ALGEBRAIC GEOMETRY

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Abstract. These are unofficial lecture notes taken by Yiming Wang for the course *Algebraic Geometry* taught by Marc Hoyois during 2025–2026 at the University of Regensburg. They are intended for personal use; any errors are solely my own. Any comments and suggestions are welcome: you can contact me by email: solvaphes[at]gmail[dot]com.

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1.1. **Overview.**

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2.1. **affine spaces.** Affine goemetry studies the solutions in affine spaces to systems of polynomial equations, while projective geometry studies the solutions in projective spaces to systems of homogeneous polynomial equations. In both cases, the solutions form a functor $\mathrm{CAlg}_k \to \mathrm{Set}$ from the category of k-algebra to the category of sets. Such functors are the basic objects of algebraic geometry:

Definition 2.1 (Algebraic functor)**.** Let k be a ring. An *algebraic kfunctor* is a functor $\mathrm{CAlg}_k \to \mathrm{Set}$. An algebraic $\mathbf{Z}$-functor is simply called an *algebraic functor.* Given an algebraic k-functor X and a k-algebra R, the elements of $X(R)$ are called the R-valued points or R-points of X.

A basic example is given by affine spaces:

Definition 2.2 (Affine space)**.** Let I be a set. The *affine I-space* over k is the algebraic k-functor

$$\mathbf{A}^I_k\colon\ \mathrm{CAlg}_k \to \mathrm{Set}, \quad R \mapsto R^I$$

We simply write $\mathbf{A}^I$ when $k = \mathbf{Z}$. For $n \geq 0$, the *affine n-space* over k is $\mathbf{A}^n_k = \mathbf{A}^{\{1,\dots,n\}}_k$. It is also called the *affine line*, if $n = 1$ and the *affine plane* if $n = 2$.

Remark 2.3. (a) $\mathbf{A}^0_k$ is a final object $*$ of $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.
(b) $\mathbf{A}^1_k$ is isomorphic to the forgetful functor $\mathrm{CAlg}_k \to \mathrm{Set}$.
(c) $\mathbf{A}^I_k$ is contravariantly functorial in the set I: a map $f\colon I \to J$ induces a natural transformation $\mathbf{A}^I_k \to \mathbf{A}^J_k$ given by precomposition with f.
(d) By the universal property of polynomial rings, the functor $\mathbf{A}^I_k$ is represented by the polynomial k-algebra $k[x_i \mid i \in I]$, i.e., there is an isomorphism

$$\mathbf{A}^I_k \simeq \mathrm{Map}(k[x_i \mid i \in I], -)\colon\ \mathrm{CAlg}_k \to \mathrm{Set}\,.$$

Indeed, given an I-tuple $(r_i)_{i\in I} \in R^I$, there is a unique k-algebra map $k[x_i \mid i \in I] \to R$ sending x_i to r_i.

2.2. **Presheaves.** We start with some categorical preliminaries on set-valued functors, also known as *presheaves.*

Definition 2.4 (Presheaves)**.** Let $\mathcal{C}$ be a category. A *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. We denote by

$$\mathrm{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

the category of presheaves on $\mathcal{C}$. More generally, given an arbitrary category $\mathcal{E}$, an *$\mathcal{E}$-valued presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{E}$.

for example, an algebraic k-functor is exactly a presheaf on $\mathrm{CAlg}_k^{\mathrm{op}}$.

Remark 2.5. The category $\mathrm{P}(\mathcal{C})$ always admits limits and colimits, which are computed *pointwise* in the category of sets. Many properties of the category of sets are thereby inherited by the category of presheaves $\mathrm{P}(\mathcal{C})$, such as the fact that filtered colimits commutes with finite limits, the fact the monomorphism and epimorphisms are effective, etc.

Definition 2.6 (Yoneda embedding)**.** Let $\mathcal{C}$ be a category. The *Yoneda embedding* of $\mathcal{C}$ is the functor

$$よ\colon \mathcal{C} \to \mathrm{P}(\mathcal{C}), \quad よ(X) = \mathrm{Map}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}\,.$$

A presheaf on $\mathcal{C}$ is called *representable*, if it lies in the essential image of よ.

Notation 2.7 (The functor Spec)**.** When $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$, the Yoneda embedding is denoted by

$$\mathrm{Spec}\colon\ \mathrm{CAlg}_k^{\mathrm{op}} \to \mathrm{P}(\mathrm{CAlg}_k^{\mathrm{op}}) = \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}).$$

Thus $\mathrm{Spec}(A)$ is the algebraic k-functor represented by the k-algebra A:

$$\mathrm{Spec}(A)(R) = \mathrm{Map}(A, R).$$

For example, $\mathbf{A}^I_k \simeq \mathrm{Spec}(k[x_i \mid i \in I])$.

Definition 2.8 (Category of elements)**.** Let $\mathcal{C}$ be a category and let $F \in \mathrm{P}(\mathcal{C})$ be a presheaf on $\mathcal{C}$. The *category of elements* $\mathrm{El}(F)$ of F is defined by the cartesian square

$$\begin{tikzcd} \mathrm{El}(F) \arrow[r] \arrow[d] & (\mathrm{Set}_*)^{\mathrm{op}} \arrow[d, "\mathrm{forget}"] \\ \mathcal{C} \arrow[r, "F"] & \mathrm{Set}^{\mathrm{op}} \end{tikzcd}$$

It is also denoted by $\int F$. Explicitly, objects of $\mathrm{El}(F)$ are pairs (X, x) with $X \in \mathcal{C}$ and $x \in F(X)$, and morphisms $(X, x) \to (Y, y)$ are morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that $f^*(y) = x$.

Theorem 2.9 (Properties of Yoneda embedding)**.** *Let $\mathcal{C}$ be a small category.*

(a) (The Yoneda lemma) Let $X \in \mathcal{C}$ and $F \in \mathrm{P}(\mathcal{C})$. Then the map

$$\mathrm{Map}(よ(X), F) \to F(X), \quad f \mapsto f(\mathrm{id}_X),$$

is a bijection with inverse $x \mapsto ((f\colon Y \to X) \mapsto f^*(x))$.

(b) The Yoneda embedding $よ\colon \mathcal{C} \to \mathrm{P}(\mathcal{C})$ is fully faithful.

(c) The Yoneda embedding $よ\colon \mathcal{C} \to \mathrm{P}(\mathcal{C})$ preserves all limits that exists in $\mathcal{C}$.

(d) Every presheaf $F \in \mathrm{P}(\mathcal{C})$ is canonically a colimit of representable presheaves:

$$\mathrm{colim}(\mathrm{El}(F) \xrightarrow{\mathrm{forget}} \mathcal{C} \xrightarrow{よ} \mathrm{P}(\mathcal{C})) \xrightarrow{\sim} F.$$

(e) (Universal property of よ) Let $\mathcal{E}$ be a cocomplete category. Then the functor

$$よ^*\colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{P}(\mathcal{C},)\mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E}),$$

is an equivalence of categories, where $\mathrm{Fun}^{\mathrm{colim}}$ denotes the category of colimit-preserving functors; the inverse is given by left Kan extension along よ. In particular, any functor $\mathcal{C} \to \mathcal{E}$ extends uniquely (up to unique isomorphism) to a colimit-preserving functor $\mathrm{P}(\mathcal{C}) \to \mathcal{E}$.

(f) If $\mathcal{E}$ is any category, then any colimit-preserving functor $K\colon \mathrm{P}(\mathcal{C}) \to \mathcal{E}$ has a right adjoint $\mathcal{E} \to \mathrm{P}(\mathcal{C})$ given by $e \mapsto \mathrm{Map}(K(よ(-)), e)$.

Remark 2.10. (a) By the Yoneda lemma, the category of elements of a presheaf $F \in \mathrm{P}(\mathcal{C})$ can be equivalently be described as the pullback

$$\begin{tikzcd} \mathrm{El}(F) \arrow[r] \arrow[d] & \mathrm{P}(\mathcal{C})_{/F} \arrow[d, "\mathrm{forgetful}"] \\ \mathcal{C} \arrow[r, "よ"] & \mathrm{P}(\mathcal{C}), \end{tikzcd}$$

whose objects are pairs (X, x) with $X \in \mathcal{C}$ and $x\colon よ(X) \to F$. Theorem 2.9 then says that every presheaf is the colimit of all representable presheaves mapping to it.

(b) By the full faithfulness of the Yoneda embedding, we have $\mathrm{El}(よ(X)) \simeq \mathcal{C}_{/X}$. Together with Theorem 2.9, this shows that a presheaf is representable if and only if its category of elements has a final object (which is then the representing object).

Corollary 2.11. *Let $F, G \in \mathrm{P}(\mathcal{C})$ be presheaves. Then*

$$\mathrm{Map}(F, G) = \lim_{(X,x) \in \mathrm{El}(F)} G(X).$$

Example 2.12. By Theorem 2.9, the functor

$$\mathrm{Spec}\colon \mathrm{CAlg}_k^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$$

preserves all limits. Moreover, limits in $\mathrm{CAlg}_k^{\mathrm{op}}$ are colimits in CAlg_k. For example k is the initial object of CAlg_k, so that $\mathrm{Spec}(k)$ is the final object of $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$. The coproduct of two k-algebras A and B is the tensor product $A \otimes_k B$, so that

$$\mathrm{Spec}(A \otimes_k B) \simeq \mathrm{Spec}(A) \times \mathrm{Spec}(B)$$

in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$. More generally, the pushout of a diagram $A \leftarrow C \to B$ in CAlg_k is the relative tensor product $A \otimes_C B$, so that

$$\mathrm{Spec}(A \otimes_k B) \simeq \mathrm{Spec}(A) \times_{\mathrm{Spec}(C)} \mathrm{Spec}(B).$$

Remark 2.13 (Algebraic structures on presheaves). Algebraic objects like monoids, groups, abelian groups, rings, modules over a ring, etc. make sense in any category with finite products. In categories of presheaves, since finite products are computed objectwise, algebraic objects are the same as presheaves valued in the category of algebraic object of the same type in Set. For example, abelian group objects in presheaves are the same as presheaves of abelian groups:

$$\mathrm{Ab}(\mathrm{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}).$$

Thus, given a presheaf $F \in \mathrm{P}(\mathcal{C})$, equipping F with an abelian group structure is equivalent to lifting F along the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$:

$$\begin{tikzcd} & \mathrm{Ab} \arrow[d, "\text{forgetful}"] \\ \mathcal{C}^{\mathrm{op}} \arrow[r, "F"'] \arrow[ur, dashed] & \mathrm{Set} \end{tikzcd}$$

2.3. **Polynomial equations.**

Definition 2.14 (System of polynomial equations)**.** Let k be a ring and let I and J be sets. A *system of J polynomial equations in I variables* over k is a J-tuple $\Sigma = (f_j)_{j\in J}$ in the polynomial ring $k[x_i \mid i \in I]$. We denote by (Σ) the ideal in $k[x_i \mid i \in I]$ generated by $(f_j)_{j\in J}$ and by $k[\Sigma]$ the k-algebra $k[x_i \mid i \in I]/(\Sigma)$.

Remark 2.15. Every k-algebra R is isomorphic to $k[\Sigma]$ for some system of polynomial equations Σ. A choice of isomorphism $R \simeq k[\Sigma]$ is exactly a *presentation* of R by generators and relations.

Given a system of polynomial equations over k, we can consider its solutions in any k-algebra. To that end, recall that there is, for any k-algebra R, an *evaluation map*

$$k[x_i \mid i \in I] \times R^I \to R, \quad (f, a) \mapsto f(a),$$

which is defined as follows: for each $a \in R^I, f \mapsto f(a)$ is the unique k-algebra map $k[x_i \mid i \in I] \to R$ sending x_i to a_i.

Definition 2.16 (Vanishing locus)**.** Let $F \subseteq k[x_i \mid i \in I]$ be a subset. The *vanishing locus* of F in $\mathbf{A}^I_k$ is the subfunctor $\mathrm{V}(F) \subseteq \mathbf{A}^I_k$ given by

$$\mathrm{V}(F)(R) = \{a \in R^I \mid f(a) = 0 \text{ for all } f \in F\} \subseteq R^I.$$

This is indeed a subfunctor: for any k-algebra map $R \to S$, the induced map $R^I \to S^I$. sends $\mathrm{V}(F)(R)$ to $\mathrm{V}(F)(S)$.

Remark 2.17. It is clear that the vanishing locus of F depends only on the ideal generated by F: if $(F) = (F')$, then $\mathrm{V}(F) = \mathrm{V}(F')$. We will see below that the converse also holds (...)

Definition 2.18 (Solution functor)**.** Let $\Sigma = (f_j)_{j\in J}$ be system of J polynomial equations in I variables over k. Its *solution functor* $\mathrm{Sol}_\Sigma\colon \mathrm{CAlg}_k \to \mathrm{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbf{A}^I_k$:

$$\mathrm{Sol}_\Sigma = \mathrm{V}(\{f_j \mid j \in J\}) \subseteq \mathbf{A}^I_k.$$

By the universal property of polynomial rings, there is a one-to-one correspondence between systems of J polynomial equations in I variables and k-algebra maps

$$k[x_j \mid j \in J] \to k[x_i \mid i \in I].$$

By the Yoneda lemma, these are in turn equivalent to natural transformations

$$\mathbf{A}^I_k \to \mathbf{A}^J_k\colon\ \mathrm{CAlg}_k \to \mathrm{Set}\,.$$

Unraveling these equivalences, the map $\mathbf{A}^I_k \to \mathbf{A}^J_k$ corresponding to a system $\Sigma = (f_j)_{j\in J}$ is given on a k-algebra R by

$$R^I \to R^J, \quad a \mapsto (f_j(a))_{j\in J}.$$

By definition, the solution functor Sol_Σ is the kernel of this map, i.e., there is a pullback square

$$\begin{tikzcd} \mathrm{Sol}_\Sigma \arrow[r, hook] \arrow[d, "0"'] & \mathbf{A}^I_k \arrow[d] \\ 0 \arrow[r, hook] & \mathbf{A}^J_k, \end{tikzcd}$$

where 0 is the subfunctor of $\mathbf{A}^J_k$ given by $0(R) = \{0\} \subseteq R^J$.

Definition 2.19 (Affine scheme)**.** A functor $\mathrm{CAlg}_k \to \mathrm{Set}$ is called an *affine k-scheme*, if it is isomorphic to Sol_Σ for some system of polynomial equations Σ over k. We denote $\mathrm{Aff}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ the full subcategory spanned by the affine k-schemes. An *affine scheme* is an affine $\mathbf{Z}$-scheme.

Example 2.20. The affine I-space $\mathbf{A}^I_k$ is an affine k-scheme, as it is the solution functor of the empty system of equations in I variables.

Lemma 2.21. *Let Σ be a system of polynomial equations over k. Then the solution functor Sol_Σ is represented by the k-algebra $k[\Sigma]$, i.e., there is an isomorphism*

$$\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma])\colon\ \mathrm{CAlg}_k \to \mathrm{Set}\,.$$

Theorem 2.22 (Characterization of affine schemes)**.** *Let k be a ring. The following conditions are equivalent for an algebraic k-functor $X\colon\ \mathrm{CAlg}_k \to \mathrm{Set}$:*

- *(a) X is an affine k-scheme.*
- *(b) X is representable, i.e., isomorphic to $\mathrm{Spec}(A)$ for some k-algebra A.*
- *(c) X preserves limits and is accessible*[1].

Corollary 2.23. *The Yoneda embedding of $\mathrm{CAlg}_k^{\mathrm{op}}$ induces an equivalence of categories*

$$\mathrm{Spec}\colon\ \mathrm{CAlg}_k^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Aff}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}).$$

Under this equivalence, the affine k-scheme Sol_Σ corresponds to the k-algebra $k[\Sigma]$.

Under the equivalence of Corollary 2.23, the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbf{A}^I_k$ of affine k-schemes corresponds to the quotient map $k[x_i \mid i \in I] \twoheadrightarrow k[\Sigma]$. This implies the following result:

[1]Accessibility of X is a technical condition saying that X is a small colimit of representables. It is equivalent to the condition that X preserves κ-filtered colimits for some infinite cardinal κ, which is usally easy to check in pratice

Corollary 2.24 (Functorial Nullstellensatz)**.** *Sending a subset $F \subseteq k[x_i \mid i \in I]$ to its vanishing locus $\mathrm{V}(F) \subseteq \mathbf{A}_k^I$ induces an order reversing bijection*

$$\mathrm{V}\colon \{\textit{ideal in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbf{A}_k^I\}.$$

Example 2.25. Consider the following systems of polynomial equations over $\mathbf{R}$ in one variables:

$$\Sigma_1 = (x^2+1), \quad \Sigma_2 = ((x^2+1)^2), \quad \Sigma_3 = (x^2+x+1), \quad \Sigma_4 = (x^4+1).$$

Then

$$\mathrm{Sol}_{\Sigma_1}(\mathbf{R}) = \varnothing, \quad \mathrm{Sol}_{\Sigma_2}(\mathbf{R}) = \varnothing, \quad \mathrm{Sol}_{\Sigma_3}(\mathbf{R}) = \varnothing, \quad \mathrm{Sol}_{\Sigma_4}(\mathbf{R}) = \varnothing,$$

and

$$\mathrm{Sol}_{\Sigma_1}(\mathbf{C}) = \{\pm i\}, \quad \mathrm{Sol}_{\Sigma_2}(\mathbf{C}) = \{\pm i\}, \quad \mathrm{Sol}_{\Sigma_3}(\mathbf{C}) = \{\zeta_3, \bar{\zeta}_3\}, \quad \mathrm{Sol}_{\Sigma_4}(\mathbf{C}) = \{\pm\zeta_8, \pm\bar{\zeta}_8\},$$

when $\zeta_n = \exp(\frac{2\pi i}{n}) \in \mathbf{C}$. All four equations have the same solutions in $\mathbf{R}$. However, as the four ideals (Σ_i) in $\mathbf{R}[x]$ are pairwise distinct, they define four different subfunctors of $\mathbf{A}_{\mathbf{R}}^1$ by Corollary 2.24. The solutions in $\mathbf{C}$ distinguish them, except for Sol_{Σ_1} and Sol_{Σ_2}. To see that $\mathrm{Sol}_{\Sigma_1} \neq \mathrm{Sol}_{\Sigma_2}$ as subfunctors of $\mathbf{A}_{\mathbf{R}}^1$, we can compute the solutions in the $\mathbf{R}$-algebra $\mathbf{C}[\varepsilon]$ of dual complex numbers (where $\varepsilon^2 = 0$):

$$\mathrm{Sol}_{\Sigma_1}(\mathbf{C}[\varepsilon]) = \{\pm i\}, \quad \mathrm{Sol}_{\Sigma_2}(\mathbf{C}[\varepsilon]) = \{\pm i + a\varepsilon \mid a \in \mathbf{C}\}.$$

On the other hand, the associated $\mathbf{R}$-algebras are

$$\mathbf{R}[\Sigma_1] \simeq \mathbf{C}, \quad \mathbf{R}[\Sigma_2] \simeq \mathbf{C}[\varepsilon], \quad \mathbf{R}[\Sigma] \simeq \mathbf{C}, \quad \mathbf{R}[\Sigma_4] \simeq \mathbf{C} \times \mathbf{C}.$$

By Lemma 2.21, Sol_{Σ_1} and Sol_{Σ_3} are both isomorphic to $\mathrm{Spec}(\mathbf{C})$. The different ideals (Σ_1) and (Σ_3) correspond to two different embeddings of the affine $\mathbf{R}$-scheme $\mathrm{Spec}(\mathbf{C})$ into $\mathbf{A}_{\mathbf{R}}^1$, and the systems Σ_1 and Σ_3 themselves are two different presentations of the $\mathbf{R}$-algebra $\mathbf{C}$.

