Slow Fibonacci Walks

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Joint with Fan Chung and Ron Graham.

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I am interested in magic tricks whose explanation requires deep mathematics. The trick should be one that would actually appeal to a layman. An example is the following: the magician asks Alice to choose two integers between 1 and 50 and add them. Then add the largest two of the three integers at hand. Then add the largest two again. Repeat this around ten times. Alice tells the magician her final number n. The magician then tells Alice the next number. This is done by computing $(1.61803398\cdots)n$ and rounding to the nearest integer. The explanation is beyond the comprehension of a random mathematical layperson, but for a mathematician it is not very deep. Can anyone do better?

soft-question big-list popularization

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Richard Stanley

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Given positive integers a_1, a_2, we define the (a_1, a_2)-Fibonacci walk to be the sequence $w_k = w_k(a_1, a_2)$ satisfying

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For example, if $w_k = w_k(10, 2)$, this gives the sequence

$$10, \; 2, \; 12, \; 14, \; 26, \; 40, \; 66 \ldots$$

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We say that w_k is an n-Fibonacci walk if $w_s = n$ for some s. For example, the above w_k is a 40-Fibonacci walk.

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For example, the following are all 40-Fibonacci walks.

$$1024,\ \underline{40},\ 1064\ldots$$

$$8,\ 8,\ 16,\ 24,\ \underline{40},\ 64\ldots$$

$$5,\ 5,\ 10,\ 15,\ 25,\ \underline{40},\ 65\ldots$$

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However, the first two can't be slow (since the next two achieve 40 with $s = 6$), and one can verify that $w_k(5, 5)$ and $w_k(10, 2)$ are (the unique) 40-slow Fibonacci walks.

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$w_3(x, y) = x + y \geq 2$, $w_3(1, n-1) = n$. $\square$

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$$6,\ 4,\ 10,\ 14, \cdots$$

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$w_k(b, a) = af_{k-1} + bf_{k-2}$, *with* f_k *the Fibonacci numbers.*

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Let $s = s(n) > 2$. If (b, a) is n-good, then (b', a') is n-good iff $a' = a + kf_{s-2} \geq 1$ and $b' = b - kf_{s-1} \geq 1$ for some k.

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For example, if we know $s(40) = 6$ and $(10, 2)$ is 40-good, then so is $(10 - kf_5, 2 + kf_4) = (10 - 5k, 2 + 3k)$, which only makes sense if $k = 0, 1$.

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Proof.

By the above lemma, every n-good pair is a solution to the diophantine equation $n = w_s(b',a') = a'f_{s-1} + b'f_{s-2}$, and the result follows since $\gcd(f_{s-1}, f_{s-2}) = 1$ for $s > 2$. □

Slow Fibonacci Walks

Theorem (Englund, Bicknell-Johnson (1997); Jones, Kiss (1998); Chung, Graham, S. (2019))

For $n > 1$ with $s = s(n)$, there exist unique integers $a = a(n),\ b = b(n)$ such that $n = af_{s-1} + bf_{s-2}$ and $1 \leq a \leq b \leq f_{s-1}$. In this case (b, a) is n-good.

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Because $w_k(x, y) = yf_{k-1} + xf_{k-2}$, by definition of s there exist a', b' such that $n = w_s(b', a') = a' f_{s-1} + b' f_{s-2}$.

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for some k such that $b', a' \geq 1$.

