Maximal Independent Sets in Clique-free Graphs

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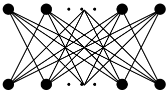
Joint with Xiaoyu He and Jiaxi Nie

History

A maximal independent set (MIS) is an independent set $I \subseteq V(G)$ which is maximal with respect to set inclusion.



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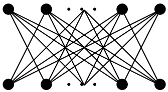

Question

Given a family of graphs $\mathcal{G}$, what's the maximum number of MIS's that a graph $G \in \mathcal{G}$ can have?

History

Let $m(n)$ denote the maximum number of MIS's in an n-vertex graph.

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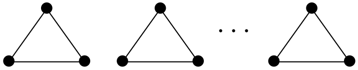
Theorem (Miller, Muller 1960; Moon, Moser 1965)

If $n \geq 2$, then

$$m(n) = \begin{cases} 3^{n/3} & n \equiv 0 \mod 3, \\ 4 \cdot 3^{(n-4)/3} & n \equiv 1 \mod 3, \\ 2 \cdot 3^{(n-2)/3} & n \equiv 2 \mod 3. \end{cases}$$



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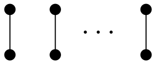
Theorem (Hujter, Tuza 1993)

If $n \geq 4$, then

$$m_3(n) = \begin{cases} 2^{n/2} & n \equiv 0 \mod 2, \\ 5 \cdot 2^{(n-5)/2} & n \equiv 1 \mod 2. \end{cases}$$



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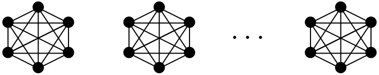
Theorem (Nielsen 2002)

If $s \in \{0,1,\ldots,k-1\}$ with $n \equiv s \mod k$, then

$$m(n,k) = \lfloor n/k \rfloor^{k-s} \lceil n/k \rceil^{s}.$$



Clique-free Graphs

Define $m_t(n,k)$ to be the maximum number of k-MIS's that an n-vertex K_t-free graph can have.

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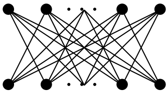
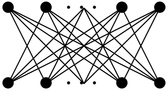
Define $m_t(n, k)$ to be the maximum number of k-MIS's that an n-vertex K_t-free graph can have. Given the previous constructions, we might expect that the maximizer for $m_t(n, k)$ will consist of the disjoint union of some "nice" graphs. One reasonable construction is a *comatching*.



Clique-free Graphs



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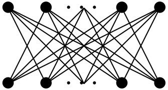

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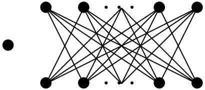

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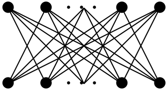

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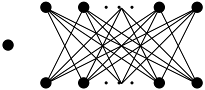

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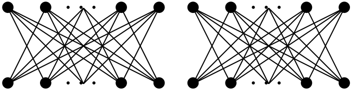

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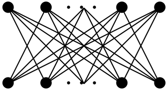

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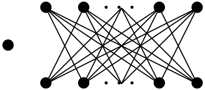

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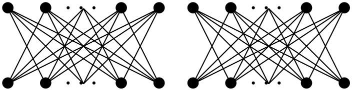

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More generally this shows $m_t(n,k) = \Omega(n^{\lfloor k/2 \rfloor})$ for fixed k.

Clique-free Graphs

Reasonable Question

Is it the case that for all k, t we have

$$m_t(n, k) = O_{k,t}(n^{\lfloor k/2 \rfloor}).$$

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For $n \geq 8$ we have

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For $n \geq 8$ we have

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and the unique graph achieving this bound is a comatching of order n. Moreover, we have

$$m_3(n, 3) = \Theta(n),$$
$$m_3(n, 4) = \Theta(n^2).$$

Better Constructions

Proposition

For all $t \geq 4$,

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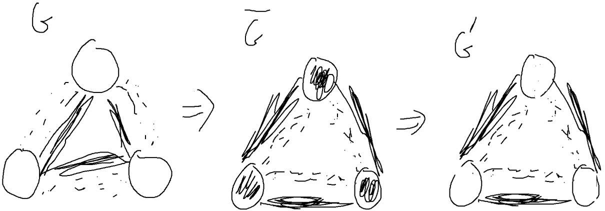
Ruzsa-Szemerédi: there exists an n-vertex tripartite graph G on $U \cup V \cup W$ with $n^{2-o(1)}$ edges such that every edge is contained in a unique triangle. Let G' be the "tripartite complement" of G, i.e. take the complement $\bar{G}$ and then delete all the edges within each of the parts U, V, W.



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Claim: every triangle $T = \{u, v, w\}$ in G is a 3-MIS in G'.