Remark 2.26. In summary, given a system of polynomial equations Σ over k, we have the following relations between Σ and Sol_Σ:

(a) The data of the pullback square below Definition 2.18 is equivalent to the data of Σ itself.
(b) The data of the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbf{A}_k^I$ is equivalent to the data of the ideal (Σ) in the polynomial ring $k[x_i \mid i \in I]$.
(c) The data of the affine k-scheme Sol_Σ alone is equivalent the k-algebra $k[\Sigma]$.

This can be compared with the following types of data in differential geometry:

(a) A smooth manifold M given as the vanishing locus of a smooth functioon $\mathbf{R}^n \to \mathbf{R}^m$.
(b) A smooth manifold M given as a closed submanifold of $\mathbf{R}^n$.
(c) A smooth manifold M.

Smooth manifolds are the basic objects of interest in differential geometry. Embedding a manifold M into a Euclidean space or realizing it as the vanishing locus of a function are often useful ways to understand M, but we do not consider this additional data to be part of the manifold M itself. The situation in algebraic geometry is entirely similar: the basic objects of interest are affine schemes. Any affine scheme X can be embedded into an affine space ($X \hookrightarrow \mathbf{A}^I$) or realized as the solution functor of a system of polynomial equations ($X \simeq \mathrm{Sol}_\Sigma$), but this data is not part of the affine scheme X itself.

A key different between differential geometry and algebraic geometry is that much easier to embed smooth manifolds into $\mathbf{R}^n$ than it is to embed schemes into $\mathbf{A}^n$. In fact, the former is always possible under mild technical assumptions[2] (which are usally taken as a part of the definition of smooth manifolds). but many interesting schemes are not affine. For example, the real projective space $\mathbf{P}^n(\mathbf{R})$ can be embedded in the Euclidean space $\mathbf{R}^{2n}$, but we will see that the algebraic projective space $\mathbf{P}^n$ with $n \geq 1$ cannot be embedded in $\mathbf{A}^N$ for any N.

[2]namely: Hausdorff, second countable, and of bounded dimension

Remark 2.27 (System of linear equations). Let us spell out the analogy with linear algebra. A system Λ of J linear equations in I variables over a ring k is J-indexed family in the free k-module $k^{(I)}$, or equivalently a k-linear map $k^{(J)} \to k^{(I)}$. If $(a_{ij})_{i\in I, j\in J}$ is the corresponding $I \times J$-matrix, a solution to Λ in a k-module M is a family $(m_i)_{i\in I}$ in M such that $\sum_{i\in I} a_{ij}m_i = 0$ for all $j \in J$. This defines a *solution functor*

$$\mathrm{Sol}_\Lambda \colon \mathrm{Mod}_k \to \mathrm{Set}\,.$$

Unraveling the definitions, $\mathrm{Sol}_\Lambda(M)$ is exactly the kernel of the map $M^I \to M^J$, obtained by appyling $\mathrm{Map}(-, M)$ to the given map $k^{(J)} \to k^{(I)}$. It follows that $\mathrm{Sol}_\Lambda \simeq \mathrm{Map}(C, -)$, where C is the cokernel of $k^{(J)} \to k^{(I)}$. Thus, we can think of a system of linear equations over k as a k-module C equipped with a presentation, and its solution functor as the k-module C itself.

2.4. **Example of affine scheme.**

Example 2.28 (The final scheme). The constant functor $\mathrm{CAlg} \to \mathrm{Set}$ sending every ring to a one-point set is isomorphic to $\mathrm{Spec}(\mathbf{Z})$ and hence is an affine3 scheme. This is the final object of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$.

Example 2.29 (The empty scheme). The functor

$$\mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \begin{cases} \varnothing & R \neq 0, \\ * & R = 0, \end{cases}$$

is an affine scheme, isomorphic to $\mathrm{Spec}(0)$. It is called the *scheme* and denoted by $\varnothing$. Note that $\varnothing$ is the initial object of Aff, but it is *not* the initial object of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, which is the constant functor with value $\varnothing$.

Example 2.30 (The idempotent classifier). Let $\mathrm{Idem} \colon \mathrm{CAlg} \to \mathrm{Set}$ be the functor sending R to the set of idempotent elements of R. Then Idem is an affine scheme, isomorphic to $\mathrm{Spec}(\mathbf{Z} \times \mathbf{Z})$. Indeed, there is a bijection

$$\mathrm{Map}_{\mathrm{CAlg}}(\mathbf{Z} \times \mathbf{Z}, R) \xrightarrow{\sim} \mathrm{Idem}(R), \quad \varphi \mapsto \varphi(1, 0),$$

which is natural in $R \in \mathrm{CAlg}$.

Example 2.31 (The multiplicative group). The functor $\mathbf{G}_m \colon \mathrm{CAlg} \to \mathrm{Ab}$ sending R to the group of units $R^\times$ is called the *multiplicative group*. It is an *affine group scheme*, meaning that the composition

$$\mathrm{CAlg} \xrightarrow{\mathbf{G}_m} \mathrm{Ab} \xrightarrow{\text{forget}} \mathrm{Set}$$

is an affine scheme. Indeed, it is isomorphic to $\mathrm{Spec}(\mathbf{Z}[u^{\pm 1}])$: for every ring R, there is a bijection

$$\mathrm{Map}_{\mathrm{CAlg}}(\mathbf{Z}[u^{\pm 1}], R) \xrightarrow{\sim} R^\times, \quad \varphi \to\mapsto \varphi(u).$$

Example 2.32 (The additive group). The functor $\mathbf{G}_a \colon \mathrm{CAlg} \to \mathrm{Ab}$ sending R to the underlying group $(R, +)$ is called the *additive group*. The composition

$$\mathrm{CAlg} \xrightarrow{\mathbf{G}_a} \mathrm{Ab} \xrightarrow{\text{forget}} \mathrm{Set}$$

is simply the forgetful functor, also known as the affine line $\mathbf{A}^1$. Hence $\mathbf{G}_a$ is an affine group scheme.

Example 2.33 (The matrix ring). Let $n \geq 0$ and let $\mathrm{Mat}_n \colon \mathrm{CAlg} \to \mathrm{Alg}$ be the functor sending R to the associated ring of $n \times n$ matrices over R. This is an associative ring object in affine schemes. Indeed, since a matrix over R is simply a family of n^2 elements of R, the composition

$$\mathrm{CAlg} \overset{\longrightarrow}{\mathrm{Mat}_n} \mathrm{Alg} \xrightarrow{\text{forget}} \mathrm{Set}$$

is isomorphic to $\mathbf{A}^{n^2}$ and hence is an affine scheme.

Example 2.34 (The general linear group). Let $n \geq 0$ and let $\mathrm{GL}_n\colon \mathrm{CAlg} \to \mathrm{Grp}$ be the functor sending R to the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices. Then GL_n is an ffine group scheme. Indeed, let $A = \mathbf{Z}[x_{ij} \mid (i,j) \in \{1,\dots,n\}^2]$ be the ring representing Mat_n, which contains the universal $n \times n$-matrix $X = (x_{ij})_{i,j}$. A matrix $M \in \mathrm{Mat}_{n\times n}(R)$ is invertible if and only if its determinant $\det(M) \in R$ is a unit. Hence, for any ring R, there is an isomorphism

$$\mathrm{Map}_{\mathrm{CAlg}}(A_{\det(X)}, R) \xrightarrow{\sim} \mathrm{GL}_n(R), \quad \varnothing \mapsto (\varphi(x_{ij}))_{i,j},$$

so that $\mathrm{GL}_n \simeq \mathrm{Spec}(A_{\det(X)})$.

Example 2.35 (The special linear group). Let $n \geq 0$ and let $\mathrm{SL}_n\colon \mathrm{CAlg} \to \mathrm{Grp}$ be the functor sending R to the special linear group $\mathrm{SL}_n(R)$ of $n \times n$ matrices with determinant 1. Let $X \in \mathrm{Mat}_n(A)$ be the universal $n \times n$ matrix as in Example 2.34. We then have

$$\mathrm{Map}_{\mathrm{CAlg}}(A/(\det(X)-1), R) \xrightarrow{\sim} \mathrm{SL}_n(R), \quad \varphi \mapsto (\varphi(x_{ij}))_{i,j},$$

so that $\mathrm{SL}_n \simeq \mathrm{Spec}(A/\det(X)-1)$. The subfunctor inclusions $\mathrm{SL}_n \subseteq \mathrm{GL}_n \subseteq \mathrm{Mat}_n$ corresponding to the ring maps

$$A \hookrightarrow A_{\det(X)} \twoheadrightarrow A/(\det(X)-1).$$

Example 2.36 (The affine space of a module). Let k be a ring and let M be a k-module. Consider the functor $\mathbf{A}(M)\colon \mathrm{CAlg}_k \to \mathrm{Mod}_k$ defined by

$$\mathbf{A}(M)(R) = \{k\text{-linear maps } M \to R\} = (M \otimes_k R)^\vee.$$

Then $\mathbf{A}(M)$ is a k-module object in affine k-schemes. Indeed, if $\mathrm{Sym}_k(M)$ is the free k-algebra on M, there is a bijection

$$\mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}_k(M), R) \xrightarrow{\sim} \mathbf{A}(M)(R), \quad \varphi \mapsto \varphi|_M,$$

so that $\mathbf{A}(M) \simeq \mathrm{Spec}(\mathrm{Sym}_k(M))$. The affine I-space is a special case of this construction: $\mathbf{A}^I_K \simeq \mathbf{A}(k^{(I)})$.

Remark 2.37. Let k be a ring and let M be a k-module. Given Example 2.36, it is tempting to consider the following *predual* of $\mathbf{A}(M)$: define $\mathbf{A}^\vee\colon \mathrm{CAlg}_k \to \mathrm{Mod}_k$ by

$$\mathbf{A}^\vee(M)(R) = M \otimes_k R.$$

There is a canonical map $\mathbf{A}^\vee(M^\vee) \to \mathbf{A}(M)$, which is an isomorphism if and only if M is a vector space (see Definition 3.3). Otherwise, $\mathbf{A}^\vee(M)$ does not preserve limits and hence is not an affine k-scheme (in fact, it is not even a scheme). For that reason, the functor $\mathbf{A}^\vee(M)$ is rarely used.

2.5. **Base change.** Given a ring map $\varphi\colon k \to k'$, we can transform any system of polynomial equations Σ over k into a system $\varphi^*(\Sigma)$ over k' by applying φ to all the coefficients. More generally, many types of data over k can be transformed into data over k' using φ, a process known as *base change*, *change of coefficients*, or *extension of scalars*. Other examples are the functor $\varphi^*\colon \mathrm{CAlg}_k \to \mathrm{CAlg}'_k$ sending a k-algebra A to the k'-algebra $A \otimes_k k'$. Unraveling these constructions, we see that there is a canonical isomorphism of k'-algebras

$$k'[\varphi^*(\Sigma)] \simeq \varphi^*(k[\Sigma]).$$

In this section, we investigate the related process of transforming an algebraic k-functor into an algebraic k'-functor.

Theorem 2.38 (Functoriality of presheaves)**.** *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let*

$$u^*\colon \mathrm{P}(\mathcal{D}) \to \mathrm{P}(\mathcal{C}), \quad F \mapsto F \circ u,$$

be the restriction along u *functor.*

(a) *The functor u^* admits a left adjoint $u_\sharp$ and a right adjoint u_* given by*

$$u_\sharp(F)(d) = \operatorname{colim}((\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}) = \operatorname*{colim}_{d \to u(c)} F(c),$$

$$u_*(F)(d) = \lim((\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}) = \lim_{u(c) \to d} F(c),$$

provided these colimits and limits exists (e.g., if $\mathcal{C}$ is small).

(b) *The functor $u_\sharp$ and u_* are fully faithful.*

(c) *If u is a localization, then u^* is fully faithful.*

(d) *If the functor u has a left adjoint u_L (resp. a right adjoint u_R), then there is a canonical isomorphism $u_\sharp \simeq u_L^*$ (resp. $u_* \simeq u_R^*$).*

Remark 2.39. (a) Given $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ the presheaves $u_\sharp(F)$ and $u_*(F)$ are special cases of *Kan extensions*: $u_\sharp(F)$ is the left Kan extension of F along u^{op}, and $u_*(F)$ is the right Kan extension of F along u^{op}.

(b) By the universal property of $よ_{\mathcal{C}}$, the functor $u_\sharp$ is the *unique* colimit preserving extension of u (up to unique isomorphism). However, there is no analogous characterization of u_*.

Corollary 2.40. *Let $\mathcal{C}$ be a category, let $Y \to X$ be a morphism in $\mathcal{C}$, and let $u \colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ be the forgetful functor.*

(a) *The functor $u^* \colon \mathrm{P}(\mathcal{C}_{/X}) \to \mathrm{P}(\mathcal{C}_{/Y})$ has a left adjoint $u_\sharp$ given by*

$$u_\sharp(F)(U \to X) = \coprod_{(U \to Y) \in \mathcal{C}_{/X}} F(U \to Y).$$

(b) *If pullback along $Y \to X$ exist in $\mathcal{C}$, the functor $u^* \colon \mathrm{P}(\mathcal{C}_{/X}) \to \mathrm{P}(\mathcal{C}_{/Y})$ has a right adjoint u_* given by*

$$u_*(F)(U \to X) = F(U \times_X Y \to Y).$$

If k is a ring, then $\mathrm{CAlg}_k \simeq \mathrm{CAlg}_{k/}$ and hence $\mathrm{CAlg}_k^{\mathrm{op}} \simeq (\mathrm{CAlg}^{\mathrm{op}})_{/k}$. Using the identification, we obtain the following special case of Corollary 2.40 with $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$:

Corollary 2.41. *Let $\varphi \colon k \to k'$ be a ring map. Then there is a triple of adjoint functors*

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{CAlg}_{k'}, \mathrm{Set}) \arrow[r, bend left, "\varphi_\sharp"] & \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}), \arrow[l, "\varphi^*"'] \arrow[l, bend left, "\varphi_*"] \end{tikzcd}$$

where:

- *φ^* is precomposition with the forgetful functor $\mathrm{CAlg}_{k'} \to \mathrm{CAlg}_k$, and it is the unique colimit preserving extension of $\varphi^* \colon \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{CAlg}_{k'}^{\mathrm{op}}$;*
- *φ_* is precomposition with $\varphi^* \colon \mathrm{CAlg}_k \to \mathrm{CAlg}_{k'}$;*
- *$\varphi_\sharp$ is given by*

$$\varphi_\sharp(X)(A) = \coprod_{k' \to A} X(A),$$

and it is the unique colimit-preserving extension of the forgetful functor $\mathrm{CAlg}_{k'}^{\mathrm{op}} \to \mathrm{CAlg}_k^{\mathrm{op}}$.

Definition 2.42. Let $\varphi \colon k \to k'$ be a ring map, giving rise to the adjoint triple of Corollary 2.41.

(a) The functor φ^* is called the *base change* or *extension of scalars* along φ and is also denoted by $X \mapsto X_{k'}$.

(b) The functor φ_* is also called *Weil restriction* or *restriction of scalars* along φ and is also denoted by R_φ or $\mathrm{R}_{k'/k}$.

Remark 2.43. Corollary 2.41 says in particular that the functors φ^* and $\varphi_\sharp$ preserve affine schemes:

(a) For a k-algebra A, $\varphi^*(\operatorname{Spec}(A)) \simeq \operatorname{Spec}(A \otimes_k k')$ in $\operatorname{Fun}(\mathrm{CAlg}_{k'}, \mathrm{Set})$. Hence, for any system of polynomial equations Σ over k, $\varphi^*(\mathrm{Sol}_\Sigma) \simeq \mathrm{Sol}_{\varphi^*(\Sigma)}$.

On the other hand, the functor φ_* does not always preserve affine schemes.

Example 2.44. Consider the affine scheme $\mathbf{G}_m \colon \mathrm{CAlg} \to \mathrm{Set}, R \mapsto R^\times$. Let $\mathbf{G}_{\mathrm{m},\mathbf{C}} \colon \mathrm{CAlg}_{\mathbf{C}} \to \mathrm{Set}$ be its base change to $\mathbf{C}$, i.e., its restriction along the forgetful functor $\mathrm{CAlg}_{\mathbf{C}} \to \mathrm{CAlg}$. The Weil restriction $\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{m,\mathbf{C}}) \colon \mathrm{CAlg}_{\mathbf{R}} \to \mathrm{Set}$ is given by

$$\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{m,\mathbf{C}})(A) = \mathbf{G}_{m,\mathbf{C}}(A \otimes_{\mathbf{R}} \mathbf{C}) = (A \otimes_{\mathbf{R}} \mathbf{C})^\times.$$

One can check that $\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{m,\mathbf{C}})$ is an affine $\mathbf{R}$-scheme: for any $\mathbf{R}$-algebra A, there is a bijection

$$\operatorname{Map}(\mathbf{R}[x,y,z,w]/(xz - yw - 1, yz + xw), A) \xrightarrow{\sim} (A \otimes_{\mathbf{R}} \mathbf{C})^\times, \quad \varphi \mapsto \varphi(x) + i\varphi(y).$$

These equations ensure that $\varphi(x)+i\varphi(y)$ is inverse to $\varphi(z)+i\varphi(w)$. More generally, if $k \subseteq k'$ is finite field extension, one can show that Weil restriction $\mathrm{R}_{k'/k}$ sends affine k'-schemes to affine k-schemes.

