1. BASIC CONCEPTS

Following tabular data shows the number of cars each dealer sold in a month.

	Dealer 1	Dealer 2	Dealer 3
Model A	4	2	5
Model B	3	4	2
Model C	6	1	3

We can organize the tabular data in the form of

$$\begin{array}{rc} & \begin{matrix} D_1 & D_2 & D_3 \end{matrix} \\ \begin{matrix} \textit{Model A} \\ A = \textit{Model B} \\ \textit{Model C} \end{matrix} & \begin{bmatrix} 4 & 2 & 5 \\ 3 & 4 & 2 \\ 6 & 1 & 3 \end{bmatrix} \end{array}$$

A **matrix** is a rectangular arrangements of numbers in rows and columns.

$$\underbrace{\left.\begin{bmatrix} a_{11} & a_{12} & a_{13}\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23}\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}\cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3}\cdots & a_{mn} \end{bmatrix}\right\} m \text{ rows}}_{n \text{ colums}}$$

rows → horizontal lines

columns → vertical lines

dimension (order) → number rows and number of columns

$m \times n$ → the matrix with m rows and n columns

$$\begin{array}{cc} & \text{3 columns} \\ A = & \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \end{bmatrix} \leftarrow \text{2 rows} \end{array}$$

Matrix A is a 2 x 3 ('two by three') matrix

entry → each number in the matrix

a_{ij} → the entry in the i^{th} row and j^{th} column of matrix

a_{13}: 1st row, 3rd column

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \end{bmatrix}$$

a_{13} is the entry in the first row and the third column: $a_{13} = 4$

Example: Write the dimension of each matrices and given entries.

Matrix	Dimension	Entry
$A = \begin{bmatrix} -1 & 3 \\ 5 & 6 \end{bmatrix}$		$a_{12} =$ $a_{22} =$
$B = \begin{bmatrix} k \\ -9 \\ 4 \end{bmatrix}$		$b_{11} =$ $b_{13} =$ $b_{21} =$
$C = \begin{bmatrix} 14 & \pi & 0 & -2.5 \end{bmatrix}$		$c_{14} =$ $c_{21} =$
$D = \begin{bmatrix} 1 & 5 & -4 & 1 & 7 \\ -3 & 7 & 7 & -3 & 8 \\ 4 & 3 & 3 & 1 & 10 \\ 12 & -1 & 9 & 4 & 4 \end{bmatrix}$		$d_{23} =$ $d_{42} =$ $d_{35} =$ $d_{41} =$

Example: Write the $A_{4\times4}$ matrix where $A = [a_{ij}]$ such that $a_{ij} = j^{i+1}$.

2. TYPES of MATRICES

Square matrix is a matrix with same number of rows and columns.

$\begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$ → 2×2 → dimension (order) is 2

$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → 3×3 → dimension (order) is 3

Rectangular matrix is a matrix with different number of rows and columns.

$\begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & 9 \end{bmatrix}$ → 2×3

Row matrix is a matrix with only one row.

$\begin{bmatrix} -2 & 4 & 8 & 0 & 3 \end{bmatrix}$ → 1×5

Column matrix is a matrix with only one column.

$\begin{bmatrix} 9 \\ -2 \\ 1 \end{bmatrix}$ → 3×1

Zero matrix is a matrix whose entries are all zero.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Identity matrix is a square matrix whose main diagonal elements are 1 and whose other elements are all zero.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not an identity

Diagonal matrix is a square matrix in which all the entries except the main diagonal entries are zero.

$\begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Triangular matrix is a square matrix in which all the entries either above or below the main diagonal are zero.

$\begin{bmatrix} 5 & 3 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 6 & 2 & 0 \\ -1 & -6 & 9 \end{bmatrix}$

3. EQUAL MATRICES

Two matrices (A and B) are equal if they have the same dimension and their corresponding entries are equal.