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Theorem (Chung, Graham, S. (2019))

Given n, let c, p be such that $n = \frac{1}{\sqrt{5}} c\phi^p$ with $\frac{1}{\sqrt{5}} \leq c < \frac{1}{\sqrt{5}}\phi$. Then

$$T(n) = \begin{cases} \frac{1}{2\sqrt{5}\phi^4 c} + O(n^{-1/2}) & p \equiv 1 \mod 2, \\ \frac{\sqrt{5}}{2}c + \frac{1+\phi^{-5}}{2\sqrt{5}c} - 1 + O(n^{-1/2}) & p \equiv 0 \mod 2,\ c \leq \frac{1+\phi^{-3}}{\sqrt{5}}, \\ 1 - \frac{\sqrt{5}}{2}\phi^{-1}c - \frac{1+\phi^{-2}}{2\sqrt{5}c} + O(n^{-1/2}) & p \equiv 0 \mod 2,\ c \geq \frac{1+\phi^{-3}}{\sqrt{5}}. \end{cases}$$

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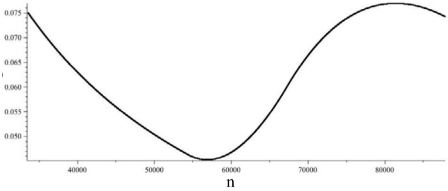

(a) Data plot of $T(n)$.

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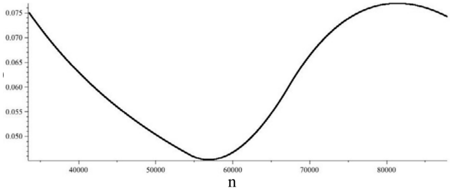

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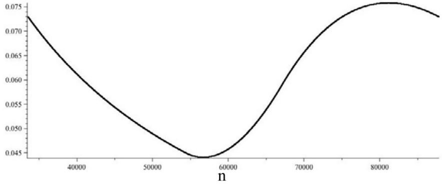

(b) Theory plot of $T(n)$.

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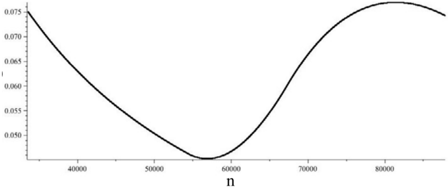

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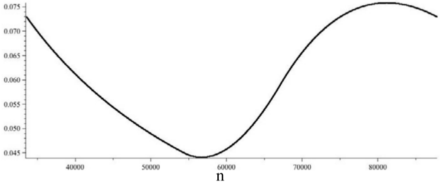

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Note that this value oscillates.

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This is true when s is odd for $1 \leq a \leq b \leq f_t$. $\square$

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Theorem (Chung, Graham, S. (2019))

$$D(n) = \begin{cases} \frac{\sqrt{5}n}{2\phi^{q+1}} + \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, \ q \equiv 1 \mod 4, \\ 1 - \frac{\sqrt{5}n}{2\phi^{q+1}} - \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, \ q \equiv 3 \mod 4. \end{cases}$$

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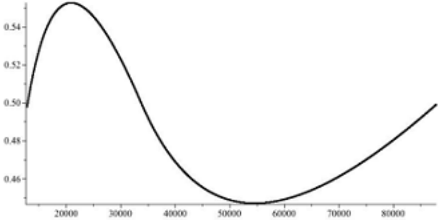
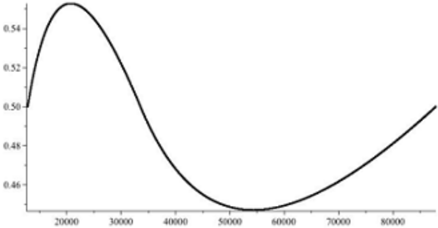
Proof.

Count triples (s, a, b) with s odd. $\square$

Applications



(a) Data plot of $D(n)$.



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$$\{d_{k+2} - d_k : k \geq 1\} = \{2, 3, 4, 5, 6, 8, 10\}.$$

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We know that $w_{s+1} = \lfloor \phi n \rfloor$ or $w_{s+1} = \lceil \phi n \rceil$. Intuitively, the smaller $\phi n - \lfloor \phi n \rfloor$ is, the more likely it is that $w_{s+1} = \lfloor \phi n \rfloor$.