A fundamental property of sets is that maps of sets are equivalent to family of sets: there is an equivalence of categories

$$\mathrm{Ar}(\mathrm{Set}) \simeq \mathrm{Fam}(\mathrm{Set}),$$

where $\mathrm{Ar}(\mathrm{Set}) = \operatorname{Fun}([1], \mathrm{Set})$ is the arrow category of Set and Fam(Set) is the category whose objects are (set-indexed) families of sets $(X_i)_{i \in I}$, where a map $(X_i)_{i\in I} \to (Y_j)_{j \in J}$ consists of a map $u \colon I \to J$ and maps $X_i \to Y_{u(i)}$ for all $i \in I$. In one direction, a map $f \colon X \to I$ corresponds to the family of its fibers $(f^{-1}\{i\})_{i \in I}$. In the other direction, a family $(X_i)_{i\in I}$ corresponds to the map $\coprod_{i \in I} X_i \to I$. If we fix the target/indexing set I, we obtain an equivalence categories

$$\mathrm{Set}_{/I} \simeq \mathrm{Set}^I\,.$$

The following proposition generalizes this fact to presheaves of sets (we recover the last equivalence by taking $\mathcal{C} = *$):

Proposition 2.45 (Slices of presheaf categories)**.** *Let $\mathcal{C}$ be a category and $F \in \mathrm{P}(\mathcal{C})$ be a presheaf on $\mathcal{C}$. Then there is an equivalence of categories*

$$\mathrm{fib}_F : \mathrm{P}(\mathcal{C})_{/F} \simeq \mathrm{P}(\mathrm{El}(F)) : \coprod_F,$$

described as follows. If $u \colon \mathrm{El}(F) \to \mathcal{C}$ is the forgetful functor, there is a tautological map $* \to u^*(F)$, whose adjoint $u_\varphi(*) \to F$ is an isomorphism.

- *Given $H \in \mathrm{P}(\mathrm{El}(F))$, the presheaf $\coprod_F H$ over F is $u_\sharp(H) \to u_\sharp(*) \simeq F$. Explicitly,*

$$\left(\coprod_F H\right)(X) = \coprod_{x \in F(X)} (H(X,x) \to *).$$

- *Given $G \in \mathrm{P}(\mathcal{C})_{/F}$, the presheaf $\mathrm{fib}_F(G)$ on $\mathrm{El}(F)$ is the pullback $u^*(G) \times_{u^*(F)} *$. Explicitly,*

$$\mathrm{fib}_F(G)(X,x) = G(X) \times_{F(X)} \{x\} \simeq \left\{ \begin{tikzcd} & G \arrow[d] \\ \text{よ}(X) \arrow[ur, dashed] \arrow[r, "x"] & F \end{tikzcd} \right\}.$$

Specializing to the case of a representable presheaf, we get:

Corollary 2.46. *Let $\mathcal{C}$ be a category, let $X \in \mathcal{C}$, and let $u \colon \mathcal{C}_{/X} \to \mathcal{C}$ be the forgetful functor. Then the functor $u_\sharp$ induces an equivalence of categories*

$$\mathrm{P}(\mathcal{C}_{/X}) \xrightarrow{\sim} \mathrm{P}(\mathcal{C})_{/\text{よ}(X)}.$$

Specializing further to $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$, we get the following key result:

Corollary 2.47. *Let k be a ring and let $\varphi\colon \mathbf{Z} \to k$ be the unique map. Then the functor $\varphi_\sharp$ induces an equivalence of categories*

$$\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/\,\mathrm{Spec}(k)}.$$

Remark 2.48. Because of Corollary 2.47, algebraic geometry over a base ring k is subsumed by algebraic geometry over $\mathbf{Z}$. In other words, working in the category $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$ of algebraic functors does not restrict the generality, and we will often do so from now on. Note that the functor Spec of Notation 2.7 is independent of k, in the sense that the following square commutes:

$$\begin{tikzcd} \mathrm{CAlg}_k^{\mathrm{op}} \arrow[r, "\sim"] \arrow[d, "\mathrm{Spec}(k)"'] & (\mathrm{CAlg}^{\mathrm{op}})_{/k} \arrow[d, "\mathrm{Spec}(k)"] \\ \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}) \arrow[r, "\sim"] & \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{\mathrm{Spec}(k)}. \end{tikzcd}$$

Remark 2.49. If $f\colon X' \to X$ is a morphism or algebraic functors, there is a triple of adjoint functors

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/X'} \arrow[r, bend right, "f_*"'] & \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/X}, \arrow[l, "f^*"'] \arrow[l, bend right, "f_\sharp"'] \end{tikzcd}$$

where $f^*(Y) = Y \times_X X'$ and $f_\sharp(Y') = Y'$. Given Proposition 2.45, this can be seen by applying Theorem 2.38 to the functor $u\colon \mathrm{El}(X') \to \mathrm{El}(X)$. This recovers Corollary 2.41 when f is $\mathrm{Spec}(\varphi)\colon \mathrm{Spec}(k') \to \mathrm{Spec}(k)$.

2.6. Functions.

Recall that the affine line $\mathbf{A}^1$ is the forgetful functor

$$\mathbf{A}^1\colon \mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto R.$$

tautologically, $\mathbf{A}^1$ has a struction of ring object in $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, given by the factorization

$$\begin{tikzcd} & \mathrm{CAlg} \arrow[d, "\mathrm{forget}"] \\ \mathrm{CAlg} \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, "\mathbf{A}^1"'] & \mathrm{Set} \end{tikzcd}$$

Recall also that $\mathbf{G}_m$ is the subfunctor of $\mathbf{A}^1$ given by $R \mapsto R^\times$, which has the structure of abelian group (see Example 2.32).

Definition 2.50 (Function and nonvanishing function)**.** Let X be an algebraic functor.

(a) A *function* on X is a map $X \to \mathbf{A}^1$. We denote by

$$\mathcal{O}(X) = \mathrm{Map}(X, \mathbf{A}^1)$$

the set of functions on X. The ring structure on $\mathbf{A}^1$ induces a ring structure on $\mathcal{O}(X)$.

(b) An *nonvanishing function* on X is a map $X \to \mathbf{G}_m$. We denote by

$$\mathcal{O}^\times(X) = \mathrm{Map}(X, \mathbf{G}_m)$$

the set of nonvanishing functions on X. The abelian group structure on $\mathbf{G}_m$ induces an abelian group structure on $\mathcal{O}^\times(X)$.

Remark 2.51. (a) Since $\mathbf{G}_m \subseteq \mathbf{A}^1$, we have $\mathcal{O}^\times(X) \subseteq \mathcal{O}(X)$.

(b) By the Yoneda lemma, there is a canonical isomorphism $\mathcal{O}(\mathrm{Spec}(R)) \simeq R$, i.e., the ring of functions on $\mathrm{Spec}(R)$ is R itself. Hence, when restricted to affine schemes, the functor $\mathcal{O}\colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{CAlg}$ is an equivalence of categories, which is inverse to Spec. Similarly, $\mathcal{O}^\times(\mathrm{Spec}(R)) \simeq R^\times$.

(c) For a general algebraic functor X, we have

$$\mathcal{O}(X) \simeq \lim_{x\colon \operatorname{Spec}(R)\to X} R, \quad f \mapsto (f\circ x)_x,$$

where the limit is indexed by the category of elements $\mathrm{El}(X)^{\mathrm{op}}$ (see Corollary 2.11). Since the unit group functor $R \mapsto R^\times$ preservess limits, it follows that $\mathcal{O}^\times$ is precisely the unit group $\mathcal{O}(X)^\times$.

(d) By definition, the functor

$$\mathcal{O}\colon \operatorname{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{CAlg}, \quad X \mapsto \mathcal{O}(X),$$

is limit-preserving. By (b) and the universal property of presheaves, it is the unique-limit preserving extension of the functor $\mathrm{CAlg} \to \mathrm{Ab}, R \mapsto R^\times$.

Remark 2.52 (Size issues). The statement (d) of Remark 2.51 is actually nonsensical due to *size issues*. Since the category CAlg is large, the limit in the formula for $\mathcal{O}(X)$ is not an object of CAlg, which is the category of small rings. There are two standard ways to rectify this issue:

(a) One can simply replace the target of $\mathcal{O}$ by the category $\widehat{\mathrm{CAlg}}$ of large rings. One may then also replace the category of sets in the source by the category $\widehat{Set}$ of large sets. We obtain the functor

$$\mathcal{O}\colon \operatorname{Fun}(\mathrm{CAlg}, \widehat{\mathrm{Set}})^{\mathrm{op}} \to \widehat{\mathrm{CAlg}},$$

which is the unique extension of the embedding $\mathrm{CAlg} \hookrightarrow \widehat{\mathrm{CAlg}}$ that preserves *large* limits.

(b) One can replace the source of $\mathcal{O}$ by the subcategory $\operatorname{Fun}^{\mathrm{acc}}(\mathrm{CAlg}, \mathrm{Set})$ of *accessible* functors, which are the functors that are *small* colimits of representables. For an accessible functor X, $\mathcal{O}(X)$ is a small limit of small rings and hence is small. We therefore have a functor

$$\mathcal{O}\colon \operatorname{Fun}^{\mathrm{acc}}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{CAlg}$$

which is the unique extension of the identity $\mathrm{CAlg} \to \mathrm{CAlg}$ that preserves *small* limits. All algebraic functors that arise in practice (including all schemes) are accessible, and all relevant constructions preserve accessibility, so that this restriction does not have any undesirable consequences.

Remark 2.53 (The adjunction between Spec and $\mathcal{O}$). Let X be an algebraic functor and A a ring. By Corollary 2.11 and the Yoneda lemma, we have a sequence of natural isomorphisms

$$\begin{aligned}
\operatorname{Map}(X, \operatorname{Spec}(A)) &\simeq \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \operatorname{Map}(\operatorname{Spec}(R), \operatorname{Spec}(A)) \\
&\simeq \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \operatorname{Map}(A, R) \\
&\simeq \operatorname{Map}(A, \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} R) \\
&\simeq \operatorname{Map}(A, \mathcal{O}(X)).
\end{aligned}$$

Thus, modulo the size issues discussed in Remark 2.52, we have an adjunction

$$\operatorname{Fun}(\mathrm{CAlg}, \mathrm{Set}) \underset{\operatorname{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathrm{CAlg}^{\mathrm{op}}.$$

2.7. **Closed and open subfunctor.** Roughly speaking, a *closed* subfunctor of an algebraic functor is a subfunctor defined by the vanishing of functions, while an *open* subfunctor is one defined by the nonvanishing of functions. This terminology is borrowed from topology, where the vanishing locus of a continuous function is closed and its nonvanishing locus is open. We will see later that there is in fact a topological interpretation of open subfunctors, though not of closed subfunctors.

Definition 2.54 (Vanishing and nonvanishing loci)**.** Let X be an algebraic functor and $F \subseteq \mathcal{O}(X)$ a set of functions on X.

(a) The *vanishing locus* of F is the subfucntor $\mathrm{V}(F) \subseteq X$ given by

$$\mathrm{V}(F)(R) = \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in F\}.$$

(b) The *nonvanishing locus* of F is the subfunctor of $\mathrm{D}(F) \subseteq X$ given by

$$\mathrm{D}(F)(R) = \{x \in X(R) \mid (f(x))_{f \in F} \text{generates the unit ideal in } R\}.$$

Remark 2.55. (a) It is clear that $\mathrm{V}(F)$ depends on the ideal (F) generated by F, and $\mathrm{D}(F)$ only on the *radical ideal* $\sqrt{(F)}$ generated by F, since an ideal is the unit ideal if and only if its radical is.

(b) We have the following implications:

$$(F) = \mathcal{O}(X) \quad \Longrightarrow \quad \mathrm{V}(F) = \varnothing \quad \Leftrightarrow \quad \mathrm{D}(F) = X.$$

Here $\varnothing_X$ is the functor with $\varnothing_X(0) = X(0)$ and $\varnothing_X(R) = \varnothing$ for $R \neq 0$. The reverse implication holds if X is affine, by Proposition 2.65 below.

Example 2.56 (Punctured affine spaces). Let I be a set. The *punctured affine I-space* $\mathbf{A}^I \setminus \{0\}$ is the nonvanishing locus of the coordinate functions $\{x_i \mid i \in I\}$ on $\mathbf{A}^I = \mathrm{Spec}(\mathbf{Z}[x_i \mid i \in I])$. Explicitly:

$$(\mathbf{A}^I \setminus \{0\})(R) = \{a \in R^I \mid (a) = R\}.$$

Note that $\mathbf{A}^1 \setminus \{0\}$ is another name for the subfunctor $\mathbf{G}_m \subseteq \mathbf{A}^1$. An I-tuple in R generating the unit ideal is also called a *unimodular row* of length I.

Remark 2.57. A set of functions $F \subseteq \mathcal{O}(X)$ induces a map $f\colon X \to \mathbf{A}^F$. By definition, we have

$$\mathrm{V}(F) = f^{-1}(0) \quad \text{and} \quad \mathrm{D}(F) = f^{-1}(\mathbf{A}^F \setminus \{0\}).$$

Warning 2.58. The terminology suggests that $\mathrm{D}(F)$ should in some sense3 be the complement of $\mathrm{V}(F)$ in X. This is true when evaluated on fields, but it is even true in the category of algebraic functors. In fact, they are not even disjoint, since $\mathrm{V}(F)(0) = \mathrm{D}(F)(0) = X(0)$. We will see later that $\mathrm{D}(F)$ is the complement of $\mathrm{V}(F)$ (i.e., the largest disjoint suboject) in various subcategories of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, such as the category of schemes. Even then, the converse fails: $\mathrm{V}(F)$ cannot be the complement $\mathrm{D}(F)$ in general, since it can happen that $\mathrm{V}(F) \neq \mathrm{V}(F')$ while $\mathrm{D}(F) = \mathrm{D}(F')$.

Proposition 2.59 (Formal properties of V and D). *Let X be an algebraic functor.*

(a) For any family $(F_i)_{i \in I}$ of subsets of $\mathcal{O}(X)$,

$$\bigcap_{i \in I} \mathrm{V}(F_i) = \mathrm{V}(\bigcup_{i \in I} F_i) \quad \textit{and} \quad \bigcup_{i \in I} \mathrm{D}(F_i) \subseteq \mathrm{D}(\bigcup_{i \in I} F_i),$$

and the inclusion is an equality on local rings.

(b) For any finite family $F_1, \ldots, F_n$ of subsets of $\mathcal{O}(X)$, we have

$$\mathrm{D}(F_1) \cap \ldots \cap \mathrm{D}(F_n) = \mathrm{D}(F_1 \cdot \ldots \cdot F_n) \quad \textit{and} \quad \mathrm{V}(F_1) \cup \ldots \cup \mathrm{V}(F_n) \subseteq \mathrm{V}(F_1 \cdot \ldots \cdot F_n),$$

and the inclusion is an equality on integral domains.

Example 2.60. Since 2 and 3 generate the unit ideal in $\mathbf{Z}$, we have $\mathrm{V}(2) \cap \mathrm{V}(3) = \mathrm{V}(1) = \varnothing$ as subfunctors of $\mathrm{Spec}(\mathbf{Z})$ (where $\varnothing$ is the scheme of Example 2.29). On the other hand, $\mathrm{D}(2) \cup \mathrm{D}(3) \neq \mathrm{D}(1) = \mathrm{Spec}(\mathbf{Z})$. For example, $\mathrm{id}_{\mathbf{Z}} \in \mathrm{Spec}(\mathbf{Z})(\mathbf{Z})$ belongs neither to $\mathrm{D}(2)(\mathbf{Z}) = \varnothing$ nor to $\mathrm{D}(3)(\mathbf{Z}) = \varnothing$.

Proposition 2.61 (Affineness of vanishing and nonvanishing loci). *Let A be a ring.*

(a) For any subset $F \subset A$, the quotient map $A \twoheadrightarrow A/(F)$ induces an isomorphism $\mathrm{Spec}(A/(F)) \xrightarrow{\sim} \mathrm{V}(F) \subseteq \mathrm{Spec}(A)$. In particular, $\mathrm{V}(F)$ is affine.

(b) For any $f \in A$, the localization map $A \to A_f$ induces an isomorphism $\mathrm{Spec}(A_f) \xrightarrow{\sim} \mathrm{D}(f) \subseteq \mathrm{Spec}(A)$. In particular, $\mathrm{D}(f)$ is affine.

If $F \subseteq A$ has more than one element, $\mathrm{D}(F) \subseteq \mathrm{Spec}(A)$ is usually not an affine scheme. For example, $\mathbf{A}^n \setminus \{0\}$ is not affine for $n \geq 2$ (see Example 2.75). This motivates the following definition:

Definition 2.62 (Quasi-affine scheme)**.** Let k be a ring. An algebraic k-functor X is a *quasi-affine k-scheme*, if there exists a k-algebra A and a *finite* subset $F \subseteq A$ such that $X \simeq \mathrm{D}(F) \subseteq \mathrm{Spec}(A)$. We denote $\mathrm{QAff}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$ the full subcategory of quasi-affine k-schemes. A *quasi-affine scheme* is a quasi-affine $\mathbf{Z}$-scheme.

Definition 2.63 (Closed and open subfunctors)**.** Let X be an algebraic fucntor.

(a) A subfunctor $Z \subseteq X$ is *closed*, if for every $x\colon \mathrm{Spec}(R) \to X$, we have $x^{-1}(Z) = \mathrm{V}(F)$ for some $F \subseteq R$.
(b) A subfunctor $U \subseteq X$ is *open*, if for every $x\colon \mathrm{Spec}(R)$, we have $x^{-1}(U) = \mathrm{D}(F)$ for some $F \subseteq R$.

Warning 2.64. Vanishing loci are always closed subfunctors and nonvanishing loci are always open subfunctors, but the converse does not hold.

The following result generalizes the functorial Nullstellensatz (see Corollary 2.24):

Proposition 2.65 (Classification of closed and open subfunctors of affine schemes)**.** *Let A be a ring.*

(a) The construction $F \mapsto \mathrm{V}(F)$ induces an order-reversing bijection
$$\{\textit{ideal in } A\} \xrightarrow{\sim} \{\textit{closed subfunctors of } \mathrm{Spec}(A)\}.$$
(b) The construction $F \mapsto \mathrm{D}(F)$ induces an order-preserving bijection
$$\{\textit{radical ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \mathrm{Spec}(A)\}.$$

Definition 2.66 (Closed and open immersions)**.** Let $f\colon Y \to X$ be a map of algebraic functors.

