Intro to Topology

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Part I

Basic Definitions and Examples

1 What is Topology?

This question is somewhat complicated to answer because people use the word "topology" to refer to two related but distinct areas of study:

- *Modern topology* is roughly speaking the study of geometric objects such as spheres, Möbius strips, Klein bottles, and so on. If a mathematician says they "study topology", this is typically what they're referring to.

- *Point set topology* (also called *general topology*) is a very general framework that includes modern topology, much of calculus, and many other areas of mathematics. **This** is what the present course is all about, and from now on whenever I say the word "topology" I'll be referring to this concept.

The central object studied in topology are mathematical objects called *topologies*. So again we're left with the question: what is a topology?

1.1 What is a Topology: the Short Answer

The definition for a topology is a follows. At this point it should **not** be obvious to you why in the world you would ever consider something like this. Here and throughout, given a set X we let $\mathcal{P}(X)$ denote the power set of X, i.e. the set of all subsets of X.

Definition 1. Given a set X, a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* of X if the following conditions are satisfied:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under (arbitrary) unions, i.e. for any $\mathcal{S} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

(c) $\mathcal{T}$ is closed under *finite* intersections, i.e. for any *finite* subset $\mathcal{S} \subseteq \mathcal{T}$, $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

Again, this is a strange definition that should seem totally bizarre. In the next subsection I'll attempt to provide motivation for why one might possibly come up with this definition. Those that aren't interested/confused by this discussion can completely ignore it without affecting their understanding of the rest of the material in this course.

1.2 What is Topology: the Long Answer

The study of topology originates with the study of calculus/real analysis. When you took courses in these areas, you learned a number of important concepts about the set of real numbers $\mathbb{R}$, as well as about functions f from $\mathbb{R}$ to $\mathbb{R}$. In particular, two very important definitions are:

1. What it means for a sequence of real numbers $(x_n)_{n \geq 1}$ to *converge* to a real number x_0.

2. What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous*.

The central aim of topology is to give a *general framework* which expands these definitions for real numbers to a much broader class of mathematical objects. In particular, it aims to answer the following two questions:

1. What does it mean for a sequence of "objects" $(x_n)_{n \geq 1}$ to *converge* to another object x_0? For example, what does it mean for a sequence of functions $(f_n)_{n \geq 1}$ to converge to another function?

2. What does it mean for a function $f : X \to Y$ between two "nice objects" X, Y to be *continuous*? For example, what does it mean for a map $f : S^2 \to S^2$ from the sphere to itself to be continuous?

We'll postpone the second question and focus on convergence. Of course, any "reasonable" answer should in particular recover the original definition of convergence from real analysis. With this in mind, let's recall this definition and then think about how we might generalize it.

Definition 2. We say that a sequence of real numbers $(x_n)_{n \geq 1}$ *converges* to a real number x_0 if for all $\varepsilon > 0$, there exists an integer N_ε such that $|x_n - x_0| < \varepsilon$ for all $n \geq N_\varepsilon$.

While this is a fine definition, it's a little difficult to generalize. It turns out (for reasons that should not be obvious at this point) that a "better" definition can be made by utilizing the language of *open intervals*, which we recall are sets $I \subseteq \mathbb{R}$ of the form $\{x : a < x < b\}$ for some $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 3. We say that a sequence of real numbers $(x_n)_{n\geq 1}$ *converges* to a real number x_0 if for every open interval I containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

Claim 1.1. *These two definitions are equivalent. That is, a sequence $(x_n)_{n\geq 1}$ and real number x_0 satisfy the conditions of Definition 2 if and only if they satisfy the conditions of Definition 3.*

Definition 3 has several advantages over Definition 2. First, it avoids any mention of the real number ε (which is mathematically nice[1] since we ultimately want to generalize things beyond real numbers). Second, it frames the definition in terms of the more "geometric" concept of open intervals.

Now, at this point it probably still isn't obvious how to generalize Definition 3 to more general objects (like sequences of functions). However, if you were to spend 30 years about this problem, then perhaps you would come up with the following idea: replace the words "open interval" in Definition 3 with the words "nice set" (where the exact definition of "nice set" depends on your exact problem at hand) and use this as your definition of convergence. Somewhat more precisely, we'll try to work with the following definition (which at this point in time you don't need to memorize since we'll forget about it a moment).

Definition 4. Given a set X, we call any set $\mathcal{T} \subseteq \mathcal{P}(X)$ a *pre-topology*[2] of X. We say that a sequence of points $(x_n)_{n\geq 1}$ with $x_n \in X$ *converges* to a point $x_0 \in X$ with respect to $\mathcal{T}$ if for every $I \in \mathcal{T}$ containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

For example, if $X = \mathbb{R}$ and $\mathcal{T}$ is the set of open intervals of $\mathbb{R}$, then this exactly recovers Definition 3. Here are a few more (extreme) examples to give some more familiarity with these definitions.

Claim 1.2. *Let X be an arbitrary set.*

*(a) If $\mathcal{T} = \emptyset$ (i.e. if $\mathcal{T}$ contains no subsets of X), then **every** sequence of points $(x_n)_{n\geq 1}$ in X converges to **every** point $x_0 \in X$ with respect to $\mathcal{T}$*

(b) If $\mathcal{T} = \mathcal{P}(X)$ (i.e. if $\mathcal{T}$ contains every subset of X), then a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point $x_0 \in X$ with respect to $\mathcal{T}$ if and only if there exists some N such that $x_n = x_0$ for all $n \geq N$ (i.e. iff x_n is "eventually constant").

(c) If $\mathcal{T} = \{\{x\} : x \in X\}$ (i.e. if $\mathcal{T}$ is the set of singletons),then a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point $x_0 \in X$ with respect to $\mathcal{T}$ if and only if there exists some N such that $x_n = x_0$ for all $n \geq N$ (i.e. iff x_n is "eventually constant").

[1] This is also psychologically nice for those who have painful memories of ε from real analysis.

[2] This name is not standard at all and we will never use it beyond this first pre-lecture.

These last two examples suggest the following definition.

Definition 5. Two pre-topologies $\mathcal{T}, \mathcal{T}'$ for the same set X are said to be *equivalent* if: a sequence $(x_n)_{n\geq 1}$ converges to x_0 with respect to $\mathcal{T}$ if and only if it converges to x_0 with respect to $\mathcal{T}'$.

For example, the claim above shows the collection of singletons $\mathcal{T}$ is equivalent to $\mathcal{P}(X)$. Given that these two collections are equivalent to each other, which one should we work with, i.e. which is "better"? A possible answer is that the larger collection $\mathcal{P}(X)$ is "better" because its extra elements give us extra flexibility. This suggests the following problem.

Question 1.3. *Given a pre-topology $\mathcal{T}$, what is the "largest" pre-topology $\mathcal{T}'$ which contains $\mathcal{T}$ and which is equivalent to $\mathcal{T}$?*

This question seems a little daunting, so instead we ask the following weaker question.

Question 1.4. *Given a pre-topology $\mathcal{T}$, are there any "obvious" sets U that we can add to $\mathcal{T}$ so that $\mathcal{T} \cup \{U\}$ is equivalent to $\mathcal{T}$?*

Again if you think about this for 30 years you might realize the following.

Claim 1.5. *Let X be a set, $\mathcal{T}$ a pre-topology, and $\mathcal{S} \subseteq \mathcal{T}$ some non-empty subset of its elements.*

(a) $\mathcal{T} \cup \{\emptyset\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{X\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{\bigcup_{U\in\mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(c) If $\mathcal{S}$ is a finite set, then $\mathcal{T} \cup \{\bigcap_{U\in\mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(d) Part (c) does not hold in general if $\mathcal{S}$ is allowed to be an infinite set[3].

Sketch of Proof. For (a), since $\emptyset$ contains no elements of X it doesn't affect whether any given element is the limit of a sequence.

For (b), one can always take $N_X = 1$ (since every sequence lies in X for all $n \geq 1$).

For (c), take $N_{\bigcup_{U\in\mathcal{S}} U}$ equal to N_U for any $U \in \mathcal{S}$.

For (d), take $N_{\bigcap_{U\in\mathcal{S}} U} = \max_U N_U$ (note how this requires the set $\mathcal{S}$ to be finite). $\square$

That is, given any pre-topology $\mathcal{T}$, we can freely add in $\emptyset$ and X, as well as (arbitrary) unions and finite intersections of elements of $\mathcal{T}$ into $\mathcal{T}$ to make a (possibly) larger pre-topology which is equivalent to $\mathcal{T}$. In particular, this means that the largest pre-topology $\mathcal{T}'$ which is equivalent to $\mathcal{T}$ must contain $\emptyset, X$ and be "closed" under taking unions and finite intersections. This is exactly the definition of a topology!

[3]Hint: take $X = \mathbb{R}$, $\mathcal{T}$ to be the set of open intervals, and $\mathcal{S} = \{(-\frac{1}{n}, \frac{1}{n})\}$.

2 Definitions and Examples

Let's again restate the definition of a topology, as well as some related definitions that will serve as a useful language for talking about topologies.

Definition 6. Given a set X, a set $\mathcal{T}$ of subsets of X is called a *topology* of X if the following hold:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcup_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

(c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcap_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. We will call the pair $(X, \mathcal{T})$ a *topological space*. When $\mathcal{T}$ is clear from context we simply write X instead of $(X, \mathcal{T})$.

