1. INTEGER EXPONENTS

$a^n \to a$ to the power of n ; n^{th} power of a ; a to the n

Definition: Let $a \in R$ and $n \in \mathbb{Z}^+$.

- $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{n times}}$

Example: Write the following terms in exponential form.

- $2 \cdot 2 \cdot 2 =$
- $\underbrace{5.5.5...5}_{25-times} =$
- $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} =$
- $(-2) \cdot (-2) \cdot (-2) =$

Example: Find the results of expressions below.

- $2^5 =$
- $(2 \cdot 3)^4 =$
- $\left(\frac{3}{5}\right)^3 =$

- $a^0 = 1$ (0^0 is undefined)

- $a^{-n} = \frac{1}{a^n}$

Example: Find the results of following expressions.

- $2005^0 =$
- $4^{-2} =$
- $\left(\frac{1}{3}\right)^{-4} =$

Note: $(-a)^n \neq -a^n$ for even n

Example: Evaluate the following expressions.

- $(-2)^3 + (-2^4) + (-2)^4 + 2^0 - 2^{-2} =$
- $\left(\frac{4}{3}\right)^{-1} + \frac{3^{-1}}{4^{-1}} =$

Properties of Integer Powers

Let $a, b \in R$ and $n, m \in \mathbb{Z}$.

1. $a^m \cdot a^n = a^{m+n}$

Proof:

Example:

- $2^3 \cdot 2^6 =$
- $9 \cdot b^6 \cdot a^5 \cdot b^{-5} \cdot a^{-2} =$
- $x^{3m+2} \cdot x^{3-4m} =$

2. $\frac{a^m}{a^n} = a^{m-n}$, $(a \neq 0)$

Proof:

Example:

- $\frac{3^7}{3^4} =$
- $\frac{x^{-7}}{x^5} =$
- $\frac{9 \cdot (x+2)^4 \cdot x^6}{x^2 \cdot (x+2)^5} =$

3. $(a^m)^n = a^{m \cdot n}$

Proof:

Example:

- $(2^4)^2 =$
- $(y^{-3})^{-6} =$

4. $(a \cdot b)^m = a^m \cdot b^m$

Proof:

Example:

- $(4^3 \cdot 3^{-2})^5 =$
- $(2x^5 \cdot y^{-2})^{-3} =$

5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $(b \neq 0)$

Proof:

Example:

- $\left(\frac{a^{-4}}{b^3}\right)^4 =$
- $\left(\frac{(m-3)^4 \cdot b^6}{4 \cdot a^{-2}}\right)^3 =$

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Assignment:
Evaluate the expressions below.

- $4^3 =$
- $\left(\frac{1}{3}\right)^3 =$
- $(-5)^2 =$
- $\left(-\frac{2}{3}\right)^3 =$
- $27^1 - (-5)^2 - \left(-\frac{6}{7}\right)^0 - 2007^0 =$
- $1^{2008} + 0^{2008} + (-1)^{2008} + 2008^1 =$
- $(abc)^3 =$
- $(4ab^2)^3 =$
- $3^5 \cdot 3^6 \cdot 3^7 =$
- $4^3 \cdot 4^{-4} \cdot 4^7 \cdot 4^{-5} =$
- $\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{-5} =$
- $\left(-\frac{3}{2}\right)^{-3} \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{3}{2}\right)^4 =$
- $\frac{5^7}{5^3} =$
- $\frac{12^{-10}}{12^{-12}} =$
- $\frac{7^{-7}}{7^{-5}} =$
- $\frac{(-2)^4}{(-2)^2} =$
- $\frac{\left(\frac{3}{7}\right)^{12}}{\left(\frac{3}{7}\right)^9} =$
- $\frac{y^4.y^5.y^7}{y^3.y^2} =$
- $\frac{x^{-3} \cdot x^{-2}}{x^5} =$
- $\frac{a^{-4} \cdot a^{-5}}{a^9} =$
- $\frac{3^{n-2}}{3^{n+4}} =$
- $\frac{(a^3)^{-2}}{a^6.a^{-9}} =$
- $\left(\frac{a^{n+1} \cdot b^{n-3}}{a^{2n} \cdot b^2}\right)^2 =$
- $\frac{81^{2x-3}}{3^{x+2}} =$
- $\frac{1000000000}{2^{11}} =$

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2. SQUARE ROOTS

Definition:

- If $a^2 = b$ then a is the square root of b.
- Observe that;

$3^2 = 9 \Rightarrow$; that is, 3 is the square root of 9

$(-3)^2 = 9 \Rightarrow$; that is, -3 is the square root of 9

To differentiate two different roots, we call the positive square root as **principal square root**.

Notation: $a^2 = b \Rightarrow a = \sqrt{b}$

After this point, we mean principal square root by square root.

Observation:
As you observe from example above, we cannot talk about square root of negative numbers.

$\sqrt{-9}$ does not exist since $3^2 = 9$ or $(-3)^2 = 9$. There is no number whose square is -9.

- Numbers whose square roots are integers or rational numbers are called **perfect squares**.

  Example:
  100 is a perfect square since $\sqrt{100} = 10 \in \mathbb{Z}$.
  $\frac{169}{25}$ is a perfect square since $\sqrt{\frac{169}{25}} = \frac{13}{5} \in \mathbb{Q}$.

Example: Evaluate the following square roots.

- $\sqrt{144} =$
- $\sqrt{0.04} =$

Definition: $\sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Example:
- $\sqrt{3^2} =$
- $\sqrt{(-5)^2} =$

Example: Simplify the expression $\sqrt{x^2} + \sqrt{y^2} - 3x + 4y$ if $x < 0$ and $y > 0$.