(a) f is a *closed immersion* or *closed embedding*, if it is a monomorphism, whose image ia a closed subfunctor of X.
(b) f is an *open immersion* or *open embedding*, if it is a monomorphism, whose image is an open subfunctor of X.

Definition 2.67 (Locally closed subfunctor, immersion)**.** Let X be an algebraic functor.

(a) A subfunctor $Y \subseteq X$ is *locally closed*, if there exists an open subfunctor $U \subseteq X$ containing Y as closed subfunctor.
(b) A map $f\colon Y \to X$ is an *immersion* if it is a monomorphism, whose image is a locally closed subfunctor of X.

Proposition 2.68 (Closure properties of immersions)**.** *(a) Consider a commutative triangle of algebraic functors*
$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "h"'] & Y \arrow[dl, "f"] \\ X & \end{tikzcd}$$
If f and g are closed immersions, so is h. If h is a closed immersion and f is a monomorphism, then g is a closed immersion. The same holds for open immersions and for immersions.

(b) Consider a cartesian square of algebraic functors
$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$
If f is a closed immersion, so is f'. The same holds for open immersion and for immersions.

Proposition 2.69. *Let X be a quasi-affine k-scheme and let $Z \hookrightarrow X$ be a closed immersion. Then Z is a quasi-affine k-scheme.*

2.8. **Zariski descent.** Let $(f_i)_{i\in I}$ be a family of element in a ring R that generates the unit ideal. Then the intersection of the vanishing loci $\mathrm{V}(f_i)$ is the empty scheme, but in general, it is not true that $\mathrm{Spec}(R)$ is the union of the nonvanishing loci $\mathrm{D}(f_i)$ (see Example 2.60). In this section, we will show that this becomes true if we compute the union in the category of affine schemes. Concretely, this means that a map $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is uniquely determined by a family of maps $\mathrm{D}(f_i) \to \mathrm{Spec}(S)$ that agrees on all the intersections $\mathrm{D}(f_i) \cap \mathrm{D}(f_j)$. This is one of the most important result in the foundations of algebraic geometry, which we will later recast as the statement that *affine schemes satisfy Zariski descent.*

Theorem 2.70 (Zariski descent for modules)**.** *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R generating the unit ideal.*

(a) (Descent for morphisms) For any R-modules M and N, the diagram

$$\mathrm{Map}(N, M) \to \prod_{i\in I} \mathrm{Map}(N_{f_i}, M_{f_i}) \rightrightarrows \prod_{i,j\in I} \mathrm{Map}(N_{f_if_j}, M_{f_if_j})$$

is an equalizer.

(b) (Descent for objects) Suppose given

- *an R_{f_i}-module M_i for each $i \in I$ and*
- *an $R_{f_if_j}$-linear isomorphism $\alpha_{ij} \colon (M_i)_{f_j} \xrightarrow{\sim} (M_j)_{f_j}$ for each $(i, j) \in I^2$,*
- *such that $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik} \colon (M_i)_{f_jf_k} \xrightarrow{\sim} (M_k)_{f_if_j}$ for each $(i, j, k) \in I^3$.*

Then there exists an R-module M with R_{f_i}-linear isomorphisms $\beta_i \colon M_{f_if_j} \xrightarrow{\sim} (M_j)_{f_i}$ for all $(i, j) \in I^2$. Moreover, this data is unique up to unique isomorphism.

Taking $N = R$ in Theorem 2.70 (a), we get the following special case:

Corollary 2.71. *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R generating the unit ideal. For any R-module M, the diagram*

$$M \to \prod_{i\in I} M_{f_i} \rightrightarrows \prod_{i,j\in I} M_{f_if_j}$$

is an equalizer in Mod_R.

Specializing further to $M = R$, we get the following result:

Corollary 2.72 (Zariski descent for affine schemes)**.** *Let R be a ring and let $(f_i)_{i\in I}$ be a family of elements of R generating the unit ideal. For any affine scheme X, the diagram*

$$X(R) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_if_j})$$

is an equalizer.

Remark 2.73. Since $X(R) \simeq \mathrm{Map}(\mathrm{Spec}(R), X)$, $\mathrm{Spec}(R_{f_i}) \simeq \mathrm{D}(f_i)$ and $\mathrm{Spec}(R_{f_if_j}) \simeq \mathrm{D}(f_i) \cap \mathrm{D}(f_j)$, Corollary 2.72 says that $\mathrm{Spec}(R)$ is the union of the open subschemes $\mathrm{D}(f_i)$ in Aff. More precisely, if $\mathrm{Glue}(I)$ is the poset with morphisms $i \leftarrow (i, j) \to j$ for all $i, j \in I$, then $\mathrm{Spec}(R)$ is the colimit of the diagram $\mathrm{Glue}(I) \to \mathrm{Aff}$ sending i to $\mathrm{D}(f_i)$ and (i, j) to $\mathrm{D}(f_i) \cap \mathrm{D}(f_j)$.

The following result is a generalization of Corollary 2.72, of which it is in fact a formal consequence:

Corollary 2.74 (Functions on nonvanishing loci)**.** *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R with image $F \subseteq R$. For any affine scheme X, there is an equalizer diagram*

$$\mathrm{Map}(\mathrm{D}(F), X) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_if_j}).$$

Example 2.75. Using Corollary 2.74 with $X = \mathbf{A}^1$, we can easily compute that the inclusion $\mathbf{A}^I \setminus \{0\} \hookrightarrow \mathbf{A}^I$ induces an isomorphism $\mathcal{O}(\mathbf{A}^I) \xrightarrow{\sim} \mathcal{O}(\mathbf{A}^I \setminus \{0\})$ as soon as $|I| \geq 2$. Since $\mathcal{O}\colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{CAlg}$ is an equivalence of categories, this implies that $\mathbf{A}^I \setminus \{0\}$ is not an affine scheme. In particular, if $n \geq 2$, $\mathbf{A}^n \setminus \{0\}$ is an example of a quasi-affine scheme that is not affine.

Definition 2.76 (Zariski-local property)**.** A property P of modules (resp. oif linear maps, of algebras, etc.) is *Zariski-local*, if for any ring R and family $(f_i)_{i\in I}$ generating the unit ideal in R, and R-module M (resp. an R-linear map $M \to N$, an R-algebra A, etc.) has property if and only if, for each $i \in I$, the R_{f_i}-module M_{f_i} (resp. the R_{f_i}-linear map $M_{f_i} \to N_{f_i}$, the R_{f_i}-algebra A_{f_i}, etc.) has property P.

Proposition 2.77 (Examples of Zariski-local properties)**.** *The following properties of modules are Zariski-local:*

(a) being zero,
(b) finite generation,
(c) finite presentation,
(d) projectivity,
(e) flatness.

The following properties of linear maps are Zariski-local:

(a) being zero,
(b) injectivity,
(c) surjectivity,
(d) bijectivity.

The following properties of algebras are Zariski-local:

(a) finite generation,
(b) finite presentation.

The following properties of sequence of modules are Zariski local:

(a) exactness.

Remark 2.78. Further Zariski-local properties of modules are: being torsion, torsion-freeness. The following properties of modules are not Zariski-local: freeness, injectivity.

Proposition 2.79 (Zariski-local nature of immersion)**.** *Let R be a ring and let $u\colon X \to \mathrm{Spec}(R)$ be a map of algebraic functors. Suppose that X satisfies Zariski descent in the sense of Corollary 2.72. Let $(f_i)_{i\in I}$ generate the unit ideal in R, and let $u_i\colon \times_{\mathrm{Spec}(R)} \mathrm{D}(f_i) \to \mathrm{D}(f_i)$ be the base change of u to $\mathrm{D}(f_i)$. For each of the following classes of maps, if each u_i belongs to the class, so does u:*

(a) monomorphisms,
(b) closed immersions,
(c) open immersions,
(d) immersions.

2.9. **Finiteness properties.** Recall that a k-algebra is of *finite presentation*, if it is isomorphic to $k[\Sigma]$, where Σ is a system of finitely many polynomial equations in finitely many variables, and it is *of finite type*, if it is isomorphic to $k[\Sigma]$, where Σ has finitely many variables (but any number of equations). We denote the respective full subcategories of CAlg_k by $\mathrm{CAlg}_k^{\mathrm{fp}}$ and $\mathrm{CAlg}_k^{\mathrm{ft}}$. It turns outs that these finiteness conditions an naturally be expressed in terms of the algebraic k-functor $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$.

Definition 2.80 (Locally of finite presentation/type)**.** Let $X\colon \mathrm{CAlg}_k \to \mathrm{Set}$ be an algebraic k-functor.

(a) X is *locally of finite presentation*, if it preserves filtered colimits.

(b) X is *locally of finite type*, it is preserves the colimits of filtered diagrams with injective transition maps.

Proposition 2.81. *Let k be a ring and A a k-algebra.*

(a) A is of finite presentation if and only if $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ *is locally of finite presentation.*
(b) A is of finite type if and only if $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ *is locally of finite type.*

Example 2.82. (a) A^I_k is locally of finite type if and only if I is a finite set, in which case it is also locally of finite presentation. The same holds for the punctured affine spaces $\mathbf{A}^I_k \setminus \{0\}$.
(b) The affine k-scheme $\mathbf{A}(M)$ is locally of finite presentation (resp. of finite type) if and only if the k-module M is of finite presentation (resp. of finite type).
(c) The algebraic k-functor $\mathbf{A}^\vee(M)$ of Remark 2.37 is locally of finite presentation for any k-module M, since the tensor product preserves colimits in each variable.
(d) The affine $\mathbf{Z}$-schemes $*, \varnothing, \mathrm{Idem}, \mathbf{G}_m, \mathbf{G}_a, \mathrm{Mat}_n, \mathrm{GL}_n, \mathrm{SL}_n$ are all locally of finite presentation.

Remark 2.83. Since filtered colimits commute with finite limits in the category of sets, the condition of being locally of finite presentation or of finite type is preserved by finite limits in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.

Remark 2.84 (Compatibility with base change). Let $\varphi\colon k \to k'$ be a ring map. Since both the forgetful functor $\mathrm{CAlg}_{k'} \to \mathrm{CAlg}_k$ and its left adjoint $\varphi^*\colon \mathrm{CAlg}_k \to \mathrm{CAlg}_{k'}$ preserves filtered colimits, it follows from Corollary 2.41 that both base change along φ and Weil restriction along φ preserve the property of being locally of finite presentation or locally of finite type. On the other hand, the third functor $\varphi_\sharp$ usually does not. For example, $\mathrm{Spec}(\mathbf{C})$ is locally of finite presentation as an affine $\mathbf{R}$-scheme, but not as an affine $\mathbf{Q}$-scheme.

Recall that an algebraic k-functor can be though of as a map of algebraic functors $X \to \mathrm{Spec}(k)$ (see Corollary 2.47). Remark 2.84 implies that, for any cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(k') \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

If f is locally of finite presentation or of finite type, so is f'. Consequently, we can extend Definition 2.80 to arbitrary maps of algebriac functors as follows:

Definition 2.85 (Morphism locally of finite presentation/type)**.** Let $f\colon X \to S$ be a map of algebraic functors.

(a) f is *locally of finite presentation*, if for every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite presentation.
(b) f is locally of finite type, if for every ring k and for every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite type.

Concretely, under the equivalence of Corollary 2.47, the base change of $f\colon X \to S$ along $s\colon \mathrm{Spec}(k) \to S$ is the algebraic k-functor $\mathrm{CAlg}_k \to \mathrm{Set}$ given by

$$(\varphi\colon k \to R) \mapsto \{x \in X(R) \mid f(x) = s(\varphi) \text{ in } S(R)\} = \left\{ \begin{tikzcd} \mathrm{Spec}(R) \arrow[r, dashed, "x"] \arrow[d, "\mathrm{Spec}(\varphi)"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & S \end{tikzcd} \right\}$$

Example 2.86. Any open immersion is locally of finite presentation, and any closed immersion is locally of finite type. A closed immersion $i\colon Z \hookrightarrow X$ is locally of finite presentation if and only if, for every ring R and every $x\colon \mathrm{Spec}(R) \to X$, there exists a *finite* subset $F \subseteq R$ such that $x^{-1}(i(Z)) = \mathrm{V}(F)$.

Proposition 2.87 (Closure properties of morphisms locally of finite presentation/type)**.** *(a) Consider a commutative triangle of algebraic functors*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "h"'] & Y \arrow[dl, "f"] \\ X. \end{tikzcd}$$

If f is locally of finite presentation, then g is locally of finite presentation if and only if h is. The same holds for locally of finite type.

(b) Consider a cartesian square of algebraic functors

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

If f is locally of finite presentation, so is f'. The same holds for locally of finite type.

2.10. **The Nullstellensatz.** Consider a monic polynomial $f \in k[x]$ over a field k. By the functorial Nullstellensatz Corollary 2.24, we know that f is determined by its zero sets in all k-algebras. On the other hand, by the elementary theory of fields, we know that f splits into linear factors over some finite field extension of k. Hence, if we know the zero sets of f over any finite field extension of k, then we know the original polynomial f provided it is separable (i.e., does not have multiple roots). In general, the zero sets of f over finite field extensions of k determine the *radical* of f, which is the product of the prime factors of f without multiplicity. The *Nullstellensatz* of Hilbert generalizes the latter statement to systems of polynomial equations in fields. In this section, we review this result while also pointing out some shortcommings of the classical perspective.

For a ring k, define the maps

$$\{\text{subsets of } k[x_1, \ldots, x_n]\} \underset{\mathrm{I}}{\overset{\mathrm{V}}{\rightleftarrows}} \{\text{subsets of } k^n\},$$

as follows:

$$\mathrm{V}(F) = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in F\}, \quad \mathrm{I}(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) \text{ for all } x \in X\}.$$

Note that both maps are order reversing and that $F \subseteq \mathrm{I}(\mathrm{V}(F))$ and $X \subseteq \mathrm{V}(\mathrm{I}(X))$ (in other words, this is an adjunction between posets). Note also that $\mathrm{I}(X)$ is always an ideal in $k[x_1, \ldots, x_n]$ and it is even a radical ideal if k is reduced (if a power of f vanishes on X, so does f). Call a subset $X \subseteq k^n$ algebraic if it lies in the image of V, or equivalently if $X = \mathrm{V}(\mathrm{I}(X))$.

Theorem 2.88 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and let $n \in \mathbf{N}$. For any subset $F \subseteq k[x_1, \ldots, x_n]$, we have*

$$\mathrm{I}(\mathrm{V}(F)) = \sqrt{(F)}.$$

Consequently, the maps V *and* I *define a one-to-one correspondence*

$$V : \{\textit{radical ideals in } k[x_1, \ldots, x_n]\} \simeq \{\textit{algebraic subsets of } k^n\}.$$

We can upgrade this result to an equivalence of categories as follows. Define the category AffSet_k of *affine algebraic set* over k as follows:

- An object of AffSet_k is a pair (n, X) with $n \geq 0$ and $X \subseteq k^n$ an algebraic subset.
- A morphism $(n, X) \to (m, X)$ is a map $f \colon X \to Y$ such that there exists a polynomial map $k^n \to k^m$ extending f.

Recall that a ring R is reduced if 0 is the only nilpotent element of R. We denote by $\mathrm{CAlg}_k^{\mathrm{red}}$ the category of reduced k-algebras.

Corollary 2.89. *Let k be an algebraically closed field. Then there is an equivalence of categories*

$$\mathrm{AffSet}_k \overset{sim}{\to} (\mathrm{CAlg}^{\mathrm{ft,red}})^{\mathrm{op}}, \quad (n, X) \mapsto k[x_1, \ldots, x_n]/\mathrm{I}(X).$$

Hence AffSet_k *is equivalent to the full subcategory of* Aff_k *spanned by the reduced affine k-schemes of finite type.*

Remark 2.90. By Hilbert's Basissatz, *finite type* and *finite presentation* are equivalent for algebras over a field (and more generally over a noetherian ring).

We can also formulate a Nullstellensatz for an arbitrary field k ad follows. Denote by $\mathrm{Field}_k^{\mathrm{fin}} \subseteq \mathrm{CAlg}_k$ the full subcategory of finite field extension of k.

Corollary 2.91. *Let k be a field and let $n \in \mathbf{N}$. Then there is an order reversing bijection*

$$\mathrm{V}\colon \{\textit{radical ideals in } k[x_1, \ldots, x_n]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbf{A}^n \colon \mathrm{Field}_k^{\mathrm{fin}} \to \mathrm{Set}\}.$$

Example 2.92 (Non-reduced intersections). Even in the context of algebraic geometry over an algebraically closed field k, there are geometric phenomena that are not captured by only considering solutions in k. Consider for example the vanishing loci $L = \mathrm{V}(y)$ and $P = \mathrm{V}(y - x^2)$ in $\mathbf{A}_k^2 = \mathrm{Spec}(k[x, y])$. Since the k-algebras $k[x, y]/(y) \simeq k[t]$ and $k[x, y]/(y - x^2) \simeq k[t]$ are reduced, these affine k-schemes are determine by the algebraic sets $L(k)$ and $P(k)$ in k^2 (by the Nullstellensatz). The intersection $L(k) \cap P(k)$ is the algebraic set $\{(0, 0)\} \subseteq k^2$, which in turn corresponds to the subfunctor $\mathrm{V}(x, y) \subseteq \mathbf{A}_k^2$. However, the functorial intersection $L \cap P \subseteq \mathbf{A}_k^2$ is the subfucntor $\mathrm{V}(x^2, y)$, which is isomorphic to $\mathrm{Spec}(k[t]/(t^2))$. One can think of $\mathrm{V}(x^2, y)$ as a first-order infinitesimal neighborhood of the origin $\mathrm{V}(x, y)$ along the x-axis; this captures the fact that the line L is tangent (to first order) to the parabola P, so that they both contain the same infinitesimal horizontal segment at the origin. This residual tangency information in the intersection can only be seen by evaluating the functor $L \cap P$ on non-reduced k-algebras.. This also resolves another issue in classical algebraic geometry, which is that intersections do not vary nicely in families. Consider for example the family of horizontal line $L_a = \mathrm{V}(y - a)$ for $a \in k$. The intersection $L_a(k) \cap P(k)$ has exactly two points for any $a \neq 0$ (since k is algebraically closed), but only a single point when $a = 0$. On the other hand, the scheme-theoretic intersection $L_a \cap P$ is given by a 2-dimensional k-algebra for *all* $a \in k$, namely $k \times k$ when $a \neq 0$ and $k[t]/(t^2)$ when $a = 0$.