Remark 2.1. For arbitrary unions, the book likes to use the notation $\bigcup_{\alpha \in J} U_\alpha$ where J is an "index set", and we will occasionally use this notation in class as well. I recommend looking at Chapter 1 Section 5 of the book to get a more detailed explanation for how this notation is used throughout the book.

Now that we have the definition of a topology in hand, let's pause for a moment and look at some examples and non-examples of topologies.

2.1 Finite Topologies

Is the following pair $(X, \mathcal{T})$ a topological space: In class write all these down and ask students what they think the answers are

$$X = \{a, b, c\},\ \mathcal{T}_1 = \{\{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! It fails to have $\emptyset \in \mathcal{T}_1$. Okay, what about

$$X = \{a, b, c\},\ \mathcal{T}_2 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

Well, we have $\emptyset, X \in \mathcal{T}_2$, and it is not difficult to check by hand that this is closed under unions and (finite) intersections (e.g. $\{a, b\} \cap \{b\} = \{b\} \in \mathcal{T}_2$; the slicker way is to note that unions/intersections of sets containing b continue to be sets containing b) so this *is* a topology! What about

$$X = \{a, b, c\},\ \mathcal{T}_3 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}?$$

No! This isn't closed under intersections $\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{T}$. What about

$$X = \{a, b, c\},\ \mathcal{T}_4 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! Again not closed under intersections $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_4$. Note that these last three examples show that topologies aren't "monotonic", i.e. if you know $\mathcal{T}_2$ is a topology and $\mathcal{T}_4 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$, you can't conclude that either $\mathcal{T}_4, \mathcal{T}_3$ are necessarily topologies.

2.2 General Families

Let's look at some more general examples.

Definition 7. For any set X, the set $\mathcal{T} = \{\emptyset, X\}$ is a topology called the *trivial topology* or *indiscrete topology*. The proof that this is a topology follows by considering all 4 of the subsets $\mathcal{S} \subseteq \mathcal{T}$ and verifying that their unions/intersections lie in $\mathcal{T}$.

Definition 8. For any set X, the power set $\mathcal{T} = \mathcal{P}(X)$ (i.e. the set of all subsets of X) is a topology called the *discrete topology*. The proof that this is a topology follows from the fact that unions/intersections of subsets of X continue to be subsets of X (and hence lie in $\mathcal{T}$.

Definition 9. For any set X, the set $\mathcal{T} = \{S \subseteq X : |X \setminus S| < \infty\} \cup \{\emptyset\}$ (i.e. the set of elements which contain all but a finite number of points from X) is a topology called the *cofinite topology* or *finite complement topology*. The proof that this is a topology follows from the fact that unions/finite intersections of cofinite sets are cofinite (this requires a bit more of an argument involving De Morgan's laws).

2.3 Euclidean and Subspace Topologies

Here we discuss possible the two most important topologies which should always be at the back of your mind.

Definition 10. For $x_0 \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, define the *open ball* $B(x_0, \varepsilon) = \{x : |x - x_0| < \varepsilon\}$. For $X = \mathbb{R}^n$, consider the set $\mathcal{T}$ consisting of all sets S such that for all $x_0 \in S$ there exists an open ball $B(x_0, \varepsilon) \subseteq S$. Then $\mathcal{T}$ is a topology called the *Euclidean topology* or *standard topology*.

Draw a picture of an open set in $\mathbb{R}^2$.

Throughout this course, whenever we consider $\mathbb{R}^n$, we will assume it is a topological space with the Euclidean topology unless stated otherwise.

We next look at a general way for generating new topologies from old ones.

Definition 11. Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, we define the *subspace topology* $\mathcal{T}' = \{U \cap Y : U \in \mathcal{T}\}$. Unless stated otherwise we will always assume subsets $Y \subseteq X$ come equipped with the subspace topology, in which case we say that Y is a *subspace* of X.

Claim 2.2. *The subspace topology is a topology.*

Proof. $\emptyset, X$ are easy. For finite intersections, if you have $V_1, \ldots, V_r$ open then $V_i = U_i \cap Y$ for some U_i, then $\bigcap V_i = \bigcap U_i \cap Y$ which is open since X is a topology. The proofs for unions is similar

Actually, the most naive proof for arbitrary unions requires invoking the axiom of choice (this is a very subtle error; it was only noticed in 2018!). Since this is an undergraduate class I'm not going to fret over this, but can talk about it in office hours for those that are interested. □

Example 2.3. Let $X = \mathbb{R}$ with the Euclidean topology and $Y = [0, 1] \subseteq \mathbb{R}$ with the subspace topology. Which of the following sets are open in X? Which are open in Y?

- $(1/4, 3/4)$
- $(1/2, 1]$
- $[1/4, 3/4)$

Around 80% of the topologies we consider in the class will either be $\mathbb{R}^n$ with the Euclidean topology or some subset $X \subseteq \mathbb{R}^n$ equippped with the subspace topology. Here are a few common examples of these sorts of spaces:

- $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ (open ball).
- D^n (open disk)
- S^{n-1} ($(n-1)$-dimensional sphere
- $I^n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$ (n-dim cube, draw some examples).

Warning: if X is a space and $Y \subseteq X$ is a subspace, it is somewhat ambiguous to talk about "open sets" (do we mean open in X or in Y?). We deal with this as follows.

Definition 12. If $Y \subseteq X$ is a subspace, we say a set A is open in Y if it belongs to the topology of Y, and similarly we define what it means for A to be open in X.

In some situations there's no ambiguity, as in the following.

Claim 2.4. *If $Y \subseteq X$ is an open set in X, then every A which is open in Y is also open in X (intersection of open sets is open).*

3 Closed Sets

- Recap: topologies (and that given a topological space $(X, \mathcal{T})$, a set $U \subseteq X$ is called open if $U \in \mathcal{T}$), examples (Euclidean, subspace)

- Idea with definition of "open" is that these generalize notion of open sets from real analysis, but equally important in real analysis is closed sets (e.g. intermediate/extreme value theorem both involve functions $f : [a, b] \to \mathbb{R}$, i.e. both involve closed intervals).

  So, what's the right way to define closed sets for general topologies?

- Definition: a set $A \subseteq X$ is called *closed* if its complement $X - A$ is open.

- Warmup: which of the following sets A are open/closed in $X = \mathbb{R}$?

  - $[0, 1]$

  - $\mathbb{R}_{>0}$

  - $\mathbb{Q}$

  - $\mathbb{R}$

  - $\emptyset$

- Def: a set $A \subseteq X$ which is both open and closed is called clopen.

- Eg in discrete topology, every set is open/closed/clopen.

- Thm: if X is a topological space then: (1) $\emptyset, X$ are closed, (2) arbitrary intersections of closed sets are closed, (3) finite unions of closed sets are closed.

  With this we see we could have defined topology via closed sets and gotten same theory; there's no real distinction.

- As in the previous lecture, if we have a subspace $Y \subseteq X$ it can be ambiguous to say A is closed. In this case we will say that A is closed in Y or closed in X as apporpriate.

  Claim: if $Y \subseteq X$ is closed, then any A which is closed in Y is closed in X.

- Given $A \subseteq X$ there's two important sets we can associate to it. Def: the interior A^o (or $intA$) is the union of all the open sets contained in A, and the closure $\bar{A}$ is intersection of closed sets containing it.

- E.g intuitively what is $[0, 1)$ interior/closure?

- Essentially, A^o is the largest open subset of A (and similar $\bar{A}$).

- Prop: Prove these
  - $A^o, \bar{A}$ are open/closed.
  - $A^o \subseteq A \subseteq \bar{A}$.
  - If A is open then $A = A^o$, and if A is closed then $A = \bar{A}$.
- In order to characterize closure we need some definitions: if U is an open set containing x, then we say that U is a *neighborhood* of x. We say that two sets A, B intersect if $A \cap B \neq \emptyset$.
- Thm: $x \in \bar{A}$ iff every neighborhood of x intersects A.
  - Equivalent to prove $x \notin \bar{A}$ iff exists neighborhood disjoint from A. Indeed, if $x \notin \bar{A}$ then there exists closed set C containing A but not x, then $X - C$ is a neighborhood disjoint from A. Reverse direction similar.
- Examples for subsets of $\mathbb{R}$:
  - $A = \{n^{-1}\}$, closure is this plus 0 (easy to see 0 is in due to neighborhood description, everything else has neighborhood disjoint from it).
  - $A = \mathbb{Q}$, closure is $\mathbb{R}$.
- Neighborhoods are one useful way to characterize closures. Another way is through limit points. Definition: given a subset $A \subseteq X$, a point x is called a *limit point* (or cluster point, or point of accumulation) of A if every neighborhood of x intersects A in some point *other than itself*. Equivalently, x is a limit point if $x \in \overline{A - \{x\}}$.
- E.g. for $X = \mathbb{R}$, $A = \{0\}$ no point is a limit point. If $A = (0, 1]$ every point in $[0, 1]$ is a limit point. If $A = \{n^{-1}\}$ then only 0 is a limit point.

  Intuitively, x is a limit point if there's a sequence in $A - x$ "converging" to x.
- Thm: if A' denotes the set of limit points of A, then $\bar{A} = A \cup A'$.
  - If $x \in A'$ then every neighborhood intersects A, so by the one theorem it's in the closure so $A \cup A' \subseteq \bar{A}$.
  - If $x \in \bar{A} \setminus A$, then every neighborhood intersects A, and necessarily $A - x$ since $x \notin A$, so $x \in A'$.
- Corollary: a set A is closed if it contains all of its limit points ($A = A \cup A'$ implies $A' \subseteq A$).

