psychologically nice for those who have painful memories of ε from real analysis.

real numbers). Second, it frames the definition in terms of the more "geometric" concept of open intervals.

Now, at this point it probably still isn't obvious how to generalize Definition 3 to more general objects (like sequences of functions). However, if you were to spend 30 years about this problem, then perhaps you would come up with the following idea: replace the words "open interval" in Definition 3 with the words "nice set" (where the exact definition of "nice set" depends on your exact problem at hand) and use this as your definition of convergence. Somewhat more precisely, we'll try to work with the following definition (which at this point in time you don't need to memorize since we'll forget about it a moment).

Definition 4. Given a set X, we call any set $\mathcal{T} \subseteq \mathcal{P}(X)$ a *pre-topology*[2] of X. We say that a sequence of points $(x_n)_{n\geq 1}$ with $x_n \in X$ *converges* to a point $x_0 \in X$ with respect to $\mathcal{T}$ if for every $I \in \mathcal{T}$ containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

For example, if $X = \mathbb{R}$ and $\mathcal{T}$ is the set of open intervals of $\mathbb{R}$, then this exactly recovers Definition 3. Here are a few more (extreme) examples to give some more familiarity with these definitions.

Claim 1.2. *Let X be an arbitrary set.*

(a) If $\mathcal{T} = \emptyset$ (i.e. if $\mathcal{T}$ contains no subsets of X), then ***every*** *sequence of points $(x_n)_{n\geq 1}$ in X converges to* ***every*** *point $x_0 \in X$ with respect to $\mathcal{T}$*

(b) If $\mathcal{T} = \mathcal{P}(X)$ (i.e. if $\mathcal{T}$ contains every subset of X), then a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point $x_0 \in X$ with respect to $\mathcal{T}$ if and only if there exists some N such that $x_n = x_0$ for all $n \geq N$ (i.e. iff x_n is "eventually constant").

(c) If $\mathcal{T} = \{\{x\} : x \in X\}$ (i.e. if $\mathcal{T}$ is the set of singletons),then a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point $x_0 \in X$ with respect to $\mathcal{T}$ if and only if there exists some N such that $x_n = x_0$ for all $n \geq N$ (i.e. iff x_n is "eventually constant").

These last two examples suggest the following definition.

Definition 5. Two pre-topologies $\mathcal{T}, \mathcal{T}'$ for the same set X are said to be *equivalent* if: a sequence $(x_n)_{n\geq 1}$ converges to x_0 with respect to $\mathcal{T}$ if and only if it converges to x_0 with respect to $\mathcal{T}'$.

For example, the claim above shows the collection of singletons $\mathcal{T}$ is equivalent to $\mathcal{P}(X)$. Given that these two collections are equivalent to each other, which one should we work with, i.e. which is "better"? A possible answer is that the larger collection $\mathcal{P}(X)$ is "better" because its extra elements give us extra flexibility. This suggests the following problem.

Question 1.3. *Given a pre-topology $\mathcal{T}$, what is the "largest" pre-topology $\mathcal{T}'$ which contains $\mathcal{T}$ and which is equivalent to $\mathcal{T}$?*

[2] This name is not standard at all and we will never use it beyond this first pre-lecture.

This question seems a little daunting, so instead we ask the following weaker question.

Question 1.4. *Given a pre-topology $\mathcal{T}$, are there any "obvious" sets U that we can add to $\mathcal{T}$ so that $\mathcal{T} \cup \{U\}$ is equivalent to $\mathcal{T}$?*

Again if you think about this for 30 years you might realize the following.

Claim 1.5. *Let X be a set, $\mathcal{T}$ a pre-topology, and $\mathcal{S} \subseteq \mathcal{T}$ some non-empty subset of its elements.*

(a) $\mathcal{T} \cup \{\emptyset\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{X\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{\bigcup_{U \in \mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(c) If $\mathcal{S}$ is a finite set, then $\mathcal{T} \cup \{\bigcap_{U \in \mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(d) Part (c) does not hold in general if $\mathcal{S}$ is allowed to be an infinite set[3].

Sketch of Proof. For (a), since $\emptyset$ contains no elements of X it doesn't affect whether any given element is the limit of a sequence.

For (b), one can always take $N_X = 1$ (since every sequence lies in X for all $n \geq 1$).

For (c), take $N_{\bigcup_{U \in \mathcal{S}} U}$ equal to N_U for any $U \in \mathcal{S}$.

