Algebraic K-Theory

Yiming Wang

2025-10-23

Contents

Preface

This document contains unofficial notes based on the lecture *Algebraic K-Theory*, delivered by Marc Hoyois at the University of Regensburg in the summer semester of 2025. The author has rewritten some of the original lecture material, and there may be some mistakes; any errors are solely my responsibility. Kindly report them to solvaphes[at]gmail[dot]com. Ironically speaking, the parts that touches scheme theory are still unreadable, since the author has learned this course before learning algebraic geometry.

Overview

Algebraic K-Theory was initially thought to be the *cohomology theories* for rings: the goal is to define functors

$$K_n : \mathsf{Ring} \to \mathsf{Ab}$$

for all $n \in \mathbb{N}$. For instance, given a ring R, we may define

$$K_0(R) = \mathbb{Z}[\text{isomorphism classes of finitely generated projective } R\text{-modules}]/\sim$$

where the equivalence relation is specified by $[P \oplus Q] \sim [P] + [Q]$. Likewise, we may also define the first K-group $K_1(R) = \mathrm{GL}_\infty(R)^{\mathrm{ab}}$, where $(-)^{\mathrm{ab}}$ is the Abelianization functor.

The idea of these definitions has an insightful perspective, which is captured by the slogan *K-theory is asymptotic linear algebra.* Although finite-dimensional vector spaces are well understood, linear algebra in general can be remarkably sophisticated. This perspective allows one to translate difficult problems in linear algebra into relatively more tractable problems in K-theory.

Algebraic K-theory for rings is initially motivated by the K-theory for schemes in algebraic geometry. Although algebraic geometry is not one of the formal prerequisites for the course, we include examples from it.

In fact, we could also try to extend the definition for K-theory to other mathematical objects. For example, there is a natural question illustrated by the following picture:

$$\begin{tikzcd}
\mathrm{CRing} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\subseteq$}"] & \mathrm{Sch}^{op} \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\subseteq$}"] \arrow[ddr, bend left, "K_n"] & \\
\mathrm{Ring} \arrow[r, phantom, "\subseteq"] \arrow[drr, bend right, "K_n"'] & X \arrow[dr] & \\
& & \mathrm{Ab}
\end{tikzcd}$$

What are the common generalizations of the K-theory of rings and schemes? In other words, what kind of category fits into X in the diagram? In fact, there are many answers to this question, stable categories were often considered the most natural or the most generalized answer. However, more recently, Efimov developed *continuous K-theory*, in which he generalized K-theory to dualizable categories.

Here is a brief history for the earlier development of the theory.

Quick History

- Grothendieck defined K_0 of schemes in 1957.
- Bass defined K_1, the negative K-groups in 1964.
- Milnor defined the *Milnor K-group* K_2 in 1967.
- Quillen defined the *higher K-groups* K_* in 1971 (see [Qui06]).

Motivation

Why should one study algebraic K-theory? One simple answer is that K-theory is ubiquitous: it is used in number theory, algebraic geometry and geometric topology (see [Mic02]). While we won't delve deeply into the motivation, K-theory is fascinating in its own right.

Known Computation

People have already made some computations in algebraic K-theory.

- Quillen computed the K-groups $K_*(\mathbb{F}_q)$ for finite fields in 1971.
- Borel computed $K_*(\mathcal{O}_F) \otimes \mathbb{Q}$, where F is a number field, in 1974.
- Suslin computed $(K/n)_*(k)$, where k is algebraically closed and $n \in k^\times$, in 1984.
- Many people up to Voevodsky have shown by 2010 that the n-th K-groups of $K_n(\mathbb{Z})$ of $\mathbb{Z}$ is almost known for all $n \not\equiv 0 \mod 4$.
- Rogues computed the 4-th K-group $K_4(\mathbb{Z}) = 0$ of $\mathbb{Z}$ in 2010.
- Knupers computed the 8-th K-group $K_8(\mathbb{Z}) = 0$ of $\mathbb{Z}$, in 2019.
- For $n \geq 3$, the $4n$-th K-groups $K_{4n}(\mathbb{Z})$ is unknown, Kummer-Vandiver conjecture states that $K_{4n}(\mathbb{Z}) = 0$ for all $n \in \mathbb{N}$.
- In the paper by Antieau-Krause-Nikolaus published in 2024, they have computed $K_*(\mathbb{Z}/p^n)$ using machinery from prismatic cohomology.

Construction

Finally, we state some of the constructions that will appear in the course:

- Quillen's plus construction
- Quillen's Q-construction
- Segal's group completion
- Waldhausen's S_*-construction

Finally, all constructions actually produce a space or in the language of ∞-categories, an *anima* $K(R)$, whose homotopy groups are precisely the K-groups: $K_n(R) = \pi_n(K(R))$, where $n \geq 0$. In fact, everything in this course can be generalized to the setting of ∞-categories.

Chapter 1.

The zero-th K-Group

1.1. Construction of K_0

1.1.1. Notation. — in this section, we denote CMon to be the category of commutative monoids and Ab to be the category of Abelian groups in the ordinary sense.

The zeroth K-group is defined as the group completion of the commutative monoid of isomorphism classes of finitely generated projective modules. To clarify the terms, we must first specify the group completion functor in the ordinary sense.

1.1.2. Proposition. — *The inclusion functor* $\mathsf{Ab} \hookrightarrow \mathsf{CMon}$ *admits a left adjoint* $\mathsf{CMon} \to \mathsf{Ab}$ *called the group completion. Let* $M \in \mathsf{CMon}$ *we denote* M^{grp} *to be the group completion of* M*.*

This means that for each $M \in \mathsf{CMon}$*, there is* $M^{\mathrm{grp}} \in \mathsf{Ab}$ *with a canonical unit map* $u\colon M \to M^{\mathrm{grp}}$ *such that for all* $A \in \mathsf{Ab}$ *and homomorphism* $f\colon M \to A$*, there is a unique homomorphism such that the following commutative diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "u"'] & A \\ M^{\mathrm{grp}} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof. The existence of adjunction can be easily seen by the adjoint functor theorem. It is also fruitful to give a explicit construction here: define

$$M^{\mathrm{grp}} = \mathbb{Z}[M]/(\text{subgroups generated by } [m+n]-[n]-[m])$$

then clearly the map $u\colon M \to M^{\mathrm{grp}}$ given by $m \mapsto [m]$ satisfies the desired property.

In fact, there is a *smaller construction*: one can define $M^{\mathrm{grp}} = M \times M/\sim$ where the equivalence relation is defined by $(m,n) \sim (m',n')$ if and only if there exists $t \in M$ such that $m+n'+t = m'+n+t$, this is much the same as the construction of localization in commutative algebra. In fact, we could think of (m,n) as $m-n$. Again, with componentwise addition, the map $u\colon M \to M^{\mathrm{grp}}$ satisfies the desired property. $\square$

1.1.3. Remark. — The group completion functor preserves finite products.

1.1.4. Remark. — The right adjoint of the inclusion functor $\mathsf{Ab} \hookrightarrow \mathsf{CMon}$ is the functor $(-)^\times\colon \mathsf{CMon} \to \mathsf{Ab}$, which sends every commutative monoid M to $M^\times$, its (commutative) *group of units*.

1.1.5. Example. — We have the following examples:

- Let $M = \{0, m\}$, we define the addition by $m + m = m$, then $M^{\mathrm{grp}} = 0$.
- With the addition operation, we have $(\mathbb{N}, +)^{\mathrm{grp}} = \mathbb{Z}$.
- With the multiplication operation, we have $(\mathbb{N}, \cdot)^{\mathrm{grp}} = 0$.
- By the fundamental theorem of arithmetic,

  $$(\mathbb{N} - \{0\}, \cdot) \cong \bigoplus_{p \text{ prime}} (\mathbb{N}, +).$$

  Therefore, we have the following isomorphism

  $$(\mathbb{N} - \{0\}, \cdot)^{\mathrm{grp}} \cong \bigoplus_{p \text{ prime}} (\mathbb{Z}, +) \cong (\mathbb{Q}_{>0}, \cdot),$$

  since the group completion functor is a left adjoint and hence preserves colimit.

1.1.6. Remark. — Let R be a ring and M be a left R-module, note that the map $u\colon M \to M^{\mathrm{grp}}$ is not injective in general. We say that a left R-module is *cancellable*, if for all $a, b, a', b' \in M$, $a + b = a' + b$ implies that $a' = a$. In fact, the unit map $u : M \to M^{\mathrm{grp}}$ is injective if and only if M is cancellable.

Now, let us recall some basic knowledge from commutative algebra.

1.1.7. Definition. — Let R be a ring and M be a left R-module, we say that M is *projective*, when the functor

$$\mathrm{Hom}_R(M, -)\colon {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}$$

is an exact functor.

1.1.8. Remark. — Let R be a ring and M a left R-module, then the following holds.

- The R-module M is projective, if and only if M is the direct summand of a free R-module.
- The R-module M is finitely generated projective, if and only if M is the direct summand of R^n for some $n > 0$.

Before we state the definition of the zeroth K-group, we need some ingredients from category theory.

1.1.9. Definition. — We have the following definitions:

1. Let $\mathcal{C}$ be a category. Its *groupoid core*, denoted $\mathcal{C}^{\simeq}$, is the maximal subgroupoid of $\mathcal{C}$; that is, the subcategory containing all objects of $\mathcal{C}$ and only the isomorphisms as morphisms.
2. Let $\mathcal{X}$ be a groupoid. Its *set of path-components*, denoted $\pi_0(\mathcal{X})$, is the set of isomorphism classes of objects in $\mathcal{X}$.

1.1.10. Remark. — The construction above assembles into functors

$$(-)^{\simeq}\colon \mathsf{Cat} \to \mathsf{Grpd} \quad \text{and} \quad \pi_0\colon \mathsf{Grpd} \to \mathsf{Set}.$$

The functor π_0 is the left adjoint of the inclusion functor $\mathsf{Set} \hookrightarrow \mathsf{Grpd}$ and the functor $(-)^{\simeq}$ is the right adjoint of the inclusion functor $\mathsf{Grpd} \hookrightarrow \mathsf{Cat}$.

1.1.11. Definition. — Let R be a ring, we define the *zeroth K-group* of R as

$$K_0(R) = (\pi_0(\mathsf{Proj}(R)^{\simeq}), \oplus)^{\mathrm{grp}},$$

where the monoid operation is the direct sum of modules.

1.1.12. Remark. — The construction above assembles into a functor $K_0\colon \mathsf{Ring} \to \mathsf{Ab}$.

1.1.13. Proposition. — *Let R be a ring. Then we have*

$$K_0(R) = (\pi_0(\mathsf{Proj}(R)^{\simeq}) \times \mathbb{N}/\sim$$

where $(P, n) \sim (P', n')$ if and only if there exists $m \in \mathbb{N}$ such that $P \oplus R^{n'+m} \cong P' \oplus R^{n+m}$. Furthermore, the right hand side can be also described as

$$\operatorname{colim}((\pi_0(\mathsf{Proj}(R)^{\simeq}) \xrightarrow{\oplus R} \pi_0(\mathsf{Proj}(R)^{\simeq}) \xrightarrow{\oplus R} \ldots),$$

where the colimit is taken in the category of sets.

Proof. It suffices to provide a canonical bijection between sets

$$((\pi_0(\mathsf{Proj}(R)^{\simeq}) \times \mathbb{N}/\sim) \to K_0(R)$$

then this bijection will induce a canonical group structure on the left hand side. Given (P, n), we may think of it as $P - R^n$ in the right hand side.

We first show that this correspondence is surjective. Every element of $K_0(R)$ has the form $P - Q$. Since Q is projective, there exists Q' such that $Q \oplus Q' \cong R^n$ for some $n \in \mathbb{N}$. Therefore we have $P - Q = P \oplus Q' - R^n$ and we may consider $P - Q$ as the image of $(P \oplus Q, n)$.

For the injectivity, suppose $P - R^n = P' - R^{n'}$ in $K_0(R)$, then by definition, there exists N such that $P \oplus R^{n'} \oplus N \cong P' \oplus R^n \oplus N$. Since N is projective, there exists N' such that $N \oplus N' \cong R^m$ for some $m \in \mathbb{N}$. But then we have $P \oplus R^{n'+m} \cong P' \oplus R^{n+m}$.

The second assertion follows from the explicit description of filtered colimit in Set. $\square$

1.1.14. Remark. — Let M be a commutative monoid, we say that $m \in M$ is an *absorbing element*, if $m + a = m$ holds for all $a \in M$. If M has an absorbing element, then M^{grp} is trivial: for all $a \in M$,

$$[m] + [a] = [m + a] = [m]$$

implies that $[a] = 0$ in M^{grp}.

1.1.15. Remark. — In the definition of K_0, the finitely generated assumption is crucial. Let κ be an infinite cardinal and write $\mathsf{Proj}_\kappa(R)$ for the category of κ-generated projective left R-modules, then

$$\pi_0(\mathsf{Proj}_\kappa(R)^{\simeq})^{\mathrm{grp}} = *.$$

In fact, we can show that $R^{(\kappa)}$ is an absorbing element: take $P \in \mathsf{Proj}_\kappa(R)$. Since P is κ-projective, there exists Q such that $P \oplus Q \cong R^{(\kappa)}$. Therefore, we have the following:

$$\begin{aligned} P \oplus R^{(\kappa)} &\cong P \oplus R^{(\kappa)} \oplus R^{(\kappa)} \oplus \dots \\ &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \dots \\ &\cong R^{(\kappa)} \oplus R^{(\kappa)} \oplus \dots \\ &\cong R^{(\kappa)}. \end{aligned}$$

This is called the ”Eilenberg-Mazur swindle” argument.

1.1.16. Example. — In the following, we will find some examples of rings whose zeroth K-group is $\mathbb{Z}$:

- Let k be a field. Then the finitely generated projective modules over k are precisely the finite dimensional vector spaces, which are characterized by their dimension. Therefore, $\pi_0(\mathsf{Proj}(R)^{\simeq}) = \mathbb{N}$, hence $K_0(k) = \mathbb{Z}$.
- Let R be a PID. Then every finitely generated projective module over R is free, see [Sta25, Tag 0ASV]. Therefore, $K_0(R) = \mathbb{Z}$.
- Let R be a PID. Then $K_0(R[x_1, \dots, x_n]) = \mathbb{Z}$. In fact, by the Quillen-Suslin Theorem, any finitely generated projective module over $R[x_1 \dots, x_n]$ is free, see [Qui76].
- Let R be a local ring. Then $K_0(R) = \mathbb{Z}$, since projective modules over a local ring is free, see [Sta25, Tag 0593].
- Let R be a division ring. Then $K_0(\mathrm{Mat}_n(R)) = \mathbb{Z}$. In fact, from Morita theory, we have an equivalence of categories $\mathsf{LMod}_{\mathrm{Mat}_n(R)} \simeq \mathsf{LMod}_R$ via the functor

  $$\mathsf{LMod}_R \to \mathsf{LMod}_{\mathrm{Mat}_n(R)}, \quad M \mapsto R^n \otimes_R M.$$

  See Corollary II.2.7.1 in [Wei13] for example.

1.1.17. Proposition. — *Let R, S be rings; then $K_0(R \times S) \cong K_0(R) \times K_0(S)$.*

Proof. We claim that the functor

$$\mathsf{Proj}(R \times S) \to \mathsf{Proj}(R) \times \mathsf{Proj}(S), \quad M \mapsto (M \otimes_{R\times S} R, M \otimes_{R\times S} S)$$

admits an inverse given by $(P, Q) \mapsto P \times Q$. Observe that

$$(P \times Q) \otimes_{R\times S} R \cong P \quad \text{and} \quad P \times Q \otimes_{R\times S} S \cong Q,$$

showing that one composite is the identity. Conversely, assume that M is a projective $R \times S$-module, we need to show that

$$M \cong (M \otimes_{R\times S} R) \times (M \otimes_{R\times S} S).$$

Note that the elements $e = (1, 0)$ and $1 - e = (0, 1)$ in $R \times S$ are complementary idempotent. Therefore, we have

$$M \otimes_{R\times S} R \cong Me \quad \text{and} \quad M \otimes_{R\times S} S \cong M(1 - e).$$

Combining this with $M \cong Me \oplus M(1-e)$ gives us the claim. By Remark 1.1.10 and Remark 1.1.3, taking group completion and groupoid core preserves finite product. The desired statement follows from the fact that the path-component functor preserves finite products, a property that can be verified directly. $\square$

1.1.18. Proposition. — *Let $(M,+,\cdot)$ be a semi-ring. Then $(M^{\mathrm{grp}},+,\cdot)$ has a unique ring structure such that the map*

$$\mu\colon M \to M^{\mathrm{grp}}$$

is a semi-ring homomorphism. Furthermore, if M is a commutative ring, then M^{grp} is also commutative.

Proof. By the definition of semi-ring and semi-ring homomorphism, the multiplication of M^{grp} is forced to be

$$[a-b]\cdot[c-d] = [(ac+bd)-(bc+ad)].$$

This is well-defined. If M is commutative, from above we see that M^{grp} is also commutative. $\square$

1.1.19. Corollary. — *Let R be a commutative ring. Then $K_0(R)$ has a canonical commutative ring structure.*

Proof. Observe that if R is a commutative ring, then $(\pi_0(\mathsf{Proj}(R)^{\simeq}),\oplus,\otimes_R)$ is a commutative semi-ring. After all, the statement follows from Proposition 1.1.18. $\square$

1.1.20. Remark. — Let R, S be semi-rings and $f: R \to S$ be a semi-ring homomorphism. The induced map

$$f^{\mathrm{grp}}: R^{\mathrm{grp}} \to S^{\mathrm{grp}}$$

is a ring homomorphism. In particular, if R and S are commutative rings and f is a ring homomorphism, then the induced map

$$K_0(f)\colon K_0(R) \to K_0(S)$$

is a ring homomorphism.

1.1.21. Example. — Let G be a finite group.

- Let Fin_G be the category of finite G-sets, then

  $$A(G) = (\pi_0(\mathsf{Fin}_G^{\simeq}),\sqcup,\times)^{\mathrm{grp}}$$

  is called the *Burnside* ring of G.
- Let $\mathsf{Rep}_{\mathbb{C}}(G)$ be the category of finite-dimensional complex linear representations of G, then

  $$R(G) = (\pi_0(\mathsf{Rep}_{\mathbb{C}}(G)^{\simeq}),\oplus,\otimes)^{\mathrm{grp}}$$

  is called the *representation ring* of G.
- We can compute these rings as Abelian groups:

  $$A(G) \cong \bigoplus_I \mathbb{Z} \quad \text{and} \quad R(G) \cong \bigoplus_J \mathbb{Z} \cong K_0(\mathbb{C}[G])$$

  Where I denotes the conjugacy classes of subgroups of G, and J denotes the conjugacy classes of elements of G. These computations came from character theory and group theory.

1.2. Trace maps

Let R be a ring. We refer to maps $K_0(R) \to A$ as *trace maps*, where A is an Abelian group. In this section, we will construct the norm map and the determinant map. Let us first collect some preliminaries from commutative algebra.

1.2.1. Remark. — Let R be a commutative ring. Recall that the prime spectrum $\mathrm{Spec}(R)$ is the topological space consisting of the prime ideals, equipped with the Zariski topology.

A basis for the open sets of $\mathrm{Spec}(R)$ is given by the distinguished open subsets

$$D(f) = \{x \in \mathrm{Spec}(R) : f \notin \mathfrak{p}_x\},$$

where $f \in R$. These are naturally identified with $\mathrm{Spec}(R[\frac{1}{f}])$, the spectrum of the localization of R at f.

1.2.2. Definition. — Let R be a commutative ring and M be a R-module.

1. We say that M is *locally free*, if we can cover $\mathrm{Spec}(R)$ by standard open subsets $D(f_i)$ for some $i \in I$ such that for each $i \in I$, the $R[\frac{1}{f_i}]$-module $M[\frac{1}{f_i}]$ is free.

2. We say that M is *finite locally free*, if we can cover $\mathrm{Spec}(R)$ by standard open subsets $D(f_i)$ for some $i \in I$ such that for each $i \in I$, the $R[\frac{1}{f_i}]$-module $M[\frac{1}{f_i}]$ is finitely generated and free.

1.2.3. Proposition. — *Let R be a commutative ring and M be a R-module, then the following are equivalent.*

1. *The R-module M is finitely generated and projective.*

2. *The R-module M is finitely generated and locally free.*

3. *The R-module M is finite locally free.*

4. *The R-module M is finitely generated and for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, the module $M_{\mathfrak{p}}$ is free and the map*

$$\mathrm{rk}(M)\colon \mathrm{Spec}(R) \to \mathbb{N}, \quad \mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p}))$$

is continuous, where $\mathbb{N}$ is equipped with the discrete topology.

In these cases, we can choose the index set I to be finite. Furthermore, if $\mathfrak{p} \in D(f)$, then we have

$$\dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(p)) = \mathrm{rk}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \mathrm{rk}_{R[\frac{1}{f}]} M[\frac{1}{f}],$$

where rk *denotes the rank of a free module.*

Proof. See [Sta25, Tag 00NX]. □

1.2.4. Definition. — Let R be a commutative ring and M be a finitely generated R-module. The *rank* of M is the map

$$\mathrm{rk}(M) : \mathrm{Spec}(R) \to \mathbb{N}, \quad \mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p})),$$

where $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. We say that M is of *constant rank* n, if the $\mathrm{rk}(M)$ is constant with value n.

1.2.5. Remark. — Let R be a commutative ring and M be a finitely generated projective R-module. By Proposition 1.2.3, the map $\mathrm{rk}(M)$ is continuous with respect to the Zariski topology, hence locally constant.

1.2.6. Notation. — Let R be a commutative ring. We denote

$$\mathbb{N}_R = \mathrm{Hom}_{\mathsf{Top}}(\mathrm{Spec}(R), \mathbb{N}) \quad \text{and} \quad \mathbb{Z}_R = \mathrm{Hom}_{\mathsf{Top}}(\mathrm{Spec}(R), \mathbb{Z}),$$

where $\mathbb{N}$ and $\mathbb{Z}$ are equipped with the discrete topology.

1.2.7. Exercise. — Let X be a topological space. Show that

$$\mathrm{Hom}_{\mathsf{Top}}(X, \mathbb{N})^{\mathrm{grp}} \simeq \mathrm{Hom}_{\mathsf{Top}}(X, \mathbb{Z}),$$

where $\mathbb{N}$ and $\mathbb{Z}$ are equipped with the discrete topology. Use this to deduce that the group completion of $\mathbb{N}_R$ is $\mathbb{Z}_R$.

A decomposition of the spectrum of a ring into disjoint open subsets corresponds precisely to a decomposition of the ring as a product, which in turn corresponds to the choice of an idempotent element in the ring. The precise statement is the following:

1.2.8. Proposition. — *Let R be a commutative ring. If $\mathrm{Spec}(R) = U \sqcup V$ for open subsets U, V, then $R \cong R_1 \times R_2$ for some rings R_1, R_2 such that $U \cong \mathrm{Spec}(R_1)$ and $V \cong \mathrm{Spec}(R_1)$.*

Proof. See [Sta25, Tag 00ED]. □

The rank map of a finitely generated projective module should not be regarded as globally constant; its value always depends on the ring action.

1.2.9. Example. — Let R_1, R_2 be commutative rings, then $R_1 \times R_2$-module R_1 is finitely generated and projective. The rank map

$$\mathrm{rk} : \mathrm{Spec}(R_1) \sqcup \mathrm{Spec}(R_2) \to \mathbb{N}$$

is constant of value 1 on $\mathrm{Spec}(R_1)$ and constant of value 0 on $\mathrm{Spec}(R_2)$.

1.2.10. Proposition. — *There is a well-defined semi-ring homomorphism*

$$\mathrm{rk} : \pi_0(\mathsf{Proj}(R)^{\simeq}) \to \mathbb{N}_R.$$

Furthermore, this induces a ring homomorphism

$$\mathrm{rk} : K_0(R) \to \mathbb{Z}_R$$

This will be referred to as the rank map.

Proof. It suffices to show that

$$\mathrm{rk}(P \oplus Q) = \mathrm{rk}(P) + \mathrm{rk}(Q) \quad \text{and} \quad \mathrm{rk}(P \otimes_R Q) = \mathrm{rk}(P) \times \mathrm{rk}(Q).$$

We will prove only the second equality, as the first is analogous. Consider a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, observe that

$$P \otimes_R Q \otimes_R \kappa(\mathfrak{p}) \cong (P \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} (Q \otimes_R \kappa(\mathfrak{p})).$$

The claim follows from the fact that the dimension of the tensor product of two vector spaces equals the product of their dimensions. □

1.2.11. Definition. — Let R be a commutative ring. We define

$$\tilde{K}_0(R) = \ker(\mathrm{rk} : K_0(R) \to \mathbb{Z}_R).$$

which is referred to as the *reduced K-theory* of R.

1.2.12. Proposition. — *Let R be a commutative ring. There is a canonical isomorphism*

$$K_0(R) \cong \tilde{K}_0(R) \oplus \mathbb{Z}_R.$$

of Abelian groups.

Proof. We will construct a splitting $\mathbb{Z}_R \to K_0(R)$. By Exercise 1.2.7, we have $\mathbb{Z}_R = (\mathbb{N}_R)^{\mathrm{grp}}$. Therefore, it suffices to construct a map $\mathbb{N}_R \to \pi_0(\mathsf{Proj}(R)^{\simeq})$.

Given $n \in \mathbb{N}_R$, let $U_i = n^{-1}(i)$. We see that U_i are disjoint open subsets of $\mathrm{Spec}(R)$ and

$$\mathrm{Spec}(R) = \bigsqcup_{i \in \mathbb{N}} U_i$$

where $U_i = \varnothing$ for all but finitely many $i \in \mathbb{N}$. By Proposition 1.2.8, this decomposition corresponds to a decomposition of rings:

$$R \cong \prod_{i=0}^{N} R_i.$$

Finally, set

$$R^n = \prod_{i=0}^{N} R_i^{n_i}$$

which defines the map

$$N(R) : \mathbb{N}_R \to \pi_0(\mathsf{Proj}(R)^{\simeq}), \quad n \mapsto R^n.$$

As in Example 1.2.9, we find that $\mathrm{rk}(R^n) = n$; hence $N(R)$ is a splitting of rk. $\square$

Let R be a commutative ring, our next goal is to construct the *determinant* map $\det : K_0(R) \to \mathsf{Pic}(R)$.

1.2.13. Notation. — We denote $\mathsf{Proj}_n(R) \subseteq \mathsf{Proj}(R)$ the full subcategory spanned by projective modules of constant rank n.

1.2.14. Definition. — Let R be a commutative ring. The group

$$\mathsf{Pic}(R) = (\pi_0(\mathsf{Proj}_1(R)^{\simeq}), \otimes).$$

is called the *Picard group* of R.

1.2.15. Remark. — Let R be a commutative ring and M be a R-module. The following are equivalent:

- The R-module M is finitely generated and projective of constant rank 1.

- The R-module M is $\otimes$-*invertible*. In other words, there exists a R-module N such that $M \otimes_R N \cong R$.

1.2.16. Definition. — Let R be a commutative ring, M be a R-module and $n \geq 0$.

1. The n-th *symmetric power* of M is defined as

$$\mathrm{Sym}_R^n(M) = M^{\otimes n}/\sim,$$

where $m_1 \otimes \ldots \otimes m_n \sim m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(n)}$ for all $\sigma \in S_n$.

2. The n-th *exterior power* of M is defined as

$$\Lambda_R^n(M) = (M^{\otimes n}/\sim),$$

where $m_1 \otimes \ldots \otimes m_n \sim 0$ if and only if there exists $i \neq j$ such that $m_i = m_j$.

3. The n-th *divided power* of M is defined as

$$\Gamma_R^n(M) = (M^{\otimes n})^{S_n},$$

where $(M^{\otimes n})^{S_n}$ denotes the invariant submodule of $M^{\otimes n}$ under the S_n-action.

1.2.17. Remark. — Let R be a commutative ring and $e_1, \ldots, e_d$ be a basis of R^d. The following holds.

- There is an isomorphism
$$\mathrm{Sym}_R^n(R^d) \cong R^{\binom{d+n-1}{n}},$$
where a basis is given by $\{e_{i_1} \otimes \ldots \otimes e_{i_n} : 1 \leq i_1 \leq \ldots \leq i_n \leq d\}$.

- There is an isomorphism
$$\Lambda_R^n(R^d) \cong R^{\binom{d}{n}},$$
where a basis is given by $\{e_{i_1} \wedge \ldots \wedge e_{i_n} : 1 \leq i_1 \leq \ldots \leq i_n \leq d\}$.

- There is an isomorphism
$$\Gamma_R^n(R^d) \cong R^{\binom{d+n-1}{n}}.$$
In this case, a basis is given by symmetrized tensors of the form
$$\frac{1}{n_1! \cdot \ldots \cdot n_d!} \sum_{\sigma \in S_n} e_{i_{\sigma(1)}} \otimes \ldots \otimes e_{i_{\sigma(n)}},$$
where $n_1 + \ldots + n_d = n$ and the multiset of indices $\{i_1, \ldots, i_n\}$ has exactly n_j copies of j.

1.2.18. Remark. — Let R be a commutative ring and M a finitely generated projective R-module. The R-modules $\mathrm{Sym}_R^n(M)$, $\Lambda_R^n(M)$ and $\Gamma_R^n(M)$ are finitely generated projective, by Remark 1.2.17 and the functoriality of the constructions.

1.2.19. Proposition. — *Let R and S be commutative rings. The following holds:*

1. *Let $f : R \to S$ be a ring homomorphism and M be a R-module. There is an induced isomorphism*
$$\mathrm{Sym}^n_R(M) \otimes_R S \cong \mathrm{Sym}^n_S(M \otimes_R S).$$
Similarly, the following induced maps
$$\Lambda^n_R(M) \otimes_R S \cong \Lambda^n_S(M \otimes_R S) \quad \textit{and} \quad \Gamma^n_R(M) \otimes_R S \cong \Gamma^n_S(M \otimes_R S)$$
are isomorphism.

2. *For R-modules M, N, we have*
$$\mathrm{Sym}^n_R(M \oplus N) \cong \bigoplus_{i+j=n} \mathrm{Sym}^i_R(M) \otimes_R \mathrm{Sym}^j_R(N).$$
Similarly, we have the following isomorphism
$$\Lambda^n_R(M \oplus N) \cong \bigoplus_{i+j=n} \Lambda^i_R(M) \otimes_R \Lambda^j_R(N) \quad \textit{and} \quad \Gamma^n_R(M \oplus N) \cong \bigoplus_{i+j=n} \Gamma^i_R(M) \otimes_R \Gamma^j_R(N)$$
for $i, j, n \geq 0$.

Proof. This is left as an exercise. □

1.2.20. Remark. — Let R be a commutative ring and $n \in \mathbb{N}_R$. In this case, we may similarly define Sym^n_R, Λ^n_R and Γ^n_R.

1.2.21. Definition. — Let R be a commutative ring and $P \in \mathsf{Proj}(R)$. The *determinant* of P is defined as
$$\det(P) = \Lambda^{\mathrm{rk}(P)}_R(P).$$
It is a projective module of rank 1. In fact, the determinant construction assembles into a functor
$$\det\colon \mathsf{Proj}(R) \to \mathsf{Proj}_1(R),$$
that is a retraction of the inclusion.

1.2.22. Remark. — Let R be a commutative ring, P a finitely generated projective R-module and $f : P \to P$ an endomorphism. Since P is projective of rank 1, the functor
$$(-) \otimes_R \det(P)\colon {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}$$
is an equivalence of categories. In particular, we have
$$\mathsf{End}_R(\det(P)) \cong \mathsf{End}_R(R) \cong R$$
and $\det(f) \in R$. Furthermore, the determinant functor is conservative.

1.2.23. Lemma. — *Let R be a commutative ring. We have*
$$\det(P \oplus Q) \cong \det(P) \otimes_R \det(Q),$$
where $P, Q \in \mathsf{Proj}(R)$.

Proof. Without losses of generality, we may assume $\mathrm{rk}(P) = n$ and $\mathrm{rk}(Q) = m$. By Proposition 1.2.19, we have

$$\Lambda_R^{m+n}(P \oplus Q) = \bigoplus_{i+j=n+m} \Lambda_R^i(P) \otimes_R \Lambda^j(Q).$$

Since $\Lambda_R^i(P) = 0$ for $i > n$ and $\Lambda_R^j(Q) = 0$ for $j > m$, the only nonzero term in the direct sum corresponds to $i = n$ and $j = m$. Therefore,

$$\Lambda_R^{m+n}(P \oplus Q) = \Lambda_R^n(P) \otimes_R \Lambda_R^m(Q)$$

which completes the proof. $\square$

1.2.24. Remark. — Let R be a commutative ring. By Lemma 1.2.23, there is an induced determinant map of the zeroth K-group: we have the following commutative diagram

$$\begin{tikzcd} \pi_0(\mathsf{Proj}(R)^{\simeq}) \arrow[r, "\det"] \arrow[d] & \mathsf{Pic}(R) \\ K_0(R) \arrow[ur, dotted, "\exists!\det"'] & \end{tikzcd}$$

such that $\det([P] \oplus [Q]) = \det([P]) \otimes_R \det([Q])$ for all $[P], [Q] \in K_0(R)$.

1.2.25. Definition. — Let R be a commutative ring. Define

$$SK_0(R) = \ker((\mathrm{rk}, \det) : K_0(R) \to \mathbb{Z}_R \times \mathsf{Pic}(R)).$$

It is sometimes called the *special* K_0 of R.