4 Convergent Sequences and Hausdorff Spaces

- Again the goal of topology is to generalize concepts from real analysis, and now that we have a lot of examples/terminology, we can finally start defining these analogs.

- One important concept that we've seen is convergent sequences. Definition: given a topological space X, we say that a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point x if for all neighborhoods U of x, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

- This definition can be used to motivate the name "limit point" from last time.

  Prop: let X be a space and $A \subseteq X$. If $x \in X$ is such that there exists a sequence $(x_n)_{\geq n}$ in $A - x$ which converges to x, then x is a limit point of A.

  - Proof is that for any neighborhood there exist infinitely many x_n in it, all of which are in A and all of which are distinct from x.

  - Converse turns out to be false (i.e. there exist limit points which are not limits) but it's not super easy to construct; see this. This is true however in nice spaces (e.g. metric spaces).

- Natural question that pops up when playing with sequences: for every sequence $(x_n)_{n\geq 1}$, does there exist at most one point x which the sequence converges to?

  Intuition with $\mathbb{R}^n$ says, yes, but this is false: in trivial topology every sequence converges to every point.

- This is a weird situation we'd like to avoid.

  Definition: a topological space is said to be *Hausdorff* or T_2 if for each pair of distinct points x, y, there exist neighborhoods U, V of x, y respectively which are disjoint. Draw picture

- Thm: if X is Hausdorff, then every sequence converges to at most one point.

  - Assume for contradiction converge to x, y let U, V be neighborhoods. Take N_U, means all $n \geq N_U$ lie in U i.e. aren't in V, contradicting the existence of N_V.

  - Note that the definition of Hausdorff is designed to be essentially the weakest condition such that this property holds.

- Examples: proofs

  - $\mathbb{R}^n$ is Hausdorff

  - Trivial with at least two points is not

  - Discrete is Hausdorff

  - Finite complement with an infinite number of points is not (every two open sets intersect in all but finitely many points).

- This next proof will use the following trick that will be used many times throughout this course:

  Neighborhood trick: a set $U \subseteq X$ is open iff for every $x \in U$ there exists an open set V_x with $x \in V_x \subseteq U$. (Proof if U is open is easy, other direction uses unions).

- Already saw Hausdorff is nice because sequences have at most one limit, which agrees with our intuition from $\mathbb{R}^n$. It also plays nicely with intuition for closed sets.

  Thm: if X is Hausdorff, then every finite subset $A \subseteq X$ is closed.

  - Not true in general: $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ doesn't have b closed.

  - Proof: suffices to prove it when $A = \{x\}$ (since unions of closed sets are closed), i.e. that $X - x$ is open. Because X is Hausdorff, each $y \in X - x$ has a neighborhood U_y disjoint from x. Hence $\bigcup U_y = X - x$ is open.

  - Note that we didn't need the full power of Hausdorff in this proof: we only used that each y has a neighborhood disjoint from each x (so e.g. it holds for finite complement topology). This disjoint neighborhood condition is called the T1 condition; we'll return to this in Chapter 4.

5 Continuous Functions

- Recap: convergence, Hausdorff (and sequences converge to at most one point).

- Again, one of the main points of topology is to generalize key concepts from real analysis.

- Recall form calculus that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for all $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exist $\delta > 0$ such that for all x with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$.

  Phew, that's a mouthful.

- Def: a map $f : X \to Y$ between two topological spaces is called *continuous* if for every open set U in Y, the set $f^{-1}(U)$ is open in X.

  - That is f is continuous if the pre-image of open sets are open.

  - Warning: The notation $f^{-1}(U) := \{x : f(x) \in U\}$ is the *pre-image* of f NOT the inverse of f (which may not exist).

- Claim: this is equivalent to the calculus definition for the Euclidean topology.

5.1 Examples

- Eg take $X = \{a, b\}$, $\mathcal{T}_x = \{\emptyset, X, \{a\}\}$, define $Y = \{1, 2\}$ with essentially the same topology. Consider all 4 possible maps $f : X \to Y$ and ask which are continuous (all but $f(a) = 2, f(b) = 1$ because $f^{-1}(1) = b$ which isn't open).

- Prop: if Y has the trivial topology, then every map $f : X \to Y$ is continuous.

  "Most" maps from trivial topology on X aren't continuous (requires every open set of Y to contain $f(X)$ or be empty).

- Prop: if X has the discrete topology, then every map $f : X \to Y$ is continuous.

  "Most" maps from discrete Y aren't continuous.

- If $\mathcal{T}, \mathcal{T}'$ are topologies on the same set X, when is the identity map $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ with $f(x) = x$ continuous? Ans: when $\mathcal{T}' \subseteq \mathcal{T}$.

  Def if $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $\mathcal{T}'$ is coarser than $\mathcal{T}$ and that $\mathcal{T}$ is finer than $\mathcal{T}'$.

  Prop: $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ with f the identity map $f(x) = x$ is continuous iff $\mathcal{T}$ is finer than $\mathcal{T}'$.

- Prop: if $A \subseteq X$ is given the subspace topology, then the inclusion map $\iota : A \to X$ defined by $\iota(a) = a$ is continuous.

  - $f^{-1}(U) = U \cap A$, which is open in A by construction of subspace topology.

- Aside: this proof shows that the subspace topology is the "weakest" topology we can put on $A \subseteq X$ so that the inclusion map is continuous.

  General theme: if you have a "natural map" $f : X \to Y$, then you should define the "weakest" topologies on X, Y such that f is continuous (eg subspace above).

- The function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ is continuous (two ways, (1) directly and (2) because calculus).

- Warning: f continuous does NOT mean it maps open sets to open sets. E.g. the previous example.

5.2 Equivalences and Constructions

- Aside: why would you ever come up with the definition of continuity?

  - Intuition of $\varepsilon - \delta$ definition definition from calculus: small changes to your input lead to small changes in output.

  - More precisely, we say that $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if for any "tolerance" $\varepsilon > 0$ we can find $\delta > 0$ sufficiently small so that $f((x - \delta, x + \delta)) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$.

  - If we tried to generalize this intuition, we might come up with the following definition:

    End of aside

- Def: a map $f : X \to Y$ is said to be continuous at $x \in X$ if for every open set $f(x) \in V \subseteq Y$, there exists an open set $x \in U \subseteq X$ such that $f(U) \subseteq V$.

  Prop: a map $f : X \to Y$ is continuous (as defined at the start of class) iff it is continuous at every point $x \in X$ (as defined above).

  - Assume f is continuous and you have some $f(x) \in V \subseteq Y$, what U should you define to have $f(U) \subseteq V$? Take $U = f^{-1}(V)$; this works by construction.

  - Assume f is continuous at each point. Let V be open and $x \in f^{-1}(V)$. Since $f(x) \in V$, exists some neighborhood U_x with $f(U_x) \subseteq V$, and hence $U_x \subseteq f^{-1}(V)$. Note that $f^{-1}(V) = \bigcup U_x$, so it's open.

- Because continuity is such a fundamental concept, it will be useful to have a few more equivalent formulations.

  Thm: X, Y be topological spaces and $f : X \to Y$. TFAE:

  1. f is continuous (i.e. preimage of open sets are open)

  2. For every closed set $B \subseteq Y$, the set $f^{-1}(B)$ is closed in X.

  3. For every subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A))}$

Proof:

- (1) to (2): assume f is continuous and let B be closed in Y. Taking $V = Y - B$, basic set theory says Then $f^{-1}(V) = X \setminus f^{-1}(B)$. Since V is open, this set is open, which means $f^{-1}(B)$ is the complement of an open set and hence open. Other direction is basically the same.

- (1) to (3): Assume f continuous and $A \subseteq X$. Aim to show $x \in \overline{A}$ implies $f(x) \in \overline{f(A)}$. Let V be neighborhood of $f(x)$, pre-image is open so neighborhood of x, thus intersects A, so $f(f^{-1}(V)) \subseteq V$ intersects $f(A)$. Since every neighborhood of $f(x)$ intersects $f(A)$, we conclude $f(x) \in \overline{f(A)}$.

- (3) to (2): Let B be closed in Y and take $A = f^{-1}(B)$; aim is to show $A = \overline{A}$. Note that $f(A) = f(f^{-1}(B)) \subseteq B$. Thus if $x \in \overline{A}$, $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$, which means $x \in f^{-1}(B) = A$. Thus $\overline{A} \subseteq A$ and they must equal each other.

- We now look at some ways of constructing new continuous functions from old ones.

  Prop: if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

- For this next result we'll want the following which we've mentioned a few times already now: let $A \subseteq Y \subseteq X$. (a) If A is open in Y and Y is open in X, then A is open in X. (b) same with closed. Prove open one

- (Pasting Lemma) Let $X = A \cup B$ with A, B either both open or both closed in X and let $f : A \to Y$ and $g : B \to Y$ be continuous maps that agree at their intersection, i.e. $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is continuous.

  - E.g. the function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x$ for $x \leq 0$ and $h(x) = x/2$ for $x \geq 0$ is continuous because of this result.

