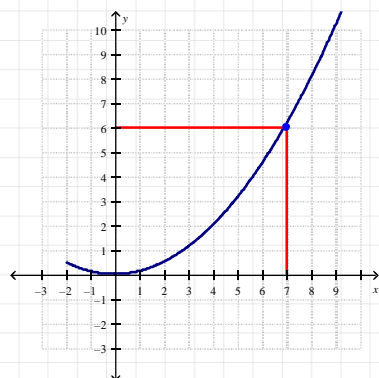


# 1. LIMIT of a FUNCTION

Limit is the value that a function **approaches** as the input approaches some certain value.

Let's understand the concept of limit by observing following function graph.

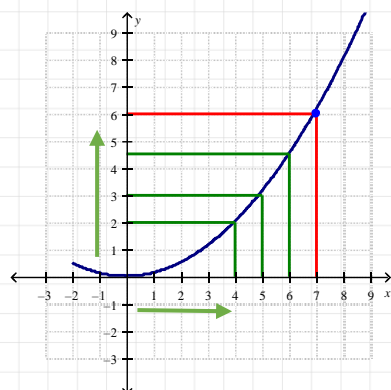


In this graph, at  $x = 7$  function has a value of  $f(7) = 6$ .

We can approach  $x = 7$  point from two directions on the x-axis.

- 1) From left (as  $x$  values getting bigger).
- 2) From right (as  $x$  values getting smaller).

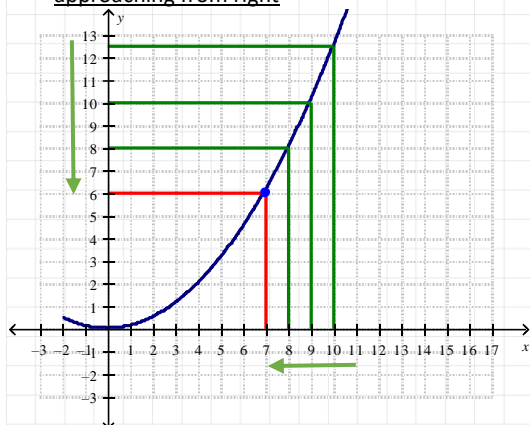
approaching from left



Observe that as  $x$  values approach to  $x = 7$ ,  $f(x)$  values approach to 6.

This is called left hand limit of  $f(x)$  at  $x = 7$  and denoted as  $\lim_{x \rightarrow 7^-} f(x) = 6$ .

approaching from right



Observe that as  $x$  values approach to  $x = 7$ ,  $f(x)$  values approach to 6.

This is called right hand limit of  $f(x)$  at  $x = 7$  and denoted as  $\lim_{x \rightarrow 7^+} f(x) = 6$ .

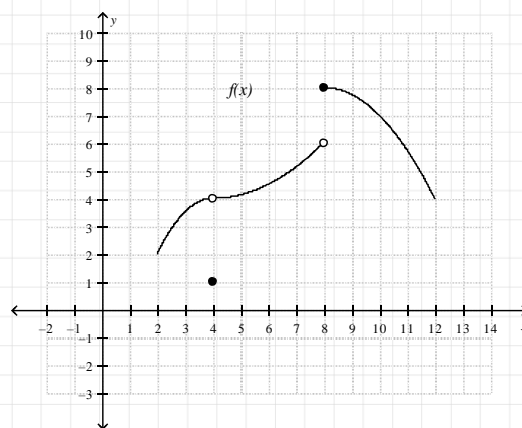
**Existence of a Limit:**

The limit of a function  $f(x)$  at a point  $x_0$  exists if and only if the right-hand and left-hand limits at  $x_0$  exist and are equal.