YW: draw picture

Example 2.93 (Geometry in mixed characteristic). Another aspect that is not captured by classical algebraic geometry over fields is algebraic geometry in *mixed characteristic*, i.e., involving rings R that do not contain any field. This is especially relevent in number theory, which studies rings of integers in finite extensions of $\mathbf{Q}$. Such rings can map to fields with different characteristics, which sometimes allows us to transport results from one characteristic to another. As a very basic example, consider the following proof that $\sqrt{2}$ is irrational (which is a reformulation of the usual argument). A positive rational number x such that $x^2 = 2$ is the same thing as an element of $X(\mathbf{Z})$, where $X \subseteq \mathbf{P}^1$ is the solution functor to the homogeneous polynomial equation $x^2 = 2y^2$. Since X is a functor, the ring map $\mathbf{Z} \to \mathbf{Z}/4$ induces a map $X(\mathbf{Z}) \to X(\mathbf{Z}/4)$. Since the squares in $\mathbf{Z}/4$ are 0 and 1, none of the six elements of $\mathbf{P}^1(\mathbf{Z}/4)$ satisfy the equation $x^2 = 2y^2$, so that $X(\mathbf{Z}/4)$ is empty. It follows that $X(\mathbf{Z})$ is also empty, i.e., that there does not exist $x \in \mathbf{Q}$ with $x^2 = 2$.

3. Projective geometry

3.1. **Projective spaces over a field.** Let k be a field. The classical projective n-space over k is the set of lines through the origin in k^{n+1}:

$$\mathbf{P}^n(k) = \{1-\text{dimensional subspaces of } k^{n+1}\}$$

Given a nonzero $(n+1)$-tuple $(a_0, \ldots, a_n) \in k^{n+1} \setminus \{0\}$, we denote by $[a_0 : \ldots : a_n]$ the 1-dimensional subspace of k^{n+1} containing $(a_0, \ldots, a_n)$. This identifies $\mathbf{P}^n(k)$ with the set of orbits of the (free)-action of $k^\times$ on $k^{n+1} \setminus \{0\}$ by scalar multiplication:

$$(k^{n+1} \setminus \{0\})/k^\times \xrightarrow{\sim} \mathbf{P}^n(k), \quad (a_0, \ldots, a_n) \mapsto [a_0, \ldots, a_n].$$

The set $\mathbf{P}^n(k)$ is the union of the $(n+1)$-subsets $U_0, \ldots, U_n$, where

$$U_i = \{[a_0, \ldots, a_n] \in \mathbf{P}^n(k) \mid a_i \text{ is a unit}\}$$

Each U_i can be identified with $\mathbf{A}^n(k) = k^n$ via

$$U_i \xrightarrow{\sim} \mathbf{A}^n(k), \quad [a_0, \ldots, a_n] \mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{\hat{a}_i}{a_i}, \ldots, \frac{a_n}{a_i}\right).$$

The complement $H_i = \mathbf{P}^n(k) \setminus U_i$ is given by

$$H_i = \{[a_0, \ldots, a_n] \in \mathbf{P}^n(k) \mid a_i = 0\}$$

and can be identified with $\mathbf{P}^{n-1}$ by dropping the i-th coordinate. We often think of $\mathbf{P}^n(k) = U_0 \coprod H_0$ as the completion of $U_0 = \mathbf{A}^n(k)$ obtained by adding a point *at infinity* on every line through the origin. These points at infinity form the *hyperplane at infinity* $H_0 = \mathbf{P}^{n-1}(k)$ in $\mathbf{P}^n(k)$.

Since the k-vector space k^{n+1} is canonically self-dual, we can identify $\mathbf{P}^n(k)$ with the set of 1-dimensional *quotient spaces of* k^{n+1}:

$$\mathbf{P}^n(k) \xrightarrow{\sim} \{1 \text{ dimensional quotient spaces of } k^{n+1}\}, \quad L \mapsto k^{n+1}/L^\perp,$$

where $L^\perp$ denotes the *orthogonal complement of* L. The *self-dualizability* lies in the fact that the line $[a_0, \ldots, a_n]$ corresponds to the *coimage* of the map $(a_0, \ldots, a_n) \colon k^{n+1} \twoheadrightarrow k$.

Proposition 3.1. *Let k be a ring and A a k-algebra.*

(a) A is of finite type if and only if, for every k-algebra R, which is a filtered union of subalgebras $(R_i)_{i\in I}$, we have

$$\mathrm{Spec}(A)(R) = \bigcup_{i \in I} \mathrm{Spec}(A)(R_i).$$

(b) A is finite presentated if and only if $\mathrm{Spec}(A) \colon \mathrm{CAlg}_k \to \mathrm{Set}$ preserves filtered colimits.

Proof. The proof is left as an exercise.To do:write the proof □

Proposition 3.2. *Let R be a ring. For an R-module V, the following conditions are equivalent:*

(a) V is finitely generated and projective;
(b) V is finitely presented and flat;
(c) V is a direct summand of R^n for some $n \in \mathbf{N}$;
(d) V is finitely presented, and for every maximal ideal $\mathfrak{m}$, the $R_\mathfrak{m}$-module $V_\mathfrak{m}$ is free;
(e) there exists some elements $f_1, \ldots, f_n \in R$ generating the unit ideal such that each V_{f_i} is a finitely generated free R_{f_i}-module;
(f) V is dualizable, i.e., there exists $V' \in \mathrm{Mod}_R$ and a map $e \colon V \otimes_R V' \to R$ such that $e \otimes (-)$ exhibits $V \otimes_R (-)$ as a left adjoint to $V' \otimes_R (-)$. In particular, we have $V' \simeq \mathrm{Hom}_R(V, -)$.

Proof. $(a) \Rightarrow (c)$: since V is finitely generated, there exists a surjective $R^n \twoheadleftarrow V$. By projectivity, this surjection splits. Therefore, V is a direct summand of R^n.

$(c) \Rightarrow (b)$: by assumption, there exists an R-module V' such that $V \oplus V' \simeq R^n$. The kernel of $R^n \twoheadrightarrow V$ is V', which is finitely generated as a direct summand of R^n. Therefore, V is finitely presented. Since the retract of every exact sequence is exact, the R-module V is flat.

$(b) \Rightarrow (d)$: a minimal generating set $R^n \twoheadrightarrow V$. After tensoring with $R_{\mathfrak{m}}$, we obtain an exact sequence

$$0 \to \ker(f) \to (R_{\mathfrak{m}})^n \xrightarrow{f} V_{\mathfrak{m}} \to 0.$$

Applying $(-) \otimes_{R_{\mathfrak{m}}} R/\mathfrak{m}$, we obtain another exact sequence

$$\mathrm{Tor}_1^{R_{\mathfrak{m}}}(V_{\mathfrak{m}}, R/\mathfrak{m}) \to \ker(f) \otimes_{R_{\mathfrak{m}}} R/\mathfrak{m} \to (R/\mathfrak{m})^n \to V/\mathfrak{m}V \to 0,$$

since $V_{\mathfrak{m}}/\mathfrak{m}V_{\mathfrak{m}} \simeq V/\mathfrak{m}V$. The flatness of V implies that $\mathrm{Tor}_1^{R_{\mathfrak{m}}}(V_{\mathfrak{m}}, R/\mathfrak{m}) \simeq \mathrm{Tor}_1^{R}(V, R/\mathfrak{m}) = 0$. By minimality of the generating set $R^n \twoheadrightarrow V$, the map $(R/\mathfrak{m})^n \to V/\mathfrak{m}V$ is an isomorphism of $R/\mathfrak{m}$-vector spaces. Therefore, we have $\ker(f) \otimes_{R_{\mathfrak{m}}} R/\mathfrak{m} = \ker(f)/\mathfrak{m}\ker(f) = 0$. By Nakayama's lemma, we have $\ker(f) = 0$. This completes the proof.

$(d) \Rightarrow (e)$: recall that we can write $V_{\mathfrak{m}}$ as a filtered colimit $V_{\mathfrak{m}} = \mathrm{colim}_{f \notin \mathfrak{m}} V_f$. Let $e_1, \ldots, e_n$ be a basis of the $R_{\mathfrak{m}}$-module $V_{\mathfrak{m}}$. By the explicit description of filtered colimits in Mod_R, there exists $e_i' \in V_{f_i}$ such that the image of e_i' under $V_{f_i} \to V_{\mathfrak{m}}$ is e_i. Take $f = f_1 \cdot \ldots \cdot f_n$, then we may assume that $e_i' \in V_f$, which becomes a basis after localization at $\mathfrak{m}$. Denote $f\colon (R_f)^n \to V_f$ be the corresponding map. Since V is finitely presented, V_f is also finitely presented, which implies that $\mathrm{coker}(f)$ and $\ker(f)$ are finitely generated. Since they become 0 after localization at $\mathfrak{m}$, by unwinding the definition, there exists $f_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ such that $R^n_{f_{\mathfrak{m}}} \to V_{f_{\mathfrak{m}}}$ is an isomorphism. It is easy to verify that $R = (f_{\mathfrak{m}})_{\mathfrak{m} \subseteq R}$, we may also choose finitely many $f_{\mathfrak{m}}$ that together generates R.

$(e) \Rightarrow (f)$: to prove that V is dualizable, it suffcies to show that the canonical natural transformation $V^\vee \otimes_R (-) \to \mathrm{Hom}_R(V, -)$ is an isomorphism, i.e., for each $N \in \mathrm{Mod}_R$, the map

$$V^\vee \otimes_R N \to \mathrm{Hom}_R(V, N)$$

is an isomorphism. Since being an isomorphism is a Zariski local property, it suffices to check that

$$(V_{f_i})^\vee \otimes_{R_{f_i}} N_{f_i} \to \mathrm{Hom}_R(V, N)_{f_i}$$

is an isomorphism. This is immediate once we have observed that

$$\mathrm{Hom}_R(V, N)_{f_i} \simeq \mathrm{Hom}_R(V, N_{f_i}) \simeq \mathrm{Hom}_{R_{f_i}}(V_{f_i}, N_{f_i}).$$

In fact, the first equality follows from the fact that localization can be written as a filtered colimit and Proposition 3.1, and the second equality follows from the universal property of localization.

$(f) \Rightarrow (a)$: since V is dualizable, $\mathrm{Hom}_R(V, -)$ preserves colimits. In particular, it preserves filtered colimits. Therefore, by Proposition 3.1, V is finitely presented, hence finitely generated. Recall that a map $f\colon X \to Y$ is an epimorphism if and only if the following diagram is a pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\sim"] \\ Y \arrow[r, "\sim"'] & Y \end{tikzcd}$$

Therefore, the functor $\mathrm{Hom}_R(V, -)$ preserves epimorphism, hence V is projective. □

Definition 3.3 (Vector space)**.** An R-module satisfying the equivalent conditions of Proposition 3.2 is called a *vector space* over R[3]. We denote by $\mathrm{Vect}_R \subseteq \mathrm{Mod}_R$ the full subcategory spanned by the vector spaces.

[3]This definition is nonstandard: when k is a field, every k-module is traditionally called a *vector space*, but only the finite dimensional ones satisfy this definition

Recall that any module M over a field admits a basis and that all bases have the same cardinality, which is called the dimension of M. To extend this notion to modules over a ring R, we consider R-fields, i.e., R-algebras that are also fields:

Definition 3.4 (Rank of a module)**.** Let R be a ring, M a R-module, and κ an R-field. The *rank* of M at κ, denoted by $\mathrm{rk}_\kappa(M)$, is the dimension of the κ-module $M \otimes_R \kappa$. We say that M has *constant rank* r, if $\mathrm{rk}_\kappa(M) = r$ for every R-field κ.

Remark 3.5 (Residue fields)*.* Let R be a ring. Recall that the *residue field* of R at a prime ideal $\mathfrak{p} \subseteq R$ is the R-field

$$\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \simeq \mathrm{Frac}(R/\mathfrak{p}).$$

The residue fields of R form an initial family of objects of the category Field_R in the following sense: for every R-field κ, there exists a unique prime ideal $\mathfrak{p}$ such that $R \to \kappa$ factors through $\kappa(\mathfrak{p})$, and the factorization is unique. In fact, if such factorization exists, the kernel of the map $R \to \kappa$ is the kernel of $R \to \kappa(\mathfrak{p})$, which is $\mathfrak{p}$, since $\kappa(\mathfrak{p}) \to \kappa$ is injective as a field extension. This implies that, the category of R-fields decomposes as

$$\mathrm{Field}_R = \coprod_{\mathfrak{p} \subseteq R} \mathrm{Field}_{\kappa(\mathfrak{p})}$$

Extensions of scalars between fields preserves dimension, i.e., if $k \to k'$ is a field extension and V is a k-module, then $\dim_k(V) = \dim_{k'}(V \otimes_k k')$. Therefore, we have $\mathrm{rk}_\kappa(M) = \mathrm{rk}_{\kappa(p)}(M)$ for any κ in the summand indexed by $\mathfrak{p}$.

Remark 3.6*.* Let R be a ring. For an R-vector space V, there exists $f_1, \ldots, f_n \in R$ generating the unit ideal such that $V_{f_i} \simeq R_{f_i}^{\oplus n_i}$, by Proposition 3.2. Let $\mathfrak{p} \subseteq R$ be a prime ideal. Then there exists f_i such that $f_i \notin \mathfrak{p}$ and $\mathrm{rk}_{\kappa(\mathfrak{p})}(V) = n_i$. In fact, localization commutes with coproduct, hence localization of V_{f_i} at $\mathfrak{p}$ gives us $V_\mathfrak{p} \simeq R_\mathfrak{p}^{n_i}$, and the claim follows from $V \otimes_R \kappa(\mathfrak{p}) \simeq V_\mathfrak{p} \otimes_{R_\mathfrak{p}} \kappa(p)$.

Remark 3.7 (Short exact sequences of vector spaces)*.* Since vector spaces are projective, every short exact sequence of vector spaces splits. Hence, for any ring map $\varphi\colon R \to R'$, the functor $\varphi^*\colon \mathrm{Vect}_R \to \mathrm{Vect}_{R'}$ preserves short exact sequence. In particular, if

$$0 \to U \to V \to W \to 0$$

is a short exact sequence in Vect_R and κ is any R-field, then $\mathrm{rk}_\kappa = \mathrm{rk}_\kappa(U) + \mathrm{rk}_\kappa(W)$.

Remark 3.8 (Modules of constant rank 0)*.* By Proposition 3.2 and **??**, a vector space is zero if and only if it has constant rank 0. Note that this is not true for more general modules. For example, the $\mathbf{Z}$-module $\mathbf{Q}/\mathbf{Z}$ has constant rank 0.

Proposition 3.9. *Let R be a ring. For an R-module L, the following conditions are equivalent:*
- *(a) L is a vector space of constant rank* 1*;*
- *(b) L is invertible, i.e., there exists $L' \in \mathrm{Mod}_R$ such that $L \otimes_R L' \simeq R$.*

Proof. $(a) \Rightarrow (b)$: Consider $L^\vee = \mathrm{Hom}_R(L, R)$. Since R_f is a (R, R_f)-bimodule, the tensor-hom adjunction gives us

$$\mathrm{Hom}_{R_f}(L \otimes_R R_f, R_f) \simeq \mathrm{Hom}_R(L, \mathrm{Hom}_{R_f}, \mathrm{Hom}_{R_f}(R_f, R_f)) \simeq \mathrm{Hom}_R(L, R_f),$$

for each $f \in R$. By Proposition A.1, we conclude that $(L^\vee)_f = (L_f)^\vee$. Therefore, if L is Zariski locally free of rank 1, then $L^\vee$ is also Zariski locally free of rank 1. Since localization commutes with tensor products, the R-module $L \otimes_R L^\vee$ is again Zariski locally free of rank 1. Therefore, the evaluation map $\mathrm{ev}\colon L \otimes_R L^\vee \to R$ is an isomorphism. In fact, this is true for $L = R$, and by **??**, being an isomorphism is a Zariski local property.

$(b) \Rightarrow (a)$: Since the R-module L is invertible, the functor $L \otimes_R (-)$ is an equivalence with an inverse $L' \otimes_R (-)$ for some R-module L'. This implies that $L \otimes_R (-)$ is left adjoint to $L' \otimes_R (-)$

with counit induced by $L \otimes_R L' \simeq R$, which implies that L is dualizable, hence a vector space. For each prime ideal $\mathfrak{p} \subseteq R$, the functor $\kappa(\mathfrak{p}) \otimes_R (-)$ commutes with tensor products. Therefore, the rank is multiplicative, since it is multiplicative for free modules. In other words, we have

$$\mathrm{rk}_{\kappa(\mathfrak{p})}(R) = \mathrm{rk}_{\kappa(\mathfrak{p})}(L \otimes_R L') \simeq \mathrm{rk}_{\kappa(\mathfrak{p})}(L) \cdot \mathrm{rk}_{\kappa(\mathfrak{p})}(L').$$

Since R is Zariski locally free of rank 1, we have $\mathrm{rk}_{\kappa(\mathfrak{p})}(L) = \mathrm{rk}_{\kappa(\mathfrak{p})}(L') = 1$ for each prime ideal $\mathfrak{p} \subseteq R$. □

Definition 3.10 (Line)**.** An R-module M satisfying the equivalent conditions of Proposition 3.9 is called a *line* over R. We denote by $\mathrm{Line}_R \subseteq \mathrm{Vect}_R$ the full subcategory of lines. A line is called *trivial* if it is isomorphic to R.

Remark 3.11. The category of vector spaces over any ring is equivalent to its opposite category, by duality. However, the symmetry between injective and surjective maps does not hold anymore.

Example 3.12. Let R be a ring and let V, W be vector spaces over R. Consider a surjection $f\colon V \twoheadrightarrow W$, then $\ker(f)$ is a vector space. However, consider an injection $f\colon V \rightarrowtail W$, then $\mathrm{coker}(f)$ is not a vector space. For example, consider the exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

In fact, surjectivity is already a universal notion. However, injectivity is not stable under scalar extensions. Therefore, we have the following stronger notion, which is equivalent to injectivity when considering vector spaces over fields.

Definition 3.13 (Universally injective map)**.** Let R be a ring and let M, N be R-modules. An R-linear map $f\colon M \to N$ is called *universally injective* if, for every ring map $R \to S$, the map $f \otimes_R S\colon M \otimes_R S \to N \otimes_R S$ is injective.

