PCMI Open Problems II

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Here is a rough description of the problems presented at the second PCMI open problem session. Please let us know if you spot any errors you'd like us to correct!

1 Noga Alon

Problem 1.1. *What is the minimum possible number of directions determined by n points in $\mathbb{R}^d$? (The points are 'genuinely' in $\mathbb{R}^d$, meaning that not all the points lie in the same hyperplane.)*

Conjecture: the minimum is $(d-1)n - C(d)$, where $C(d)$ is a constant depending only on d. This is realized by placing all but $d-1$ points on a line, and having the rest in general position.

János Pach said this conjecture appears in a book by Brass, Moser, and Pach ('05). Some versions raised earlier by Jamison ('85), Blokhuis and Seress ('04).

True for dimensions $d = 2$ (Ungar) and $d = 3$ (Pach, Pinchasi, Shamir).

Alon came to it from a distinct distances question in typical norms. Alon and Pinchasi earlier this year proved a lower bound of $\Omega(dn)$.

2 János Pach

Given a graph G and a subset $U \subseteq V(G)$, we define the *profile* of U to be the multiset of $\binom{|U|}{2}$ distances between two points of U (with the distance being with respect to G, not $G[U]$). Let $h(n)$ denote the maximum h such that every n-vertex graph G has 2 disjoint h-element sets with the same profile.

Problem 2.1. *Determine (bounds for) $h(n)$.*

At present, the best known bounds are

$$\Omega\left(\frac{\log^2 n}{\log\log^2 n}\right) \leq h(n) \leq n/4,$$

with the lower bound due to Alon and the upper bound due to Albertson-Pach-Young. In particular, the upper bound comes from taking a $K_{n/2}$ and attaching a path of length $n/2$ to one of its vertices.

3 Mathias Schacht

The typical Ramsey property is denoted as follows: for graphs G and H, we say that $G \to (H)_r$ if every r-coloring of the edges of G contains a monochromatic copy of H.

The canonical Ramsey property is as follows: for a vertex-ordered graphs G (meaning that $V(G)$ comes with a prescribed total linear order), we say that $G \overset{*}{\to} H$ if for every coloring of $E(G)$ by $\mathbb{N}$, there exists either a monochromatic copy of H, a rainbow copy of H, a 'min-colored' copy of H, or a 'max-colored' copy of H, where, by a 'min(max)-colored' copy of H, we mean there is some ordering $v_1, \dots, v_k$ of the vertices of H such that the color of the edge $v_i v_j$ is determined by i (j), and distinct for different i (j).

This question is about understanding the relationship between Ramsey properties and canonical Ramsey properties.

Problem 3.1. *Prove that for any r, there exists graph G on $[n]$ such that $G \to (K_1 0)_r$ (10 not important, just not too small) but G does not have the canonical Ramsey property $G \overset{*}{\to} (K_{10})$.*

Often Ramsey and canonical Ramsey come together: the bounds of the same shape, the thresholds in random graphs are of same order, etc. It would be really surprising if Ramsey implied canonical Ramsey, so Mathias believes the opposite, hence this question.

Apparently the bounds on Ramsey and canonical Ramsey are not enough to answer the question. However, it is possible that taking G to be some suitable complete graph could work.

This is known to be true if one replaces K_{10} by a cycle, or an arithmetic progression, making the appropriate changes. We note that canonical Ramsey is too restrictive for triangles, hence the question was asked for K_{10}.

4 Cosmin Pohoata

Define $R'_3(n)$ to be the smallest N such that in every 3-coloring of $E(K_N^{(3)})$ there exists a color-avoiding copy of $K_n^{(3)}$ (ie a copy which avoids at least one of the three colors).

Trivially we have $R'_3(n) \leq R_3(n)$. We wonder if significantly better bounds hold.

Problem 4.1. *Prove that $R'_3(n) \leq 2^{n^c}$ for some $c > 0$.*

5 Sam Spiro

Given a graph F, we define its expansion F^+ to be the 3-uniform hypergraph obtained by inserting a new vertex inside each edge of F. For example, C_ℓ^+ is the loose cycle of length ℓ. We also recall that given an r-uniform hypergraph we define $\mathrm{ex}(n, F)$ to be the maximum number of edges that an n-vertex F-free r-uniform hypergraph can have.

Mubayi and Verstraëte proved that if F is a graph satisfying $\mathrm{ex}(n, F) = O(n^{\phi})$ (where $\phi = 1.61\ldots$ is the golden ratio), then $\mathrm{ex}(n, F^+) = O(n^2)$, and this upper bound is tight for any F which is not a subgraph of a star. We suspect that the following strengthening also holds.

