Problems that I would like Somebody to Solve

Sam Spiro

May 17, 2021

Contents

1 What is this?

This is a (very informal) collection of problems that are of personal interest to me, and it is definitely not supposed to be an exhaustive list of the hardest/most interesting problems in math. Feel free to reach out to me if you have any questions regarding these questions and/or if you spot any glaring typos.

Organization. Section 2 contains the main problems that I'm interested in. Solving any of them will put you in the "Hall of Fame" of Section 4 as thanks for putting the problem to rest. Section 3 consists of some other problems that I would still appreciate being resolved, but not to the level that I'll sleep better at night knowing that they've been solved. In particular, many of these problems are relatively elementary to state, and I would not be surprised if several of them were easy to solve.

The order that the problems appear is essentially chronological. Note that problems can jump between being labeled "main" or "small" depending on my current mood, and I may remove problems from the list if I start actively working on them again.

2 The Main Problems

2.1 Small Quasikernels

Let D be a digraph. Given a set S, we define $N^+(S) = \bigcup_{v \in S} N^+(v)$, where $N^+(v)$ is the out-neighborhood of v. We say that a set $K \subseteq V(D)$ is a *kernel* of D if (1) $N^+(K) \cap K = \emptyset$ (that is, K is an independent set of the underlying graph of D), and (2) $N^+(K) \cup S = V(D)$ (that is, every vertex is either in K or can be reached by a vertex in K in one step).

Not every digraph has a kernel (take any directed cycle of odd length), but it is not too hard to prove that every digraph has a *quasikernel*. This is a set $Q \subseteq V(D)$ such that (1) $N^+(Q) \cap Q = \emptyset$ and such that (2) $N^+(N^+(Q)) \cup N^+(Q) \cup Q = V(D)$. That is, it is an independent set such that every vertex can be reached from Q in at most two steps.

Given that every digraph has a quasikernel, it is natural to ask how small of a quasikernel one can find. One quickly realizes that it can be quite large: any source of D must belong to a quasikernel of D. Thus the most natural setting to consider is when D has no sources, and in this case the following was conjectured by Erdős and Székely.

Conjecture 2.1. *Every digraph D with no sources has a quasikernel of size at most $|V(D)|/2$.*

Some progress has been made, see [5], but overall the conjecture is far from being resolved.

2.2 The Turán problem in Random Hypergraphs

Given two hypergraphs H, F, we define $\mathrm{ex}(H, F)$ to be the maximum number of edges in an F-free subgraph of H. We're particularly interested in the case when $H = H^r_{n,p}$, which is the random r-uniform hypergraph obtained by keeping each edge of K^r_n independently and with probability p. There are many, many open problems related to determining $\mathrm{ex}(H^r_{n,p}, F)$, see for example [8, 12, 13]; but here we'll focus on just one case.

Let C^3_4 be the 3-uniform loose 4-cycle, which can be defined by having edges

$$\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 7\}, \{7, 8, 1\}.$$

That is, it's obtained from the graph C_4 by inserting an extra vertex into each edge. A standard deletion argument shows that

$$\mathbb{E}[\mathrm{ex}(H^3_{n,n^{-2/3}}, C^3_4)] = \Omega(n^{4/3}),$$

and work of Mubayi and Yepremyan [7] shows

$$\mathbb{E}[\mathrm{ex}(H_{n,n^{-2/3}}, C^3_4)] \leq n^{13/9+o(1)}.$$

Problem 2.2. *Improve upon either of these bounds for* $\mathbb{E}[\mathrm{ex}(H^3_{n,n^{-2/3}}, C^3_4)]$.

Mubayi and Yepremyan [7] conjecture that the lower bound from the deletion argument is essentially best possible, and they provide some evidence that points in this direction. On the other hand, Nie, Verstraëte, and myself [8] proved that the analogous problem for loose triangles has a stronger lower bound than what you get from a deletion argument, which suggests that improvements might be possible for all loose cycles. Overall it's very unclear what the correct answer should be here.

3 Some Smaller Problems

3.1 Polynomial Relations Between Matrices of Graphs

The general question is: given two matrices X, Y associated to a graph G, does there exist an integer r and a polynomial f such that $X^r = f(Y)$? For example, let A be the adjacency matrix of G, D its diagonal matrix of degrees (i.e. $D_{uu} = d_u$), and $L = D - A$ its combinatorial Laplacian. If G is d-regular, then we have $A = dI - L$, so we get a positive answer with $r = 1$ and $f(x) = d - x$. As another exampmle, if G is (d_1, d_2)-biregular, meaning it's bipartite with every vertex of one part having degree d_1 and the other all having degree d_2, then one can check that $A^2 = (d_1 I - L)(d_2 I - L)$, so this gives another family of examples.