Remark 3.14.
- (a) If the R-linear map $f\colon M \to N$ is universally injective, then for every R-module P, then for every R-module P, the map $f \otimes_R P\colon M \otimes_R P \to N \otimes_R P$ is injective. This follows from applying the definition of universally injectivity to the ring map $R \to \mathrm{Sym}_R(P)$ and using that P is a direct summand of $\mathrm{Sym}_R(P)$.
- (b) We need not consider the dual notion of *universally surjective map*, since surjective maps are preserved by base change, as base change preserves cokernels. In other words, every surjective map is universally surjective.

Example 3.15.
- (a) By functoriality of $- \otimes_R -$, any map with a retraction (equivalently, any direct summand inclusion is universally injective).
- (b) The inclusion $\bigoplus_{\mathbf{N}} \mathbf{Z} \hookrightarrow \prod_{\mathbf{N}} \mathbf{Z}$ is an example of a universally injective map of abelian groups that does not admit a retraction.
- (c) Let $f \in R$. The multiplication by f map $R \to R$ is injective if and only if f is not a zero divisor, but it is universally injective if and only if f is a unit (as it becomes the zero map after base change to R/f).

Remark 3.16. By Zariski local property, *vector space* and *line* are Zariski-local properties of modules, and *universally injective* is a Zariski local property of linear maps. In fact, by Proposition 3.2, *vector space* and *line* are precisely the minimal weakenings of the properties *free of finite rank* and *free of rank* 1 that are Zariski-local.

Definition 3.17 (Subspace and quotient space)**.** Let V be an R-module.
- (a) A *subspace* of V is a submodule $U \subseteq V$ such that U is a vector space and the inclusion map $U \hookrightarrow V$ is universally injective.
- (b) A *quotient space* of V is a quotient module $V \twoheadrightarrow W$ such that W is a vector space.

The definition is mostly used when V itself as vector space. In this case, there are several useful characterizations of subspaces and of quotient spaces:

Proposition 3.18 (Characterization of subspaces)**.** *Let R be a ring and let U and V be vector spaces over R. For a map $f\colon U \to V$, the following are equivalent:*

(a) f is universally injective;
(b) for every R-field κ, the map $f \otimes_R \kappa$ is injective;
(c) f is injective and the cokernel of f is a vector space.
(d) f admits a retraction.
(e) the dual map $f^\vee\colon V^\vee \to U^\vee$ is surjective.

Proof. $(a) \Rightarrow (b)$: this is clear.

$(b) \Rightarrow (e)$: since V is an R-vector space, we observe that

$$\mathrm{Hom}_\kappa(V \otimes_R \kappa, \kappa) \simeq \mathrm{Hom}_R(V, \kappa) \simeq V^\vee \otimes_R \kappa.$$

Under this isomorphism, we have $(f \otimes_R \kappa)^\vee = f^\vee \otimes_R \kappa$. The assumption of (b) implies that $f^\vee \otimes_R \kappa$ is surjective for all field κ. In particular, this implies that $f^\vee \otimes_R \kappa(\mathfrak{m})$ is surjective for all maximal ideal $\mathfrak{m} \subseteq R$. It is clear that $V^\vee$ is an R-vector space, hence finitely generated. This implies that for each maximal ideal $\mathfrak{m}$, there exists some $s \notin \mathfrak{m}$ such that $f^\vee \otimes_R R_s$ is surjective. Since surjectivity is a Zariski local property, the map $f^\vee$ is surjective.

$(e) \Rightarrow (d)$: assume that $f^\vee\colon V^\vee \to U^\vee$ is surjective. Since tensor products commute with colimits, the map $V^\vee \otimes_R U \to U^\vee \otimes_R U$ is surjective. This implies that $\mathrm{Hom}_R(V, U) \to \mathrm{Hom}_R(U, U)$ is surjective. In other words, the map f has a retraction.

$(d) \Rightarrow (c)$: since f has a retraction, the map f is injective and the exact sequence

$$0 \to U \to V \to \mathrm{coker}(f) \to 0$$

splits. Therefore, $\mathrm{coker}(f)$ is a direct summand of W. Since V is an R-vector space, it is a summand of a finitely generated free module, hence $\mathrm{coker}(f)$ is also a direct summand of a finitely generated free module. In other words, $\mathrm{coker}(f)$ is an R-vector space.

$(c) \Rightarrow (a)$: since $\mathrm{coker}(f)$ is an R-vector space and f is injective, the exact sequence

$$0 \to U \to V \to \mathrm{coker}(f) \to 0$$

splits. Therefore $f\colon U \to V$ is an inclusion of direct summand. By Example 3.15, the map f is universally injective. □

Proposition 3.19 (Characterization of quotient spaces)**.** *Let R be a ring and let U and V be vector spaces over R. For a map $f\colon U \to V$, the following are equivalent:*

(a) f is surjective;
(b) for every R-field κ, $f \otimes_R \kappa$ is surjective;
(c) f is surjective and the kernel of f is a vector space;
(d) f admits a section;
(e) the dual map $f^\vee\colon W^\vee \to V^\vee$ is universally injective.

Proof. The proof is analogous to that of Proposition 3.18. □

Corollary 3.20. *Let R be a ring and let V and W be vector spaces over R. A map $f\colon V \to W$ is an isomorphism if and only if, for every R-field κ, $f \otimes_R \kappa$ is an isomorphism.*

Corollary 3.21. *Let R be a ring and let V and W be vector spaces over R such that $\mathrm{rk}_\kappa(V) = \mathrm{rk}_\kappa(W)$ for all R-fields κ. For a map $f\colon V \to W$, the following are equivalent:*

(a) f is an isomorphism;
(b) f is surjective;
(c) f is universally injective.

Proof. Since universal injectivity and surjectivity is stable under base change, it suffices to check this in the case where R is a field. □

Corollary 3.22. *Let R be a ring and let V be a vector space over R. There is a bijection:*

$$\{\textit{subspaces of } V\} \simeq \{\textit{quotient spaces of } V^\vee\}, \quad (U \hookrightarrow V) \mapsto (V^\vee \twoheadrightarrow U^\vee).$$

Conversely, there is also a bijection:

$$\{\textit{quotient spaces of } V\} \simeq \{\textit{subspaces of } V^\vee\}, \quad (V \twoheadrightarrow W) \mapsto (W^\vee \hookrightarrow V^\vee).$$

Remark 3.23. In practice, we do not distinguish between *submodules* of M and *monomorphisms* into M, nor between *quotient modules* of M and *epimorphisms* out of M: we may always identify a monomorphism with its image and an epimorphism with its coimage. Two monomorphisms into M have the same image if and only if they are isomorphic over M (in which case the isomorphism is unique). Similarly, two epimorphisms out of M have the same coimage if and only if they are (uniquely) isomorphic under M. These identifications are happening for example in the statement of Corollary 3.22.

Remark 3.24. Let $f : V \to W$ be a map between vector spaces. In general, neither $\ker(f)$ nor $\operatorname{coker}(f)$ is a vector space. By Proposition 3.18 and Proposition 3.19, $\ker(f)$ is a vector space if f is surjective, and $\operatorname{coker}(f)$ is vector space if f is universally injective (but injectivity does not suffice).

3.2. Projective spaces.

Definition 3.25. Let k be a ring and I a set. The *projective I-space* over k is the algebraic k-functor

$$\mathbf{P}^I_k : \mathrm{CAlg}_k \to \mathrm{Set}, \quad R \mapsto \{\text{quotient lines of } R^{(I)}\}.$$

We simply write $\mathbf{P}^I$, when $k = \mathbf{Z}$. For $n \geq -1$, the *projective n-space* over k is $\mathbf{P}^n_k = \mathbf{P}^{0,\dots,n}_k$. It is also called the *projective line* if $n = 1$ and the *projective plane* if $n = 2$.

Remark 3.26. (a) The projective (-1)-space $\mathbf{P}^{-1}_k = \mathbf{P}^\emptyset$ is the empty k-scheme $\emptyset$ (**??**).

(b) The projective 0-space $\mathbf{P}^0_k$ is the final k-scheme $\operatorname{Spec}(k)$

(c) If I is a *finite* set, Corollary 3.22 provides a natural bijection

$$\mathbf{P}^I_k(R) \simeq \{\text{sublines of } R^I\}.$$

In fact, this follows from $(R^I)^\vee \simeq R^I$ and if W is a line, then $W^\vee$ is also a line. However, if I is infinite or more generally if we replace the R-module $R^{(I)}$ by some R-module that is not a vector space (see find reference), then it is necessary to use quotient lines instead of sublines in the definition.

Notation 3.27 (Projective coordinates)**.** Let L be a line over R and let $(a_0, \dots, a_n) \in L^{n+1}$ be a family generating L as an R-module. The induced map $R^{n+1} \to L$ is then surjective and its coimage is a quotient line of R^{n+1}, which we denote by

$$[a_0 : \dots : a_n] \in \mathbf{P}^n(R)$$

Given another line M and generating family $(b_0, \dots, b_n) \in M$, we have $[a_0 : \dots : a_n] = [b_0 : \dots : b_n]$ if and only if there is a (necessarily unique) isomorphism $L \xrightarrow{\sim} M$ sending each a_i to b_i. When k is a field, we have

$$\mathbf{P}^n(k) \xrightarrow{\sim} k^{n+1} \setminus \{0\}, \quad (a_0, \dots, a_n) \mapsto [a_0, \dots, a_n].$$

The affine and projective spaces $\mathbf{A}^I_k$ and $\mathbf{P}^I_k$ are related via the punctured affine space $\mathbf{A}^I_k \setminus \{0\}$ of find reference. Recall that $\mathbf{A}^I_k \setminus \{0\}$ is a subfunctor of $\mathbf{A}^I_k$ given by

$$(\mathbf{A}^I_k \setminus \{0\})(R) = \{a \in R^I \mid (a) = R\}.$$

Under the identification of R^I with the R-linear dual of $R^{(I)}$, I-tuples generating the unit ideal correspond to surjective maps $R^{(I)} \twoheadrightarrow R$. In particular, there is a canonical map $\mathbf{A}^I_k \setminus \{0\} \to \mathbf{P}^I_k$ sending an I-tuple to the corresponding quotient of $R^{(I)}$. We therefore have a zigzag of morphisms

$$\mathbf{A}^I_k \hookleftarrow \mathbf{A}^I_k \setminus \{0\} \to \mathbf{P}^I_k$$

relating the affine space and the projective space. For $n \geq -1$, this specializes to a zigzag

$$\mathbf{A}_k^{n+1} \hookleftarrow \mathbf{A}_k^{n+1} \setminus \{0\} \to \mathbf{P}_k^n.$$

Proposition 3.28. *Let I be a set and R a ring. The map $\mathbf{A}^I \setminus \{0\} \to \mathbf{P}^I$ induces an injective map*

$$(\mathbf{A}^I \setminus \{0\})(R)/R^\times \hookrightarrow \mathbf{P}^I(R),$$

where $R^\times$ acts on $(\mathbf{A}^I \setminus \{0\})(R)$ by multiplication, whose image consists exactly of the trivial quotient lines of $R^{(I)}$ In particular, if every line over R is trivial (e.g., if R is a local ring or a principal ideal domain), then this map is bijective.

Proof. Assume that $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ define the same line in $\mathbf{P}^I(R)$. Then by definition, there exists some isomorphism such that the diagram

$$\begin{tikzcd} R^{(I)} \arrow[r, "(a_i)_{i\in I}"] \arrow[d, "(b_i)_{i\in I}"'] & R \\ R \arrow[ur, "\simeq"'] & \end{tikzcd}$$

commutes, which is equivalent to an element $\lambda \in R^\times$ such that $\lambda a_i = b_i$ for each $i \in I$. □

Remark 3.29. The scaling action of $R^\times$ on $(\mathbf{A}^I \setminus \{0\})(R)$ is functorial in R, i.e., it is an action of the affine group scheme $\mathbf{G}_m$ on the algebraic functor $\mathbf{A}^I \setminus \{0\}$. By Proposition 3.28, the map $\mathbf{A}^I \setminus \{0\} \to \mathbf{P}^I$ induces a monomorphism of algebraic functors $(\mathbf{A}^I \setminus \{0\})/\mathbf{G}_m \hookrightarrow \mathbf{P}^I$. This map is Zariski locally surjective, in the sense that for each ring R, there exists $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and the map is surjective on R_{f_i}.

Remark 3.30.
(a) Unlike the affine space construction $I \mapsto \mathbf{A}^I$, the constructions $I \mapsto \mathbf{P}^I$ and $I \mapsto \mathbf{A}^I \setminus \{0\}$ are not contravariantly functorial on the whole category of sets, but only on the subcategory of sets and *surjections*.
(b) A technical difference between affine and projective spaces is that $\mathbf{A}^I$ is quasi-compact for all I (this property will be defined in ??), whereas $\mathbf{P}^I$ and $\mathbf{A}^I \setminus \{0\}$ are quasi-compact if and only if I is finite. For this reason, many basic results in projective geometry assume a finite number of variables.

3.3. **Graded rings.** In the affine case, each family of polynomial equations Σ corresponds to an algebraic functor Sol_Σ, which is isomorphic to the affine scheme $\mathrm{Spec}(k[\Sigma])$. In the projective setting, we would like to construct an analogous correspondence, but this is only possible for homogeneous polynomials, i.e. polynomials lying in a single graded component of the standard graded polynomial ring.

Definition 3.31 (Homogeneous polynomial)**.** Let k be a ring, I a set and $d \in \mathbf{N}$. A polynomial in $k[x_i \mid i \in I]$ is *homogeneous of degree* d if it is a k-linear combination of monomials of the form $\prod_{i\in I} x_i^{n_i}$, where $\sum_{i\in I} n_i = d$. We denote by $k[x_i \mid i \in I]_d \subseteq k[x_i \mid i \in I]$ the k-submodule of homogeneous polynomials of degree d.

Note that there is a direct sum decomposition

$$k[x_i \mid i \in I] = \bigoplus_{d\in\mathbf{N}} k[x_i \mid i \in I]_d,$$

which is compatible with multiplication in the sense that the product of a homogeneous polynomial of degree d with one of degree e gives a homogeneous polynomial of degree $d + e$. These properties are abstracted in the notion of *graded ring*:

Definition 3.32 (Graded ring)**.** Let $(\Gamma, +, 0)$ be a commutative monoid (typically $\mathbf{N}$ or $\mathbf{Z}$). A Γ-*graded ring* is a ring A together with subgroups $A_\gamma \subseteq A$ for all $\gamma \in \Gamma$ such that:

(a) A is the direct sum of the subgroups A_γ: $\bigoplus_{\gamma\in\Gamma} A_\gamma \xrightarrow{\sim} A$;
(b) $1 \in A_0$, and for all $\gamma, \delta \in \Gamma$, we have $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$.

Elements of A_γ are called *homogeneous of degree* γ. If k is a ring, a Γ-graded k-algebra is a Γ-graded ring A with a ring map $k \to A$ that lands in A_0.

Definition 3.33 (Graded module)**.** Let A be a Γ-graded ring. A Γ-*graded A-module* is an A-module M together with subgroups $M_\gamma \subseteq M$ for all $\gamma \in \Gamma$ such that:

(a) M is the direct sum of the subgroups M_γ: $\bigoplus_{\gamma\in\Gamma} M_\gamma \xrightarrow{\sim} M$;
(b) For all $\gamma, \delta \in \Gamma$, we have $A_\gamma M_\delta \subseteq M_{\gamma+\delta}$.

An ideal $I \subseteq A$ is called *homogeneous* if it is a Γ-graded A-module with $I_\gamma = I \cap A_\gamma$.

Remark 3.34. Let A be a Γ-graded ring and M a Γ-graded A-module.

(a) A_0 is a subring of A, each A_γ is an A_0-module, and each M_γ is an A_0-module.
(b) Let $\Gamma \to \Delta$ be a map of commutative monoids. Then A is also a Δ-graded ring with $A_\delta = \bigoplus_{\gamma\mapsto\delta} A_\gamma$ and M is similarly a Δ-graded A-module. For example, any $\mathbf{N}$-graded ring A is a $\mathbf{Z}$-graded ring with $A_n = 0$ for $n < 0$.
(c) An ideal in A is homogeneous if and only if it is generated by homogeneous elements.
(d) If $I \subseteq A$ is homogeneous ideal, the quotient ring A/I inherits a unique Γ-grading given by $(A/I)_\gamma = A_\gamma/I_\gamma$ such that $A \twoheadrightarrow A/I$ is a graded map. Moreover, the A/I-module M/IM has a unique Γ-grading such that $M \to M/IM$ is a graded map.
(e) Let $S \subseteq A$ be a set of homogeneous elements whose degrees are invertible in Γ. Then the localized ring $A[S^{-1}]$ inherits a unique Γ-grading such that $A \to A[S^{-1}]$ is a graded map: if $a \in A_\gamma$ and $s \in S \cap A_\delta$, then $a/s \in A[S^{-1}]_{\gamma-\delta}$. Moreover, the $A[S^{-1}]$-module $M[S^{-1}]$ has a unique Γ-grading such that $M \to M[S^{-1}]$ is a graded map.

Example 3.35. Let k be a ring.

(a) For any set I, the polynomial algebra $k[x_i \mid i \in I]$ is an $\mathbf{N}$-graded k-algebra with $k[x_i \mid i \in I]_d$ the subgroup of homogeneous polynomials of degree d.
(b) The Laurent polynomial algebra $k[x^{\pm 1}]$ is a $\mathbf{Z}$-graded k-algebra with x in degree 1. More generally, inverting any set of homogeneous polynomials in $k[x_i \mid i \in I]$ yields a $\mathbf{Z}$-graded k-algebra.
(c) Let M be a k-module. The symmetric algebra $\mathrm{Sym}_k(M)$ has a canonical $\mathbf{N}$-grading

$$\mathrm{Sym}_k(M) = \bigoplus_{d\in\mathbf{N}} \mathrm{Sym}_k^d(M),$$

where $\mathrm{Sym}_k^d(M)$ is the d-th symmetric power of M. This recovers (a) for $M = k^{(I)}$.
(d) Let L be a line over k. Then $\bigoplus_{d\in\mathbf{Z}} L^{\otimes d}$ is naturally a $\mathbf{Z}$-graded k-algebra.

Notation 3.36. Let M be a Γ-graded A-module, and $\gamma \in \Gamma$.

