PCMI Open Problems

Transcribed by Sam Spiro*

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Here is a rough description of the problems[1] presented at the PCMI open problem session. The problem ordering has been moved slightly compared to the order that they were presented in order to put similar problems closer together. Please let me know if you spot any errors you'd like me to correct!

1 Imre Leader

A "Maze" is an infinite chessboard which has some "start" square and a "treasure" square together with some set of blocked off paths, with these obstacles defined in such a way that there does exist a path from the start to the treasure that doesn't pass through any obstacle. The main question is as follows.

Question 1.1. *Does there exist a sequence of "instructions" (e.g. "Go north, then go east, then go east,..."') such that regardless of the maze, following this sequence of instruction gets you from the start to the tresure square?*

Even the followign case remains open.

Question 1.2. *Can you solve the problem if you are further told that there is a path of length 1 from the start square to the treasure square?*

Note that for compactness reason, if there exists such a sequence, then there exists a sequence of finite length which does this.

2 Blair Sullivan

We say that a graph is c-closed if whenever two vertices u, v have at least c common neighbors, then u, v are adjacent. The main question is the following.

*Unless you think the writing is horrible, in which case it was written by Bob.

[1]My own problem has an unbiased amount of polish only because I wrote up the writeup previously, not because I'm trying to make my problem see cooler than everyone else's!

Question 2.1. *How many maximal cliques can a c-closed graph have?*

Fox, Roughgarden, Seshadhri, Wei, and Wein proved that the number is at most $O(n^{2-2^{1-c}})$ and at least $\Omega(n^{3/2})$. Can you improve upon either of these bounds?

3 Sam Spiro

Let D be a digraph. Given a set S, we define $N^+(S) = \bigcup_{v \in S} N^+(v)$, where $N^+(v)$ is the out-neighborhood of v. We say that a set $K \subseteq V(D)$ is a *kernel* of D if (1) $N^+(K) \cap K = \emptyset$ (that is, K is an independent set of the underlying graph of D), and (2) $N^+(K) \cup S = V(D)$ (that is, every vertex is either in K or can be reached by a vertex in K in one step).

Not every digraph has a kernel (take any directed cycle of odd length), but it is not too hard to prove that every digraph has a *quasikernel*. This is a set $Q \subseteq V(D)$ such that (1) $N^+(Q) \cap Q = \emptyset$ and such that (2) $N^+(N^+(Q)) \cup N^+(Q) \cup Q = V(D)$. That is, it is an independent set such that every vertex can be reached from Q in at most two steps.

Given that every digraph has a quasikernel, it is natural to ask how small of a quasikernel one can find. One quickly realizes that it can be quite large: any source of D must belong to a quasikernel of D. Thus the most natural setting to consider is when D has no sources, and in this case the following was conjectured by P.L. Erdős and Székely.

Conjecture 3.1. *Every digraph D with no sources has a quasikernel of size at most $|V(D)|/2$.*

Overall very little is known here. There are a few special classes of digraphs for which this is known, and only a very weak bound of roughly $|V(D)| - \lfloor |V(D)|^{1/2} \rfloor$ is known in general due to Spiro. This is a somewhat natural barrier given the current known proofs showing that quasikernels exist, which motivates the following.

Question 3.2. *Can you come up with a new proof showing that quasikernels exist?*

4 Maya Sankar

The following surprising fact is known: any Cayley graph over $(\mathbb{Z}/2\mathbb{Z})^d$ has chromatic number not equal to 3. This result has a number of natural followup questions.

Question 4.1. *Are there any chromatic numbers forbidden (other than 2) for Cayley graphs over $(\mathbb{Z}/p\mathbb{Z})^d$?*

The answer to this is not clear even for $p = 3$.

Question 4.2. *Are there other forbidden chromatic numbers for $(\mathbb{Z}/2\mathbb{Z})^d$?*

It is known that every power of 2 is possible, and it is also recently shown that other chromatic numbers are also possible, notably 7. The following is still open.

