Polynomial Relations Between Matrices of Graphs

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Spectral Graph Theory

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G

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$$G \to M_G$$

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$$G \to M_G \to \{\lambda_1, \ldots, \lambda_n\}$$

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$$G \to M_G \to \{\lambda_1, \ldots, \lambda_n\} \to \text{Properties of } G$$

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$$A_{ij} = \begin{cases} 1 & ij \in E(G) \\ 0 & ij \notin E(G) \end{cases}$$

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Theorem

If $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of A, then the total number of closed walks (walks from a vertex to itself) of length k in G is

$$\lambda_1^k + \cdots + \lambda_n^k.$$

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Example

If G is the path on 3 vertices, then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

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If G is biregular (i.e. bipartite where every vertex on the same side has the same degree d_i), then

$$A^2 = (d_1 I - L)(d_2 I - L),$$

and using this one can also translate between the eigenvalues of A and L.

Polynomial Relations

Question

Do there exist "triregular graphs", i.e. those with

$$A^3 = f(L)$$

for some polynomial f?

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Theorem (S. 2018)

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Polynomial Relations

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Let G be a connected graph. If there exists a positive integer r and polynomial f such that

$$A^r = f(L),$$

then G is either regular or biregular.

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Table: Graphs Satisfying $X^r = f(Y)$

X/Y	A	L	Q	$\mathcal{L}$
A		Reg, Bireg	Reg, Bireg	Reg, Bireg
L	Reg		Reg	Reg
Q	Reg	Reg		Reg
$\mathcal{L}$	Reg, Bireg	Reg	Reg	

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For this project, we decided to focus on (proper) relations $f(A) = g(L)$ when at least one of f, g has low degrees.

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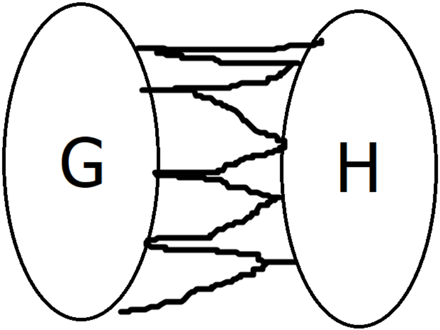
Theorem (FHLS 2023+)

If G is connected and $f(A) = g(L)$ is proper with $\deg f, \deg g \leq 2$, then G is either regular or biregular.

Low Degree Relations

We define the *join* $G \vee H$ by taking $G \cup H$ and adding all edges between G and H



Low Degree Relations

Theorem (FHLS 2023+)

Let G be a k-regular m-vertex graph and H a d-regular n-vertex graph such that $G \vee H$ is not regular. Let A_G, A_H be the adjacency matrices of G, H and let A, L be the adjacency matrix and Laplacian matrix of $G \vee H$.

There exist polynomials f, g such that $f(A) = g(L)$ with $\deg g \leq 2$ if and only if there exists no $\mu \neq k, d$ which is an eigenvalue of both A_G and A_H and no eigenvalue of A_G or A_H is equal to $\frac{k+d-\sqrt{(k-d)^2+4mn}}{2}$.

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Informally, this says the join of two regular graphs G, H has $f(A) = g(L)$ with $\deg(g) \leq 2$ if and only if G, H share no eigenvalues and neither of their eigenvalues equal $\frac{k+d-\sqrt{(k-d)^2+4mn}}{2}$.

Proofs

Theorem (FHLS 2023+)

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$$f(x) = ax^2 + bx + c, \qquad g(x) = \alpha x^2 + \beta x + \gamma.$$

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We thus have $d(u) + d(v)$ equal to a common value for all $uv \in E(G)$. This common value must be $\delta + \Delta$, which means every vertex of minimum degree is only adjacent to vertices of maximum degree and vice versa. This means G is either regular or biregular. $\square$

Proofs

Proposition

For $G \vee H$, under reasonable conditions there exist a proper relation $f(A) = g(L)$ with $\deg f = 2, \ \deg g = 3$.

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Question

If $f(A) = g(L)$ proper with $\deg f = 2$, does G have at most 2 degrees?

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If $f(A) = g(L)$ is proper with $\deg f \leq 2,\ g \leq 3$, *then G is either regular, biregular, or the join of two regular graphs.*

Question

If $f(A) = g(L)$ proper with $\deg f = 2$, does G have at most 2 degrees? More generally, does G have at most $\deg f$ degrees?

Open Problems

Problem

Can you come up with a guess as to which graphs have $f(A) = g(L)$ with $\deg f = \deg g = 3$? What about $\{\deg f, \deg g\} = \{2, 4\}$?

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If $f(A) = g(\mathcal{L})$ with $\deg f, \deg g \leq 2$, then G is either regular or biregular.