(a) We denote by $M(\gamma)$ the Γ-graded A-module whose underlying A-module is M with $M(\gamma)_\delta = M_{\gamma+\delta}$. This defines an endofunctor $M \mapsto M(\gamma)$, which is an equivalence if γ is invertible in Γ.
(b) We denote by $A^{(\gamma)}$ the $\mathbf{N}$-graded ring $\bigoplus_{n\in\mathbf{N}} A_{n\gamma}$ with $(A^{(\gamma)})_n = A_{n\gamma}$. Similarly, $M^{(\gamma)}$ is an $\mathbf{N}$-graded $A^{(\gamma)}$-module.
(c) Let $A = \bigoplus_{n\in\mathbf{Z}} A_n$ be a $\mathbf{Z}$-graded ring and f be a invertible element in degree 1. Then there exists an isomorphism

$$A_0[x^{\pm 1}] \to A, \quad x \mapsto f.$$

Intuitively, multiplication by f shifts degrees: $A_n \simeq f^n \cdot A_0$, hence every element $x = A_n$ can be uniquely written as $x = a_0 f^n$, where $a_0 \in A_0$.

(d) Suppose γ is invertible in Γ. Given $f \in A_\gamma$, then A_f is a Γ-graded ring and we write $A_{(f)} = (A_f)_0$. Similarly, $M_{(f)} = (M_f)_0$ is an $A_{(f)}$-module. Note that there is an isomorphism of $\mathbf{Z}$-graded rings
$$A_{(f)}[x^{\pm 1}] \xrightarrow{\sim} A_f^{(\gamma)}, \quad x \mapsto f.$$
In fact, explicit descriptions of $A_{(f)}$ and $A_f^{(\gamma)}$ are
$$A_{(\gamma)} = \{\frac{a}{f^n} \mid a \in A_{n\gamma}\} \quad \text{and} \quad (A_f^{(\gamma)})_k = \{\frac{a}{f^n} \mid a \in A_{(k+m)\gamma}\}.$$
In particular, we find that $A_{(f)} \simeq (A_f^{(\gamma)})_0$. The isomorphism then follows from part (c).

3.4. **Homogeneous polynomial equations.**

Definition 3.37 (System of homogeneous polynomial equations)**.** Let k be a ring and let I and J be sets. A *system of J homogeneous polynomial equations in I variables* over k is a J-tuple $\Sigma = (f_j)_{j\in J}$ of homogeneous polynomials in $k[x_i \mid i \in I]$. We denote by (Σ) the homogeneous ideal in $k[x_i \mid i \in I]$ generated by $(f_j)_{j\in J}$ and by $k[\Sigma]$ the $\mathbf{N}$-graded k-algebra $k[x_i \mid i \in I]/(\Sigma)$.

The point of considering homogeneous polynomial equations is that they have well-defined solutions in projective space, as we now explain.

Remark 3.38 (Symmetric powers of lines). Let L and L' be lines over a ring R such that $L \otimes_R L' \simeq R$. We define the *negative tensor powers* of L by $L^{\otimes(-d)} = (L')^{\otimes d}$ for all $d > 0$.

(a) The canonical quotient map $L^{\otimes d} \to \mathrm{Sym}^d_R(L)$ is an isomorphism. This follows from the observation that a d-linear map $L^{\times d} \to M$ is automatically symmetric. Indeed, by Zariski local property, equality of linear maps can be checked locally. Therefore, it suffices to check the case for $L = R$, which is obviously symmetric.
(b) There exists an R-algebra structure on $\bigoplus_{d\in\mathbf{Z}} L^{\otimes d}$, making it the initial among R-algebras A with an R-linear map $L \to A$ whose image generates the unit ideal.To do: explain this in more detail

Construction 3.39 (Evaluation of homogeneous polynomials)**.** Let k be a ring, I a set, $d \in \mathbf{N}$, R a k-algebra, and L a line over R. We construct an evaluation map.
$$k[x_i \mid i \in I]_d \times L^I \to L^{\otimes d}, \quad (f, a) \mapsto f(a),$$
as follows. An element $a \in L^I$ is equivalently an R-linear map $a\colon R^{(I)} \to L$. The map $f \mapsto f(a)$ is then the composite
$$k[x_i \mid i \in I]_d = \mathrm{Sym}^d_k(k^{(I)}) \to \mathrm{Sym}^d_R(R^{(I)}) \xrightarrow{\mathrm{Sym}^d_R(L)} \mathrm{Sym}^d_R(L) = L^{\otimes d}.$$
Concretely, this is the unique k-linear map sending a degree d monomial $\prod_i x_i^{n_i}$ to the tensor product $\bigotimes_i a_i^{\otimes n_i} \in L^{\otimes d}$. Note that this recovers the usual evaluation when $L = R$, in which case also $L^{\otimes d} = R$.

Definition 3.40 (Vanishing locus)**.** Let $F \subseteq k[x_i \mid i \in I]$ be a set of homogeneous polynomials. The *Vanishing locus* of F in $\mathbf{P}^I_k$ is the subfunctor $\mathrm{V}(F) \subseteq \mathbf{P}^I_k$ given by
$$\mathrm{V}(F)(R) = \{a\colon R^{(I)} \twoheadrightarrow L \mid f(a) = 0 \text{ for all } f \in F\} \subseteq \mathbf{P}^I(R).$$
This is indeed a subfunctor: for any k-algebra map $R \to S$, the induced map $\mathbf{P}^I(R) \to \mathbf{P}^I(S)$ sends $\mathrm{V}(F)(R)$ to $\mathrm{V}(F)(S)$.

It is clear that the subfunctor $\mathrm{V}(F) \subseteq \mathbf{P}^I_k$ depends only on the homogeneous ideal (F). However, unlike in the affine case, different homogeneous ideals can still have the same vanishing locus; we will discuss this later.

Definition 3.41 (Solution functor)**.** Let $\Sigma = (f_j)_{j \in J}$ be a system of J homogeneous polynomial equations in I variables over k. Its *solution functor* $\mathrm{hSol}_\Sigma \colon \mathrm{CAlg}_k \to \mathrm{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbf{P}^I_k$:

$$\mathrm{hSol}_\Sigma = \mathrm{V}(\{f_j \mid j \in J\}) \subseteq \mathbf{P}^I_k.$$

Definition 3.42 (Projective scheme)**.** A functor $\mathrm{CAlg}_k \to \mathrm{Set}$ is a *projective k-scheme* if it is isomorphic to hSol_Σ for some system of homogeneous polynomial equations Σ in finitely many variables over k. We denote $\mathrm{Proj}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ the full subcategory spanned by the projective k-schemes.

Next, we want to show that the solution functor hSol_Σ depends only on the $\mathbf{N}$-graded k-algebra $k[\Sigma]$, analogously to $\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma])$ (reference: solution functor for affine) We first note that not every $\mathbf{N}$-graded k-algebra A is isomorphic to one of the form $k[\Sigma]$:

Proposition 3.43. *Let k ba a ring. An $\mathbf{N}$-graded k-algebra A is isomorphic to $k[\Sigma]$ if and only if A is generated by A_1, i.e. the canonical map $\mathrm{Sym}_k(A_1) \to A$ is surjective.*

Proof. Assume that $\mathrm{Sym}_k(A_1) \twoheadrightarrow A$ is surjective. Choose a generating set $k^{(I)} \twoheadrightarrow A_1$ of A_1. Since $\mathrm{Sym}_k(-)$ is a right adjoint, it preserves colimits. In particular, the map $\mathrm{Sym}_k(k^{(I)}) \to \mathrm{Sym}_k(A_1)$ is surjective. After composition, we obtain a $\mathbf{N}$-graded surjection $k[x_i \mid i \in I] \twoheadrightarrow \mathrm{Sym}_k(A)$, the kernel of which is a homogeneous ideal.

Conversely, the elements $(x_i)_{i \in I}$ lies in degree 1 and generates $k[\Sigma]$. □

The following construction will be generalized to arbitrary $\mathbf{N}$-graded algebra later, but the case of algebras generated in degree 1 is by far the most important case in practice.

Construction 3.44 (Proj of $\mathbf{N}$-graded algebras generated in degree 1)**.** Let k be a ring and A an $\mathbf{N}$-graded k-algebra generated by A_1. We define the algebraic k-functor $\mathrm{Proj}(A) \colon \mathrm{CAlg}_k \to \mathrm{Set}$ by

$$\mathrm{Proj}(A)(R) = \{\text{quotient lines } A_1 \otimes_k R \twoheadrightarrow L \text{ such that } \mathrm{Sym}_k(A_1) \to \mathrm{Sym}_R(L) \text{ factors through } A\}$$

Proposition 3.45. *Let Σ be a system of homogeneous polynomial equations over k. Then there exists a canonical isomorphism*

$$\mathrm{hSol}_\Sigma \simeq \mathrm{Proj}(k[\Sigma]) \colon \mathrm{CAlg}_k \to \mathrm{Set}\,.$$

If $\mathrm{Sym}_k(M)$ is the free k-algebra on M with its natural grading, we have $\mathbf{P}(M) \simeq \mathrm{Proj}(\mathrm{Sym}_k(M))$. In particular, if M is finitely generated, then $\mathbf{P}(M)$ is a projective k-scheme. The projective I-space is a special case of this construction $\mathbf{P}^I_k \simeq \mathbf{P}(k^{(I)})$.

Proof. Let $\Sigma = (f_j)_{j \in J}$ be a family of J homogeneous polynomials in I variables and $a \colon R^{(I)} \twoheadrightarrow L$ be a quotient line such that $f_j(a) = 0$ for each $j \in J$. We first note that there is a correspondence:

$$\mathrm{Sym}_R(-) : \{\text{quotient line } R^{(I)} \twoheadrightarrow L\} \simeq \{\text{graded maps } R[x_i \mid i \in I] \twoheadrightarrow \mathrm{Sym}_R(L)\} : (-)_1$$

The condition $f_j(a) = 0$ for all $j \in J$ is then equivalent to the condition that $\mathrm{Sym}(a)$ factors through $k[\Sigma] \otimes_k R$. This gives us a surjection $k[\Sigma] \otimes_k R \twoheadrightarrow \mathrm{Sym}(L)$. Since both $\mathrm{Sym}_R(L)$ and $k[\Sigma] \otimes_k R$ are generated by degree 1, this is equivalent to a diagram of the form

$$\begin{tikzcd}
\mathrm{Sym}_R(k[\Sigma]_1 \otimes_k R) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Sym}_R(L) \arrow[d, "\sim"] \\
k[\Sigma] \otimes_k R \arrow[r, two heads] & \mathrm{Sym}_R(L)
\end{tikzcd}$$

Observe that $\mathrm{Sym}_R((-)\otimes_k R)\simeq \mathrm{Sym}_k(-)\otimes_k R$, since both are left adjoints of the forgetful functor. Therefore, the above diagram is equivalent to the following diagram

$$\begin{tikzcd} \mathrm{Sym}_k(k[\Sigma]_1)\otimes_k R \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Sym}_R(L) \\ k[\Sigma]\otimes_k R \arrow[ur] & \end{tikzcd}$$

which is, by adjunction, furthermore equivalent to a diagram of the form

$$\begin{tikzcd} \mathrm{Sym}_k(k[\Sigma]_1) \arrow[r] \arrow[d] & \mathrm{Sym}_R(L) \\ k[\Sigma] \arrow[ur] & \end{tikzcd}$$

Unwinding the definition of hSol_Σ and $\mathrm{Proj}(k[\Sigma])$, we find that we have proved the statement. $\square$

Example 3.46 (The projective space of a module). Let k be a ring and M a k-module. Consider the functor $\mathbf{P}(M)\colon \mathrm{CAlg}_k\to \mathrm{Set}$ defined by

$$\mathbf{P}(M)(R)=\{\text{quotient lines of } M\otimes_k R\}$$

If $\mathrm{Sym}_k(M)$ is the free k-algebra on M with its natural grading, we have $\mathbf{P}(M)\simeq \mathrm{Proj}(\mathrm{Sym}_k(M))$. In particular, if M is finitely generated, then $\mathbf{P}(M)$ is a projective k-scheme. The projective I-space is a special case of the construction $\mathbf{P}^I_k\simeq \mathbf{P}(k^{(I)})$.

Analogously to reference: predual for affine, we can also define a dual functor $\mathbf{P}^\vee(M)\colon \mathrm{CAlg}_k\to \mathrm{Set}$ by

$$\mathbf{P}^\vee(M)(R)=\{\text{sublines of } M\otimes_k R\}.$$

If M is a vector space, then $\mathbf{P}^\vee(M^\vee)\simeq \mathbf{P}(M)$. This follows from Corollary 3.22. In general, however, $\mathbf{P}^\vee(M)$ is not a scheme.

Remark 3.47. If A is an $\mathbf{N}$-graded k-algebra generated by A_1, then $\mathrm{Proj}(A)$ is a subfunctor $\mathbf{P}(A_1)$, by definition.

3.5. **Loci associated with linear maps.** Consider the projective n-space $\mathbf{P}^n=\mathrm{Proj}(\mathbf{Z}[x_0,\dots,x_n])$, whose R-points are quotient lines $R^{\{0,\dots,n\}}\twoheadrightarrow L$. Given a homogeneous polynomial $f\in \mathbf{Z}[x_0,\dots,x_n]_d=\mathrm{Sym}^d(\mathbf{Z}^{\{0,\dots,n\}})$, we can consider the subfunctor $\mathrm{D}(f)$ of $\mathbf{P}^n$ consisting of the quotient lines $R^{\{0,\dots,n\}}\twoheadrightarrow L$ such that $L^{\otimes d}$ is generated by the image of f, or equivalently such that the composite

$$R\xrightarrow{f}\mathrm{Sym}^d_R(R^{\{0,\dots,n\}})\twoheadrightarrow L^{\otimes d}$$

is an isomorphism. This is similar to nonvanishing locus of a function, except that f is not quite a function on $\mathbf{P}^n$ (as we will see later, it is a section of the line bundle $\mathcal{O}(d)$ on $\mathbf{P}^n$). For example, $\mathrm{D}(x_i)$ can be identified with the affine n-space $\mathbf{A}^{\{0,\dots,\hat{i},\dots,n\}}$. The goal of this section is to show that $\mathrm{D}(f)\subseteq \mathbf{P}^n$ is an open subfunctor, which is moreover relatively affine, in the sense that its preimage in any affine scheme is affine. We start with some generalities. Let P be a property of modules that it preserved by base change. For any ring A and A-module M, we can then consider the locus where M has property P, which refers to the following subfunctor of $\mathrm{Spec}(A)$:

$$R\mapsto\{\varphi\colon A\to R\mid \text{the } R\text{-module } \varphi^*(M) \text{ has property } P\}$$

For example, M has a zero locus, a finite presentation locus, a flateness locus, etc. Similarly, properties of linear maps, algebras,etc., that are preserved by base change have associated loci.

Definition 3.48 (Loci associated with a linear map)**.** Let A be a ring and let $f\colon M\to N$ be a map of A-modules. We define the following subfunctors of $\mathrm{Spec}(A)$:

(a) The *vanishing locus* $\mathrm{V}(f)$ of f is the locus where f is zero;
(b) The *epimorphism locus* $\mathrm{Epi}(f)$ of f is the locus where f is surjective;
(c) The *monomorphism locus* $\mathrm{Mono}(f)$ is the locus where f is universally injective[4];
(d) The *isomorphism locus* $\mathrm{Iso}(f)$ of f is the locus where f is bijective.

If N is a line (resp. if M is a line), the epimorphism locus (resp. the monomorphism locus) of f is also called the *nonvanishing locus* of f and denoted by $\mathrm{D}(f)$.

Remark 3.49. Vanishing and nonvanishing loci for $X = \mathrm{Spec}(A)$ can be viewed as a special case of Definition 3.48: if $F \subseteq A$ is a subset and $f\colon A^{(F)} \to A$ is the induced linear map, then $\mathrm{V}(F) = \mathrm{V}(f)$ and $\mathrm{D}(F) = \mathrm{D}(f)$.

Remark 3.50. Let $f\colon V \to W$ be a map of vector spaces. By Proposition 3.18, we have

$$\mathrm{Mono}(f) = \mathrm{Epi}(f^\vee) \quad \text{and} \quad \mathrm{Iso}(f) = \mathrm{Epi}(f) \cap \mathrm{Epi}(f^\vee).$$

If moreover V and W have the same rank (e.g, are both lines), Corollary 3.21 implies that

$$\mathrm{Iso}(f) = \mathrm{Epi}(f) = \mathrm{Mono}(f)$$

Proposition 3.51. *Let A be a ring and let $f\colon M \to N$ be a map of A-modules.*
(a) if N is a vector space, then $\mathrm{V}(f) \subseteq \mathrm{Spec}(A)$ is a closed subfunctor.
(b) If N is a vector space, then $\mathrm{Epi}(f) \subseteq \mathrm{Spec}(A)$ is an open subfunctor.
(c) If M and N are vector spaces, then $\mathrm{Mono}(f) \subseteq \mathrm{Spec}(A)$ is an open subfunctor.
(d) If M and N are vector spaces, then $\mathrm{Iso}(f) \subseteq \mathrm{Spec}(A)$ is an open subfunctor.

Proof. (a) Let $f(x)\colon A \xrightarrow{x} M \xrightarrow{f} N$. It is easy to check that $\mathrm{V}(f) = \bigcap_{x \in M} \mathrm{V}(f(x))$. Since in $\mathrm{Spec}(A)$, the intersection of closed functors is still closed, we may assume that $M = A$. Since $(-)^\vee\colon \mathrm{Vect}_A \to \mathrm{Vect}_A$ is an equivalence of categories, the map f is zero if and only if $f^\vee$ is zero. This implies that $\mathrm{V}(f) = \mathrm{V}(f^\vee)$. We now claim that $\mathrm{V}(f^\vee) = \mathrm{V}(\mathrm{im}(f^\vee))$: an A-algebra $\varphi\colon A \to R$ is in $\mathrm{V}(f^\vee)$ if and only if the induced map $f \otimes \mathrm{id}_R\colon N^\vee \otimes_A R \to R$ is zero, which happens if and only if $\varphi(\mathrm{im}(f^\vee))$ is zero. In fact, we have the following commutative square:

$$\begin{tikzcd} N^\vee \arrow[r, "f^\vee"] \arrow[d] & A \arrow[d, "\varphi"] \\ N^\vee \otimes_A R \arrow[r, "f\otimes \mathrm{id}_R"'] & R \end{tikzcd}$$

Note that the elements of the form $n \otimes 1$ generates $N^\vee \otimes_A R$, where $n \in N^\vee$.