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix} \neq \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ because } (1\times 3)\neq(3\times 1)$$

Example: Find x, y, z

$$\begin{bmatrix}x-y & 2\\ 5 & 1\end{bmatrix}=\begin{bmatrix}4 & 2\\ z & y-2\end{bmatrix}$$

Example: (UN 2010 PAKET A)

$$A=\begin{pmatrix}4a & 8 & 4\\ 6 & -1 & -3b\\ 5 & 3c & 9\end{pmatrix},\ B=\begin{pmatrix}12 & 8 & 4\\ 6 & -1 & -3a\\ 5 & b & 9\end{pmatrix}$$

If $A=B$, then $a+b+c=$

4. OPERATIONS ON MATRICES

A. Addition

Only matrices with equal dimensions can be added (subtracted)

To add (subtract) matrices, just add (subtract) the corresponding elements.

Example: Given

$$A=\begin{bmatrix}2 & -1 & 7\\ -3 & 0 & 4\end{bmatrix},\ B=\begin{bmatrix}0 & 4 & -5\\ 5 & -3 & 2\end{bmatrix},\ C=\begin{bmatrix}2 & 1 & -4\\ 8 & 0 & 1\end{bmatrix}$$

$A+B=$

$A-B=$

$A+B+C=$

$A-C+B=$

Example: (UN 2012/B25)

$$A=\begin{pmatrix}3 & y\\ 5 & -1\end{pmatrix},\ B=\begin{pmatrix}x & 5\\ -3 & 6\end{pmatrix},\ C=\begin{pmatrix}-3 & -1\\ y & 9\end{pmatrix}$$

If $A+B-C=\begin{pmatrix}8 & 5x\\ -x & -4\end{pmatrix}$ then $x+2xy+y=\ldots$

Properties of Addition

Let A, B, C be matrices with $m\times n$ dimension.

1. $A+B$ is also $m\times n$ matrix.
2. $A+B=B+A$
3. $A+(B+C)=(A+B)+C$
4. $A+0=A$
5. $A+(-A)=(-A)+A=0$

B. Multiplication by Scalar

To multiply matrices by a scalar, just multiply each entries by the scalar.

Example: Given

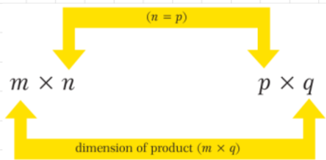
$$A=\begin{bmatrix}0 & -2\\ 1 & 4\\ -5 & 2\end{bmatrix},\ B=\begin{bmatrix}3 & 12\\ 6 & 4\\ 0 & -1\end{bmatrix}$$

$3A=$ $\qquad\qquad\qquad$ $-B=$

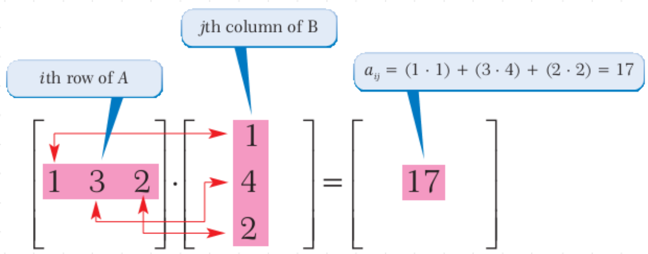
$2A-3B=$

C. Multiplication of Matrices

If matrix A has dimension $m\times n$ and B has dimension $p\times q$, then $A\cdot B$ only exists if $n=p$.



For $A\cdot B$, we use rows of A and columns of B as follows:



Example: Given

$$A=\begin{bmatrix}-1 & 5 & 1\\ -3 & 2 & 0\end{bmatrix},\ B=\begin{bmatrix}3 & -2\\ 2 & 5\\ 1 & -1\end{bmatrix}$$

$A\cdot B=$

$B\cdot A=$

Example: Given

$$A=\begin{bmatrix}-2 & 7\\ 1 & 5\end{bmatrix},\ B=\begin{bmatrix}3 & 1\\ -2 & 2\end{bmatrix}$$

$A\cdot B=$ $\qquad\qquad$ $B\cdot A=$

Example: (UN 2010 PAKET B)

$$A=\begin{pmatrix}-c & 2\\ 1 & 0\end{pmatrix},\ B=\begin{pmatrix}4 & a\\ b+5 & -6\end{pmatrix},\ C=\begin{pmatrix}-1 & 3\\ 0 & 2\end{pmatrix},\ D=\begin{pmatrix}4 & b\\ -2 & 3\end{pmatrix}$$

If $2A-B=CD$, then $a+b+c=$

Properties of Matrix Multiplication

Let A,B,C be matrices whose products are defined and $k\in\mathbb{R}$.