Somewhat surprisingly, the only graphs which have $A^r = f(L)$ for some r, f are the regular and biregular graphs [10]. In fact, I was able to prove that a similar phenomenon holds for any $X, Y \in \{A, L, Q, \mathcal{L}\}$ where $Q = D + A$ is the signless Laplacian and $\mathcal{L} = D^{-1/2} L D^{-1/2}$ is the normalized Laplacian, ***except*** for one irksome case.

Conjecture 3.1 ([10])**.** *If G is a graph and $A^r = f(\mathcal{L})$ for some integer $r \geq 1$ and polynomial f, then G is either regular or biregular.*

I wrote [10] after knowing only some very basic spectral graph theory, so every once in a while I think "this problem probably wasn't really that hard. I should be able to solve it without much effort." As you might have guessed, this has not turned out to be the case!

The most promising approach to solving Conjecture 3.1 that I came up with was the following.

Conjecture 3.2 ([10])**.** *Let* $\mathbf{1}$ *denote the all 1's vector. If G is connected and $D^{1/2}\mathbf{1}$ is an eigenvector of A^2, then G is regular or biregular.*

In [10] I prove that this conjecture implies Conjecture 3.1, and computational data strongly suggests that this result is true.

3.2 Slow Tribonacci Walks

Given a triple of positive integers (w_1, w_2, w_3), recursively define $w_k = w_{k-1} + w_{k-2} + w_{k-3}$ for $k \geq 4$. We say that (w_1, w_2, w_3) is an *n-tribonacci walk* if $w_s = n$ for some s. There are infinitely many n-tribonacic walks, e.g. those of the form $(42, n, x)$ for any x. To make things more interesting, we say that (w_1, w_2, w_3) is an *n-slow tribonacci walk* if $w_s = n$ with s as large as possible. For example, $(1,1,1)$ and $(42,3,42)$ are both 3-tribonacci walks, but only the first one is slow. Let $p(n)$ denote the number of n-slow tribonacci walks. For example, it's easy to check that $p(3) = 1$, and $p(1) = p(2) = \infty$

Question 3.3. *Does there exist some absolute constant c such that either $p(n) = \infty$ or $p(n) \le c$ for all n?*

If we instead look at Fibonacci walks (which are defined using the Fibonacci recurrence $w_k = w_{k-1} + w_{k-2}$), then one can show that $p(n) \le 2$ for all $n \ge 2$ [2]. More generally if one looks at walks following a two-term recurrence of the form $w_k = \alpha w_{k-1} + \beta w_{k-2}$ with $\alpha, \beta \ge 1$ relatively prime, then $p(n) \le \alpha^2 + 2\beta - 1$ for all but finitely many n [11].

Computational data made it easy to conjecture the correct answer for two-term recurrences, but the situation is less clear for slow tribonacci walks. For example, $p(61) = 9$, which is fairly large given how small 61 is.

I'll note that there are many other interesting problems related to the behavior of slow recurrences that were left unanswered in [2, 11]. However, as is often the case in number theory, these relatively easy to state questions are likely very difficult to solve. This being said, I do think that this tribonacci problem is tractable.

3.3 An Adversarial Chernoff Bound

Persi Diaconis, Ron Graham, Xiaoyu He, and myself [3] proved the following.

Theorem 3.4 ([3])**.** *Let X_i be independent Bernoulli random variables with $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$. Let $S_t = \sum_{i \le t} X_i$. There exist absolute constants c_0, c_1 such that for all $\lambda > 0$ and integers $k_1 \ge k_0 \ge 2\lambda^{-1}$,*

$$\Pr[\exists t \in [k_0, k_1] : |S_t - pt| \ge \lambda pt] \le \frac{c_0 k_1}{k_0} e^{-c_1 \lambda^3 p k_0}.$$

That is, with high probability, for every t in the interval $[k_0, k_1]$, every partial sum S_t is close to its expectation. This is immediate for any given value of t by the Chernoff bound (since each S_t is a binomial random variable), but it does *not* follow from the Chernoff bound and a naive application of the union bound (this gives a bound like $k_1 e^{-\lambda^2 p k_0}$, which is much weaker if k_0 is very large).

Question 3.5. *Does the bound of Theorem 3.4 hold with λ^2 instead of λ^3?*

Note that λ^2 would be best possible because this is what one gets if $k_0 = k_1$. While the statement of Theorem 3.4 is fairly technical, the proof itself only required a slightly clever application of the union bound together with the Chernoff bound, so my hope is that more sophisticated probabilistic tools can be used to solve Question 3.5 without too much difficulty.

Secretly I'm interested in this because it would improve upon the error term for our main result in [3], but also I just think it's of independent interest to determine how much concentration one can get for an "adversarial" binomial distribution.