Example: Simplify the expression $\sqrt{(x-5)^2} + \sqrt{x^2}$ if $0 < x < 5$ and $y < 0$.

Example: Find x, y satisfying the equation of $\sqrt{x^2} + \sqrt{y^2 - 4y + 4} = 0$.

Properties of Square Root

Let $a \geq 0, b \geq 0$ and $n \in \mathbb{Z}$.

1. $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$

Example:
- $\sqrt{3} \cdot \sqrt{27} =$
- $\sqrt{5} \cdot \sqrt{5} =$
- $\sqrt{25 \cdot 16} =$
- If $a > 0$, then $\sqrt{36 \cdot a^2} =$

2. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Example:
- $\frac{\sqrt{24}}{\sqrt{6}} =$
- $\sqrt{\frac{1}{49}} =$
- $\sqrt{\frac{16}{81}} =$
- $\frac{\sqrt{72a^3}}{\sqrt{18a}} =$

3. $\left(\sqrt{a}\right)^n = \sqrt{a^n}$

Example:
- $\left(\sqrt{5}\right)^3 =$
- $\left(\sqrt{2}\right)^8 =$

4. $\sqrt{a^2 b} = a\sqrt{b}$

Example:
- $\sqrt{8} =$
- $\sqrt{27} =$
- $\sqrt{50} =$
- $\sqrt{8} + 2\sqrt{32} - \sqrt{18} + \sqrt{72} - \sqrt{98} =$

5. $m\sqrt{x} + n\sqrt{x} - k\sqrt{x} = (m + n - k)\sqrt{x}$ where $m, n, k \in \mathbb{R}$

Example:
- $2\sqrt{5} + \sqrt{5} =$
- $10\sqrt{3} - 4\sqrt{3} =$
- $5\sqrt{x} - 9\sqrt{x} + \sqrt{64x} =$

6. If $a < b$ then $\sqrt{a} < \sqrt{b}$

Example: Compare the numbers below.

- $\sqrt{7}$ and 3
- $3\sqrt{5}$ and $2\sqrt{10}$
- $-2\sqrt{3}$ and $-3\sqrt{2}$

Rationalizing a Denominator

Choose the appropriate number for multiplying both numerator and denominator so that denominator becomes rational.

Example: Rationalize the denominator of each fraction.

- $\frac{\sqrt{3}}{\sqrt{2}} =$
- $\frac{3}{\sqrt{3}} =$
- $\frac{3\sqrt{5}}{2\sqrt{2}} =$

In order to rationalize the denominator of the form $\sqrt{a} \pm \sqrt{b}$, we will use the identity of $(x-y)(x+y) = x^2 - y^2$.

Definition: $(x-y)$ is the **conjugate** of $(x+y)$ and vice versa.

Example: Rationalize the denominator of each fraction.

- $\frac{1}{\sqrt{3}+\sqrt{2}} =$
- $\frac{3\sqrt{2}-2}{5+2\sqrt{5}} =$
- $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}-1} =$
- $\frac{\sqrt{6}+\sqrt{2}}{1-\sqrt{3}} =$
- $\frac{\sqrt{3}}{\sqrt{3}+2\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{3}-2\sqrt{2}} =$
- $\frac{1}{\sqrt{\sqrt{6}+\sqrt{2}}} =$

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Square Root of an Irrational Sum

$\sqrt{(m+n) \pm 2\sqrt{m \cdot n}} = \sqrt{m} \pm \sqrt{n}$ where $m > n$

Proof:

Example: Find the results of following expressions.

- $\sqrt{3+2\sqrt{2}} =$
- $\sqrt{5+2\sqrt{6}} =$
- $\sqrt{6+\sqrt{32}} =$
- $\sqrt{6-4\sqrt{2}} =$

- $\sqrt{5+\sqrt{21}} =$

- $\sqrt{5+\sqrt{21+\sqrt{13+\sqrt{9}}}} =$

- $\sqrt{6\sqrt{6\sqrt{72\sqrt{\frac{1}{16}\sqrt{16}}}}} =$

Infinite Forms

Example: Find the results of the expressions below.

- $\sqrt{2\sqrt{2\sqrt{2\ldots}}} =$

- $\sqrt{a\sqrt{a\sqrt{a\ldots}}} = 7 \Rightarrow a = ?$

- $\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}} = 5 \Rightarrow x = ?$

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Assignment:

1. Evaluate the following expressions.

- $\sqrt{4} =$
- $\sqrt{100} =$
- $\sqrt{0} =$
- $\sqrt{4^2} =$
- $\sqrt{(-4)^2} =$
- $\sqrt{-4} =$
- $\sqrt{7} \cdot \sqrt{7} =$
- $\sqrt{2} \cdot \sqrt{8} =$
- $\sqrt{10} \cdot \sqrt{90} =$
- $\sqrt{5} \cdot \sqrt{4} \cdot \sqrt{20} =$
- $2\sqrt{2} \cdot 3\sqrt{2} =$
- $2\sqrt{x} \cdot 3\sqrt{x} =$
- $\sqrt{\frac{1}{4}} =$
- $\sqrt{\frac{25}{9}} =$
- $\frac{\sqrt{24a^3}}{\sqrt{6a}} =$
- $\frac{\sqrt{x \cdot y}}{\sqrt{x^3 \cdot y^3}}$
- $(\sqrt{2})^3 =$
- $\left(\sqrt{5}\right)^3 =$
- $(\sqrt{3})^2 + (\sqrt{5})^4 - (\sqrt{3})^2 =$
- $10\sqrt{5} - 3\sqrt{5} =$
- $2\sqrt{5} + \sqrt{5} =$
- $3\sqrt{9a} + 5\sqrt{16a} =$
- $5\sqrt{x} - \sqrt{9x} + \sqrt{64x} =$
- $\sqrt{50} =$