  - Result is false if one of A, B is open and the other closed, e.g. $X = \mathbb{R}$, $A = (-\infty, 0]$ and $B = (0, \infty)$ with $f(x) = -1$ and $g(x) = 1$.

  - Proof: only prove case when A, B both closed. Let C be a closed set. Not difficult to argue $h^{-1}(C) = f^{-1}(V) \cup g^{-1}(C)$. Since f, g continuous, these two sets are *closed in* A, B. Since A, B are closed in X, the lemma above implies these two sets are closed in X. Thus intersection is closed, proving the result by equivalent formulation of continuity.

5.3 Homeomorphisms

- Let $X = \{a, b\}$, $\mathcal{T}_x = \{\emptyset, X, \{a\}\}$ and similarly define $Y = \{1, 2\}$ with $\mathcal{T}_y$ basically the same. Ask if $(X, \mathcal{T}_x) = (Y, \mathcal{T}_y)$? Answer is no, but they are "equivalent".

- Definition: a map $f : X \to Y$ is said to be a *homeomorhism* if (a) f is a bijection, (b) f is continuous, and (c) f^{-1} is continuous (this exists because f is a bijection); equivalently $f(U)$ is open whenever U is open.
  - If there exists a homeomorphism between X, Y we say these spaces are *homeomorphic* and write $X \cong Y$.
  - Note for those familiar with algebra that although this sounds like "homomorphism" its much closer to isomorphism.
- Eg are the X, Y at the start of this subsection homeomorphic?
  - What homeomorphism shows this?
  - Check that that this works: draw a column on the right listing the open sets of Y with the open sets of X on the other side, draw arrows from Y backwards labeled f^{-1} to their corresponding sets, then arrows going the other way labeled f.
  - Aside: a map f being a homeomorphism is equivalent to saying it "induces" a bijection between $\mathcal{T}_x$ and $\mathcal{T}_y$ (as the example above demonstrates), i.e. that the two topologies are just "relabelings" of each other. This relabeling definition is perhaps more intuitive, but the homeomrphism definition is easier to work with in practice.
- Eg let $B_n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ denote the ball of radius r centered at x.

  Claim: $B_n(x, r) \cong B_n(x + a, r)$ for all a (i.e. translates of the same space are homeomorphic)
  - What's the homeomorphism? $f(z) = z + a$.
  - That f is a bijection is straightforward.
  - That f and its inverse $g(z) = z - a$ are continuous follows "from calculus" (i.e. we know the topological definition of continuity is equivalent to the calculus definition, and we know from real analysis that translations are continuous functions).
  - Convention: throughout this course, if you have a function $f : X \to Y$ with X, Y subspaces of Euclidean space, you are allowed to say f is continuous "by calculus" whenever it follows from basic real analysis that f is continuous.
  - Aside: one can also prove f is continuous by hand (which is what I originally planned to do), but it is a real pain. The proof will become a lot easier once we have the tools from next lecture. Maybe sketch this out.
- Claim: if $r, c > 0$, then $B_n(0, r) \cong B_n(0, cr)$ (i.e. dilates of the same space are homeomorphic). Proof: $f(x) = cx$ is a homeomorphism "by calculus".

- The two statements above imply that any two balls in $\mathbb{R}^n$ of finite radius are homeomorphic to each other. In fact, this continues to hold even for infinite radiuses:

  Claim: $B_n(0,1) \cong \mathbb{R}^n$. Proof: take $f(x) = \frac{x}{1-|x|}$ draw picture of arrows going out, with arrows expanding more farther away, this and its inverse $g(y) = \frac{y}{1+|y|}$ are continuous "by calculus".

- Variant: $[0,1)$ and $\mathbb{R}_{\geq 0}$ with subspace topologies are homoemorphic ($f : [0,1) \to \mathbb{R}_{\geq 0}$ with $f(x) = \frac{x}{1-x}$ is continuous by calculus, its inverse $g(y) = \frac{y}{y+1}$ also continuous by calculus).

- Prop: $X = S^1 = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = 1\}$ and $Y = \{(x,y) \in \mathbb{R}^2 : \max(|x|,|y|) = 1\}$ (square) are homeomorphic draw picture, map is $f(x,y) = (\frac{x}{\max(|x|,(|y|)}, \frac{y}{\max(|x|,|y|)})$ and inverse $g(x,y) = (\frac{x}{\sqrt{|x|^2+|y|^2}}, \frac{y}{\sqrt{|x|^2+|y|^2}})$

- More generally, any two subspaces of $\mathbb{R}^n$ are homeomorphic if you can can "twist/bend" one into the other.

  - E.g. S^1 and some wild non-intersecting looking curve.

  - E.g. donut and coffee cup.

- Warning: f being continuous and bijective doesn't imply inverse is continuous, e.g. $[0,1)$ to circle via $f(x) = (\cos(2\pi x), \sin(2\pi x))$ (is a continuous bijection, but its inverse isn't continuous because of preimages around 0)

- Sometimes a map can be a "local" homeomorphism. Definition: let $f : X \to Y$ be an injective continuous map. If the restricted map $f' : X \to f(X)$ is a homeomorphism, then we say that the original map $f : X \to Y$ is an *imbedding*.

- Prop: the relation of being homeomorphic is an equivalence relation.

- Aside: say that a property is a *topological property* if the property is preserved under homeomorphisms.

  - E.g. Cardinality (if have two homeomorphic spaces then necessarily same cardinality because f bijection).

  - E.g. connectedness (see later).

  - Non-e.g.: location (translations), size, boundedness

  - Non-e.g: "smoothness" (e.g. can't distinguish circle vs square). That is, topology is too loose to understand curvature, but this can be resolved through "differential topology".

5.4 Padding for Time

- Various general continuous maps:
  - Constant functions.
  - Restricting domain.
  - Expanding codomain.
- Aside: the two most important definitions in any field of math is (1) what are the objects of study, (2) what are the "nice maps" between these objects? E.g. topological spaces/continuous, vector spaces/linear, sets/functions, groups/homomorphisms. More generally category theory.

6 Basis

- Problem: it can be hard to show that relatively simple maps f are continuous "by hand", e.g. showing the translation $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = x + a$ is tricky.

- Part of the difficulty above is that the definition of open sets in $\mathbb{R}^n$ is complicated: recall that a say U is open in $\mathbb{R}^n$ iff for every $x \in U$ there exists a ball $B_x \subseteq U$ containing x; this means weird shapes can be open draw one.

  Observation: general open sets $U \subseteq \mathbb{R}^n$ can be complex, but they're made up of simple building blocks (i.e. balls). Can we extend this idea?

- Idea: given a collection of sets $\mathcal{B}$, we want to define a topology $\mathcal{T}$ by having $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$ (by the neighborhood trick, this is the same as saying every open set is the union of elements of $\mathcal{B}$).

  Problem: $\mathcal{T}$ won't be a topology for arbitrary sets $\mathcal{B}$, so we need to figure out some conditions on $\mathcal{B}$ which makes this work out.

- Definition: given a set X, a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* of X if (1) for every $x \in X$, there exists some $B \in \mathcal{B}$ containing x and (2) for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.

  - Eg open balls in Euclidean topology (if distance from x to x_i is d_i, then you can take $B = B(x, \min\{\varepsilon_i - d_i\})$
  - Eg $X = \mathbb{R}^2$ and $\mathcal{R} = \{(a, b) \times (c, d)\} \subseteq \mathbb{R}^2$ (intersection itself is open rectangle).
  - Eg $X = \mathbb{R}$ and $\mathcal{H} = \{[a, b)\}$ (again intersection just works).
  - $\mathcal{D} = \{\{x\} : x \in X\}$ always works.

- Definition: if $\mathcal{B}$ is a basis for X, the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ is defined by having $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$ (note that this implies $\mathcal{B}$ are all open sets in $\mathcal{T}$).

  - Note this recovers Euclidean if $\mathcal{B}$ is open balls.
  - This is a topology: $\emptyset$ easy, X by (1). Pairwise intersection: there exists $B_i \subseteq U_i$ containing x each time, take intersection, by (2) there's some B contained in this containing x. Arbitrary union, take any i with $x \in U_i$ and then its corresponding basis element.
  - Aside: conditions (1) and (2) for $\mathcal{B}$ being a basis turn out to be equivalent to the condition that $\mathcal{T}$ is a topology, so this really is the "right" definition for a basis to make.

- What topologies do previous examples generate? Claim (will see soon) $\mathcal{R}$ generates Euclidean, i.e. same as balls $\mathcal{B}$ (despite the two having no elements in common). Topology generated by $\mathcal{H}$ is something other than Euclidean called "lower limit topology". $\mathcal{D}$ is discrete.

- The exact definition of the topology generated by $\mathcal{B}$ is somewhat complicated. Here's a cleaner formulation. Lemma: if $\mathcal{B}$ is a basis, then the topology $\mathcal{T}$ generated by $\mathcal{B}$ equals the set of all possible unions of elements of $\mathcal{B}$ (this includes the empty union).

  Proof: Note that $\mathcal{B} \subseteq \mathcal{T}$, and because $\mathcal{T}$ is a topology, it necessarily contains all possible unions of $\mathcal{B}$. Conversely, if $U \in \mathcal{T}$ then for each $x \in U$ there exists $B_x \in \mathcal{B}$ containing x and contained in U, so $U = \bigcup_{x \in U} B_x$.

- Now we get to one of the most useful consequences of basis.