Question 4.3. *Is 5 a forbidden chromatic number for $(\mathbb{Z}/2\mathbb{Z})^d$?*

5 Matija Bucić

Given a graph G, define $\tau(G)$ to be the number of spanning trees of G. People are interested in studying the possible range of this, i.e. in determining the growth rate of $|\{\tau(G) : v(G) = n\}|$.

Question 5.1. *Is* $|\{\tau(G) : v(G) = n\}| = n^{\Omega(n)}$*?*

Note that $\tau(G) \leq \tau(K_n) = n^{n-2}$, so this result is best possible if true. It might even be the case that this result is true when G is restricted to be planar.

6 Rajko Nenadov

Let $\mathcal{C}(G)$ denote the set of cycle lengths in G. Erdős asked to consider

$$c(n) := |\{\mathcal{C}(G) : v(G) = n\}|.$$

Faudree showed that $c(n) \geq 2^{n/2}$. Verstraëte proved $c(n) \leq 2^{n-n^{1/10}}$ and Nenadov proved $c(n) \leq 2^{n-\sqrt{n}}$. This seems like a natural barrier to the problem.

Question 6.1. *Can you prove* $c(n) \leq 2^{n-n^{1/2+\varepsilon}}$ *for some* $\varepsilon > 0$*?*

Question 6.2. *Can you improve the lower bound?*

7 Santiago Morales

An Ulam word is defined as follows: 0 and 1 are Ulam word. A longer string is an Ulam word if it can be written uniquely as the concatenation of two distinct Ulam words. The main question is to determine how many Ulam words there are. Computer evidence suggests the following conjecture.

Conjecture 7.1. *The number of Ulam words of length n divided by 2^n is asymptotic to* $.526n^{-3/10}$.

8 David Conlon

A graph is d-degenerate if there is an ordering of the vertices $v_1, \ldots, v_n$ such that each v_i has at most d neighbors in $\{v_1, \ldots, v_{i-1}\}$.

An old question of Erdős asks if H is bipartite and d-degenerate, then $\mathrm{ex}(n, H) = O(n^{2-1/d})$. This seems hard. An even harder problem is the following.

Conjecture 8.1 (Conlon)**.** *If H is bipartite and d-degeenerate and has no $K_{d,d}$, then there exists some $\varepsilon_H > 0$ such that* $\mathrm{ex}(n, H) = O(n^{2-1/d-\varepsilon_H})$.

This is known if $d = 2$ but otherwise is mostly wide open.

9 Patrick Morris

Let $\mathcal{H}(n,d)$ be the set of all n-vertex graphs H with maximum degree at most d. Note that a K_{d+1}-factor is an element of $\mathcal{H}(n,d)$, and it is a meta-conjecture that this is the "hardest' element to embed in a given graph.

For example, it is conjectured that if G is an n-vertex graph and $\delta(G) \geq \frac{d}{d+1}n$ then G contains a copy of every element in $\mathcal{H}(n,d)$. A similar sort of conjecture is open in the random setting. Here's a lesser known conjecture that generalizes both of these. For this, let G_p denote the graph obtained by keeping each edge of G independently and with probability p.

Conjecture 9.1. *If $\delta(G) \geq \frac{d}{d+1}n$ and if $p \gg p^*_{K_{d+1}}$ (the threshold for a clique factor), then G_p contains a copy of every element of $\mathcal{H}(n,d)$ with high probability.*

While this is likely difficult in general, the $d=2$ case is likely doable (and in particular, the other two conjectures have been solved in this case).

10 Anita Liebenau

Question 10.1. *What is the minimum number of edges in a graph G which is "universal" for all d-degenerate graphs, i.e. such that $H \subseteq G$ for all n-vertex d-degenerate graphs H?*

A counting argument shows this is at least $\Omega(n^{2-1/d})$, and the best upper bound is $O(n^{2-1/d}\log^{2/d}(n))$. The question then is if you can remove these log factors via a better construction.