- $\sqrt{8}+2\sqrt{32}-\sqrt{18}+\sqrt{72}-\sqrt{98}=$
- $\sqrt{27}=$
- $\sqrt{12}-\sqrt{48}+\sqrt{108}=$
- $\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)=$
- $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)=$
- $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)=$
- $\sqrt{3+\sqrt{5}}\cdot\sqrt{3-\sqrt{5}}=$
- $\sqrt{7+\sqrt{4+\sqrt{0}}}=$
- $\sqrt{8\cdot\sqrt{16\cdot\sqrt{\frac{1}{96}\cdot\sqrt{9\cdot\sqrt{16}}}}}=$

2. Compare the following numbers.

- $\sqrt{7}....3$
- $3\sqrt{5}....2\sqrt{10}$
- $2\sqrt{7}....3\sqrt{3}$
- $-2\sqrt{3}....-3\sqrt{2}$

3. Evaluate the following expressions.

- $\sqrt{3+2\sqrt{2}}=$
- $\sqrt{6+2\sqrt{8}}=$
- $\sqrt{9-4\sqrt{5}}=$
- $\sqrt{7-4\sqrt{3}}=$
- $\sqrt{2\sqrt{2\sqrt{2\sqrt{2.....}}}}=$
- $\sqrt{3\sqrt{3\sqrt{3....}}}=$
- $\sqrt{x\sqrt{x\sqrt{x\sqrt{x....}}}}=$

4. Rationalize the denominators of the expressions below.

- $\frac{3}{\sqrt{3}}=$
- $\frac{1}{\sqrt{3}+\sqrt{2}}=$
- $\frac{\sqrt{6}+\sqrt{2}}{\sqrt{3}-1}=$
- $\frac{1}{\sqrt{5}+\sqrt{2}}+\frac{1}{\sqrt{5}-\sqrt{2}}=$
- $\frac{\sqrt{2}}{\sqrt{3}-2\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{3}+2\sqrt{2}}=$

3. RADICAL EXPRESSIONS

Definition:

- If $a^n = b$ then a is the n^{th} root of b.

Notation: $a^n = b \Rightarrow a = \sqrt[n]{b}$ (n is index, b is radicand)

- Observe that;

$3^2 = 9 \Rightarrow 3 = \sqrt{9}$; that is, 3 is the square root of 9

$(-2)^3 = -8 \Rightarrow -2 = \sqrt[3]{-8}$; that is, -2 is the cubic root of -8

Observation:
An even root is always non-negative, but odd root can be negative or positive.

There is no real number whose even power is negative.
$\sqrt[6]{-64}$ is undefined.

Example: Find the results of following expressions.

- $\sqrt[3]{27} =$
- $\sqrt[9]{-1} =$
- $\sqrt[3]{-0.008} =$
- $\sqrt[4]{-81} =$

Definition: $\sqrt[n]{x^n} = \begin{cases} |x|, & \text{if n is even} \\ x, & \text{if n is odd} \end{cases}$

Example: Find the results of following expressions.

- $\sqrt[3]{(-5)^3} =$
- $\sqrt[4]{(-5)^4} =$
- $\sqrt[7]{(2x-6)^7} =$

Example: Simplify the expression $\sqrt{x^2} + \sqrt[5]{y^5} + \sqrt[6]{z^6}$ if $x < 0, y < 0, z > 0$.

Properties of Radical Expressions

Let $a \geq 0, b \geq 0$ and $k, m, n \in \mathbb{Z}^+$ where $m \neq 1, n \neq 1$.
Note that if m and n are odd, the following properties also hold for $a < 0, b < 0$.

1. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$
2. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
3. $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$
5. $\sqrt[n \cdot k]{a^{m \cdot k}} = \sqrt[n]{a^m}$
6. $\sqrt[n]{a^n b} = a\sqrt[n]{b}$

Example:

- $\sqrt[3]{8 \cdot 27 \cdot 125} =$
- $\sqrt[5]{\dfrac{32}{243}} =$
- $\left(\sqrt[5]{k^2}\right)^4 =$
- $\sqrt[3]{\sqrt[4]{7}} =$
- $\sqrt[48]{m^{16}} =$
- $\sqrt[3]{108b^4} =$
- $\sqrt[4]{x^3 \sqrt[5]{x}} =$
- $\dfrac{\sqrt{a} \cdot \sqrt[4]{a}}{\sqrt[3]{a}} =$

Basic Operations on Radical Expressions

- Simplify the roots.
- Add/Subtract the expression with the same index and same radicand.
- Multiply/Divide the expression with the same index.

Example:

- $\sqrt[3]{x^4} - \sqrt[3]{27x^4} + \sqrt[3]{125x^4} =$
- $\sqrt[3]{128} + 4\sqrt[3]{16} + 2\sqrt[3]{54} =$
- $\sqrt[4]{x^3} \cdot \sqrt{x} =$
- $\dfrac{\sqrt[3]{a} \cdot \sqrt[4]{a^3}}{\sqrt{a^7}} =$
- $\dfrac{\sqrt[3]{a^2 \cdot \sqrt[5]{a^2 \cdot \sqrt{a}}}}{\sqrt[3]{a}} =$

Rationalizing a Denominator

Choose the appropriate number for multiplying both numerator and denominator so that denominator becomes rational.