  Thm: if Y is generated by a basis $\mathcal{B}$, then $f : X \to Y$ is continuous iff $f^{-1}(B)$ is open for $B \in \mathcal{B}$.

  - Continuous implies this condition.
  - This condition plus U equal to union of basis elements gives other direction.

- E.g. to check that the translation map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = x + a$ is continuous, it suffices to prove that $f^{-1}(B)$ is open whenever B is an open ball, and this holds since $f^{-1}(B)$ is an open ball.

- Basis play nicely with subspaces.

  Prop: if $\mathcal{B}$ is a basis for X, then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for Y. (every element of $Y \subseteq X$ is in some element, if you look at the intersection, lift to X, then project back down you get the thing).

- Basis make it easier to check if sets are closed.

  Thm: if X has a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.

  - Recall: $x \in \bar{A}$ iff every neighborhood of x intersects A.
  - It suffices to show this latter condition is equivalent to having every neighborhood intersect A, easy because B is open and because neighborhoods always contain a basis sub-neighborhood.

- We know how to go from basis to topology. Sometimes it will be useful to go the other way, i.e. given a topology $\mathcal{T}$ how do we find a basis for it?

  - Prop: let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ a family of subsets of X. If (a) every element of $\mathcal{B}$ is open and (b) For every open set $U \subseteq X$ and every $x \in U$ there is an element $B \in \mathcal{B}$ with $x \in B \subseteq U$, then $\mathcal{B}$ is a basis which generate the topology on X.

- Proof (if $\mathcal{B}$ satisfies these conditions then it is a basis): Taking $U = X$ implies (1) of basis. Since $\mathcal{B}$ are open sets, $B_1 \cap B_2$ is open so can find a B to satisfy (2), so this is a basis.
- Proof (that the topology generates $\mathcal{T}$): Let $\mathcal{T}'$ be the topology generated by $\mathcal{B}$. Every element $W \in \mathcal{T}'$ is a union of elements of $\mathcal{B}$ which are open sets, so $W \in \mathcal{T}$. On the other hand, each $U \in \mathcal{T}$ can be written as the union of basis elements so $U \in \mathcal{T}'$.

• Corollary: $\mathcal{R}$ generates $\mathbb{R}^2$.

• Basis requires two relatively weak conditions, but sometimes its useful to relax even these.

- Def: a set $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a *sub-basis* (or pre-basis) if for every $x \in X$, there exists some $B \in \mathcal{B}$ containing x (so it has (1) of the definition of the basis but not necessarily (2)).
- Claim: the set of finite intersections of a sub-basis $\mathcal{S}$ is a basis. We define the topology generated by $\mathcal{S}$ to be the topology generated by this basis.
- Claim: if Y is generated by a subbasis $\mathcal{S}$, then $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open for all $U \in \mathcal{S}$.

7 Product Topologies

7.1 Finite Products

- The next few lectures explore forming new topologies by performing "operations" on old ones.

- Definition: given topological spaces X, Y, define the *product topology* on $X \times Y$ as the topology generated by the basis $\mathcal{B} = \{U \times V : U \text{ open in } X, V \text{ open in } Y\}$.

  Claim: $\mathcal{B}$ is a basis (and hence does indeed generate a topology).

- Warning $\mathcal{B}$ is *not* a topology, i.e. the open sets of $X \times Y$ are *not* (the only) elements of $\mathcal{B}$. Insert picture of union of two rectangles

- Claim: if $\mathbb{R}$ has the standard topology, then the product topology $\mathbb{R} \times \mathbb{R}$ equals the standard topology on $\mathbb{R}^2$ (equivalently, the topology generated by the basis of open rectangles is the same as the topology generated by open balls).

- Examples of products:

  - $S^1 \times I^1$ is cyllinder.

  - $S^1 \times S^1$ is torus.

  - $S^1 \times D^2$ is solid torus.

- Claim: if $\mathcal{B}, \mathcal{C}$ are basis that generate the topologies for X, Y, then $\mathcal{D} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$ generates the product topology $X \times Y$.

  - Recall lemma from before: a set $\mathcal{D}$ is a basis generating a topology $\mathcal{T}$ if (1) $\mathcal{D}$ is a set of open sets, (2) for every $x \in U \in \mathcal{T}$ there exists $x \in D \subseteq U$.

- Subspace and product topologies "commute": if you take $X \times Y$ and give $A \times B$ the subspace topology, it's the same as if you first gave A, B the subspace topology and then took the product topology.

- Def: the proection map $\pi_1 : X \times Y \to X$ has $\pi_1(x, y) = x$, and we similarly define π_2.

  These are the most important maps related to product topologies. In particular, they can be used to generate the product topology.

  Claim: the sets $\{\pi_1^{-1}(U) : U \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathcal{T}_Y\}$, i.e. the sets $U \times Y$ and $X \times V$ form a subbasis for the product topology $X \times Y$.

- Prop: the projection maps π_i are continuous (proof: $\pi_1^{-1}(U) = U \times Y$)

In fact, the product topology on $X \times Y$ is the "weakest" topology such that the projection maps are continuous (analogous to how subspace topology was the weakest so that the inclusion map $\iota : A \to X$ was continuous). This is essentially why we defined things this way.

- Prop: if $f : A \to X$ and $g : A \to Y$ are continuous, then the map $h : A \to X \times Y$ with $h(a) = (f(a), g(a))$ is continuous (suffices to check basis elements, $U \times V$ whose preimage is $f^{-1}(U) \cap g^{-1}(V)$).
  - Converse also true: if $h : A \to X \times Y$ with $h(a) = (f(a), g(a))$ is continuous, then f, g are continuous (suffices to look at preimage of $U \times Y$).
  - Warning: no useful way for saying that a function $h : A \times B \to X$ is continuous.

7.2 General Products

- How do you define product topology for three spaces? Same basic idea, also works for any finite number of products. What about infinite products? This is more subtle.
- First we need to figure out how to define infinite products.
  - Let's recall finite products. Literally $X \times X = \{(x_1, x_2) : x_i \in X\}$. What does it mean for $x = (x_1, x_2)$ to be an ordered pair? It's equivalent to having a map $x : \{1, 2\} \to X$ with $x(1) = x_1$ and $x(2) = x_2$.
  - Given sets J, X, we define a J-tuple of elements of X to be a function $\mathbf{x} : J \to X$. If $\alpha \in J$ we often denote the value $\mathbf{x}(\alpha)$ by x_α and denote $\mathbf{x}$ by the symbol $(x_\alpha)_{\alpha \in J}$.
  - Given an indexed family of sets $\{A_\alpha\}_{\alpha \in J}$, we define the cartesian product $\prod_{\alpha \in J} A_\alpha$ to be the set of all J-tuples of $X = \bigcup A_\alpha$ such that $x_\alpha \in A_\alpha$ for all $\alpha \in J$.
    If $A_\alpha = X$ for all $\alpha \in J$, then we will write this product as X^J (equivalent to set of all functions from J to X), and if $J = \mathbb{Z}_{>0}$ we use the shorthand X^ω.
  - Eg if $J = \mathbb{Z}_{>0}$ and $A_\alpha = \mathbb{R}$ for all α what is $\prod A_\alpha = \mathbb{R}^{\mathbb{Z}_{>0}} = \mathbb{R}^\omega$? Formally this is all functions $f : \mathbb{Z}_{>0} \to \mathbb{R}$, which (in tuple notation) is the set of sequences of real numbers (e.g. $(n^2)_{n \geq 1}$ is in this set).
  - Most examples of cartesian product will be of the form X^ω, i.e. this will just be infinite sequences.
- Topology? Naive attempt: given a family of topological spaces $\{X_\alpha\}_{\alpha \in J}$ we define the *box topology* on $\prod X_\alpha$ as having the basis consisting of sets $\prod U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open.
  - Lem: this is a basis (and hence a well defined topology).

– This is the simplest thing to do, but it has some serious issues. In particular, nice properties that held for finite products don't hold in general here.

– Define $f : \mathbb{R} \to \mathbb{R}^\omega$ via $f(t) = (t, t, \ldots)$. Simple map that's continuous in each coordinate, but f is not continuous (take preimage of $(-1, 1) \times (-1/2, 1/2) \times \cdots$, get $\{0\}$).

– This isn't ideal: we'd like to say like before that if we have a map $f : Y \to X_1 \times X_2 \cdots$ which is continuous in each coordinate then f is continuous.

- Non-obvious solution: given a family of topological spaces $\{X_\alpha\}_{\alpha \in J}$ we define the *product topology* on $\prod X_\alpha$ as having the basis consisting of sets $\prod U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open **and** where $U_\alpha = X_\alpha$ for all but finitely many α.

  – Lem: this is a basis (and hence a well defined topology).

  – Eg for $\mathbb{R}^\omega$, the set $(-1, 1) \times (-1/2, 1/2) \times \cdots$ is open in the box topology (since it's a basis element), but it is **not** open in the product topology (since every non-empty open set must contain a basis element)

  – Note that the box and product agree on finite products but for infinite one's the product topology is coarser than box.

  – Product topology, while less obvious, turns out to be way nicer, so from now on whenever we look at infinite product spaces we'll assume they have the product topology unless stated otherwise. One of main reasons is the following.

- How might you come up with this definition? One way is through projection maps $\pi_\alpha : \prod X_\beta \to X_\alpha$ defined by $\pi_\alpha(x) = x_\alpha$.