1.2.26. Remark. — The map $(\mathrm{rk}, \det)$ is always surjective: the map

$$\mathbb{Z}_R \times \mathrm{Pic}(R) \to K_0(R), \quad (n, L) \mapsto [R^n + L - R].$$

is a non-additive section.

1.2.27. Exercise. — Let R be a commutative ring. Show that $SK_0(R)$ is an ideal of $K_0(R)$.

1.2.28. Remark. — The following inclusion

$$SK_0(R) \subseteq \tilde{K}_0(R) \subseteq K_0(R)$$

is the beginning of the *motivic filtration* on algebraic K-theory:

$$\ldots \subseteq \mathcal{F}^2 K_*(R) \subseteq \mathcal{F}^1 K_*(R) \subseteq \mathcal{F}^0 K_*(R) = K_*(R),$$

which is a multiplicative filtration $\mathcal{F}$ on $K_*(R)$.

1.3. Stably free modules

1.3.1. Definition. — Let R be a commutative ring.

- A R-module M is called *stably free*, if there exists $n, m \in \mathbb{N}$ such that $M \oplus R^n \cong R^m$.
- Let M, N be R-modules. We say that M and N are *stably isomorphic*, if there exists $n, m \in \mathbb{N}$ such that $M \oplus R^n \cong N \oplus R^m$.

Let M, N be finitely generated projective R-modules. By Proposition 1.1.13, M, N are stably isomorphic if and only if $[M] = [N] \in K_0(R)$.

1.3.2. Remark. — A stably free module is not necessarily free. However, stably isomorphic modules of rank 1 are in fact isomorphic. Observe that

$$L \cong \det(L \oplus R^m) \cong \det(L' \oplus R^n) \cong L',$$

where L and L' are finitely generated projective R-modules of rank 1.

1.3.3. Definition. — Let X be a topological space, the *Krull dimension* of X is

$$\dim(X) = \sup\{n \in \mathbb{N} : \text{exists a chain } Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \text{ of irreducible closed subsets}\}.$$

Let R be a commutative ring. The *Krull dimension* of R is defined as the Krull dimension of $\operatorname{Spec}(R)$.

1.3.4. Example. — Let R be a commutative ring.

1. We have $\dim(k) = 0$, whenever R is a field.
2. We have $\dim(R) = 1$, whenever R is a PID but not a field.

1.3.5. Theorem (Bass-Serre Cancellation Theorem)**.** — *Let R be a noetherian commutative ring of finite Krull dimension d and P a finitely generated projective R-module of rank $n > d$, then the following holds.*

- *The R-module P has a rank 1 summand. Furthermore, we have $P \cong P' \oplus R^{n-d}$, where P' is of constant rank d.*
- *If P is stably isomorphic to some finitely generated projective module Q, then $P \cong Q$.*

The following is a counterexample where stable isomorphism does not imply isomorphism.

1.3.6. Example. — Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and $P = \ker(\sigma)$, where

$$\sigma : R^3 \to R, \ \sigma(f, g, h) = xf + yg + zh.$$

The Krull dimension of R is 2 and P is finitely generated projective of rank 2. Additionally, there is an isomorphism of R-modules $P \oplus R \cong R^3$, yet P is not free.

1.3.7. Corollary. — *Let R be a noetherian commutative ring, where $\dim(R) \leq 1$. If $P \in \mathsf{Proj}(R)$ with $\operatorname{rk}(P) \geq 1$, then $P \cong \det(P) \oplus R^{\operatorname{rk}(P)-1}$.*

Proof. By Theorem 1.3.5, we have $P \cong L \oplus R^m$, where $\operatorname{rk}(L) = 1$. Applying det and rk on both sides yields $L \cong \det(P)$ and $m = \operatorname{rk}(P) - 1$. $\square$

1.3.8. Corollary. — *Let R be a notherian commutative ring, where* $\dim(R) \leq 1$. *We have* $SK_0(R) = 0$ *and the map*

$$(\mathrm{rk}, \det) : K_0(R) \to \mathbb{Z}_R \times \mathrm{Pic}(R)$$

is an isomorphism.

Proof. Let $[P] \in SK_0(R)$, then $\det(P) \cong R$. By Corollary 1.3.7, the R-module P is free. The statement follows from the fact that a free module of rank 0 is the zero module. □

1.3.9. Remark. — Let $R = \mathcal{O}_K$ be the ring of integers of a number field K. From algebraic number theory, we know that $\mathrm{Pic}(\mathcal{O}_K)$ is finite. The *Bass Conjecture* states that all K-groups $K_i(R)$ are finitely generated Abelian groups if R is regular of finite type over $\mathbb{Z}$. Quillen proved this for rings R with $\dim(R) \leq 1$.

1.4. The Hattori-Stallings trace

In this section, let R be an arbitrary ring.

1.4.1. Remark. — Let R be a non-commutative ring. It is possible for $R^n \cong R^m$ even when $n \neq m$. For example, let R be any ring and $E = \mathsf{End}_R(R^{(\mathbb{N})})$ be the ring of matrices with infinite column. We have $E \cong E^2$ as left E-modules and therefore $K_0(E) = 0$.

1.4.2. Remark. — Let R be a ring and M a left R-module. We have an evaluation map

$$\mathrm{ev} : \mathrm{Hom}_R(M, R) \otimes_{\mathbb{Z}} M \to R, \quad \varphi \otimes x \mapsto \varphi(x),$$

where $\mathrm{Hom}_R(M, R)$ is a right R-module, with ring action $\varphi \cdot r = \varphi(-) \cdot r$. However, the map ev is not necessarily R-balanced: we have

$$\varphi r \otimes x \mapsto \varphi(x) r, \quad \text{but} \quad \varphi \otimes rx \mapsto \varphi(rx) = r\varphi(x).$$

Let $r, s \in R$, define $[r, s] = rs - sr$ and let $[R, R]$ be the subgroup generated elements of the form $[r, s]$. Now, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(M,R) \otimes_{\mathbb{Z}} M \arrow[r, "\mathrm{ev}"] \arrow[d] & R \arrow[d] \\
\mathrm{Hom}_R(M,R) \otimes_R M \arrow[r, dashed, "\exists! \overline{\mathrm{ev}}"] & R/[R,R]
\end{tikzcd}$$

Furthermore, we have the map defined as

$$\alpha_M : \mathrm{Hom}_R(M, R) \otimes_R M \to \mathsf{End}_R(M), \quad \varphi \otimes x \mapsto \varphi(-)x,$$

one can check that this is well-defined.

1.4.3. Lemma. — *Let R be a ring and M a finitely generated projective R-module. The map α_M defined above is an isomorphism.*

Proof. The case $M = R^n$ is trivial. In general, since M is projective, we can write M as a retract of R^n:
$$M \xrightarrow{s} R^n \xrightarrow{r} M,$$
where $r \circ s = \mathrm{id}_M$. Therefore, we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}_R(M,R) \otimes_R M \arrow[r] \arrow[d, "\alpha_M"'] & \mathrm{Hom}_R(R^m,R) \otimes_R R^m \arrow[r] \arrow[d, "\alpha_{R^m}"] & \mathrm{Hom}_R(M,R) \otimes_R M \arrow[d, "\alpha_M"'] \\
\mathrm{End}_R(M) \arrow[r] & \mathrm{End}_R(R^m) \arrow[r] & \mathrm{End}_R(M)
\end{tikzcd}$$

Since the compositions of the horizontal maps are identity maps, the map α_M is a retract of the map α_{R^n}, implying that α_M is an isomorphism. $\square$

1.4.4. Definition. — Let R be a ring and P a finitely generated projective left R-module. The following composition

$$\mathrm{tr}_P : \mathrm{End}_R(P) \xrightarrow{\alpha_P^{-1}} \mathrm{Hom}_R(P,R) \otimes_R P \xrightarrow{\overline{\mathrm{ev}}} R/[R,R]$$

is called the *trace map* of P.

1.4.5. Remark. — Let R be a ring and P a finitely generated projective left R-module. If $P \oplus Q \cong R^n$ for some left R-module Q, then we have

$$\mathrm{tr}_P(f) = \mathrm{tr}_{R^n}(f \oplus 0_Q),$$

where $f : P \to P$ is an endomorphism of P.

1.4.6. Proposition. — *Let R be a ring. The trace maps have the following properties.*

1. *Cyclic Invariance. We have*
$$\mathrm{tr}_P(g \circ f) = \mathrm{tr}_Q(f \circ g),$$
where $f : P \to Q$ and $g : Q \to P$ are maps between finitely generated projective left R-modules.

2. *Additivity. We have*
$$\mathrm{tr}_{P \oplus Q} = \mathrm{tr}_P(f) + \mathrm{tr}_Q(g),$$
where $f : P \to P$ and $g : Q \to Q$ are maps between finitely generated projective left R-modules.

Proof. 1. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{End}_R(Q) \arrow[r, "\simeq"] & \mathrm{Hom}_R(Q,R) \otimes_R Q \arrow[r, "\overline{\mathrm{ev}}"] & R/[R,R] \\
\mathrm{Hom}_R(Q,P) \arrow[r, "\simeq"] \arrow[u, "f_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_R(Q,R) \otimes_R P \arrow[r] \arrow[u] \arrow[d] & R/[R,R] \arrow[u] \arrow[d] \\
\mathrm{End}_R(P) \arrow[r, "\simeq"] & \mathrm{Hom}_R(P,R) \otimes_R P \arrow[r, "\overline{\mathrm{ev}}"] & R/[R,R]
\end{tikzcd}$$

Since both $\mathrm{tr}_P(g \circ f)$ and $\mathrm{tr}_Q(f \circ g)$ correspond to the middle horizontal composition, we conclude $\mathrm{tr}_P(g \circ f) = \mathrm{tr}_Q(f \circ g)$.

2. Since $f \oplus g = (f \oplus 0) + (0 \oplus g)$, we have the following equality:

$$\mathrm{tr}_{P\oplus Q}(f \oplus g) = \mathrm{tr}_{P\oplus Q}(f \oplus 0) + \mathrm{tr}_{P\oplus Q}(0 \oplus g).$$

   The desired result follows from applying cyclic invariance to $(f, 0)\colon P \to P\oplus Q$ and the projection $\mathrm{pr}_P\colon P\oplus Q \to P$, and similarly for $(0, Q)\colon Q \to P\oplus Q$ and $\mathrm{pr}_Q\colon P\oplus Q \to Q$. $\square$

If R is a commutative ring, then the Hattori-Stalling trace is almost the rank map.

1.4.7. Definition. — Let R be a ring. The following map

$$\mathrm{tr}\colon \pi_0(\mathsf{Proj}(R)^{\simeq}) \to R/[R, R], \quad P \mapsto \mathrm{tr}_P(\mathrm{id}_P)$$

is a well-defined morphism between commutative monoids. Therefore, we have an induced map

$$\mathrm{tr}\colon K_0(R) \to R/[R, R]$$

called the *Hattori-Stalling trace map.*

1.4.8. Proposition. — *Let R be a commutative ring, then we have the following commutative diagram*

$$\begin{tikzcd} K_0(R) \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{rk}"'] & R \\ \mathbb{Z}_R \arrow[ur, "r"'] & \end{tikzcd}$$

where $r : \mathbb{Z}_R \to R$ is the evaluation map at the ideal (0) *composed with the unique map $\mathbb{Z} \to R$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R, then we have the following commutative square:

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "\mathrm{rk}"'] & K_0(R_{\mathfrak{p}}) \arrow[d, "\mathrm{rk}"] \\ \mathbb{Z}_R \arrow[r] & \mathbb{Z}_{R_{\mathfrak{p}}} \end{tikzcd}$$

Since $R_{\mathfrak{p}}$ is local, the right vertical map is an isomorphism. Therefore, we have another commutative square

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "\mathrm{rk}"'] & \prod_{\mathfrak{p}\subseteq R} K_0(R_{\mathfrak{p}}) \arrow[d, "\mathrm{rk}"] \\ \mathbb{Z}_R \arrow[r] & \prod_{\mathfrak{p}\subseteq R} \mathbb{Z}_{R_{\mathfrak{p}}} \end{tikzcd}$$

where the right vertical map is an isomorphism and the lower horizontal map is injective. This implies that the kernel of the upper horizontal map is $\tilde{K}_0(R)$. Now, consider the following commutative square

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "\mathrm{tr}"'] & \prod_{\mathfrak{p}\subseteq R} K_0(R_{\mathfrak{p}}) \arrow[d, "\prod_{\mathfrak{p}\subseteq R} \mathrm{tr}"] \\ R \arrow[r] & \prod_{\mathfrak{p}\subseteq R} R_{\mathfrak{p}} \end{tikzcd}$$

We know that the lower horizontal map is injective and the kernel of the upper horizontal map is $\tilde{K}_0(R)$. By the universal property of quotient, we have an induced map $r : \mathbb{Z}_R \to R$ with the desired property. $\square$

1.4.9. Remark. — Let R be a ring. More generally, $R/[R, R] = \mathrm{HH}_0(R)$, where HH_* is the *Hochschild homology*; it is the homology of the chain complex

$$R \xleftarrow{[\cdot,\cdot]} R^{\otimes 2} \longleftarrow R^{\otimes 3} \longleftarrow \dots$$

where $[\cdot, \cdot]$ is defined by $r \otimes s \mapsto [r, s]$. The Hattori-Stalling trace can be generalized to the *Denis trace* $K_*(R) \to \mathrm{HH}_*(R)$.

Chapter 2.

The first K-group

2.1. The construction of K_1

2.1.1. Definition. — Let R be a ring. The *first K-group* of R is defined as

$$K_1(R) = \mathrm{GL}(R)^{\mathrm{ab}} \quad \text{where} \quad \mathrm{GL}(R) = \mathrm{colim}_n \, \mathrm{GL}_n(R).$$

An alternative description is $K_1(R) = \mathrm{GL}(R)/\mathrm{E}(R)$, where $\mathrm{E}_n(R) \subseteq \mathrm{GL}_n(R)$ is the subgroup generated by the elementary matrices.

Recall that $K_0(R)$ is defined as the group completion of $\pi_0(\mathsf{Proj}(R)^{\simeq})$. The idea of the construction of $K_1(R)$ lies in the following diagram.

$$\begin{array}{ccc}
\pi_0(\mathsf{Proj}(R)^{\simeq}) & \xrightarrow{\text{grp}} & K_0(R) \\
{\scriptstyle \pi_0}\uparrow & & \uparrow{\scriptstyle \pi_0} \\
\mathsf{Proj}(R)^{\simeq} & \xrightarrow{\text{grp}} & X
\end{array}$$

A reasonable candidate for $K_1(R)$ should be something that fits into X in the diagram above, where grp denotes the group completion functor. In fact, $(\mathsf{Proj}(R)^{\simeq}, \oplus)$ is a commutative monoid in the category of groupoids, therefore, we can take its group completion. In order to make this precise, we first give some recollections on groupoids and symmetric monoidal structure.

2.1.2. Definition. — A *groupoid* is a category in which all morphisms are invertible.

2.1.3. Example. — The following are typical examples of groupoids:

- If X is a set, then one can view X as a groupoid with only the identity morphisms.
- If G is a group, there is a groupoid BG with one object $*$ and $\mathrm{Aut}_{BG}(*) = G$. We call BG the *classifying groupoid* of G.
- If X is a G-set, the *action groupoid* $X//G$ is defined as follows: the objects are the elements of X. For $x, y \in X//G$, the set of morphisms is given by

  $$\mathrm{Map}_{X//G}(x, y) = \{g \in G : gx = y\}.$$

  As an example, a set Y can be viewed as $Y//e$ and the classifying groupoid BG can be viewed as $*//G$.

2.1.4. Remark. — Let Γ be a groupoid.

1. Recall that we have defined

$$\pi_0\Gamma = \{\text{isomorphism classes of objects}\}$$

   as the *path component* of Γ.

2. Let x be an object of Γ. We call $\pi_1(\Gamma, x) = \mathrm{Aut}_\Gamma(x)$ the *first homotopy group* of Γ.

3. Let $\mathcal{C}, \mathcal{D}$ be categories. There is a category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ called the *functor category*. The objects are functors $\mathcal{C} \to \mathcal{D}$. Given functors $F, G \in \mathsf{Fun}(\mathcal{C}, \mathcal{D})$, the morphisms between them are exactly the natural transformations $F \Rightarrow G$.

2.1.5. Remark. — Let Γ, Γ' be groupoids. The functor category $\mathsf{Fun}(\Gamma, \Gamma')$ is again a groupoid. In fact, the category of groupoids Grpd is a $(2,1)$-category enriched in Grpd.

2.1.6. Definition. — Let Γ, Γ' be groupoids. A functor $\Gamma \to \Gamma'$ is an *equivalence*, if there exists $g : \Gamma' \to \Gamma$ such that $f \circ g \simeq \mathrm{id}_{\Gamma'}$ in $\mathsf{Fun}(\Gamma', \Gamma)$ and $g \circ f \simeq \Gamma$ in $\mathsf{Fun}(\Gamma, \Gamma')$.

2.1.7. Remark. — Let $f : \Gamma \to \Gamma'$ be a functor between groupoids. It is an equivalence if and only if it satisfies the following conditions.

- It is *fully faithful*, in the sense that for all $x, y \in \Gamma$, the map

$$f : \mathrm{Map}(x, y) \to \mathrm{Map}(f(x), f(y))$$

  is an isomorphism.

- It is *essentially surjective*, in the sense that the map

$$\pi_0(f) : \pi_0(\Gamma) \to \pi_0(\Gamma')$$

  is surjective.

2.1.8. Remark. — Let Γ and Γ' be groupoids. All of the constructions $\pi_0, \pi_1(-, x), \mathsf{Fun}(-, \Gamma')$ and $\mathsf{Fun}(\Gamma, -)$ preserve equivalences. By contrast, the "set of objects" is not equivalence-invariant. For example, consider the following picture:

$$\begin{tikzcd} * & \simeq & (* \arrow[r, bend left] & * \arrow[l, bend left]) \end{tikzcd}$$

Although the two groupoids above have different numbers of objects, they are equivalent.

In fact, we have the structure theorem for groupoids.

2.1.9. Theorem. — *Let Γ, Γ' be groupoids. The following holds.*

1. *A functor $f : \Gamma \to \Gamma'$ is an equivalence if and only if the following holds.*
   - *The map $\pi_0(f) : \pi_0(\Gamma) \to \pi_0(\Gamma')$ is an isomorphism.*
   - *For all $x \in \Gamma$, the induced map*

$$\pi_1(f, x) : \pi_1(\Gamma, x) \to \pi_1(\Gamma', f(x))$$

     is an equivalence.

2. *We have an equivalence of groupoids*

$$\Gamma \simeq \coprod_{i \in I} BG_i,$$

where I is an index set and G_i are groups for $i \in I$.

Proof. We prove the statement as follows.

1. One direction is straightforward. We will assume the assumption and prove that f is an equivalence. By the second condition, f is essentially surjective. Therefore it suffices to check that f is fully faithful. Assume that $x, y \in \Gamma$ and consider the map

$$\mathrm{Map}_\Gamma(x, y) \to \mathrm{Map}_{\Gamma'}(f(x), f(y)).$$

If $x \not\simeq y$ then $f(x) \not\simeq f(y)$ and both sides are $\varnothing$. Now assume that $x \simeq y$ and choose an isomorphism $\alpha : y \simeq x$, then we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Map}_\Gamma(x, y) \arrow[r] \arrow[d, "\alpha \circ -"'] & \mathrm{Map}_{\Gamma'}(f(x), f(y)) \arrow[d, "- \circ f(\alpha)"] \\
\mathrm{Aut}(x) \arrow[r] & \mathrm{Aut}(y)
\end{tikzcd}$$

The vertical maps are clearly isomorphisms. By the second condition, the lower horizontal map is an isomorphism. Therefore, it follows that the upper horizontal map is also an isomorphism.

2. Choose $x_i \in \Gamma$ for each $i \in \pi_0(\Gamma)$ and let $G_i = \mathrm{Aut}_\Gamma(x_i)$. The map

$$\coprod_{i \in \pi_0(\Gamma)} B\mathrm{Aut}_\Gamma(x_i) \to \Gamma$$

is an equivalence by the first statement.

This finishes the proof. $\square$

Groupoids can be viewed as truncated spaces within the framework of algebraic topology. The above statement has a profound connection to the homotopy theory of topological spaces.

2.1.10. Definition. — A continuous map $f : X \to Y$ between topological spaces is called a *weak equivalence* if the following holds.

- The map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism.
- For $i \geq 1$ and all $x \in X$, the map $\pi_i(X, x) \to \pi_i(Y, f(x))$ is an isomorphism.

In the above, $\pi_0(X) = [*, X]$ is the set of *path components* of X, and $\pi_i(X, x) = [(S^i, *), (X, x)]_*$ is the *i-th homotopy group* of X.

2.1.11. Definition. — A topological space X is *n-truncated*, if $\pi_i(X, x) = 0$ for $i > n$ and $x \in X$.

2.1.12. Remark. — Denote $\mathsf{Top}_{\leq n} \subseteq \mathsf{Top}$ the full subcategory of n-truncated topological spaces. The following holds.

- The localization at the weak equivalences

$$\mathsf{Top}_{\leq 0}[\mathrm{weq}^{-1}] \simeq \mathsf{Set}, \quad X \mapsto \pi_0(X)$$

  is an equivalence of categories.

- The localization at the weak equivalences

$$\mathsf{Top}_{\leq 1}[\mathrm{weq}^{-1}] \simeq \mathrm{Grpd}, \quad X \mapsto \Pi_1(X)$$

  is an equivalence of $(2,1)$-categories, where $\Pi_1(X)$ is the fundamental groupoid of X.

- More generally, for $-2 \leq n \leq \infty$, the localization at the weak equivalences

$$\mathsf{Top}_{\leq n}[\mathrm{weq}^{-1}] \simeq n\text{-Grpds}, \quad X \mapsto \Pi_n(X)$$

  is an equivalence of $(n,1)$-categories, where $\Pi_n(X)$ is the *groupoid of n-type* of X.

In fact, there is a notion of ∞-groupoid that is equivalent to the localization of Top at the weak equivalences. We have the following asccending inclusions of categories:

$$\{*\} = \mathsf{Grpd}_{-2} \subseteq \{\varnothing \to x\} \subseteq \mathsf{Set} \subseteq \ldots \subseteq \mathsf{Grpd}_\infty = \mathsf{An}$$

where An denotes the ∞-category of *anima*, another name for ∞-groupoids. The functor $\mathsf{Grpd}_n \to \mathsf{Grpd}_\infty$ has a left adjoint $\tau_{\leq n}$ called the *n-th truncation*.

In the modern perspective, the goal of K-theory is to define an anima $K(R)$, such that $\tau_{\leq 0}(K(R)) = K_0(R)$.

2.2. Symmetric monoidal categories

2.2.1. Definition. — A *monoidal category* is a category $\mathcal{C}$, equipped with

- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *monoidal tensor*,
- an object 1 of $\mathcal{C}$,
- an isomorphism $\alpha_{X,Y,Z}$

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z).$$

  for every triple (X,Y,Z) of objects of $\mathcal{C}$,

- isomorphisms λ_X and ρ_X

$$\lambda_X : 1 \otimes X \simeq X \text{ and } \rho_X : X \otimes 1 \simeq X.$$

  for each object X of $\mathcal{C}$.

These data must also satisfy the *unit axiom*: the following diagram

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[r, "\alpha"] \arrow[d, "\rho\otimes Y"'] & X \otimes (1 \otimes Y) \\
X \otimes Y \arrow[ur, "\mathrm{id}_X \otimes \lambda"'] &
\end{tikzcd}$$

commutes for every objects $X, Y \in \mathcal{C}$. We also have the *pentagon axiom*: the following diagram

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[r, "\alpha"] \arrow[d, "\alpha"] & (X \otimes (Y \otimes Z)) \otimes W \arrow[r, "\alpha"] & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "\alpha"] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[rr, "\alpha"] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

commutes for every object $X, Y, Z, W \in \mathcal{C}$.

2.2.2. Definition. — A *braided category* is a monoidal category $\mathcal{C}$ such that the following holds.

- For each object X, Y of $\mathcal{C}$, there is a natural isomorphism $\gamma_{X,Y} : X \otimes Y \simeq Y \otimes X$ called *braiding*. Moreover, it must satisfy the *hexagon identities*: the following two diagram

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha"] \arrow[d, "\gamma\otimes\mathrm{id}_Z"'] & X \otimes (Y \otimes Z) \arrow[r, "\gamma"] & (Y \otimes Z) \otimes X \arrow[d, "\gamma"] \\
(Y \otimes X) \otimes Z \arrow[r, "\alpha"] & Y \otimes (X \otimes Z) \arrow[r, "\mathrm{id}_Y \otimes\gamma"] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_X \otimes\gamma"'] & (X \otimes Y) \otimes Z \arrow[r, "\gamma"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha"] \\
X \otimes (Z \otimes Y) \arrow[r, "\alpha"] & (X \otimes Z) \otimes Y \arrow[r, "\gamma\otimes Y"] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

commutes for every $X, Y, Z \in \mathcal{C}$.

- From these identities, we can deduce that the following diagram

$$\begin{tikzcd}
X \otimes 1 \arrow[r, "\gamma"] \arrow[d, "\rho"'] & 1 \otimes X \\
X \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

commutes for each $X \in \mathcal{C}$.

2.2.3. Definition. — A *symmetric monoidal category* $\mathcal{C}$ is a braided monoidal category such that there exist an natural equivalence $\gamma^2 \simeq \mathrm{id}$.

2.2.4. Definition. — Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, then a *monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor equipped with

- a natural isomorphism $\mu_{X \otimes Y} : F(X \otimes Y) \simeq F(X) \otimes F(Y)$, and
- a natural isomorphism $\epsilon : F(1_{\mathcal{C}}) \simeq F(1_{\mathcal{D}})$.

Furthermore, they must satisfy *associativity*: the following diagram

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\alpha"] \arrow[d, "\mu \otimes \mathrm{id}_{F(Z)}"] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{id}_{F(Z)} \otimes \mu"'] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "\mu"] & F(X) \otimes F(Y \otimes Z) \arrow[d, "\mu"'] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha)"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

commutes for every $X, Y, Z \in \mathcal{C}$, and *unitality*: the following diagram

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes F(X) \arrow[r, "\epsilon \otimes \mathrm{id}_{F(X)}"] \arrow[d, "\lambda"'] & F(1_{\mathcal{C}}) \otimes F(X) \arrow[d, "\mu"] \\
F(X) & F(1_{\mathcal{C}} \otimes X) \arrow[l, "F(\lambda)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes 1_{\mathcal{D}} \arrow[r, "\mathrm{id}_{F(X)} \otimes \epsilon"] \arrow[d, "\rho"'] & F(X) \otimes F(1_{\mathcal{C}}) \arrow[d, "\mu"] \\
F(X) & F(X \otimes 1_{\mathcal{C}}) \arrow[l, "F(\rho)"]
\end{tikzcd}$$

commutes for each $X \in \mathcal{C}$.

2.2.5. Definition. — Let $\mathcal{C}, \mathcal{D}$ be braided monoidal categories, then a *braided monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor, such that the following diagram

$$\begin{tikzcd}
F(X \otimes Y) \arrow[r, "\mu"] \arrow[d, "F(\gamma)"'] & F(X) \otimes F(Y) \arrow[d, "\gamma"] \\
F(Y \otimes X) \arrow[r, "\mu"] & F(Y) \otimes F(X)
\end{tikzcd}$$

commutes for every $X, Y \in \mathcal{C}$.

2.2.6. Definition. — Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal categories, then a *symmetric monoidal functor* is a braided monoidal functor.

2.2.7. Definition. — Let $\mathcal{C}, \mathcal{D}$ be monoidal categories, $F, G : \mathcal{C} \to \mathcal{D}$ be monoidal functors, a *monoidal natural transformation* $\varphi : F \Rightarrow G$ is a natural transformation such that the following diagram

$$\begin{tikzcd}
F(X \otimes Y) \arrow[r, "\varphi"] \arrow[d, "\mu"'] & G(X \otimes Y) \arrow[d, "\mu"] \\
F(X) \otimes F(Y) \arrow[r, "\varphi \otimes \varphi"] & G(X) \otimes G(Y)
\end{tikzcd}$$

commutes for every $X, Y \in \mathcal{C}$, and the following diagram

$$\begin{tikzcd}
F(1_{\mathcal{C}}) \arrow[r] \arrow[d, "\epsilon"'] & G(1_{\mathcal{D}}) \arrow[dl, "\epsilon"] \\
1_{\mathcal{D}} &
\end{tikzcd}$$

commutes.

2.2.8. Remark. — We denote $\mathsf{Mon}(\mathsf{Cat})$, $\mathsf{BMon}(\mathsf{Cat})$ and $\mathsf{Mon}(\mathsf{Cat})$ the 2-category of monoidal categories, braided monoidal category and symmetric monoidal category respectively. Furthermore, there are forgetful functors

$$\mathsf{CMon}(\mathsf{Cat}) \subseteq \mathsf{BMon}(\mathsf{Cat}) \subseteq \mathsf{Mon}(\mathsf{Cat}) \subseteq \mathsf{Cat}.$$

Similarly, there is a notion of *symmetric monoidal groupoids* and *symmetric monoidal set* and so on. A *monoidal set* is simply a monoid whereas a symmetric monoidal set is simply a commutative monoid and we can form the corresponding 2-categories. The embedding

$$\mathsf{Set} \subseteq \mathsf{Grpd} \subseteq \mathsf{Cat}$$

is preserved after taking $\mathsf{Mon}(-)$, $\mathsf{BMon}(-)$ and $\mathsf{CMon}(-)$ respectively.

2.2.9. Example. — The following are some examples of (symmetric) monoidal categories.

1. Let R be a ring. The category of left R-modules equipped with the tensor product $(\mathsf{LMod}, \otimes, 0)$ is a symmetric monoidal category and the category $(\mathsf{Proj}(R)^{\simeq}, \otimes, 0)$ is even a symmetric monoidal groupoid.
2. Let R be a commutative ring. The category of R-modules equipped with coproduct $({}_R\mathsf{Mod}(R), \oplus, R)$ is a symmetric monoidal category, and $(\mathsf{Proj}(R)^{\simeq}, \oplus, R)$ is even a symmetric monoidal groupoid.
3. Let $\mathcal{C}$ be a category with finite coproduct. The symmetric monoidal category $(C, \sqcup, \varnothing)$ is said to be equipped with the *cocartesian symmetric monoidal structure*.
4. Let $\mathcal{C}$ be a category. The category $(\mathsf{Fun}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$ is a monoidal category.
5. Let R be a ring. The category of R-bimodules equipped with the tensor product $({}_R\mathrm{Mod}_R, \otimes, R)$ is a monoidal category.

Next, we will discuss one of the most important types of (symmetric) monoidal categories: the *free (symmetric) monoidal categories*.

2.2.10. Remark. — Here are some of the simplest examples of free monoidal categories.

- The symmetric monoidal category $\mathbb{F} = (\mathrm{Fin}^{\simeq}, \sqcup, \varnothing)$ is the free symmetric monoidal category on one object: for every symmetric monoidal category $\mathcal{C}$, the evaluation at the singleton
$$\mathrm{ev}_* : \mathsf{Fun}^{\mathrm{sym}}(\mathbb{F}, \mathcal{C}) \to \mathcal{C}$$
is an equivalence of categories, where $\mathsf{Fun}^{\mathrm{sym}}(-,-)$ denotes the category of symmetric monoidal functors. The inverse is given by $X \mapsto (I \mapsto X^{\otimes I})$.

  One may think of this as an analog of the fact that given a commutative monoid $M \in \mathsf{CMon}(\mathsf{Set})$, the evaluation at 1
$$\mathrm{ev}_1 : \mathrm{Map}_{\mathsf{CMon}}(\mathbb{N}, M) \to M$$
is an isomorphism of commutative monoids.

- The free monoidal category on one object is $(\mathbb{N}, +)$: for every monoidal category $\mathcal{C}$, the evaluation at 0
$$\mathrm{ev}_0 : \mathsf{Fun}^{\mathrm{mon}}(\mathbb{N}, \mathcal{C}) \to \mathcal{C},$$
is an equivalence of categories, where $\mathsf{Fun}^{\mathrm{mon}}(-,-)$ denotes the category of monoidal functors.

- The free braided monoidal category on one object is the groupoid of braided finite sets, which is
$$\Pi_1(\bigsqcup)_{n\geq 0}\mathrm{Conf}_n(\mathbb{R}^2)) \simeq \bigsqcup_{n\geq 0} BBr_n.$$
The left hand side is the fundamental groupoid of the coproduct of n-th configuration spaces of $\mathbb{R}^2$, the right hand side is the coproduct of the classifying space of the braided groups.

2.2.11. Definition. — A *Picard groupoid* $\mathcal{X}$ is a symmetric monoidal groupoid in which every object is $\otimes$-invertible: for every $x \in \mathcal{X}$, there exists $x' \in \mathcal{X}$ such that $x \otimes x' \simeq 1_{\mathcal{X}}$.

2.2.12. Notation. — We denote $\mathsf{Ab}(\mathsf{Grpd}) \subseteq \mathsf{CMon}(\mathsf{Grpd})$ the full subcategory spanned by the Picard groupoids.