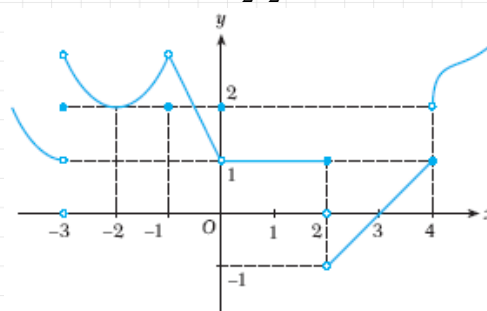
That is;

$$\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = L \text{ and } \lim_{x \rightarrow x_0^+} f(x) = L$$

**Example:** Examine the limit of following  $f(x)$  where  $f: [2, 12] \rightarrow \mathbb{R}$  function at  $x = 2$ ,  $x = 4$ ,  $x = 7$ ,  $x = 8$ ,  $x = 12$ .



**Example:** The graph of  $f$  is shown in the figure. At which integer values of  $x$  in the interval  $(-\frac{7}{2}, \frac{9}{2})$  does the limit exist?



**Example:** Examine the limit of

$$f(x) = \begin{cases} 2x + 1; & x > -3 \\ -2 + x; & x < -3 \end{cases} \text{ at } x = -3.$$

## Limit of a Polynomial Function:

Limit of any polynomial function  $f(x)$  as  $x$  approaches to  $x_0$  is  $f(c)$ .

That is,

For any  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

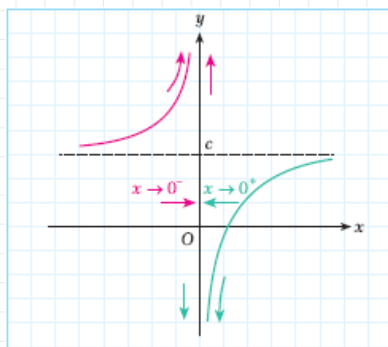
$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

**Example:** Find the limit of  $f(x) = x^3 + 2x^2 - 4x + 1$  at  $x = 2$ .

**Example:** Find the limit of  $f(t) = 1 - 3t^2 + 2t$  at  $-1$ .

## Limits Involving Infinity

Let us study on the limits of the function graphed in the figure.

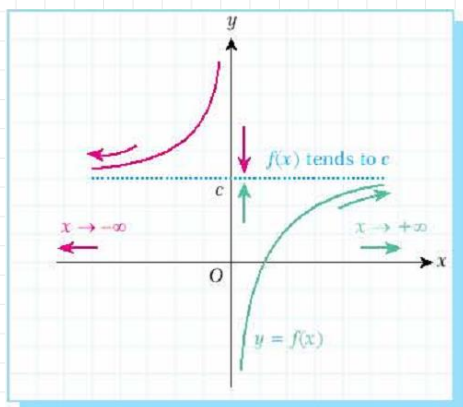


$$f : \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{c\}$$

$$\lim_{x \rightarrow 0^+} f(x) = \dots$$

$$\lim_{x \rightarrow 0^-} f(x) = \dots$$

**Note:** "infinity -  $\infty$ " is NOT a real number. It is a concept which describes the situation in which a function continues without end in a positive or negative direction.



$$f : \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{c\}$$

$$\lim_{x \rightarrow \infty} f(x) = \dots$$

$$\lim_{x \rightarrow -\infty} f(x) = \dots$$

## Remark:

For some functions the following limits are possible.

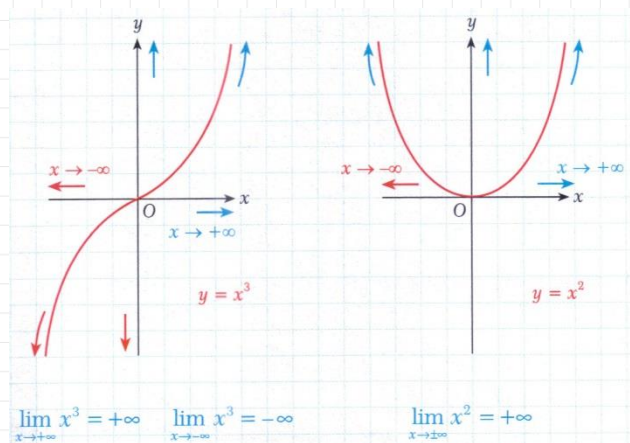
$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow +\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