- For $\frac{a}{\sqrt[m]{b^n}}$, use $\sqrt[m]{b^{m-n}}$
- For one of the following right hand side factors, use the other one.

  $a^3+b^3=(a+b)(a^2-ab+b^2)$
  $a^3-b^3=(a-b)(a^2+ab+b^2)$

Example: Rationalize the denominator of each fraction.

- $\frac{3}{\sqrt[8]{3^5}}=$
- $\frac{1}{\sqrt[3]{2}-1}=$
- $\frac{5}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}=$

Infinite Forms

Example: Find the results of following expressions.

- $\sqrt[3]{10\sqrt[3]{10\sqrt[3]{10\sqrt[3]{\ldots}}}}=$
- $\sqrt[4]{3^6\sqrt[4]{3^6\sqrt[4]{3^6\sqrt[4]{\ldots}}}}=$

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Assignment:

1. Evaluate the following expressions.

- $\sqrt[3]{-27}=$
- $\sqrt[3]{8}=$
- $\sqrt[4]{-16}=$
- $\sqrt[5]{0}=$
- $\sqrt[5]{(-3)^5}=$
- $\sqrt[4]{(-2)^4}=$
- $\sqrt[4]{16}=$
- $\sqrt[7]{-128}=$
- $\sqrt[5]{x^2}\cdot\sqrt[5]{x^3}=$
- $\sqrt[3]{3}\cdot\sqrt[3]{9}=$
- $\sqrt{3}\cdot\sqrt{12}=$
- $\sqrt[3]{x}\cdot\sqrt[3]{x^2\cdot y^3}=$
- $\frac{\sqrt[3]{625}}{\sqrt[3]{5}}=$
- $\frac{\sqrt{2}}{\sqrt{8}}=$
- $\frac{\sqrt[4]{x^5}}{\sqrt[4]{x}}=$
- $\frac{3}{\sqrt[3]{3}}=$
- $\sqrt[3]{40}=$
- $\sqrt[3]{81}=$
- $\sqrt[4]{32}=$
- $\sqrt[4]{x^5\cdot y^6}=$
- $\sqrt[3]{a^2}\cdot\sqrt[6]{a^3}=$
- $\sqrt[3]{x}\cdot\sqrt[4]{x^3}=$

- $\sqrt{2}\cdot\sqrt[3]{4}=$
- $\sqrt[4]{32}\cdot\sqrt[3]{16}=$
- $\dfrac{\sqrt[4]{a^3}}{\sqrt[5]{a^2}}=$
- $\dfrac{\sqrt[3]{6}}{\sqrt{2}}=$
- $\dfrac{\sqrt[3]{4}}{\sqrt[4]{3}}=$
- $\dfrac{\sqrt[6]{x^4}}{\sqrt[8]{x^3}}=$

2. Evaluate the following expressions.

- $\sqrt[4]{512}-3\sqrt[4]{16}+5\sqrt[4]{162}=$
- $\sqrt[5]{16\sqrt[5]{16\sqrt[5]{16\sqrt[5]{\ldots}}}}=$
- $\sqrt[3]{24\cdot\sqrt[3]{24\cdot\sqrt[3]{24\cdot\sqrt[3]{\ldots}}}}=$

3. Rationalize the denominators of following expressions.

- $\dfrac{1}{\sqrt[6]{5^2}}=$
- $\dfrac{1}{3+\sqrt[3]{2}}=$
- $\dfrac{1}{1-\sqrt[3]{3}+\sqrt[3]{9}}=$

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4. RATIONAL EXPONENTS

Definition: Let $a\in R$, $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$.

$a^{m/n}=\sqrt[n]{a^m}$ (if n is even, a^m must be non-negative)

Example: Evaluate the following expressions.

- $81^{1/2}=$
- $(-128)^{3/7}=$
- $25^{-3/2}=$

Properties of Rational Exponents

The properties of integer exponents are also true for rational exponents.

Let $a>0, b>0$ and $n,m\in\mathbb{Q}$.

1. $a^m\cdot a^n=a^{m+n}$
2. $\dfrac{a^m}{a^n}=a^{m-n}$
3. $\left(a^m\right)^n=a^{m\cdot n}$
4. $(a\cdot b)^m=a^m\cdot b^m$
5. $\left(\dfrac{a}{b}\right)^m=\dfrac{a^m}{b^m}$

Example: Evaluate the following expressions $(x>0)$.

- $x^{1/3}\cdot x^{1/4}=$
- $\dfrac{x^{1/3}}{x^{1/4}}=$
- $\left(x^{1/3}\right)^{1/4}=$
- $\dfrac{x^{1/3}\cdot x^{-1/4}}{x^{-1/6}}=$
- $\left(x^{0.6}\right)^{0.3}\cdot x^{0.4}=$
- $\left(3^{-1/9}\right)^{1.8}\cdot 9^{0.1}=$

Example: Simplify the following expressions.