2.2.13. Example. — Let $\mathcal{C}$ be a symmetric monoidal category. Its *Picard groupoid* is $\underline{\mathsf{Pic}}(\mathcal{C}) \subseteq \mathcal{C}^{\simeq}$, the full subcategory spanned by the $\otimes$-invertible objects.

2.2.14. Remark. — Let R be a commutative ring. The *Picard groupoid* of R is defined as
$$\underline{\mathsf{Pic}}(R) = \underline{\mathsf{Pic}}((\mathsf{Mod}_R, \oplus)) = (\mathsf{Proj}_1(R), \oplus).$$
In this case, we have $\pi_0(\underline{\mathsf{Pic}}(R)) = \mathsf{Pic}(R)$. Furthermore, we have
$$\pi_1(\underline{\mathsf{Pic}}(R), L) = \mathsf{Aut}_R(L) = R^{\times},$$
This follows from a similar argument as in Remark 1.2.22.

2.2.15. Remark. — In a Picard groupoid, all π_1 are isomorphic: let $\Gamma \in \mathsf{Ab}(\mathsf{Grpd})$ and $x \in \Gamma$, then the endomorphism functor $x \otimes - : \Gamma \to \Gamma$ is an equivalence, since x is invertible. Therefore, the induced map
$$x \otimes -\colon \pi_1(\Gamma, 1) \to \pi_1(\Gamma, x)$$
is an equivalence for all $x \in \Gamma$.

2.2.16. Proposition (Eckman-Hilton Argument)**.** — *Let* $(\mathcal{C}, \otimes, 1)$ *be a monoidal category, then* $(\mathrm{End}_{\mathcal{C}}(1), \circ, \mathrm{id}_1)$ *is a commutative monoid.*

Proof. We see that $\mathrm{End}_{\mathcal{C}}(1)$ is a monoid in two ways, either via composition $\circ$ and the unit id_1 or via $\otimes$: for each $f, g \in \mathrm{End}_{\mathcal{C}}(1)$, $f \otimes g \in \mathrm{End}_{\mathcal{C}}(1 \otimes 1) \simeq \mathrm{End}_{\mathcal{C}}(1)$. Furthermore, one has $f \otimes \mathrm{id}_1 = f$ and $\mathrm{id}_1 \otimes f = f$ for all $f \in \mathrm{End}_C(1)$. This implies that the unit of the two operations coincides. By the functoriality of $- \otimes -$ in both variables, we also has
$$(f \circ g) \otimes (h \circ k) = (f \otimes h) \circ (g \otimes k).$$
Finally, the classical Eckmann-Hilton argument gives us the desired result. $\square$

2.2.17. Corollary. — *Let Γ be a monoidal groupoid. The set $\pi_1(\Gamma, 1)$ is an Abelian group. Furthermore, if Γ is even a Picard group, then $\pi_1(\Gamma, x)$ is an Abelian group for each $x \in \Gamma$.*

2.2.18. Remark. — In the classical setting, the Eckmann-Hilton argument implies that

$$\mathsf{Mon}(\mathsf{Mon}(\mathsf{Set})) \simeq \mathsf{CMon}(\mathsf{Set}) \quad \text{and} \quad \mathsf{Mon}(\mathsf{CMon}(\mathsf{Set})) \simeq \mathsf{CMon}(\mathsf{Set}).$$

In fact, for any category $\mathcal{C}$, we have the following equivalence

$$\mathsf{Mon}(\mathsf{CMon}(\mathcal{C})) \simeq \mathsf{CMon}(\mathcal{C}).$$

If $\mathcal{C}$ is a 2-category, then we have following equivalences

$$\mathsf{Mon}(\mathsf{Mon}(\mathcal{C})) \simeq \mathsf{BMon}(\mathcal{C}),\ \mathsf{Mon}(\mathsf{BMon}(\mathcal{C})) \simeq \mathsf{CMon}(\mathcal{C}),\ \mathsf{Mon}(\mathsf{CMon}(\mathcal{C})) \simeq \mathsf{CMon}(\mathcal{C}).$$

If $\mathcal{C}$ is an ∞-category, then $\mathsf{Mon}^{(n)}(\mathcal{C})$ is called *E_n-monoids* in $\mathcal{C}$, and roughly speaking, we have

$$\mathsf{CMon}(\mathcal{C}) \simeq \lim_{n\to\infty} \mathsf{Mon}^{(n)}(\mathcal{C}).$$

We simply mean that a commutative monoid in $\mathcal{C}$ admits infinitely many compatible monoid operations. In fact, $\mathsf{CMon}(\mathcal{C})$ is called the category of *$\mathbb{E}_\infty$-monoid* in $\mathcal{C}$.

2.2.19. Remark. — The inclusion functor $\mathsf{Ab}(\mathsf{Grpd}) \subseteq \mathsf{CMon}(\mathsf{Grpd})$ has a left adjoint $\mathrm{grp} : \mathsf{CMon}(\mathsf{Grpd}) \to \mathsf{Ab}(\mathsf{Grpd})$, by the 2-categorical adjoint functor theorem. In other words, for all $M \in \mathsf{CMon}(\mathsf{Grpd})$, there exists a Picard groupoid M^{grp} and a symmetric monoidal functor $\mu : M \to M^{\mathrm{grp}}$ such that for all $A \in \mathsf{Ab}(\mathsf{Grpd})$,

$$\mu^* : \mathsf{Fun}^{\mathrm{sym}}(M^{\mathrm{grp}}, A) \to \mathsf{Fun}^{\mathrm{sym}}(M, A)$$

is an equivalence. Furthermore, by Remark 1.1.10, we have the following picture:

$$\begin{tikzcd}
\mathsf{Ab}(\mathsf{Grpd}) \arrow[r, hook, shift right] \arrow[d, "\pi_0"', shift right] \arrow[r, phantom, "\perp"] & \mathsf{CMon}(\mathsf{Grpd}) \arrow[l, dashed, "\mathrm{grp}"', shift right] \arrow[d, "\pi_0"', shift right] \\
\mathsf{Ab}(\mathsf{Set}) \arrow[r, hook, shift right] \arrow[u, hook, shift right] \arrow[r, phantom, "\perp"] \arrow[u, phantom, "\dashv"] & \mathsf{CMon}(\mathsf{Set}) \arrow[l, "\mathrm{grp}"', shift right] \arrow[u, hook, shift right] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

In other words, this provides an enhancement of the group completion functor at the level of groupoids.

In fact, we can say more about this diagram:

2.2.20. Proposition. — *Let $M \in \mathsf{CMon}(\mathsf{Grpd})$. We have $\pi_0(M^{\mathrm{grp}}) \simeq \pi_0(M)^{\mathrm{grp}}$.*

Proof. It follows from Remark 2.2.19, that for all $A \in \mathsf{Ab}$, we have

$$\mathrm{Hom}_{\mathsf{Ab}}(\pi_0(M^{\mathrm{grp}}), A) \simeq \mathsf{Fun}^{\mathrm{sym}}(M^{\mathrm{grp}}, A) \simeq \mathsf{Fun}^{\mathrm{sym}}(M, A).$$

The above shows that $\pi_0 \circ \mathrm{grp}$ is a left adjoint of the inclusion $\mathsf{Ab}(\mathsf{Set}) \subseteq \mathsf{CMon}(\mathsf{Grpd})$. Similarly, one can show that $\mathrm{grp} \circ \pi_0$ is also such left adjoint. The claim follows from the fact that left adjoint is unique up to isomorphism. $\square$

2.3. The group completion theorem

Having defined the group completion functor for symmetric monoidal groupoids, we now turn to methods for computing it. Group completion is a powerful tool for such calculations. Before developing the abstract framework, we begin with a toy example.

2.3.1. Example. — Let $\mathbb{F} = (\mathrm{Fin}^{\simeq}, \sqcup, \varnothing)$, we will compute its group completion. First, observe that $\pi_0(\mathbb{F}) = \mathbb{N}$ and $\pi_1(\mathbb{F}, \{1, \dots, n\}) \simeq S_n$. By Theorem 2.1.9, we have

$$\mathbb{F} \simeq \coprod_{n \in \mathbb{N}} BS_n.$$

By Proposition 2.2.20, $\pi_0(\mathbb{F}^{\mathrm{grp}}) = \mathbb{Z}$. Since $\mathbb{F}^{\mathrm{grp}}$ is a Picard groupoid, by Remark 2.2.15, all of its automorphism groups are isomorphic. This implies that

$$\mathbb{F}^{\mathrm{grp}} \simeq \coprod_{n \in \mathbb{Z}} BA,$$

for some Abelian group A. In fact, we will see that $A = S_\infty^{\mathrm{grp}}$, where $S_\infty = \mathrm{colim}_{n \in \mathbb{N}} S_n$. Furthermore, the sign map

$$\mathrm{sgn}\colon S_n \to \mathbb{Z}/2\mathbb{Z}$$

induces an isomorphism $S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$. But then, for $n \geq 2$, we have $A_n \cong [S_n, S_n]$, which gives us $S_n^{\mathrm{grp}} \cong \mathbb{Z}/2\mathbb{Z}$ and hence $S_\infty^{\mathrm{grp}} \cong \mathbb{Z}/2\mathbb{Z}$.

2.3.2. Definition. — The *Picard groupoid* $\mathbb{S}$ is defined as follows.

- Its objects are $n \in \mathbb{Z}$.
- For $m, n \in \mathbb{S}$, we have

  $$\mathrm{Map}_{\mathbb{S}}(n, m) = \begin{cases} \varnothing & n \neq m \\ \mathbb{Z}/2\mathbb{Z} & n = m \end{cases}$$

  as the set of morphisms from n to m.
- The unit object is 0.

There is a monoidal structure $+\colon \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ on $\mathbb{S}$, given by $(m, n) \mapsto m + n$ on object and

$$\mathrm{Aut}_{\mathbb{S}}(m) \times \mathrm{Aut}_{\mathbb{S}}(n) \to \mathrm{Aut}_{\mathbb{S}}(m+n) \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot} \mathbb{Z}/2\mathbb{Z}$$

on morphisms. If we take $\gamma_{m,n} : m \times n \simeq n \times m$ to be $\gamma_{m,n} = (-1)^{mn}$, then this upgrades $\mathbb{S}$ into symmetric monoidal category $(\mathbb{S}, +, 0)$. One can check that it is even Picard groupoid.

2.3.3. Remark. — In the above setting, one can also take the identity morphism to be the braiding. This will give us the Picard groupoid $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In fact, we have $\mathbb{S} \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in $\mathsf{Mon}(\mathsf{Grpd})$, but they are not equivalent as Picard groupoids. In fact, the functor

$$\mathbb{S} \to \pi_0 \mathbb{S} = \mathbb{Z}$$

is a symmetric monoidal functor. However, the functor

$$\mathbb{Z} \to \mathbb{S}, \ n \to n$$

is not even braided monoidal.

2.3.4. Theorem. — *There is a symmetric monoidal functor $\eta : \mathbb{F} \to \mathbb{S}$, which exhibits $\mathbb{S}$ as group completion of $\mathbb{F}$. In other words, for every Picard groupoid A, the following diagram commutes:*

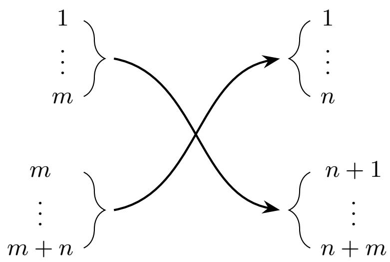
$$\begin{tikzcd}
\mathsf{Fun}^{\mathrm{sym}}(\mathbb{S}, A) \arrow[r, "\simeq"] \arrow[dr, "\mathrm{ev}_1"'] & \mathsf{Fun}^{\mathrm{sym}}(\mathbb{F}, A) \arrow[d, "\mathrm{ev}_1"] \\
& A
\end{tikzcd}$$

where the upper morphism is given by precomposition with η.

Proof. We will first construct the functor: let

$$\mathbb{F} = \bigsqcup_{n \in \mathbb{N}} BS_n.$$

It admits a symmetric monoidal structure with $\alpha, \lambda, \rho = \mathrm{id}$ and $\gamma_{m,n} : m + n \simeq n + m$ defined as swapping the first $1, \ldots, m$ with $n+1, \ldots, n+m$ and the first $m+1, \ldots, m+n$ with $n+1, \ldots, n+m$, which is illustrated as follows:



Let us now define $\eta : \mathbb{F} \to \mathbb{S}$: it is given by $n \mapsto n$ on object and sign: $S_n \to \mathbb{Z}/2\mathbb{Z}$ and take $\mu_{m,n} = \epsilon = \mathrm{id}$. It suffices to check that the following diagram commute:

$$\begin{tikzcd}
m + n \arrow[r, "\simeq"] \arrow[d, "\eta(\gamma^{\mathbb{F}}_{m,n})"'] & n + m \arrow[d, "\gamma^{\mathbb{S}}_{m,n}"] \\
n + m \arrow[r, "\simeq"] & n + m
\end{tikzcd}$$

In other words, we need to check that $\eta(\gamma^{\mathbb{F}}_{m,n}) = \gamma^{\mathbb{S}}_{m,n}$. This is left as an easy exercise.

Now, we will check the universal property: let $\varphi \colon \mathbb{F} \to A$ be a symmetric monoidal functor. By symmetric monoidality, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "+1"] \arrow[d, "\varphi"'] & \mathbb{F} \arrow[r, "+1"] \arrow[d, "\varphi"] & \ldots \\
A \arrow[r, "+\varphi(1)"] & A \arrow[r, "+\varphi(1)"] & \ldots
\end{tikzcd}$$

After applying $\pi_1(-, 0)$ to this diagram, we will get the following commutative diagram:

$$\begin{tikzcd}
S_0 \arrow[r, hook] \arrow[d] & S_1 \arrow[r, hook] \arrow[d] & S_2 \arrow[r, hook] \arrow[d] & \ldots \\
\pi_1(A, 0) \arrow[r, "\simeq"] & \pi_1(A, \varphi(1)) \arrow[r, "\simeq"] & \pi_1(A, 2\varphi(1)) \arrow[r, "\simeq"] & \ldots
\end{tikzcd}$$

Since $\pi_1(A,0)$ is Abelian, each $\varphi\colon S_n \to \pi_1(A,0)$ factors through $S_n^{\mathrm{ab}} = \mathbb{Z}/2\mathbb{Z}$. In other words, we have the following commutative diagram:

$$\begin{tikzcd} S_n \arrow[r] \arrow[d, "\mathrm{sgn}"'] & \pi_1(A,0) \\ \mathbb{Z}/2\mathbb{Z} \arrow[ur, dashed, "\exists!\hat{\varphi}"'] & \end{tikzcd}$$

We define $\tilde{\varphi} : \mathbb{S} \to A$ by $n \mapsto n \cdot \varphi(1)$, the n-fold tensor product of φ, on objects. For $n \geq 0$, let $\epsilon \in \mathrm{Aut}_{\mathbb{S}}(n)$, define $\tilde{\varphi}(\epsilon) = \hat{\varphi}(\sigma)$ for any $\sigma \in S_n$ such that $\mathrm{sgn}(\sigma) = \epsilon$. For $n \leq 0$, take $\sigma \in S_{-n}$ instead. It is easy to see that $\tilde{\varphi} \circ \eta = \varphi$ and the functor

$$\eta^*\colon \mathsf{Fun}^{\mathrm{sym}}(\mathbb{S}, A) \to \mathsf{Fun}^{\mathrm{sym}}(\mathbb{F}, A)$$

is essentially surjective.

Finally, to prove that η^* is an equivalence, it suffice to show that it is fully faithful, which amounts to show that the two evaluation maps are fully faithful. We will prove the case for

$$\mathrm{ev}_1 : \mathsf{Fun}^{\mathrm{sym}}(\mathbb{S}, A) \to A.$$

Let $\varphi, \psi \in \mathsf{Fun}^{\mathrm{sym}}(\mathbb{S}, A)$ and $\alpha \in \mathsf{Nat}^{\mathrm{sym}}(\varphi, \psi)$, with $\alpha_n : \varphi(n) \to \psi(n)$, then ev_1 is fully faithful if and only if $\{\alpha_n\}_{n\in\mathbb{Z}}$ is uniquely determined by $\alpha_1 : \varphi(1) \to \psi(1)$. Observe that the diagram

$$\begin{tikzcd} \varphi(2) \arrow[r, "\alpha_2"] \arrow[d, "\mu"'] & \psi(2) \arrow[d, "\mu"] \\ \varphi(1)+\varphi(1) \arrow[r, "\alpha_1+\alpha_1"] & \psi(1)+\psi(1) \end{tikzcd}$$

commutes, where the two vertical maps are isomorphism. By induction, for $n \geq 0$, α_n is completely determined by α_1. We also have the following commutative diagram:

$$\begin{tikzcd} \varphi(n)+\varphi(-n) \arrow[r] \arrow[d, "\simeq"] & \psi(n)+\psi(-n) \arrow[d, "\simeq"] \\ \varphi(0) \arrow[r] & \psi(0) \end{tikzcd}$$

Since $\varphi(0) \to \psi(0)$ is the identity map of the symmetric monoidal unit of A, the upper horizontal map corresponds to a specified element in $\pi_1(A,0)$. By Remark 2.2.15, the map $\alpha(-n)$ is completely determined by $\alpha(n)$. The case for $\mathrm{ev}_1\colon \mathsf{Fun}^{\mathrm{sym}}(\mathbb{F}, A) \to A$ is similar. $\square$

2.3.5. Remark. — The notation $\mathbb{S}$ usually denotes *sphere spectrum.* The Picard group defined above should actually be the 1-truncation of the sphere spectrum $\tau_{\leq 1}\mathbb{S}$. We have the following diagram:

$$\begin{tikzcd} \mathsf{CMon}(\mathsf{Set}) \arrow[r, "\mathrm{grp}"] \arrow[d, hook] & \mathsf{Ab}(\mathsf{Set}) \arrow[d, hook] \\ \mathsf{CMon}(\mathsf{Grpd}) \arrow[r, "\mathrm{grp}"] \arrow[d, hook] & \mathsf{Ab}(\mathsf{Grpd}) \arrow[d, hook] \\ \mathsf{CMon}(\mathsf{Grpd}_\infty) \arrow[r, "\mathrm{grp}"] & \mathsf{Ab}(\mathsf{Grpd}_\infty) \end{tikzcd}$$

However the lower square does not actually commute.

With little additional work, this discussion generalizes to a considerably broader context:

2.3.6. Definition. — Let M be a commutative monoid, a submonoid L is called cofinal, if for all $x \in M$, there exists $y \in M$ such that $x + y \in L$.

2.3.7. Remark. — Let M be a commutative monoid and L a submonoid. The following holds.

- If $L \subseteq M$ is cofinal, then the induced morphism $L^{\mathrm{grp}} \to M^{\mathrm{grp}}$ is injective.
- The map $M \times L \to M^{\mathrm{grp}}$ which sends (x, y) to $x - y$ exhibits M^{grp} as the quotient of $M \times L$ by the equivalence relation $(x, y) \sim (x', y')$ if and only if there exists $z \in L$ such that $x + y' + z = x' + y + z$.
- If $L = \mathbb{N}$ is a cofinal submonoid of M, then

  $$M^{\mathrm{grp}} \simeq \operatorname{colim}(M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \dots)$$

  where $+1 : M \to M$ is given by $x \mapsto x + 1$.

2.3.8. Theorem (The Group Completion Theoreom)**.** — *Let $M \in \mathsf{CMon}(\mathsf{Grpd})$ and $(x_i)_{i \in I} \in M$ be a family of objects that generates a cofinal submonoid of $\pi_0(M)$. For $n \in \mathbb{N}^I$, define $G_n = \mathrm{Aut}_M(\bigotimes_{i \in I} x_i^{\otimes n_i})$, then we have*

$$M^{\mathrm{grp}} \simeq \coprod_{\pi_0(M)^{\mathrm{grp}}} BG_\infty^{\mathrm{ab}}.$$

where $G_\infty = \operatorname{colim}_{n \in \mathbb{N}^I} G_n$.

2.3.9. Definition. — Let R be a ring. We denote

$$\tau_{\le 1} K(R) = (\mathsf{Proj}(R)^{\simeq}, \oplus)^{\mathrm{grp}}$$

and then we define the *first K-group* of R as

$$K_1(R) = \pi_1(\tau_{\le 1} K(R), 0).$$

Later we will introduce the K-theory anima $K(R)$, whose homotopy groups are the higher K-groups $K_n(R)$. Its 1-truncation agrees with $\tau_{\le 1} K(R)$ given above, this justifies the notation In the same spirit, we may regard $K_0(R)$ as $\pi_0(\tau_{\le 1} K(R))$.

2.3.10. Exercise. — Let R be a ring. Let $\mathrm{E}_n(R) \subseteq \mathrm{GL}_n(R)$ be the subgroup generated by the elementary matrices $e_{i,j}(r)$ for $i \neq j, r \in R$.

1. Show that $\mathrm{E}_n(R)$ is perfect for $n \geq 3$. In other words, $\mathrm{E}_n(R) = [\mathrm{E}_n(R), \mathrm{E}_n(R)]$. (Hint: use the following formula).

   $$[e_{ij}(r), e_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l, \\ e_{ij}(rs) & \text{if } j = k \text{ and } i \neq l, \\ e_{kj}(-sr) & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

   This is easy to check.

2. Let $g, h \in \mathrm{GL}_n(R)$. Show that $[g,h] \oplus 1_n \in \mathrm{GL}_{2n}(R)$ belongs to $\mathrm{E}_{2n}(R)$. (Hint: use the identities

$$\begin{pmatrix} [g,h] & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}$$
$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and show that every triangular matrix in $\mathrm{GL}_n(R)$ with 1's on the diagonal belongs to $\mathrm{E}_n(R)$.

3. Deduce from 1. and 2. that $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$.

2.3.11. Corollary. — *Let R be a ring. Then $K_1(R) \simeq \mathrm{GL}(R)^{\mathrm{ab}} = \mathrm{GL}(R)/\mathrm{E}(R)$.*

Proof. This is a simple application of Theorem 2.3.8. The element R generates the cofinal submonoid of free modules. We see that $G_n = \mathrm{Aut}_M(R^n)$ and $G_\infty = \mathrm{GL}(R)$. Putting these togther gives us

$$M^{\mathrm{grp}} \simeq \coprod_{\pi_0(M)^{\mathrm{grp}}} BG_\infty^{\mathrm{ab}} \simeq \coprod_{\pi_0(M)^{\mathrm{grp}}} B\,\mathrm{GL}(R)^{\mathrm{ab}}$$

Aftering taking homotopy group we get the first equality. The second equality follows from Exercise 2.3.10. ☐

2.3.12. Remark. — Let $M \in \mathsf{CMon}(\mathsf{Set})$ with a cofinal map $\mathbb{N} \to M$. We can compute the group completion of M as

$$M^{\mathrm{grp}} = \mathrm{colim}(M \overset{+1}{\to} M \overset{+1}{\to} M \overset{+1}{\to} \dots).$$

However, this does not work for $M \in \mathsf{CMon}(\mathsf{Grpd})$. For example, if $M = \mathsf{Proj}(R)^{\simeq}$, then the above would imply that

$$M^{\mathrm{grp}} \simeq \mathrm{colim}(M \overset{\oplus R}{\to} M \overset{\oplus R}{\to} M \overset{\oplus R}{\to} \dots) \simeq \coprod_{K_0(R)} B\,\mathrm{GL}(R).$$

However, the group $\mathrm{GL}(R)$ is not Abelian, which implies that the right hand side does not admit a symmetric monoidal structure.

2.3.13. Remark. — The group completion theorem implies that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{CMon}(\mathsf{Set}) & \xrightarrow{\mathrm{grp}} & \mathsf{Ab}(\mathsf{Set}) \\ \downarrow & & \downarrow \\ \mathsf{CMon}(\mathsf{Grpd}) & \xrightarrow{\mathrm{grp}} & \mathsf{Ab}(\mathsf{Grpd}) \end{array}$$

Note that this is not formal: for associative monoids,

$$\begin{array}{ccc} \mathsf{Mon}(\mathsf{Set}) & \xrightarrow{\mathrm{grp}} & \mathsf{Grp}(\mathsf{Set}) \\ \downarrow & & \downarrow \\ \mathsf{Mon}(\mathsf{Grpd}) & \xrightarrow{\mathrm{grp}} & \mathsf{Grp}(\mathsf{Grpd}) \end{array}$$

the above diagram does not commute.

2.4. K_1 and the trace map

Let R be a commutative ring and $P \in \mathsf{Proj}(R)$. We have an induced map

$$\det\colon \operatorname{End}_R(P) \to \operatorname{End}_R(\det(P)).$$

But note that $\operatorname{End}_R(\det(P)) \cong R$, since $\det(P)$ is invertible of rank 1. This gives us the *determinant map* $\det\colon \operatorname{End}_R(P) \to R$.

2.4.1. Definition. — Let R be a commutative ring and $P \in \mathsf{Proj}(R)$. The determinant map restrict to

$$\det\colon \mathrm{GL}(P) \to R^\times,$$

where $\mathrm{GL}(P) = \operatorname{Aut}_R(P)$. We denote $\mathrm{SL}(P)$ the kernel of this map.

2.4.2. Remark. — Let R be a commutative ring. We have

$$\mathrm{SL}(R^n) = \mathrm{SL}_n(R) \quad \text{and} \quad E_n(R) \subseteq \mathrm{SL}_n(R).$$

For each $n \in \mathbb{N}$, the following diagram commutes

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r] \arrow[dr, "\det"'] & \mathrm{GL}_{n+1}(R) \arrow[d, "\det"] \\ & R^\times \end{tikzcd}$$

Therefore we have a well-defined map

$$\det : \mathrm{GL}_n(R) \to R^\times$$

where the kernel is given by

$$\mathrm{SL}(R) = \operatorname{colim}_n \mathrm{SL}_n(R),$$

since finite limit commutes with filtered colimit. Therefore, there exists a unique determinant map such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{GL}(R) \arrow[r, "\det"] \arrow[d] & R^\times \\ \mathrm{GL}(R)^{\mathrm{ab}} \arrow[ur, dashed, "\det"'] & \end{tikzcd}$$

which follows from the commutativity of R.

2.4.3. Definition. — Let R be a commutative. The map

$$\det\colon K_1(R) \to R^\times$$

is again called the *determinant map*. We denote $SK_1(R)$ the kernel of this map.

2.4.4. Remark. — Let R be a commutative ring. The following holds:

$$SK_1(R) = \mathrm{SL}(R)/E(R) = \mathrm{SL}(R)^{\mathrm{ab}}.$$

Furthermore, the exact sequence

$$0 \to SK_1(R) \to K_1(R) \to R^\times \to 0$$

splits, which implies that $K_1(R) = SK_1(R) \oplus R^\times$.

2.4.5. Remark. — Let R be a commutative ring. One might attempt to define the determinant map of $K_1(R)$ using the map

$$\det\colon (\mathsf{Proj}(R)^{\simeq}, \oplus) \to (\mathrm{Pic}(R), \otimes).$$

However, this is not a braided monoidal functor: let

$$\gamma : R \oplus R \to R \oplus R, \quad \gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If this is a braided monoidal functor, then it would induce the identity map

$$\mathrm{id}_{\det(R)\otimes\det(R)}\colon \det(R) \otimes \det(R) \to \det(R) \otimes \det(R).$$

But this is impossible, since

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

This shows that $\mathrm{Pic}(R)$ is not the right target of det.

2.4.6. Definition. — Let R be a commutative ring. We define $\mathrm{Pic}^{\mathbb{Z}}(R)$ to be the following Picard groupoid: the underlying monoid is $(\mathbb{Z}_R \times \mathrm{Pic}(R), (+, \otimes))$ with the braiding:

$$\gamma\colon (n + n', L \otimes L') \simeq (n + n', L' \otimes L).$$

where $\gamma = (-1)^{n \cdot n'} \gamma_{L,L'} \in \mathbb{Z}_R$. This defines a symmetric monoidal functor

$$(\mathrm{rk}, \det) : \mathsf{Proj}(R)^{\simeq} \to \mathrm{Pic}^{\mathbb{Z}}(R).$$

By the universal property of $\tau_{\leq 1}K(R)$, we have an induced functor

$$(\mathrm{rk}, \det) : \tau_{\leq 1}K(R) \to \mathrm{Pic}^{\mathbb{Z}}(R).$$

After applying π_0, this gives us the construction for $K_0(R)$:

$$(\mathrm{rk}, \det) : K_0(R) \to \mathbb{Z}_R \times \mathrm{Pic}(R).$$

After applying π_1, this gives the determinant map

$$\det : K_1(R) \to R^{\times}.$$

This resolves the issue noted in Remark 2.4.5.

2.4.7. Remark. — In fact, we have $\mathrm{Pic}^{\mathbb{Z}}(R) = \mathrm{Pic}(\mathcal{D}(R), \otimes)$, where the $\mathcal{D}(R)$ is the derived category of R.

2.5. Computation of K_1

Let k be a field. We can show that $SL_n(k) = E_n(k)$ for each $n \geq 1$. This shows that $SK_1(R) = 0$ and the determinant functor gives us an isomorphism $K_1(k) \simeq k^{\times}$. Futhermore, for a division ring D, we can show that $K_1(D) = (D^{\times})^{\mathrm{ab}}$, see Example III.1.3.5 in [Wei13]. Therefore, we may ask the following question: for what kind of ring R do we have $K_1(R) \simeq R^{\times}$? A sufficient condition is $SL_n(R) = E_n(R)$ for $n \geq 1$.

2.5.1. Proposition. — *Let R be a commutative ring. Furthermore, assume that one of the following is true.*

- *The ring R is a local ring.*
- *The ring R is an Euclidean domain.*

Then $SL_n(R) = E_n(R)$ for all $n \geq 1$, hence $K_1(R) \simeq R^\times$.

Proof. It is left as an exercise to show that

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & \lambda & * \\ 0 & 0 & 1 \end{bmatrix} \in E_n(R).$$

which implies that $n \times n$ upper triangular matrices with determinant 1 belongs to $E_n(R)$. Therefore, it suffices to show that any invertible matrice is equivalent to a upper triangle matrix by elementary operations.

- Assume R is local and $A \in \mathrm{GL}_n(R)$. We can form the following sequence of operations

  $$A = \begin{bmatrix} * & \vdots \\ u & * \\ * & \vdots \end{bmatrix} \mapsto \begin{bmatrix} u & \vdots \\ \vdots & * \\ * & \vdots \end{bmatrix} \mapsto \begin{bmatrix} u & \vdots \\ 0 & * \\ 0 & \vdots \end{bmatrix}.$$

  This follows from the fact that every column contains a unit, since over the residue field of R, every column has a non-zero element.

- Assume R is an Euclidean domain and $A \in \mathrm{GL}_n(R)$. Pick $a_{i,1}$ in the first column with minimal degree, then we can use it to decrease the degree of the first column. Repeat this process, if we obtain an element of degree 0, then it is an unit. If at some step, all element in the column have the same degree. Since A is invertible, we have $(a_{1,1}, \dots, a_{n,1}) = R$. Therefore, there exists $j \neq i$ such that $a_{j,1} \notin (a_{i,1})$, then we may produce an element with a lower degree. Finally, we may repeat this process and argue like the first part.

□

2.5.2. Example. — Proposition 2.5.1 has some immediate consequences:

- We may conclude that
  $$K_1(\mathbb{Z}) = \mathbb{Z}^\times = \mathbb{Z}/2\mathbb{Z},$$
  since $\mathbb{Z}$ is an Euclidean domain with the valuation $\deg(n) = |n|$.

- Let k be a field. We may conclude that
  $$K_1(k[t]) = k^\times.$$
  Recall that $k[t]$ is an Euclidean with the degree of polynomial as the valuation.

- The Gaussian integer $Z[i]$ and the Eisenstein integer are euclidean domains. Therefore, we have $K_1(\mathbb{Z}[i]) = \mathbb{Z}/4\mathbb{Z}$ and $K_1(\mathbb{Z}[w]) = \mathbb{Z}/6\mathbb{Z}$ respectively.
- Recall that for imaginary quadratic fields, $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is PID if and only if

  $$d = -1, -2, -3, -7, -11, -19, -43, -67, -163.$$

  Also note that $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is euclidean, whenever $|d| \leq 11$. However, we actually have

  $$[K_1(\mathcal{O}_{\mathbb{Q}(\sqrt{d})}) = (\mathcal{O}_{\mathbb{Q}(\sqrt{d})})^{\times},$$

  for all d above.

2.5.3. Remark. — Consider the ring

$$R = \mathbb{Z}[t][t^{-1}, (t^n - 1)^{-1}, n \geq 1].$$

It is a PID, but $SK_1(R) \neq 0$.

2.6. K_1 of a number field

Let $\mathbb{Q} \subseteq F$ be a finite field extension, i.e. F is a *number field*. Recall that the map

$$(\mathrm{rk}, \det)\colon K_0(\mathcal{O}_F) \xrightarrow{\simeq} \mathbb{Z} \times \mathrm{Pic}(\mathcal{O}_F)$$

is an isomorphism and $\mathrm{Pic}(\mathcal{O}_F)$ is a finite Abelian group. For higher K-groups, we have the following theorems:

2.6.1. Theorem. — *Let F be a number field and $\mathcal{O}_F$ be the ring of integers of F. Then we have $SK_1(\mathcal{O}_F) = 0$, and*

$$\det : \tau_{\leq 1} K(\mathcal{O}_F) \to \mathrm{Pic}^{\mathbb{Z}}(\mathcal{O}_F)$$

is an equivalence of Picard groupoids.