For (d), take $N_{\bigcap_{U \in \mathcal{S}} U} = \max_U N_U$ (note how this requires the set $\mathcal{S}$ to be finite). □

That is, given any pre-topology $\mathcal{T}$, we can freely add in $\emptyset$ and X, as well as (arbitrary) unions and finite intersections of elements of $\mathcal{T}$ into $\mathcal{T}$ to make a (possibly) larger pre-topology which is equivalent to $\mathcal{T}$. In particular, this means that the largest pre-topology $\mathcal{T}'$ which is equivalent to $\mathcal{T}$ must contain $\emptyset, X$ and be "closed" under taking unions and finite intersections. This is exactly the definition of a topology!

2 Definitions and Examples

Let's again restate the definition of a topology, as well as some related definitions that will serve as a useful language for talking about topologies.

Definition 6. Given a set X, a set $\mathcal{T}$ of subsets of X is called a *topology* of X if the following hold:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcup_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

[3]Hint: take $X = \mathbb{R}$, $\mathcal{T}$ to be the set of open intervals, and $\mathcal{S} = \{(-\frac{1}{n}, \frac{1}{n})\}$.

(c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcap_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. We will call the pair $(X, \mathcal{T})$ a *topological space*. When $\mathcal{T}$ is clear from context we simply write X instead of $(X, \mathcal{T})$.

Remark 2.1. For arbitrary unions, the book likes to use the notation $\bigcup_{\alpha \in J} U_\alpha$ where J is an "index set", and we will occasionally use this notation in class as well. I recommend looking at Chapter 1 Section 5 of the book to get a more detailed explanation for how this notation is used throughout the book.

Now that we have the definition of a topology in hand, let's pause for a moment and look at some examples and non-examples of topologies.

2.1 Finite Topologies

Is the following pair $(X, \mathcal{T})$ a topological space: In class write all these down and ask students what they think the answers are

$$X = \{a, b, c\},\ \mathcal{T}_1 = \{\{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! It fails to have $\emptyset \in \mathcal{T}_1$. Okay, what about

$$X = \{a, b, c\},\ \mathcal{T}_2 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

Well, we have $\emptyset, X \in \mathcal{T}_2$, and it is not difficult to check by hand that this is closed under unions and (finite) intersections (e.g. $\{a, b\} \cap \{b\} = \{b\} \in \mathcal{T}_2$; the slicker way is to note that unions/intersections of sets containing b continue to be sets containing b) so this *is* a topology! What about

$$X = \{a, b, c\},\ \mathcal{T}_3 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}?$$

No! This isn't closed under intersections $\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{T}$. What about

$$X = \{a, b, c\},\ \mathcal{T}_4 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! Again not closed under intersections $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_4$. Note that these last three examples show that topologies aren't "monotonic", i.e. if you know $\mathcal{T}_2$ is a topology and $\mathcal{T}_4 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$, you can't conclude that either $\mathcal{T}_4, \mathcal{T}_3$ are necessarily topologies.

2.2 General Families

Let's look at some more general examples.

Definition 7. For any set X, the set $\mathcal{T} = \{\emptyset, X\}$ is a topology called the *trivial topology* or *indiscrete topology*. The proof that this is a topology follows by considering all 4 of the subsets $\mathcal{S} \subseteq \mathcal{T}$ and verifying that their unions/intersections lie in $\mathcal{T}$.

Definition 8. For any set X, the power set $\mathcal{T} = \mathcal{P}(X)$ (i.e. the set of all subsets of X) is a topology called the *discrete topology*. The proof that this is a topology follows from the fact that unions/intersections of subsets of X continue to be subsets of X (and hence lie in $\mathcal{T}$.

Definition 9. For any set X, the set $\mathcal{T} = \{S \subseteq X : |X \setminus S| < \infty\} \cup \{\emptyset\}$ (i.e. the set of elements which contain all but a finite number of points from X) is a topology called the *cofinite topology* or *finite complement topology*. The proof that this is a topology follows from the fact that unions/finite intersections of cofinite sets are cofinite (this requires a bit more of an argument involving De Morgan's laws).

2.3 Euclidean and Subspace Topologies

Here we discuss possible the two most important topologies which should always be at the back of your mind.

Definition 10. For $x_0 \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, define the *open ball* $B(x_0, \varepsilon) = \{x : |x - x_0| < \varepsilon\}$. For $X = \mathbb{R}^n$, consider the set $\mathcal{T}$ consisting of all sets S such that for all $x_0 \in S$ there exists an open ball $B(x_0, \varepsilon) \subseteq S$. Then $\mathcal{T}$ is a topology called the *Euclidean topology* or *standard topology*.