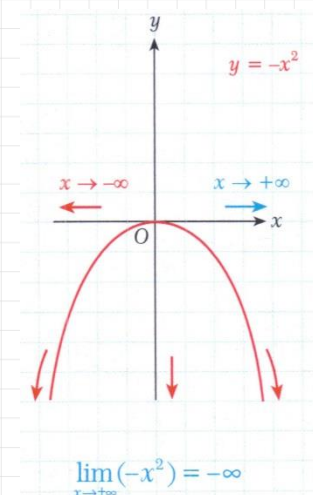
## Example:



$$\lim_{x \rightarrow +\infty} x^3 = +\infty$$

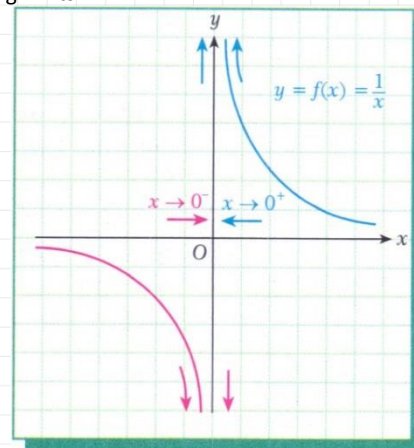
$$\lim_{x \rightarrow -\infty} x^3 = -\infty$$

$$\lim_{x \rightarrow \pm\infty} x^2 = +\infty$$



$$\lim_{x \rightarrow \pm\infty} (-x^2) = -\infty$$

**Example:** Given  $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{x}$  then calculate the following limits.



$$\lim_{x \rightarrow 0^+} f(x) = \dots$$

$$\lim_{x \rightarrow 0^-} f(x) = \dots$$

$$\lim_{x \rightarrow +\infty} f(x) = \dots$$

$$\lim_{x \rightarrow -\infty} f(x) = \dots$$

**Example:** Given  $f : \mathbb{R} - \{2\} \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{x-2}$  then calculate the following limits.

$$\lim_{x \rightarrow 2^+} f(x) =$$

$$\lim_{x \rightarrow 2} f(x) =$$

$$\lim_{x \rightarrow 2^-} f(x) =$$

**Example:** Find the limit of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \frac{3x^2 + x}{x^2 - 5} \text{ as } x \text{ approaches to } +\infty.$$

### Limit Combination Theorem:

Let  $f(x)$  and  $g(x)$  be functions such that  $\lim_{x \rightarrow x_0} f(x) = a$  and

$$\lim_{x \rightarrow x_0} g(x) = b. \text{ Then}$$

$$\text{a) } \lim_{x \rightarrow x_0} [f(x) + g(x)] = a + b$$

$$\text{b) } \lim_{x \rightarrow x_0} [f(x) - g(x)] = a - b$$

$$\text{c) } \lim_{x \rightarrow x_0} [f(x) \cdot g(x)] = a \cdot b$$

$$\text{d) } \lim_{x \rightarrow x_0} \left[ \frac{f(x)}{g(x)} \right] = \frac{a}{b} (b \neq 0)$$

$$\text{e) } \lim_{x \rightarrow x_0} k \cdot f(x) = k \cdot a (k \in \mathbb{R})$$

**Example:** Given  $f(x) = 3$  and  $g(x) = \frac{1}{x} (x \neq 0)$  then evaluate the following limits.

$$\lim_{x \rightarrow -2} [f(x) + g(x)] = \dots$$

$$\lim_{x \rightarrow +\infty} [f(x) \cdot g(x)] = \dots$$

$$\lim_{x \rightarrow -3} \left[ \frac{f(x)}{g(x)} \right] = \dots$$

**Example:** Perform the following limits

$$\bullet \lim_{x \rightarrow 6} (25^{\frac{1}{x-5}}) = \dots$$

$$\bullet \lim_{x \rightarrow 2} \left( \frac{2 + 3^{\frac{1}{x}}}{1 + 4^{\frac{1}{x}}} \right) = \dots$$

$$\bullet \lim_{x \rightarrow 0} (\sqrt{2^{1-x^2}})$$

$$\bullet \lim_{x \rightarrow -3} (2^{\sqrt{1-x}})$$

$$\bullet \lim_{x \rightarrow 0} (5^{\frac{1}{x}})$$

**Example:** Given  $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{1 + 2^{\frac{2}{x}}}$  and  $g(x) = \frac{1}{x}$  then find the following limits.