2.6.2. Theorem (Dirichlet Unit Theorem). — *Let F be a number field and $\mathcal{O}_F$ be the ring of integers of F. Then*

$$\mathcal{O}_F^{\times} = \mu_F \oplus \mathbb{Z}^{r_1 + r_2 - 1},$$

where μ_F denotes the root of unity in F, r_1 is the number of real embeddings $F \hookrightarrow \mathbb{R}$ and r_2 is $\frac{1}{2}$ of the number of non-real embedding $F \hookrightarrow \mathbb{C}$.

A consequence of the Dirichtlet Unit Theorem is the following:

2.6.3. Theorem. — *Let F be a number field an $\mathcal{O}_F$ be the ring of integers of F. Then we have*

$$K_1(\mathcal{O}_F) \cong \mu(F) \oplus \mathbb{Z}^{r_1 + r_2 - 1}$$

where r_1 and r_2 are as in Theorem 2.6.2.

2.7. A quick view of K_2

Let R be a ring. Recall that $K_1(R) = \mathrm{GL}(R)/E(R)$. The idea of defining $K_2(R)$ is that it should measure the non-trivial relation between the elementary operations.

2.7.1. Definition. — Let R be a ring and $n \geq 3$. Then the n-th *Steinberg group* $\mathrm{St}_n(R)$ is defined as follows:

- The generators are $x_{i,j}(r)$, where $1 \leq i, j \leq n$, $i \neq j$ and $r \in R$.
- The relations are $x_{i,j}(r) \cdot x_{i,j}(s) = x_{i,j}(r+s)$ and

$$[x_{i,j}(r), x_{k,l}(s)] = \begin{cases} 1 & \text{if } j \neq k,\ i \neq l \\ x_{i,l}(rs) & \text{if } j = k,\ i \neq l \end{cases}$$

  They are called the *Steinberg relations*.

There exists a surjective group homomorphism $\mathrm{St}_n(R)(R) \to E_n(R)$ given by $x_{i,j}(r) \mapsto e_{i,j}(r)$. We define the *Steinberg group*

$$\mathrm{St}(R) = \mathrm{colim}_{n \in \mathbb{N}} \mathrm{St}_n(R).$$

Then we have a map:

$$\mathrm{St}(R) \to E(R).$$

This map is surjective, since colimit commutes with cokernel.

2.7.2. Definition (Milnor)**.** — The *second K group* of a ring R is defined by

$$K_2(R) = \ker(\mathrm{St}(R) \twoheadrightarrow (R)),$$

where the map is defined as in Definition 2.7.1.

2.7.3. Remark. — Milnor have shown that $K_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

2.8. Universal central extension

2.8.1. Definition. — Let G be a group. An *extension* of G is a surjective group homomorphism $\tau : \hat{G} \twoheadrightarrow G$. A *morphism of extension* of G is a diagram of the following form:

$$\begin{tikzcd} \hat{G} \arrow[r, "\tau"] \arrow[d, "\varphi"'] & G \\ \hat{G}' \arrow[ur, "\tau'"'] & \end{tikzcd}$$

where τ and τ' are extensions. An extension τ is said to be *central* if $\ker(\tau) \subseteq Z(\hat{G})$.

2.8.2. Proposition. — *Let G be a group. The group G is perfect if and only if the category of central extension of G has an initial object called the universal central extension.*

Proof. See Lemma III.5.3.2 and Theorem III.5.4 in [Wei13]. □

2.8.3. Proposition (Hopf)**.** — *Let G be a perfect group. Assume that we have a presentation $G = F/R$, where F is a free group, then the extension*

$$\tau_{\text{univ}} : [F, F]/[R, F] \to G$$

is the universal central extension.

Proof. See Theorem III.5.4 in [Wei13] □

2.8.4. Proposition. — *The map $\mathrm{St}_n(R) \to E_n(R)$ is the universal central extension of $E_n(R)$ for $5 \leq n \leq \infty$.*

Proof. See Proposition III.5.5.1 in [Wei13] □

2.8.5. Remark (Group Homology)**.** — Let G be a group. The $\mathbb{Z}$-coefficient *group homology* of G is defined as

$$H_*(G, \mathbb{Z}) = H_*(BG, \mathbb{Z}) = \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}).$$

where $H_*(BG, \mathbb{Z})$ denotes the $\mathbb{Z}$-coefficient singular homology of BG. In fact, we have $H_0(G, \mathbb{Z}) = \mathbb{Z}$, $H_1(G, \mathbb{Z}) = G^{\text{ab}} = \text{coker}(\tau_{\text{univ}})$ and $H_2(G, \mathbb{Z}) = \ker(\tau_{\text{univ}})$. This futhermore shows that $K_1(R) = H_1(GL(R), \mathbb{Z})$, $K_2(R) = H_2(GL(R), \mathbb{Z})$, Quillen also showed that $K_3(R) = H_3(St(R), \mathbb{Z})$. However, note that $K_0(R) \ncong H_0(R)$.

2.8.6. Remark. — Let G be a group. We have

$$H_0(G, \mathbb{Z}) = \mathbb{Z}, \quad H_1(G, \mathbb{Z}) = G^{\text{ab}}, \quad \text{and} \quad H_2(G, \mathbb{Z}) = \ker(\tau_{\text{univ}}).$$

By Proposition 2.8.4, this furthermore shows that

$$K_1(R) = H_1(\mathrm{GL}(R), \mathbb{Z}), \quad \text{and} \quad K_2(R) = H_2(\mathrm{GL}(R), \mathbb{Z}).$$

Quillen have also shown that $K_3(R) \cong H_3(\mathrm{St}(R), \mathbb{Z})$. However, note that $K_0(R) \ncong H_0(R)$.

2.8.7. Theorem (Matsumoto)**.** — *Let k be a field. We have*

$$K_2(k) = (k^\times \otimes_{\mathbb{Z}} k^\times)/\langle a \otimes (1-a), a \rangle$$

where $a \in k \backslash \{0, 1\}$.

Milnor defined the higher K-groups using the observation above.

2.8.8. Definition. — Let k be a field. We can define the following graded ring:

$$K_*^M(k) = \bigoplus_{n \geq 0} ((k^\times)^{\otimes n}/(a \otimes (1-a))$$

This is called the *Milnor K-theory* of k. We call the piece of the grading

$$K_n^M(k) = (k^\times)^{\otimes n}/(a \otimes (1-a))$$

the n-th *Milnor K-groups* of k.

2.8.9. Remark. — Let k be a field. Note that

$$K_n^M(k) \cong K_n(k), \quad \text{for} \quad 0 \leq n \leq 2.$$

Later we will define the higher K-theory of rings in the sense of Quillen. We note that the comparision map

$$K_n^M(k) \to K_n(k)$$

is not an isomorphism for $n \geq 3$.

2.8.10. Remark. — Let k be a field. It turns out that $K_*^M(k)$ is a piece of the associated grading of the motivic filtration of $K_*(k)$.

Chapter 3.

Higher K-theory of rings

The idea behind defining higher K-theory is to associate to every ring R a suitable *space* (or, in modern terms, an *anima*) $K(R)$. The higher K-groups are then defined simply as the homotopy groups of this space. To achieve this, we regard $(\mathrm{Pic}(R)^{\simeq}, \oplus)$ as an object in An, the ∞-category of anima (that is, ∞-groupoids). We will then define a group completion functor and establish the group completion theorem. Before turning to this topic, we first provide some brief remarks on ∞-categories.

If we define strict 1-categories as the ordinary categories, then we can define strict n-categories inductively as the categories enriched in $(n-1)$-categories. IN other words,

3.0.1. Definition. — Let $1 \leq n < \infty$. A strict n-category C consists of

- A set of objects.
- For every object X, Y of C, there is a strict $n-1$-category $\mathrm{Map}(X, Y)$.

However, this is not the correct notion of n-categories. Even the classical notion of ordinary categories is somewhat unsatisfactory, since isomorphisms are not exactly the same as equivalences. This classical framework causes no issues as long as we restrict attention to equivalence-invariant phenomena. After modding out by this distinction, it is still possible to give a correct definition of n-categories for finite n.

It is also worth remarking that every n-category is equivalent to a strict n-category whenever $n \leq 2$. However, this is no longer true in general for $n \geq 3$.

We now turn to ∞-categories and ∞-groupoids. From the modern perspective, ∞-groupoids are the primitive notions of ∞-categories; they should be regarded as the most basic objects, playing a role analogous to that of sets in ordinary mathematics.

As mentioned earlier, any n-category with $n \leq 2$ can be modeled by a strict n-category. Similarly, there are many possible models for ∞-groupoids and ∞-categories. For instance, one model of ∞-groupoids is given by topological spaces. In fact, the category An can be realized as the localization of Top at the weak equivalences.

3.1. Simplicial homotopy theory

Sets should be thought of as discrete ∞-groupoids. Another way to think of ∞-groupoids is the following:

3.1.1. Remark. — Every ∞-groupoid is the colimit of a diagram in Set of the shape

$G : \Delta^{\mathrm{op}} \to \mathsf{Set}$, here the colimit is taken in the ∞-categorical sense.

$$\begin{tikzcd} \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set}) \arrow[r, hook] \arrow[dr, two heads, "\mathrm{colim}"'] & \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Grpd}_\infty) \arrow[d, two heads, "\mathrm{colim}"] \\ & \mathsf{Grpd}_\infty \end{tikzcd}$$

In fact, the colimit functor exhibits Grpd_∞ as the a localization of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$.

3.1.2. Definition. — The *simplex category* Δ is the category consisting of non-empty finite ordered sets and the order-preserving maps between them. To be more precise:

- The objects are $[n] = \{0 < 1 < \ldots < n\}$, where $n \geq 0$.
- The morphisms between $[n], [m]$ are the ordered preserving maps between them.

A *simplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. Similarily, a *co-simplicial object* in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$.

3.1.3. Remark. — There are two special types of morphisms in Δ. For every $n \in \mathbb{N}$, the *i-th face map* is the map

$$\delta_i : [n] \to [n+1] \qquad 0 \leq i \leq n+1$$

defined by skipping the i-th element. The *degeneracy map* $b_i : [n] \to [n+1]$ is the map

$$b_i : [n] \to [n+1] \qquad 0 \leq i \leq n-1$$

defined by hitting the value i twice. Every map $f : I \to J$ in Δ factors uniquely up to isomorphism as a composition $f : I \to I' \to J$, where the epimorphism $I \to I'$ can be written as a unique composition of the degeneracy maps and the monomorphism $I' \to J$ can be written as a unique composition of the face maps.

3.1.4. Definition. — We denote the category

$$s\mathcal{C} = \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}).$$

the *category of simplicial objects* in a category $\mathcal{C}$.

3.1.5. Remark. — We have a presentation of the category Δ as follows:

1. The objects are $[n]$, for $n \geq 0$.
2. The generators of the morphisms are the face maps and the degeneracy maps.
3. We have the following simplicial relations:
   - If $j \leq i$, then $\delta_i \delta_j = \delta_{i+1} \delta_j$.
   - If $j < i$, then $b_j b_i = b_{i-1} b_j$.
   - We have

$$b_j \delta_i = \begin{cases} \delta_i b_{i-1} & i < j \\ \mathrm{id} & i = j \text{ or } i = j+1 \\ \delta_{i-1} b_j & i > j+1 \end{cases}$$

The face maps and the degeneracy maps as well as the relations between them encodes all the information of the category Δ.

3.1.6. Notation. — We have the following notation:

- If $X \in s\mathcal{C}$, then we write $X_n = X([n])$.
- We write $d_i : X_{n+1} \to X_n$ for the face map $\delta_i^* = X(\delta_i)$.
- We write $s_i : X_{n+1} \to X_n$ for the degeneracy map $b_i^* = X(b_i)$.
- If $\mathcal{C} = \mathsf{Set}$. The elements of X_n are called the *the n-simplicies* of X. A 0-simplex is also called a *vertex*, a 1-simplex is called an *edge*, and a 2-simplex is called a *face*.

3.1.7. Remark. — The simplicial objects in Set assemble into a category

$$\mathsf{sSet} = \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$$

called the category of *simplicial sets*. Since in functor categories limit and colimit are computed pointwise, the category sSet admits finite limit and colimit.

Simplicial sets can be thought of as the combinatorial data of triangulations of topological spaces.

3.1.8. Definition. — Let $I \in \mathsf{Fin}$. We define

$$|\Delta^I| = \{(t_i)_{i\in I} \in \mathbb{R}^I : \sum_{i\in I} t_i = 1\} \subseteq \mathbb{R}^I$$

the *topological I-simplicies*. This construction is functorial: given a map $\varphi : I \to J$ in Fin, then we get a map $\varphi_* : |\Delta^I| \to |\Delta^J|$ defined by extending $e_i \mapsto e_{\varphi(i)}$ linearly.

In fact, this defines a cosimplicial topological space

$$\Delta \to \mathsf{Fin} \to \mathsf{Top}.$$

At the level of objects, this functor sends each n to $|\Delta^n|$, and the face maps are sent to the maps of the form $\delta_i : |\Delta^n| \to |\Delta^{n+1}|$ for $0 \leq i \leq n+1$, it is the inclusion of the face opposite to the vertex i, and the degeneracy maps are sent to the maps of the form $b_i : |\Delta^n| \to |\Delta^{n-1}|$ for $0 \leq i \leq n-1$, by collapsing the edge $[i, i+1]$.

3.1.9. Definition. — The *geometric realization* $|\cdot| : \mathsf{sSet} \to \mathsf{Top}$ is the left Kan-extension of the cosimplicial topological space functor along the Yoneda embedding. It is also the unique colimit preserving functor such filling the following diagram:

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, hook, "よ_\Delta"'] & \mathsf{Top} \\ \mathsf{sSet} \arrow[ur, dashed, "|\cdot|"'] & \end{tikzcd}$$

Recall that every presheaf can be written as the colimit of representable presheaves. Therefore, if $X \in \mathsf{sSet}$, then

$$X = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}, x\in X_n} \mathrm{Map}_\Delta(-, [n]).$$

By the defining property of geometric realization, we have

$$|X| = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}},x\in X_n} |\Delta^n| = (\bigsqcup_{n\geq 0}\bigsqcup_{x\in X_n} |\Delta^n|)/\sim$$

where $\sim$ is defined as follows: for every $\varphi : [n] \to [m]$, $x \in X_n$ and $v \in |\Delta^n|$, force $(\varphi^*(x), v) \sim (x, \varphi_*(v))$. We can also write

$$|X| = (\bigsqcup_{n\geq 0}\bigsqcup_{x\in X_n^{\mathrm{nd}}} |\Delta^n|)/\sim,$$

where $X_n^{\mathrm{nd}} = X_n \backslash (\bigcup_{i=0}^{n-1} s_i(X_{i-1}))$.

3.1.10. Remark. — The geometric realization functor $|\cdot|\colon \mathsf{sSet} \to \mathsf{Top}$ admits a right adjoint $\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{sSet}$ given by $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, X)$ for $X \in \mathsf{Top}$ and $n \geq 0$.

3.1.11. Example. — We have the following examples:

- Let $\Delta^n = \mathrm{Map}_{\Delta}(-, n)$. We usually denote

  $$\Delta^n_m = \mathrm{Map}_{\Delta}([m], [n]),$$

  the set of non-decreasing maps from $[m]$ to $[n]$. For example, we have

  $$\Delta^n_0 = \{i : 0 \leq i \leq n\} \quad \text{and} \quad \Delta^n_1 = \{(i, j) : 0 \leq i \leq j \leq n\}$$

  The latter has an alternative description:

  $$\Delta^n_1 = \{(i, i) : 0 \leq i \leq n\} \cup \{(i, j) : 0 \leq i < j \leq n\}.$$

- We have the *boundary* $\partial\Delta^n \subseteq \Delta^n$ defined by

  $$\partial\Delta^n_m = \{\text{non surjective map } f : [m] \to [n]\},$$

  alternatively, we have

  $$\partial\Delta^n_m = \mathrm{coeq}(\bigsqcup_{i<j} \mathrm{Map}_{\Delta}([m], [n] - \{i, j\}) \rightrightarrows \bigsqcup_{i\in[n]} \mathrm{Map}([m], [n] - \{i\}).$$

  This implies that

  $$\partial\Delta^n = \mathrm{coeq}(\bigsqcup_{i<j} \Delta^{n-2} \rightrightarrows \bigsqcup_{i\in[n]} \Delta^{n-1}).$$

  After taking the geometric realization, we see that

  $$|\partial\Delta^n| = \partial|\Delta^n| \subseteq \mathbb{R}^{n-1},$$

  since as left $|\cdot|$ preserves colimits.

- We have the k-th *inner horn* $\Lambda^n_k \subseteq \partial\Delta^n$ for $0 \leq k \leq n$ defined by

  $$\Lambda^n_k = \{f : [m] \to [n] : [n] - \{k\} \not\subseteq \mathrm{im}\, f\}.$$

  Similarly, we have

  $$\Lambda^n_k = \mathrm{coeq}(\bigsqcup_{i,j\neq k} \Delta^{n-2} \rightrightarrows \bigsqcup_{i\neq k} \Delta^{n-1}).$$

  This shows that the geometric realization $|\Lambda^n_k|$ is the same as $\partial|\Delta^n|$ minus the face opposite to the k-th vertex.

- We define $\Delta^1/\partial\Delta^1$ via the following pushout:

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\ * \arrow[r] & \Delta^1/\partial\Delta^1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

  More explicitly, if we compute this pushout pointwise, we will get

$$(\Delta^1/\partial\Delta^1)_m = \{[m] \xrightarrow{\text{surj}} [1]\} \sqcup *$$

  where $*$ corresponds to the non-surjective maps. There are $m+1$ elements in this set. There are only two non degenerate simplices, namely the vertex $*$ and $\mathrm{id}_{[1]}$. This shows that $|\Delta^1/\partial\Delta^1| \simeq S^1$. In fact, we can see this in another way:

$$|\Delta^1/\partial\Delta^1| = |\Delta^1|/|\partial\Delta^1| = S^1,$$

  since $|\cdot|$ preserves colimit. More generally, we can define $|\Delta^n/\partial\Delta^n|$ and show that $|\Delta^n/\partial\Delta^n| \simeq S^n$.

- Since $\mathbb{R}$ can be viewed as the pushout of infinitely many copies of the unit interval and that $|\cdot|$ preserves colimit, we have

$$\mathbb{R} = |\ldots \sqcup \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup \ldots|$$

  where the pushout is taken along δ_1 and δ_0.

In fact, the theory of simplicial sets has many similarities with the theory of CW-complexes in the following sense.

3.1.12. Remark. — Let $X, Y \in \mathsf{sSet}$. The following holds.

1. The topological space $|X|$ is always Hausdorff, paracompact and *compactly generated*, that is, a subset $U \subseteq |X|$ is open if and only if for every map $f : K \to |X|$, where K is compact Hausdorff, the inverse image $f^{-1}(U) \subseteq K$ is open.

2. The topological space $|X|$ is compact if and only if it is *finite*, that is, there are only finitely many non-degenerate simplices.

3. The topological space $|X|$ is locally compact if and only X is *locally finite*, that is, every $x \in X_0$ is contained in finitely many non-degenerate simplices.

4. Assume that X or Y is locally finite. We have

$$|X \times Y| = |X| \times |Y|,$$

   where the product is taken in the category of compactly generated topological space. It is equipped with the product topology.

3.1.13. Definition. — Let $X, Y \in \mathsf{sSet}$ and $f, g : X \to Y$ be maps between simplicial sets.

1. A *homotopy* from f to g is a map $X \times \Delta^1 \to Y$ such that

$$\begin{tikzcd} X \arrow[d, "\delta_0"'] \arrow[dr, "f"] & \\ X \times \Delta^1 \arrow[r, "h"] & Y \\ X \arrow[u, "\delta_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

   the above diagram commutes.

2. The maps f, g are called *homotopic* if they are equivalent under the equivalence relation generated by the existence of a homotopy.

3. A map $f : X \to Y$ is called a *homotopy equivalence* if there exists $h : Y \to X$ such that $h \circ f \simeq \mathrm{id}_X$ and $f \circ h \simeq \mathrm{id}_Y$.

3.1.14. Definition. — A map $f : X \to Y$ between simplicial sets is called a *weak equivalence*, whenever $|f| : |X| \to |Y|$ is a weak equivalence.

3.1.15. Remark. — Let $f : X \to Y$ be a map between simplicial sets. If f is a homotopy equivalence, then after applying the geometric realization functor we see that $|f| \colon |X| \to |Y|$ is again a homotopy equivalence. In particular f is a weak equivalence.

3.1.16. Remark. — Simplicial sets can also be a model for infinity groupoids in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathsf{sSet}[\mathrm{weq}^{-1}] \arrow[rr, "|\cdot|"] \arrow[dr, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"'] & & \mathsf{Top}[\mathrm{weq}^{-1}] \arrow[dl, "\mathrm{Sing}"] \\ & \mathsf{Grpd}_\infty & \end{tikzcd}$$

where every arrow in the diagram is an equivalence of infinity categories.

3.1.17. Notation. — We have the following notation:

- We denote stCat the category of strict categories.
- The category Pos of posets is a full subcategory of stCat.

Observe that there is an embedding $\Delta \to \mathsf{stCat}$ given by $(I, \leq) \mapsto (I, \leq)$.

3.1.18. Definition. — The *nerve functor* $N : \mathsf{stCat} \to \mathsf{sSet}$ is defined as follows: let $\mathcal{C} \in \mathsf{stCat}$, then we define the simplicial set $N(\mathcal{C})$ by setting

$$N(\mathcal{C})_n = \mathrm{Map}_{\mathsf{stCat}}([n], \mathcal{C}).$$

The simplicial identities are encoded in the first variable. Pictorially, $N(\mathcal{C})$ is the string of n-composable arrows. For example, $N(\mathcal{C})_0$ encodes the information of the objects of $\mathcal{C}$ and $N(\mathcal{C})_1$ encodes the information of the morphisms of $\mathcal{C}$. Let $x, y \in N(\mathcal{C})_0$ and $f \in N(\mathcal{C})_1$ such that $d_0(f) = y$ and $d_1(f) = x$. Then x is the *source* of f, y is the *target* of f and we write $f \colon x \to y$. Furthermore, $d_0(x) = \mathrm{id}_x$. It also follows that

$$N(\mathcal{C})_2 \simeq N(\mathcal{C})_1 \times_{N(\mathcal{C})_0} N(\mathcal{C})_1.$$

We define the composition map to be the inverse of this map.

3.1.19. Proposition. — *The nerve functor $N\colon \mathsf{stCat} \to \mathsf{sSet}$ preserves coproduct.*

Proof. For strict categories $\mathcal{C}$ and $\mathcal{D}$, we have

$$N(\mathcal{C} \sqcup \mathcal{D})_n = \mathsf{Fun}([n], \mathcal{C} \sqcup \mathcal{D}) = \mathsf{Fun}([n], \mathcal{C}) \sqcup \mathsf{Fun}([n], \mathcal{D}) = N(\mathcal{C})_n \sqcup N(\mathcal{D})_n.$$

The proposition follows from the fact that the above isomorphism is natural in n. □

3.1.20. Remark. — A natural transformation between functors $\alpha\colon f \Rightarrow g$ is the same as a functor $\mathcal{C} \times [1] \to \mathcal{D}$. On object, it takes $(c, 0)$ to $f(c)$ and $(c, 1)$ to $g(c)$. It is clear that the nerve functor preserves limit. Therefore, if you take the nerve functor, you will get a simplicial homotopy from $N(f)$ to $N(g)$:

$$N(\mathcal{C}) \times \Delta^1 \to N(\mathcal{D}).$$

and after taking geometric realization, we will get

$$|N(\mathcal{C})| \times |\Delta^1| \to |N(\mathcal{D})|,$$

since Δ^1 is locally finite.

3.1.21. Proposition. — *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor. Assume that f has an right (left) adjoint $g\colon \mathcal{D} \to \mathcal{C}$, then f is an equivalence.*

Proof. Since f is left adjoint to g, we have the unit and counit

$$\mu\colon \mathrm{id} \Rightarrow f \circ g \quad \text{and} \quad \epsilon\colon g \circ f \Rightarrow \mathrm{id}\,.$$

By Remark 3.1.20, after taking geometric realizstion, they become homotopies to the identity. □

3.1.22. Corollary. — *The nerve functor $N : \mathsf{stCat} \to \mathsf{sSet}$ sends equivalences to weak equivalences, hence induces a functor*

$$N : \mathsf{stCat}[E] \to \mathsf{sSet}[W].$$

which induces a functor $N : \mathsf{Cat} \to \mathsf{Grpd}_\infty$, where we denote Cat as the 2-category of categories, with functors as morphisms and natural isomorphisms as 2-morphisms.

3.1.23. Exercise. — Show that the nerve functor is fully faithful and a simplicial set $X \in \mathsf{sSet}$ is in the essential image, iff for all $n \geq 0$, and $0 < k < n$, every lifting problem as follows has a unique solution.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

In other words, X has the unique right lifting propery with respect to inner horn inclusion. Furthermore, $X \in \mathsf{sSet}$ is the nerve of a 1-groupoid if and only if the above holds for all $0 \leq k \leq n$.

In the strict case, the lifting solution is unique. However, when working up to homotopy, uniqueness may fail, so we must relax the condition. This motivates the following definition:

3.1.24. Definition. — Let $X \in \mathsf{sSet}$.

- We say X is a *quasi-catgeory*, if it has the right lifting property with respect to inner horn inclusions.
- We say X is a *Kan complex*, if it has the right lifting property with respect to all horn inclusions.

3.1.25. Example. — A *simplicial groupoid* $\mathcal{X}$ is a category enriched in simplicial sets in which every morphism is invertible up to composition. Concretely:

- We have a collection of objects $\mathrm{ob}(\mathcal{X})$.
- For two object x, y of $\mathcal{X}$, we have a simplicial set $\mathrm{Map}_{\mathcal{X}}(x, y)$.
- For three object x, y, z of $\mathcal{X}$, there are simplicial maps

$$- \circ -\colon \mathrm{Map}_{\mathcal{X}}(y, z) \times \mathrm{Map}_{\mathcal{X}}(x, y) \to \mathrm{Map}_{\mathcal{X}}(x, z)$$

  that are associative and unital.
- Every 0-vertex of $\mathrm{Map}_{\mathcal{X}}(x, y)$ is invertible up to composition.

The category of simplicial groupoid is denote by sGrpd. In fact, we can define a nerve functor $N\colon \mathsf{sGrpd} \to \mathsf{sSet}$ as follows: let $\mathcal{X}$ be a simplicial groupoid, the n-simplices $N(\mathcal{X})_n$ consists of objects $(x_1, \ldots, x_n)$ together with composable simplicies $f_i \in \mathrm{Map}_{\mathcal{X}}(x_i, x_{i+1})$ for each $i \in \{1, \ldots, n\}$. The face and degeneracy maps are defined in the usual way, by composition and insertion of identities, respectively. Moreover, the nerve functor factors through Kan, yielding a functor $N\colon \mathsf{sGrpd} \to \mathsf{Kan}$.

3.1.26. Proposition. — *Let $X \in \mathsf{Top}$. The singular simplicial set $\mathrm{Sing}(X)$ is a Kan complex.*

Proof. For $n \geq 0$ and $0 \leq k \leq n$, the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

corresponds to a lifting problem as follows, by Remark 3.1.10.

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

This can be solved, since $|\Lambda^n_k| \subseteq |\Delta^n|$ has a continuous retraction. $\square$

3.1.27. Remark. — Here are some properties of Kan complexes:

- Let $X, Y \in \mathsf{Kan}$. Every weak equivalence $X \to Y$ is a homotopy equivalence.

- Let $X \in \mathsf{sSet}$. There exists a weak equivalence $\eta\colon X \to X'$, where X' is a Kan complex. In fact, the inclusion map induces an equivalence of ∞-categories

$$\mathrm{inc}\colon \mathsf{Kan}[\mathrm{heq}^{-1}] \simeq \mathsf{sSet}[\mathrm{weq}^{-1}],$$

  where the left hand side denote the localization of Kan at the homotopy equivalences.

- Let $X \in \mathsf{Kan}$, $x \in X_0$ and $n \geq 0$. We have

$$|\cdot|\colon \mathrm{Map}_{\mathsf{sSet}_*}(\Delta^n \backslash \partial\Delta^n, (X, x))/\sim \to \mathrm{Map}_{\mathsf{Top}_*}((S^n, *), (|X|, x))/\sim,$$

  where we quotient out the homotopy equivalences on both sides.

3.2. Classifying spaces

3.2.1. Definition. — Let G be a group. The *classifying space* BG is defined as $|N(BG)|$.

3.2.2. Proposition. — *Let Γ be a groupoid and $X = |N(\Gamma)|$. The following holds:*

- *We have $\pi_0(X) \simeq \pi_0(\Gamma)$.*
- *We have $\pi_1(X, x) \simeq \pi_1(\Gamma, x)$ for all $x \in \Gamma$.*
- *We have $\pi_n(X, x) = 0$ if $n \geq 2$. In other words, X is 1-truncated.*

Proof. By Theorem 2.1.9, we may choose an equivalence

$$\Gamma \simeq \bigsqcup_{i \in I} BG_i.$$

By Proposition 3.1.19, the functor $|N(-)|$ takes equivalence to homotopy equivalence and preserves coproduct. Therefore, without losses of generality we may assume $\Gamma \simeq BG$. The group G is canonically a G-set with left multiplication as the G-action. Therefore, by Example 2.1.3, we can define $EG = G//G$. By definition, we have $BG = *//G$, the G-equivariant map $p\colon G \to *$ induces a map

$$p//G\colon EG \to BG.$$

Since $\pi_0(EG) = *$ and $\mathrm{Aut}_{EG}(x) = *$, by Theorem 2.1.9, we have $EG \simeq *$. We also see that G acts on EG via left multiplication. After taking the nerve, we see that $N(EG)_n \cong G^n$ and $N(BG)_n \cong G^{n-1}$, where we use G^0 to denote $*$. Degreewise the map $N(EG)_n \to N(BG)_n$ is given by projection onto the last $n-1$-factor and we act G on $N(EG)_n$ by acting on the first factor. After taking the geometric realization, we see that

$$|N(EG)| = (\bigsqcup_{G} * \sqcup \bigsqcup_{G \times G} \Delta^1 \sqcup \bigsqcup_{G \times G \times G} \Delta^2 \sqcup \ldots)/\sim$$

Since G acts degreewise freely on $N(EG)$ and G acts on $|N(EG)|$ by permuting the cells, the action of G on $|N(EG)|$ is *properly discontinuous*. In other words, for all $x \in |N(EG)|$, there exists a neighborhood U of x such that for all $g \in G \backslash \{1\}$, then $g \cdot U \cap U = \varnothing$. Furthermore, the map $|N(EG)|/G \to |N(BG)|$ is an equivalence, which implies that $|N(EG)| \to |N(BG)|$ is a Galois covering map with the Galois group G. As we've shown before, $|N(EG)|$ is contractible, the claim then follows by applying the long exact sequence for homotopy groups. $\square$

3.2.3. Remark. — Let $\Gamma \in \mathsf{Grpd}$. By Proposition 3.2.2, we have an equivalence

$$\Pi_1(|N(\Gamma)|) \simeq \Gamma.$$

In fact, this implies the homotopy hypothesis for 1-groupoid:

$$\mathsf{Top}_{\leq 1}[\mathrm{weq}^{-1}] \underset{N}{\overset{\Pi_1}{\rightleftarrows}} \mathsf{Grpd}$$

the above defines an equivalence of 2-categories.

3.2.4. Example. — We have the following examples of classifying space:

- We have $|N(B\mathbb{Z})| \simeq S^1$, this recovers the usual universal covering $\mathbb{R} \to S^1$.
- We have $|N(BC_2)| \simeq \mathbb{RP}^\infty$, this recovers the universal covering $S^\infty \to \mathbb{RP}^\infty$.

3.2.5. Warning. — Let $\mathcal{C}$ be a category. The topological space $|N(\mathcal{C})|$ is usually not 1-truncated. Furthermore, Thomason showed in [Tho80] that the functor

$$|N(-)| \colon \mathsf{stCat} \to \mathsf{Grpd}_\infty$$

is essentially surjective. In fact, it even exhibits Grpd_∞ as a localization of stCat.

3.3. CW complexes

3.3.1. Definition. — A *CW-complex* is a topological space X equipped with a filtration

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X$$

such that $\operatorname{colim}_{n\to\infty} X^{(n)} = X$ in Top, and for all $n \geq 0$, there exists pushout square

$$\begin{tikzcd} \coprod_\alpha S^{n-1} \arrow[r, "\tilde{\varphi}_\alpha"] \arrow[d, hook] & X^{(n-1)} \arrow[d, hook] \\ \coprod_\alpha D^n \arrow[r, "\varphi_\alpha"] & X^{(n)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We call $X^{(n)}$ the *n-skeleton* of X. A map $f \colon X \to Y$ between CW-complexes is a collection of continuous maps $\{f_n \colon X_n \to Y_n\}_n$. This defines a category CW with an forgetful functor $\mathsf{CW} \to \mathsf{Top}$.

3.3.2. Remark. — Lets recall some general facts from algebraic topology.

- The underlying space of a CW-complex is Hausdorff, paracompact and compactly generated.
- Every smooth manifold has a CW structure.
- Every topological manifold of dim $\neq 4$ has a CW structure. It is still an open question for the case dim $= 4$.

