

Generalized Turán Problems for Trees and More

Sam Spiro, Georgia State University

Based on Joint Work with Sean English



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Leaderboard

Current Rankings

My Points

**Sam Spiro**

15400 pts



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**Jacob Platnick**

12100 pts



(4)

4

**Irena Swanson**

11000 pts



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5

**Shannon Auckly**

9800 pts



(0)

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**Sammy Luo**

15600 pts

**Sam Spiro**

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4

**Shannon Auckly**

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5

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Turán Problems

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Theorem

$$\text{ex}(n, K_2) = 0.$$

Turán Problems

Theorem (Turán 1941)

$$\text{ex}(n, K_r) = \left\lfloor \binom{r-1}{2} \frac{n^2}{(r-1)^2} \right\rfloor$$

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$$\text{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

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Theorem (Kővari-Sós-Turán 1954)

$$\text{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

Theorem (Folklore)

If T is a tree with $e(T) \geq 2$, then

$$\text{ex}(n, T) = \Theta(n).$$

Turán Problems

Given a family of graphs \mathcal{F} , we say G is \mathcal{F} -free if G is F -free for all $F \in \mathcal{F}$. We define $\text{ex}(n, \mathcal{F})$ to be the maximum number of edges in an n -vertex \mathcal{F} -free graph.

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Conjecture (Compactness Conjecture)

If \mathcal{F} is a finite family of graphs, then $\text{ex}(n, \mathcal{F}) = \Theta(\text{ex}(n, F))$ for some $F \in \mathcal{F}$.

Generalized Turán Problems

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Definition (Alon-Shikhelman 2016)

Given a graph H and a family of graphs \mathcal{F} , we define the generalized Turán number $\text{ex}(n, H, \mathcal{F})$ to be the maximum number of copies of H in an n -vertex \mathcal{F} -free graph.

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See the recent survey by Gerbner and Palmer for way more history than I'm going to give here.

Generalized Turán Problems

Theorem (Gerbner-Palmer 2019)

For all $r \geq 2$ and families \mathcal{F} we have

$$\text{ex}(n, K_r, \mathcal{F}) \leq \text{ex}(n, \mathcal{F})^{r/2}.$$

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Theorem (Füredi-Kündgen 2006)

If $\text{ex}(n, \mathcal{F}) = \Theta(n^{2-\beta})$ and \mathcal{F} does not contain a star, then

$$\text{ex}(n, K_{1,t}, \mathcal{F}) = \tilde{\Theta}(\max\{n^t, n^{t+1-t\beta}\}).$$

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Question

Can we prove bounds on $\text{ex}(n, T, \mathcal{F})$ for arbitrary trees T (for some possibly fixed \mathcal{F})?

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Theorem (Alon-Shikhelman 2016)

If T is a tree and F is a tree, then $\text{ex}(n, T, F) = \Theta(n^k)$ for some integer k .

Theorem (Letzter 2019)

If H is any graph and F is a tree, then $\text{ex}(n, H, F) = \Theta(n^k)$ for some integer k .

A General Theorem for Trees

Theorem (English-S. 2025+)

For any tree T , integer $k \geq 1$, and family of graphs \mathcal{F} , either

$$\text{ex}(n, T, \mathcal{F}) = \Omega(n^k),$$

or

$$\text{ex}(n, T, \mathcal{F}) = O(\text{ex}(n, \mathcal{F})^{k-1}).$$

Moreover, we can determine which of these two cases happen for a given T, k, \mathcal{F} .

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Corollary

If T is a tree and \mathcal{F} contains a forest, then $\text{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k .

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Proof.

Let k be the smallest integer such that $\text{ex}(n, T, \mathcal{F}) = O(n^k)$.