- $\left(\sqrt[3]{a^2}\right)^4 \cdot \left(\sqrt{a^6}\right)^{1/9} =$
- $\dfrac{\sqrt{x^3y^2}+\sqrt{x^2y^3}}{\sqrt{x^2y}+\sqrt{xy^2}} =$
- $a^{\frac{1}{2}} - \dfrac{a-a^{-2}}{a^{\frac{1}{2}}-a^{-\frac{1}{2}}} + \dfrac{1-a^{-2}}{a^{\frac{1}{2}}+a^{-\frac{1}{2}}} + \dfrac{2}{a^{\frac{3}{2}}} =$

Assignment:

1. Evaluate the expressions below.

- $36^{-1/2} =$
- $2.125^{1/3} =$
- $-4.001^{-3/2} =$
- $(-128)^{3/7} =$
- $\left(y^{0.7}\right)^{0.5} \cdot y^{0.15} =$
- $\left(2^{0.5}\right)^{-0.5} \cdot \left(0.5^{0.5}\right)^{-1.25} =$
- $\left(3^{-\frac{1}{9}}\right)^{-0.5} \cdot 9^{0.1} =$
- $16^{0.125} \cdot 8^{-\frac{5}{6}} \cdot 4^{2.5} =$
- $\left(y^{\frac{5}{7}}\right)^{1.4} \cdot \left(y^{-\frac{3}{8}}\right)^{2.4} =$
- $\left(\sqrt[5]{a^{\frac{4}{3}}}\right)^{\frac{3}{8}} =$

2. Simplify the following expressions.

- $\dfrac{p^{\frac{2}{7}}-q^{\frac{2}{7}}}{p^{\frac{1}{7}}-q^{\frac{1}{7}}} =$
- $\left(b^{0.4}-4\right) \div \left(\dfrac{b^{0.6}+8}{b^{0.2}+2} - 2b^{0.2}\right) =$
- $\left(\dfrac{a^{1.5}-b^{1.5}}{a^{0.5}-b^{0.5}} + a^{0.5} \cdot b^{0.5}\right) \div \left(a^{0.5}+b^{0.5}\right) =$

5. EXPONENTIAL EQUATIONS

Let's have two equal exponential expressions.

If the powers (bases) are same then bases (powers) must be same also.

Example: Solve the following equations.

- $3^x = 9 \Rightarrow x =$
- $2^{3x-1} = 32 \Rightarrow x =$
- $3^{x+4} = \dfrac{1}{81} \Rightarrow x =$
- $3^x + 3^{x+2} = 3^3 \cdot 2 \cdot 5 \Rightarrow x =$
- $\dfrac{1}{25^x} = 0.000064 \Rightarrow x =$

Assignment:

Solve the following equations.

- $5^x = 5^3 \Rightarrow x =$
- $3^{2x} = 3^{12} \Rightarrow x =$
- $\left(\dfrac{2}{3}\right)^{4y-5} = \left(\dfrac{2}{3}\right)^{35} \Rightarrow y =$
- $\left(\dfrac{115}{13}\right)^{12} = \left(\dfrac{115}{13}\right)^{2y+4} \Rightarrow y =$
- $5^x = 25^3 \Rightarrow x =$
- $8^y = 4096 \Rightarrow y =$
- $2^x \cdot 2^{x+2} = 256 \Rightarrow x =$
- $25^{2x+5} = 125^{3x} \Rightarrow x =$
- $x^{x+4} = 1 \Rightarrow x =$

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Review Test

1. $\left(\frac{6}{0.75}-\frac{5}{0.2}+\frac{4}{0.25}\right)^{19}=?$

A) –21 B) –1 C) 1 D) 12 E) 19

2. $\frac{(-1)^{-1}\cdot 1^{-1}\cdot(-2)^{3}}{2^{-3}\cdot(-2)^{-2}\cdot 1^{-14}}=?$

A) 4 B) $\frac{1}{12}$ C) $\frac{11}{8}$ D) –4 E) -8

3. $(-0.008)^{-2/3}\times(0.5)^{-2}=?$

A) $\frac{1}{10}$ B) 25 C) 50 D) 75 E) 100

4. $\{[(-2)^{-1}]^{2}\}^{\frac{3}{2}}=?$

A) –16 B) –8 C) $-\frac{1}{8}$ D) $\frac{1}{16}$ E) 16

5. $(-x^{-3})^{-4}\cdot(-x^{2})^{-4}\cdot(x^{2})^{-3}=?$

A) x^{-2} B) $-x^{3}$ C) x^{5} D) $-x^{6}$ E) x^{7}

6. $\frac{(-a)^{-4}\cdot(-a)^{3}\cdot(-a)^{-3}}{(-a)^{2}\cdot(-a^{-3})}=?$

A) a^{-3} B) $-a^{-2}$ C) 2^{a} D) $-a^{-5}$ E) a^{4}

7. $\frac{1}{1-2^{a}}+\frac{2^{a}}{1-2^{-a}}=?$

A) $1+a^{2}$ B) 1 C) 2^{a} D) $1-2^{a}$ E) 2^{-a}

8. $\frac{4.2^{x}+2^{x-1}}{2^{x+1}-2^{x}-2^{x-1}}=?$

A) 2^{x} B) 2^{-x} C) 1 D) 6 E) 9

9. $\sqrt{4.9}-\sqrt{22.5}+\sqrt{8.1}=?$

A) $\frac{1}{\sqrt{10}}$ B) $\sqrt{10}$ C) $\frac{1}{\sqrt{5}}$ D) $\sqrt{10}$ E) $\frac{\sqrt{10}}{2}$

10. $\sqrt{15+\sqrt[3]{-108}\cdot\sqrt[3]{-\frac{1}{64}}}=?$

A) $3\sqrt{2}$ B) $2\sqrt{3}$ C) $\sqrt{3}$ D) $\frac{\sqrt{3}}{2}$ E) 1

11. $\left(\dfrac{1}{\sqrt{a}+y}+\dfrac{1}{\sqrt{a}-y}\right)\cdot\dfrac{y^2-a}{\sqrt{a^{-1}}}=?$