- Every topological manifold is homotopy equivalent to a CW-complex of the corresponding dimension.
- Let $X \in \mathsf{sSet}$. The topological space $|X|$ has a canonical CW-structure, where

$$|X|^{(n)} = (\bigsqcup_{0\le k\le n} \bigsqcup_{x\in X_k^{nd}} |\Delta^k|)/\sim = |\operatorname{sk}_n X|,$$

where the n-th *skeleton* $\operatorname{sk}_n X \subseteq X$ is the simplicial subset generated by X_k for $0 \le k \le n$. The following diagram commutes:

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, "|\cdot|"] \arrow[dr, "|\cdot|"'] & \mathsf{Top} \\ & \mathsf{CW} \arrow[u, hook] \end{tikzcd}$$

In other words, the geometric realization factors through CW.

3.3.3. Remark. — Let $X, Y \in \mathsf{CW}$. The topological space $X \times Y$ is compactly generated and admits a CW-structure

$$(X \times Y)^{(n)} = \bigcup_{p+q=n} X^{(p)} \times Y^{(p)}.$$

In particular, this allows us to define homotopies in CW as maps $X \times |\Delta^1| \to Y$, and hence to speak of homotopy equivalences in the category CW.

3.3.4. Theorem (CW Approximation)**.** — *Let X be a topological space. The following holds.*

- *There exists $X' \in \mathsf{CW}$ and a weak equivalence $\eta\colon X' \to X$.*
- *Let $X, Y \in \mathsf{CW}$. Any map $f\colon X \to Y$ in Top is homotopic to a map in CW.*

3.3.5. Theorem. — *A map $f : X \to Y$ in CW is a weak equivalence if and only if it is a homotopy equivalence.*

In summary, the following diagram commutes:

$$\begin{tikzcd} \mathsf{CW} \arrow[r, hook] \arrow[drr, "\Pi_\infty"'] & \mathsf{Top} \arrow[r, "|\cdot|"] \arrow[dr, "\Pi_\infty"] & \mathsf{sSet} \arrow[d, "\Pi_\infty"] & \mathsf{Kan} \arrow[l, hook'] \arrow[dl] \\ & & \mathsf{Grpd}_\infty & \end{tikzcd}$$

where Π_∞ denotes the localization functor.

3.4. Group completion

3.4.1. Notation. — In this section, whenever we refer to $\mathcal{C}$ as a category, we mean a strict category, as opposed to an ∞-category.

Let R be a ring. In analogy with the construction of $K_1(R)$, we must introduce an appropriate notion of group completion in the setting of ∞-categories. Conceptually, this should arise as the left adjoint to the inclusion functor:

$$\mathsf{CMon}(\mathsf{Grpd}_\infty) \underset{\longleftarrow}{\overset{(-)^{\mathrm{grp}}}{\dashrightarrow}} \mathsf{Ab}(\mathsf{Grpd}_\infty)$$

But before doing so, we must first introduce the appropriate notion of a monoid object in Grpd_∞. Ordinarily, in a strict category, one defines a monoid object by specifying a binary operation together with the commutative diagrams encoding the required compatibilities. However the usual definition does not generalize to higher categories, since in the infinity setting, there are infinitely many of these commutative diagrams to write down. The better approach is the following:

3.4.2. Remark. — Let $\mathcal{C}$ be a category with finite product and $M \in \mathsf{CMon}(\mathcal{C})$. Let I be a finite set. We can assign I to the finite product M^I, consisting of map $I \to M$, which we may write as $i \mapsto x_i$. Via the multiplication map $m\colon M^I \to M$, we may think of this as a sum $\sum_{i\in I} x_i$. A morphism $I \to J$ gives rise to a coproduct decomposition $I = \bigsqcup_{j\in J} I_j$, where $I_j = \varphi^{-1}(j)$ and we have the following commutative diagram:

$$\begin{tikzcd}
\prod_{j\in J} M^{I_j} \arrow[r, "m"] \arrow[d, "\simeq"'] & \prod_{j\in J} M \arrow[d, "m"] \\
M^I \arrow[r, "m"] & M
\end{tikzcd}$$

In other words, given a map $\varphi\colon I \to J$, there is a map

$$\mu_\varphi\colon M^I \to M^J, \quad (x_i)_{i\in I} \mapsto (\sum_{i\in I_j} x_i)_{j\in J}.$$

This construction gives us a functor $M\colon \mathsf{Fin} \to \mathcal{C}$. However, this functor does not capture all the structure of a commutative monoid—for instance, it does not encode commutativity. To remedy this, we need to enlarge Fin by adding additional morphisms.

Let Fin' be the category of finite sets and partially defined maps. Explicitly, a morphism $\varphi : I \to J$ in Fin' is given by a map $D(f) \to J$ in Set, where $D(f) \subseteq I$ is called the *domain* of f. The composition is defined as follows: given partially defined maps $f : I \to J$ with domain $D(f)$ and $g : J \to K$ with domain $D(g)$, then $g \circ f : I \to K$ is defined by $g \circ f$ on the domain $f^{-1}(D(g))$. For every commutative monoid $M \in \mathsf{CMon}(\mathcal{C})$, we get a functor $\mathsf{Fin}' \to C$ by $I \mapsto M^I$ and $M^I \to M^J$ given by $M^I \to M^{D(f)} \to M^J$.

3.4.3. Definition. — The maps $\rho_i : I \to *$ in Fin' with domain $\{i\}$ are called *Segal maps.* Let $\mathcal{C}$ be a category. A functor $F : \mathsf{Fin}' \to \mathcal{C}$ is called *Segal* if for all $I \in \mathsf{Fin}$ the induced map

$$(\rho_i)_{i\in I} : F(I) \to \prod_{i\in I} F(*)$$

is an isomorphism in $\mathcal{C}$.

3.4.4. Proposition. — *Let $\mathcal{C}$ be a category. There is a functor*

$$\mathsf{CMon}(\mathcal{C}) \to \mathsf{Fun}(\mathsf{Fin}', \mathcal{C})$$

whose essential image consists of the Segal functors.

3.4.5. Remark. — There is also a variant of the story above. One can define *span category of finite sets* $\mathsf{Span}(\mathsf{Fin})$ as follows:

- The objects of $\mathsf{Span}(\mathsf{Fin})$ are finite sets.
- Let $I, J \in \mathsf{Span}(\mathsf{Fin})$. A morphism from I to J is a diagram of the form $I \leftarrow K \rightarrow J$.
- Let $I \leftarrow L \rightarrow J$ and $J \leftarrow P \rightarrow K$ be two morphism, their composite is defined by taking pullback:

$$\begin{tikzcd} & & Q \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & L \arrow[dl] \arrow[dr] & & P \arrow[dl] \arrow[dr] & \\ I & & J & & K \end{tikzcd}$$

  The composition is associative and unital.

Observe that Fin' is a subcategory of $\mathsf{Span}(\mathsf{Fin})$, where the morphisms are spans whose left leg is an injection. Let $\mathcal{C}$ be a category. In this case, we have a fully faithful functor

$$\mathsf{CMon}(\mathcal{C}) \to \mathsf{Fun}(\mathsf{Span}(\mathsf{Fin}), \mathcal{C}),$$

whose essential image consists of the functors that preserves finite product.

3.4.6. Remark. — The target of the functor in Proposition 3.4.4 generalizes nicely to ∞-categories. For example, one can show that there exists a fully faithful functor of 2-categories

$$\mathsf{CMon}(\mathsf{Cat}) \hookrightarrow \mathsf{Fun}(\mathsf{Fin}', \mathsf{Cat})$$

whose image consists of functors $F : \mathsf{Fin}' \to \mathsf{Cat}$ such that for all I, the induced map

$$(\rho_i)_{i\in I} \colon F(I) \to \prod_{i\in I} F(*)$$

is an equivalence of categories.

Now we will define the commutative monoid objects in Grpd_∞.

3.4.7. Definition. — An E_∞*-space* is a functor $X : \mathsf{Fin}' \to \mathcal{C}$, where $\mathcal{C}$ is Top or sSet, such that for all $I \in \mathsf{Fin}'$, the induced map

$$(\rho_i)_{i\in I} : X(I) \to \prod_{i\in I} X(*)$$

is a weak equivalence. We call $X(*)$ the *underlying space* of X. A morphism between E_∞-spaces $X \to Y$ is a natural transformation between functors. We say that it is a *weak equivalence* if $X(*) \to Y(*)$ is a weak equivalence. In this case, since weak equivalences are invariant under product, the map $X(I) \to Y(I)$ is again a weak equivalence for all $I \in \mathsf{Fin}'$.

The localization functor $\Pi_\infty \colon \mathcal{C} \to \mathsf{Grpd}_\infty$ preserves finite product, see Lemma 6.2.5 in [Cno24]. Therefore, passing to Grpd_∞, the choice of $\mathcal{C}$ won't matters for our purpose.

3.4.8. Remark. — Let $X \colon \mathsf{Fin}' \to \mathsf{Top}$ be an E_∞-space. The functor $\mathrm{fgt} \circ X \colon \mathsf{Fin}' \to \mathsf{Set}$ is a commutative monoid.

Next, we will generalize the notion of associative monoids to ∞-categories.

3.4.9. Remark. — Let $\mathsf{Fin}_{\mathrm{ord}}$ be the *category of finite ordered set*, $\mathcal{C}$ be a category with finite product and $M \in \mathsf{Mon}(\mathcal{C})$. We may assign each $I \in \mathsf{Fin}_{\mathrm{ord}}$ to a finite product M^I, consisting of elements of the form $(x_i)_{i\in I}$. There is also a multiplication map

$$\mu_I \colon M^I \to M, \quad (x_i)_{i\in I} \mapsto \prod_{i\in I} x_i$$

Given a map $I \to J$ in $\mathsf{Fin}_{\mathrm{ord}}$, we have an ordered decomposition of I, which then gives us a map $\mu_f : M^I \to M^J$.

The *category of pointed finite ordered sets* $\mathsf{Fin}'_{\mathrm{ord}}$ is defined as follows:

- The objects of $\mathsf{Fin}'_{\mathrm{ord}}$ are finite sets.
- Let $I, J \in \mathsf{Fin}'_{\mathrm{ord}}$. A morphism $f : I \to J$ in $\mathsf{Fin}_{\mathrm{ord}}$ is an ordered map $I \to J \cup \{\pm\infty\}$.

The intuition is that some elements at both ends are sent to $-\infty$ and ∞ respectively. For each $i \in I$, we also have the *Segal maps* $\rho_i : I \to * \cup \{\pm\infty\}$, which are defined by sending i to $*$, $k < i$ to $-\infty$ and $k > i$ to ∞.

3.4.10. Proposition. — *Let $\mathcal{C}$ be a category. There is a fully faithful functor*

$$\mathsf{Mon}(\mathcal{C}) \hookrightarrow \mathsf{Fun}(\mathsf{Fin}'_{\mathrm{ord}}, \mathcal{C})$$

whose essential image consists of functors such that for all $I \in \mathsf{Fin}_{\mathrm{ord}}$, the induced map

$$F(I) \to \prod_{i\in I} F(*)$$

is an isomophism in $\mathcal{C}$.

3.4.11. Definition. — An E_1-*space* or a A_∞-*space* X is a functor $X : \mathsf{Fin}'_{\mathrm{ord}} \to \mathcal{C}$, where $\mathcal{C}$ is Top or sSet, such that the induced map

$$(\rho_i)_{i\in I} : F(I) \to \prod_{i\in I} F(*)$$

is a weak equivalence. Again, after localization, the particular choice of $\mathcal{C}$ no longer matters.

3.4.12. Remark. — There are equivalences of categories:

$$\mathsf{Fin}' \simeq \mathsf{Fin}_* \quad \text{and} \quad \mathsf{Fin}_{\mathrm{ord}} \simeq \Delta^{\mathrm{op}}$$

where Fin_* is the *category of pointed finite sets*. Therefore we can view E_1-objects as simplicial objects. In particular, for a category $\mathcal{C}$,

$$\mathsf{Mon}(\mathcal{C}) \hookrightarrow \mathsf{stCat} \hookrightarrow \mathsf{sSet}$$

is an embedding of the category of monoids in $\mathcal{C}$ into the category of simplicial sets.

3.4.13. Remark. — Let $X : \mathsf{Fin}' \to \mathcal{D}$, where $\mathcal{D}$ is Kan or CW, be an E_∞-space. In these cases, the weak equivalences are automatically homotopy equivalences. Therefore we can choose an inverse:

$$\begin{tikzcd} \prod_{i\in I} X(*) \arrow[r, "\simeq"] \arrow[dr, dashed, "m"'] & X(I) \arrow[d, "\mu_I"] \\ & X(*) \end{tikzcd}$$

This endows $X(*)$ with a E_∞-multiplication.

3.4.14. Example. — Let $\mathcal{C}$ be a symmetric monoidal category, which can be viewed as a functor $\mathsf{Fin}' \to \mathsf{Cat}$. The nerve $N(\mathcal{C})$ is an E_∞-space, since N preserves finite product and sends equivalence to weak equivalences.

3.4.15. Definition. — An E_∞- or a E_1-space is called an *E_∞-group* or a *E_1-group* if $\pi_0 X$ is a group. An E_∞-group is also called a *Picard-∞-groupoid.*

We will take the next theorem as an black box.

3.4.16. Theorem. — *We have the following adjunction:*

$$(-)^{\mathrm{grp}} : \mathsf{CMon}(\mathsf{Grpd}_\infty) \rightleftarrows \mathsf{Ab}(\mathsf{Grpd}_\infty) : \mathrm{inc}$$

Furthermore, we also have

$$\pi_0(M^{\mathrm{grp}}) = \pi_0(M)^{\mathrm{grp}} \quad \textit{and} \quad \pi_1(M^{\mathrm{grp}}) = \pi_1(M)^{\mathrm{grp}},$$

where M is an E_∞-space.

3.4.17. Definition. — Let R be a ring. The *K-theory space* of R is defined as

$$K(R) = (N(\mathsf{Proj}(R)^{\simeq}, \oplus))^{\mathrm{grp}}$$

Furthermore, the *n-th K-group* is defined as $K_n(R) = \pi_n(R)$, for all $n \in \mathbb{N}$.

3.4.18. Example. — The group completion of $N(\mathsf{Fin}^{\simeq}, \sqcup)$ is called the *sphere spectrum* $\mathbb{S}$. This terminology comes from the Barrat-Priddy-Quillen theorem: the n-th homotopy group $\pi_n(\mathbb{S})$ is in fact the n-th stable homotopy groups of spheres! One can use the Hopf fibration $\eta : S^3 \to S^2$ to compute $\pi_0(\mathbb{S}) = \mathbb{Z}$ and $\pi_1(\mathbb{S}) = \mathbb{Z}/2$.

3.5. Homology

3.5.1. Notation. — We denote $\mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ the category of chain complexes of non-negative degrees, which takes value in the category of Abelian groups.

3.5.2. Definition. — Given a chain complex

$$A = (A_0 \xleftarrow{\partial_0} A_1 \xleftarrow{\partial_1} A_2 \xleftarrow{\partial_2} \dots)$$

where $\partial^2 = 0$. We can define the *n-homology* $H_n(A) = \ker(\partial_n)/\operatorname{im}(\partial_n)$. A morphism $f \colon A \to B$ in $\mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ is called an *quasi-isomorphism*, whenever it induces isomorphisms in homology $H_n(A) \simeq H_n(B)$, for each $n \geq 0$.

3.5.3. Definition. — We define the *category of simplicial Abelian groups* $\mathsf{sAb} = \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Ab})$. Let $A \in \mathsf{sAb}$, we will define functors $C_*,\ N_* : \mathsf{sAb} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ as follows:

- The *Moore chain complex* $C_*(A)$ is defined as
$$C_*(A) : A_0 \xleftarrow{\partial_0} A_1 \xleftarrow{\partial_1} A_2 \xleftarrow{\partial_2} \dots,$$
where $\partial : A_n \to A_{n-1}$ is given by $\partial(a) = \sum_{i=0}^{n} (-1)^i d_i(a)$.
- The *normalized chain complex* $N_*(A) \subseteq C_*(A)$ is defined as
$$N_n(A) = \{x \in A_n : d_i(x) = 0 \text{ for all } i \geq 1\} = \bigcap_{i=1}^{n} \ker(d_i).$$
In this case, the boundary map is $\partial = d_0$.
- Let $D_*(A) \subseteq C_*(A)$ be the subcomplex where $D_n(A) \subseteq A_n$ is the subgroup generated by the degenerate n-simplices.

3.5.4. Proposition. — *Let $A \in \mathsf{sAb}$, then the following holds:*

1. *The composition $N_*(A) \hookrightarrow C_*(A) \twoheadrightarrow C_*(A)/D_*(A)$ is an isomorphism.*
2. *The map $N_*(A) \hookrightarrow C_*(A)$ is a quasi-isomorphism.*
3. *The homology $H_n(C_*(A)) = \pi_n(A, 0)$, where A is viewed as a simplicial set.*

Proof. The first claim is Theorem III.2.1 in [GJ99]. The second claim is Theorem III.2.4 in [GJ99]. The third claim is Corollary III.2.7 in [GJ99]. □

3.5.5. Theorem. — *The normalized chain complex functor*
$$N_* : \mathsf{sAb} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$$
is an equivalence of categories, under which simplicial homotopies corresponds to chain homotopies and weak equivalences correspond to quasi-isomorphisms. In fact, we have
$$\mathsf{sAb}[\mathrm{weq}^{-1}] \simeq \mathsf{Ch}_{\geq 0}(\mathsf{Ab})[\mathrm{weq}^{-1}],$$
where we localize both sides at the weak equivalences.

Proof. See Corollary III.2.3 in [GJ99]. □

3.5.6. Definition. — Let $X \in \mathsf{sSet}$. The *free simplicial Abelian group* $\mathbb{Z}[X]$ generated by X is defined by $\mathbb{Z}[X]_n = \mathbb{Z}[X_n]$, where $\mathbb{Z}[X_n]$ denotes the free Abelian group generated by the set X_n. The face maps and the degeneracy maps are defined by extending those on X_n linearly.

3.5.7. Definition. — Let $A \in \mathsf{Ab}$. We have the following definition:

1. Let $X \in \mathsf{sSet}$. We can define the *chain complex* of X with value in A as
$$C_*(X, A) = C_*(\mathbb{Z}[X] \otimes A) \in \mathsf{Ch}_{\geq 0} \quad \text{and} \quad H_*(X, A) = H_*(C_*(X, A)).$$
where the tensor product is taken degreewise in Ab.

2. Let $X \in \mathsf{Top}$. Recall that $\mathrm{Sing}(X)_n = \mathrm{Map}_{\mathsf{Top}}(|\Delta^n|, X)$. Therefore, we can define

$$C_*^{\mathrm{sing}}(X, A) = C_*(\mathrm{Sing}(X), A) = (\bigoplus_{\mathrm{Sing}(X)_0} A \xleftarrow{\partial_0} \bigoplus_{\mathrm{Sing}(X)_1} A \xleftarrow{\partial_1} \dots)$$

and we can also define its homology:

$$H_*^{\mathrm{sing}}(X, A) = H_*(C_*^{\mathrm{sing}}(X, A)).$$

This is the *singular homology* of X with coefficients in A.

There is another notion of homology that is often easier to compute, namely the cellular homology.

3.5.8. Definition. — Let X be a CW-complex. The *cellular chain complex with $\mathbb{Z}$-coefficients* $C_*^{\mathrm{cell}}(X)$ is defined as follows:

$$C_n^{\mathrm{cell}}(X, \mathbb{Z}) = \bigoplus_{I_n} \mathbb{Z}$$

where I_n denote the set of n-cells of X. The boundary map

$$\partial \colon C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X)$$

is given by a $I_n \times I_{n-1}$ matrix of integers. For $\alpha \in I_n$ and $\beta \in I_{n-1}$, the integer $\partial_{\alpha,\beta}$ is defined via the degree of the following map:

$$S^{n-1} \xrightarrow{\alpha} X^{(n-1)} \twoheadrightarrow X^{(n-1)}/(X^{(n-1)} - e_\beta^\circ) \cong D^{n-1}/\partial D^{n-1} \cong S^{n-1}$$

while choosing an orientation of the cells. The construction $C_*^{\mathrm{cell}}(-)$ defines a functor

$$C_*^{\mathrm{cell}}(-) : \mathsf{CW} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab}).$$

More generally, for $A \in \mathsf{Ab}$, we can define the *cellular chain complex of X with coefficients in A* as

$$C^{\mathrm{cell}}(X; A) = C^{\mathrm{cell}}(X) \otimes A,$$

where the tensor product is again taken degreewise.

3.5.9. Definition. — Let X be a topological space and $\mathcal{C}$ be a category. A $\mathcal{C}$-valued local system on X is a functor $A \colon \Pi_1(X) \to \mathcal{C}$ denoted by $x \mapsto A_x$.

3.5.10. Remark. — Let X be a topological space. Assume X is path connected and $x \in X$, by Theorem 2.1.9, we have

$$\Pi_1(X) \simeq B\pi_1(X, x).$$

Therefore, a local system on X is just the choice of a object in $\mathcal{C}$ with a $\pi_1(X, x)$-action. Let $A \colon \Pi_1(X) \to \mathsf{Ab}$ be a local system. We can define the singular chain complex $C_*^{\mathrm{sing}}(X, A)$, and, when X is a CW-complex, the cellular chain complex $C_*^{\mathrm{cell}}(X, A)$. For example,

$$C_n^{\mathrm{sing}}(X, A) = \bigoplus_{b \colon \Delta^n \to X} A_b,$$

where A_b is the evaluation of the local system at the barycenter of $b\colon \Delta^n \to X$ and

$$C_*^{\mathrm{cell}}(X, A) = \bigoplus_{c\colon D^n \to X} A_c,$$

where A_c is the evaluation of the local system at the center of $c\colon D^n \to X$.

3.5.11. Remark. — Let X be a topological space. Assume X is path-connected and there exists a universal cover $p\colon (\tilde{X}, \tilde{x}) \to (X, x)$. Choose $x \in X$, then we see that $\pi_1(X, x)$ acts freely on $\tilde{X}$ such that $\tilde{X}/\pi_1(X, x) \simeq X$. Since $\pi_1(X, x)$ permutes the n-simplices and the action commutes with the boundary map, $\pi_1(X, x)$ acts on $C_*^{\mathrm{sing}}(\tilde{X})$. Let A be a local system of Abelian groups on X, that is, A is a $\mathbb{Z}[\pi_1(X, x)]$-module. The above shows that

$$C_*^{\mathrm{sing}}(X, A) = C_*^{\mathrm{sing}}(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X,x)]} A.$$

Furthermore, if X is a CW-complex, then $\tilde{X}$ has a canonical CW-structure such that $\tilde{X}^{(n)} = p^{-1}(X^{(n)})$. Observe that the action of $\pi_1(X, x)$ on $\tilde{X}$ permutes the n-cells freely. Therefore, we have

$$C_*^{\mathrm{cell}}(X, A) \simeq C_*^{\mathrm{cell}}(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X,x)]} A,$$

since $\pi_1(X, x)$ acts on $C_n^{\mathrm{cell}}(\tilde{X})$.

3.5.12. Example. — Recall that we have a universal covering $S^2 \to \mathbb{R}P^2$. By covering theory, this computes the fundamental group $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$. The projective plane $\mathbb{R}P^2$ has an CW-filtration given by

$$* \subseteq \mathbb{R}P^1 \subseteq \mathbb{R}P^2.$$

By Remark 3.5.11, there is an induced CW-structure on S^2 as follows:

$$(S^2)^{(0)} = \{x, y\}, \quad (S^2)^{(1)} = \{\alpha, \beta\} \quad \text{and} \quad (S^2)^{(2)} = \{\rho, \tau\}.$$

Therefore the cellular chain complex is given by

$$C_*^{\mathrm{cell}}(S^2) : \mathbb{Z}_x \oplus \mathbb{Z}_y \xleftarrow{\partial} \mathbb{Z}_\alpha \oplus \mathbb{Z}_\beta \xleftarrow{\partial} \mathbb{Z}_\rho \oplus \mathbb{Z}_\tau.$$

More precisely, the boundary map is given by

$$\partial(\alpha) = y - x, \quad \partial(\beta) = x - y, \quad \partial(\rho) = \alpha + \beta \quad \text{and} \quad \partial(\tau) = \alpha + \beta.$$

Let A be a local system of Abelian groups on X. We have

$$C_*^{\mathrm{cell}}(\mathbb{R}P^2, A) = C_*^{\mathrm{cell}}(S^2) \otimes_{\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]} A,$$

since $\mathbb{Z}/2\mathbb{Z}$ acts on $C_*^{\mathrm{cell}}(S^2)$, by Remark 3.5.11.

3.6. Refined whitehead theorem

3.6.1. Theorem. — *A map $f : X \to Y$ between topological spaces is a weak equivalence if and only if the following holds.*

1. *The map f induces an equivalence of groupoids $\Pi_1(X) \to \Pi_1(Y)$. In other words, the map f induces isomorphisms at the level of π_0 and π_1.*

2. *For all local system $A : \Pi_1(Y) \to \mathsf{Ab}$, the induced map*
$$f_* : H_n(X, f^*A) \to H_n(Y, A)$$
is an isomorphism, for each $n \geq 0$.

Proof. See [HH79]. □

3.6.2. Definition. — A map $f : X \to Y$ between topological spaces is called *acyclic*, if it induces an isomorphism of homology with arbitrary coefficients.

3.6.3. Proposition. — *Let $f\colon X \to Y$ be a map between topological spaces.*

1. *The map $f\colon X \to Y$ is a weak equivalence if and only if f is acyclic and induces an isomorphism at the level of π_1.*
2. *The map f is acyclic if and only if for all $y \in Y$, we have $\tilde{H}_*(\mathrm{hofib}_y(f)) = 0$.*
3. *The map $p : X \to *$ is acyclic if and only if $\tilde{H}_*(X) = 0$. In this case, we call X an acyclic space.*

Proof. See [HH79]. □

3.6.4. Example. — We have the following examples:

- The Poincaré 3–sphere P can be constructed as a solid dodecahedron with opposite faces identified under a $3\pi/5$-rotation. There is an action of $\mathbb{Z}/2\mathbb{Z}$ on ∂P. Moreover, $H_*(P) = H_*(S^3)$, while $\pi_1(P)$ is a non-trivial extension of A_5 by $\mathbb{Z}/2\mathbb{Z}$. In particular, $P \setminus \{*\}$ is an acyclic space with $|\pi_1| = 120$.
- A group G is called *acyclic*, if $BG = |N(BG)|$ is acyclic. An example of an acyclic group is
$$\langle a_i \ (i \in \mathbb{Z}/4\mathbb{Z}) \mid [a_{i+1}, a_i] = a_i \rangle.$$
This is usually called the *Higman group.*

3.6.5. Theorem (Quillen's plus construction)**.** — *Let X be a CW-complex, $x \in X$ and P a perfect subgroup of $\pi_1(X, x)$. There exists a CW-complex X_P^+ and a map $\eta : X \to X_P^+$ such that the following holds:*

1. *The map η is acyclic and identifies $\pi_1(X_P^+, \eta(x))$ with $\pi_1(X, x)/P$.*
2. *For every map $f\colon X \to Y$ between topological spaces such that $\pi_1(f, x)$ kills P, there is an unique map $\hat{f}\colon X_P^+ \to Y$ such that the diagram*
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] & Y \\ X_P^+ \arrow[ur, dashed, "\exists!\hat{f}"'] & \end{tikzcd}$$
commutes up to homotopy.

Proof. Without losses of generality, we may assume that X is connected and $x \in X^{(0)}$. We denote $G = \pi_1(X, x)/P$ and construct X_P^+ as follows:

Step 1: Attach 2-cells to kill P: for all $\alpha \in P$, choose a representative $\partial e_\alpha : S^1 \to X^{(1)}$ and define X' by the following pushout square:

$$\begin{tikzcd} \bigvee_{\alpha\in P} S^1 \arrow[r, "(\partial e_\alpha)_\alpha"] \arrow[d, hook] & X \arrow[d, hook, "\iota"] \\ \bigvee_{\alpha\in P} D^2 \arrow[r, "(De_\alpha)_\alpha"'] & X' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By van Kampen theorem, after taking $\pi_1(-)$ to the diagram above, we have

$$\begin{tikzcd} \mathbb{Z}^{|P|} \arrow[r] \arrow[d] & \pi_1(X,x) \arrow[d, "\iota_*"] \\ * \arrow[r] & \pi_1(X',x) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $x' = \iota(x)$. This implies that $\pi_1(X',x') = \pi_1(X,x)/P$. By the Hurewicz theorem, we know that $\iota\colon X \to X'$ is an isomorphism on H_n for $n \neq 2,3$, but it doesn't necessarily preserve H_2 and H_3.

Step 2: Attach 3-cells to fix the homology: let $p'\colon \tilde{X}' \to X'$ be the universal covering of (X',x') and let $X \to X'$ be the map defined above. We define $\tilde{X}$ via the pullback square

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "\tilde{\iota}"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X}' \arrow[d, "p'"] \\ X \arrow[r, hook, "\iota"] & X' \end{tikzcd}$$

which implies that $\tilde{X} \to X$ is a covering space. Therefore, we have the following long exact sequence:

$$1 \to \pi_1(\tilde{X},\tilde{x}) \to \pi_1(X,x) \to \pi_0(\mathrm{hofib}_x(p)) \to 1.$$

By using the pasting law for homotopy pullback and passing to the long exact sequence associated to $\mathrm{hofib}_x(p) \to \tilde{X}' \to X'$, we conclude that $\pi_0(\mathrm{hofib}_x(p)) \cong \pi_1(X',x')$. This shows that $\pi_1(\tilde{X},\tilde{x}) \cong P$. For each $\alpha \in P$, let $\tilde{e^2_\alpha}\colon D^2 \to \tilde{X}'$ be a lift of $e^2_\alpha\colon D^2 \to X'$, as the following diagram depicts:

$$\begin{tikzcd} & \tilde{X}' \arrow[d, "p'"] \\ D^2 \arrow[ur, dashed, "\tilde{e^2_\alpha}"] \arrow[r, "e^2_\alpha"'] & X' \end{tikzcd}$$

Furthermore, $\tilde{X}'$ is obtained from $\tilde{X}$ by attaching the 2-cells $g \cdot \tilde{e^2_\alpha}$, where $g \in G$. This gives us a short exact sequence of chain-complexes:

$$0 \to C^{\mathrm{cell}}_*(\tilde{X}) \to C^{\mathrm{cell}}_*(\tilde{X}') \to \bigoplus_{G\times P} \mathbb{Z}[2] \to 0,$$

where $\mathbb{Z}[2]$ means that $\mathbb{Z}$ lives in degree 2. By Hurewicz theorem, we have $H_1(\tilde{X}) = \pi_1(\tilde{X})^{\mathrm{ab}}$. Since P is perfect, this implies that $H_1(\tilde{X}) = 0$. Therefore, we have the short exact sequence

$$0 \to H_2(\tilde{X}) \to H_2(\tilde{X}') \to \bigoplus_{G\times P} \mathbb{Z} \to 0,$$

By Hurewicz theorem again, there is a surjection

$$\pi_2(\tilde{X}, \tilde{x}) \twoheadrightarrow \bigoplus_{G\times P} \mathbb{Z}$$

For each 2-cell $[\tilde{e^2_\alpha}] \in \bigoplus_{G\times P} \mathbb{Z}$, we may choose a map $\partial \tilde{e^3_\alpha} \colon S^2 \to (\tilde{X}')^{(2)}$ that lift $[\tilde{e^2_\alpha}]$. We define X^+ by the following pushout square:

$$\begin{tikzcd} \bigvee_{\alpha\in P} S^2 \arrow[r, "(\partial \tilde{e^2_\alpha})_P"] \arrow[d, hook] & X' \arrow[d, hook] \\ \bigvee_{\alpha\in P} D^3 \arrow[r, "(\tilde{e^2_\alpha})_P"'] & X^+ \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By van Kampen theorem, we conclude that $\pi_1(X', x') \cong \pi_1(X^+, x^+)$.