Draw a picture of an open set in $\mathbb{R}^2$.

Throughout this course, whenever we consider $\mathbb{R}^n$, we will assume it is a topological space with the Euclidean topology unless stated otherwise.

We next look at a general way for generating new topologies from old ones.

Definition 11. Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, we define the *subspace topology* $\mathcal{T}' = \{U \cap Y : U \in \mathcal{T}\}$. Unless stated otherwise we will always assume subsets $Y \subseteq X$ come equipped with the subspace topology, in which case we say that Y is a *subspace* of X.

Claim 2.2. *The subspace topology is a topology.*

Proof. $\emptyset, X$ are easy. For finite intersections, if you have $V_1, \ldots, V_r$ open then $V_i = U_i \cap Y$ for some U_i, then $\bigcap V_i = \bigcap U_i \cap Y$ which is open since X is a topology. The proofs for unions is similar

Actually, the most naive proof for arbitrary unions requires invoking the axiom of choice (this is a very subtle error; it was only noticed in 2018!). Since this is an undergraduate class I'm not going to fret over this, but can talk about it in office hours for those that are interested. □

Example 2.3. Let $X = \mathbb{R}$ with the Euclidean topology and $Y = [0, 1] \subseteq \mathbb{R}$ with the subspace topology. Which of the following sets are open in X? Which are open in Y?

- $(1/4, 3/4)$
- $(1/2, 1]$
- $[1/4, 3/4)$

Around 80% of the topologies we consider in the class will either be $\mathbb{R}^n$ with the Euclidean topology or some subset $X \subseteq \mathbb{R}^n$ equippped with the subspace topology. Here are a few common examples of these sorts of spaces:

- $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ (open ball).
- D^n (open disk)
- S^{n-1} ($(n-1)$-dimensional sphere
- $I^n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$ (n-dim cube, draw some examples).

Warning: if X is a space and $Y \subseteq X$ is a subspace, it is somewhat ambiguous to talk about "open sets" (do we mean open in X or in Y?). We deal with this as follows.

Definition 12. If $Y \subseteq X$ is a subspace, we say a set A is open in Y if it belongs to the topology of Y, and similarly we define what it means for A to be open in X.

In some situations there's no ambiguity, as in the following.

Claim 2.4. *If $Y \subseteq X$ is an open set in X, then every A which is open in Y is also open in X (intersection of open sets is open).*

3 Closed Sets

- Recap: topologies (and that given a topological space $(X, \mathcal{T})$, a set $U \subseteq X$ is called open if $U \in \mathcal{T}$), examples (Euclidean, subspace)

- Idea with definition of "open" is that these generalize notion of open sets from real analysis, but equally important in real analysis is closed sets (e.g. intermediate/extreme value theorem both involve functions $f : [a, b] \to \mathbb{R}$, i.e. both involve closed intervals).

  So, what's the right way to define closed sets for general topologies?

- Definition: a set $A \subseteq X$ is called *closed* if its complement $X - A$ is open.

- Warmup: which of the following sets A are open/closed in $X = \mathbb{R}$?

  - $[0, 1]$

  - $\mathbb{R}_{>0}$

  - $\mathbb{Q}$

  - $\mathbb{R}$

  - $\emptyset$

- Def: a set $A \subseteq X$ which is both open and closed is called clopen.

- Eg in discrete topology, every set is open/closed/clopen.

- Thm: if X is a topological space then: (1) $\emptyset, X$ are closed, (2) arbitrary intersections of closed sets are closed, (3) finite unions of closed sets are closed.

  With this we see we could have defined topology via closed sets and gotten same theory; there's no real distinction.

- As in the previous lecture, if we have a subspace $Y \subseteq X$ it can be ambiguous to say A is closed. In this case we will say that A is closed in Y or closed in X as apporpriate.

  Claim: if $Y \subseteq X$ is closed, then any A which is closed in Y is closed in X.

- Given $A \subseteq X$ there's two important sets we can associate to it. Def: the interior A^o (or $intA$) is the union of all the open sets contained in A, and the closure $\bar{A}$ is intersection of closed sets containing it.

- E.g intuitively what is $[0, 1)$ interior/closure?

- Essentially, A^o is the largest open subset of A (and similar $\bar{A}$).

