NORM MAPS AND THE TATE CONSTRUCTION

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ABSTRACT. These are the notes prepared by Yiming Wang for the talk *Norm maps and the Tate construction* for the seminar on *Goodwillie Calculus* organized by Marc Hoyois during the winter semester $2025 - 2026$ at the University of Regensburg. These notes aim to give a detailed treatment of the construction of the norm maps in the setting of stable categories and the classification of n-excisive functors. Throughout these notes, we use *category* to mean ∞-category.

CONTENTS

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1. Norm maps and the Tate construction

In the literature, *ambidexterity* is typically formulated as a property of a map of *animae* relative to a target ∞-category $\mathcal{C}$: namely, that for the induced precomposition functor on $\mathcal{C}$*-valued local systems*, the left and right adjoints coincide (equivalently, the canonical comparison between them is an equivalence). When ambidexterity holds for a class of maps (for example, finite sets), it often forces strong structural constraints on $\mathcal{C}$ (for example, semiadditivity). By progressively increasing the truncatedness of the maps under consideration, one is led to the notion of *higher semiadditivity*. Owing to time constraints, we will not develop these topics in the talk.

In these notes, we will construct a canonical comparison map between the colimit and limit of a G-equivariant object in a suitably well-behaved category, where G is a finite group. Concretely, we define the *norm map* relating *orbits* and *fixed points*, and we introduce the *Tate construction*, which measures the failure of the norm map to be an equivalence.

The term *norm map* arises from the following classical considerations:

Remark 1.1 (Classical norm maps).

(a) Let L/K be a finite separable field extension. The *norm map* $\mathrm{N}_{L/K}\colon L^\times \to K^\times$ is a multiplicative group homomorphism given by

$$\mathrm{N}_{L/K}(x) = \prod_{g\in G} g(x),$$

where $G = \mathrm{Gal}(L/K)$ is the Galois group of the field extension.

(b) Let G be a finite group and M be a G-module. We denote M_G the G*-orbits* of M and M^G the G*-invariant* of M. Then there is a well-defined *norm map* $\mathrm{Nm}_G\colon M_G \to M^G$ given by

$$\mathrm{Nm}_G(x) = \sum_{g\in G} gx.$$

Viewing a G-module M as a functor $M\colon BG \to \mathrm{Ab}$ in the classical sense, its orbits and invariants are given by the colimit and limit of this functor respectively, and the norm map gives a canonical comparison.

The significance of the norm map construction is that it allows one to define *Tate cohomology*, which links group homology and group cohomology into a single functorial construction that satisfies the desired cohomological properties, e.g. short exact sequences of G-modules induces long exact sequences of cohomology.

Definition 1.2. Let G be a finite group, and let M be a G-module. The *Tate cohomology of* G with coefficients in M is defined as $\hat{H}^{i-1}(G,M) = H_{-i}(G,M)$ for $i<0$, $\hat{H}^i(G,M) = H^i(G,M)$ for $i>0$ and there is an exact sequence

$$0 \to \hat{H}^{-1}(G,M) \to M_G \xrightarrow{\mathrm{Nm}_G} M^G \to \hat{H}^0(G,M) \to 0.$$

In particular, the zero-th Tate cohomology is the cokernel of the norm map.

We would like to adopt an analogue of the above, called the *Tate construction* associated to the norm map. Before doing so, we will slightly increase the generality. Note that the colimit and limit of functors $BG \to \mathrm{Ab}$ are special instances of the left and right Kan extensions of f^*, where $f\colon BG \to *$ is the unique map to the point.

Remark 1.3. Let $f\colon X \to Y$ be a map between animae, and $\mathcal{C}$ be a category with limits and colimits indexed by $X_{/y}$ and $X_{y/}$ respectively, for each $y \in Y$. Then Kan extensions along f exist and are computed pointwise. In particular, the precomposition functor

$$f^*\colon \mathrm{Fun}(Y,\mathcal{C}) \to \mathrm{Fun}(X,\mathcal{C})$$

admits a left adjoint $f_! = \mathrm{Lan}_f(-)$ and a right adjoint $f_* = \mathrm{Ran}_f(-)$.