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We have $P(n,r) = 0$ if $r \geq 1 - \frac{1}{\sqrt{5}}\phi^{-1} \approx .72$. Otherwise, given n, let c, p be such that $n = \frac{1}{\sqrt{5}} c\phi^p$ with $\frac{1}{\sqrt{5}} \leq c < \frac{1}{\sqrt{5}}\phi$. Then $P(n,r)$ satisfies

$$\begin{cases} -\frac{1}{2}\phi^{-1}c + (1-r) + \left(r^2 - r + \frac{1}{2\sqrt{5}}\phi^{-1}\right)c^{-1} + O(n^{-1/2}) & p \text{ odd, } c \leq (1-r)\phi, \\ \frac{\sqrt{5}}{2}\phi\left(r - \frac{1}{\sqrt{5}}\phi\right)^2 c^{-1} + O(n^{-1/2}) & p \text{ odd, } c \geq (1-r)\phi, \\ -\frac{1}{2}c + (1-r) + \left(\phi^{-1}r^2 - \phi^{-1}r + \frac{1}{2\sqrt{5}}\phi^{-2}\right)c^{-1} + O(n^{-1/2}) & p \text{ even, } c \leq 1-r, \\ \frac{\sqrt{5}}{2}\left(r - \frac{1}{\sqrt{5}}\phi\right)^2 c^{-1} + O(n^{-1/2}) & p \text{ even, } 1-r \leq c \leq r \\ \frac{1}{2}c - r + \left(\phi r^2 - \phi r + \frac{1}{2\sqrt{5}}\phi^2\right)c^{-1} + O(n^{-1/2}) & p \text{ even, } c \geq r. \end{cases}$$

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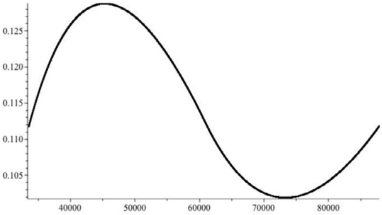

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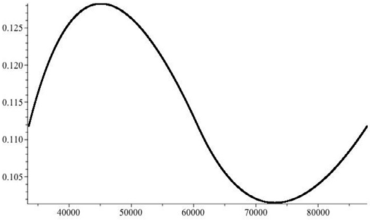

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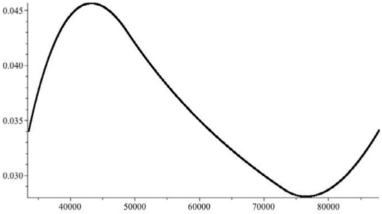
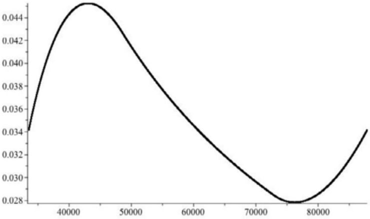


(c) Data plot of $P(n, .6)$.



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Note that when $\alpha = \beta = 1$ we have $g_k = f_k, \; \gamma = \phi, \; \lambda = -\phi^{-1}$.

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and only finitely many n with
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$$c^{-1}\left(\frac{(2\beta - 2d - 1)\gamma(\alpha - 2\delta + \alpha^{-1}\delta^2)}{2\beta^2(\gamma^2 - 1)} + \frac{\gamma^2}{\gamma^2 - 1}\sum_{q=d+1}^{\beta-1}\frac{\beta - q}{\beta^2}\right) + O(\gamma^{-r} + (\beta\gamma^{-2})^r),$$

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When $\beta = 1$ the proof is essentially the same as before, otherwise one has to be careful about the divisibility condition.

