Generalized Turán Problems for Trees and More

Sam Spiro, Georgia State University

Based on Joint Work with Sean English

Turán Problems

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Theorem

$$\mathrm{ex}(n, K_2) = 0.$$

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Theorem (Turán 1941)

$$\mathrm{ex}(n, K_r) = \left\lfloor \binom{r-1}{2} \frac{n^2}{(r-1)^2} \right\rfloor$$

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Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

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Theorem (Folklore)

If T is a tree with $e(T) \geq 2$, then

$$\mathrm{ex}(n, T) = \Theta(n).$$

Turán Problems

Given a family of graphs $\mathcal{F}$, we say G is $\mathcal{F}$-free if G is F-free for all $F \in \mathcal{F}$. We define $\text{ex}(n, \mathcal{F})$ to be the maximum number of edges in an n-vertex $\mathcal{F}$-free graph.

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Conjecture (Compactness Conjecture)

If $\mathcal{F}$ is a finite family of graphs which does not include both a star and a matching, then $\mathrm{ex}(n, \mathcal{F}) = \Theta(\mathrm{ex}(n, F))$ for some $F \in \mathcal{F}$.

Generalized Turán Problems

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Definition (Alon-Shikhelman 2016)

Given a graph H and a family of graphs $\mathcal{F}$, we define the generalized Turán number $\mathrm{ex}(n, H, \mathcal{F})$ to be the maximum number of copies of H in an n-vertex $\mathcal{F}$-free graph.

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See the recent survey by Gerbner and Palmer for way more history than I'm going to give here.

Generalized Turán Problems

Theorem (Gerbner-Palmer 2019)

For all $r \geq 2$ and families $\mathcal{F}$ we have

$$\mathrm{ex}(n, K_r, \mathcal{F}) \leq \mathrm{ex}(n, \mathcal{F})^{r/2}.$$

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Theorem (Füredi-Kündgen 2006)

If $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^{2-\beta})$ and $\mathcal{F}$ does not contain a star, then

$$\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \tilde{\Theta}(\max\{n^t, n^{t+1-t\beta}\}).$$

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Theorem (Letzter 2019)

If H *is any graph and* F *is a tree, then* $\mathrm{ex}(n, H, F) = \Theta(n^k)$ *for some integer* k*.*

A General Theorem for Trees

Theorem (English-S. 2025+)

For any tree T, integer $k \geq 1$, and family of graphs $\mathcal{F}$, either

$$\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^k),$$

or

$$\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^{k-1}).$$

Moreover, we can determine which of these two cases happen for a given $T, k, \mathcal{F}$.

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Proof.

Let k be the smallest integer such that $\mathrm{ex}(n, T, \mathcal{F}) = O(n^k)$.

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$$\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^{k-1}) = O(n^{k-1}).$$

□

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Corollary

If $T \neq K_2$ is a tree with $\ell \geq 2$ leaves, then any family $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = O(n^\ell)$ has $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k.

In particular, if $\mathrm{ex}(n, T, \mathcal{F}) = o(n^\ell)$ then $\mathrm{ex}(n, T, \mathcal{F}) = O(n^{\ell-1})$.

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Proof.

If $\mathcal{F}$ contains a forest then we are done by the previous result.

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If $\mathcal{F}$ contains a forest then we are done by the previous result. Otherwise, the graph G obtained by duplicating each leaf of T a total of n/ℓ times is $\mathcal{F}$-free

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Proof.

If $\mathcal{F}$ contains a forest then we are done by the previous result. Otherwise, the graph G obtained by duplicating each leaf of T a total of n/ℓ times is $\mathcal{F}$-free and contains $\Omega(n^\ell)$ copies of T. $\square$

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Theorem (English-S. 2025++)

If T is a tree with $\ell \geq 2$ leaves, then every family $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = o(n^{\ell+1})$ satisfies

$$\mathrm{ex}(n, T, \mathcal{F}) = O\left(n^{\ell + \frac{\ell^2 - \ell}{e(T) - 1}}\right).$$

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Theorem (English-S. 20??++)

For the path graph P_t, every graph F with $\mathrm{ex}(n, P_t, F) = O(n^{\alpha(P_t)})$ has $\mathrm{ex}(n, P_t, F) = \Theta(n^k)$ for some integer k.

Stability for Generalized Turán Problems

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Proposition (English-Halfpap-Krueger 2024)

For stars $K_{1,t}$, every family of graphs $\mathcal{F}$ either satisfies $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$ *or* $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O(n)$.

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If $\mathcal{F}$ does not contain a subgraph of a star, then $G = K_{1,n-1}$ is $\mathcal{F}$-free and shows $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$.

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We want to generalize this idea by saying that for *any* graph H, there exists some "simple" family $\mathcal{F}_H$ such that the behavior of $\mathrm{ex}(n, H, \mathcal{F})$ depends on how $\mathcal{F}$ "interacts" with $\mathcal{F}_H$.