In this setting, our goal is to produce a natural transformation $\mathrm{Nm}_f\colon f_! \to f_*$. Before doing so, we need to recall the basic machinery of adjointable diagrams.

Definition 1.4 (Adjointable diagrams)**.** A commutative square of functors of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f^*"] \arrow[d, "g^*"'] & \mathcal{D} \arrow[d, "g'^*"] \\ \mathcal{C}' \arrow[r, "f'^*"'] & \mathcal{D}' \end{tikzcd}$$

is *right adjointable*, if the functors f^* and f'^* admits right adjoint f_* and f'_* respectively, and the canonical *Beck-Chevalley transformation*

$$\mathrm{BC}_*\colon g^* \circ f_* \to f'_* \circ f'^* \circ g^* \circ f_* \xrightarrow{\sim} f'_* \circ g'^* \circ f^* \circ f_* \to f'_* \circ g'^*$$

is a natural isomorphism.

Lemma 1.5 (Right adjointablility)**.** *Given a pullback diagram of animae as follows:*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Let $\mathcal{C}$ be a category that admits limits indexed by $X_{/y}$ and $X'_{/y'}$, where $y \in Y$ and $y' \in Y'$. Then the induced commutative square

$$\begin{tikzcd} \mathrm{Fun}(X, \mathcal{C}) \arrow[r, "f^*"] \arrow[d, "g^*"'] & \mathrm{Fun}(Y, \mathcal{C}) \arrow[d, "g'^*"] \\ \mathrm{Fun}(X', \mathcal{C}) \arrow[r, "f'^*"'] & \mathrm{Fun}(Y', \mathcal{C}) \end{tikzcd}$$

is right adjointable.

Proof. By the pointwise criterion for natural isomorphisms, it suffices to show that, for every functor $F \in \mathrm{Fun}(Y, \mathcal{C})$, the natural transformation $g^* f_* F \to f'_* g'^* F$ is an isomorphism in $\mathrm{Fun}(X', \mathcal{C})$. Unwinding the definitions, it suffices to show that, for each $x' \in X'$, the canonical map

$$\lim(Y_{/g(x')} \to Y \xrightarrow{F} \mathcal{C}) \to \lim(Y'_{/x'} \to Y \xrightarrow{F} \mathcal{C})$$

is an isomorphism. Therefore, we need to show that the map $Y'_{/x'} \to Y_{/g(x')}$ is an isomorphism over Y. By definition, $Y'_{/x'}$ can be written as a pullback as follows:

$$\begin{tikzcd} Y'_{/x'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "f'"] \\ X'_{/x'} \arrow[r] & X' \end{tikzcd}$$

Since X' is an anima, $X'_{/x'}$ is again an anima with a terminal object, hence contractible. Therefore, we may idenfity $Y'_{/x'}$ with the fiber $\mathrm{fib}_{x'}(f')$. Similarly, we may idenfity $Y_{/g(x')}$ with the fiber $\mathrm{fib}_{g(x')}(f)$. The claim then follows from the original diagram being a pullback square, since it induces an equivalence on fibers. $\square$

Remark 1.6 (Left adjointablility). Similar to Definition 1.4 and Lemma 1.5, there is a notion of *left adjointable diagram*, and the Beck-Chevalley transformation $\mathrm{BC}_!$.

We will construct the norm map inductively, by first constructing that of the diagonal map.

Proposition 1.7. *Let $f\colon X \to Y$ be a map between animae, $\Delta\colon X \to X \times_Y X$ be the diagonal map and $\mathrm{pr}_0, \mathrm{pr}_1\colon X \times_Y X \to X$ be the projections. Any comparision map*

$$\mathrm{Nm}_\Delta\colon \Delta_! \xrightarrow{\sim} \Delta_*.$$

induces a canonical comparison map

$$\mathrm{Nm}_f\colon f_! \to f_*$$

called the norm map *determined by f.*

Proof. The natural transformation Nm_Δ induces a natural transformation

$$\mathrm{pr}_0^* \to \Delta_*\Delta^* \mathrm{pr}_0^* \simeq \Delta_* \overset{\mathrm{Nm}_\Delta^{-1}}{\simeq} \Delta_! \simeq \Delta_!\Delta^* \mathrm{pr}_1^* \to \mathrm{pr}_1^*,$$

since $\mathrm{pr}_0 \circ \Delta = \mathrm{id}_X = \mathrm{pr}_1 \circ \Delta$. By adjunction, this gives us a natural transformation

$$\mathrm{id}_{\mathrm{Fun}(X,\mathcal{C})} \to (\mathrm{pr}_0)_* \circ \mathrm{pr}_1^*$$

