# Random Turán Problems

Sam Spiro, Rutgers University

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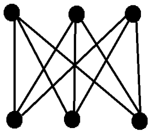
**Theorem (Mantel 1907)**

$$\mathrm{ex}(n, K_3) = \left\lfloor n^2/4 \right\rfloor .$$



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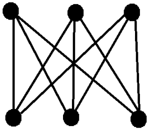

**Theorem (Erdős-Stone, Simonovits 1946)**

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}.$$

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The lower bound is tight when $p = 1$. The upper bound is tight if $p$ is "small."

$$\frac{1}{2}p\binom{n}{2} \lesssim \mathrm{ex}(G_{n,p}, K_3) \lesssim p\binom{n}{2},$$

with the lower bound tight for $p = 1$ and the upper bound tight for $p \ll n^{-1/2}$.

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### Theorem (Frankl-Rödl 1986)

*Whp,*

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**Theorem (Conlon-Gowers, Schacht 2010)**

*Whp,*

$$\mathrm{ex}(G_{n,p}, F) = p \cdot \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2} \qquad p \gg n^{-1/m_2(F)},$$

*where* $m_2(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subseteq F\}$.

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**Conjecture**

*If $F$ is a bipartite graph which is not a forest, then whp*

$$\mathrm{ex}(G_{n,p}, F) = \begin{cases} \Theta(p \cdot \mathrm{ex}(n, F)) & p \gg n^{-1/m_2(F)}, \\ (1+o(1))p\binom{n}{2} & p \ll n^{-1/m_2(F)}. \end{cases}$$

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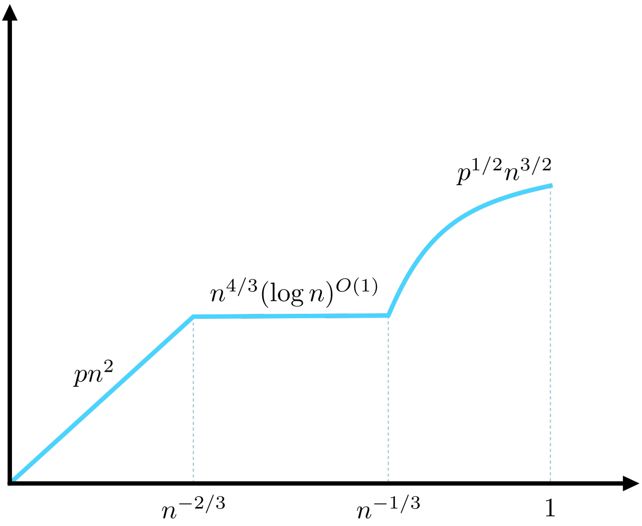
This conjecture turns out to be completely false!




Plot of $\mathsf{ex}(G_{n,p}, C_4)$ (Füredi 1991)

## Conjecture (McKinley-S.)

*If $F$ is a graph with* $\mathrm{ex}(n, F) = \Theta(n^\alpha)$ *for some* $\alpha \in (1, 2]$, *then whp*

$$\mathrm{ex}(G_{n,p}, F) = \max\{\Theta(p^{\alpha-1} n^\alpha), n^{2-1/m_2(F)}(\log n)^{O(1)}\},$$

*provided* $p \gg n^{-1/m_2(F)}$.

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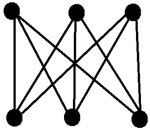
**Theorem (Nie-S. 2023 (Informal))**

*This conjecture (essentially) implies Sidorenko's conjecture.*

## Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$



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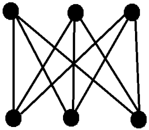

### Theorem (Morris-Saxton 2013)

$$\mathrm{ex}(G_{n,p}, K_{s,t}) = O(p^{1-1/s} n^{2-1/s}) \textit{ for } p \textit{ large.}$$

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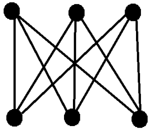

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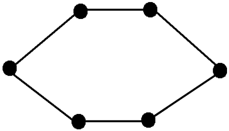
*Moreover, this bound is tight whenever* $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$.