- Prop: Prove these
  - $A^o, \bar{A}$ are open/closed.
  - $A^o \subseteq A \subseteq \bar{A}$.
  - If A is open then $A = A^o$, and if A is closed then $A = \bar{A}$.
- In order to characterize closure we need some definitions: if U is an open set containing x, then we say that U is a *neighborhood* of x. We say that two sets A, B intersect if $A \cap B \neq \emptyset$.
- Thm: $x \in \bar{A}$ iff every neighborhood of x intersects A.
  - Equivalent to prove $x \notin \bar{A}$ iff exists neighborhood disjoint from A. Indeed, if $x \notin \bar{A}$ then there exists closed set C containing A but not x, then $X - C$ is a neighborhood disjoint from A. Reverse direction similar.
- Examples for subsets of $\mathbb{R}$:
  - $A = \{n^{-1}\}$, closure is this plus 0 (easy to see 0 is in due to neighborhood description, everything else has neighborhood disjoint from it).
  - $A = \mathbb{Q}$, closure is $\mathbb{R}$.
- Neighborhoods are one useful way to characterize closures. Another way is through limit points. Definition: given a subset $A \subseteq X$, a point x is called a *limit point* (or cluster point, or point of accumulation) of A if every neighborhood of x intersects A in some point *other than itself*. Equivalently, x is a limit point if $x \in \overline{A - \{x\}}$.
- E.g. for $X = \mathbb{R}$, $A = \{0\}$ no point is a limit point. If $A = (0, 1]$ every point in $[0, 1]$ is a limit point. If $A = \{n^{-1}\}$ then only 0 is a limit point.

  Intuitively, x is a limit point if there's a sequence in $A - x$ "converging" to x.
- Thm: if A' denotes the set of limit points of A, then $\bar{A} = A \cup A'$.
  - If $x \in A'$ then every neighborhood intersects A, so by the one theorem it's in the closure so $A \cup A' \subseteq \bar{A}$.
  - If $x \in \bar{A} \setminus A$, then every neighborhood intersects A, and necessarily $A - x$ since $x \notin A$, so $x \in A'$.
- Corollary: a set A is closed if it contains all of its limit points ($A = A \cup A'$ implies $A' \subseteq A$).

4 Convergent Sequences and Hausdorff Spaces

- Again the goal of topology is to generalize concepts from real analysis, and now that we have a lot of examples/terminology, we can finally start defining these analogs.

- One important concept that we've seen is convergent sequences. Definition: given a topological space X, we say that a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point x if for all neighborhoods U of x, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

- This definition can be used to motivate the name "limit point" from last time.

  Prop: let X be a space and $A \subseteq X$. If $x \in X$ is such that there exists a sequence $(x_n)_{\geq n}$ in $A - x$ which converges to x, then x is a limit point of A.

  - Proof is that for any neighborhood there exist infinitely many x_n in it, all of which are in A and all of which are distinct from x.

  - Converse turns out to be false (i.e. there exist limit points which are not limits) but it's not super easy to construct; see this. This is true however in nice spaces (e.g. metric spaces).

- Natural question that pops up when playing with sequences: for every sequence $(x_n)_{n\geq 1}$, does there exist at most one point x which the sequence converges to?

  Intuition with $\mathbb{R}^n$ says, yes, but this is false: in trivial topology every sequence converges to every point.

- This is a weird situation we'd like to avoid.

  Definition: a topological space is said to be *Hausdorff* or T_2 if for each pair of distinct points x, y, there exist neighborhoods U, V of x, y respectively which are disjoint. Draw picture

- Thm: if X is Hausdorff, then every sequence converges to at most one point.

  - Assume for contradiction converge to x, y let U, V be neighborhoods. Take N_U, means all $n \geq N_U$ lie in U i.e. aren't in V, contradicting the existence of N_V.

  - Note that the definition of Hausdorff is designed to be essentially the weakest condition such that this property holds.

- Examples: proofs

  - $\mathbb{R}^n$ is Hausdorff

  - Trivial with at least two points is not

  - Discrete is Hausdorff

  - Finite complement with an infinite number of points is not (every two open sets intersect in all but finitely many points).

- Already saw Hausdorff is nice because sequences have at most one limit, which agrees with our intuition from $\mathbb{R}^n$. It also plays nicely with intuition for closed sets.

  Thm: if X is Hausdorff, then every finite subset $A \subseteq X$ is closed.

  - Not true in general: $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ doesn't have b closed.

  - Proof: suffices to prove it when $A = \{x\}$ (since unions of closed sets are closed), i.e. that $X - x$ is open. Because X is Hausdorff, each $y \in X - x$ has a neighborhood U_y disjoint from x. Hence $\bigcup U_y = X - x$ is open.

  - Note that we didn't need the full power of Hausdorff in this proof: we only used that each y has a neighborhood disjoint from each x (so e.g. it holds for finite complement topology). This disjoint neighborhood condition is called the T1 condition; we'll return to this in Chapter 4.