Consider the pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

By Lemma 1.5, there is a canonical natural isomorphism

$$\mathrm{BC}_*\colon f^* f_* \xrightarrow{\sim} (\mathrm{pr}_0)_* \mathrm{pr}_1^*.$$

This gives us a natural transformation $\mathrm{id}_{\mathrm{Fun}(X,\mathcal{C})} \to f^* f_*$, which induces by adjunction, the map

$$\mathrm{Nm}_f\colon f_! \to f_*.$$

We will refer this to the *norm map determined by f.* □

We now construct more generally the norm map by induction on the truncatedness of the fibers.

Definition 1.8 (Animae of finite n-type)**.** Let X be an anima, we say that it is n-finite for $n \geq -1$, if the following conditions are satisfied:

(a) For each $m > n$ and any base point $x \in X$, the homotopy groups $\pi_m(X, x)$ vanish for $m > n$.
(b) For all $m \in \mathbf{N}$ and any base point $x \in X$, the homotopy groups $\pi_*(X, x)$ are finite.

We say that X is (-2)-finite, if it is contractible.

Remark 1.9 (Examples of n-finite animae). We collect some basic facts about n-finite animae:

(a) (-2)-finite anima is the same as a point $*$.
(b) (-1)-finite animae are $*$ and $\varnothing$.
(c) 0-finite animae are finite sets.
(d) Let $f\colon X \to Y$ be a map between animae. If the fibers of f are all m-finite, then the fibers of the diagonal Δ are all $(m-1)$-finite, for $m > -2$.
(e) Note that m-finite animae need not be finite animae (i.e. animae generated from the point under finite colimits), and the converse implication also fails.

Remark 1.10 (Base step). Let $f\colon X \to Y$ be a map between animae, whose fibers are (-1)-finite. Then the fibers of the diagonal map Δ are contractible, hence Δ is an equivalence. The norm map Nm_Δ is then the canonical natural isomorphism that identify Δ_* and $\Delta_!$ as the inverse to Δ^*. We claim that when $\mathcal{C}$ is semiadditive, the map Nm_f is an equivalence.

Proposition 1.11 (Characterization of pointedness)**.** *Let $\mathcal{C}$ be a category with an initial and a final object. The following conditions are equivalent:*

(a) For each map $f\colon X \to Y$ between animae, whose fibers are (-1)-finite animae, the natural transformation

$$\mathrm{Nm}_f\colon f_! \to f_*$$

is an equivalence.

(b) For each map $f\colon X \to *$, where X is a (-1)-finite anima, the natural transformation

$$\mathrm{Nm}_f\colon f_! \to f_*$$

is an equivalence.

(c) The category $\mathcal{C}$ is pointed.

Proof. $(a) \Rightarrow (c)$: let $Y \simeq *$ and $X \simeq \varnothing$. By the pointwise criterion for natural isomorphisms, for each $F \in \mathrm{Fun}(\varnothing, \mathcal{C}) \simeq *$, the map $f_!F \to f_*F$ is an isomorphism in $\mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$. Unwinding the definition, the map $\varnothing \to *$ is an isomorphism, where $\varnothing$ is the initial object of $\mathcal{C}$ and $*$ is the terminal object. This implies that $\mathcal{C}$ is pointed.

$(c) \Rightarrow (b)$: if $X \simeq \varnothing$, then the proof is identical as above. If $X \simeq *$, then up to contractible choice, the diagonal map can be identified as $\Delta \simeq \mathrm{id}_*$, and everything in the diagram degenerates to the point, hence the norm map $\mathrm{Nm}_f \simeq \mathrm{id}_{\mathcal{C}}$, which is clearly an isomorphism.