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Let k be the smallest integer such that $\text{ex}(n, T, \mathcal{F}) = O(n^k)$. If $\text{ex}(n, T, \mathcal{F}) = \Omega(n^k)$ then we're done

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Let k be the smallest integer such that $\text{ex}(n, T, \mathcal{F}) = O(n^k)$. If $\text{ex}(n, T, \mathcal{F}) = \Omega(n^k)$ then we're done, otherwise our theorem implies that

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$$\text{ex}(n, T, \mathcal{F}) = O(\text{ex}(n, \mathcal{F})^{k-1}) = O(n^{k-1}).$$



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Corollary

If $T \neq K_2$ is a tree with $\ell \geq 2$ leaves, then any family \mathcal{F} with $\text{ex}(n, T, \mathcal{F}) = O(n^\ell)$ has $\text{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k .

In particular, if $\text{ex}(n, T, \mathcal{F}) = o(n^\ell)$ then $\text{ex}(n, T, \mathcal{F}) = O(n^{\ell-1})$.

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Proof.

If \mathcal{F} contains a forest then we are done by the previous result. Otherwise, the graph G obtained by duplicating each leaf of T a total of n/ℓ times is \mathcal{F} -free and contains $\Omega(n^\ell)$ copies of T . □

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Theorem (English-S. 2025++)

If T is a tree with $\ell \geq 2$ leaves, then every family \mathcal{F} with $\text{ex}(n, T, \mathcal{F}) = o(n^{\ell+1})$ satisfies

$$\text{ex}(n, T, \mathcal{F}) = O\left(n^{\ell + \frac{\ell^2 - \ell}{e(T) - 1}}\right).$$

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Theorem (English-S. 20??++)

For the path graph P_t , every graph F with $\text{ex}(n, P_t, F) = O(n^{\alpha(P_t)})$ has $\text{ex}(n, P_t, F) = \Theta(n^k)$ for some integer k .

Stability for Generalized Turán Problems

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Proposition (English-Halfpap-Krueger 2024)

For stars $K_{1,t}$, every family of graphs \mathcal{F} either satisfies $\text{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$ or $\text{ex}(n, K_{1,t}, \mathcal{F}) = O(n)$.

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If \mathcal{F} does not contain a subgraph of a star, then $G = K_{1,n-1}$ is \mathcal{F} -free and shows $\text{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$.

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We want to generalize this idea by saying that for *any* graph H , there exists some “simple” family \mathcal{F}_H such that the behavior of $\text{ex}(n, K_{1,t}, \mathcal{F})$ depends on how \mathcal{F} “interacts” with \mathcal{F}_H .

Stability for Generalized Turán Problems

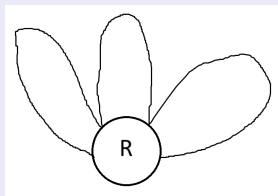
Definition

Given a graph H , a subset $R \subseteq V(H)$, and an integer q , we define the sunflower-power H_R^q to be the graph obtained by taking q copies of H which all agree on R and which are otherwise disjoint.

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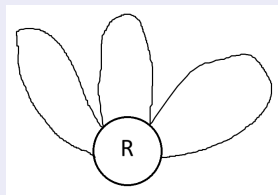
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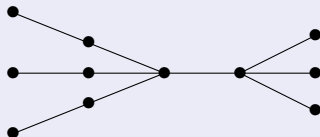
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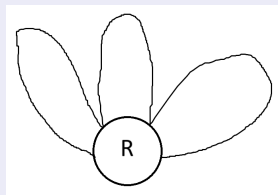
$$(P_5)_{\{x_3, x_4\}}^3$$



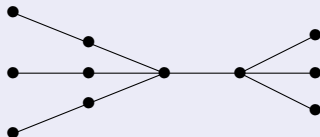
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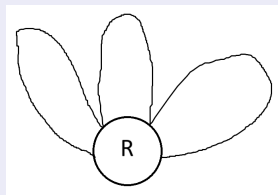
We define

$$\mathcal{F}_{H,k}^q = \{H_R^q : H - R \text{ has at least } k \text{ connected components}\}.$$

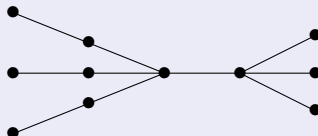
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Claim

Every graph in $\mathcal{F}_{H,k}^q$ has at least q^k copies of H .