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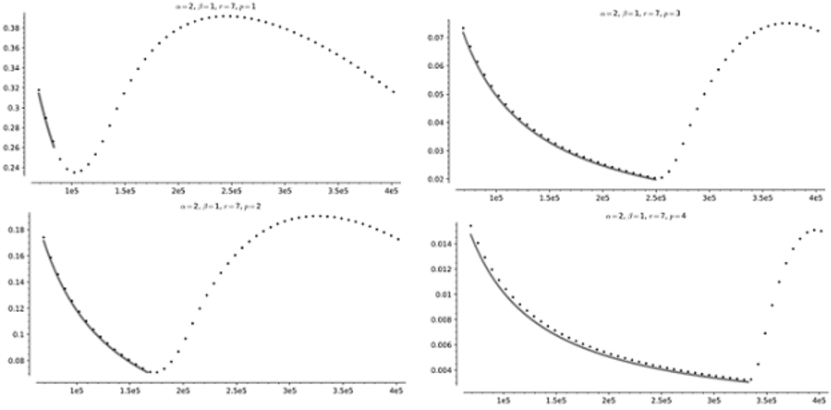
$\alpha = 2, \; \beta = 1, \; p = 1, 2, 3, 4.$



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Corollary: the Fibonacci sequence *is* special!

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More generally, given a set of relatively prime pairs T, define
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For all n with $s^{\alpha,\beta}(n) > 2$, we have

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The two extreme cases are $n = g_{s-1} + \beta g_{s-2} \approx \gamma^s$ and $n = \beta g_{s-1} g_s \approx \gamma^{2s}$.

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In particular this implies $\log_\Gamma \gamma < 4$. Since $\gamma_{\alpha,\beta}$ is monotonically increasing in α and β, the set $R_T = \{(\alpha, \beta) : \log_\Gamma \gamma_{\alpha,\beta} < 4\} \cap T$ is finite. $\square$

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Proposition

Let T be a set of pairs. Given $(\alpha, \beta) \in T$, let $\gamma = \gamma_{\alpha,\beta}$ and $c = \log_\Gamma \gamma$. Then

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In particular, for $(1, 4)$-walks we have $c \approx 1.95$, which explains why they were so hard to find.

Slowest Slow Walks

Corollary

If T is such that there exists a unique pair $(\alpha', \beta') \in T$ with $\gamma_{\alpha',\beta'} = \Gamma$, then almost every n has $\mathbf{S}_T(n) = \{(\alpha', \beta')\}$.

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As $|R_T|$ is finite, the density of $m \leq n$ with some other $(\alpha, \beta) \in \mathbf{S}(m)$ is at most $o(1)$. $\square$

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What if there isn't such a unique pair?

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What if there isn't such a unique pair? For example, if $T = \{(1, 6), (2, 3)\}$ we have $\gamma_{1,6} = \gamma_{2,3} = 3$.

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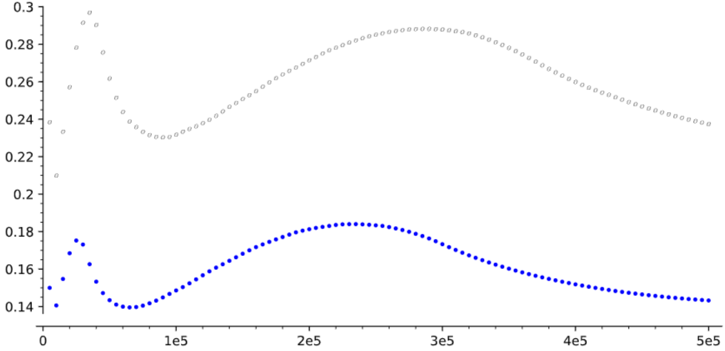
Let $e_{\alpha,\beta}(n) = n^{-1}|\{m : m \leq n, \mathbf{S}_T(m) = \{(\alpha,\beta)\}\}|$ for $T = \{(1,6),(2,3)\}$.



Here the open black circles are $e_{2,3}(n)$ and the solid blue dots are $e_{1,6}(n)$.

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- What happens if you require a slow walk to hit two prescribed numbers n_1 and n_2? Note that $w_1 = n_1, \ w_2 = n_2$ works, so this is well defined.

The End

Thank You!