# Eulerian Polynomials of Digraphs

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Joint with Kyle Celano and Nicholas Sieger

# Permutation Statistics

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A *descent* of a permutation $\sigma$ on the set $[n] := \{1, 2, \ldots, n\}$ is an index $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. An *inversion* is a pair of integers $(i, j)$ with $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$.

$$\sigma = 23154$$

$$\mathrm{Des}(\sigma) = \{2, 4\}, \ \mathrm{Inv}(\sigma) = \{(1,3), (2,3), (4,5)\}$$

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The generating functions

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} \quad M_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}(\sigma)}$$

are called Eulerian polynomials and Mahonian polynomials, respectively.

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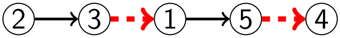
A *permutation* of an $n$-vertex digraph $D = (V, E)$ is a bijection $\sigma : V \to [n]$. A *descent* of such a permutation is an arc $u \to v$ such that $\sigma(u) > \sigma(v)$.

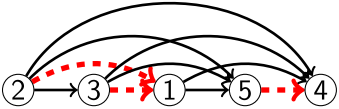
# A Common Generalization

## Claim (Foata-Zeilberger)

*If* $D = \overrightarrow{P}_n$*, then* $D$*-descents are exactly descents. Similarly* $D = \overrightarrow{K}_n$ *corresponds to inversions.*




(a) $\mathrm{des}(23154) = 2$

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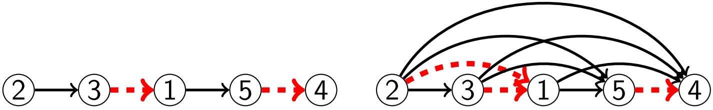


(a) $\mathrm{des}(23154) = 2$ (b) $\mathrm{inv}(23154) = 3$

For a digraph $D = (V, E)$, we define

$$A_D(t) = \sum_{\sigma \in \mathfrak{S}_D} t^{\mathrm{des}_D(\sigma)}. \tag{1}$$

In particular, $A_{\overrightarrow{P_n}}(t) = A_n(t)$ and $A_{\overrightarrow{K_n}}(t) = M_n(t)$.

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and the later being the number of *correct proofs of the Riemann hypothesis*[1].

[1] As of the time of writing.

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**Question (Kalai 2002)**

What can be said about $\nu(G)$?

For example, $\nu(P_n) = |A_{\vec{P}_n}(-1)|$ is the number of alternating permutations.

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**Definition**

Given an $n$-vertex graph $G$, we say that an ordering $\pi = (\pi_1, \ldots, \pi_n)$ of the vertex set $V(G)$ is an *even sequence* if each of the subgraphs $G[\pi_1, \ldots, \pi_i]$ induced by the first $i$ vertices of $\pi$ have an even number of edges for all $1 \leq i \leq n$.


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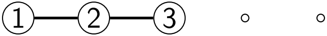

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*We have $\nu(G) = \eta(G)$ whenever $G$ is bipartite*

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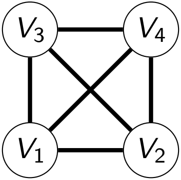


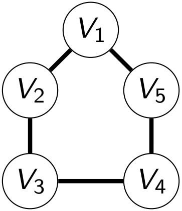

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*No!*

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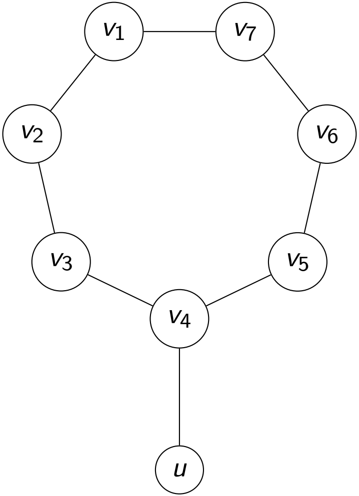
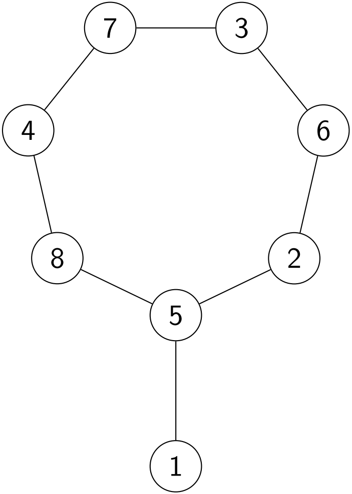
The following graph has $0 = \nu(G) < \eta(G) = 1088$.




(a) $C_7^*$




(b) An even sequence

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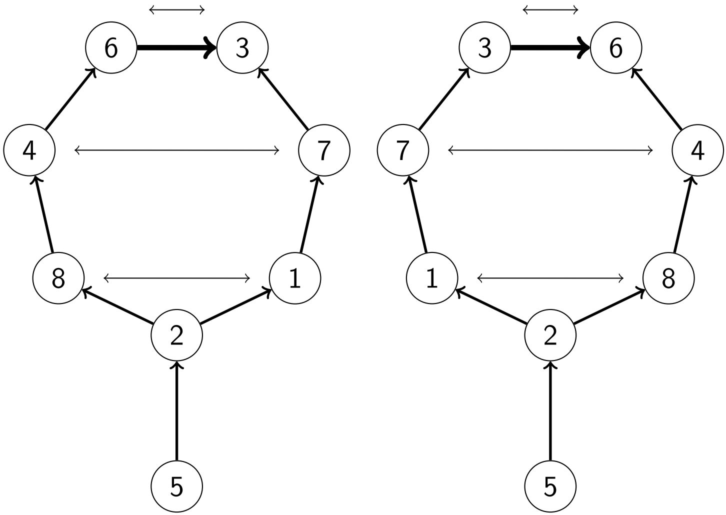