$(b) \Rightarrow (a)$: it suffices to show that for each $F \in \mathrm{Fun}(X, \mathcal{C})$ and $y \in Y$, the map

$$\mathrm{Nm}_f(F(y))\colon f_!F(y) \to f_*F(y)$$

is an isomorphism. We have the following pullback square:

$$\begin{tikzcd} \mathrm{fib}_y(f) \arrow[r, "\iota_y"] \arrow[d, "p_y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ * \arrow[r, "y"'] & Y \end{tikzcd}$$

By Lemma 1.5, we may identify $y^*\mathrm{Nm}_f$ with

$$(p_y)_!\iota_y^* \xrightarrow{\mathrm{BC}_!^{-1}} y^*f_! \xrightarrow{y^*\mathrm{Nm}_f} y^*f_* \xrightarrow{\mathrm{BC}_*} (p_y)_*\iota_y^*.$$

In fact, we can show that this can be identified with the norm map $\mathrm{Nm}_{p_y}\iota_y^*$. The verification is by calculation with $f_!$ and f_*. We remark that an inductive proof with full generality can be found in [HL13, Proposition 4.2.1]. Therefore, the case reduces to the case $Y \simeq *$. ☐

Remark 1.12. Let $\mathcal{C}$ be a pointed category, $f\colon X \to Y$ be a map between animae with 0-finite fibers. Then the diagonal map $\Delta\colon X \to X \times_Y X$ has (-1)-finite fibers. By Proposition 1.11, the map Nm_Δ exists and is an equivalence. Therefore, by Proposition 1.7, we obtain a norm map

$$\mathrm{Nm}_f\colon f_! \to f_*.$$

We now claim that if $\mathcal{C}$ is semiadditive, then the norm map constructed above is an equivalence.

Proposition 1.13 (Characterization of semiadditivity)**.** *Let $\mathcal{C}$ be a pointed category with finite products and coproducts. The following statements are equivalent:*

(a) For each map $f\colon X \to Y$ between animae, whose fibers are 0-finite, the norm map

$$\mathrm{Nm}_f\colon f_! \to f_*$$

is an equivalence.

(b) For each map $f\colon X \to *$, where X is a 0-finite anima, the norm map

$$\mathrm{Nm}_f\colon f_! \to f_*$$

is an equivalence.

(c) The category $\mathcal{C}$ is semiadditive.

Proof. $(a) \Rightarrow (c)$: let S be a finite set, $Y \simeq *$ and $f\colon S \to *$. In this case, each functor $F \in \mathrm{Fun}(S, \mathcal{C})$ corresponds to a family of objects $(F(s))_{s\in S}$, and the norm map is a comparison map

$$\mathrm{Nm}_f(F)\colon \coprod_{s\in S} F(s) \to \prod_{s\in S} F(s)$$

in $\mathcal{C}$. We will first compute the norm map of the diagonal. Let $G \in \mathrm{Fun}(S, \mathcal{C})$ corresponds to the collection of objects $(G(s))_{s\in S}$. By the same virtue as Proposition 1.11, we can take the pullback along a point $(s,t)\colon * \to S \times S$ as follows:

$$\begin{tikzcd} \mathrm{fib}_{(s,t)}(\Delta) \arrow[r, "\iota"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "\Delta"] \\ * \arrow[r, "(s,t)"'] & S \times S \end{tikzcd}$$

Pullback diagrams of sets are computed in the usual sense, if $s = t$, then $\mathrm{fib}_{(s,t)}(\Delta) \simeq *$, and the norm map of the diagonal is given by the identity map:

$$\mathrm{Nm}_\Delta(G)(s,s) \simeq \mathrm{Nm}_p(\iota^*(G)) \simeq \mathrm{id}_{G(s)}$$

If $s \neq t$, then $\mathrm{fib}_{(s,t)}(\Delta) \simeq \varnothing$, and the norm map is given by the zero map. Therefore, unwinding the definition, for $H \in \mathrm{Fun}(S, \mathcal{C})$ by $H = (H(s))_{s\in S}$, the map

$$\mathrm{pr}_0^*(F)(s,t) \to \mathrm{pr}_1^*(F)(s,t)$$

is given by the zero map $0\colon G(s) \to G(t)$ when $s \neq t$, and the identity map $\mathrm{id}_{G(s)}\colon G(s) \to G(s)$ when $s = t$. Unwinding the definition furthermore, we find that the map

$$\mathrm{Nm}_f(F)\colon \coprod_{s\in S} F(s) \to \prod_{s\in S} F(s)$$

is given by the $(|S| \times |S|)$-identity matrix. Since $\mathcal{C}$ is semi-additive, the norm map Nm_f is an natural isomorphism.