Step 3: The map $X \to X^+$ is acyclic: let $p^+ \colon \widetilde{X^+} \to X^+$ be the universal covering of (X^+, x^+). Since $\pi_1(X', x') \cong \pi_1(X^+, x^+)$, we have the following pullback square:

$$\begin{tikzcd} \tilde{X}' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X^+} \arrow[d] \\ X' \arrow[r, hook] & X^+ \end{tikzcd}$$

As before, $\widetilde{X^+}$ is obtained from $\tilde{X}'$ by attaching 3-cells $g \cdot \tilde{e^3_\alpha}$, where $\alpha \in P$ and $g \in G$. This gives us an exact sequence

$$0 \to C_*^{\mathrm{cell}}(\tilde{X}) \overset{\iota}{\hookrightarrow} C_*^{\mathrm{cell}}(\widetilde{X^+}) \to \mathrm{coker}(\iota) \to 0$$

Explicitly, this gives us the following diagram:

$$\begin{tikzcd} \bigoplus_{I_4} \mathbb{Z} \arrow[r, "="] \arrow[d] & \bigoplus_{I_4} \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \bigoplus_{I_3} \mathbb{Z} \arrow[r, hook] \arrow[d] & \bigoplus_{I_3 \sqcup (G\times P)} \mathbb{Z} \arrow[r, two heads] \arrow[d] & \bigoplus_{G\times P} \mathbb{Z} \arrow[d, "d"] \\ \bigoplus_{I_2} \mathbb{Z} \arrow[r, hook] \arrow[d] & \bigoplus_{I_2 \sqcup (G\times P)} \mathbb{Z} \arrow[r, two heads] \arrow[d] & \bigoplus_{G\times P} \mathbb{Z} \arrow[d] \\ \bigoplus_{I_1} \mathbb{Z} \arrow[r, "="] & \bigoplus_{I_1} \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

We will show that d is an isomorphism: first, observe that d is G-equivariant. For 3-cell $\tilde{f}_\alpha \in \bigoplus_{G\times P}\mathbb{Z}$ and 2-cell $\tilde{e}_\beta \in \bigoplus_{G\times P}\mathbb{Z}$, the following composite

$$S^2 \xrightarrow{\partial\tilde{f}_\alpha} \tilde{X}'^{(1)} \twoheadrightarrow \tilde{X}'^{(2)}/(\tilde{X}'^{(2)} - \tilde{e}^\circ_\beta) \simeq S^2$$

has degree 1, if $\alpha = \beta$ and degree 0 otherwise. By the long exact sequence of homology, the map

$$\iota\colon C^{\mathrm{cell}}_*(\tilde{X}) \hookrightarrow C^{\mathrm{cell}}_*(\widetilde{X}^+)$$

is a quasi-isomorphism. For each local system A on X, after applying $(-)\otimes_{\mathbb{Z}[G]} A$, we see that the map

$$\iota'\colon C^{\mathrm{cell}}_*(X, A) \to C^{\mathrm{cell}}_*(X^+, A)$$

is a quasi-isomorphism. This proves that the map $X \to X^+$ is acyclic.

This completes the proof of Quillen's plus construction. □

3.6.6. Remark. — In fact, the acyclic maps are exactly the epimorphisms in Grpd_∞. Furthermore, the epimorphisms $X \to Y$ correspond precisely to the perfect normal subgroups $P \subseteq \pi_1(X, x)$. See [Rap19].

3.6.7. Proposition. — *Let $f : X \to Y$ be an acyclic map between topological spaces. For all $x \in X$, the map*

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

is surjective and has a perfect kernel.

Proof. Let $F = \mathrm{hofib}_{f(x)}(f)$. Since $f\colon X \to Y$ is acyclic, we have $\tilde{H}_*(F) = 0$, by Proposition 3.6.3. In particular, F is connected and $\pi_1(F, y)$ is perfect. Therefore, we have a long exact sequence of homotopy groups:

$$\pi_1(F, y) \to \pi_1(X, x) \to \pi_1(Y, f(x)).$$

This shows that the map $\pi_1(X, x) \to \pi_1(Y, f(x))$ is surjective and the kernel is a quotient of a perfect group, hence perfect. □

3.7. Another version of the group completion theorem

Let R be a ring. Recall that we have defined the *K-theory space of R* as

$$K(R) = (N(\mathsf{Proj}(R)^{\simeq}, \oplus)^{\mathrm{grp}},$$

where N denotes the nerve functor. We now apply the plus construction to obtain an explicit description of the K-theory of rings.

3.7.1. Theorem. — *Let X be an E_∞-space. Suppose there is a cofinal embedding $\mathbb{N} \subseteq \pi_0(X)$. Let*

$$X_\infty = \mathrm{colim}(X \xrightarrow{+1} X \xrightarrow{+1} X \xrightarrow{+1} \ldots).$$

Then $X^{\mathrm{grp}} \simeq X^+_\infty$, where $X_\infty \to X^+_\infty$ is the initial map out of X_∞ that kills the commutator subgroups of the fundamental groups of X_∞. Moreover, we have $\pi_1(X^+_\infty) \cong \pi_1(X_\infty)^{\mathrm{grp}}$.

Proof. See [Nik17]. □

3.7.2. Remark. — Let R be a ring. Let

$$X = N(\mathsf{Proj}(R)^{\simeq}).$$

By Theorem 3.7.1, we have

$$X_\infty = \coprod_{K_0(R)} B\,\mathrm{GL}(R).$$

Since $X^{\mathrm{grp}} \simeq X_\infty^+$, we conclude

$$K(R) \simeq \coprod_{K_0(R)} B\,\mathrm{GL}(R)^+.$$

This implies that the map

$$K_0(R) \times B\,\mathrm{GL}(R) \to K(R).$$

is acyclic. For $i > 0$, we can define the *i-th K-group of R* as

$$K_i(R) = \pi_i(B\,\mathrm{GL}(R)^+).$$

This was Quillen's original definition of the higher K-groups of rings.

Chapter 4.

K-theory of Exact Categories

4.1. Definition of exact categories

The K-theory of exact categories first appeared in [Qui06].

4.1.1. Definition. — We have the following definition:

1. A *zero object* 0 in a category is an object that is both initial and terminal.

2. A category is called *pointed*, if it admits a zero object.

3. A category $\mathcal{C}$ is called *semi-additive* or *pre-additive*, if it is pointed and the comparision map
$$\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix} : X \sqcup Y \to X \times Y$$
is an isomorphism for each $X, Y \in \mathcal{C}$. In this case, X and Y have coincident product and coproduct, called the *biproduct* $X \oplus Y$.

4. A category $\mathcal{C}$ is called *additive*, if it is semi-additive and the *shear map*
$$\begin{bmatrix} \mathrm{id}_X & \mathrm{id}_X \\ \mathrm{id}_X & 0 \end{bmatrix} : X \sqcup X \to X \times X$$
is an isomorphism for all $X \in \mathcal{C}$.

4.1.2. Remark. — Let $\mathcal{C}$ be a category. The following holds:

- Assume that $\mathcal{C}$ is pointed, then $\mathcal{C}$ is canonically enriched in Set_*, *the category of pointed sets*. For all X, Y, $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is pointed with the zero-morphism as the base point, and the composition preserves the zero-morphism.

- Assume that $\mathcal{C}$ is semi-additive, then it is canonically enriched in $\mathsf{CMon}(\mathsf{Set})$. let $f, g \in \mathrm{Map}_{\mathcal{C}}(X, Y)$, then we can define $f + g$ to be the composite
$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$
The composition map
$$\mathrm{Map}_{\mathcal{C}}(X, Y) \oplus \mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$
is clearly bi-additive.

- The category $\mathcal{C}$ is additive if and only if $\mathcal{C}$ is semi-additive and $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is an Abelian group, for each $X, Y \in \mathcal{C}$.

4.1.3. Example. — One can easily verify the following examples:

- The category of sets Set is not pointed.
- The categories $\mathsf{Set}_*, \mathsf{Mon}, \mathsf{Grp}$ are pointed.
- The category CMon is semi-additive, but not additive.
- The categories $\mathsf{Ab}, \mathsf{Mod}_R, \mathsf{Proj}_R$ are additive for all rings $R \in \mathsf{Ring}$.

4.1.4. Definition. — An exact category is an additive category $\mathcal{C}$ with two classes of morphisms, called *admissible monomorphisms* and *admissible epimorphisms*. Let $X, Y \in \mathcal{C}$. We will use $X \twoheadrightarrow Y$ to denote an admissible epimorphism and $X \rightarrowtail Y$ to denote an admissible monomorphism. The two classes of morphisms must satisfy the following properties:

1. Both classes are closed under isomorphisms in $\mathsf{Fun}([1], \mathcal{C})$.
2. Both classes contain all the identities and are closed under composition.
3. For all $X \in \mathcal{C}$, the unique map $0 \to X$ is an admissible monomorphism and unique $X \to 0$ is an admissible epimorphism.
4. Admissible monomorphisms are preserved by cobase change: for any admissible monomorphism $X \rightarrowtail Y$ and any morphism $X \to X'$, the pushout square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X' \sqcup_X Y \end{tikzcd}$$

   exists, and $X' \rightarrowtail X' \sqcup_X Y$ is again an admissible monomorphism. Similarly, admissible epimorphisms are preserved by base change.

5. Let $\iota \colon X \rightarrowtail Y$ be an admissible monomorphism. Then $\mathrm{coker}(\iota)$ is an admissible epimorphism, and $\iota = \ker(\mathrm{coker}(\iota))$.

$$\begin{tikzcd} X \arrow[r, tail, "\iota"] \arrow[d] & Y \arrow[d, two heads, "\mathrm{coker}(\iota)"] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y/X \end{tikzcd}$$

   Let $p \colon X \twoheadrightarrow Y$ be an admissible epimorphism. Then $\ker(p)$ is admissible monomorphism and $(\mathrm{coker}(\ker(p)) = p$.

4.1.5. Remark. — The following are some easy consequences of the definition.

- Let $\mathcal{C}$ be an exact category. The opposite category $\mathcal{C}^{\mathrm{op}}$ is again exact, with admissible monomorphism and epimorphisms exchanged.

- By Definition 4.1.4, the classes of admissible monomorphism and admissible epimorphisms determine each other, and admissible epimorphisms are precisely cokernels of admissible monomorphism, whereas admissible monomorphism are kernels of admissible epimorphisms.

4.1.6. Definition. — Let $\mathcal{C}$ be an exact category. An *exact sequence in* $\mathcal{C}$ is a sequence of the form

$$X \rightarrowtail Y \twoheadrightarrow Z,$$

where $\iota\colon X \rightarrowtail Y$ is an admissible monomorphim and $p\colon Y \twoheadrightarrow Z$ is an admissible epimorphism, $p \circ i = 0$ and $X \simeq \ker(p)$ and $\operatorname{coker}(\iota) \simeq Z$. In other words, the commutative square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & Z \end{tikzcd}$$

is both a pullback square and a pushout square.

4.1.7. Definition. — Let $\mathcal{C}, \mathcal{D}$ be exact categories. A functor $\mathcal{C} \to \mathcal{D}$ is called *exact* if the following holds.

1. The functor F preserves finite products.

2. The functor F preserves admissible monomorphism and admissible epimorphisms.

3. The functor F preserves pushouts along admissible monomorphism and pullbacks along admissible epimorphisms. In particular, the functor F preserves exact squares.

Exact categories and exact functors form a category ExCat.

4.1.8. Definition. — Let $\mathcal{C}$ be an exact category. A (full) subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called an *exact subcategory* if $\mathcal{C}'$ carries an exact structure, in which a morphism in $\mathcal{C}'$ is an admissible monomorphism (resp. admissible epimorphism) if and only if it is an admissible monomorphism (resp. admissible epimorphism) in $\mathcal{C}$.

4.1.9. Example. — We have the following examples:

- Every Abelian category carries *the maximal exact structure*, in which the admissible monomorphisms are precisely the monomorphisms and the admissible epimorphisms are the epimorphisms.

- Every additive category carries *the minimal exact structure*, in which the admissible monomorphisms are of the form $X \rightarrowtail X \oplus Y$ and the admissible epimorphisms are of the form $X \oplus Y \twoheadrightarrow X$.

- Let $\mathcal{C}$ be an exact category and $\mathcal{C}' \subseteq \mathcal{C}$ a full-subcategory *closed under extension*: whenever we have an exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$, where $X, Z \in \mathcal{C}'$, then $Y \in \mathcal{C}'$. Then $\mathcal{C}'$ admits the induced exact structure in which an admissible monomorphism is exactly an admissible monomorphism in $\mathcal{C}$ with cokernel in $\mathcal{C}'$, and an admissible epimorphism is exactly an admissible epimorphism in $\mathcal{C}$ with kernel in $\mathcal{C}'$.

- The category of Abelian groups Ab is exact with the maximal exact structure. Consider the full subcategory $\mathsf{Ab}_{\mathrm{tor}} \subseteq \mathsf{Ab}$ of torsion abelian groups, its admissible monomorphisms are monomorphisms and admissible epimorphisms are epimorphisms.
- Consider $\mathsf{Ab}_{\mathrm{torf}}$, the full subcategory of Ab spanned by the torsion free abelian groups. Then $\mathsf{Ab}_{\mathrm{torf}}$ is exact: an admissible monomorphism is a monomorphism, whose cokernel is still torsion free. For example the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is an admissible monomorphism. However, the maps $\mathbb{Z} \hookrightarrow n \cdot \mathbb{Z}$ and $\mathbb{Z} \hookrightarrow \mathbb{Q}$ are not admissible monomorphism. An admissible epimorphism is just an epimorphisms.
- Let R be a commutative ring. The *category of torsion modules* $\mathsf{Mod}_R^{\mathrm{tor}}$ and the *category of torsion free modules* $\mathsf{Mod}_R^{\mathrm{torf}}$ are exact subcategories of Mod_R.
- Let X be a scheme. The *category of quasi-coherent sheaves* $\mathsf{QCoh}(X)$ on X is an Abelian category. A quasi-coherent sheaf is a *vector bundle*, whenever it is locally free of finite rank. Then $\mathsf{Vect}(X) \subseteq \mathsf{QCoh}(X)$, the full-subcategory spanned by the vector bundles over X, is closed under extensions and hence admits a canonical exact structure. A sequence in $\mathsf{Vect}(X)$ is exact if and only if it is exact in $\mathsf{QCoh}(X)$. Assume that $X = \mathrm{Spec}(R)$ is an affine scheme, then we have the following commutative diagram

$$\begin{tikzcd} \mathsf{Vect}(X) \arrow[r, "\simeq"] \arrow[d, hook] & \mathsf{Proj}(R) \arrow[d, hook] \\ \mathsf{QCoh}(X) \arrow[r, "\simeq"] & \mathsf{Mod}_R \end{tikzcd}$$

  In $\mathsf{Vect}(X) \simeq \mathsf{Proj}(R)$, every exact sequence

$$0 \to M \rightarrowtail N \twoheadrightarrow P \to 0$$

  splits, since P is projective. Therefore, when X is affine, $\mathsf{Vect}(X)$ is equipped with the minimal exact structure. However, when $X = \mathbb{P}^1$, the following sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}^{\otimes 2} \to \mathcal{O}_{\mathbb{P}}(1) \to 1$$

  is exact but does not split. This shows that the exact structure on $\mathsf{Vect}(X)$ is usually larger than the minimal one.

4.1.10. Proposition. — *Let $\mathcal{C}$ be an exact category. There exists a fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{A}$, where $\mathcal{A}$ is an Abelian category and $\mathcal{C}$ is closed under extensions in $\mathcal{A}$. Furthermore, admissible monomorphisms in $\mathcal{C}$ are monomorphisms in $\mathcal{A}$ with cokernel in $\mathcal{C}$ and admissible epimorphisms in $\mathcal{C}$ are the epimorphisms in $\mathcal{A}$ with kernel in $\mathcal{C}$.*

Proof. We will give a sketch of the proof. Let

$$A = \mathsf{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathsf{Ab}) \subseteq \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Ab})$$

be the full-subcategory spanned by the *left exact functors*: the functors that preserves finite products and send exact squares to pullback squares in Ab. Let

$$よ_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{A}, \quad X \mapsto \mathrm{Map}_{\mathcal{C}}(-, X)$$

be the Yoneda embedding. The functor $\mathrm{Map}_{\mathcal{C}}(-, X)$ is left exact, since it takes colimits in $\mathcal{C}$ to limits in Ab. It suffices to show that $\mathcal{A}$ is abelian and $\mathcal{C}$ is closed under extensions. See Proposition A.2 in [Kel90] $\square$

4.1.11. Definition. — Let $\mathcal{C}$ be an exact category. We define the *zeroth K-group of* $\mathcal{C}$ as

$$K_0(\mathcal{C}) = \mathbb{Z}[\pi_0(\mathcal{C}^{\simeq})]/\sim$$

where $[X] + [Z] \sim [Y]$ if and only if there exists an exact square $X \rightarrowtail Y \twoheadrightarrow Z$. The construction $K_0(-)$ defines a functor $K_0(-)\colon \mathsf{ExCat} \to \mathsf{Ab}$.

4.1.12. Example. — We have following examples:

1. Let $\mathcal{C}$ be an exact category equipped with the minimal exact structure. Then we have

$$K_0(\mathcal{C}) \simeq \pi_0(\mathcal{C}^{\simeq})^{\mathrm{grp}}.$$

   For example, we have $K_0(\mathsf{Proj}(R)) = K_0(R)$.

2. Let $\mathsf{Ab}^{\mathrm{fg}}$ be the Abelian category of finite Abelian groups equipped with the maximal exact structure. One can show that

$$K_0(\mathsf{Ab}^{\mathrm{fg}}) \cong \mathbb{Z}$$

   generated by $[\mathbb{Z}]$.

4.2. The Q-construction

4.2.1. Construction. — Let $\mathcal{C}$ be an exact category. The *Q-construction of* $\mathcal{C}$ is defined as follows:

- The objects of $Q(\mathcal{C})$ are the same as the objects of $\mathcal{C}$.
- Let $X, Y \in Q(\mathcal{C})$, a morphism from X to Y is a span in $\mathcal{C}$ of the form

$$X \twoheadleftarrow Z \rightarrowtail Y,$$

  where the left pointing morphisms are the admissible epimorphisms in $\mathcal{C}$ and the right pointing morphisms are the admissible momomorphisms in $\mathcal{C}$. Two spans are viewed as the same, whenever there exists a commutative diagram of the form

$$\begin{tikzcd} & Z \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, "\simeq"'] & \\ X & & Y \\ & Z' \arrow[ul, two heads] \arrow[ur, tail] & \end{tikzcd}$$

- Let $X \twoheadleftarrow P \rightarrowtail Y$ and $Y \twoheadleftarrow Q \rightarrowtail Z$ be two spans in $\mathcal{C}$, their composition is given by taking the pullback

$$\begin{tikzcd} & & W \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & P \arrow[dl, two heads] \arrow[dr, tail] & & Q \arrow[dl, two heads] \arrow[dr, tail] & \\ X & & Y & & Z \end{tikzcd}$$

  The composition is unital and associative.

4.2.2. Lemma (Leftover lemma). — *Consider a pullback square as follows:*

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "q"'] & V \arrow[d, two heads, "p"] \\ U \arrow[r, tail, "i"] & Y \end{tikzcd}$$

where i is an admissible monomorphism and p is an admissible epimorphism. Then j is an admissible monomorphism and the commutative square is a pushout square.

Proof. We have the following diagram:

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, two heads, "p"] \\ U \arrow[r, tail, "i"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & Y \arrow[d, two heads] \\ 0 \arrow[r, tail] & Y/U \end{tikzcd}$$

By pasting lemma, the following commutative square is a pullback square:

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, two heads] \\ 0 \arrow[r, tail] & Y/U \end{tikzcd}$$

As the kernel of the admissible epimorphism $V \twoheadrightarrow Y/U$, the map j is an admissible monomorphism. To proof that the given commutative square is a pushout square, we take the kernel of the map q:

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & W \arrow[r, "j"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, two heads, "p"] \\ 0 \arrow[r, tail] & U \arrow[r, tail, "i"] & Y \end{tikzcd}$$

Again, by pasting lemma, the outer square above is a pullback square. By definition of exact categories, it is also a pushout square. Using the pasting lemma again gives us the desired result. □

4.2.3. Remark. — Let $\mathcal{C}$ be an exact category. A span

$$X \twoheadleftarrow Z \rightarrowtail Y$$

is an isomorphism in $Q(\mathcal{C})$ if and only if $Z \twoheadrightarrow X$ and $Z \rightarrowtail Y$ are isomorphisms in $\mathcal{C}$. This implies that $Q(\mathcal{C}^{\simeq}) \simeq \mathcal{C}^{\simeq}$.

4.2.4. Definition. — Let X be a topological space and $x \in X$. The *loop space of X at the base point x* is defined as

$$\Omega_x X = \{\gamma \colon I \to X : \gamma(0) = \gamma(1) = x\},$$

where $\Omega_x X$ is equipped with the subspace topology of the compact open topology.

4.2.5. Definition. — Let $\mathcal{C}$ be an exact category. We define the *K-theory of $\mathcal{C}$* as

$$K(\mathcal{C}) = \Omega_0 |N(Q(\mathcal{C}))|,$$

where 0 is the zero object of $\mathcal{C}$. For all $n \geq 0$, we define the *n-the K-group of $\mathcal{C}$* as

$$K_n(\mathcal{C}) = \pi_n(K(\mathcal{C})) = \pi_{n+1}|N(Q)\mathcal{C}))|,$$

where the last equality follows from the suspension loop space adjunction.

4.2.6. Remark. — The K-theory construction for exact categories assembles into a functor

$$K(-)\colon \mathsf{ExCat} \to \mathsf{Top}.$$

Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between exact categories. Then there is a canonical functor $Q(f)\colon Q(\mathcal{C}) \to Q(\mathcal{D})$, by applying f to the spans in C directly. This gives us $f_*\colon K(\mathcal{C}) \to K(\mathcal{D})$, by the functoriality of rest of the constructions. Suppose $\alpha\colon f \simeq g$ is a natural isomorphism between exact functors, then $Q(\alpha)\colon Q(f) \simeq Q(g)$ defines a natural isomorphism. Therefore, we will get a homotopy $K(\alpha)\colon K(f) \simeq K(g)$. Assume that $f\colon \mathcal{C} \to \mathcal{D}$ is an equivalence, then $f_*\colon K(\mathcal{C}) \to K(\mathcal{D})$ is a homotopy equivalence.

4.2.7. Remark. — Let $\mathcal{C}$ and $\mathcal{D}$ be exact categories. The following holds:

1. We have $K(\mathcal{C}^{\mathrm{op}}) \simeq K(\mathcal{C})$.

2. The functor $K(-)$ preserves finite products and the terminal object.

3. The functor $K_n(-)\colon \mathsf{ExCat} \to \mathsf{Grp}$ preserves filtered colimits.

4.2.8. Corollary. — *There exists a functor*

$$K\colon \mathsf{ExCat} \to \mathsf{Alg}_{E_\infty}(\mathsf{Top})$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{ExCat} \arrow[r, dashed, "K"] \arrow[dr, "K"'] & \mathsf{Alg}_{E_\infty}(\mathsf{Top}) \arrow[d, "\mathrm{fgt}"] \\
& \mathsf{Top}
\end{tikzcd}$$

where $\mathsf{Alg}_{E_\infty}(\mathsf{Top}) \subseteq \mathsf{Fun}(\mathsf{Fin}', \mathsf{Top})$ denotes the full-subcategory spanned by the E_∞-spaces.

Proof. Since ExCat admits finite product, we can associate to each $\mathcal{C} \in \mathsf{ExCat}$ a functor

$$\mathcal{C}^\times\colon \mathsf{Fin}' \to \mathsf{ExCat}, \quad I \mapsto \mathcal{C}^I.$$

This defines a functor

$$(-)^\times\colon \mathsf{ExCat} \to \mathsf{Fun}(\mathsf{Fin}', \mathsf{ExCat}).$$

It is straightforward to check that the objects in its essential image satisfy the Segal condition. Moreover, by Remark 4.2.8, the functor $K(-)$ preserves finite products. Hence, post-composing with $K(-)$ yields the desired functor. $\square$

4.3. Fibrations

4.3.1. Definition. — Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor and let $d \in \mathcal{D}$. We define the *over-category* of f at d as

$$f/d = \mathcal{C} \times_{\mathcal{D}} (\mathcal{D}/d).$$

Explicitly, an object of f/d is a pair $(c,\, f(c) \xrightarrow{\alpha} d)$ consisting of an object $c \in \mathcal{C}$ together with a morphism $\alpha\colon f(c) \to d$ in $\mathcal{D}$. A morphism

$$(c, f(c) \xrightarrow{\alpha} d) \longrightarrow (c', f(c') \xrightarrow{\alpha'} d)$$

is a morphism $g\colon c \to c'$ in $\mathcal{C}$ such that $\alpha = \alpha' \circ f(g)$.

Similarly, we define the *under-category* of f at d as

$$d\backslash f = \mathcal{C} \times_{\mathcal{D}} (d\backslash\mathcal{D}).$$

An object of $d\backslash f$ is a pair $(c,\, d \xrightarrow{\beta} f(c))$ with $c \in \mathcal{C}$ and a morphism $\beta\colon d \to f(c)$ in $\mathcal{D}$. A morphism

$$(c, d \xrightarrow{\beta} f(c)) \longrightarrow (c', d \xrightarrow{\beta'} f(c'))$$

is a morphism $g\colon c \to c'$ in $\mathcal{C}$ such that $\beta' = f(g) \circ \beta$.

4.3.2. Definition. — A functor $p : E \to B$ in stCat is called a *locally cartesian fibration* or a *Grothendieck prefibration*, whenever for every $b \in B$, the fully faithful functor

$$p^{-1}(b) \to b\backslash p, \quad e \mapsto (e, b \xrightarrow{\mathrm{id}} p(e))$$

has a right adjoint. We denote the right adjoint by

$$f^*\colon b\backslash p \to p^{-1}(b), \quad (e, b \xrightarrow{f} p(e)) \mapsto f^*(e)$$

More explicitly, for every $f : b \to b'$, we get a functor $f^* : p^{-1}(b') \to p^{-1}(b)$. Let $g\colon b' \to b''$ be another map and consider $g \circ f\colon b \to b''$. Then by the adjunction property, there is a natural transformation $\mu\colon f^*g^* \to (gf)^*$.

4.3.3. Definition. — A functor $p : E \to B$ is called a *cartesian fibration* or a *Grothendieck fibration*, if it is a locally cartesian fibration such that the canonical natural transformation

$$\mu\colon f^*g^* \to (gf)^*$$

is a natural isomorphism.

4.3.4. Definition. — A functor $p : E \to B$ is called a *locally cocartesian fibration*, whenever $p^{\mathrm{op}} : E^{\mathrm{op}} \to B^{\mathrm{op}}$ is a locally cartesian fibration.

4.3.5. Example. — Let Mod be the *category of modules* defined as follows:

- The objects of Mod are pairs of the form (A, M), where $A \in \mathsf{Ring}$ and $M \in {}_R\mathsf{Mod}$.
- Let $(A, M), (B, N) \in \mathsf{Mod}$. A morphism $f\colon (A, M) \to (B, N)$ consists of a ring homomorphism $f\colon A \to B$ and an A-linear map $f\colon M \to N$.

- The composition is defined in a similar way, and it satisfies the usual unitality and associativity conditions.

There is a canonical functor

$$p\colon \mathsf{Mod} \to \mathsf{Ring}, \quad (A, M) \mapsto A.$$

The functor p is a cocartesian fibration. Let $f : B \to B'$ be a homomorphism of rings. The restriction of scalar gives us the functor

$$f^*\colon {}_B\mathsf{Mod} \to {}_{B'}\mathsf{Mod}, \quad M \mapsto M \otimes_B B'$$

In fact, p is also a cartesian fibration. Let $f\colon B \to B'$ be a homomorphism of rings. The forgetful functor gives us the functor ${}_{B'}\mathsf{Mod} \to {}_B\mathsf{Mod}$.

4.3.6. Remark. — Let $p : E \to B$ be both a locally cocartesian fibration and a locally cartesian fibration and f be a morphism in B. Then the functor $f_!$ is the left adjoint of f^*.

4.3.7. Definition. — We have the following definition:

1. A map $p : E \to B$ in sSet is a *Kan fibration*, if it has the right lifting property for $\Lambda^k_n \subseteq \Delta^n$, for all $n \in \mathbb{N}$ and $0 \leq k \leq n$. It is a *locally trivial fibration* if for all $\Delta^n \to B$, there exists some F and an isomorphism $E \times_B \Delta^n \xrightarrow{\simeq} F \times \Delta^n$ such that the following diagram commute:
$$\begin{tikzcd} E \times_B \Delta^n \arrow[r, "\simeq"] \arrow[d, "\mathrm{pr}_{\Delta^n}"'] & F \times \Delta^n \arrow[dl, "\mathrm{pr}_{\Delta^n}"] \\ \Delta^n & \end{tikzcd}$$
The map p is a *covering map*, whenever it is locally trivial fibration such that F is a set.
2. A map $p : E \to B$ between topological spaces is a *trivial fibration*, if for all $b \in B$, there exists an open set $U \subseteq B$ of b and an isomorphism such that the following diagram commutes:
$$\begin{tikzcd} E \times_B U \arrow[r, "\simeq"] \arrow[d, "\mathrm{pr}_{\Delta^n}"'] & F \times U \arrow[dl, "\mathrm{pr}_{\Delta^n}"] \\ U & \end{tikzcd}$$
The map p is a *covering map*, whenever all the fibers are discrete.

4.3.8. Remark. — Let $p : E \to B$ be a Serre fibration. The comparison map

$$p^{-1}(b) \to \mathrm{hofib}_b(p)$$

is a weak equivalence.

4.3.9. Proposition. — *We will take the following as black boxes:*

1. *Locally trivial fibration in* Top *are Serre fibrations. Locally trivial fibration in* sSet *with fibers in* Kan *are Kan fibrations.*

2. *The geometric realization functor sends Kan fibrations to Serre fibrations, locally trivial fibrations in* sSet *to locally trivial fibrations in* Top *and covering maps in* sSet *to coverings mapsin* Top.

3. *Let* $p\colon E \to B$ *be a (co)cartesian fibration* stCat *with groupoid fibers. If for all* $f : b \to b'$ *in* B*, the functor* $f^* : p^{-1}(b') \to p^{-1}(b)$ *is an equivalence, then* $N(p)$ *is a Kan fibration. Furthermore, if the fibers of* $p\colon E \to B$ *are sets, then* $N(p)$ *is a covering map.*

4.3.10. Lemma. — *Let* $\mathcal{C}$ *be a category. There is an equivalence of categories*

$$\mathsf{Cov}(|N(\mathcal{C})|) \simeq \mathsf{Fun}_{\mathrm{inv}}(\mathcal{C}, \mathsf{Set}),$$

where $\mathsf{Cov}(|N(\mathcal{C})|)$ *is the full-subcategory of* $\mathsf{Top}_{/|N(\mathcal{C})|}$ *spanned the covering maps and* $\mathsf{Fun}_{\mathrm{inv}}(\mathcal{C}, \mathsf{Set})$ *denotes the full-subcategory of* $\mathsf{Fun}(\mathcal{C}, \mathsf{Set})$ *spanned by the functors that send every morphism to an isomorphism.*

Proof. Let $p\colon E \to |N(\mathcal{C})|$ be a covering map. Let $f\colon c \to c'$ be a morphism in $\mathcal{C}$. Then we can view f as a fuctor $f\colon [1] \to \mathcal{C}$. Applying $|N(-)|$ gives us a path $\tilde{f}\colon x \to y$ in $|N(\mathcal{C})|$. By the theory of covering spaces, this induces an isomorphism $f_*\colon p^{-1}(c) \to p^{-1}(c')$. This gives us a functor

$$F_f\colon \mathcal{C} \to \mathsf{Set}, \quad c \mapsto p^{-1}(c).$$

It remains to verify that this construction is functorial.

Conversely, let $F : \mathcal{C} \to \mathsf{Set}$ be a functor. We can form the *Grothendieck construction*

$$\int_{\mathcal{C}} F = \{(c, x) : c \in \mathcal{C}, x \in F(c)\}.$$

A morphism $(c, x) \mapsto (c', x')$ is a morphism $f : c \to c'$ in $\mathcal{C}$ such that $F(f)(x) = x'$. Then the functor

$$p : \int_{\mathcal{C}} F \to \mathcal{C}, \quad (c, x) \mapsto c$$

is a cocartesian fibration with discrete fibers $p^{-1}(c) = F(c)$. Since F sends every morphism to an isomorphism, by Proposition 4.3.9, the map $N(p)$ is a covering. □

4.3.11. Remark. — Let $X \in \mathsf{CW}$. By the fundamental theorem of covering space theory, we have the following equivalence of categories

$$\mathsf{Cov}(X) \simeq \mathsf{Fun}(\Pi_1(X), \mathsf{Set}).$$

Lemma 4.3.10 is equivalent to $\Pi_1|N(\mathcal{C})|$ is the groupoid completion of $\mathcal{C}$.