  Claim: the sets $\bigcup_\alpha \{\pi_\alpha^{-1}(U) : U \in \mathcal{T}_{X_\alpha}\}$ form a subbasis for the product topology $\prod X_\alpha$ (i.e. their finite intersections are exactly the basis elements of the product topology).

  Claim: the product topology is the "weakest" topology such that each projection map π_α are continuous.

- Thm: let $\{X_\alpha\}_{\alpha \in J}$ be a family of spaces, A a space, $f_\alpha : A \to X_\alpha$ a family of functions, and $f : A \to \prod X_\alpha$ defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$. If $\prod X_\alpha$ is given the product topology, then f is continuous iff f_α is continuous for all α.

  – Proof: if some f_α is not continuous then $f_\alpha^{-1}(U_\alpha)$ is not open in A for some open set $U_\alpha \subseteq X_\alpha$. Note that the set $U = \prod U_\beta$ with $U_\beta = X_\beta$ for $\beta \neq \alpha$ is open (since it's a basis element) and

  $$f^{-1}(U) = \bigcap f^{-1}(U_\beta) = f^{-1}(U_\alpha) \bigcap_{\beta \neq \alpha} A = f^{-1}(U_\alpha)$$

  which isn't open by assumption.

- Assume now each f_α is continuous, and recall to prove f is continuous it suffices to show $f^{-1}(U)$ is open for all basis elements $U = \prod U_\alpha$. For such a basis element we have
$$f^{-1}(\prod U_\alpha) = \bigcap f_\alpha^{-1}(U_\alpha).$$
Note that all but finitely many terms in this intersection equal A by definition of U being a basis element. Thus this set is equal to the finite intersection of sets of the form $f_\alpha^{-1}(U_\alpha)$. These sets are all open since each f_α is continuous, so their finite intersection is also open.

- In particular, the map $f : \mathbb{R} \to \mathbb{R}^\omega$ with $f(t) = (t, t, \ldots)$ is continuous under the product topology.

- Aside: the above deals with continuity of (infinite) product spaces, what about sequences in product spaces?

  - E.g. consider $\mathbb{R}^\mathbb{R} = \{f : \mathbb{R} \to \mathbb{R}\}$. Claim that $f_n \to f$ iff $f_n(x) \to f(x)$ for all x, i.e. product topology is the topology of pointwise convergence.
    This is a HW problem.

- Closed sets play nicely with both kinds of product spaces:

  Thm: Let $\{X_\alpha\}$ be a family of spaces and $A_\alpha \subseteq X_\alpha$ for all α. If $\prod X_\alpha$ is given either the product or the box topology, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

  - Let $x \in \prod \overline{A}_\alpha$, we want to show $x \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element containing x. Since $x_\alpha \in \overline{A}_\alpha$ and each $U_\alpha \subseteq X_\alpha$ is open, we have that $U_\alpha \cap A_\alpha \neq \emptyset$, and hence $\prod U_\alpha \cap \prod A_\alpha \neq \emptyset$. Since U was arbitrary, it follows that x is in the closure of $\prod A_\alpha$.

  - Now assume $x \in \overline{\prod A_\alpha}$. We want to show $x \in \overline{A_\beta}$ for all β. Fix any neighborhood $U_\beta \subseteq X_\beta$ of x_β. Observe that $U = \prod U_\alpha$ with $U_\alpha = X_\alpha$ for $\alpha \neq \beta$ is a neighborhood of x in the product space, so by hypothesis it intersects $\prod A_\alpha$ at some point y. But then $y_\beta \in U_\beta \cap A_\beta$. This implies every neighborhood of x_β intersects A_β, so $x_\beta \in \overline{A_\beta}$ and hence $x \in \prod \overline{A_\beta}$.

8 Quotient Topology

- Idea: we want to take a space X and "glue" points of X to create a new space.
  - E.g. if you have a square I^2, then gluing two opposite sides gives a cylinder $I \times S^1$.
  - E.g. if you glue the circles of the cylinder you get a torus $S^1 \times S^1$ (same thing happens if you take I^2 and glue both pairs of opposite sides)
- Need to formally define how to "glue" things.

  Aside: There are two different ways of doing this: quotient maps and equivalence relations. The book does the former, we'll mostly be doing the latter (see the supplement on the website).
- Def: an equivalence relation $\sim$ on a set X is a binary relation satisfying reflexivity, symmetry, and transitivity.

  Given $x \in X$, the equivalence class $[x]$ of X is the subset of X with $[x] = \{y \in X : x \in y\}$. We let $X/\sim$ denote the set of equivalence classes:

$$(X/\sim) = \{[x] : x \in X\}.$$

- Some examples (Question: what "space" do these equivalence classes define?)
  - Let $X = \mathbb{R}$ and define $\sim$ by $x \sim y$ iff $x - y \in \mathbb{Z}$. The equivalence classes are

$$[x] = \{\ldots, x-2, x-1, x, x+1, x+2, \ldots\}.$$

    E.g. $[1/2] = \{, \ldots, -1/2, 1/2, 3/2, \ldots\} = [-1/2] = [5/2] = \cdots$.
    In particular we can write

$$\mathbb{R}/\sim = \{[x] : 0 \leq x < 1\} = \{[y] : 99.5 < y \leq 100.5\}.$$

    When asking about the space, draw $\mathbb{R}$ as a spiral projecting onto S^1, identify the points of $[0]$ and note that they map to the same thing.
  - Let $A \subseteq X$ be sets, define $\sim$ by $x \sim y$ if $x = y$ or if $x, y \in A$. Equivalence classes are $[x] = \{x\}$ if $x \notin A$ and $[x] = A$ if $x \in A$.

    E.g. if X is a circle and A is two arcs then this turns into a figure eight.

    E.g. if X is a cyllinder and A is one of the faces this turns into a cone.
  - Let $X = [0,1]^2$, define $\sim$ by $(x, 0) \sim (x, 1)$ for $0 \leq x \leq 1$ and $(0, y) \sim (1, y)$ (draw diagram of what this represents). Equivalence classes: $[(x, y)] = \{(x, y)\}$ if $0 < x, y < 1$, $[(x, 0)] = \{(x, 0), (x, 1)\}$ if $0 < x < 1$, similar for y, $[(0, 0)] = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$

- Question: if X is not just a set but a topological space, what's the right way to define open sets on $X/\sim$?
  - Idea: similar to product spaces, there's a natural "projection map" $\pi : X \to X/\sim$ defined by $\pi(x) = [x]$. Just like for products, the "right" notion of topology should be the smallest collection of sets such that π is continuous.
  - To get a handle on this, given $U \sum X/\sim$, what is $\pi^{-1}(U)$? Answer: $\bigcup_{[x]\in U}[x]$ (i.e. U is a set of equivalence classes $[x]$, and an element of X maps to an equivalence class of U iff x lies in an equivalence class of U).
  - In particular, if we want π to be continuous, then every open set $U \subseteq X/\sim$ needs to have that $\bigcup_{[x]\in U}[x]$ is open.
- Def: Let X be a topological space and $\sim$ an equivalence relation on X. The quotient topology on $X/\sim$ consists of all sets $U \subseteq (X/\sim)$ such that $\bigcup_{[x]\in U}[x] \subseteq X$ is open in X.

  To emphasize: U is a set of equivalence classes of X, so this union is over subsets of X (and hence does indeed lie in X).
- Prop: the quotient topology is a topology.
  - If $U = \emptyset$ then the union is $\emptyset$ which is empty. If $U = (X/\sim)$ then the union is X.
  - Let $\bigcup_\alpha U_\alpha$ be an arbitrary union of open sets in $X/\sim$. Then

    $$\bigcup_{[x]\in\bigcup U_\alpha}[x] = \bigcup_\alpha \bigcup_{[x]\in U_\alpha}[x],$$

    which is the union of open sets in X by definition of U_α being open.
  - Similarly for a finite intersection

    $$\bigcup_{[x]\in\bigcap U_i}[x] = \bigcap\left(\bigcup_{[x]\in U_i}[x]\right),$$

    which is the finite intersection of open sets.
- Claim: the canonical map $\pi : X \to (X/\sim)$ is continuous with respect to the quotient topology (and the quotient topology is the largest topology for which this holds).
- Problem: how do we show that $X/\sim$ with the quotient topology is homeomorphic to some space Y we care about? This was theoretically the whole reason we're doing all this in the first place.
- Caution: when dealing with maps f from equivalence classes, we have to make sure f is well defined.

E.g. Say $X = \mathbb{R}$ and $x \sim y$ iff $x - y \in \mathbb{Z}$; we intuited that this should be homeomorphic to S^1. Consider the map $f : (\mathbb{R}/\sim) \to S^1$ defined by $f([x]) = (\cos(x), \sin(x))$. What is $f([0])$? $f([1])$? Problem: $[0] = [1]$, so this map isn't well defined.

- We can get around this issue by considering "nice" maps from X and the following.

  Thm (universal property of the quotient topology): let X be a topological space and $\sim$ an equivalence relation on X. Endow $X/\sim$ with the quotient topology and let $\pi : X \to X/\sim$ be the canonical projection.

  Let Y be another topological space and $f : X \to Y$ a continuous function such that $f(x) = f(x')$ whenever $x \sim x'$ in X. Then there exists a unique continuous function $\bar{f} : (X/\sim) \to Y$ such that $f = \bar{f} \circ \pi$.