## Theorem (Bondy-Simonovits 1974)

$$\mathrm{ex}(n, C_{2b}) = O(n^{1+1/b}).$$



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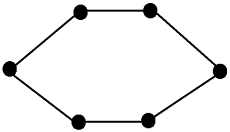

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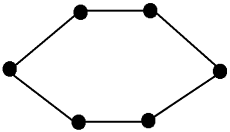

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*Moreover, this is tight whenever* $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2b}\}) = \Theta(n^{1+1/b})$.

**Theorem (Jiang-Longbrake 2022)**

*If $F$ satisfies "mild conditions", then*

$$\mathrm{ex}(G_{n,p}, F) = O(p^{1-m_2^*(F)(2-\alpha)} n^{\alpha}) \text{ for } p \text{ large},$$

*where* $m_2^*(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subsetneq F,\ e(F') \geq 2\}$.

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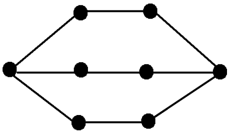

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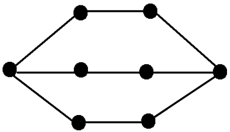

**Theorem (Corsten-Tran 2021)**

$$\mathrm{ex}(G_{n,p}, \theta_{a,b}) = O(p^{\frac{2}{ab}} n^{1+1/b}) \textit{ for } p \textit{ large.}$$

Note: our conjecture predicts $p^{\frac{1}{b}} n^{1+1/b}$.

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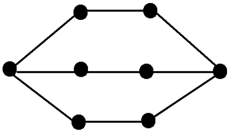

### Theorem (McKinley-S. 2023)

*For $a \geq 100$,*

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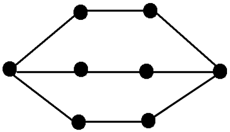

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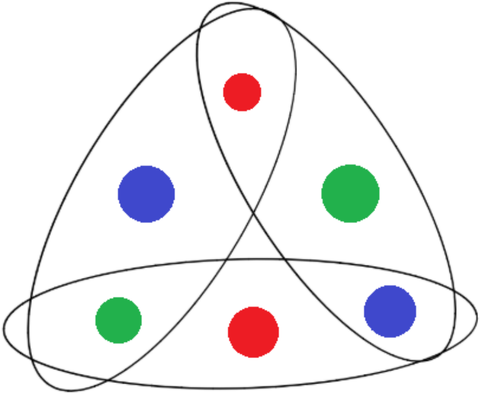
### Theorem (S. 2022)

*This bound is tight whenever $a$ is sufficiently large in terms of $b$.*

# Hypergraphs



**Theorem (S.-Verstraëte 2021)**

*Let $K^r_{s_1,\dots,s_r}$ denote the complete $r$-partite $r$-graph with parts of sizes $s_1,\dots,s_r$. There exist constants $\beta_1,\beta_2,\beta_3,\gamma$ depending on $s_1,\dots,s_r$ such that, for $s_r$ sufficiently large in terms of $s_1,\dots,s_{r-1}$, we have whp*

$$\mathrm{ex}(G^r_{n,p},K^r_{s_1,\dots,s_r})=\begin{cases}\Theta\left(pn^r\right) & n^{-r}\ll p\le n^{-\beta_1},\\ n^{r-\beta_1+o(1)} & n^{-\beta_1}\le p\le n^{-\beta_2}(\log n)^{\gamma},\\ \Theta(p^{1-\beta_3}n^{r-\beta_3}) & n^{-\beta_2}(\log n)^{\gamma}\le p\le 1.\end{cases}$$

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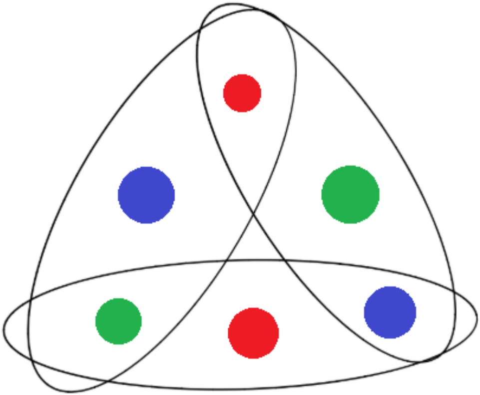
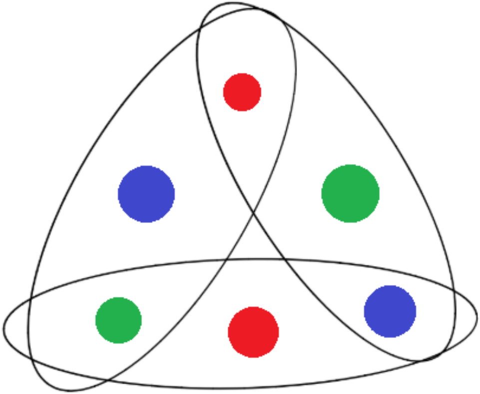
**Theorem (Nie-S. 2023 (Informal))**