$$\lim_{x \rightarrow 0} f(x) = \dots$$

$$\lim_{x \rightarrow 0} g(x) = \dots$$

$$\lim_{x \rightarrow 0} (f(x) \cdot g(x)) = \dots$$

### Exercises 2.1 – Page 71 in Zambak

Part A, B, D, F





## 2. INDETERMINATE FORMS

The situation in which the value of a function of a point may not be defined in real numbers is called **indeterminate form**.

### A. $\frac{0}{0}$ As a Limit

If  $f(x_0) = 0$  and  $g(x_0) = 0$ , then  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ .

In this case, there exists a function  $h(x)$  which is a common factor of  $f$  and  $g$  such that  $h(x_0) = 0$ . So, we get

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f_1(x) \cdot h(x)}{g_1(x) \cdot h(x)}$$

Since  $x \neq x_0$  we can cancel the factors. So,

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f_1(x) \cdot h(x)}{g_1(x) \cdot h(x)} = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x_0)}{g_1(x_0)}$$

**Example:** Calculate the limits below.

- $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} =$

- (UN 2011 PAKET 21)

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} =$$

- $\lim_{x \rightarrow -1} \frac{x^2 - 5x - 6}{x + 1} =$

- (UN 2008 PAKET A/B)

$$\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^2 + 2x - 8} =$$

- (UN 2012/D49)

$$\lim_{x \rightarrow 1} \frac{1 - x}{2 - \sqrt{x + 3}} =$$

- (UN 2010 PAKET A)

$$\lim_{x \rightarrow 0} \left( \frac{3x}{\sqrt{9 + x} - \sqrt{9 - x}} \right) =$$

- $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1 + x} - 1}{x} =$

### The Indeterminate Form $\frac{0}{0}$ in Trigonometric Functions

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{0}{0}$$

**Theorem:**  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

**Example:** Find the following limits based on the theorem.

- $\lim_{x \rightarrow 0} \frac{\tan x}{x} = \dots$

- $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \dots$

**Generalization:** The following results can be obtained also by solving as in the last two examples.

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \lim_{x \rightarrow 0} \frac{ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$$

$$\lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \lim_{x \rightarrow 0} \frac{ax}{\tan bx} = \lim_{x \rightarrow 0} \frac{\tan ax}{\tan bx} = \frac{a}{b}$$

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\tan bx} = \lim_{x \rightarrow 0} \frac{\tan ax}{\sin bx} = \frac{a}{b}$$

**Example:** Find the following limits.

- $\lim_{x \rightarrow 0} \frac{\sin 2x}{3x} =$

- $\lim_{x \rightarrow 0} \frac{\sin 2x}{\tan x} =$

- $\lim_{x \rightarrow 0} \frac{2 \sin \frac{x}{2}}{\tan \frac{x}{3}} =$

- $\lim_{x \rightarrow 0} \frac{(x+2) \tan 6x}{\sin 3x} =$

- $\lim_{x \rightarrow 0} \frac{\sin(x-3)}{x-3} =$

- $\lim_{x \rightarrow 1} \frac{\tan(x-1)}{4x-4} =$

**Example:** Calculate the limits below.

- (UN 2010 PAKET B)

$$\lim_{x \rightarrow 0} \left( \frac{\sin x + \sin 5x}{6x} \right) =$$

- $\lim_{x \rightarrow -2} \frac{4-x^2}{\sin(2-x)} =$

- (UN 2007 PAKET B)

$$\lim_{x \rightarrow 2} \frac{\sin(x-2)}{x^2 - 3x + 2} =$$

- $$\lim_{x \rightarrow 0} \frac{5x^2}{\sin^2\left(\frac{x}{3}\right)} =$$

- $$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\pi - 2x} =$$

- (UN 2011 PAKET 12)

$$\lim_{x \rightarrow 0} \left( \frac{1 - \cos 2x}{2x \cdot \sin 2x} \right) =$$

- (UN 2012/C37)

$$\lim_{x \rightarrow 0} \left( \frac{1 - \cos 2x}{x \cdot \tan 2x} \right) =$$

### B. $\frac{\infty}{\infty}$ As a Limit

Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  and

$g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$  be two polynomials.