  - Proof sketch: define $\bar{f}([x]) = f(x)$. This is a well defined map from $(X/\sim)$ to Y, so it remains to show it's continuous.
  - Let $U \subseteq Y$ be open, its preimage $\bar{f}^{-1}(U)$ is open in $X/\sim$ iff the following set is open in X:
  $$\bigcup_{[x] \in \bar{f}^{-1}(U)} [x] == f^{-1}(U),$$
  where the equality used that $x \in f^{-1}(U)$ iff $[x] \subseteq f^{-1}(U)$ (since $f(x) = f(x')$ if $x \sim x'$) which holds iff $[x] \in \bar{f}^{-1}(U)$. This set is open in X since f is continuous.

- Strategy for proving $X/\sim$ is homeomorphic to Y:
  - (1) Find a candidate continuous function $f : X \to Y$.
  - (2) Prove $f(x) = f(x')$ whenever $x \sim x'$; then $\bar{f} : (X/\sim) \to Y$ defined by $\bar{f}([x]) = f(x)$ is well defined and continuous by universal property.
  - (3) Find a candidate inverse continuous function $g : Y \to (X/\sim)$.
  - (4) Prove $f \circ g = id_Y$ and $g \circ f = id_X$.

- Prop: $\mathbb{R}/\sim$ with $x \sim y$ iff $x - y \in \mathbb{Z}$ is homeomorphic to S^1.
  - (1) Find candidate function $f : X \to Y$. Take $f(x) = (\cos(2\pi x), \sin(2\pi x))$. This is continuous from calculus.
  - (2) If $x \sim x'$ then $x - x' \in \mathbb{Z}$, thus
  $$f(x) = (\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi x + 2\pi(x' - x)), \sin(2\pi x + 2\pi(x' - x))) = (\cos(2\pi x'), \sin(2\pi x'))$$
  Thus the induced map $\bar{f}([x]) = \cos(2\pi x), \sin(2\pi x))$ is well defined and continuous.
  - (3) Construct inverse. We will use the pasting lemma: Let $Y = A \cup B$ with A, B both closed in Y and let $g_1 : A \to Z$ and $g_2 : B \to Z$ be continuous maps that agree

at their intersection. Then the function $g : Y \to Z$ defined by $g(x) = g_1(x)$ for $x \in A$ and $g(x) = g_2(x)$ for $x \in B$ is continuous.

Let $A = \{(x, y) \in S^1 : y \leq 0\}$ and $B = \{(x, y) \in S^1 : y \geq 0\}$. For $z \in A$ there exists a unique $0 \leq x \leq 1/2$ with $z = (\cos(2\pi x), \sin(2\pi x)$, define $g_1' : A \to \mathbb{R}$ by $g_1'(z) = x$. Similarly $z \in B$ can be written uniquely as $(\cos(2\pi x), \sin(2\pi x))$ with $1/2 \leq x \leq 1$. These functions are continuous by calculus but do *not* agree on $A \cap B$ (e.g. they map $(1, 0)$ to 0 and 1).

Can fix this: define $g_1 : A \to (R/\sim)$ by $g_1 = \pi \circ g_1'$ and similarly define g_2'. Since g_i' and π are continuous, their compositions g_i are also continuous. Moreover, g_1, g_2 agree on their intersection $\{(1, 0), (-1, 0)\}$, so the pasting lemma gives some continuous function $g : S^1 \to (\mathbb{R}/\sim)$.

  - (4) Basic calculations shows $f \circ g$ and $g \circ f$ are identity maps.

- That was a lot, let's take a breather and play with pictures for a little bit.
  - Quotients through diagrams. Idea is that equivalence relations are a pain to write down in full, so often people will just draw pictures to indicate what they mean.
  - Draw square with vertical sides identified. The arrows mean $(0, y) \sim (1, y)$. Physically take a strip of papers with arrows on two sides and glue together; emphasize that this makes it so the arrows line up.
  - Draw previous thing but with one arrow reversed; what equivalence relation does this correspond to? What does this shape look like? Again physically do an example, trace out a line to show it's one-sided.
  - Draw square for torus and ask them what this represents.
  - Draw previous with one pair of arrows reversed; this is called a Klein bottle
  - Draw with both reversed; this is projective space.
- Aside: alternative persepective through quotient maps.
  - Let $q : X \to Y$ be a surjective map. If X is a topolgoical space, then we define the "quotient topology" on Y (with respect to q) by making $U \subseteq Y$ open iff $q^{-1}(U)$ is open in X (note that this is the weakest topology so that q is continuous).
  - Claim: this is a topology.
  - How is this related to previous definition? Given such a q, deifne an equivalence relation $\sim$ on X by having $x \sim x'$ iff $q(x) = q(x')$.

    Claim: $X/\sim$ with the quotient topology is homeomorphic to Y with the "quotient topology".

  – The two perspectives (equivalence relations and quotient maps) are entirely equivalent to each other; you should feel free to use whatever makes the most sense.

Post numberphile video on Klein bottles after this lecture.

9 Metric Spaces

9.1 Basics and Examples

- Again want to generalize ideas from real analysis, i.e. from Euclidean topology, so let's take a closer look at this.

- This is the topology generated by the basis of open balls $B_n(x, \varepsilon) = \{y \in \mathbb{R}^n : |x - y| < \varepsilon\}$ where $|x - y|$ is the Euclidean distance from x to y, i.e. $\sqrt{\sum |x_i - y_i|^2}$.

  Idea: what if we replaced Euclidean distance with some other form of "distance"?

- Eg say you're in Manhattan and your friend is one block up and to the right from you. How far away are you two?

  - One answer is $\sqrt{2}$ blocks (this is Euclidean distance, i.e. the distance "as the crow flies" since it's only useful if you can ignore the building between you).

  - Another answer is 2 blocks away since that's in practice how far you have to travel.

  - Def: given two points $x, y \in \mathbb{R}^n$, we define the *Manhattan distance* to be $\sum |x_i - y_i|$ (i.e. this is the distance if you can only travel along the axis without cutting corners).

- Broad question: what are other reasonable "distance functions" $d(x, y)$ we can consider between two objects x, y of a set X? In particular, what are reasonable axioms to impose on such a function d?

  - E.g. what should $d(x, x)$ be, i.e. the distance from x to itself? Intuitively this is 0.

  - If you think about things some more you might come up with the following definition.

- Def: given a set X, a function $d : X \times X \to \mathbb{R}$ is a *metric* if (1) $d(x, y) = 0$ for all $x, y \in X$ with equality iff $x = y$, (2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and (3) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. Discuss why these are reasonable/cases where maybe these don't quite hold.

- These axioms, in addition to being relatively intuitive, generalize the key properties of the Euclidean distance function, and with these axioms alone we can generalize much of the theory from this case. To do this we need some more definitions analogous to what we had in the Euclidean case.

  - Def: given a metric d, a point $x \in X$, and a real number $\varepsilon > 0$, we define the open ball $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. We will also refer to this as the ε-*neighborhood* of x.

- Claim: the set of open balls is a basis (every point is in at least one. If x is in the intersection of two balls of y, z then you can take the minimum $\varepsilon_y - d(x,y), \varepsilon_z - d(x,z)$ and this will be contained in both by the triangle inequality or something).

- Here we actually proved a useful fact: if x is a ball B, then there exists an $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subseteq B$.

- Def: We define the metric topology (induced by d) is the topology generated by the basis of open balls $B_d(x,\varepsilon)$.

  That is, it is the collection of sets U such that for every $x \in U$ there exists a ball B with $x \in B \subseteq U$.

  By the lemma, this is equivalently the set of U such that every $x \in U$ we have some $B_d(x,\varepsilon) \subseteq U$.

- Examples.

  - For any set X, the function $d(x,y) = 1$ if $x \neq y$ and $d(x,x)$ is a metric (check). What are balls here? Either single points or the whole space. This generates the discrete topology.

  - Claim L2-metric on $\mathbb{R}^n$ is a metric. Balls are balls.

  - Claim L1-metric d_M is a metric. Balls are diamonds.

  - Define square metric on $\mathbb{R}^n$ by $d_s(x,y) = \max\{|x_i - |y_i|\}$. Claim metric, only tricky part is triangle inequality which you can do by noting

    $$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq d_s(x,y) + d_s(x,z).$$

    Balls are squares.

- Question: what do these last three topologies generate? Claim is they're all Euclidean. More generally:

  Thm: let d, d' be metrics on the set X and $\mathcal{T}, \mathcal{T}'$ the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ iff for each $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$.

  - (Proof that $\varepsilon - \delta$ implies finer) A set U is open in $\mathcal{T}$ iff for each $x \in U$ we can find a neighborhood $B_d(x,\varepsilon) \subseteq U$ (via that ball centering lemma we proved). Note that every such U also has $B_{d'}(x,\delta) \subseteq U$, so every set open in $\mathcal{T}$ must also be open in $\mathcal{T}'$.

  - Other direction is exercise.

- Claim: Euclidean, square, and Manhattan metric all generate the same topology.

  - Proof (just of Euclidean vs square). Easy to show $d_s(x,y) \leq d_e(x,y) \leq \sqrt{n} d_s(x,y)$. This implies $B_e(x,\varepsilon) \subseteq B_s(x,\varepsilon)$ and that $B_s(x,\varepsilon/\sqrt{n}) \subseteq B_e(x,\varepsilon)$. The previous lemma gives things.