$(c) \Rightarrow (b)$: as above, we have computed the norm map for the case $f\colon S \to *$, where S is a finite set. This is an isomorphism if and only if $\mathcal{C}$ is semiadditive.

$(b) \Rightarrow (a)$: this follows by the same virtue as Proposition 1.11. □

Remark 1.14. Let $\mathcal{C}$ be a semiadditive category with limits and colimits indexed by 1-finite animae. Assume that $f\colon X \to Y$ is a map between animae with 1-truncated fibers. Then the diagonal Δ has 0-truncated fibers. By Proposition 1.13, the norm map Nm_Δ is an isomorphism. This gives us a map

$$\mathrm{Nm}_f\colon f_! \to f_*.$$

Again we refer to it as the *norm map determined by f*.

Remark 1.15 (Well-definedness). The notion of norm maps defined in Remark 1.10, Remark 1.12, Remark 1.14 coincides.

Definition 1.16. Let $\mathcal{C}$ be a semiadditive category with all BG-indexed limits and colimits. In Remark 1.14, take $f\colon BG \to *$, we denote the resulting natural transformation as

$$\mathrm{Nm}_G\colon (-)_G \to (-)^G.$$

We will refer to it as the *norm map determined by G*.

Definition 1.17. Let $\mathcal{C}$ be a stable category with all BG-indexed limits and colimits. We call the functor

$$(-)^{tG}\colon \mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}, \quad X \mapsto \mathrm{cofib}(\mathrm{Nm}_G(X))$$

the *Tate construction*.

2. Characterization of n-excisive functors

We conclude with an application to the classification of n-excisive functors in Goodwillie calculus. Recall that we have the following theorem:

Theorem 2.1. *Let $\mathcal{C}$ be a category which admits finite colimits and a final object, let $\mathcal{D}$ be a differentiable category. Let $\mathcal{E} \subseteq \mathrm{Fun}([1] \times_{\{1\}} [1], \mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ be the full subcategory spanned by diagrams of functors of the form:*

$$E \to H \leftarrow *,$$

where $*$ is the terminal object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, E is $(n-1)$-excisive and reduced and H is n-homogeneous. Then the pullback functor induces an equivalence of categories $\mathcal{E} \simeq \mathrm{Exc}^n_*(\mathcal{C}, \mathcal{D})$. Let F be an n-excisive functor, then we may choose E to be $\mathrm{P}_{n-1}(F)$ and $H = \mathrm{fib}(F \to \mathrm{P}_{n-1}(F))$.

Let $\mathcal{C}$ and $\mathcal{D}$ be stable categories and $\mathcal{D}$ admits countable limits and colimits. In order to classify n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$, we need to classify natural transformations $E \to H$.

In this case, every n-homogeneous functor can be written as

$$(h\Delta)_{\Sigma_n} \colon \mathcal{C} \to \mathcal{D}, \quad X \mapsto h(X, \dots, X)_{\Sigma_n},$$

where h is a symmetric n-linear functor, see [Lur17, Theorem 6.1.4.14]. By [Lur17, Corollary 6.1.2.7], the suspension functor induces an equivalence of stable categories

$$[1] \colon \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}).$$

Therefore, natural transformations $E \to H$ corresponds to natural transformation $E \to (h\Delta)_{\Sigma_n}[1]$.