Then,

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty \text{ And } \lim_{x \rightarrow \pm\infty} g(x) = \pm\infty. \text{ So, } \lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

In this case,

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = \frac{x^n \left( a_n + \frac{a_{n-1}}{x} + \dots + \frac{a_1}{x^{n-1}} x + \frac{a_0}{x^n} \right)}{x^m \left( b_m + \frac{b_{m-1}}{x} + \dots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m} \right)}$$

$$= \lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = \frac{a_n x^n}{b_m x^m} = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \\ \pm\infty & \text{if } n > m \end{cases}$$



**Example:** Calculate the limits below.

$$\bullet \lim_{x \rightarrow \infty} \frac{x^4 + 2x}{x^3 + 5} =$$

$$\bullet \lim_{x \rightarrow \infty} \frac{4x^2 + 5x + 3}{x^3 + 2x - 1} =$$

$$\bullet \lim_{x \rightarrow -\infty} \frac{5x^4 + 11x^2 + 2}{3x^4 + 5x^3 - 2} =$$

$$\bullet \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 - 2}}{3x + 1} =$$

$$\bullet \lim_{x \rightarrow \infty} \frac{(2x^3 - 3x^2 + 1)(3x^2 + x + 11)}{(4x^4 + x^3 - 2x + 1)(5x - 2)} =$$

$$\bullet \lim_{x \rightarrow -\infty} \frac{(2x^3 - 3 + 1)^2 (x^2 - x + 1)^3}{(x^4 + x + 2)^2 (3x^2 + 2x + 5)^2} =$$

$$\bullet \lim_{x \rightarrow 0} \frac{\cot 4x}{\cot 5x} =$$

$$\bullet \lim_{x \rightarrow \infty} \frac{2^x - 2^{-x}}{2^x + 2^{-x}} =$$

$$\bullet \lim_{x \rightarrow \infty} \frac{2^x + 2 \cdot 3^x + 1}{3^x + 5 \cdot 2^x - 1} =$$

$$\bullet \lim_{x \rightarrow \infty} \frac{2^{x+1} + 2^x}{2^{x+2} + 2^x} =$$

### C. $0 \cdot \infty$ As a Limit

If  $\lim_{x \rightarrow x_0} f(x) = 0$  and  $\lim_{x \rightarrow x_0} g(x) = \pm\infty$ , then  $\lim_{x \rightarrow x_0} [f(x) \cdot g(x)] = 0 \cdot \infty$ .

In this case, transform  $0 \cdot \infty$  into the indeterminate form of  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ .

**Example:** Calculate the limits below.

$$\bullet \lim_{x \rightarrow \infty} \left( x \cdot \sin \frac{1}{x} \right) =$$

$$\bullet \quad \lim_{x \rightarrow 0} (3x \cdot \cot 2x) =$$

$$\bullet \quad \lim_{x \rightarrow -\infty} (2x - 5) \cdot \frac{1}{x - 2} =$$

$$\bullet \quad \lim_{x \rightarrow -\pi} \sin x \cdot \frac{1}{\pi + x} =$$

$$\bullet \quad \lim_{x \rightarrow \infty} \left( x^2 \cdot \left( 1 - \cos^2 \frac{1}{x} \right) \right) =$$

#### D. $\infty - \infty$ As a Limit

If  $\lim_{x \rightarrow x_0} f(x) = \infty$  and  $\lim_{x \rightarrow x_0} g(x) = \infty$ , then

$$\lim_{x \rightarrow x_0} [f(x) - g(x)] = \infty - \infty.$$

In this case, transform  $\infty - \infty$  into the indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ .

**Example:** Calculate the limits below.

• (UN 2010 PAKET B)

$$\lim_{x \rightarrow 0} \left( \frac{2}{x - 2} - \frac{8}{x^2 - 4} \right) =$$

$$\bullet \quad \lim_{x \rightarrow \pi} \left( \frac{1}{\sin x} - \frac{1}{\tan x} \right) =$$

$$\bullet \quad \lim_{x \rightarrow \infty} \sqrt{x^2 + 2x + 5} - \sqrt{x^2 + 6x + 1} =$$

- $\lim_{x \rightarrow \infty} \sqrt{3x^2 - x + 4} - \sqrt{3x^2 + 2x + 1} =$

- $\lim_{x \rightarrow \infty} (x + 1 - \sqrt{x^2 - 4x - 1}) =$

- $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} + x + 4) =$

- $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x + 4) =$

- (UN 2005)  
 $\lim_{x \rightarrow \infty} (\sqrt{x(4x + 5)} - 2x + 1) =$

- (UN 2009 PAKET A/B)  
 $\lim_{x \rightarrow \infty} \frac{\sqrt{5x + 4} - \sqrt{3x + 9}}{4x} =$

*Exercises 2.2 – Page 92 in Zambak*

1-a,b,e,f,i

2-a,b,c,d,e,h,j

3-a,b,c,f,h,k

4-a,b,d,e

5-a,d,e

6-a,b,d,f

7-a,d,e,h,i

### 3. CONTINUITY

#### Continuity at a Point

If  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$  then we say  $f$  is **continuous** at  $x_0$ .

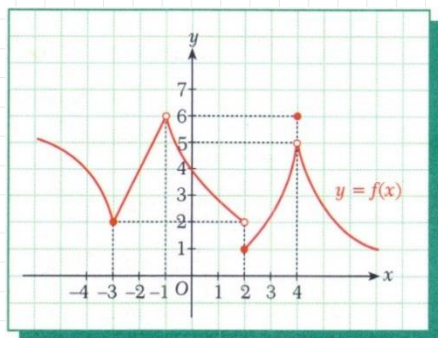
Otherwise,  $f$  is **discontinuous** at  $x_0$ .

For a function  $f$  to be continuous at  $x_0$ , followings are necessary:

1.  $\lim_{x \rightarrow x_0} f(x)$  must exist.
2.  $f(x_0)$  must exist.
3.  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$  must be satisfied.

In roughly speaking, if we can draw the graph of a function without lifting our hand then the function is continuous.

**Example:** Examine the continuity of  $f(x)$  at  $x = -3$ ,  $x = -1$ ,  $x = 2$ ,  $x = 4$ .



**Example:** Examine the continuity of  $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ 2 - x & \text{if } x > 1 \end{cases}$  at  $x = 1$ .

**Example:** Examine the continuity of  $f(x) = \begin{cases} 2x & \text{if } x = 2 \\ x^2 - 4 & \text{if } x \neq 2 \end{cases}$  at  $x = 2$ .

**Example:** The function  $f(x)$  is defined as

$$f(x) = \begin{cases} ax + b, & x > -1 \\ 4, & x = -1 \\ 2b, & x < -1 \end{cases}$$

If the function  $f(x)$  is continuous at  $x = -1$ , then  $a + b = ?$

**Theorem:** If  $f$  and  $g$  are two continuous functions at  $x_0$ , then so are

1.  $\alpha \cdot f(x)$  where  $(\alpha \in \mathbb{R})$
2.  $f(x) \pm g(x)$
3.  $f(x) \cdot g(x)$
4.  $\frac{f(x)}{g(x)}$  where  $(g(x) \neq 0)$