Stability for Generalized Turán Problems

Proposition (Key Observation)

For every H, k, \mathcal{F} , either $\text{ex}(n, H, \mathcal{F}) = \Omega(n^k)$ or there exists some q such that every \mathcal{F} -free graph is $\mathcal{F}_{H,k}^q$ -free.

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Proof.

If some $H_R^q \in \mathcal{F}_{H,k}^q$ is always \mathcal{F} -free, then the previous claim with $q \approx n/v(H)$ shows $\text{ex}(n, H, \mathcal{F}) = \Omega(n^k)$.

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Corollary (General Stability for Generalized Turán)

If $\text{ex}(n, H, \mathcal{F}_{H,k}^q) = O_q(n^\beta)$, then every family \mathcal{F} either satisfies $\text{ex}(n, H, \mathcal{F}) = \Omega(n^k)$ or $\text{ex}(n, H, \mathcal{F}) = O(n^\beta)$.

Stability for Generalized Turán Problems

Proposition

For every graph H and family \mathcal{F} , either $\text{ex}(n, H, \mathcal{F}) = \Omega(n)$ or $\text{ex}(n, H, \mathcal{F}) = O(1)$.

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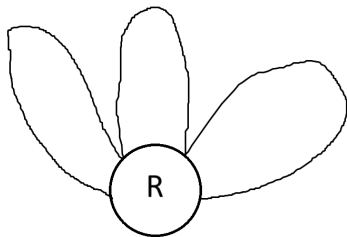
By the previous corollary, it suffices to prove $\text{ex}(n, H, \mathcal{F}_{H,1}^q) = O_q(1)$.

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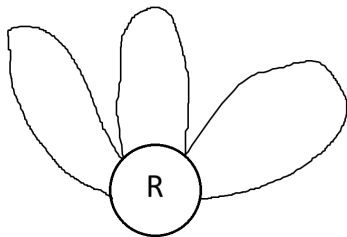


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This is essentially equivalent to the Erdős-Rado Sunflower lemma. □

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Theorem (English-S. 2025+)

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Theorem (English-S. 2025+)

If T is a tree and if G is $\mathcal{F}_{T,k}^q$ -free, then the number of copies of T in G is at most $O(e(G)^{k-1})$.

Proof Sketch

Proof Vibes

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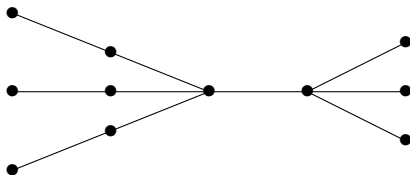
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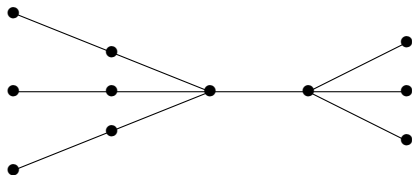


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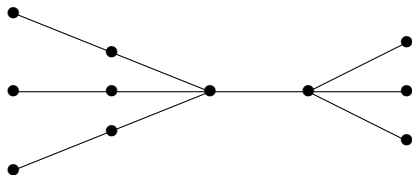
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For this example, to specify a given copy of P_5 , we **must** identify its last edge. More generally, whatever set of edges E we choose to identify a given copy K in a graph, the edges of E **must** intersect every subtree $K' \subseteq K$ which has “many extensions.”

Proof Vibes

Question

Given a set of subtrees \mathcal{T} of a tree T , when can we guarantee that there exist a set of $k - 1$ edges E which intersect all of these subtrees?

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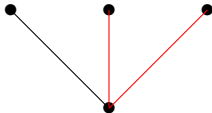
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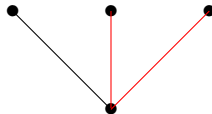
Theorem (Helly Property for Trees)

If \mathcal{T} is a set of subtrees of a tree T such that the vertex sets of the subtrees pairwise intersect, then there exists a vertex $v \in V(T)$ which intersects the vertex set of every subtree of \mathcal{T} .