- For Manhattan vs square observe $d_s(x,y) \leq d_M(x,y) \leq n d_s(x,y)$ and a similar proof works.

9.2 Metrizable Spaces and Properties of Metric Spaces

- Def: we say that a topological space X is metrizable if there exists a metric d on X which induces the topology of X. A pair (X, d) is a metric space if X is a metrizable topology and d is a metric inducing the topology on X.

- Prop: if X is metrizable, then every subspace $A \subseteq X$ is metrizable (one can take the metric for X and restrict it to A; this is still a metric and it gives right topology).

- We've seen $\mathbb{R}^n$ is metrizable for all n. Are infinite product spaces metrizable?

- Thm: let $\bar{d}(a,b) = \min\{|a-b|, 1\}$. For $x, y \in \mathbb{R}^\omega$, define

$$D(x,y) = \sup \frac{\bar{d}(x,y)}{i}.$$

  This is a metric which induces the product topology on $\mathbb{R}^\omega$.

  - Proof of metric: triangle inequality holds for each term in the sup, so the inequality holds for the sup.

  - Induces product: let $U = \prod U_j$ be a basis element in product topology with $\alpha_1, \dots, \alpha_n$ the finitely many indices with $U_j \neq \mathbb{R}$ and consider $x \in U$. Since $U_{\alpha_i} \subseteq \mathbb{R}$ is open, there exists a $0 < \varepsilon_i \leq 1$ with $B(x_{\alpha_i}, \varepsilon_i) \subseteq U_i$. Define $\varepsilon = \min \varepsilon_i / i$, we claim that $B_D(x, \varepsilon) \subseteq U$. Indeed, if y is in this ball then by deifnition $\varepsilon > D(x,y) \geq \frac{\bar{d}(x,y)}{i}$ for all i. Since $\varepsilon \leq \varepsilon_i / i$, we have $\bar{d}(x_{\alpha_i}, y_{\alpha_i}) < \varepsilon_i \leq 1$, so $|x_{\alpha_i} - y_{\alpha_i}| < \varepsilon_i$. It follows that $y_{\alpha_i} \in U_{\alpha_i}$ for all i, proving $y \in U$ as desired.

  - Let $B = B_D(x, \varepsilon)$ with $\varepsilon < 1$, we want to find a basis neighborhood of x in B. Let N be a large enough integer such that $N^{-1} < \varepsilon$. Let V be the basis element with

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \cdots .$$

    Claim that $V \subseteq B$. Indeed, observe that for any $y \in \mathbb{R}^\omega$ and $i \geq N$ we have $\frac{\bar{d}(x_i, y_i)}{i} \leq N^{-1}$. Thus

$$D(x,y) \leq \max\{\frac{\bar{d}(x_1, y_1)}{1}, \dots, N^{-1}\}.$$

    If $y \in V$, then this expression is less than ε (since each $\bar{d}$ is at most ε), proving $y \in B_D(x, \varepsilon)$ as desired.

- What about product for other index sets J? Or box topology?

- Idea: to show X is metrizable you just need to construct a metric. To show X *isn't* metrizable, we show that it fails to have some property that every metric space must have.

  - E.g. Prop: every metric space is Hausdorff. Proof

  - Corollary: confinite topology with X infinite is not metrizable.

- (Sequence lemma) Let X be a topological space and $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is metrizable.

  - Note that the first part is similar to something we proved earlier (sequence in $A - x$ converging to x means $x \in A'$) and proof is basically the same.

  - Proof 1: Suppose $x_n \to x$ with $x_n \in A$. Then every neighborhood U of x intersects A, so $x \in \overline{A}$ by Theorem 17.5.

  - Proof 2: Suppose X is metrizable, say with d generating its topology and let $x \in \overline{A}$. Consider the balls $B_d(x, 1/n)$. Because $x \in \overline{A}$, there exists a point x_n which intersects A and $B_d(x, 1/n)$. We claim that this sequence converges to x_n. Indeed, any open set U containing x contains an open ball $B_d(x, \varepsilon)$. for $N \geq \varepsilon^{-1}$ we have $x_n \in B_d(x, \varepsilon) \subseteq U$ for all $n \geq N$, proving the claim/result.

  - Note: we didn't really use the full power of the metric space here, only that there exists a "nice" countable family of basis neighborhoods for each point. We will look more at this weaker notion in chapter 4.

- With this we can prove non-metrizability of some topologies on products.

  Prop: The box topology on $\mathbb{R}^\omega$ is not metrizable.

  - $A = \{(x_1, x_2, \ldots,) | x_i > 0 \ \forall i\}$. Claim that $0 = (0, 0, \ldots) \in \overline{A}$. Equivalent to saying every basis element $B = (a_1, b_1) \times \cdots$ containing 0 intersects A. Indeed, $a_i < 0 < b_i$ and hence the point $(\frac{1}{2}b_1, \ldots) \in A \cap B$.

  - Claim no sequence in A converges to 0 let a_n be a sequence with $a_n = (x_{1n}, x_{2n}, \ldots, x_{i,n}, \ldots)$. Since $x_{in} > 0$, we can take the basis element $B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \cdots$. Note that $0 \in B'$ but it contains no a_n point (since $x_{nn} \notin (-x_{nn}, x_{nn})$).

- If J is uncountable then $\mathbb{R}^J$ with the product topology is not metrizable.

  - Let $A \subseteq \mathbb{R}^J$ be the points (x_α) with $x_\alpha = 1$ for all but finitely many α. Claim $0 \in \overline{A}$. Take $B = \prod U_\alpha$ a basis element containing 0 and let $\alpha_1, \ldots, \alpha_n$ be the indices with $U_{\alpha_i} \neq \mathbb{R}$. Then the point (x_α) with $x_{\alpha_i} = 0$ and $x_\alpha = 1$ otherwise lies in $A \cap B$.

  - Claim no sequence converges. Indeed, let a_n be a sequence and let $J_n \subseteq J$ be the indices with $(a_n)_\alpha \neq 1$. Note that $\bigcup J_n$ is a countable union of finite sets, so there is some $\beta \notin \bigcup J_n$. This means $(a_n)_\beta = 1$ for all n.

Let $U = \bigtimes U_\alpha$ with $U_\beta = (-1, 1)$ and $U_\alpha = \mathbb{R}$ otherwise. Then no point of a_n is contained in U, so 0 can not be a limit.

9.3 Continuity

- It turns out metric spaces play particularly nicely with continuity. In particular, most definitions from real analysis generalize.

- Let $f : X \to Y$ with X, Y metrizable with metrics d_X, d_Y. f is continuous iff for all $x \in X, \varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$.

  Proof is similar to showing $\varepsilon - \delta$ definition for Euclidean space is equivalent (which they already did for HW).

- Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

  Proof.

  Assume f is continuous and let $x_n \to x$. We wish to show $f(x_n) \to f(x)$.

  Let V be a neighborhood of $f(x)$. Continuity means $f^{-1}(V)$ is a neighborhood of x, so for some N we have $x_n \in f^{-1}(V)$ for all $n \geq N$. Thus $f(x_n) \in V$ for $n \geq N$ as well.

  Now assume X is metrizable and that the convergent condition is satisfied. We aim to show that for any $A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$ (which we proved is equivalent to f being closed).

  If $x \in \overline{A}$, then by the sequence lemma there is a sequence of points $x_n \in A$ converging to x. By assumption $f(x_n) \to f(x)$. The other direction of the sequence lemma implies $f(x) \in \overline{f(A)}$, giving $f(\overline{A}) \subseteq \overline{f(A)}$. $\square$

- In metric spaces we can recover other important notions from real analysis.

  Def: let $f_n : X \to Y$ be a sequence of functions with (Y, d) a metric space. We say that (f_n) *converges uniformly* to a function $f : X \to Y$ if for all $\varepsilon > 0$ there exists an N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$ and all $x \in X$.

  Warning: the definition depends not only on the topology of Y, but also the specific metric d which induces it.

- Uniform limit theorem: let $f_n : X \to Y$ be a sequence of continuous functions with Y a metric space. If (f_n) converges uniformly to f, then f is continuous.

  - General strategy: try to remember how you proved analogous results from real analysis.

- Recall that the classic real analysis proof for $X = Y = \mathbb{R}$ goes via a $\varepsilon/3$ argument, i.e. to show f is continuous at x_0 you write

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

- Let $V \subseteq Y$ be open and $x_0 \in f^{-1}(V)$. We aim to find a neighborhood U of x_0 such that $f(U) \subseteq V$.

- Let $y_0 = f(x_0)$ and choose ε so that $B(y_0, \varepsilon) \subseteq V$. Using uniform convergence, we can choose N so that $d(f_n(x), f(x)) < \varepsilon/3$ for all $n \geq N$ and all x.

- Because f_N is continuous at x_0, there exists a neighborhood U of x_0 such that $f_N(U) \subseteq B(f_N(x_0), \varepsilon/3)$.

- Note that for all $x \in U$ we have $d(f(x), f_N(x)) < \varepsilon/3$, $d(f_N(x), f_N(x_0)) < \varepsilon/3$ and $d(f_N(x_0), f(x_0)) < \varepsilon/3$. By triangle inequality we find $d(f(x), f(x_0)) < \varepsilon$ for all $x \in U$, proving the result.