A) $2a$ B) $-2a$ C) $a\sqrt{b}$ D) $2b$ E) $b\sqrt{a}$

12. $\left(\sqrt{2}+2\right)\cdot\left(\sqrt{6-4\sqrt{2}}\right)=?$

A) $\sqrt{2}$ B) $2\sqrt{2}$ C) 2 D) $\dfrac{\sqrt{2}}{2}$ E)1

13. If $8^x = 9$ and $3^y = 64$, what is x . y ?

A) 2 B) 4 C)12 D) 14 E) 16

14. $x<2 \Rightarrow |x-3| + x + \sqrt{(x-2)^2}=?$

A) x+1 B) x-1 C) -x+5 D) 3x-1 E) 3x+1

15. If a<0<b<c, then what is $\sqrt{(a-b)^2}-|b-c|+|c-a|=?$

A) 2b-a B)2c-2a C) b-2a D) 2b2a E) c-a

16. $\dfrac{\sqrt{3^x}}{\sqrt[3]{9^{x+1}}}=0.3 \Rightarrow x=?$

A)5 B)2 C) $\dfrac{4}{3}$ D) $\dfrac{3}{2}$ E) $\dfrac{1}{2}$

17. $\dfrac{1}{\sqrt{3}-3}-\dfrac{2-\sqrt{3}}{\sqrt{3}}+\dfrac{1}{2-\sqrt{3}}=?$

A) $\dfrac{15-5\sqrt{3}}{6}$ B) $\dfrac{15+\sqrt{3}}{6}$ C) $\dfrac{5+\sqrt{3}}{3}$ D) $2\sqrt{3}$ E) $\dfrac{15-\sqrt{3}}{3}$

18. $\sqrt{\sqrt{2+1}}\cdot\sqrt{\sqrt[4]{2-1}}\cdot\sqrt{\sqrt[4]{2+1}}=?$

A)1 B)2 C) $\sqrt{2}$ D) $\sqrt[4]{2}$ E) $\dfrac{\sqrt{2}}{2}$

19. $\sqrt{5\sqrt{5\sqrt{5......}}}\;\sqrt{2\sqrt[3]{4\sqrt{2\sqrt[3]{4}}}}=?$

A)13 B)12 C)9 D)7 E)6

20. $\dfrac{5^{x+3}.5^{4x-3}}{5^3(5^{2x-1})^3.25^{3x}}=\dfrac{1}{5^{16-x}} \Rightarrow x=?$

A)-4 B)-3 C) $\dfrac{1}{2}$ D)2 E)4

21. $\dfrac{5^{x+3}\cdot 5^{4x-3}}{5^3(5^{2x-1})^3\cdot 25^{3x}}=\dfrac{1}{5^{16-x}}\Rightarrow x=?$

A)-4 B)-3 C) $\frac{1}{2}$ D)2 E)4

22. $\dfrac{\sqrt[5]{32^{2x+y}}}{\sqrt[3]{8^{x+y}}}=\left(\dfrac{1}{16}\right)^{-x+2}\Rightarrow x=?$

A)16 B10 C)4 D) $\frac{4}{3}$ E) $\frac{8}{3}$

23. $\dfrac{1}{3^x}+\dfrac{1}{3^{x-y}}+\dfrac{1}{3^{x-2}}=a.3^{-x}\Rightarrow a=?$

A)3 B)5 C)13 D) $\frac{1}{3}$ E) $\frac{8}{3}$

24. If $a=\sqrt{3}, b=\sqrt[3]{9}$ and $c=\sqrt[6]{27}$, then which one is true?

A)c>b>a B)a=c<b C)a=c>b D)c<a<b E)c>a>b

25. $\sqrt{8-2\sqrt{15}}+\left|2\sqrt{3}-2\sqrt{5}\right|=?$

A) $\sqrt{3}$ B) $\sqrt{3}-\sqrt{5}$ C) $\sqrt{5}-\sqrt{3}$
D) $3\sqrt{5}-3\sqrt{3}$ E) $3\sqrt{3}-3\sqrt{5}$

26. (UN 2012/A13)

If $a=4, b=2, c=\dfrac{1}{2}$, then $\left(a^{-1}\right)^2\cdot\dfrac{b^4}{c^{-3}}=\ldots$

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{1}{8}$ D) $\frac{1}{16}$ E) $\frac{1}{32}$

27. (UN 2011 PAKET 12)

$\dfrac{7x^3y^{-4}z^{-6}}{84x^{-7}y^{-1}z^{-4}}=\ldots$

A) $\dfrac{x^{10}z^{10}}{12y^3}$ B) $\dfrac{z^2}{12x^4y^3}$ C) $\dfrac{x^{10}y^5}{12z^2}$ D) $\dfrac{y^3z^2}{12x^4}$ E) $\dfrac{x^{10}}{12y^3z^2}$

28. (UN 2010 PAKET A)

$\left(\dfrac{27a^{-5}b^{-3}}{3^5a^{-7}b^{-5}}\right)^{-1}=\ldots$

A) $(3ab)^2$ B) $3\,(ab)^2$ C) $9\,(ab)^2$ D) $\dfrac{3}{(ab)^2}$ E) $\dfrac{9}{(ab)^2}$

29. (UN 2011 PAKET 12)

$\dfrac{\sqrt{5}+2\sqrt{3}}{\sqrt{5}-3\sqrt{3}}=\ldots$

A) $\dfrac{20+5\sqrt{15}}{22}$ B) $\dfrac{23-5\sqrt{15}}{22}$ C) $\dfrac{20-5\sqrt{15}}{-22}$

