Problems that I would like Somebody to Solve

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October 14, 2022

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1 What is this?

This is a (very informal) collection of problems that are of personal interest to me, most of which are lesser known problems. Feel free to reach out to me if you have any questions regarding these questions and/or if you spot any glaring typos.

Warning: if you start to seriously think about any of these problems, I recommend telling me this so that I don't accidentally end up working on it at the same time (which has happened before!).

Organization. Section 2 contains the main problems that I'm interested in. Solving any of them will put you in the "Hall of Fame" of Section 4 as thanks for putting the problem to rest. Section 3 consists of some other problems that I would still appreciate being resolved, but not to the level that I'll sleep better at night knowing that they've been solved. In particular, many of these problems are relatively elementary to state, and I would not be surprised if several of them were easy to solve.

The order that the problems appear is essentially chronological. Note that problems can jump between being labeled "main" or "small" depending on my current mood, and I may remove problems from the list if I start actively working on them again.

2 The Main Problems

2.1 Coloring mod p

Given a graph G and an integer p, we say that $I \subseteq G$ is an *independent set mod p* if every vertex in the induced graph $G[I]$ has degree $0 \mod p$. For example, independent sets are always independent sets mod p. We define the *mod p independence number* $\alpha_p(G)$ to be the size of a largest independent set mod p. Similarly we define the *mod p chromatic number* $\chi_p(G)$ to be the smallest integer k such that there exists a partition $V_1 \cup \cdots \cup V_k$ of $V(G)$ such that V_i is an independent set mod p for all i.

Conjecture 2.1. *For all primes p, there exists a constant $C = C(p)$ such that for all graphs G, $\chi_p(G) \leq C$.*

It's quite plausible that the conjecture is true without having to restrict to primes, but focusing on primes is probably a good place to start since one can most easily use algebraic techniques in this case.

Gallai proved that Conjecture 2.1 is holds with $C = 2$ when $p = 2$, see [9] for a simple proof, as well as [6] for two other proofs written in a different language [1] Caro, Krasikov, and Roditty [1] proved a weaker version of Conjecture 2.1, showing that G can be partitioned into C induced subgraphs $G[V_1], \ldots, G[V_C]$ such that $e(G[V_i]) \equiv 0 \mod p$ for all i. Ferber, Hadiman and Krivelevich [5] showed that there exists a C such that almost every graph has $\chi_p(G) \leq C$.

Overall Conjecture 2.1 seems pretty hard, and there are a couple of weaker versions of this conjecture that might be provable.

Conjecture 2.2. *For all primes p, there exists a constant $C = C(p)$ such that for all graphs G, $\alpha_p(G) \leq |V(G)|/C$.*

Conjecture 2.3. *For all primes p, there exists a constant $C = C(p)$ such that for all graphs G, one can partition $V(G)$ into C sets $V_1 \cup \cdots \cup V_C$ such that no $G[V_i]$ contains a vertex of degree $1 \mod p$.*

It also natural to conjecture this for $-1 \mod p$, since in both cases we know the result holds for $p = 2$.

Lastly, we note that a trivial lower bound on the $C(p)$ in Conjecture 2.1 is $C(p) \geq p$ by considering $G = K_p$. However, for odd p one can prove that we must have $C(p) \geq p + 1$ (there

[1]This reference gives three proofs that there exists a solution to the "Lights Out!" game. It is relatively easy to show that this implies the stated result by considering a graph G' with a leaf attached to each vertex.

are many examples; the simplest is to take a circulant graph on $2p+2$ vertices such that every vertex has degree $p+1$). It would be interesting to know if one could find constructions which give significantly stronger bonds.

2.2 Small Quasikernels

Let D be a digraph. Given a set S, we define $N^+(S) = \bigcup_{v \in S} N^+(v)$, where $N^+(v)$ is the out-neighborhood of v. We say that a set $K \subseteq V(D)$ is a *kernel* of D if (1) $N^+(K) \cap K = \emptyset$ (that is, K is an independent set of the underlying graph of D), and (2) $N^+(K) \cup S = V(D)$ (that is, every vertex is either in K or can be reached by a vertex in K in one step).

Not every digraph has a kernel (take any directed cycle of odd length), but it is not too hard to prove that every digraph has a *quasikernel*. This is a set $Q \subseteq V(D)$ such that (1) $N^+(Q) \cap Q = \emptyset$ and such that (2) $N^+(N^+(Q)) \cup N^+(Q) \cup Q = V(D)$. That is, it is an independent set such that every vertex can be reached from Q in at most two steps.