5 Continuous Functions

- Recap: convergence, Hausdorff (and sequences converge to at most one point).

- Again, one of the main points of topology is to generalize key concepts from real analysis.

- Recall form calculus that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for all $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exist $\delta > 0$ such that for all x with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$.

  Phew, that's a mouthful.

- Def: a map $f : X \to Y$ between two topological spaces is called *continuous* if for every open set U in Y, the set $f^{-1}(U)$ is open in X.

  - Warning: The notation $f^{-1}(U) := \{x : f(x) \in U\}$ is the *pre-image* of f NOT the inverse of f (which may not exist).

- Claim: this is equivalent to the calculus definition for the Euclidean topology.

5.1 Examples

- Eg take $X = \{a, b\}$, $\mathcal{T}_x = \{\emptyset, X, \{a\}\}$, define $Y = \{1, 2\}$ with essentially the same topology. Consider all 4 possible maps $f : X \to Y$ and ask which are continuous (all but $f(a) = 2, f(b) = 1$ because $f^{-1}(1) = b$ which isn't open).

- Prop: if Y has the trivial topology, then every map $f : X \to Y$ is continuous.

  "Most" maps from trivial topology on X aren't continuous (requires every open set of Y to contain $f(X)$ or be empty).

- Prop: if X has the discrete topology, then every map $f : X \to Y$ is continuous.

  "Most" maps from discrete Y aren't continuous.

- If $\mathcal{T}, \mathcal{T}'$ are topologies on the same set X, when is the identity map $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ with $f(x) = x$ continuous? Ans: when $\mathcal{T}' \subseteq \mathcal{T}$.

  Def if $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $\mathcal{T}'$ is coarser than $\mathcal{T}$ and that $\mathcal{T}$ is finer than $\mathcal{T}'$.

  Prop: $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ with f the identity map $f(x) = x$ is continuous iff $\mathcal{T}$ is finer than $\mathcal{T}'$.

- Prop: if $A \subseteq X$ is given the subspace topology, then the inclusion map $\iota : A \to X$ defined by $\iota(a) = a$ is continuous.

  - $f^{-1}(U) = U \cap A$, which is open in A by construction of subspace topology.

- Aside: this proof shows that the subspace topology is the "weakest" topology we can put on $A \subseteq X$ so that the inclusion map is continuous.

  General theme: if you have a "natural map" $f : X \to Y$, then you should define the "weakest" topologies on X, Y such that f is continuous (eg subspace above).

- The function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ is continuous (two ways, (1) directly and (2) because calculus).

- Warning: f continuous does NOT mean it maps open sets to open sets. E.g. the previous example.

5.2 Equivalences and Constructions

- Aside: why would you ever come up with the definition of continuity?

  - Intuition of $\varepsilon - \delta$ definition definition from calculus: small changes to your input lead to small changes in output.

  - More precisely, we say that $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if for any "tolerance" $\varepsilon > 0$ we can find $\delta > 0$ sufficiently small so that $f((x-\delta, x+\delta)) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$.

  - If we tried to generalize this intuition, we might come up with the following definition:

    End of aside

- Def: a map $f : X \to Y$ is said to be continuous at $x \in X$ if for every open set $f(x) \in V \subseteq Y$, there exists an open set $x \in U \subseteq X$ such that $f(U) \subseteq V$.

  Prop: a map $f : X \to Y$ is continuous (as defined at the start of class) iff it is continuous at every point $x \in X$ (as defined above).

  - Assume f is continuous and you have some $f(x) \in V \subseteq Y$, what U should you define to have $f(U) \subseteq V$? Take $U = f^{-1}(V)$; this works by construction.

  - Assume f is continuous at each point. Let V be open and $x \in f^{-1}(V)$. Since $f(x) \in V$, exists some neighborhood U_x with $f(U_x) \subseteq V$, and hence $U_x \subseteq f^{-1}(V)$. Note that $f^{-1}(V) = \bigcup U_x$, so it's open.

- Because continuity is such a fundamental concept, it will be useful to have a few more equivalent formulations.

  Thm: X, Y be topological spaces and $f : X \to Y$. TFAE:

  1. f is continuous (i.e. preimage of open sets are open)

  2. For every closed set $B \subseteq Y$, the set $f^{-1}(B)$ is closed in X.

  3. For every subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A))}$

Proof:

- (1) to (2): assume f is continuous and let B be closed in Y. Taking $V = Y - B$, basic set theory says Then $f^{-1}(V) = X \setminus f^{-1}(B)$. Since V is open, this set is open, which means $f^{-1}(B)$ is the complement of an open set and hence open. Other direction is basically the same.