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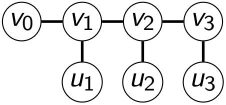
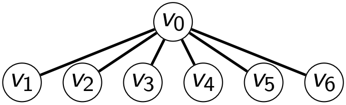
**Theorem (Celano, Sieger, S. 2023)**

*If $T$ is a tree on $2n+1$ vertices, then*

$$n!2^n \leq \nu(T) = \eta(T) \leq (2n)!$$

*Moreover, equality holds in the lower bound if and only if $T$ is a hairbrush, and equality holds in the upper bound if and only if $T$ is a star.*



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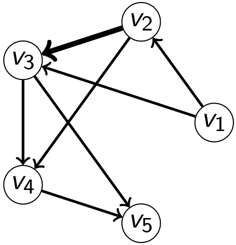


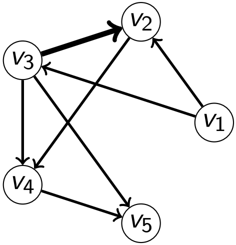


$A_{D_1}(t) = (1+t)^3(1+t+11t^2+t^3+t^4)$ $\quad$ $A_{D_2}(t) = (1+t)(1+5t+16t^2+16t^3+16t^4+5t^5+t^6)$

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Is $\lfloor \frac{n}{2} \rfloor$ the largest $\operatorname{mult}(A_D(t), -1)$ can be?

Note that tournaments have the largest (potential) degree for $A_D(t)$.

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**Theorem (Celano, Sieger, S. 2023)**

*If $D$ is an $n$-vertex digraph, then*

$$\mathrm{mult}(A_D(t), -1) \leq n - s_2(n),$$

*where $s_2(n)$ denotes the number of 1's in the binary expansion of $n$. Moreover, for all $n$, there exist $n$-vertex digraphs $D$ with*

$$A_D(t) = \frac{n!}{2^{n-s_2(n)}}(1+t)^{n-s_2(n)}.$$

The only extremal examples we know are "impartial digraphs", which is weird.

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**Claim**

*If $G$ has an even number of edges, then*

$$\eta(G) = \sum_v \eta(G - v).$$

# Proof Ideas

## Proposition

*If $G$ is a graph such that every induced subgraph $G' \subseteq G$ with an even number of edges satisfies*

$$\nu(G') = \sum \nu(G' - v),$$

*then $\nu(G) = \eta(G)$.*

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*For any digraph,*

$$A_D(t) = \sum_{v \in V} \frac{t^{\deg_D^+(v)} + t^{\deg_D^-(v)}}{2} A_{D-v}(t)$$

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The corollary will thus apply if we can show for any bipartite graph $G$, any/some orientation $D$ has the same sign for all of the terms above.

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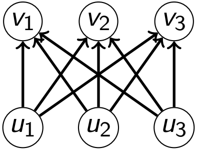

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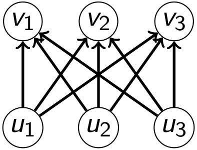


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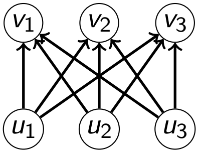


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$$\nu(G) = A_D(-1) = \sum_{v \in V(D)} A_{D-v}(-1) = \sum \nu(G - v).$$

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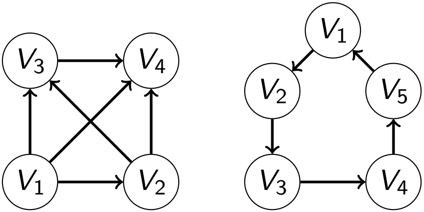
*There exists "natural" orientations for complete multipartite graphs/blowup of a cycles which makes it easy to predict the sign of $A_D(t)$.*



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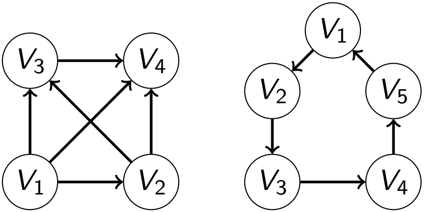


## Theorem (Celano, Sieger, S. 2023)

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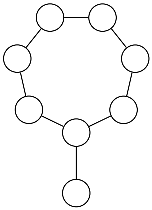

**Proposition (Celano, Sieger, S. 2023)**

*If $G$ is a connected graph, then $G$ is induced odd pan-free if and only if it is either bipartite, complete multipartite, or a blowup of a cycle.*

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Note that $\nu(G) \leq \eta(G)$ in general, so perhaps this is some special subset of even sequences.

**Conjecture**

*If $G$ is an Eulerian graph, then $\nu(G) = \sum_v \nu(G - v)$.*

# Open Problems

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**Problem**

For any bipartite graph $G = ([n], E)$ and orientation $D$ of $G$, construct an explicit involution $\phi : \mathfrak{S}_n \to \mathfrak{S}_n$ such that

(a) The set of fixed points $\mathcal{F}_\phi$ of $\phi$ is the set of (inverses of) even sequences of $G$, and

(b) $(-1)^{\mathrm{des}_D(\sigma)} = -(-1)^{\mathrm{des}_D(\phi(\sigma))}$ for all $\sigma \notin \mathcal{F}_\phi$.

# Open Problems

**Conjecture**

*If $D$ is the orientation of a complete multipartite graph which has $r$ parts of odd size, then* $\mathrm{mult}(A_D(t), -1) = \left\lfloor \frac{r}{2} \right\rfloor$.

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## Question

Does there exist a digraph $D$ such that $A_D(t)$ has an integral root which is not equal to either 0 or $-1$?

No such digraph exists on at most 5 vertices, and there exist digraphs with real roots of magnitude larger than 2.