4.3.12. Theorem. — *Let* $\mathcal{C}$ *be an exact category. We have* $\pi_0(K(\mathcal{C})) \simeq K_0(\mathcal{C})$.

Proof. By definition, we have

$$\pi_0(K(C)) = \pi_1(|N(Q(C))|, 0).$$

Since for every $X \in \mathcal{C}$, there exists an admissible epimorphism $X \twoheadrightarrow 0$, the topological space $|N(Q(\mathcal{C}))|$ is path connected. By Theorem 2.1.9, this implies that

$$\Pi_1|N(Q(\mathcal{C}))| \simeq B\pi_1(|N(Q(\mathcal{C}))|, 0) \simeq B\pi_0 K(\mathcal{C}).$$

By Lemma 4.3.10 and Remark 4.3.11, we have the following equivalence of categories

$$\mathsf{Fun}(B\pi_0 K(\mathcal{C}), \mathsf{Set}) \simeq \mathsf{Cov}(|N(Q(\mathcal{C}))|) \simeq \mathsf{Fun}_{\mathrm{inv}}(Q(\mathcal{C}), \mathsf{Set}).$$

Note that a group G is determined by the category $\mathsf{Fun}(BG, \mathsf{Set})$: the G-set G is the unique object of $\mathsf{Fun}(BG, \mathsf{Set})$ such that the functor

$$\mathrm{Map}(G, -)\colon \mathsf{Fun}(BG, \mathsf{Set}) \to \mathsf{Set}$$

preserves colimits and G is the group of automorphisms of the G-set G. Therefore, it suffices to show that

$$\mathsf{Fun}(BK_0(\mathcal{C}), \mathsf{Set}) \simeq \mathsf{Fun}_{\mathrm{inv}}(Q(\mathcal{C}), \mathsf{Set}).$$

We first want to define the functor

$$\mathsf{Fun}_{\mathrm{inv}}(Q(\mathcal{C}), \mathsf{Set}) \to \mathsf{Fun}(BK_0(\mathcal{C}), \mathsf{Set}), \quad F \mapsto F(0)$$

with some action of $K_0(\mathcal{C})$ on $F(0)$. Recall that

$$K_0(\mathcal{C}) = \mathbb{Z}[\pi_0(\mathcal{C}^{\simeq})]/\sim,$$

where for every exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ we have $[X]+[Z] = [Y]$. Also note that $K_0(\mathcal{C})$ is Abelian, since for $X, Y \in \mathcal{C}$ we have the spans

$$X \rightarrowtail X \oplus Y \twoheadrightarrow Y \quad \text{and} \quad Y \rightarrowtail X \oplus Y \twoheadrightarrow X.$$

Let $F \in \mathsf{Fun}_{\mathrm{inv}}(Q(\mathcal{C}), \mathsf{Set})$ and $X \in \mathcal{C}$, define the following spans

$$i_X \;=\; (0 = 0 \rightarrowtail X) \quad \text{and} \quad \rho_X \;=\; (0 \twoheadleftarrow X = X)$$

By the assumption, this gives us two isomorphisms

$$F(i_X)\colon F(0) \to F(X) \quad \text{and} \quad F(\rho_X)\colon F(X) \to F(0)$$

We define $\alpha_X = F(\rho_X) \circ F(i_X)^{-1}$, then we see that α_X only depends on the isomorphism class of X. This gives us a group homomorphism

$$\mathbb{Z}\pi_0(\mathcal{C}^{\simeq}) \to \mathrm{Aut}(F(0)), \quad [X] \mapsto \alpha_X.$$

In order to get a map $K_0(\mathcal{C}) \to \mathrm{Aut}(F(0))$, we need to show that, for every exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{C}$, we have $\alpha_X \alpha_Z = \alpha_Y$. In other words, we have to show that

$$F(\rho_Z) \circ F(i_Z)^{-1} \circ F(\rho_X) \circ F(i_X)^{-1} = F(\rho_Y) \circ F(i_Y)^{-1}$$

Since we have the following decomposition of spans:

$$\rho_Y = \rho_Z \circ (Z \twoheadleftarrow Y = Y) \quad \text{and} \quad i_Y = i_X \circ (X = X \rightarrowtail Y)$$

we are reduced to showing:

$$(0 \twoheadleftarrow X = X) \circ (X = X \rightarrowtail Y) = (0 = 0 \rightarrowtail Z) \circ (Z \twoheadleftarrow Y = Y).$$

But this is clear, since both compositions are $0 \twoheadleftarrow X \rightarrowtail Y$.

$$\mathsf{Fun}_{lc}(Q\mathcal{C}, \mathsf{Set}) \to \mathsf{Fun}(BK_0(\mathcal{C}), \mathsf{Set}).$$

Conversely, let $K_0(\mathcal{C})$ act on S. Define the functor

$$F_S\colon Q(\mathcal{C}) \to \mathsf{Set}, \quad X \mapsto S \quad \text{and} \quad (X \overset{p}{\twoheadleftarrow} Y \rightarrowtail Z) \mapsto [\ker(p)]$$

where $[\ker(p)] \in K_0(\mathcal{C})$ can be viewed as an element in $\mathrm{Aut}(S)$. The composition is given by

$$\begin{tikzcd}
 & & W \arrow[dl, two heads, "q'"'] \arrow[dr, tail] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & U \arrow[dl, two heads, "p"'] \arrow[dr, tail] & & V \arrow[dl, two heads, "q"'] \arrow[dr, tail] & \\
X & & Y & & Z
\end{tikzcd}$$

Therefore, we need to show that

$$[\ker(pq')] = [\ker(p)] + [\ker(q')].$$

This follows from the exact sequence

$$\ker(q) \rightarrowtail \ker(pq') \twoheadrightarrow \ker(p).$$

Finally, one can easily check that the two functors defined above are inverses to each other.

$\square$

4.4. Quillen's Theorem A and B

Recall that we have define the geometric realization functor $|\cdot|\colon \mathsf{sSet} \to \mathsf{Top}$. In fact, the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set}) \arrow[r, "|\cdot|"] \arrow[d, hook] & \mathsf{Top} \arrow[d, "\Pi_\infty"] \\
\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Grpd}_\infty) \arrow[r, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"] & \mathsf{Top}[\mathrm{weq}^{-1}]
\end{tikzcd}$$

Thus, we may regard the lower horizontal map as a kind of geometric realization functor. Since Top and sSet are models for Grpd_∞, we define the *geometric realization for simplicial spaces* as any functor $|\cdot|\colon \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{X}) \to \mathcal{X}$, where $\mathcal{X}$ is sSet of Top, such that the following condition holds.

- The functor $|\cdot|$ preserves weak equivalences.
- After localizing at the weak equivalences, we obtain a functor

  $$|\cdot|\colon \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Grpd}_\infty) \to \mathsf{Grpd}_\infty,$$

  that is left adjoint to the constant functor.

4.4.1. Remark. — We can define the following functor:

$$|\cdot| : \mathsf{Fun}(\Delta^{\mathrm{op}}, s\mathsf{Set}) \simeq \mathsf{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathsf{Set}) \overset{\Delta}{\to} \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})) = s\mathsf{Set},$$

where Δ is the diagonal map. Since Δ^{op} is sifted, the diagonal is cofinal, and this functor satisfies the condition stated above. In the following, we will use this functor as our model of geometric realization of simplicial sets.

4.4.2. Lemma. — *Let* $\chi_{\bullet,\bullet} : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to X$ *be a bisimplicial space, where* X *is* $s\mathsf{Set}$ *or* Top*. After currying, this gives us a functor* $\overline{\chi}_{-,\bullet} : \Delta^{\mathrm{op}} \to \mathsf{Fun}(\Delta^{\mathrm{op}}, X)$ *given by* $\overline{\chi}_{-,\bullet}(n) = \chi_{n,\bullet}$*. Composing with the geometric realization, we get a functor* $\chi_{-,\bullet} : \Delta^{\mathrm{op}} \to X$*. Similarly, we can swap the component and define a functor* $\chi_{\bullet,-} : \Delta^{\mathrm{op}} \to X$*, we also have the functor* $\chi^{\Delta} : \Delta^{\mathrm{op}} \to X$ *defined by* $\chi^{\Delta} = \chi \circ \Delta$*. The geometric realization of* $\chi_{-,\bullet}$*,* $\chi_{\bullet,-}$ *and* χ^{Δ} *are weakly equivalent.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
 & & \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \arrow[d, "{(1,2,2)}"] \arrow[dr, "\overline{\chi}_{-,\bullet}"] & \\
\Delta^{\mathrm{op}} \arrow[r, "\Delta"] \arrow[urr, bend left, "\Delta"] \arrow[drr, bend right, "\Delta"'] & \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \arrow[r, "{(1,1,2)}"] \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \arrow[r, "\chi"] & \mathsf{Set} \\
 & & \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \arrow[u, "{(2,1,2)}"'] \arrow[ur, "\overline{\chi}_{\bullet,-}"'] &
\end{tikzcd}$$

where the numbers represent the projection maps on to the corresponding components and we have slightly abused the notation to denote the uncurrying of χ also by χ. Since taking the geometric realization is really the same as pre-composing with the diagonal, we eventually see that the three composition in the diagram corresponds to the geometric realization of $\overline{\chi}_{-,\bullet}$, $\overline{\chi}_{\bullet,-}$ and χ^{Δ} respectively. $\square$

We now introduce Quillen's Theorems A and B, which provide powerful tools for computations in algebraic K-theory.

4.4.3. Theorem (A). — *Let* $\mathcal{C}, \mathcal{D}$ *be strict categories and* $f : \mathcal{C} \to \mathcal{D}$ *a functor. Assume that* $N(d/f)$ *is weakly contractible for every* $d \in \mathcal{D}$*, then* $N(f) : N(\mathcal{C}) \to N(\mathcal{D})$ *is a weak equivalence.*

Proof. Define a category $S(f)$ as follows:

- The objects of $S(f)$ are triples (X, Y, v), where $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $v \colon Y \to f(X)$.
- A morphism $(u, w) \colon (X, Y, v) \to (X', Y', v')$ consists of a morphisms $u \colon X \to X'$ and $w \colon Y' \to Y$ such the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, "v"] & f(X) \arrow[d, "f(u)"] \\
Y' \arrow[u, "w"] \arrow[r, "v'"'] & f(X')
\end{tikzcd}$$

- Composition of morphisms is defined in the same way and is associative and unital.

There is a forgetful functor $S(f) \to \mathcal{C} \times \mathcal{D}^{\mathrm{op}}$, which is a cocartesian fibration. In fact, passing to fibers, this gives us the Hom functor

$$\mathrm{Hom}_D(-, f(-)) : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}.$$

There is a commutative diagram of the form:

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{p_1} & S(f) & \xrightarrow{p_2} & \mathcal{D}^{\mathrm{op}} \\ \Big\downarrow f & & \Big\downarrow f' & & \Big\downarrow = \\ \mathcal{D} & \xleftarrow{p_1} & S(\mathrm{id}_{\mathcal{D}}) & \xrightarrow{p_2} & \mathcal{D}^{\mathrm{op}} \end{array}$$

where $f'(X,Y,v) = (f(X),Y,v)$. By the 2 out of 3 property of weak equivalences, it suffices to show that $N(p_1)$ and $N(p_2)$ are weak equivalences. We define the functor $T(f) : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}$ given by

$$T(f)(p,q) = \{(X_0 \to X_1 \to \ldots \to X_p \in \mathcal{C},\ Y_q \to \ldots \to Y_0 \to f(X_0) \in \mathcal{D})\}$$

Adopting the notation of Lemma 4.4.1, we obtain

$$T(f)_{p,\bullet} = \coprod_{X_0 \to \ldots \to X_p} N(\mathcal{D}/f(X_0)) \quad \text{and} \quad T(f)_{\bullet,q} = \coprod_{Y_q \to \ldots \to Y_0} N(Y_0/f).$$

Furthermore, we have $T(f)^{\Delta} = N(S(f))$. Denote

$$\pi_1, \pi_2 : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$$

to be the projection map on to the corresponding factor, then

$$\pi_i^{\Delta} = \mathrm{id}_{\Delta^{\mathrm{op}}} \quad \text{for} \quad i \in \{1,2\}.$$

Aftering precomposing $N(\mathcal{C})$ with π_1, we define the map

$$\varphi\colon T(f) \to \pi_1^* N(\mathcal{C}), \quad \varphi_{p,q}\colon T(f)_{p,q} \to N(\mathcal{C})_p,$$

where $N(\mathcal{C})_p$ denotes $\mathsf{Fun}([p],\mathcal{C})$ and $\varphi_{p,q}$ is given by the projection to $\mathcal{C}$. But then by taking the diagonal, we can write the map $N(p_1)$ as

$$\varphi^{\Delta}\colon N(S(f)) \to N(\mathcal{C}).$$

Therefore, it suffices to show that $|\varphi^{\Delta}|$ is a weak equivalence. By Lemma 4.4.1, this boils down to showing that $|\varphi_{p,\bullet}|$ is a weak equivalence for a fixed $p \in \Delta^{\mathrm{op}}$. Since geometric realization preserves colimits, we can write

$$|\varphi_{p,\bullet}|\colon \coprod_{X_0 \to \ldots \to X_p} |N(\mathcal{D}/f(X_0))| \to \coprod_{X_0 \to \ldots \to X_p} *$$

Since the category $\mathcal{D}/f(X_0)$ has initial objects, the geometric realization $|N(\mathcal{D}/f(X_0))|$ is contractible, which implies that $|\varphi_{p,\bullet}|$ is a weak equivalence. In order to show that the map $N(p_2)$ is a weak equivalence, we similarly define

$$\psi\colon T(f) \to \pi_2^* N(\mathcal{D}^{\mathrm{op}})$$

and use the same argument as before, now using that $N(Y_0/f)$ is weakly contractible. $\square$

4.4.4. Theorem (B). — *Let $\mathcal{C}, \mathcal{D}$ be strict categories and $f\colon \mathcal{C} \to \mathcal{D}$ a functor. Assume that for every morphism $d \to d' \in \mathcal{D}$, the induced map $|N(d'/f)| \to |N(d/f)|$ is a weak equivalence, then for every $d \in \mathcal{D}$, the canonical map $|N(d/f)| \to \mathrm{hofib}_d\,|N(f)|$ is a weak equivalence.*

Proof. See [Qui06]. $\square$

4.4.5. Remark. — Let $\mathcal{C}$ and $\mathcal{D}$ be strict categories and $f\colon \mathcal{C} \to \mathcal{D}$ be a functor. Choose $d \in \mathcal{D}$, then theorem B says that under some condition, the nerve functor sends the pullback square

$$\begin{tikzcd} d/f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "f"] \\ * \arrow[r, "d"] & \mathcal{D} \end{tikzcd}$$

to a homotopy cartesian square in sSet.

4.4.6. Remark. — Theorem B implies theorem A. Let $f\colon X \to Y$ be a map between topological spaces, $x \in X$ and $y \in Y$. The long exact sequence of homotopy groups

$$\ldots \to \pi_1(X,x) \to \pi_1(Y,y) \to \pi_0 \operatorname{hofib}_y(f) \to \pi_0(X) \to \pi_0(Y)$$

shows that the map $f\colon X \to Y$ is a weak equivalence, if and only if for all $y \in Y$, the homotopy fiber $\operatorname{hofib}_y(f)$ is weakly contractible. We will not prove theorem B, since we will not use it in the following sections.

4.5. Fundamental theorem of the Q-construction

The fundamental theorem of the Q-construction are the theorem of *additivity, resolution, devissage, localization* and *cofinality.*

4.5.1. Definition. — Let $\mathcal{C}$ be an exact category. We denote $E\mathcal{C} \subseteq \mathsf{Fun}([2],\mathcal{C})$ the *full subcategory spanned by the exact sequences.* The degeneracy maps define functors s,e,q : $E\mathcal{C} \to \mathcal{C}$ and natural transformations $s \Rightarrow e \Rightarrow q$ such that for all $E \in E\mathcal{C}$, the sequence

$$s(E) \to e(E) \to q(E)$$

is exact. Furthermore, there is an exact structure on $E\mathcal{C}$ such that $f \in E\mathcal{C}$ is an admissible (mono/epi)morphism, if and only if $s(f)$, $e(f)$ and $q(f)$ are admissible (mono/epi)morphisms. Euipped with this exact structure, the functors s,e,q are exact.

4.5.2. Remark. — Let $\mathcal{C}$ be a category and $\mathcal{D}$ an exact category. There is a canonical exact structure on $\mathsf{Fun}(\mathcal{C},\mathcal{D})$: a natural transformation $\alpha\colon F \Rightarrow G$ is an admissible (mono/epi)morphsim, if and only if for all $X \in \mathcal{C}$, the morphism $\alpha(X)\colon F(X) \to G(X)$ is an admissible (mono/epi)morphism.

4.5.3. Theorem (Additivity theorem)**.** — *Let $\mathcal{C}$ be an exact category. The map*

$$(s_*,q_*)\colon K(E\mathcal{C}) \to K(\mathcal{C}) \times K(\mathcal{C})$$

is a weak equivalence.

Proof. By definition, it suffices to show that the map

$$(s,q)\colon |N(Q(E\mathcal{C}))| \to |N(Q(\mathcal{C}) \times Q(\mathcal{C}))|$$

is an weak equivalence. By theorem A, it suffices to show that for all $(X,Y) \in Q(\mathcal{C}) \times Q(\mathcal{C})$, the slice category $N((s,q)/(X,Y))$ is weakly contractible. The category $\mathcal{A} = (s,q)/(X,Y)$ is decribed as follows:

- The objects of $\mathcal{A}$ consist of pairs (E, u, v), where $E \in Q(E\mathcal{C})$ and
$$u\colon s(E) \to X \quad \text{and} \quad v\colon q(E) \to Y$$
are morphisms in $Q(\mathcal{C})$.
- Let $(E, u, v), (E', u', v') \in \mathcal{A}$, a morphism $f\colon (E, u, v) \to (E', u', v')$ consists of a morphism $f\colon E \to E'$ in $Q(E\mathcal{C})$, a morphism $s(f)\colon s(E) \to s(E')$ over X and a morphism $q(f)\colon q(E) \to q(E')$.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be the full subcategory spanned by the objects such that u is of the form
$$s(E) \twoheadleftarrow Z \overset{\simeq}{\rightarrowtail} X$$
and $\mathcal{A}'' \subseteq \mathcal{A}'$ be the full subcategory spanned by the objects such that v is of the form
$$q(E) \overset{\simeq}{\twoheadleftarrow} Z \rightarrowtail Y.$$
The theorem will be proved through the following steps.

Step 1: The inclusion $\mathcal{A}' \subseteq A$ admits a left adjoint. Let (E, u, v) be an object of $\mathcal{A}$, where $E\colon A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in $\mathcal{C}$ and $u\colon A \twoheadleftarrow X' \rightarrowtail X$, $v\colon C \twoheadleftarrow Y' \rightarrowtail Y$ are morphisms in $Q(\mathcal{C})$. After taking pushout, we obtain the following commutative diagram
$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner" near end] & X \arrow[d, two heads] \\ A \arrow[r, tail] & A' \end{tikzcd}$$
Afterwards, we can construct the following pushout square:
$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner" near end] & B \arrow[r, two heads] \arrow[d, tail] & C \arrow[d, "\simeq"] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd}$$
where C' is defined as the cokernel of the map $A' \rightarrowtail B'$. Since the left square is a pushout square, there is an induced isomorphism $C \xrightarrow{\simeq} C'$. We define an object (E', u', v') in $\mathcal{A}'$, where
$$E'\colon A' \rightarrowtail B' \twoheadrightarrow C', \quad u'\colon A' \twoheadleftarrow X = X, \quad \text{and} \quad v'\colon C \twoheadleftarrow Y' \rightarrowtail Y.$$
One can verify that there is a universal morphism $(E, u, v) \to (E', u', v')$, which shows that $\mathcal{A}' \subseteq \mathcal{A}$ admits a left adjoint. Similarly, the inclusion $\mathcal{A}'' \subseteq \mathcal{A}'$ admits a right adjoint.

Step 2: The category $\mathcal{A}''$ has an initial object. Consider the object $(0, \iota, \mu)$, where
$$0\colon 0 \twoheadleftarrow 0 \rightarrowtail 0, \quad \iota\colon 0 \twoheadleftarrow X = X, \quad \text{and} \quad \mu\colon 0 = 0 \rightarrowtail Y.$$
Let $(E, u.v)$ be an object of $\mathcal{A}''$, where
$$E\colon A \rightarrowtail B \twoheadrightarrow C, \quad u\colon A \twoheadleftarrow X = X \quad \text{and} \quad v\colon C = C \rightarrowtail Y.$$

We will first show that there exists an unique morphism from 0 to A over X in $Q(\mathcal{C})$. Let $0 \twoheadleftarrow A' \rightarrowtail A$ be a span, assume that we have the following diagram:

$$\begin{tikzcd}
 & & X \arrow[dl, two heads] \arrow[dr, "="] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & A' \arrow[dl, two heads] \arrow[dr, tail] & & X \arrow[dl] \arrow[dr, "="] & \\
0 & & A & & X
\end{tikzcd}$$

Since pullback preserves isomorphism, the map $A' \rightarrowtail A$ is an isomorphism. Similarly, there exists an unique morphism from 0 to C over Y in $Q(\mathcal{C})$. By the pasting law for pullback squares, there is an unique map $f\colon 0 \to E$ with the desired property.

Since an adjoint pair induces a homotopy equivalence on nerves, the proof is complete. $\square$

4.5.4. Corollary. — *Let $\mathcal{C}$ and $\mathcal{C}'$ be exact categories and*

$$F \Rightarrow F' \Rightarrow F''$$

be natural transformations between exact functors from $\mathcal{C}$ to $\mathcal{C}'$. Assume that for each object $X \in \mathcal{C}$, the following sequence

$$F(X) \rightarrowtail F'(X) \twoheadrightarrow F''(X)$$

is exact, then we have an equivalence of maps in Grpd_∞ as follows:

$$F'_* \simeq F_* + F''_*,$$

where F_*, F'_* and F''_* are maps from $K(\mathcal{C})$ to $K(\mathcal{C}')$ and the addition uses the E_∞-structure of $K(\mathcal{C})$.

Proof. In fact, the natural transformations define a functor

$$\tilde{F}\colon \mathcal{C} \to E\mathcal{C}, \quad X \mapsto (F(X) \rightarrowtail F'(X) \twoheadrightarrow F''(X))$$

In other words, we have $(F, F', F'') = (s, e, q) \circ \tilde{F}$. Therefore, without losses of generality, we may assume that $F = s, F' = e$ and $F'' = q$. We define the functor

$$f\colon \mathcal{C} \times \mathcal{C} \to E\mathcal{C}, \quad (X, Z) \mapsto (X \rightarrowtail X \oplus Z \twoheadrightarrow Z),$$

then $(s, q) \circ f = \mathrm{id}_{\mathcal{C}\times\mathcal{C}}$ and $e \circ f = - \oplus -$. This implies that

$$(e \circ f)_* \simeq (s \circ f)_* + (q \circ f)_*$$

Then consider the following diagram:

$$\begin{tikzcd}
K(\mathcal{C}) \times K(\mathcal{C}) & K(\mathcal{C}) \times K(\mathcal{C}) \arrow[l, "="'] \arrow[r, "+"] \arrow[d, "f_*"] & K(\mathcal{C}) \\
 & K(E\mathcal{C}) \arrow[ul, "{(s_*, q_*)}"] \arrow[ur, "e_*"'] &
\end{tikzcd}$$

By Theorem 4.5.3, the map (s_*, q_*) is a weak equivalence. By 2 out of 3 property of weak equivalences, the map f_* is a weak equivalence. This implies that e_* is equivalent to $s_* + q_*$ in Grpd_∞. $\square$

4.5.5. Corollary. — *Let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor. Assume that we have a finite filtration by exact subfunctors:*

$$0 = F_0 \rightarrowtail F_1 \rightarrowtail \ldots \rightarrowtail F_n = F.$$

We define the quotient functor $\mathrm{gr}^i F = F_i/F_{i-1}$ *by*

$$F_i/F_{i-1} \colon \mathcal{C} \to \mathcal{C}', \quad F_i/F_{i-1}(X) = \mathrm{coker}(F_i(X) \to F_{i-1}(X)).$$

Then for $1 \le i \le n$, the functor $\mathrm{gr}^i \colon \mathcal{C} \to \mathcal{C}'$ is exact, and we have

$$F_* \simeq \sum_{i=1}^{n} (\mathrm{gr}^i F)_*$$

as functors from $K(\mathcal{C})$ to $K(\mathcal{C}')$.

Proof. This is left as an exercise. □

4.5.6. Example. — We have the following examples:

1. Let X be a scheme and $\mathcal{C} = \mathsf{Vect}(X)$. Assume that

   $$0 \to E' \to E \to E'' \to 0$$

   is a short exact sequence in $\mathsf{Vect}(X)$. Then we have an exact sequence of endofunctors

   $$(E' \otimes -) \rightarrowtail (E \otimes -) \twoheadrightarrow (E'' \otimes -)$$

   in $\mathsf{Fun}(\mathsf{Vect}(X), \mathsf{Vect}(X))$. By Corollary 4.5.4, there is an equivalence of maps

   $$(E \otimes -)_* \simeq (E' \otimes -)_* + (E'' \otimes -)_*$$

   from $K(\mathsf{Vect}(X))$ to $K(\mathsf{Vect}(X))$ in Grpd_∞.

2. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$ be a graded ring and $\mathsf{Proj}^{\mathrm{gr}}(A)$ be the *category of projective $\mathbb{Z}$-graded modules*, the full subcategory the category of $\mathbb{Z}$-graded modules spanned by the those that are finitely generated projective as A-modules. The category $\mathsf{Proj}^{\mathrm{gr}}(A)$ is exact. Let $P \in \mathsf{Proj}^{\mathrm{gr}}(A)$. Then we can write

   $$P = \bigoplus_{i \in \mathbb{Z}} P_i, \text{ where } \quad A_i P_j \subseteq P_{i+j}.$$

   for $i, j \in \mathbb{Z}$, which implies that $P_i \in \mathsf{Proj}(A_0)$. We denote

   $$t \colon \mathsf{Proj}^{\mathrm{gr}}(A) \to \mathsf{Proj}^{\mathrm{gr}}(A)$$

   be the shift functor. Since t is exact, it exhibits $K_n(\mathsf{Proj}^{\mathrm{gr}}(A))$ as a $\mathbb{Z}[t^1]$-module. Therefore, the exact functor $(-)\otimes_{A_0} : \mathsf{Proj}(A_0) \to \mathsf{Proj}^{\mathrm{gr}}(A)$ induces a map

   $$K_n(\mathsf{Proj}(A_0)) \otimes_{\mathbb{Z}} \mathbb{Z}[t^1] \to K_n(\mathsf{Proj}^{\mathrm{gr}}(A)).$$

   for each $n \in \mathbb{N}$. In fact, this map is an isomorphism. Let $P \in \mathsf{Proj}^{\mathrm{gr}}(A)$ and $F_i P$ be the submodule generated by P_k for $k \le i$. We denote the functor

   $$T \colon \mathsf{Proj}^{\mathrm{gr}}(A) \to \mathsf{Proj}(A), \quad P \mapsto P \otimes_A A_0.$$

One can check that

$$F_iP/F_{i+1}P \simeq A(-i) \otimes_{A_0} T(P)_i,$$

where $A(-i)_k = A_{k-i}$. For $n \in \mathbb{N}$, we define $\mathsf{P}_n \subseteq \mathsf{Proj}^{\mathrm{gr}}(A)$ the full-subcategory spanned by the objects P such that $F_{-n-1}P = 0$ and $F_nP = P$, then

$$\mathsf{Proj}^{\mathrm{gr}}(A) = \bigcup_{n\geq 0} P_n.$$

Note that the category P_n is exact and F_iP assembles into an exact functor $F_i : \mathsf{P}_n \to \mathsf{P}_n$. This gives us a filtration by admissible monomorphisms

$$0 \subseteq F_{-n} \subseteq \ldots \subseteq F_{n-1} \subseteq F_n = \mathrm{id}_{\mathsf{P}_n}$$

in $\mathsf{Fun}(\mathsf{P}_n, \mathsf{P}_n)$. By Corollary 4.5.5, we have an equivalence of maps

$$\mathrm{id}_{P_n} \simeq \sum_{i=-n}^{n} (A(-i) \otimes_{A_0} T(-)_i)_*$$

from $K(\mathsf{P}_n)$ to $K(\mathsf{P}_n)$ in Grpd_∞. This shows that we have a map

$$\alpha\colon (\bigoplus_{i=-n}^{n} t^i\mathbb{Z}) \otimes_{\mathbb{Z}} K_*(\mathsf{Proj}(A_0)) \to K_*(\mathsf{P}_n)$$

with an inverse β. They are given by

$$\alpha(\sum_{i=-n}^{n} t_i \otimes x) = (A(-i) \otimes_{A_0} T(-i))_n(X) \quad \text{and} \quad \beta(y) = \sum_{i=-n}^{n} t^i \otimes (T_i)_n(y).$$

By definition, we have $\beta \circ \alpha = \mathrm{id}$ and $\alpha \circ \beta = \mathrm{id}$, by the Theorem 4.5.3. To conclude the claim, we take colimits on both sides. Assume that A is commutative, then

$$\mathsf{Proj}^{\mathrm{gr}}(A) = \mathsf{Vect}(\mathrm{Spec}(A)/\mathbb{G}_n).$$

This is the geometric interpretation of the example above.

4.6. Resolution theorem

Let $\mathcal{C}$ be an exact category and $\mathcal{P} \subseteq \mathcal{C}$ be a full subcategory containing 0 and closed under extension. We know from Remark 4.1.9, the subcategory $\mathcal{P}$ is exact, and the inclusion $\mathcal{P} \subseteq \mathcal{C}$ is an exact functor. When is the induced map $K(\mathcal{P}) \to K(\mathcal{C})$ a weak equivalence?

4.6.1. Example. — Let $\mathcal{C} = \mathsf{Ab}^{\mathrm{fg}}$ and $\mathcal{P} = \mathsf{Proj}(\mathbb{Z})$. The inclusion induces an isomorphism

$$\mathrm{inc}_*\colon K_0(\mathcal{P}) \xrightarrow{\simeq} K_0(\mathcal{C}),$$

since both sides are $\mathbb{Z}$. Let $X \in \mathcal{C}$. By definition, we may find a surjection $p\colon P_0 \twoheadrightarrow X$ such that $P_0 \in \mathcal{P}$. But then we have an short exact sequence:

$$0 \to \ker(p) \rightarrowtail P_0 \twoheadrightarrow X \to 0.$$

Since $\mathbb{Z}$ is a PID, the submodule $\ker(p) \in \mathcal{P}$.

4.6.2. Definition. — Let $\mathcal{C}$ be an exact category, $\mathcal{P} \subseteq \mathcal{C}$ be a full-subcategory that contains 0 and is closed under extension, and $X \in \mathcal{C}$. A *$\mathcal{P}$-resolution* of X is an exact complex

$$\ldots \to P_2 \to P_1 \to P_0 \to X \to 0$$

where $P_i \in \mathcal{P}$ for each $i \in \mathbb{N}$. We say that it has *length* n if $P_i = 0$ for all $i > n$. Let $\mathcal{P}_n \subseteq \mathcal{C}$ be the full-subcategory spanned by the objects that have a $\mathcal{P}$-resolution of length less than n. Furthermore, we denote $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_\infty = \bigcup_{n>0} \mathcal{P}_n \subseteq \mathcal{C}$.

4.6.3. Theorem (Resolution Theorem). — *Let $\mathcal{C}$ be an exact category and $\mathcal{P} \subseteq \mathcal{C}$ be a full-subcategory that contains* 0 *and is closed under extension. Assume that the following holds:*

1. *For each exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{C}$, if $Y, Z \in \mathcal{P}$, then $X \in \mathcal{P}$.*

2. *For each $X \in \mathcal{C}$, there exists an admissible epimorphism $p\colon P \twoheadrightarrow X$, where $P \in \mathcal{P}$.*

Then for $0 \leq n \leq \infty$, the subcategory $\mathcal{P}_n$ is closed under extension in $\mathcal{C}$ and the map

$$\mathrm{inc}_*\colon K(\mathcal{P}) \to K(\mathcal{P}_n)$$

induced by the inclusion $\mathcal{P} \subseteq \mathcal{P}_n$ is a weak equivalence.

Proof. We will first black blox the following fact:

Step 1: For each $n \in \mathbb{N}$, the subcategory $\mathcal{P}_n \subseteq \mathcal{C}$ is closed under extension. Let $X \rightarrowtail Y \twoheadrightarrow Z$ be an exact sequence in $\mathcal{C}$ such that $Z \in \mathcal{P}_{n+1}$ and $Y \in \mathcal{P}_n$. Then $X \in \mathcal{P}_n$.

By Remark 4.2.7, the functor $K(-)_*$ preserves filtered colimits. Therefore, we have

$$K_*(\mathcal{P}_\infty) = \mathrm{colim}_{n\in\mathbb{N}} K_*(\mathcal{P}_n).$$

Without losses of generality, we may assume that $n < \infty$. It now suffices to show that the map induced by the inclusion

$$\mathrm{inc}_*\colon K(\mathcal{P}_n) \to K(\mathcal{P}_{n+1})$$

is a weak equivalence. In other words, the map

$$\mathrm{inc}_*\colon N(Q(\mathcal{P}_n)) \to N(Q(\mathcal{P}_{n+1}))$$

is a weak equivalence. We may consider the following factorization

$$\begin{tikzcd} Q(\mathcal{P}_n) \arrow[d, "g"'] \arrow[dr, "\mathrm{inc}_*"] & \\ \mathcal{A} \arrow[r, "f"'] & Q(\mathcal{P}_{n+1}) \end{tikzcd}$$

where g is essentially surjective and f is fully faithful.

Step 2: The map $N(g)$ is a weak equivalence.

By theorem A, it suffices to show that for every $\mathcal{P} \in \mathcal{A}$, the nerve $N(g/p)$ is weakly contractible. The category g/p can be described as follows:

- The objects of g/p are of the form (Q, q), where $Q \in \mathcal{P}_n$ and q is a span of the form $Q \twoheadleftarrow R_0 \rightarrowtail P$. We define $R_1 \to R_0$ to be the kernel of the map $R_0 \twoheadrightarrow Q$, then equivalently, an object of g/p is a $\mathcal{P}_{n+1}$-*admissible pair* (R_0, R_1) such that

$$R_1 \rightarrowtail R_0 \rightarrowtail P \quad \text{and} \quad R_0/R_1 \in \mathcal{P}_n$$

- Let (R_0, R_1) and $(R_0', R_1') \in g/p$. There is a unique morphism $(R_0, R_1) \leq (R_0', R_1')$ if and only if there exists a sequence of admissible monomorphisms

$$R_0' \rightarrowtail R_0 \rightarrowtail R_1 \rightarrowtail R_1'$$

But then by the first step, we have $R_0 \in \mathcal{P}_n$, hence $(R_0, 0) \in g/p$. Consider the following diagram in g/p:

$$(R_0, R_1) \geq (R_0, 0) \leq (0, 0).$$

This gives us a zig-zag of natural transformations between $\mathrm{id}_{g/p}$ and the constant functor. Therefore, $|N(g/p)|$ is contractible.