Proof Vibes



Proof Vibes



Definition

We call a subtree $T' \subseteq T$ leaf-cuttable if $e(T) \geq 1$ and if every edge in $E(T) \setminus E(T')$ which intersects T' intersects a leaf of T' .

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Theorem (English-S. 2025+)

If \mathcal{T} is a set of leaf-cuttable subtrees of a tree T such that every k subtrees from \mathcal{T} contains two subtrees with intersecting edge sets, then there exists a set $E \subseteq E(T)$ of at most $k - 1$ edges which intersects the edge set of every subtree of \mathcal{T} .

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This result implies the vertex-Helly result (and also König's Theorem for trees).

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- Verify that the “highly extendable” subtrees \mathcal{T}_K are leaf-cuttable (easy),
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- Verify that finding this set of $k - 1$ edges for each \mathcal{T}_K is enough to give the desired bound (annoying but doable).

Going Further

Theorem (English-S. 2025+)

For any tree T , integer $k \geq 1$, and family of graphs \mathcal{F} , either

$$\text{ex}(n, T, \mathcal{F}) = \Omega(n^k),$$

or every \mathcal{F} -free graph is $\mathcal{F}_{T,k}^q$ -free and

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and this holds for bipartite graphs without isolated edges by König's Theorem.

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Is it the case that for every bipartite graph H without isolated vertices, integer k , and family of graphs \mathcal{F} that either

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Fact

This is false for C_4 and $k = 2$:(

Going Further

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Theorem (Bukh-Conlon 2018)

Every rational in $[1, 2]$ is realizable for $H = K_2$.

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Every rational in $[t, t + 1]$ is realizable for $H = K_{1,t}$ and no rational in $(1, t)$ is.

Going Further

Theorem (English-S. 2025+)

For every graph H of maximum degree Δ , every rational in the range

$$\left[v(H) - \frac{e(H)}{2\Delta^2}, v(H) \right]$$

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This is best possible for stars.

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We believe we can verify that this is true for all graphs on at most 4 vertices through various ad hoc techniques.

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The idea here is that any F of this form must be a subgraph of every member of $\mathcal{F}_{H, \alpha(H)}^q \neq \emptyset$ for some q . For any fixed q this is a finite number of possibilities, and I don't think arbitrarily large q should give new behaviors (analogous to $F \subseteq K_{2,t}$ implying $\text{ex}(n, F) = \Theta(n^r)$ for some $r \in \{0, 1, 3/2\}$).

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Can one say anything about generalized Turán numbers of trees T satisfying

$$n^{\ell+1} \ll \text{ex}(n, T, \mathcal{F}) \ll n^{\ell+2}?$$

For this it would suffice to prove upper bounds on $\text{ex}(n, T, \mathcal{F}_{T, \ell+2}^q)$.

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For a given graph H , what's the smallest number r such that $\text{ex}(n, H, \mathcal{F}) = O(\text{ex}(n, \mathcal{F})^r)$ for all families \mathcal{F} ? What if I further tell you that $\text{ex}(n, H, \mathcal{F}) = o(n^k)$?

Open Problems

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I'm willing to offer up to \$1 for a proof* or disproof of either of these questions.

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If G contains $kn^{2-1/s}$ edges with $k = \omega(1)$, then it contains at least $\Omega(k^{st}n^s)$ copies of $K_{s,t}$.

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Does the same phenomenon hold if G contains many copies of K_r ?

Open Problems: Supersaturation

Conjecture (Dubroff-Gunby-Narayanan-S.)

If $2 \leq r \leq s \leq t$ and if G contains at least $kn^{r - \binom{r}{2}/s}$ copies of K_r , then it contains at least $k^{st/\binom{r}{2}} n^{s - o(1)}$ copies of $K_{s,t}$.

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Conjecture (Dubroff-Gunby-Narayanan-S.)

For all $1 \leq k \leq n^{1/2t}$, there exist n -vertex graphs with $kn^{3/2}$ triangles and at most $k^t n^{3/2+o(1)}$ copies of $K_{2,t}$.