Given that every digraph has a quasikernel, it is natural to ask how small of a quasikernel one can find. One quickly realizes that it can be quite large: any source of D must belong to a quasikernel of D. Thus the most natural setting to consider is when D has no sources, and in this case the following was conjectured by Erdős and Székely.

Conjecture 2.4. *Every digraph D with no sources has a quasikernel of size at most $|V(D)|/2$.*

Some progress has been made, see [7], but overall the conjecture is far from being resolved.

3 Some Smaller Problems

3.1 Slow Tribonacci Walks

Given a triple of positive integers (w_1, w_2, w_3), recursively define $w_k = w_{k-1} + w_{k-2} + w_{k-3}$ for $k \geq 4$. We say that (w_1, w_2, w_3) is an *n-tribonacci walk* if $w_s = n$ for some s. There are infinitely many n-tribonacic walks, e.g. those of the form $(42, n, x)$ for any x. To make things more interesting, we say that (w_1, w_2, w_3) is an *n-slow tribonacci walk* if $w_s = n$ with s as large as possible. For example, $(1, 1, 1)$ and $(42, 3, 42)$ are both 3-tribonacci walks, but only the first one is slow. Let $p(n)$ denote the number of n-slow tribonacci walks. For example, it's easy to check that $p(3) = 1$, and $p(1) = p(2) = \infty$

Question 3.1. *Does there exist some absolute constant c such that either $p(n) = \infty$ or $p(n) \leq c$ for all n?*

If we instead look at Fibonacci walks (which are defined using the Fibonacci recurrence $w_k = w_{k-1} + w_{k-2}$), then one can show that $p(n) \leq 2$ for all $n \geq 2$ [3]. More generally if one looks at walks following a two-term recurrence of the form $w_k = \alpha w_{k-1} + \beta w_{k-2}$ with $\alpha, \beta \geq 1$ relatively prime, then $p(n) \leq \alpha^2 + 2\beta - 1$ for all but finitely many n [12].

Computational data made it easy to conjecture the correct answer for two-term recurrences, but the situation is less clear for slow tribonacci walks. For example, $p(61) = 9$, which is fairly large given how small 61 is.

I'll note that there are many other interesting problems related to the behavior of slow recurrences that were left unanswered in [3, 12]. However, as is often the case in number theory, these relatively easy to state questions are likely very difficult to solve. This being said, I do think that this tribonacci problem is tractable.

3.2 An Adversarial Chernoff Bound

Persi Diaconis, Ron Graham, Xiaoyu He, and myself [4] proved the following.

Theorem 3.2 ([4])**.** *Let X_i be independent Bernoulli random variables with $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$. Let $S_t = \sum_{i \le t} X_i$. There exist absolute constants c_0, c_1 such that for all $\lambda > 0$ and integers $k_1 \ge k_0 \ge 2\lambda^{-1}$,*

$$\Pr[\exists t \in [k_0, k_1] : |S_t - pt| \ge \lambda pt] \le \frac{c_0 k_1}{k_0} e^{-c_1 \lambda^3 p k_0}.$$

That is, with high probability, for every t in the interval $[k_0, k_1]$, every partial sum S_t is close to its expectation. This is immediate for any given value of t by the Chernoff bound (since each S_t is a binomial random variable), but it does *not* follow from the Chernoff bound and a naive application of the union bound (this gives a bound like $k_1 e^{-\lambda^2 p k_0}$, which is much weaker if k_0 is very large).

Question 3.3. *Does the bound of Theorem 3.2 hold with λ^2 instead of λ^3?*

Note that λ^2 would be best possible because this is what one gets if $k_0 = k_1$. While the statement of Theorem 3.2 is fairly technical, the proof itself only required a slightly clever application of the union bound together with the Chernoff bound, so my hope is that more sophisticated probabilistic tools can be used to solve Question 3.3 without too much difficulty.

Secretly I'm interested in this because it would improve upon the error term for our main result in [4], but also I just think it's of independent interest to determine how much concentration one can get for an "adversarial" binomial distribution.