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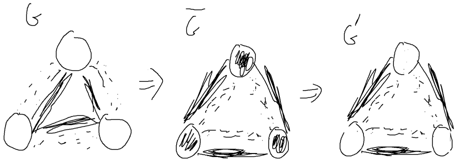

Claim: every triangle $T = \{u, v, w\}$ in G is a 3-MIS in G'.
Since G contains $n^{2-o(1)}$ triangles, and since the tripartite graph G' is K_t-free for $t \geq 4$, we conclude the result. $\square$

Better Constructions

Using generalization of the Ruzsa-Szemerédi construction due to Gowers and Janzer gives:

Theorem (He, Nie, S. 2021)

For all fixed k, t, we have

$$m_t(n, k) \geq n^{\left\lfloor \frac{(t-2)k}{t-1} \right\rfloor - o(1)}.$$

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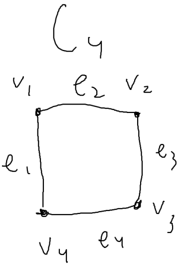
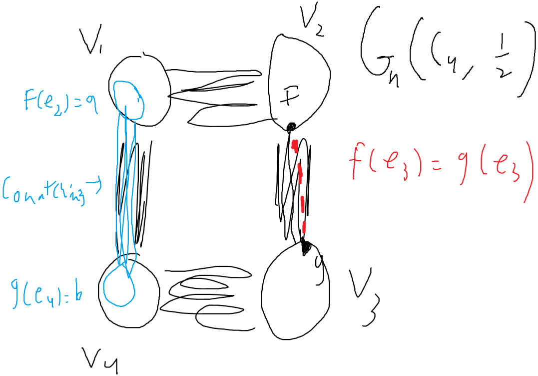
Reasonable Question

Is this bound essentially tight? In particular, for triangle-free graphs do we have

$$m_3(n,k) = \Theta(n^{\lfloor k/2 \rfloor}).$$

Better Construction: Blowups





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Theorem (He, Nie, S. 2021)

$t \geq 3$ *and* $k \geq 2(t-1)$, *then*

$$m_t(n,k) \geq n^{\frac{(t-2)k}{t-1} - o(1)}.$$

Upper Bounds

We think these lower bounds are essentially best possible:

Conjecture (He, Nie, S.; S.)

For all fixed k, t, we have

$$m_t(n,k) = O(n^{\frac{(t-2)k}{t-1}}).$$

Moreover, for $k < 2(t-1)$ we have

$$m_t(n,k) = O(n^{\left\lfloor \frac{(t-2)k}{t-1} \right\rfloor}).$$

Upper Bounds

Proposition

For all $k < t$ we have

$$m_t(n,k) = O(n^{\left\lfloor \frac{(t-2)k}{t-1} \right\rfloor}) = O(n^{k-1}).$$

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Theorem (He, Nie, S. 2021)

For all $k \leq 4$, we have

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Open Problems: Order of Magnitude

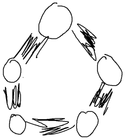
Conjecture

$$m_3(n,5) = \Theta(n^{5/2}).$$

Open Problems: Order of Magnitude

Proposition (He, Nie, S. 2021)

If G is an n-vertex graph which is the subgraph of a blowup of C_5, then it contains at most $O(n^{5/2})$ 5-MIS's.

$G \subseteq$ 

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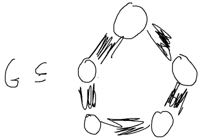

Conjecture

If G is an n-vertex subgraph of a blowup of a k-vertex triangle-free graph H, then G contains at most $O(n^{k/2})$ k-MIS's.

Open Problems: Order of Magnitude

Question

Are the $o(1)$ terms in our exponents necessary when $t \geq 4$? In particular, is it true that

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Proposition

If G is an n-vertex tripartite graph, then G has at most $n^{2-o(1)}$ 3-MIS's.

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If G is an n-vertex K_4-free graph with "many" k-MIS's, is it true that G has chromatic number $O_k(1)$?

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If G is an n-vertex K_4-free graph with "many" k-MIS's, is it true that G has chromatic number $O_k(1)$?

Note that for K_3-free graphs it is easy to prove that if G has at least 1 k-MIS, then $\chi(G) \leq k+1$

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- The classical functions $m(n), m_3(n), m(n,k)$ have relatively simple answers, but combining them into $m_t(n,k)$ seems to give a much more complex problem.

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- All of our constructions utilize Rusza-Szemerédi type graphs as building blocks, together with "twisted blowups" of these graphs.
- We think these constructions are essentially best possible, but upper bounds seem very difficult (partially because there are so many constructions).
- Many, many open problems remain!