Lemma 2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be stable categories and $\mathcal{D}$ admits countable limits and colimits. The exact sequence*

$$(h\Delta)^{t\Sigma_n} \to (h\Delta)_{\Sigma_n}[1] \to (h\Delta)^{\Sigma_n}[1]$$

induces an equivalence of mapping spectra:

$$\mathrm{Map}(E, (h\Delta)^{t\Sigma_n}) \simeq \mathrm{Map}(E, (h\Delta)_{\Sigma_n}[1]).$$

Proof. Since $\mathcal{C}$ and $\mathcal{D}$ are stable, a functor f is n-excisive if and only if f^{op} is n-excisive, see [Lur17, Corollary 6.1.1.17]. Since $(B\Sigma_n)^{\mathrm{op}} \simeq B\Sigma_n$, and $(-)^{\mathrm{op}}$ swaps colimits and limits, we have

$$((h\Delta)^{\Sigma_n})^{\mathrm{op}} \simeq ((h\Delta)^{\mathrm{op}})_{\Sigma_n}.$$

Clearly $(h\Delta)^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is symmetric n-linear. Therefore, the functor $(h\Delta)^{\Sigma_n}$ is n-*cohomogeneous*, i.e. $\mathrm{Map}(E, (h\Delta)^{\Sigma_n}) \simeq *$ for each $(n-1)$-excisive functor E. This gives us the following exact sequence

$$\mathrm{Map}(E, (h\Delta)^{t\Sigma_n}) \to \mathrm{Map}(E, (h\Delta)_{\Sigma_n}[1]) \to *.$$

since mapping spectra preserves limits in both variables. This implies that the left map is an isomorphism, by the functoriality of the pullback construction. □

Therefore, we have reduced the problem of classifying n-excisive functors to classify natural transformations $E \to (h\Delta)^{t\Sigma_n}$. In fact, this is already covered by the induction hypothesis, since $(h\Delta)^{t\Sigma_n}$ is $(n-1)$-excisive. Before we prove this, we will need the following fact:

Lemma 2.3. *Let G be a finite group and let $\mathcal{C}$ be a semiadditive category which admits all BG-indexed limits and colimits. Denote $i \colon * \to BG$ to be the inclusion of the base point and $f \colon BG \to *$ the unique map to the point. Assume that $N \colon BG \to \mathcal{C}$ is left Kan extended along i from $M \in \mathcal{C}$, then we have*

$$N \simeq i_! M \simeq \coprod_{g \in G} M \simeq \prod_{g \in G} M \simeq i_* M$$

Then the norm map is given by

$$\mathrm{Nm}_G(N) \colon f_! N \simeq f_! i_! M \simeq \mathrm{id}_! M \simeq \mathrm{id}_* M \simeq f_* i_* N \simeq f_* N.$$

In particular, the norm map $\mathrm{Nm}_G(N)$ *is an equivalence.*

Lemma 2.4. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be stable categories, where* $\mathcal{D}$ *admits countable limits. Let* h *be a symmetric* n*-linear functor, then* $(h\Delta)^{t\Sigma_n}$ *is* $(n-1)$*-excisive.*

Proof. The excisiveness of a multivariable functor can be given by the excisiveness in each variables, see [Lur17, Corollary 6.1.4.14]. Therefore, the functor $(h\Delta)$ is n-excisive. Since the n-excisive approximation $\mathrm{P}_n\colon \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C},\mathcal{D})$ exhibits $\mathrm{Exc}^n(\mathcal{C},\mathcal{D})$ as a Bousfield localization of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, the full-subcategory $\mathrm{Exc}^n(\mathcal{C},\mathcal{D})$ is closed under all limits. It is also closed under all colimits, since F is n-excisive if and only if F^{op} is n-excisive, whenever $\mathcal{C}$ and $\mathcal{D}$ are stable. Therefore, the functor $(h\Delta)^{\Sigma_n}$ and $(h\Delta)_{\Sigma_n}$ are n-excisive. Since there is the following cofiber sequence of n-excisive functors

$$(h\Delta)_{\Sigma_n} \to (h\Delta)^{\Sigma_n} \to (h\Delta)^{t\Sigma_n},$$

the Tate construction $(h\Delta)^{t\Sigma_n}$ is also n-excisive.