3.3 Saturation Games

For a family of graphs $\mathcal{F}$, we say that a graph G is $\mathcal{F}$-saturated if G contains no graph of $\mathcal{F}$ as a subgraph, but adding any edge to G would create a subgraph of $\mathcal{F}$. The $\mathcal{F}$-saturation game consists of two players, Max and Mini, who alternate turns adding edges to an initially empty graph G on n vertices (say with Max starting), with the only restriction being that G is never allowed to contain a subgraph that lies in $\mathcal{F}$. The game ends when G is $\mathcal{F}$-saturated. The payoff for Max is the number of edges in G when the game ends, and Mini's payoff is the opposite of this. We let $\mathrm{sat}_g(n, \mathcal{F})$ denote the number of edges that the graph in the $\mathcal{F}$-saturation game

C	A	T	A	C
C	A	T	A	C
C	A	T	A	C
C	A	T	A	C
C	A	T	A	C

Figure 1: A grid containing 22 CAT's.

ends with when both players play optimally, and we call this quantity the game $\mathcal{F}$-saturation number.

Bounding $\mathrm{sat}_g(\mathcal{F}; n)$ seems to be pretty hard in general, and even the original problem of determining $\mathrm{sat}_g(n, C_3)$ is still wide open. See [11] for further history and known bounds. In [11] I proved $\mathrm{sat}_g(n, \mathcal{C}_\infty^o \setminus \{C_3\}) = O(n)$, where $\mathcal{C}_\infty^o$ is the set of all odd cycles. I also proved (somewhat indirectly) that $\mathrm{sat}_g(n, \mathcal{C}_\infty^o \setminus \{C_{2k+1}\}) = \Omega(n^2)$ for $k \geq 3$. Given this, it is natural to ask the following.

Problem 3.4. *Prove non-trivial bounds on* $\mathrm{sat}_g(n, \mathcal{C}_\infty^o \setminus \{C_5\})$*, where* $\mathcal{C}_\infty^o = \{C_3, C_5, C_7, \ldots\}$*.*

I'd be happy to have even an $\omega(n)$ lower bound or any non-trivial asymptotic upper bound for this problem. Possibly a more tractable problem is the following.

Problem 3.5. *Prove non-trivial bounds on* $\mathrm{sat}_g(n, C_k)$ *for* $k > 3$*.*

For odd k, I proved an asymptotic upper bound of $\mathrm{sat}(n, C_k) \leq \frac{4}{27}n^2 + o(n^2)$. In [2] the authors proved a non-trivial lower bound for C_4 if you play a "bipartite" version of the game, but I'd still like to see bounds proved in the original setting. Lastly, I'd like to know the following.

Question 3.6. *Does there exist a finite set of (odd) cycles* $\mathcal{C}$ *such that* $\mathrm{sat}_g(n, \mathcal{C}) = O(n)$*?*

3.4 Words in Grids

In Figure 1, there are 22 occurrences of the word "CAT" if we allow the word to be written in rows, columns, diagonals, and either forwards or backwards. Indeed, each C is in one row and at least one diagonal line containing CAT, and the middle two C's are in two such diagonals.

If you generalize this pattern to an $n \times n$ grid where each row goes "CATACATACAT...", then it's not too hard to see that there exist roughly $\frac{3}{2}n^2$ many copies of the word CAT (intuitively this is because there are about $n/4$ C's in each of the n rows, and most C's are an endpoint of 6 different instances of the word CAT).

Conjecture 3.7. *In any* $n \times n$ *grid, there are asymptotically at most* $\frac{3}{2}n^2$ *CAT's.*

If the reader prefers, essentially the same conjecture holds for the word DOG instead of CAT.

More generally, for any word w with k distinct letters, Patchell and I [10] conjectured that every $n \times n$ grid contains at most $\frac{3}{k-1}n^2$ copies of w asymptotically. In general the best we were able to prove is an upper bound of $\frac{4}{k-1}n^2$, though at one point I convinced myself that I could solve the problem when $k = 2$. More generally we proved the following.

Proposition 3.8 ([10])**.** *Let $f(w, n)$ be the maximum number of occurrences a word w can appear in an $n \times n$ grid, and let $f(w, n, 1)$ denote the maximum number of occurrences of w in an $n \times 1$ grid. If $f(w, n, 1) \sim \alpha n$, then*

$$3\alpha n^2 \lesssim f(w, n) \lesssim 4n^2.$$

This result suggests the following.

Question 3.9. *Is $f(w, n, 1)$ always asymptotic to some $\alpha_w n$? If so, is α_w easy to compute?*

I strongly suspect that the first half of this question has a positive answer and shouldn't be hard to prove, but the second half is unclear.

4 Hall of Fame

A list of people who have successfully solved any of my main problems.

- Wang and Zhao [13] for solving my original conjecture on ballot permutations, and Lin, Wang, and Zhao [8] for solving an even stronger version.

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