*Many hypergraphs fail to have a flat middle range. More precisely, any hypergraph which isn't Sidorenko fails to have a flat middle range.*



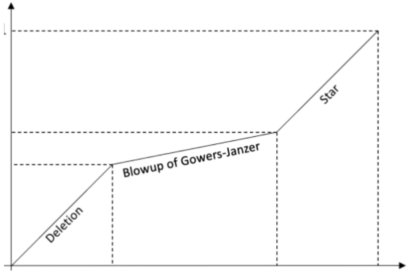


We define the *loose cycle* $C_\ell^r$ to be the $r$-uniform hypergraph obtained by inserting $r-2$ distinct vertices into each edge of the graph cycle $C_\ell$.

## Theorem (Nie-S.-Verstaëte 2020; Nie 2023)

*For $r \geq 3$, if $p \gg n^{-r+3/2}$ then whp*

$$\mathrm{ex}(G^r_{n,p}, C^r_3) = \max\{p^{\frac{1}{2r-3}} n^{2+o(1)}, pn^{r-1+o(1)}\}.$$




Picture due to Jiaxi Nie.

**Theorem (Mubayi-Yepremyan 2020; Nie 2023)**

*For $r \geq 4$, if $p \gg n^{-r+1+\frac{1}{2\ell-1}}$ then whp*

$$\mathrm{ex}(G^r_{n,p}, C^r_{2\ell}) = \max\{n^{1+\frac{1}{2\ell-1}}, pn^{r-1+o(1)}\}.$$

## Theorem (Mubayi-Yepremyan 2020; Nie 2023)

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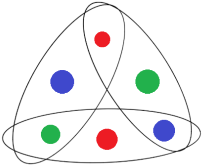
Generalizations of these results were obtained by Nie-S. for expansions of hypergraphs.



Given a $k$-graph $F$, we define its $r$-expansion $F^{+r}$ to be the $r$-graph obtained by inserting $r-k$ new vertices into each edge of $F$.

# Upper Bound Techniques

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**Proof.**

Containers. $\square$

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Hypergraph containers. $\square$

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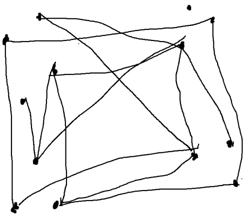
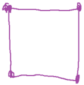


$ex(p_n, F) =$



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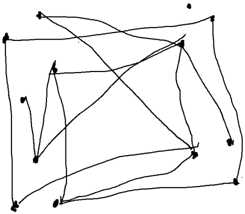
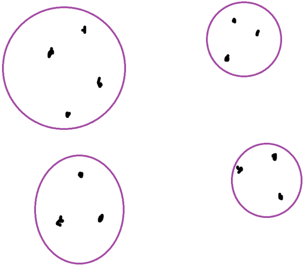
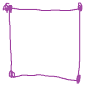
$G_{n,p} =$ 



$ex(pn, F) =$ 

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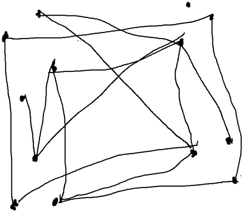
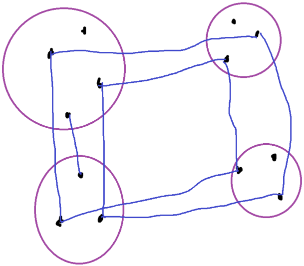
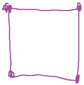
$G_{n,p} =$ 



$\mathrm{ex}(P_n, F) =$ 

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We say that a hypergraph $F$ is *Sidorenko* if for all $r$-graphs $H$, we have

$$t_F(H) \geq t_{K_r^r}(H)^{e(F)}.$$

# Sidorenko's Conjecture

**Conjecture (Sidorenko 1986)**

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**Theorem (Conlon-Lee-Sidorenko 2023)**