3.4 Saturation Games

For a family of graphs $\mathcal{F}$, we say that a graph G is $\mathcal{F}$-saturated if G contains no graph of $\mathcal{F}$ as a subgraph, but adding any edge to G would create a subgraph of $\mathcal{F}$. The $\mathcal{F}$-saturation game consists of two players, Max and Mini, who alternate turns adding edges to an initially empty graph G on n vertices (say with Max starting), with the only restriction being that G is never allowed to contain a subgraph that lies in $\mathcal{F}$. The game ends when G is $\mathcal{F}$-saturated. The payoff for Max is the number of edges in G when the game ends, and Mini's payoff is the opposite of this. We let $\mathrm{sat}_g(n,\mathcal{F})$ denote the number of edges that the graph in the $\mathcal{F}$-saturation game ends with when both players play optimally, and we call this quantity the game $\mathcal{F}$-saturation number.

Bounding $\mathrm{sat}_g(\mathcal{F};n)$ seems to be pretty hard in general, and even the original problem of determining $\mathrm{sat}_g(n,C_3)$ is still wide open. See [9] for further history and known bounds. In [9] I proved $\mathrm{sat}_g(n,\mathcal{C}_\infty^o\setminus\{C_3\})=O(n)$, where $\mathcal{C}_\infty^o$ is the set of all odd cycles. I also proved (somewhat indirectly) that $\mathrm{sat}_g(n,\mathcal{C}_\infty^o\setminus\{C_{2k+1}\})=\Omega(n^2)$ for $k\ge 3$. Given this, it is natural to ask the following.

Problem 3.6. *Prove non-trivial bounds on* $\mathrm{sat}_g(n,\mathcal{C}_\infty^o\setminus\{C_5\})$*, where* $\mathcal{C}_\infty^o=\{C_3,C_5,C_7,\ldots\}$.

I'd be happy to have even an $\omega(n)$ lower bound or any non-trivial asymptotic upper bound for this problem. Possibly a more tractable problem is the following.

Problem 3.7. *Prove non-trivial bounds on* $\mathrm{sat}_g(n,C_k)$ *for* $k>3$.

For odd k, I proved an asymptotic upper bound of $\mathrm{sat}(n,C_k)\le\frac{4}{27}n^2+o(n^2)$. In [1] the authors proved a non-trivial lower bound for C_4 if you play a "bipartite" version of the game, but I'd still like to see bounds proved in the original setting. Lastly, I'd like to know the following.

Question 3.8. *Does there exist a finite set of (odd) cycles* $\mathcal{C}$ *such that* $\mathrm{sat}_g(n,\mathcal{C})=O(n)$*?*

4 Hall of Fame

A list of people who have successfully solved any of my main problems.

- Wang and Zhao [14] for solving my original conjecture on ballot permutations, and Lin, Wang, and Zhao [6] for solving an even stronger version.

References

[1] James M Carraher, William B Kinnersley, Benjamin Reiniger, and Douglas B West. The game saturation number of a graph. *Journal of Graph Theory*, 85(2):481–495, 2017.

[2] Fan Chung, Ron Graham, and Sam Spiro. Slow fibonacci walks. *Journal of Number Theory*, 210:142–170, 2020.

[3] Persi Diaconis, Ron Graham, Xiaoyu He, and Sam Spiro. Card guessing with partial feedback. *arXiv preprint arXiv:2010.05059*, 2020.

[4] J. D. Horton and A. Kurn. Counting sequences with complete increasing subsequences. *Congressus Numerantium*, 33:75–88, 1981.

[5] Alexandr Kostochka, Ruth Luo, and Songling Shan. Towards the small quasi-kernel conjecture. *arXiv preprint arXiv:2001.04003*, 2020.

[6] Zhicong Lin, David G.L. Wang, and Tongyuan Zhao. A decomposition of ballot permutations, pattern avoidance and gessel walks. *arXiv preprint arXiv:2103.04599*, 2021.

[7] Dhruv Mubayi and Liana Yepremyan. Random turán theorem for hypergraph cycles. *arXiv preprint arXiv:2007.10320*, 2020.

[8] Jiaxi Nie, Sam Spiro, and Jacques Verstraete. Triangle-free subgraphs of hypergraphs. *arXiv preprint arXiv:2004.10992*, 2020.

[9] Sam Spiro. Saturation games for odd cycles. *arXiv preprint arXiv:1808.03696*, 2018.

[10] Sam Spiro. Polynomial relations between matrices of graphs. *Journal of Graph Theory*, 90(3):288–303, 2019.

[11] Sam Spiro. Slow recurrences. *arXiv preprint arXiv:1909.06517*, 2019.

[12] Sam Spiro and Jacques Verstraëte. Counting hypergraphs with large girth. *arXiv preprint arXiv:2010.01481*, 2020.

[13] Sam Spiro and Jacques Verstraëte. Relative turán problems for uniform hypergraphs. *arXiv preprint arXiv:2009.02416*, 2020.

[14] David GL Wang and Tongyuan Zhao. The peak and descent statistics over ballot permutations. *arXiv preprint arXiv:2009.05973*, 2020.