Stability for Generalized Turán Problems

Definition

Given a graph H, a subset $R \subseteq V(H)$, and an integer q, we define the sunflower-power H_R^q to be the graph obtained by taking q copies of H which all agree on R and which are otherwise disjoint.

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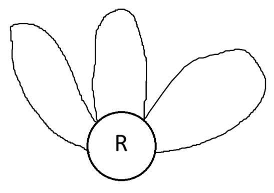

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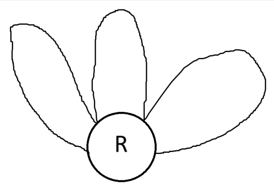
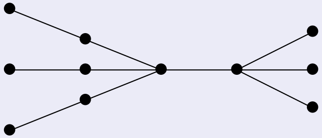


$(P_5)^3_{\{x_3,x_4\}}$



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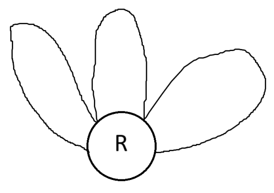

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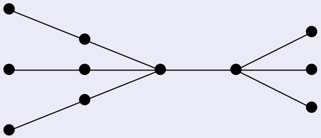

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$$\mathcal{F}^q_{H,k} = \{H_R^q : H - R \text{ has at least } k \text{ connected components}\}.$$

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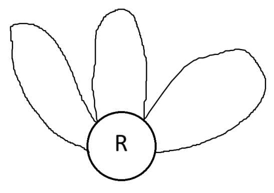

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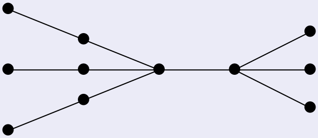

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Claim

Every graph in $\mathcal{F}^q_{H,k}$ has at least q^k copies of H.

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Proposition (Key Observation)

For every $H, k, \mathcal{F}$, either $\mathrm{ex}(n, H, \mathcal{F}) = \Omega(n^k)$ or there exists some q such that every $\mathcal{F}$-free graph is $\mathcal{F}^q_{H,k}$-free.

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Proof.

If some $H^q_R \in \mathcal{F}^q_{H,k}$ is $\mathcal{F}$-free for all q, then the previous claim with $q \approx n/v(H)$ shows $\mathrm{ex}(n, H, \mathcal{F}) = \Omega(n^k)$.

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Corollary (General Stability for Generalized Turán)

If $\mathrm{ex}(n, H, \mathcal{F}^q_{H,k}) = O_q(n^\beta)$, then every family $\mathcal{F}$ either satisfies $\mathrm{ex}(n, H, \mathcal{F}) = \Omega(n^k)$ or $\mathrm{ex}(n, H, \mathcal{F}) = O(n^\beta)$.

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Proposition

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By the previous corollary, it suffices to prove $\mathrm{ex}(n, H, \mathcal{F}^q_{H,1}) = O_q(1)$.

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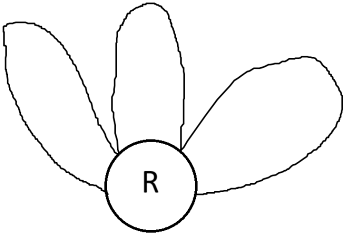

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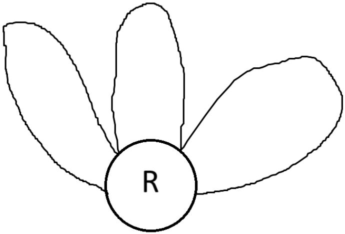

This is essentially equivalent to the Erdős-Rado Sunflower lemma. $\square$

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Theorem (English-S. 2025+)

If T is a tree and if G is $\mathcal{F}^q_{T,k}$-free, then the number of copies of T in G is at most $O(e(G)^{k-1})$.

Proof Sketch

Proof Vibes

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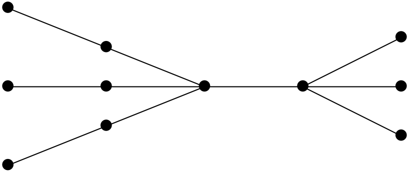

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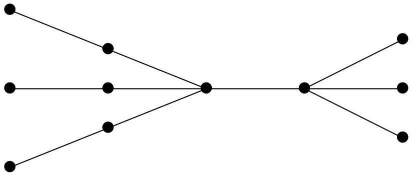

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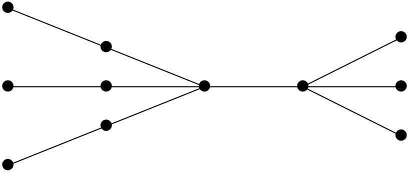

For this example, to specify a given copy of P_5, we **must** identify its last edge. More generally, whatever set of edges E we choose to identify a given copy K in a graph, the edges of E **must** intersect every subtree $K' \subseteq K$ which has "many extensions."