**Exercises 3.1 – Page 121 in Zambak**

Part A

## Review Test

1.  $\lim_{x \rightarrow 6} (3x - 4) = ?$

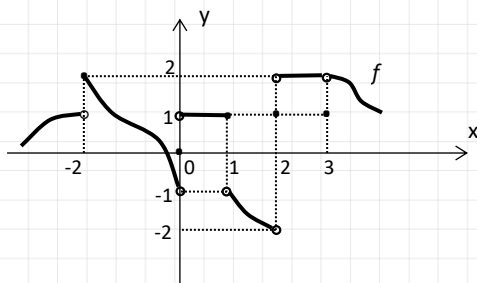
- A) 0    B) -4    C) 14    D) 36    E)
- $+\infty$

2.  $\lim_{x \rightarrow 0} (5x^2 - 3x + 4) = ?$

- A) -4    B) -1    C) 0    D) 1    E) 4

3. If  $\lim_{x \rightarrow a-1} (x^2 + a) = 3$ , then find the sum of the values of  $a$ .

- A) -3    B) -1    C) 0    D) 1    E) 3

 4. At which points does the function  $f$  have no limit?


- A)
- $\{-2, 2\}$
- B)
- $\{-2, 0, 1, 2, 3\}$
- C)
- $\{0, 1, 2\}$
- 
- D)
- $\{-2, 0, 1, 2\}$
- E)
- $\{-2, 0, 1\}$

5.  $f(x) = \begin{cases} 2x^2 - 1, & x \geq 0 \\ 3x - 1, & x < 0 \end{cases} \Rightarrow \lim_{x \rightarrow 0} f(x) = ?$

- A) 0    B) -1    C) 1    D)
- $\infty$
- E) Not Exist

6.  $f(x) = \begin{cases} 2x - 1, & x \geq 2 \\ x^2 - 1, & -2 < x < 2 \\ x - 1, & x < -2 \end{cases}$

 What is  $\lim_{x \rightarrow 2} f(x)$ ?

- A) 1    B) -2    C) 3    D) -3    E) Not Exist

7. (UN 2003)

$$\lim_{x \rightarrow 2} \frac{4 - x^2}{3 - \sqrt{x^2 + 5}} = ?$$

- A) -12    B) -6    C) 0    D) 6    E) 12

8.  $\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} = ?$

- A) 0    B) 1    C) 2    D) 4    E) 8

9.  $\lim_{x \rightarrow \sqrt{2}} \frac{x - \sqrt{2}}{x^2 - 2} = ?$

- A)
- $\frac{1}{2\sqrt{2}}$
- B) 0    C)
- $\infty$
- D)
- $\frac{1}{\sqrt{2}}$
- E)
- $-\frac{1}{\sqrt{2}}$

10.  $\lim_{x \rightarrow 64} \frac{\sqrt[3]{x} - 4}{\sqrt{x} - 8} = ?$

- A) 3    B)
- $\frac{2}{3}$
- C)
- $\frac{3}{2}$
- D)
- $\frac{1}{3}$
- E) 0

11. (UN 2009 PAKET A/B)

$$\lim_{x \rightarrow -2} \frac{x+2}{\sqrt{5x+14}-2} = ?$$

- A) 4    B) 2    C) 1.2    D) 0.8    E) 0.4

12. (UN 2006)

$$\lim_{x \rightarrow 0} \frac{\sqrt{4+2x} - \sqrt{4-2x}}{x} = ?$$

- A) 4    B) 2    C) 1    D) 0    E) -1

13.  $\lim_{x \rightarrow 0} \frac{\sin 2x}{\tan 5x} = ?$

- A)  $\frac{5}{2}$     B)  $\frac{2}{5}$     C) 0    D) 2    E) 5

14. (UN 2010 PAKET A)

$$\lim_{x \rightarrow 0} \left( \frac{\cos 4x \cdot \sin 3x}{5x} \right) = ?$$

- A)  $\frac{5}{3}$     B) 1    C)  $\frac{3}{5}$     D)  $\frac{1}{5}$     E) 0

15.  $\lim_{x \rightarrow \pi} \frac{\sin x \cdot \sin 2x}{2 - 2\cos^2 x} = ?$

- A) 1    B)  $\frac{1}{2}$     C) 0    D)  $-\frac{1}{2}$     E) -1

16. (UN 2007 PAKET A)

$$\lim_{x \rightarrow 0} \left( \frac{2x \cdot \sin 3x}{1 - \cos 6x} \right) = ?$$