- (1) to (3): Assume f continuous and $A \subseteq X$. Aim to show $x \in \overline{A}$ implies $f(x) \in \overline{f(A)}$. Let V be neighborhood of $f(x)$, pre-image is open so neighborhood of x, thus intersects A, so $f(f^{-1}(V)) \subseteq V$ intersects $f(A)$. Since every neighborhood of $f(x)$ intersects $f(A)$, we conclude $f(x) \in \overline{f(A)}$.

- (3) to (2): Let B be closed in Y and take $A = f^{-1}(B)$; aim is to show $A = \overline{A}$. Note that $f(A) = f(f^{-1}(B)) \subseteq B$. Thus if $x \in \overline{A}$, $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$, which means $x \in f^{-1}(B) = A$. Thus $\overline{A} \subseteq A$ and they must equal each other.

- Here's a way to construct new continuous functions from old ones.

  (Pasting Lemma) Let $X = A \cup B$ with A, B either both open or both closed in X and let $f : A \to Y$ and $g : B \to Y$ be continuous maps that agree at their intersection, i.e. $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is continuous.

  - E.g. the function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x$ for $x \leq 0$ and $h(x) = x/2$ for $x \geq 0$ is continuous because of this result.

  - Result is false if one of A, B is open and the other closed, e.g. $X = \mathbb{R}$, $A = (-\infty, 0]$ and $B = (0, \infty)$ with $f(x) = -1$ and $g(x) = 1$.

  - Proof: only prove case when A, B both closed. Let C be a closed set. Not difficult to argue $h^{-1}(C) = f^{-1}(V) \cup g^{-1}(C)$. Since f, g continuous, these two sets are *closed in* A, B. Since A, B are closed in X, some lemma implies these two sets are closed in X. Thus intersection is closed, proving the result by equivalent formulation of continuity.

5.3 Homeomorphisms

- Let $X = \{a, b\}$, $\mathcal{T}_x = \{\emptyset, X, \{a\}\}$ and similarly define $Y = \{1, 2\}$ with $\mathcal{T}_y$ basically the same. Ask if $(X, \mathcal{T}_x) = (Y, \mathcal{T}_y)$? Answer is no, but they are "equivalent".

- Definition: a map $f : X \to Y$ is said to be a *homeomorhism* if (a) f is a bijection, (b) f is continuous, and (c) f^{-1} is continuous (this exists because f is a bijection); equivalently $f(U)$ is open whenever U is open.

  - If there exists a homeomorphism between X, Y we say these spaces are *homeomorphic* and write $X \cong Y$.

- Note for those familiar with algebra that although this sounds like "homomorphism" its much closer to isomorphism.

- Eg are the X, Y at the start of this subsection homeomorphic?
  - What homeomorphism shows this?
  - Check that that this works: draw a column on the right listing the open sets of Y with the open sets of X on the other side, draw arrows from Y backwards labeled f^{-1} to their corresponding sets, then arrows going the other way labeled f.
  - Aside: a map f being a homeomorphism is equivalent to saying it "induces" a bijection between $\mathcal{T}_x$ and $\mathcal{T}_y$ (as the example above demonstrates), i.e. that the two topologies are just "relabelings" of each other. This relabeling definition is perhaps more intuitive, but the homeomrphism definition is easier to work with in practice.

- Eg let $B_n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ denote the ball of radius r centered at x.

  Claim: $B_n(x, r) \cong B_n(x + a, r)$ for all a (i.e. translates of the same space are homeomorphic)
  - What's the homeomorphism? $f(z) = z + a$.
  - That f is a bijection is straightforward.
  - That f and its inverse $g(z) = z - a$ are continuous follows "from calculus" (i.e. we know the topological definition of continuity is equivalent to the calculus definition, and we know from real analysis that translations are continuous functions).
  - Convention: throughout this course, if you have a function $f : X \to Y$ with X, Y subspaces of Euclidean space, you are allowed to say f is continuous "by calculus" whenever it follows from basic real analysis that f is continuous.
  - Aside: one can also prove f is continuous by hand (which is what I originally planned to do), but it is a real pain. The proof will become a lot easier once we have the tools from next lecture. Maybe sketch this out.

- Claim: if $r, c > 0$, then $B_n(0, r) \cong B_n(0, cr)$ (i.e. dilates of the same space are homeomorphic). Proof: $f(x) = cx$ is a homeomorphism "by calculus".