D) $\dfrac{20+5\sqrt{15}}{-22}$ E) $\dfrac{23+5\sqrt{15}}{-22}$

30. (UN 2010 PAKET A)

$\dfrac{4(2+\sqrt{3})(2-\sqrt{3})}{(3+\sqrt{5})}=\ldots$

A) $-(3-\sqrt{5})$ B) $-\frac{1}{4}(3-\sqrt{5})$ C) $\frac{1}{4}(3-\sqrt{5})$

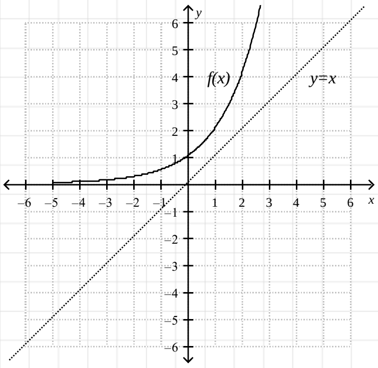
D) $(3-\sqrt{5})$ E) $(3+\sqrt{5})$

6. LOGARITHM

Definition: $f(x)=a^x$ where $a>0,\ a\neq 1$ is called an **exponential function.**

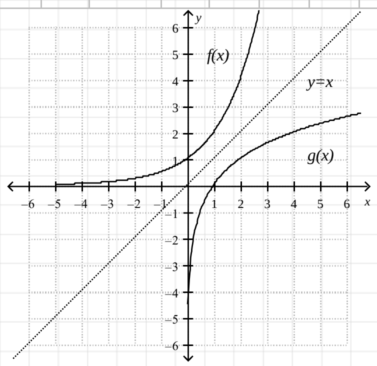
Let's draw the graph of $f(x)=2^x$ function.

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	1/8	1/4	1/2	1	2	4	8



Then get the symmetry with respect to $y=x$ line. That is, change x and y values.

x	1/8	1/4	1/2	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3



$f(x)$ and $g(x)$ are called inverse of each other. To find inverse, change x and y, then leave y alone.

That is, $y=2^x \Rightarrow x=2^y$

Then we need to leave y alone.

How is it possible?

Definition:

$x=a^y \Leftrightarrow y=\log_a x$ where $a>0$, x>0, $a\neq 1$

Example: Write the following terms in logarithmic form.

- $64=2^6 \Rightarrow$
- $243=3^5 \Rightarrow$
- $\frac{1}{8}=2^{-3} \Rightarrow$

Example: Write the following terms in exponential form.

- $\log_{11} 121=2 \Rightarrow$
- $\log_3 81=4 \Rightarrow$
- $\log_{10}\frac{1}{100}=-2 \Rightarrow$

Properties of Logarithm

Let $a>0, x>0, y>0$.

1. $\log_a 1=0$

Example:

- $\log_3 1=$
- $\log_9 1=$
- $\log_2(x^2-x-1)=0 \Rightarrow x=$

2. $\log_a a=1$

Example:

- $\log_8 8=$
- $\log_{\frac{1}{4}}\frac{1}{4}=$
- $\log_6(x^2-5x)=1 \Rightarrow x=$

3. $a^{\log_a x}=x$

Example:

- $7^{\log_7 x}=$
- $\left(\frac{1}{3}\right)^{\log_{\frac{1}{3}} x}=$
- $(x^2-2x)^{\log_3 5}=5 \Rightarrow x=$

4. $\log_a(x\cdot y)=\log_a x+\log_a y$

Example:

- $\log_4(4\cdot y)=$
- $\log_2 6=$

5. $\log_a\left(\dfrac{x}{y}\right)=\log_a x-\log_a y$

Example:

- $\log_3\left(\dfrac{81}{5}\right)=$
- $\log_3 60-\log_3 4=$

6. $\log_a x^n=n\cdot\log_a x$

Example:

- $\log_5 6^{10}=$
- $\log_7 49=$
- $\log_a\dfrac{x^m y^n}{z^p t^q}=$
- $\log_3(81\cdot\sqrt[3]{9})=$

7. $\log_a b\cdot\log_b c=\log_a c$

Example:

- $\log_5 7\cdot\log_7 125=$
- $\log_3 2\cdot\log_2 25\cdot\log_5 3=$

8. $\log_a b=\dfrac{1}{\log_b a}$

Example:

- $\log_{36} 6=$
- $\dfrac{1}{\log_{64} 2}=$
- $\dfrac{1}{\log_2 x}+\dfrac{1}{\log_3 x}+\dfrac{1}{\log_6 x}=2\Rightarrow x=$

9. $\log_a b=\dfrac{\log_c b}{\log_c a}$

Example:

- $\log_2(\log_3 4\cdot\log_4 81)=$

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10. $\log_{a^m} x=\dfrac{1}{m}\cdot\log_a x$

Example:

- $\log_{\sqrt{2}}\sqrt[3]{3}=$

Example: Solve the equation of $9^{\log_3\sqrt{x^2+2x}}=8$.

Example: If $\log_3(8!)=a$, then find $\log_3(9!)$ in terms of a.

Example: If $\log 2=a$ and $\log 7=b$, then find $\log\sqrt{140}$ in terms of a and b.

Example: If $\log_6 2=a$, then find $\log_{18} 36$ in terms of a.

Example: If $y=\dfrac{e^x-1}{2e^x}$, then find x in terms of y.