11 Matthew Jenssen

A covering code of radius R is a set $\mathcal{C} \subseteq \{0,1\}^n$ such that $\bigcup_{x\in\mathcal{C}} B_x(R) = \{0,1\}^n$ (where here $B_x(R)$ denotes the Hamming ball centered at x of radius R). The density of a code is defined to be $\frac{|\mathcal{C}||B_x(R)|}{2^n}$ (i.e. this is the average number of times a point is covered by one of these balls). Note that the density must always be at least 1 by definition.

Let $\mu(n,r)$ denote the minimum density of a covering code of radius R, and also define $\mu^*(R) := \limsup_{n\to\infty}\mu(n,R)$.

Conjecture 11.1. $\mu^*(R) = 1$ *for any fixed R.*

That is, in high dimensions one can essentially perfectly tile with Hamming balls. The current best bound is $\mu^*(R) < eR\log R$, and any quantitative improvement (even on the constant) would be interesting.

12 Noah Kravitz

Let p be a large prime and $A \subseteq \mathbb{Z}/p\mathbb{Z}$. Consider the set $A+A-2\cdot A = \{a+b-2c : a,b,c \in A\}$.

Conjecture 12.1. *If $A + A - 2 \cdot A \neq \mathbb{Z}/p\mathbb{Z}$, then $|A| \leq (\frac{1}{4} + o(1))p$.*

It's known that $|A| \leq p/3$ by Cauchy-Davenport, and even that $|A| \leq (1/3 - \varepsilon)p$ for some small ε by a more complicated argument.

13 Rob Morris

Let $f : \{-1,1\}^n \to \{-1,1\}^n$ be a bijection and consider $\Pr[\langle x, y\rangle \geq 0 \text{ and} \langle f(x), f(y)\rangle \geq 0]$ with x, y chosen independently and uniformly.

Question 13.1. *Is this probability at least $\frac{1}{4} - o(1)$ as $n \to \infty$?*

Note that if f is chosen uniformly random then this is true.

14 Wojciech Samotij

Let V be a set, $\mathcal{H} \subseteq 2^V$ a set system, $p \in (0,1)$, and V_p the random subset of V obtained by including each element independently with probability p. We want to study the probability. $\Pr[\mathcal{H}[V_p] = \emptyset]$. If $\mathcal{G}$ is a set system which covers $\mathcal{H}$, then one can show that this probability is at least $\Pi_{A \in \mathcal{G}}(1 - p^{|A|})$. Moreover, if we define $w_p(\mathcal{H}) := \mathbb{E}|\mathcal{G}[V_p]|$ then it is known that this probability is at most $\exp(-w_{p/u(\mathcal{H})^2}(\mathcal{H}))$ where $u(\mathcal{H})$ is the uniformity of $\mathcal{H}$.

Question 14.1. *Can one replace $u(\mathcal{H})^2$ in this result with a smaller value? In particular, can one prove this with $\log u(\mathcal{H})$ instead?*

Note that it is known that one can not replace $u(\mathcal{H})^2$ here with a constant.

15 Elad Tzalik

Given a graph G, we say that $H \subseteq G$ is a $(2k-1)$-spanner if $\text{dist}_H(x, y) \leq (2k-1)\text{dist}_G(x, y)$ for all x, y. The main question in this area is the following.

Conjecture 15.1. *There exists a graph G such that every $(2k-1)$-spanner $H \subseteq G$ satisfies $e(H) = \Omega(n^{1+1/k})$.*

This is equivalent to the girth conjecture of Erdős. Another direction[2] for this problem is to consider an analog for directed graphs, where here the notion of distance we use is $\text{dist}(u \leftrightarrow v) := \text{dist}(u \to v) + \text{dist}(v \to u)$.

Question 15.2. *Does the same sort of conjecture hold for directed graphs under this notion of distance?*

[2] Pun intended.