- The two statements above imply that any two balls in $\mathbb{R}^n$ of finite radius are homeomorphic to each other. In fact, this continues to hold even for infinite radiuses:

  Claim: $B_n(0, 1) \cong \mathbb{R}^n$. Proof: take $f(x) = \frac{x}{1-|x|}$ draw picture of arrows going out, with arrows expanding more farther away, this and its inverse $g(y) = \frac{y}{1+|y|}$ are continuous "by calculus".

- Variant: $[0,1)$ and $\mathbb{R}_{\geq 0}$ with subspace topologies are homeomorphic ($f : [0,1) \to \mathbb{R}_{\geq 0}$ with $f(x) = \frac{x}{1-x}$ is continuous by calculus, its inverse $g(y) = \frac{y}{y+1}$ also continuous by calculus).

- Prop: $X = S^1 = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = 1\}$ and $Y = \{(x,y) \in \mathbb{R}^2 : \max(x,y) = 1\}$ (square) are homeomorphic draw picture, map is $f(x,y) = (\frac{x}{|x|}, \frac{y}{|y|})$ and inverse $g(x,y) = (\frac{x}{\sqrt{|x|^2+|y|^2}}, \frac{y}{\sqrt{|x|^2+|y|^2}})$

- More generally, any two subspaces of $\mathbb{R}^n$ are homeomorphic if you can can twist/bend one into the other.

  - E.g. S^1 and some wild non-intersecting looking curve.

  - E.g. donut and coffee cup.

- Warning: f being continuous and bijective doesn't imply inverse is continuous, e.g. $[0,1)$ to circle via $f(x) = (\cos(2\pi x), \sin(2\pi x))$ (is a continuous bijection, but its inverse isn't continuous because of preimages around 0)

- Sometimes a map can be a "local" homeomorphism. Definition: let $f : X \times Y$ be an injective continuous map. If the restricted map $f' : X \to f(X)$ is a homeomorphism, then we say that the original map $f : X \to Y$ is an *imbedding*.

- Prop: the relation of being homeomorphic is an equivalence relation.

- Aside: say that a property is a *topological property* if the property is preserved under homeomorphisms.

  - E.g. Cardinality (if have two homeomorphic spaces then necessarily same cardinality because f bijection).

  - E.g. connectedness (see later).

  - Non-e.g.: location (translations), size, boundedness

  - Non-e.g: "smoothness" (e.g. can't distinguish circle vs square). That is, topology is too loose to understand curvature, but this can be resolved through "differential topology".

5.4 Padding for Time

- Various general continuous maps:

  - Constant functions.

  - Composition of continuous functions.
  - Inclusion maps.
  - Restricting domain.
  - Expanding codomain.

- Aside: the two most important definitions in any field of math is (1) what are the objects of study, (2) what are the "nice maps" between these objects? E.g. topological spaces/continuous, vector spaces/linear, sets/functions, groups/homomorphisms. More generally category theory.

6 Basis

- Recap: continuous functions, homeomorphisms.
- Problem: it can be hard to show that relatively simple maps f are continuous "by hand", e.g. showing the translation $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = x + a$ is tricky.
- Part of the difficulty above is that the definition of open sets in $\mathbb{R}^n$ is complicated: recall that a say U is open in $\mathbb{R}^n$ iff for every $x \in U$ there exists a ball $B_x \subseteq U$ containing x; this means weird shapes can be open draw one.

  Observation: general open sets $U \subseteq \mathbb{R}^n$ can be complex, but they're made up of simple building blocks (i.e. balls). Can we extend this idea?
- Idea: given a collection of sets $\mathcal{B}$, we want to define a topology $\mathcal{T}$ by having $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$.

  Problem: $\mathcal{T}$ won't be a topology for arbitrary sets $\mathcal{B}$, so we need to figure out some conditions on $\mathcal{B}$ which makes this work out.
- Definition: given a set X, a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* of X if (1) for every $x \in X$, there exists some $B \in \mathcal{B}$ containing x and (2) for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.
  - Eg open balls in Euclidean topology (if distance from x to x_i is d_i, then you can take $B = B(x, \min\{\varepsilon_i - d_i\})$)
  - Eg $X = \mathbb{R}^2$ and $\mathcal{R} = \{(a, b) \times (c, d)\} \subseteq \mathbb{R}^2$ (intersection itself is open rectangle).
  - Eg $X = \mathbb{R}$ and $\mathcal{H} = \{[a, b)\}$ (again intersection just works).
  - $\mathcal{D} = \{\{x\} : x \in X\}$ always works.
- Definition: if $\mathcal{B}$ is a basis for X, the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ is defined by having $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$ (note that this implies $\mathcal{B}$ are all open sets in $\mathcal{T}$).
  - Note this recovers Euclidean if $\mathcal{B}$ is open balls.
  - This is a topology: $\emptyset$ easy, X by (1). Pairwise intersection: there exists $B_i \subseteq U_i$ containing x each time, take intersection, by (2) there's some B contained in this containing x. Arbitrary union, take any i with $x \in U_i$ and then its corresponding basis element.
  - Aside: conditions (1) and (2) for $\mathcal{B}$ being a basis turn out to be equivalent to the condition that $\mathcal{T}$ is a topology, so this really is the "right" definition for a basis to make.