Example: (UN 2012/C37) If ${}^5\log 3=a$ and ${}^3\log 4=b$, then ${}^4\log 15=\ldots$

Example: (UN 2010 PAKET A)

$$\frac{{}^{3}\log\sqrt{6}}{\left({}^{3}\log 18\right)^{2}-\left({}^{3}\log 2\right)^{2}}=\ldots$$

Example: (UN 2005)

$${}^{r}\log\frac{1}{p^{5}}\cdot{}^{q}\log\frac{1}{r^{3}}\cdot{}^{p}\log\frac{1}{q}=\ldots$$

Example: Solve the following equations.

- $2^{x}+2^{x-1}=48\Rightarrow x=$
- $2^{x-1}=3\Rightarrow x=$
- $\dfrac{11+\sqrt[x]{16}}{1+\sqrt[2x]{16}}=5\Rightarrow x=$
- $\log_3(x+17)-2=\log_3(2x)\Rightarrow x=$
- $3^{\log x^{2}}-4\cdot 3^{\log x}+3=0\Rightarrow x=$
- $\log_3 x-2\log_x 3=1\Rightarrow x=$
- $x^{\log_2 x}=16\Rightarrow x=$

Review Test

1. What is the logarithmic form of $y=5^{x-1}$?

A) $x=\log_y 5-y$ B) $x=\log_y 5+1$ C) $x=\log_5 y+1$ D) $x=\log_5 y-1$ E) $x=\log_5 y^{-1}$

2. What is the solution of the equation $2^x=5$?

A) $\log_2 5$ B) $\log_5 2$ C) $\log 2$ D) $\log 5$ E) $\log 2^5$

3. $\log_{0.5} 32=?$

A) -5 B) -4 C) -3 D) 3 E) 4

4. $\log_{0.5} 0.25=?$

A) -2 B) -1 C) 1/2 D) 2 E) 5

5. $\log_2 10+\log_2 4+\log_2(0.2)=?$

A) 1 B) 2 C) 3 D) 4 E) 5

6. $\log_2 x=3 \Rightarrow x=?$

A) 6 B) 7 C) 8 D) 16 E) 32

7. $\log_x 125=3 \Rightarrow x=?$

A) 125^3 B) 3^{125} C) $125/3$ D) 25 E) 5

8. $2^{2(\log 100)-3}=?$

A) 2 B) 1 C) 0.25 D) 1/2 E) 4

9. If $\log_{10}\sqrt{5}=a$, then find $\log_{10}\sqrt[3]{0.005}$ in terms of a .

A) $\frac{2a-3}{3}$ B) $2a-3$ C) $3a$ D) $2a$ E) $\frac{1}{3}$

10. $\log_7 4\cdot\log_3 49\cdot\log_{16} 3=?$

A) $\frac{-1}{2}$ B) $\frac{1}{4}$ C) 1 D) 3 E) 4

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11. $\log_3 \frac{1}{9} + \log_{\frac{1}{9}} 3 = ?$

A) $-\frac{5}{2}$ B) $-\frac{3}{2}$ C) 0 D) $\frac{7}{2}$ E) 3

12. $\log_2 25 \cdot \log_3 2 \cdot \log_5 3 = ?$

A) 1 B) 2 C) 3 D) 4 E) 5

13. $e^{\ln 6 - \ln 2} = ?$

A) 1 B) 2 C) 3 D) e^2 E) e^3

14. $\log_2 0.25 + 27^{\log_3 2} = ?$

A) 2 B) 4 C) 6 D) 8 E) 10

15. $\frac{1}{\log_5 20} + \frac{1}{\log_8 20} + \frac{1}{\log_{10} 20} + = ?$

A) 1 B) 2 C) 3 D) 4 E) 5

16. If $\log_x(ab) = 8$ and $\log_x(\frac{a}{b}) = -2$, which one of the following terms is a ?

A) x B) x^2 C) x^3 D) x^4 E) x^5

17. (UN 2012/B25)

If $^2\log 3 = x$ and $^2\log 10 = y$, then $^6\log 120 = \ldots$

A) $\frac{x+y+2}{x+1}$ B) $\frac{x+1}{x+y+2}$ C) $\frac{x}{xy+2}$ D) $\frac{xy+2}{x}$ E) $\frac{2xy}{x+1}$

18. (UN 2010 PAKET B)

$$\frac{{}^{27}\log 9 + {}^{2}\log 3 \cdot {}^{\sqrt{3}}\log 4}{{}^{3}\log 2 - {}^{3}\log 18} = \ldots$$

A) $-\frac{14}{3}$ B) $-\frac{14}{6}$ C) $-\frac{10}{6}$ D) $\frac{14}{6}$ E) $\frac{14}{3}$

19. (UN 2008 PAKET A/B)

If $^7\log 2 = a$ and $^2\log 3 = b$, then $^6\log 14 = \ldots$

A) $\frac{a}{a+b}$ B) $\frac{a+1}{b+1}$ C) $\frac{a+1}{a(b+1)}$ D) $\frac{b+1}{a+1}$ E) $\frac{b+1}{b(a+1)}$

20. (UN 2004)

If $^2\log 5 = x$ and $^2\log 3 = y$, then $^2\log 300^{\frac{3}{4}} = \ldots$

A) $\frac{2}{3}x + \frac{3}{4}y + \frac{3}{2}$ B) $\frac{3}{2}x + \frac{3}{2}y + 2$ C) $2x + y + 2$

D) $2x + \frac{3}{4}y + \frac{3}{2}$ E) $2x + \frac{3}{2}y + 2$