Therefore, to show that $(h\Delta)^{t\Sigma_n}$ is $(n-1)$-excisive, it suffices to show that the cross effects cr_n carries each object to the zero object, since the n-th cross effect controlls excisiveness at level n, see [Lur17, Corollary 6.1.4.10]. Consider the functor $(h\Delta)\colon B\Sigma_n \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$, then $\mathrm{cr}_n(h\Delta)$ determines a functor $B\Sigma_n \to \mathrm{Fun}(\mathcal{C}^n,\mathcal{D})$, which admits a limit and a colimit, by the assumption on $\mathcal{D}$. Since $\mathcal{D}$ is stable and limits and colimits in functors categories are computed pointwise, the functor category $\mathrm{Fun}(\mathcal{C}^n,\mathcal{D})$ is stable and there is a canonical norm map

$$\mathrm{Nm}\colon (\mathrm{cr}_n(h\Delta))_{\Sigma_n} \to (\mathrm{cr}_n(h\Delta))^{\Sigma_n}$$

The cross-effect is given by $\mathrm{cr}_n = \mathrm{Red}\circ(\oplus^*)$, where $\mathrm{Red}\colon \mathrm{Fun}(\mathcal{C}^n,\mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}^n,\mathcal{D})$ is the reduction functor and $\oplus\colon \mathcal{C}^n \to \mathcal{C}$ is the coproduct functor. Since limits and colimits in functor categories are computed pointwise, precomposition functors always preserves all limits and colimits, in particular, $(\oplus^*)$ preserves all limits and colimits. The reduction functor is built from finite limits of limits and colimits preserving functors, hence preserves all limits and colimits, since $\mathcal{D}$ is stable. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
(\mathrm{cr}_n(h\Delta))_{\Sigma_n} \arrow[r, "\mathrm{Nm}"] \arrow[d, "\simeq"'] & (\mathrm{cr}_n(h\Delta))^{\Sigma_n} \arrow[d, "\simeq"] \\
\mathrm{cr}_n((h\Delta)_{\Sigma_n}) \arrow[r, "\mathrm{cr}_n(\mathrm{Nm})"] & \mathrm{cr}_n((h\Delta)^{\Sigma_n})
\end{tikzcd}$$

Verifying the commutativity of this diagram requires a modest amount of 2-categorical diagram chasing. We omit the details and refer the reader to [CSY22, Theorem 3.2.3]. By above, we have

$$\mathrm{cr}_n((h\Delta)^{t\Sigma_n}) \simeq (\mathrm{cr}_n(h\Delta))^{t\Sigma}.$$

To show that the left hand side vanishes, it suffcies to show that the norm map Nm is an natural isomorphism. By the virtue of Lemma 2.3, it suffices to show that the functor $\mathrm{cr}_n(h\Delta)$ is left Kan extended along $* \to B\Sigma_n$. In other words, an induced representation of the symmetric group Σ_n. This follows from canonical equivalence as follows:

$$\mathrm{cr}_n(h\Delta) \simeq \bigoplus_{\sigma\in\Sigma_n} \mathrm{Red}(h^\sigma) \simeq \bigoplus_{\sigma\in\Sigma} h^\sigma \simeq \bigoplus_{\sigma\in\Sigma} h,$$

since h is symmetric n-linear, see [Lur17, Proposition 6.1.4.13]. □

Theorem 2.5 (Classification of n-excisive functors)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be stable categories, where* $\mathcal{D}$ *admits countable colimits. An* n*-excisive functor* $F\colon \mathcal{C}\to\mathcal{D}$ *is determined by the following data:*

- *(a) an* $(n-1)$ *excisive functor* E*, which can be take as* $\mathrm{P}_{n-1}(F)$*.*
- *(b) a symmetric* n*-linear functor* $h\colon \mathcal{C}^n\to\mathcal{D}$*, which corresponds to the* n*-homogeneous functor* $\mathrm{fib}(F\to \mathrm{P}_{n-1}(F))$*.*

(c) A natural transformation between $(n-1)$-excisive functors $E \to (h\Delta)^{t\Sigma_n}$, which is well-understood by induction hypothesis.

Proof. The proof is to combine Theorem 2.1, Lemma 2.2 and Lemma 2.4. By the pasting lemma, the functor F can be recovered by taking the pullback of $(h\Delta)^{\Sigma_n} \to (h\Delta)^{t\Sigma}$ along $E \to (h\Delta)^{t\Sigma}$. $\square$

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