1. $A\cdot(B\cdot C)=(A\cdot B)\cdot C$
2. $A\cdot(B+C)=A\cdot B+A\cdot C$ and $(A+B)\cdot C=A\cdot C+B\cdot C$
3. $k\cdot(A\cdot B)=(k\cdot A)\cdot B=A\cdot(k\cdot B)$
4. In general, $A\cdot B\neq B\cdot A$
5. If $A\cdot B=A\cdot C$, then in general $B\neq C$.
6. If A is a square matrix and $n\in\mathbb{N}$ then $A^0=I$, $A^1=A$,
   $A^2=AA$, $A^3=AA^2$, ... , $A^n=AA^{n-1}$
7. $A\cdot I=I\cdot A$

Example: $X=\begin{bmatrix}-2 & 0\\ 1 & 5\end{bmatrix}\Rightarrow X^2=$

Example: $A=\begin{bmatrix}0 & 3\\ 2 & 0\end{bmatrix}\Rightarrow A^{206}=$

5. THE TRANSPOSE OF A MATRIX

The **transpose** of a matrix A is formed by writing its columns as rows and denoted by A^T (or A').

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\cdots & a_{1n}\\ a_{21} & a_{22} & a_{23}\cdots & a_{2n}\\ a_{31} & a_{32} & a_{33}\cdots & a_{3n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{m1} & a_{m2} & a_{m3}\cdots & a_{mn}\end{bmatrix}_{m\times n}\Rightarrow A^T=\begin{bmatrix}a_{11} & a_{21} & a_{31}\cdots & a_{m1}\\ a_{12} & a_{22} & a_{32}\cdots & a_{m2}\\ a_{13} & a_{23} & a_{33}\cdots & a_{m3}\\ \vdots & \vdots & \vdots & \vdots\\ a_{1n} & a_{2n} & a_{3n}\cdots & a_{mn}\end{bmatrix}_{n\times m}$$

$$A=\begin{bmatrix}-5 & 7 & 1\\ 3 & 6 & -2\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}-5 & 3\\ 7 & 6\\ 1 & -2\end{bmatrix}$$

Example: Write the transpose of each matrices.

$$A=\begin{bmatrix}-4 & 1 & -2\\ 3 & 6 & 9\\ 5 & -4 & 7\end{bmatrix}\Rightarrow A^T=$$

$$B=\begin{bmatrix}4\\ -1\\ 5\end{bmatrix}\Rightarrow B^T=$$

$$C=\begin{bmatrix}3 & -7 & 2\end{bmatrix}\Rightarrow C^T=$$

Example: (UN 2007 PAKET B)

If $A=\begin{pmatrix}x+y & x\\ y & x-y\end{pmatrix}$, $B=\begin{pmatrix}1 & -\frac{1}{2}x\\ -2y & 3\end{pmatrix}$, and $A^T=B$, then

$x+2y=$ ...

Properties of Matrix Transposition

$c\in\mathbb{R}$

1. $(A+B)^T=A^T+B^T$
2. $(A^T)^T=A$
3. $(c\cdot A)^T=c\cdot A^T$
4. $(A\cdot B)^T=B^T\cdot A^T$

6. INVERSE OF A MATRIX

The inverse of A is denoted by A^{-1}.

$AA^{-1}=A^{-1}A=I$

A matrix which has an inverse is called an **invertible matrix**.

A matrix which does not have an inverse is called a **noninvertible (or singular) matrix**.