- A) -1    B)  $-\frac{1}{3}$     C) 0    D)  $\frac{1}{3}$     E) 1

17. (UN 2005)

$$\lim_{x \rightarrow 0} \frac{\sin 12x}{2x(x^2 + 2x - 3)} = ?$$

- A) -4    B) -3    C) -2    D) 2    E) 6

18. (UN 2012/D49)

$$\lim_{x \rightarrow 0} \frac{\cos 4x - 1}{x \cdot \tan 2x} = ?$$

- A) 4    B) 2    C) -1    D) -2    E) -4

19. (UN 2012/B25)

$$\lim_{x \rightarrow 0} \left( \frac{x \cdot \tan x}{1 - \cos 2x} \right) = ?$$

- A)  $-\frac{1}{2}$     B) 0    C)  $\frac{1}{2}$     D) 1    E) 2

20.  $\lim_{x \rightarrow -\infty} \frac{5x^2 + 3x - 1}{x^3 + 2x^2 - x} = ?$

- A) 0    B)  $\infty$     C)  $-\infty$     D) 5    E) Not Exist

21.  $\lim_{n \rightarrow \infty} \left( \frac{(n+1)(2n-1)}{6n^2-5} \right) = ?$

- A) 0    B)
- $\frac{2}{3}$
- C)
- $\frac{1}{4}$
- D)
- $\frac{1}{3}$
- E)
- $\frac{1}{2}$

22.  $\lim_{x \rightarrow \infty} \frac{(2x+3)^3 \cdot (3x-2)^2}{x^5+5} = ?$

- A) 16    B) 60    C) 48    D) 36    E) 72

23.  $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2+2x+1}}{4x+1} = ?$

- A)
- $-\frac{3}{4}$
- B)
- $\frac{2}{3}$
- C)
- $-\frac{2}{3}$
- D) 1    E)
- $\frac{3}{4}$

24. (UN 2004)

$$\lim_{x \rightarrow 3} \left( \frac{1}{x-3} - \frac{6}{x^2-9} \right) = ?$$

- A)
- $-\frac{1}{6}$
- B)
- $\frac{1}{6}$
- C)
- $\frac{1}{3}$
- D)
- $\frac{1}{2}$
- E) 1

25. (UAN 2003)

$$\lim_{x \rightarrow 0} \left( (2x+1) - \sqrt{4x^2-3x+6} \right) = ?$$

- A)
- $\frac{3}{4}$
- B) 1    C)
- $\frac{7}{4}$
- D) 2    E)
- $\frac{5}{2}$

26.  $\lim_{x \rightarrow \infty} \left( \sqrt{2x^2+2x} - \sqrt{2x^2-3} \right) = ?$

- A) 0    B) 2    C)
- $\frac{1}{2\sqrt{2}}$
- D)
- $\frac{1}{\sqrt{2}}$
- E)
- $\infty$

27.  $\lim_{x \rightarrow -\infty} (x+1 + \sqrt{x^2+x+1}) = ?$

- A)
- $\frac{1}{2}$
- B)
- $-\frac{1}{2}$
- C) -1    D) 1    E) 0

28.  $\lim_{x \rightarrow \infty} (\sqrt{x^2+ax+15} - \sqrt{x^2+2x+5}) = 1 \Rightarrow a = ?$

- A) 4    B) 6    C) 8    D) 10    E) 12

29.  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} x+1, & x < 0 \\ a, & x = 0 \\ x^3+1, & x > 0 \end{cases}$

If the function  $f(x)$  is continuous at the point  $x=0$ , what is the value of  $a$ ?

- A) 1    B) 2    C) 3    D) 4    E) 5

30.  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} \frac{ax+2}{x}, & x < 2 \\ ax-1, & x \geq 2 \end{cases}$

If the function  $f(x)$  is continuous at the point  $x=2$ , what is the value of  $a$ ?

- A) 1    B) 2    C) 3    D) 4    E) 5