- What topologies do previous examples generate? Claim (will see soon) $\mathcal{R}$ generates Euclidean, i.e. same as balls $\mathcal{B}$ (despite the two having no elements in common). Topology generated by $\mathcal{H}$ is something other than Euclidean called "lower limit topology". $\mathcal{D}$ is discrete.

- The exact definition of the topology generated by $\mathcal{B}$ is somewhat complicated. Here's a cleaner formulation. Lemma: if $\mathcal{B}$ is a basis, then the topology $\mathcal{T}$ generated by $\mathcal{B}$ equals the set of all possible unions of elements of $\mathcal{B}$ (this includes the empty union).

  Proof: Note that $\mathcal{B} \subseteq \mathcal{T}$, and because $\mathcal{T}$ is a topology, it necessarily contains all possible unions of $\mathcal{B}$. Conversely, if $U \in \mathcal{T}$ then for each $x \in U$ there exists $B_x \in \mathcal{B}$ containing x and contained in U, so $U = \bigcup_{x \in U} B_x$.

- Now we get to one of the most useful consequences of basis.

  Thm: if Y is generated by a basis $\mathcal{B}$, then $f : X \to Y$ is continuous iff $f^{-1}(B)$ is open for $B \in \mathcal{B}$.

  - Continuous implies this condition.
  - This condition plus U equal to union of basis elements gives other direction.
  - Result continues to hold if Y is generated by a sub-basis.

- E.g. to check that the translation map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = x + a$ is continuous, it suffices to prove that $f^{-1}(B)$ is open whenever B is an open ball, and this holds since $f^{-1}(B)$ is an open ball.

- Basis play nicely with subspaces.

  Prop: if $\mathcal{B}$ is a basis for X, then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for Y. (every element of $Y \subseteq X$ is in some element, if you look at the intersection, lift to X, then project back down you get the thing).

- Basis make it easier to check if sets are closed.

  Thm: if X has a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.

  - Recall: $x \in \bar{A}$ iff every neighborhood of x intersects A.
  - It suffices to show this latter condition is equivalent to having every neighborhood intersect A, easy because B is open and because neighborhoods always contain a basis sub-neighborhood.

- We know how to go from basis to topology. Sometimes it will be useful to go the other way, i.e. given a topology $\mathcal{T}$ how do we find a basis for it?

  - Prop: let X be a topological space and $\mathcal{B}$ a family of subsets of X. If (a) every element of $\mathcal{B}$ is open and (b) For every open set $U \subseteq X$ and every $x \in U$ there is an element $B \in \mathcal{B}$ with $x \in B \subseteq U$, then $\mathcal{B}$ is a basis which generate the topology on X.

- Proof part I: $X = U$ implies (1) of basis. Since $\mathcal{B}$ are open sets, $B_1 \cap B_2$ is open so can find a B to satsisfy (2), so this is a basis.

- Proof part II: Let $\mathcal{T}'$ be the topology generated by $\mathcal{B}$, note $\mathcal{T} \subseteq \mathcal{T}'$ by the way things are defined. Conversely, if $W \in \mathcal{T}'$ then by the previous lemma W is the union of elements of $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}$ and since $\mathcal{T}$ is closed under unions, this means $W \in \mathcal{T}$.

- Corollary: $\mathcal{R}$ generates $\mathbb{R}^2$.

- Basis requires two relatively weak conditions, but sometimes its useful to relax even these.

  - Def: a set $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a *sub-basis* (or pre-basis) if for every $x \in X$, there exists some $B \in \mathcal{B}$ containing x (so it has (1) of the definition of the basis but not necessarily (2)).

  - Claim: the set of finite intersections of a sub-basis $\mathcal{S}$ is a basis. We define the topology generated by $\mathcal{S}$ to be the topology generated by this basis.