Step 3: The map $N(f)$ is a weak equivalence.

By theorem A, it suffices to show that for each $X \in Q(\mathcal{P}_{n+1})$, the topological space $|N(X/f)|$ is weakly contractible. The category $\mathcal{F} = X/f$ is described as follows:

- The objects of X/f are pairs of the form (P, u), where $P \in \mathcal{A}$ and u is a span of the form $X \twoheadleftarrow P' \rightarrowtail P$ in $Q(\mathcal{P}_{n+1})$.
- The morphisms of X/f are given in the obvious way.

By the first step, we have $P' \in \mathcal{P}_n$. Let $\mathcal{F}' \subseteq \mathcal{F}$ be the full-subcategory spanned by the spans whose right leg is an isomorphism. Observe that the inclusion $\mathcal{F}' \subseteq \mathcal{F}$ has a right adjoint given by sending $(P, (X \twoheadleftarrow P' \rightarrowtail P))$ to $(P', (X \twoheadleftarrow P' \rightarrowtail P'))$. Therefore, it suffices to show that $N(\mathcal{F}')$ is weakly contractible. In fact, we can describe the category $\mathcal{F}'$ as follows:

- The objects of $\mathcal{F}'$ are pairs of the form (P, u), where $P \in \mathcal{A}$ and $u\colon P \twoheadrightarrow X$ is an admissible epimorphism.
- Let (P, u) and $(P', u') \in \mathcal{F}'$. A morphism $f\colon (P, u) \to (P', u')$ consists of a $\mathcal{P}_{n+1}$-admissible morphism $f\colon P \to P'$ over X.

By the first step, the category $\mathcal{F}'$ is non-empty. Let (P', u') be a fixed object in $\mathcal{F}'$ and $(P, u) \in \mathcal{F}'$. Consider the pair $(P \times_X P', v)$, where $v\colon P \times_X P' \twoheadrightarrow X$. Since there exists an exact sequence

$$\ker(u) \rightarrowtail P \times_X P' \twoheadrightarrow P'$$

and $\ker(u)' \in \mathcal{P}_n$, we have $P \times_X P'$, by the first step. Furthermore, the zig-zag of morphism

$$(P, u) \to (P \times_X P', v) \leftarrow (P', u')$$

gives us a zig-zag of natural transformations from $\mathrm{id}_{\mathcal{F}'}$ to $\mathrm{const}_{(P',u')}$. This implies that $N(X/f)$ is a weak equivalence.

As a composition of weak equivalences, the map $\mathrm{inc}_*\colon N(Q(\mathcal{P}_n)) \to N(Q(\mathcal{P}_{n+1}))$ is a weak equivalence. This finishes the proof. $\square$

4.6.4. Example. — We have the following examples:

1. By Theorem 4.6.3, we obtain a weak equivalence
$$\mathrm{inc}_*\colon K(\mathsf{Proj}(\mathbb{Z})) \to K(\mathsf{Ab}^{\mathrm{fg}}).$$
This follows from Example 4.6.1.

2. Let X be a Noetherian scheme. A quasi-coherent sheaf $\mathcal{F}$ on X is *coherent*, if it has finite presentation. Let $\mathsf{Coh}(X) \subseteq \mathsf{QCoh}(X)$ be the category of quasi-coherent sheaves. The category $\mathsf{Coh}(X)$ is Abelian, and $\mathsf{Vect}(X) \subseteq \mathsf{Coh}(X)$ is closed under extension.

4.6.5. Definition. — Let R be a commutative Noetherian ring. We say R is *regular*, if every coherent R-module has a finite resolution by finitely generated projective R-module. In other words, we have $\mathsf{Proj}(R)_\infty = \mathsf{Coh}(R)$.

4.6.6. Remark. — Let R be a commutative Noetherian ring. The following are equivalent:

1. The ring R is regular.
2. For all $\mathfrak{p} \in \mathrm{Spec}(R)$, the ring $R_\mathfrak{p}$ is regular.
3. For every $\mathfrak{p} \in \mathrm{Spec}(R)$, we have $\dim(R_\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(\mathfrak{m}/\mathfrak{m}^2)$, where $\mathfrak{m} = \mathfrak{p}R\mathfrak{p}$. In this case, $\mathfrak{m}/\mathfrak{m}^2$ is called the *cotangent space* to $\mathrm{Spec}(R)$ at $\mathfrak{p}$.

4.6.7. Definition. — A Noetherian scheme X is *regular*, whenever for all affine open subset $U \subseteq X$, we have $\mathcal{O}_X(U)$ is regular.

4.6.8. Remark. — Let X be a Noetherian scheme. The following are equivalent:

1. The scheme X is regular.
2. For every $x \in X$, each local ring $\mathcal{O}_{X,x}$ is regular.
3. For every $x \in X$, we have
$$\dim(\mathcal{O}_{X,x}) = \dim_{k(x)}(T_x^*(X))$$
where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ denotes the residue field at x.

4.6.9. Remark. — Let X be a Noetherian scheme. The following holds.

1. Assume that X is regular and $Y \to X$ is a smooth morphism of finite presentation. Then Y is regular.
2. Let k be a perfect field and X is of finite presentation over k. Then X is regular if and only if X is smooth over k.

4.6.10. Example. — We have the following examples of regular rings:

1. Every field is regular.
2. Every PID is regular.
3. Every Dedekind domain is regular.

4. Let R be a regular ring. The ring of polynomial $R[X]$ is again regular.
5. Let k be a field. The ring $R = k[X]/(X^2)$ is not regular, since the R-module k does not have a finite resolution by finitely generated projective R-module.

4.6.11. Definition. — A scheme X has the *resolution property*, if for all $\mathcal{F} \in \mathsf{QCoh}(X)$ of finite presentation, there exists a vector bundle ξ and a epimorphism $\xi \to F$.

4.6.12. Remark. — The following holds, by Serre and Illusie respectively.

1. Any quasi-projective scheme over any ring has the resolution property.
2. Let X be a regular and separated scheme. Then X has the resolution property and we have $\mathsf{Vect}(X)_\infty = \mathsf{Coh}(X)$.

4.6.13. Corollary. — *Let X be a Noetherian scheme that has the resolution property. The induced map is a weak equivalence*

$$K(\mathsf{Vect}(X)) \simeq K(\mathsf{Vect}(X)_\infty).$$

If furthermore X is regular, then $K(\mathsf{Vect}(X)) \simeq K(\mathsf{Coh}(X))$.

4.6.14. Example. — Let X be the affine plane with double origins over a field k. Then X is regular but has no resolution property. In fact, one can show that $K(\mathsf{Vect}(X)) = K(k)$, but $K(\mathsf{QCoh}(X)) = K(k) \times K(k)$.

4.6.15. Notation. — Let X be a scheme. We define the *naive K-theory of X* as

$$K^{\text{naive}}(X) = K(\mathsf{Vect}(X))$$

If furthermore X is Noetherian, we denote $G(X) = K(\mathsf{Coh}(X))$.

4.6.16. Remark. — Let X be a Noetherian scheme. The notion of $K^{\text{naive}}(X)$ is the correct notion for K-theory of schemes, only when X is noetherian and has the resolution property. However, it is not a well-behaved invariant in general. Instead, one should define $K(X) = G(X)$ for all regular scheme X.

4.6.17. Theorem (Comparison with Group Completion)**.** — *Let $\mathcal{C}$ be an additive category with the minimal exact structure. We have $K(\mathcal{C}) \simeq |N(\mathcal{C}^{\simeq}, \oplus)|^{\text{grp}}$.*

Proof. The idea is the following: for an E_∞-space M we have $M^{\text{grp}} \simeq \Omega|B_\bullet M|$, where $B_\bullet M : \Delta^{\text{op}} \to \mathsf{Top}$ is the underlying E_1-space of M. Therefore, it suffices to compare $|N(Q\mathcal{C})|$ and $|B_\bullet N(\mathcal{C}^{\simeq}, \oplus)|$. For further details, see [Gra76]. $\square$

4.6.18. Theorem (Cofinality Theorem)**.** — *Let $\mathcal{C}$ be an exact category and $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full-subcategory closed under extension and contains 0. Furthermore, assume that for every $X \in \mathcal{C}$, there exists $Y \in \mathcal{C}$ such that $X \oplus Y \in \mathcal{C}_0$. The the map induced by the inclusion*

$$\text{inc}_* \colon K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$$

is injective and the induced map

$$\text{inc}_* \colon K_i(\mathcal{C}_0) \to K_i(\mathcal{C})$$

is an isomorphism for each $i \geq 1$.

Proof. This follows immediately from Theorem 4.6.17 □

4.6.19. Example. — Let R be a ring. The category $\mathsf{Free}(R)$ of free modules over R satisfies the condition in Theorem 4.6.18. Therefore, $K_0(\mathsf{Free}(R))$ is an acyclic subgroup generated by $[R]$ and we have an isomorphism $K_i(\mathsf{Free}(R)) \simeq K_i(R)$ for $i \geq 1$.

4.6.20. Definition. — Let $\mathcal{C}$ be an exact category and $X, Y \in \mathcal{C}$. We say that X is a *sub-object* of Y, when there exists an admissible monomorphism $X \rightarrowtail Y$. An object $Z \in \mathcal{C}$ is *simple*, whenever the only subobject of Z is 0 and Z itself.

4.6.21. Theorem (Dévissage). — *Let $\mathcal{A}$ be an Abelian category and $\mathcal{B} \subseteq \mathcal{A}$ be a full-subcategory closed under finite sum, sub-objects and quotients. The category $\mathcal{B}$ is Abelian and the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is exact. Assume that every $A \in \mathcal{A}$ admits a finite filtration*

$$0 \subseteq A_0 \subseteq \ldots \subseteq A_n = A,$$

such that $A_i/A_{i-1} \in \mathcal{B}$ for all $i \in \{1, \ldots, n\}$, then the induced map $K(\mathcal{B}) \to K(\mathcal{A})$ is an weak equivalence. However, note that the subcategory $\mathcal{B}$ is not necessarily closed under extension.

Proof. We first observe that the map $f : Q\mathcal{B} \to Q\mathcal{A}$ is fully faithful. By Theorem A, in order to show that the induced map

$$\mathrm{inc}_* : |N(Q(\mathcal{B})) \to N(Q(\mathcal{A}))|$$

is a weak equivalence, it suffices to show that $N(f/A)$ is weakly contractible for all $A \in Q\mathcal{A}$.

Similar to the proof of Theorem 4.6.3, we can describe the category f/A as follows:

- The objects of f/A are pairs sub-objects $A_0 \subseteq A_1 \subseteq A$ such that $A_1/A_0 \in \mathcal{B}$.
- For (A_0, A_1) and $(A_0', A_1') \in f/A$, there exists a unique morphism $(A_0, A_1) \leq (A_0', A_1')$ if and only if there exists a sequence of sub-objects

$$A_0' \subseteq A_0 \subseteq A_1 \subseteq A_1'$$

We know that there exists a chain of sub-objects

$$0 = A_0 \subseteq \ldots \subseteq A_n = A,$$

such that $A_i/A_{i-1} \in \mathcal{B}$ and $N(f/0)$ is weakly contractible. By induction, it suffices to show that, if $A' \subseteq A$ where $A/A' \in \mathcal{B}$, then the map

$$N(i) : N(f/A') \to N(f/A)$$

defines a weak equivalence. In fact, we can prove this by constructing a homotopy inverse by

$$r : f/A \to f/A', \quad (A_0, A_1) \mapsto (A_0 \cap A', A_1 \cap A'),$$

where the intersection $A_0 \cap A'$ is defined via the pullback:

$$\begin{tikzcd} A_0 \cap A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A' \arrow[d, tail] \\ A_0 \arrow[r, tail] & A \end{tikzcd}$$

This is well-defined, since $(A_1 \cap A')/(A_0 \cap A')$ is again a sub-object of $A_1/A_0 \in \mathcal{B}$. One can check that $N(r) \circ N(i) = \mathrm{id}_{N(f/A')}$. For $i \circ r$, we have the following zig-zag of morphisms

$$(A_0, A_1) \leq (A_0 \cap A', A_1) \geq (A_0 \cap A', A_1 \cap A')$$

There is a canonical map

$$A_1/A_0 \cap A' \rightarrowtail A_1/A_0 \oplus A/A',$$

showing that $A_1/(A_0 \cap A') \in \mathcal{B}$, and hence the zig-zag of morphisms define a zig-zag of natural tranformations from $\mathrm{id}_{f/A}$ to $i \circ r$. After taking the nerve, we obtain $N(i) \circ N(r) = \mathrm{id}_{N(f/A)}$. □

4.6.22. Example. — Let X be a Noetherian scheme and $Z \hookrightarrow X$ a closed immersion. Define the full-subcategory

$$\mathsf{Coh}_Z(X) = \{\mathcal{F} \in \mathsf{Coh}(X) : \mathcal{F}|_{X-Z} = 0\}.$$

Then the map induced by the inclusion

$$i_* : \mathsf{Coh}(Z) \hookrightarrow \mathsf{Coh}_Z(X)$$

satisfies the assumption of Theorem 4.6.21. Therefore, the induced map

$$i_* \colon G(Z) \to K(\mathsf{Coh}_Z(X))$$

is an isomorphism. Let $X = \mathrm{Spec}(\mathbb{Z})$ and $Z = \mathrm{Spec}(\mathbb{F}_p)$. Then we have

$$\mathsf{Coh}(X) = \mathsf{Ab}^{\mathrm{fg}} \quad \text{and} \quad \mathsf{Coh}_Z(X) = \{M \in \mathsf{Ab}^{\mathrm{fg}} : \text{exist } r \geq 0 \text{ such that } p^r M = 0\}.$$

4.6.23. Corollary. — *Let X be a Noetherian scheme and $Z \hookrightarrow X$ be a nil-immersion. In other words, it is defined by a nilpotent ideal. Then the map $G(Z) \to G(X)$ is an equivalence.*

4.6.24. Remark. — We've seen that K_0 is nil-invariant. However this is not true for K_i, whenever $i \geq 1$. For example, consider the map $K_1(k) \to K_1(k[\varepsilon]/\varepsilon^2)$. The left hand side is $k^\times$, whereas the left hand side is $k^\times \oplus k$.

4.6.25. Corollary. — *Let $\mathcal{A}$ be an Abelian category such that for all $A \in \mathcal{A}$, there exists a filtration of sub-objects as follows:*

$$0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$$

such that A_i/A_{i+1} is simple. The induced map is an isomorphism:

$$K_*(\mathcal{A}) \simeq \bigoplus_{s \in S} K_*(\mathrm{End}_{\mathcal{A}}(s)),$$

where S is a set of representatives of simple objects.

Proof. An object of $\mathcal{A}$ is *semi-simple*, whenever it can be written as a direct sum of simple objects in $\mathcal{A}$. We denote $\mathcal{B} \subseteq \mathcal{A}$ the full-subcategory spanned by the semi-simple objects of $\mathcal{A}$. By Theorem 4.6.21, the induced map

$$\mathrm{inc}_* \colon K(\mathcal{B}) \to K(\mathcal{A})$$

is an equivalence. Furthermore, observe that

$$\mathcal{B} = \mathrm{colim}_{F \subseteq S} \langle F \rangle_{\oplus}$$

where F is a finite subset of S and $\langle F \rangle_{\oplus}$ denotes the additive subcategory generated by F. In other words, the category $\langle F \rangle_{\oplus}$ is the smallest full-subcategory of $\mathcal{A}$ containing F and is closed under finite direct sums. Observe that the direct sum map defines an equivalence

$$\oplus \colon \prod_{s \in F} \langle \{s\} \rangle_{\oplus} \xrightarrow{\simeq} \langle F \rangle_{\oplus},$$

where fully-faithfulness follows from the fact that $\mathrm{Hom}_{\mathcal{A}}(s, s') = 0$ if $s \not\simeq s'$. Therefore there is an equivalence

$$K(\langle F \rangle_{\oplus}) \simeq \prod_{s \in F} K(\langle \{s\} \rangle_{\oplus})$$

Finally, one can show that there is an equivalence

$$\langle \{s\} \rangle_{\oplus} \simeq \mathsf{Proj}(\mathrm{End}_{\mathcal{A}}(s)^{\mathrm{op}})$$

which implies that $K(\langle \{s\} \rangle_{\oplus}) \simeq K(\mathrm{End}_{\mathcal{A}}(s))$. The claim follows from combining all the ingredients above. □

4.7. Localization of abelian categories

4.7.1. Definition. — Let $\mathcal{A}$ be an Abelian category. A *Serre subcategory* of $\mathcal{A}$ is a full-subcategory $\mathcal{B} \subseteq \mathcal{A}$ containing 0 and is closed under sub-objects, quotients and extension.

4.7.2. Remark. — Let $\mathcal{A}$ be an Abelian category. A Serre subcategory $\mathcal{B}$ of $\mathcal{A}$ is always Abelian, and the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is exact.

4.7.3. Proposition. — *Let $\mathcal{A}$ be an Abelian category and $\mathcal{B} \subseteq \mathcal{A}$ be a Serre subcategory. There exists an Abelian category $\mathcal{A}/\mathcal{B}$ equipped with an exact functor $q \colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ such that for all Abelian category $\mathcal{C}$, the induced functor*

$$q^* : \mathsf{Fun}^{\mathrm{ex}}(\mathcal{A}/\mathcal{B}, \mathcal{C}) \to \mathsf{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{C})$$

is fully faithful with essential image consisting of functors whose restriction to $\mathcal{B}$ is 0. *Moreover, we have* $\ker(q) = \mathcal{B}$.

Proof. The category of $\mathcal{A}/\mathcal{B}$ has the same object as $\mathcal{A}$. Given X, Y in $\mathcal{A}/\mathcal{B}$, we define

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) = \mathrm{colim}_{(X', Y') \in \mathcal{A}^2} \mathrm{Hom}_{\mathcal{A}}(X', Y').$$

One can check that this construction satisfies the desired properties. □

4.7.4. Example. — Let X be a Noetherian scheme, $\gamma \colon Z \hookrightarrow X$ a closed immersion and $d \colon U \hookrightarrow X$ is its open complement, then the induced functor

$$\gamma^* \colon \mathsf{Coh}(X)/\mathsf{Coh}_Z(Z) \to \mathsf{Coh}(U)$$

defines an equivalence.

4.7.5. Theorem (Localization)**.** — *Let $\mathcal{A}$ be an Abelian category and $\mathcal{B} \subseteq \mathcal{A}$ be a Serre subcategory. The sequence*

$$\mathcal{B} \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

induces a homotopy fiber sequence

$$K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{B}).$$

Therefore we have the long exact sequence of K-groups.

$$\ldots \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to K_{n-1}(\mathcal{B}) \to \ldots$$

which follows immediately from the long exact sequence for homotopy groups.

Proof. We will give a sketch of the proof. Since the functor Ω preserves homotopy fiber sequence, we can reduce to showing that the following is a homotopy fiber sequence:

$$|N(Q(\mathcal{B}))| \to \mathrm{hofib}_0(|N(Q(\mathcal{A}))| \to N(Q(\mathcal{A}/\mathcal{B})).$$

For every $u \colon X \to Y$ in $Q(\mathcal{A}/\mathcal{B})$, one can show that $u^* \colon Y/Q(q) \to X/Q(q)$ is a weak equivalence. By Theorem B, one can show that

$$\mathrm{hofib}_0 |N(Q(\mathcal{A}))| \simeq |N(0, Qq)|.$$

But then everything follows from the fact that the map

$$Q(\mathcal{B}) \to 0/Q(q)$$

is a weak equivalence, which follows from Theorem A. $\square$

4.7.6. Corollary. — *Let X be a Noetherian scheme, $i \colon Z \hookrightarrow X$ a closed immersion, $j \colon U \hookrightarrow X$ its open complement. The following holds.*

1. *The following sequence*

$$G(Z) \xrightarrow{i_*} G(X) \xrightarrow{j_*} G(U)$$

   is a homotopy fiber sequence.

2. *Assume that X and Z are regular and separated. The sequence*

$$K^{\mathrm{naive}}(Z) \to K^{\mathrm{naive}}(X) \to K^{\mathrm{naive}}(U)$$

   is an exact sequence.

Proof. We prove the statement as follows.

1. This follows from Theorem 4.7.5 and Theorem 4.6.21.

2. This follows from the first claim, since $K^{\mathrm{naive}}(X) = G(X)$ in this case.

$\square$

4.8. $\mathbb{A}^1$-invariance of G-theory

4.8.1. Theorem. — *Let R be a Noetherian commutative ring. The functor*

$$- \otimes_R R[t] \colon \mathsf{Coh}(R) \to \mathsf{Coh}(R[t])$$

induces a weak equivalence $G(R) \simeq G(R[t])$.

4.8.2. Corollary. — *Let R be a regular ring. Then we have $K(R) \simeq K(R[t])$.*

4.8.3. Remark. — We have the following remark.

1. Corollary 4.8.2 fails, whenever R is not regular. For example, consider $R = k[X]/(X^2)$.
2. Recall that the K-theory of rings is defined as

   $$K(R) = (\mathsf{Proj}(R)^{\simeq}, \oplus)^{\mathrm{grp}}.$$

   However, the functor $\mathsf{Proj}(-)^{\simeq}$ is not $\mathbb{A}^1$-invariant. Assume that $R \neq 0$, then

   $$\mathsf{Proj}(R)^{\simeq} \to \mathsf{Proj}(R[t])^{\simeq}$$

   is not an equivalence.
3. The Bass-Serre conjecture says that the above map induces an isomorphism on π_0 if R is regular. We know this after group completion. This is also true when R contains a field.
4. Let X be a Noetherian scheme. The induced map $G(X) \to G(X \times \mathbb{A}^1)$ is an equivalence.

4.8.4. Remark. — The G-theory has functoriality in some cases. Let $f : R \to S$ be a map between Noetherian rings. This induces a map

$$f^* \colon \mathsf{Coh}(R) \to \mathsf{Coh}(S), \quad M \mapsto M \otimes_R S.$$

The map f^* is exact if and only if f is flat, that is, S is a flat-R-module. In this case, we obtain a map $f_* \colon G(R) \to G(S)$.

4.8.5. Definition. — We have the following definition.

1. Let R be a ring. A left R-module M has *Tor-amplitude* $\leq n$, whenever $\mathrm{Tor}_i^R(N, M) = 0$ for all $i > n$ and all right R-module N.
2. Let $f : R \to S$ be a ring homomorphism. We say that it has *Tor-amplitude* $\leq n$ if S has Tor-amplitude $\leq n$ as a R-module.

4.8.6. Proposition. — *Let R be a ring and M a left R-module. Then M has Tor-amplitude $\leq n$, if and only if there exists a resolution*

$$0 \to F_n \to \ldots \to F_0 \to M \to 0$$

where F_i is a left R-module for $i \in \{0, \ldots, n\}$. In particular, M has Tor-amplitude ≤ 0, if and only if it is flat.

4.8.7. Example. — Let R be a ring. Assume that $r \in R$ is not a zero divisor, then the map $R \to R/(r)$ has Tor-amplitude ≤ 1, since we have the following exact sequence

$$0 \to R \xrightarrow{r} R \to R/(r) \to 0$$

More generally, if $(r_1, \dots, r_n)$ is a regular sequence in R, then the map

$$R \to R/(r_1, \dots, r_n)$$

has Tor-amplitude $\leq n$, since we have the exact sequence

$$0 \to \Lambda^n_R(R^n) \to \dots \to \Lambda^2_R(R^n) \to R^n \xrightarrow{(r_1,\dots,r_n)} R \to R/(r_1, \dots, r_n) \to 0$$

In particular, the map

$$R[t_1, \dots, t_n] \to R, \quad t_i \mapsto 0$$

has Tor-amplitude $\leq n$.

4.8.8. Remark. — There is a well-defined functor

$$G : \mathsf{Ring}^{\mathrm{noet}}_{\mathrm{finTor}} \to \mathsf{Alg}_{E_\infty}(\mathsf{Top})$$

from the full-subcategory of Ring spanned by the Noetherian rings with finite Tor-amplitude to the category of E_∞-spaces. Let $f\colon R \to S$ be a ring map of Tor-amplitude $\leq n$ and $\mathcal{C}_i \subseteq \mathsf{Coh}(R)$ be the full-subcategory spanned by the objects M such that $\mathrm{Tor}^R_j(S, M) = 0$ for all $j > i$. But then we see that $\mathcal{C}_n = \mathsf{Coh}(R)$, $\mathsf{Proj}(R) \subseteq \mathcal{C}_0$ and the map

$$f^*\colon \mathcal{C}_0 \to \mathsf{Coh}(S)$$

is exact. Furthermore, each $\mathcal{C}_{i-1} \subseteq \mathcal{C}_i$ satisfies the assumption of the resolution theorem. Given a short exact sequence

$$0 \to M \to N \to P \to 0 \quad \text{in} \quad \mathsf{Coh}(R)$$

By the left exactness of Tor_*, if $M, P \in \mathcal{C}_i$, then $N \in \mathcal{C}_i$ and if $N, P \in \mathcal{C}_i$, then $M \in \mathcal{C}_i$. Since for all $M \in \mathrm{Coh}(R)$, there exists a map $R^n \twoheadrightarrow M$, this implies that

$$\begin{tikzcd} K(C_0) \arrow[r, "f^*"] \arrow[d, "\simeq"'] & G(S) \\ G(R) \arrow[ur, dashed] & \end{tikzcd}$$

there is an induced map $G(R) \to G(S)$ such that the above diagram commutes.

4.8.9. Example. — We have the following examples:

1. Let k be a field. The following map

$$p\colon k[X]/(X^2) \twoheadrightarrow k$$

has ∞-Tor-amplitude, that is, it has Tor-amplitude $\leq n$ for all $n \in \mathbb{N}$.

2. The map $\pi_0 G(R) \to \pi_0 G(S)$ is given by

$$[M] \mapsto \sum_{i \geq 0} (-1)^i [\mathrm{Tor}_i^R(S, M)].$$

   This follows from unpacking the proof of Theorem 4.6.3.

3. Let $f\colon X \to Y$ be a map of Noetherian schemes of finite Tor-amplitude. Assume that X has the resolution property, then we obtain a map $f^*\colon G(X) \to G(Y)$ by the same argument.

4. Let $p\colon Y \to X$ be a proper map between Noetherian schemes. We obtain a map $p_*: G(Y) \to G(X)$: denote $\mathcal{C}_0 \subseteq \mathsf{Coh}(Y)$ the full-subcategory spanned by the objects $\mathcal{F}$ such that $R^i p_* \mathcal{F} = 0$, then the map $p_*\colon \mathcal{C}_0 \to \mathsf{Coh}(X)$ is exact. Applying Theorem 4.6.3 to the opposite category gives us the desired map.

5. Let $p\colon Y \to X$ be a proper map of finite Tor-dimension between Noetherian schemes. Assume that X, Y are quasi-projective over a Noetherian ring, then we obtain a map

$$p_*\colon K^{\mathrm{naive}}(Y) \to K^{\mathrm{naive}}(X).$$

   The key point is that, if $\mathcal{F} \in \mathrm{Vect}(Y)$ such that $R^i p_* \mathcal{F} = 0$ for all $i > 0$, then we have $p_* \mathcal{F} \in \mathrm{Vect}(X)_\infty$.

4.9. Analogy with cohomology of topological spaces

Recall from algebraic topology that we have two functors $H^*, H_*\colon \mathsf{Top} \to \mathsf{grAb}$, namely the *singular cohomology* and *singular homology.* In this course we have also defined functors $K_*, H^*\colon \mathsf{Sch} \to \mathsf{grAb}$. One should think K_* as an analog to H^* and G_* as an analog to H_*^{BM}, where H_*^{BM} is the same as H_* except that it also allows locally finite sum of simplicies. Recall that we have an natural isomophism of functors $H_*^{BM} \simeq H^*$, when X is an oriented manifold, and $K_* \simeq G_*$, when X is regular.

4.9.1. Lemma. — *Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring A. Equivalently, the ring A_0 is Noetherian and $\bigoplus_{n \geq 1} A_n$ is a finitely generated ideal. Assume that the map $A_0 \to A$ is flat and $A \to A_0$ has finite Tor-amplitude, then we have a map*

$$\oplus_{A_0} A\colon \mathsf{Coh}(A_0) \to \mathsf{Coh}^{\mathrm{gr}}_{\geq 0}(A),$$

where $\mathsf{Coh}^{\mathrm{gr}}_{\geq 0}(A)$ denotes the category of coherent A-modules concentrated in non-negative degrees. This induces an isomorphism

$$\mathbb{Z}[X] \otimes G_*(A_0) \simeq K_*(\mathsf{Coh}^{\mathrm{gr}}_{\geq 0}(A)),$$

where the variable X corresponds to the degree shift functor.

Proof. Denote $\mathcal{N} \subseteq \mathsf{Coh}^{\mathrm{gr}}_{\geq 0}(A)$ the full-subcategory spanned by the objects A such that $\mathrm{Tor}_i^A(A_0, -) = 0$ for all $i > 0$. By Theorem 4.6.3, the following map

$$\otimes_A A_0 : K(\mathcal{N}) \to K(\mathsf{Coh}^{\mathrm{gr}}_{\geq 0}(A))$$

is an isomophism. We observe that $\mathcal{N} = \bigcup_{n\geq 0} \mathcal{N}_n$, where $\mathcal{N}_n$ denotes the full-subcategory spanned by the A-modules generated in degree $\leq n$. We will now prove that

$$K_*(\mathcal{N}_n) \simeq \bigoplus_{i=0}^{n} K_*(\mathsf{Coh}(A_0))$$

In fact, there is a map

$$\alpha\colon \prod_{i=0}^{n} \mathsf{Coh}(A_0) \to \mathcal{N}_n, \quad (M_i)_{i=0}^n \mapsto \bigoplus_{i=0}^{n} A(-i) \otimes_{A_0} M_i,$$

and a inverse map

$$\beta\colon \mathcal{N}_n \to \prod_{i=0}^{n} \mathsf{Coh}(A_0), \quad M \mapsto (M \otimes_A A_0)_{i=0}^n.$$

One can check that $\beta \circ \alpha = \mathrm{id}$. Conversely, $\alpha \circ \beta$ is the same as taking the associated graded of finite filtration of $\mathrm{id}_{\mathcal{N}_n}$, which implies that $K(\alpha \circ \beta) \simeq K(\mathrm{id})$, by Theorem 4.5.3. □

4.9.2. Theorem. — *Let R be a Noetherian ring. Then there is an weak equivalence $G(R) \simeq G(R[t])$.*

Proof. Omitted. □

4.9.3. Theorem. — *Let X be a Noetherian scheme and $\eta \in \mathsf{Vect}_r(X)$. The map*

$$\prod_{i=0}^{n-1} \mathsf{Coh}(X) \to \mathsf{Coh}(\mathbb{P}(\eta)), \quad (\mathcal{F}_i) \mapsto \bigoplus_{i=0}^{r-1} \pi^*(\mathcal{F}_\rangle) \otimes \mathcal{O}(-i)$$

induces an weak equivalence $G(\mathbb{P}(\eta)) \simeq G(X)^r$.

4.9.4. Remark. — Theorem 4.9.3 also holds for K-theory. It also has an analog in topology: let $\eta\colon V \to X$ be a complex vector bundle, then we have

$$H^*(\mathbb{P}(V), \mathbb{Z}) = \bigoplus_{i=0}^{r-1} H^{*-2i}(X, \mathbb{Z}).$$

Bibliography

[Cno24] Bastiaan Cnossen. Stable homotopy theory and higher algebra (notes). Lecture notes / book project, 2024. In-progress; model-independent treatment via ∞-categories.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser, Basel, 1999.

[Gra76] Daniel R. Grayson. Higher algebraic k-theory. ii (after daniel quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, volume 551 of *Lecture Notes in Mathematics*, pages 217–240. Springer, Berlin, 1976.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[HH79] Jean-Claude Hausmann and Dale Husemoller. Acyclic maps. *L'Enseignement Mathématique*, 25(1–2):53–75, 1979.

[Kel90] Bernhard Keller. Chain complexes and stable categories. *Manuscripta Mathematica*, 67(4):379–417, 1990.

[Lan25] Marcus Land. Algebraic k-theory. `https://www.mathematik.uni-muenchen.de/~land/K-theory.pdf`, 2025. Lecture notes, SS25.

[Mic02] Atiyah Michael. Mathematics in the 20th century. *Bulletin of the London Mathematical Society*, 34(1):1–15, 2002.

[Nik17] Thomas Nikolaus. The group completion theorem via localizations of ring spectra. `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Group_completion.pdf`, 2017. Expository note.

[Qui76] Daniel Quillen. Projective modules over polynomial rings. *Inventiones mathematicae*, 36(1):167–171, 1976.

[Qui06] Daniel Quillen. Higher algebraic k-theory: I. In *Higher K-Theories: Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institute, from August 28 to September 8, 1972*, pages 85–147. Springer, 2006.

[Rap19] George Raptis. Some characterizations of acyclic maps. *Journal of Homotopy and Related Structures*, 14(3):773–785, 2019.

[Sta25] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2025.

[Tho80] R. W. Thomason. Cat as a closed model category. *Journal of Pure and Applied Algebra*, 17(2):129–143, 1980.

[Wei13] Charles A. Weibel. *The K-book: An Introduction to Algebraic K-theory*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013.