Example: Show that A and B are inverses of each other.

$$A=\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}$$

Example: Find the inverse of

$A=\begin{bmatrix}-1 & 3\\ 1 & -2\end{bmatrix}$ (by assuming $A^{-1}=\begin{bmatrix}x & y\\ z & t\end{bmatrix}$)

Inverse of 2 x 2 Matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$.

If the inverse exists, then $A^{-1} = \dfrac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example: Find the inverse of $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

Example: For which value of x does the matrix $A = \begin{bmatrix} x-1 & 2 \\ 6 & -3 \end{bmatrix}$ have no inverse?

7. DETERMINANT OF A MATRIX

Every square matrix can be assigned a real number which is called the **determinant** of the matrix.

Determinant of 2 x 2 Matrix

Determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\boldsymbol{ad - bc}$.

$$\overset{+\qquad\quad -}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = ad - bc$$

Example: Evaluate the determinant of each matrices.

- $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$
- $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$
- $\begin{bmatrix} 2-k & -3 \\ k & 3 \end{bmatrix}$
- $\begin{bmatrix} 103 & 101 \\ 102 & 100 \end{bmatrix}$

Determinant of 3 x 3 Matrix

3×3 Matrix determinant can be found by Sarrus Method as following.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$- \quad - \quad - \qquad\qquad + \quad + \quad +$$

Example: Evaluate the determinant of matrices below.

- $\begin{bmatrix} -1 & 0 & 2 \\ 4 & -2 & 3 \\ 1 & 5 & 4 \end{bmatrix}$
- $\begin{bmatrix} -1 & 0 & 2 \\ 4 & -2 & 3 \\ 1 & 5 & 4 \end{bmatrix}$

8. LINEAR EQUATION BY MATRICES

Let's solve the following system of linear equations

$$\begin{aligned} x - 2y &= 5 \\ x + y &= 8 \end{aligned} \quad \text{(by elimination)}$$

Matrix representation of system above is:

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

That is, $A \cdot X = B$

So, $X = A^{-1} \cdot B$

Example: Solve the following system of equations

$$\begin{aligned} 2x - y &= 3 \\ -x + 3y &= 1 \end{aligned}$$

Alternatively, Sarrus can be used to find solution.

$$\begin{aligned} 2x - y &= 3 \\ -x + 3y &= 1 \end{aligned} \Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ then}$$

$$x = \frac{\det \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}}{\det \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}} \qquad y = \frac{\det \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}}{\det \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}}$$

Example: Solve the following system of equations.

- $$\begin{aligned} x-y+3z&=4 \\ x+2y-2z&=10 \\ 3x-y+5z&=14 \end{aligned}$$

- $$\begin{aligned} x-2y+3z&=5 \\ 2x-4y+6z&=3 \\ 2x-3y+z&=9 \end{aligned}$$

Active Note Book

Review Test

1. If $A=\begin{pmatrix} 2 & -3 & 4 \\ 5 & 0 & -7 \\ 2 & 4 & 0 \end{pmatrix}$, then find the value of $(a_{21}+a_{32}-a_{11})$.

A) 2 B)4 C) 5 D) 7 E) 10

2. $A=\begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix} \Rightarrow 3A-2I_{2x2}=?$

A) $\begin{bmatrix} 9 & 6 \\ 3 & -6 \end{bmatrix}$ B) $\begin{bmatrix} 7 & 4 \\ 1 & -8 \end{bmatrix}$ C) $\begin{bmatrix} 7 & 6 \\ 3 & -8 \end{bmatrix}$

D) $\begin{bmatrix} 11 & 8 \\ 5 & -4 \end{bmatrix}$ E) $\begin{bmatrix} 9 & 4 \\ 1 & -6 \end{bmatrix}$

3. $A=\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ are given. If $A\cdot C=C+2\cdot B$,

then which one the following is the matrix C ?

A) $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$ B) $\begin{bmatrix} 10 \\ -6 \end{bmatrix}$ C) $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$

D) $\begin{bmatrix} -6 & 8 \\ 10 & 6 \end{bmatrix}$ E) $\begin{bmatrix} -3 & 4 \\ 5 & 3 \end{bmatrix}$

4. $A=\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ and *x+y+z+t*=3. If $A^T+A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

a+b+c+d=?