Conjecture 5.1. *If F is a graph with $\mathrm{ex}(n, F) = O(n^{5/3})$, then $\mathrm{ex}(n, F^+) = O(n^2)$.*

The value of 5/3 is best possible, as one can prove prove using random polynomial graphs that for all $\varepsilon > 0$ there exist $\delta > 0$ and a graph F with $\mathrm{ex}(n, F) = O(n^{5/3+\varepsilon})$ and $\mathrm{ex}(n, F^+) = \Omega(n^{2+\delta})$.

6 Nathan Tung

Given a directed graph D and a tournament T, we define $t_D(T) = \frac{\#\,\mathrm{hom}(D \to T)}{v(T)^{v(D)}}$. We say a digraph D is tournament anti-Sidorenko if $t_D(T) \le 2^{-e(D)} (= t_{\vec{K}_2}(T)^{e(D)})$ for all tournaments T, and we say that D is tournament Sidorenko if $t_D(T) \ge 2^{-e(D)}$ for all tournaments T.

It is known that if D is a directed path then it is tournament anti-Sidorenko, and that if D is an alternating path then it is tournament Sidorenko. We also know that if D is a directed cycle of length ℓ, then it is tournament anti-Sidorenko if ℓ is not divisible by 4 (and if $4|\ell$ then it is neither tournament Sidorenko nor tournament anti-Sidorenko). A theorem of He-Mani-Nie-Tung shows that the fraction of orientations of P_k that are tournament Sidorenko is at most $.99998^k$ for large k. Given this, it is natural to ask about anti-Sidorenko, for which it is conjectured that the following holds.

Conjecture 6.1. *Amongst orientations of P_k, a $\frac{1}{2} - o_k(1)$ fraction of them are tournament anti-Sidorenko.*

7 Huy Pham

Problem 7.1. *Let G be a group (not necessarily abelian). Suppose $A \subseteq G$ has no nontrivial solution to $xy^{-1}xy^{-1} = zt^{-1}$ (that is, $x \neq y$ and $z \neq t$). Prove that $|AA^{-1}| > |A|^{1+\varepsilon}$ for some constant $\varepsilon > 0$ (or even more weakly $> |A| \log |A|$).*

Problem 7.2. *Consider the really nice group $G = \mathbb{F}_2^d$. Can we 2-color G such that for every subspace H of size Cd, $H \setminus \{0\}$ is not monochromatic?*

8 Liana Yepremyan

The following problem[1] is motivated by a question of Buratti from 2013.

[1] Possibly this exact question is not quite true as written ala some comments from Noga Alon regarding the case when n is prime and when exactly $n - 1$ of the Hamiltonian cycles have jumps of size 1.

Question 8.1. *Given n Hamiltonian cycles in K_n such that any two are either edge disjoint or the same, does there exist a rainbow Hamiltonian cycle (i.e. one which uses a distinct edge from each Hamiltonian cycle)?*

Here's a maybe easier question.

Question 8.2. *Given n Hamiltonian cycles in K_n such that any two are either edge disjoint or the same, does there exist a rainbow path of length $n - o(n)$?*

Even the following is not known.

Question 8.3. *Given n Hamiltonian cycles in K_n such that any two are either edge disjoint or the same, does there exist a rainbow path forest with $n - o(n)$ edges and "few" components?*

9 Yuval Wigderson

We are interested in the growth rate hypergraph Ramsey functions of the following form: $r(H, K_n^k)$ where H is a fixed k-uniform hypergraph, and n grows. These can be anywhere between a polynomial and a tower of height $k - 1$.

Question 9.1. *For $k = 3$ and H being the Fano plane, is $r(H, K_n^{(3)})$ polynomial or superpolynomial? (Fano plane minus an edge is also interesting, perhaps the smallest unknown case.)*

Question 9.2. *For $k = q + 1$, where q is a prime power, and H being a projective plane over $\mathbb{F}_q$, denoted $\mathbb{P}^2\mathbb{F}_q$, is $r(H, K_n^{(3)})$ an tower of height which increases in k?*

There was a recent breakthrough in uniformity 3: there exists linear hypergraphs for which the growth rate is superpolynomial. Yuval et al. showed that in higher uniformity can get increasing tower heights almost matching the upper bounds for some linear hypergraphs.

For $k = 4$, we can show that $r(\mathbb{P}^2\mathbb{F}_3, K_n^{(4)}) > 2^{n-2}$. The construction demonstrating this is as follows: split the vertex set into two equal parts, and color those 4-tuples red which have exatly two vertices in each part. Repeat this iteratively inside of the parts. Color blue all the other 4-tuples. There is no blue clique of size logarithmic in the size of the vertex set, and there is no red $\mathbb{P}^2\mathbb{F}_3$: for every bipartition of points of $\mathbb{P}^2\mathbb{F}_3$, there exists a line hitting an odd number of some side.