Proof Vibes

Question

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Theorem (Helly Property for Trees)

If $\mathcal{T}$ is a set of subtrees of a tree T such that the vertex sets of the subtrees pairwise intersect, then there exists a vertex $v \in V(T)$ which intersects the vertex set of every subtree of $\mathcal{T}$.

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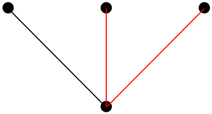

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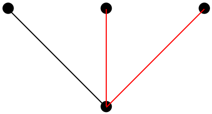

Definition

We call a subtree $T' \subseteq T$ leaf-cuttable if $e(T) \geq 1$ and if every edge in $E(T) \setminus E(T')$ which intersects T' intersects a leaf of T'.

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Theorem (English-S. 2025+)

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If $\mathcal{T}$ is a set of <u>leaf-cuttable</u> subtrees of a tree T such that every k subtrees from $\mathcal{F}$ contains two subtrees with intersecting edge sets, then there exists a set $E \subseteq E(T)$ of at most $k-1$ edges which intersects the edge set of every subtree of $\mathcal{T}$.

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This result implies the vertex-Helly result (and also König's Theorem for trees).

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- Verify that finding this set of $k-1$ edges for each $\mathcal{T}_K$ is enough to give the desired bound (annoying but doable).

Going Further

Theorem (English-S. 2025+)

For any tree T, integer $k \geq 1$, and family of graphs $\mathcal{F}$, either

$$\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^k),$$

or every $\mathcal{F}$-free graph is $\mathcal{F}^q_{T,k}$-free and

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and this holds for bipartite graphs without isolated edges by König's Theorem.

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Is it the case that for every bipartite graph H without isolated vertices, integer k, and family of graphs $\mathcal{F}$ that either

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Fact

This is false for C_4 and $k = 2$:(

Going Further

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Every rational in $[1, 2]$ *is realizable for* $H = K_2$.

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Theorem (English-Halfpap-Krueger 2024)

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Theorem (English-Halfpap-Krueger 2024)

Every rational in $[t, t+1]$ *is realizable for* $H = K_{1,t}$ *and no rational in* $(1, t)$ *is.*

Going Further

Theorem (English-S. 2025+)

For every graph H of maximum degree Δ, every rational in the range

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This is best possible for stars.

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We believe we can verify that this is true for all graphs on at most 4 vertices through various ad hoc techniques.

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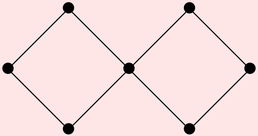
We have $\mathrm{ex}(n, P_7, F) = O(n^4)$ *where* F *is two* C_4*'s sharing a vertex.*



This is easy if we avoid just C_4: we can specify the last two edges of P_7 and x_4 in $(n^{3/2})^2 \cdot n = n^4$ ways, and this uniquely determines the P_7.

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The idea here is that any F of this form must be a subgraph of every member of $\mathcal{F}^q_{H,\alpha(H)} \neq \emptyset$ for some q. For any fixed q this is a finite number of possibilities, and I don't think arbitrarily large q should give new behaviors (analogous to $F \subseteq K_{2,t}$ implying $\mathrm{ex}(n, F) = \Theta(n^r)$ for some $r \in \{0, 1, 3/2\}$).

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Question

Can one prove that if $T_1, \ldots$ are "reasonable" subtrees of a tree T which pairwise intersect in at least one path of length 2, then there exists a path of length 2 common to each T_i?

Open Problems

Question

Can one say anything about generalized Turán numbers of trees T satisfying

$$n^{\ell+1} \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{\ell+2}?$$

For this it would suffice to prove upper bounds on $\mathrm{ex}(n, \mathcal{F}^q_{T,\ell+2})$.

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Proposition

If G contains $kn^{2-1/s}$ edges with $k = \omega(1)$, then it contains at least $\Omega(k^{st}n^s)$ copies of $K_{s,t}$.

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Question

Does the same phenomenon hold if G contains many copies of K_r?

Open Problems: Supersaturation

Conjecture (Dubroff-Gunby-Narayanan-S.)

If $2 \leq r \leq s \leq t$ *and if* G *contains at least* $kn^{r-\binom{r}{2}/s}$ *copies of* K_r*, then it contains at least* $k^{st/\binom{r}{2}}n^{s-o(1)}$ *copies of* $K_{s,t}$*.*

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Conjecture (Dubroff-Gunby-Narayanan-S.)

For all $1 \leq k \leq n^{1/2t}$, there exist n-vertex graphs with $kn^{3/2}$ triangles and at most $k^t n^{3/2+o(1)}$ copies of $K_{2,t}$.