A)3 B)4 C)5 D)6 E)7

5. $A=\begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$ are given. If $X=2A-B$,

find X^{-1}.

A) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ B) $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ C) $\begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$

D) $\begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}$ E) $\begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}$

6. If $A=\begin{bmatrix}2 & 1\\ 0 & x\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}\frac{1}{2} & -\frac{3}{2}\\ 0 & 3\end{bmatrix}$, then $x=?$

A)-1/2 B)-1/3 C)1/3 D)1/2 E) 2/3

7. If $A=\begin{bmatrix}x+1 & \frac{2}{3}\\ -\frac{2}{3} & y\end{bmatrix}$, $A^{-1}=A^{T}$ then what is the value of $y-x$?

A)1 B) $\sqrt{5}$ C) $1-\sqrt{5}$ D) $\frac{\sqrt{5}}{3}$ E) $\frac{\sqrt{5}-1}{3}$

8. $A=\begin{bmatrix}5 & 0\\ a & -5\end{bmatrix} \Rightarrow A^{50}=?$

A) $5^{50}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ B) $5^{100}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ C) $25^{50}\begin{bmatrix}1 & 0\\ a & 1\end{bmatrix}$

D) $\begin{bmatrix}5^{50} & 0\\ a^{50} & 5^{50}\end{bmatrix}$ E) $5^{50}\begin{bmatrix}5 & 0\\ a & -5\end{bmatrix}$

9. $\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix}+x\begin{bmatrix}2\\ -1\end{bmatrix}=\begin{bmatrix}6\\ 5\end{bmatrix} \Rightarrow x+y=?$

A)3 B)4 C)5 D) 6 E)7

10. If $\begin{vmatrix}x & 3\\ y & -1\end{vmatrix}=4$ and $\begin{vmatrix}3 & -4\\ y & x\end{vmatrix}=3$ then $x-y=?$

A)2 B)5 C)7 D)8 E)9

11. If $A=\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}$, what is $(A+B)^2$?

A) $\begin{pmatrix}4 & 0\\ 6 & 9\end{pmatrix}$ B) $\begin{pmatrix}4 & 0\\ 6 & -9\end{pmatrix}$ C) $\begin{pmatrix}4 & 0\\ -12 & 16\end{pmatrix}$

D) $\begin{pmatrix}4 & 0\\ -6 & -9\end{pmatrix}$ E) $\begin{pmatrix}-4 & 0\\ 6 & 9\end{pmatrix}$

12. If $A=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ satisfies the equation of $A^2=pA+qI$, then $p-q=?$

A) 16 B)9 C) 8 D) 3 E) -1

13. $A=\begin{pmatrix}5+x & x\\ 5 & 3x\end{pmatrix}$ and $B=\begin{pmatrix}9 & -x\\ 7 & 4\end{pmatrix}$ are given. If the determinant of A and B are equal, then find the possible values of x.

A) 3 or 4 B)-3 or 4 C) -4 or 3 D) -4 or 5 E) -5 or 3

14. Find the determinant of $\begin{pmatrix}1 & 0 & 2\\ 3 & 1 & -1\\ 2 & 5 & 1\end{pmatrix}$.

A) 24 B) 26 C) 28 D) 30 E) 32

15. Given matrices $A=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$, $B=\begin{pmatrix}-1 & 2\\ 5 & 6\end{pmatrix}$ and $C=\begin{pmatrix}a & -1\\ 2 & 3\end{pmatrix}$. If the determinant of the matrix of $2A-B+3C$ is 10, find the value of *a*.

A) -5 B)-3 C) -2 D) 2 E) -5

16. $\left.\begin{aligned}3x-y+z&=9\\ -2x+y-2z&=-8\\ x+2y+5z&=3\end{aligned}\right\}$ then $x+y+z=?$

A)0 B)1 C)2 D)3 E) None of them