*If $F$ is an $r$-graph which is not Sidorenko, then there exists $\epsilon = \epsilon(F) > 0$ such that*

$$\mathrm{ex}(n, F) = \Omega(n^{r - \frac{v(F)-r}{e(F)-1} + \epsilon}).$$

# Sidorenko's Conjecture

For an $r$-graph $F$, define

$$s(F) := \sup\{s : \exists H \neq \emptyset,\ t_F(H) = t_{K_r^r}(H)^{s+e(F)}\}.$$

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**Theorem (Nie-S. 2023)**

*If $F$ is an $r$-graph with $e(F) \geq 2$ and $\frac{v(F)-r}{e(F)-1} < r$, then for any $p = p(n) \geq n^{-\frac{v(F)-r}{e(F)-1}}$, we have whp*

$$\mathrm{ex}(G_{n,p}^r, F) \geq n^{r - \frac{v(F)-r}{e(F)-1} - o(1)} \left(pn^{\frac{v(F)-r}{e(F)-1}}\right)^{\frac{s(F)}{e(F)-1+s(F)}}.$$

# Proof of Main Theorem

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Let $\mathcal{N}_F(G)$ denote the number of copies of $F$ in $G$.

**Lemma**

*If $F$ is an $r$-graph such that there exists an $r$-graph $H$ with $t_{K_r^r}(H) = \alpha$ and $t_F(H) = \beta$, then for all $r$-graphs $G$ and integers $N \geq 1$ we have*

$$\mathrm{ex}(G, F) \geq \alpha^N e(G) - \beta^N \mathcal{N}_F(G).$$

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Given two $r$-graphs $H, H'$, we define the *tensor product* $H \otimes H'$ to be $r$-graph on $V(H) \times V(H')$ where $((x_1, y_1), \ldots, (x_r, y_r)) \in E(H \otimes H')$ if and only if $(x_1, \ldots, x_r) \in E(H)$ and $(y_1, \ldots, y_r) \in E(H')$.

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$$\mathrm{ex}(G, F) \geq \alpha^N e(G) - \beta^N \mathcal{N}_F(G).$$

Given two $r$-graphs $H, H'$, we define the *tensor product* $H \otimes H'$ to be $r$-graph on $V(H) \times V(H')$ where $((x_1, y_1), \ldots, (x_r, y_r)) \in E(H \otimes H')$ if and only if $(x_1, \ldots, x_r) \in E(H)$ and $(y_1, \ldots, y_r) \in E(H')$.
Fact: for any $r$-graphs $F, H$ and $N \geq 1$, we have

$$t_F(H^{\otimes N}) = t_F(H)^N.$$

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One gets the result by deleting an edge from each copy of $F$ in $G'$. $\square$

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**Theorem (Nie-S. 2023)**

$$s(F^{+r}) \leq \frac{v(F)-k}{v(F)-k+(r-k)(s(F)+e(F)-1)} \cdot s(F).$$

*In particular, expansions of Sidorenko hypergraphs are Sidorenko.*

# Open Problems

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*For every bipartite graph $F$, there exists an $r \geq 2$ such that $F^{+r}$ is Sidorenko.*

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Is it true that $F$ is Sidorenko if and only if there exists an expansion $F^{+r}$ which is Sidorenko?

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*For every bipartite graph $F$, there exists an $r \geq 2$ such that $F^{+r}$ is Sidorenko.*

**Question**

Is it true that $F$ is Sidorenko if and only if there exists an expansion $F^{+r}$ which is Sidorenko? In particular, are all expansions of non-bipartite graphs not Sidorenko?

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$$r^{-2\ell} \ll s(C^r_{2\ell+1}) \leq \frac{2\ell - 1}{r - 2}.$$

Our best guess is

$$s(C^r_{2\ell+1}) = \frac{\ell}{(r-1)\ell - 1}.$$

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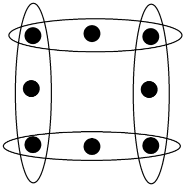
Prove tight bounds for the 3-uniform loose 4-cycle.



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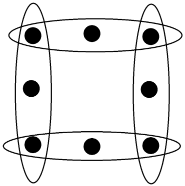

**Problem**

Prove tight bounds for subdivisions of complete bipartite graphs.