(b) Since N is a vector space, there exists some vector space N' such that $N \oplus N' \simeq A^n$. Recall that a map between A-modules $g\colon P \to P'$ is surjective if and only if $g \oplus \mathrm{id}_Q\colon P \oplus Q \to P' \oplus Q$ is surjective, one can show this by computing the cokernel of each map. Therefore, $\mathrm{Epi}(f) = \mathrm{Epi}(f \oplus \mathrm{id}_{N'})$ and we may assume that $N = R^n$. Furthermore, we have $\mathrm{Epi}(f) = \mathrm{Epi}(\Lambda^n f)$ and can assume that $N = R$Todo: explain why. In this case, for each ring map $\varphi\colon A \to R$, we have a commutative square of A-modules of the form

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\mathrm{id}_M \otimes \varphi"'] & A \arrow[d] \\ M \otimes_A R \arrow[r, "f\otimes \mathrm{id}_R"'] & R \end{tikzcd}$$

Since the image of $\mathrm{id}_M \otimes \varphi$ generates $M \otimes_A R$ as an R-module, the map $f \otimes \mathrm{id}_R$ is surjective if and only if $f \otimes \mathrm{id}_R(\mathrm{im}(\mathrm{id}_M \otimes \varphi))$ generates R, which happens if and only if $\varphi(\mathrm{im}(f))$

[4] We need to use universal injectivity to define the monomorphism locus, as injectivity is not preserved by base change

generates R. In other words, we have $\mathrm{Epi}(f) = \mathrm{D}(\mathrm{im}(f))$, which implies that $\mathrm{Epi}(f)$ is an open subfunctor of $\mathrm{Spec}(A)$.

(c) Since M and N are vector spaces, by Remark 3.50, we have $\mathrm{Mono}(f) = \mathrm{Epi}(f^\vee)$, which is an open subfunctor of $\mathrm{Spec}(A)$.

(d) Since $\mathrm{Iso}(f) = \mathrm{Mono}(f) \cap \mathrm{Epi}(f)$, and the intersection of open subfunctors of $\mathrm{Spec}(A)$ is again open, we see that $\mathrm{Iso}(f)$ is open.

□

We now study more closely the nonvanishing locus of a map of lines $s\colon L' \to L$ over R. Since L' is invertible, we can tensor s with the inverse of L' without changing its nonvanishing locus, so it suffices to consider the case $L' = R$. We can then identify the R-linear map $s\colon R \to L$ with an element $s \in L$. In the special case $L = R$, the theory of localization of rings tell us that the nonvanishing locus of any $s \in R$ is isomorphic to $\mathrm{Spec}(R_s)$ (see **??**). This theory can be generalized to nontrivial lines as follows.

Definition 3.52 (Periodic module)**.** Let R be a ring, L a line over R, and $s \in L$. An R-module M is *s-periodic* if the map

$$\mathrm{id}_M \otimes s\colon M \otimes_R R \to M \otimes_R L$$

is an isomorphism.

Proposition 3.53. *Let R be a ring, L a line over R, and $s \in L$.*

(a) There exists an initial s-periodic R-algebra R_s.

(b) The forgetful functor $\mathrm{Mod}_{R_s} \to \mathrm{Mod}_R$ identifies Mod_{R_s} with the full subcategory of s-periodic R-modules in Mod_R. Hence, the inclusion of this subcategory has a left adjoint $M \mapsto M_s$ given by $M_s = M \otimes_R R_s$.

(c) The R-module M_s can be computed as the colimit of the sequence

$$M \xrightarrow{s} M \otimes_R L \xrightarrow{s} M \otimes_R L^{\otimes 2} \xrightarrow{s} \dots.$$

Proof. We define

$$M_s = \mathrm{colim}(M \xrightarrow{s} M \otimes_R L \xrightarrow{s} M \otimes_R L^{\otimes 2} \xrightarrow{s} \dots).$$

In fact, an explicit description of M_s is the following:

$$M_s = \{\frac{m}{s^n} \mid n \in \mathbf{N},\ m \in M \otimes_R L^{\otimes n}\}/\sim,$$

where $\frac{m}{s^n} \sim \frac{sm}{s^{n+1}}$. Consider the following diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \arrow[dl, "\mathrm{id}"'] \\
M \otimes_R L^{\otimes(n-1)} \arrow[r, "\mathrm{id}\otimes s"] \arrow[d] & M \otimes_R L^{\otimes(n-1)} \otimes_R L \arrow[d] \arrow[dl, "\mathrm{id}"'] \\
M \otimes_R L^{\otimes n} \arrow[r, "\mathrm{id}\otimes s"] \arrow[d] & M \otimes_R L^{\otimes n} \otimes_R L \arrow[d] \arrow[dl, "\mathrm{id}"'] \\
\vdots & \vdots
\end{tikzcd}$$

Since the following diagram commutes:

$$\begin{tikzcd}
M \otimes_R L^{\otimes(n-1)} \arrow[r, "\mathrm{id}\otimes s"] \arrow[d] & M \otimes_R L^{\otimes(n-1)} \otimes_R L \arrow[d] \\
M \otimes_R L^{\otimes n} \arrow[r, "\mathrm{id}\otimes s"] & M \otimes_R L^{\otimes n} \otimes_R L
\end{tikzcd}$$

we obtain a map between the colimits $s\colon M_s \to M_s \otimes_R L$. Consider the following diagram:

$$\begin{tikzcd}
M \otimes_R L^{\otimes(n-2)} \otimes_R L \arrow[r, "\mathrm{id}"] \arrow[d] & M \otimes_R L^{\otimes(n-1)} \arrow[d] \\
M \otimes_R L^{\otimes(n-1)} \otimes_R L \arrow[r, "\mathrm{id}"] & M \otimes_R L^{\otimes n}
\end{tikzcd}$$

$$\begin{tikzcd}
m \otimes l_1 \otimes \ldots \otimes l_{n-2} \otimes l \arrow[r, mapsto] \arrow[d, mapsto] & m \otimes l_1 \otimes \ldots \otimes l_{n-2} \otimes l \arrow[d, mapsto] \\
m \otimes l_1 \otimes \ldots \otimes l_{n-2} \otimes s \otimes l \arrow[r, mapsto] & m \otimes l_1 \otimes \ldots \otimes l_{n-2} \otimes l \otimes s
\end{tikzcd}$$

Recall that we have $L^d \simeq \mathrm{Sym}^d_R(L)$, i.e. Σ_d acts trivially on $L^{\otimes d}$. Take $d = 2$, we find that the above diagram commutes, hence gives us a map $t\colon M_s \otimes_R L \to M_s$. To show that $t \circ s = \mathrm{id}$, observe that the following diagram commutes:

$$\begin{tikzcd}
M \otimes_R L^{\otimes(n-1)} \arrow[r] \arrow[d] & M \otimes_R L^{(n-2)} \otimes_R L \arrow[dl, "\mathrm{id}"] \\
M \otimes_R L^{\otimes n} &
\end{tikzcd}$$

$$\begin{tikzcd}
m \otimes l_1 \otimes \ldots \otimes l_{n-1} \arrow[r, mapsto] \arrow[d, mapsto] & m \otimes l_1 \otimes \ldots \otimes l_{n-1} \otimes s \arrow[dl, mapsto] \\
m \otimes l_1 \otimes \ldots \otimes l_{n-1} \otimes s &
\end{tikzcd}$$

To show that $s \circ t = \mathrm{id}$, observe that the following diagram commutes:

$$\begin{tikzcd}
& M \otimes_R L^{\otimes(n-1)} \otimes_R L \arrow[dl, "\mathrm{id}"'] \arrow[d] \\
M \otimes_R L^{\otimes n} \arrow[r, "\mathrm{id} \otimes s"'] & M \otimes_R L^{\otimes n} \otimes_R L
\end{tikzcd}$$

$$\begin{tikzcd}
& m \otimes l_1 \otimes \ldots \otimes l_{n-1} \otimes l \arrow[dl, mapsto] \arrow[d, mapsto] \\
m \otimes l_1 \otimes \ldots \otimes l_{n-2} \otimes s \otimes l & m \otimes l_1 \otimes \ldots \otimes l_{n-2} \otimes l \otimes s \arrow[l, mapsto]
\end{tikzcd}$$

This implies that M_s is s-periodic. Similarly, one can construct R_s. That $M \otimes_R R_s = M_s$ follows from the fact that tensor product commutes with colimits. It is clear that the construction $M \mapsto M_s$ assembles into a functor. It now suffcies to show that it is the left adjoint of the inclusion $\mathrm{Mod}_{R_s} \hookrightarrow \mathrm{Mod}_R$: look at the comparison map $\mathrm{Map}(M \otimes_R L, N) \to \mathrm{Map}(M, N)$, by adjunction, this is equivalent to the map

$$(s^\vee \otimes \mathrm{id}_N)_*\colon \mathrm{Map}(M, L^\vee \otimes_R N) \to \mathrm{Map}(M, N)$$

It suffices to show that $s^\vee \otimes \mathrm{id}_N$ is an isomorphism. In fact, we have the following commutative diagram

$$\begin{tikzcd} L \otimes_R L^\vee \arrow[r, "\mathrm{id}_L \otimes s^\vee"] \arrow[d, "\mathrm{ev}"'] & L \\ R \arrow[ur, "s"'] & \end{tikzcd}$$

After tensoring with N, we find that the map $s^\vee \otimes \mathrm{id}_N$ is an isomorphism. This proves that $(-)_s$ is a left adjoint. Now, for each R-module M and N, we have an isomorphism

$$(M \otimes_R N)_s \simeq M_s \otimes_R N_s,$$

since tensor product preserves colimits in both variables. In particular, if A is a R-algebra, then A_s is an R-algebra with the multiplication map given by $A_s \otimes_R A_s \simeq (A \otimes_R A)_s \to A_s$. In particular, R_s is an R-algebra. Since left adjoint preserves initial object, R_s is initial as s-periodic R-module. Also, there exists a fully faithful embedding of the category of s-periodic R-algebra into the category of s-periodic modules, which implies that R_s is initial as a s-periodic R-algebra. $\square$

Corollary 3.54. *Let R be a ring, L a line over R, and $s \in L$. Then there exists a canonical isomorphism* $\mathrm{Spec}(R_s) \xrightarrow{\sim} \mathrm{D}(s) \subseteq \mathrm{Spec}(R)$. *In particular,* $\mathrm{D}(s)$ *is an affine scheme.*

Remark 3.55 (Improved Zariski descent). All the Zariski descent results from before can be generalized by replacing the family $(f_i)_{i\in I}$ generating the unit ideal in R with a family $(s_i)_{i\in I}$ generating some line L over R. For example, the generalized version of **??** says that, for any affine scheme X, the diagram

$$X(R) \to \prod_{i\in I} X(R_{s_i}) \rightrightarrows \prod_{i,j\in I} X(R_{s_i,s_j})$$

is an equalizer diagram (where $s_is_j \in L^{\otimes 2}$). For each result, we can either repeat the proof with no essential changes, or reduce the new statement to the old one using that $\mathrm{D}(s) = \mathrm{D}(F_s)$, where $F_s \subseteq R$ is the image by $s^\vee \colon L^\vee \to R$ of a generating family of $L^\vee$.To do: write more about the details

Example 3.56 (Projective completion of affine spaces). Let k be a ring and M a k-module. Recall the algebraic k-functors $\mathbf{A}(M)$ and $\mathbf{P}(M)$ from Example 2.37 and 3.42. There is a canonical embedding

$$\mathbf{A}(M) \hookrightarrow \mathbf{P}(M \oplus k), \quad (a \colon M \otimes_k R \to R) \mapsto ((a, \mathrm{id}_R) \colon (M \otimes_k R) \oplus R \twoheadleftarrow R).$$

which is open in $\mathrm{Spec}(R)$ by Proposition 3.47, and also affine by Corollary 3.50. Taking $M = k^{(k)}$ we obtain a canonical open immersion

$$\mathbf{A}^I \hookrightarrow \mathbf{P}^{I \sqcup \{0\}},$$

identifying $\mathbf{A}^I$ with $\mathrm{D}(x_0) \subseteq \mathbf{P}^{I\sqcup\{0\}}$. For $I = \{1, \dots, n\}$, this is an open immersion $\mathbf{A}^n \hookrightarrow \mathbf{P}^n$.

3.6. **The functor Proj.** In this section, we define $\mathrm{Proj}(A)$ for any $\mathbf{N}$-graded ring A. If A is finitely generated as an A_0-algebra, which is usually the case in practice, we will see that $\mathrm{Proj}(A)$ reduces to Corollary 3.44. Thus, the increase in generality is minimal. Instead, the main result of this section is that Corollary 3.44 satisfies

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}).$$

We will see that this isomorphism gives rise to the *Veronese embedding*, which is a central feature of projective geometry.

Definition 3.57 (Eventually surjective map)**.** A map of $\mathbf{N}$-graded rings $A \to B$ is called *eventually surjective* if for every $d \geq 1$ and every $b \in B_d$, there exists $n \geq 1$ such that $A_{nd} \otimes_{A_0} B_0 \to B_{nd}$ hits b^n. We denote by $\mathrm{CAlg}^{\mathbf{N},\mathrm{es}}$ the category of $\mathbf{N}$-graded rings and eventually surjective maps.

Recall the notation $A^{(d)} = \bigoplus_{n\in\mathbf{N}} A_{nd}$ and $A_{(f)} = (A_f)_0$ from Notation 3.36.

Construction 3.58. We construct a functor

$$\mathrm{Proj}\colon (\mathrm{CAlg}^{\mathbf{N},\mathrm{es}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$$

with a natural transformation $\mathrm{Proj}(-) \to \mathrm{Spec}((-)_0)$. If A is an $\mathbf{N}$-graded k-algebra, we can thus view $\mathrm{Proj}(A)$ as an algebraic k-functor, and we will see in Theorem 3.57(ii), that is recovers Corollary 3.44, when A is generated by A_1.

Let A be an $\mathbf{N}$-graded ring. We first define

$$\mathrm{Proj}_1(A) \in \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/\,\mathrm{Spec}(A_0)} \simeq \mathrm{Fun}(\mathrm{CAlg}_{A_0}, \mathrm{Set})$$

as follows. Given an A_0-algebra R, $\mathrm{Proj}_1(A)(R)$ is the set of pairs (L, φ), where L is a quotient line of the R-module $A_1 \otimes_{A_0} R$ and $\varphi\colon A \otimes_{A_0} R \to \mathrm{Sym}_R(L)$ is a map of graded R-algebras extending the quotient map $A_1 \otimes_{A_0} R \twoheadrightarrow L$. Note that φ is automatically surjective, so that we can alternatively describe $\mathrm{Proj}_1(A)(R)$ as the set of quotient $\mathbf{N}$-graded algebras of $A \otimes_{A_0} R$ that are isomorphic to symmetric algebras of lines.

For $d \in \mathbf{N}$, we define $\mathrm{Proj}_d(A) = \mathrm{Proj}_1(A^{(d)})$. For any $n \in \mathbf{N}$, there is then a canonical map

$$\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A), \quad (L, \varphi) \mapsto (L^{\otimes}, \varphi^{(n)}),$$

where $L^{\otimes n}$ is a quotient of $A_{nd} \otimes_{A_0} R$ via φ_n. This defines a functor

$$\mathbf{N}^{\mathrm{div}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}), \quad d \mapsto \mathrm{Proj}_d(A),$$

where $\mathbf{N}^{\mathrm{div}}$ is the poset of natural numbers under divisiblility. We then set

$$\mathrm{Proj}(A) = \operatorname*{colim}_{d \in \mathbf{N}^{\mathrm{div}}_{>0}} \mathrm{Proj}_d(A).$$

Since 0 is the final object of $\mathbf{N}^{\mathrm{div}}$, this comes with a canonical map

$$\mathrm{Proj}(A) \to \mathrm{Proj}_0(A) = \mathrm{Proj}_1(A^{(0)}) = \mathrm{Proj}_1(A_0[x]) = \mathbf{P}^0_{A_0} = \mathrm{Spec}(A_0).$$

Let $\alpha\colon A \to B$ be a map of $\mathbf{N}$-graded rings and let (L, φ) be an R-point of $\mathrm{Proj}_d(B)$ for some $d \geq 1$. Then there exists $b_1, \dots, b_r \in B_d$ whose images by φ_d generate L. If for some $n \geq 1$, the powers $b_1^n, \dots, b_r^n$ are in the image of $\alpha_{nd}\colon A_{nd} \otimes_{A_0} B_0 \to B_{nd}$, then $(L^{\otimes n}, \varphi^{(n)} \circ \alpha^{(nd)})$ is an R-point of $\mathrm{Proj}_{nd}(A)$. Thus, if α is eventually surjective, precomposition with α defines a map $\mathrm{Proj}(\alpha)\colon\ \mathrm{Proj}(B) \to \mathrm{Proj}(A)$.

Remark 3.59. For every $d \geq 1$, the inclusion $d\mathbf{N}^{\mathrm{div}}_{>0} \subseteq \mathbf{N}^{\mathrm{div}}_0$ is cofinal and hence induces an isomorphism

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}).$$

Remark 3.60 (Degree d maps). We can enlarge the domain of definition of Proj as follows. If A and B are $\mathbf{N}$-graded rings and $d \in \mathbf{N}$, a *degree d map* from A to B is a graded ring map $A \to B^{(d)}$. If it is eventually surjective and $d \geq 1$, then by Remark 3.59, it induces

$$\mathrm{Proj}(B) \simeq \mathrm{Proj}(B^{(d)}) \to \mathrm{Proj}(A).$$

This also works if $d = 0$: $\mathrm{Proj}(B) \to \mathrm{Spec}(B_0) = \mathrm{Proj}(B^{(0)}) \to \mathrm{Proj}(A)$. We can compose a degree d with a degree e map to obtain a degree $d + e$ map. Thus, there is an enlargement of the category $\mathrm{CAlg}^{\mathbf{N},\mathrm{es}}$ with the same objects and in which the set of maps from A to B is $\coprod_{d \in \mathbf{N}} \mathrm{Map}_{\mathrm{CAlg}^{\mathbf{N},\mathrm{es}}}(A, B^{(d)})$, and the functor Proj extends to this category.

Proposition 3.61 (Zariski descent for Proj)**.** *Let A be an $\mathbf{N}$-graded ring, let $d \in \mathbf{N}$, and let X be either* $\mathrm{Proj}_d(A)$ *of* $\mathrm{Proj}(A)$*. For every ring R, every line L over R, and every generating family $(s_i)_{i\in I}$ of L, the diagram*

$$X(R) \to \prod_{i \in I} X(R_{s_i}) \rightrightarrows \prod_{i,j \in I} X(R_{s_i s_j})$$

is an equalizer.

Proof. The follow from Zariski descent for module. Let $(L_i, \varphi_i) \in X(R_{s_i})$ such that $(L_i, \varphi_i)_{s_j} = (L_j, \varphi_j)_{s_i}$. We have $A_1 \otimes_{A_0} R_{s_j} \to L_i$. This implies that we have a commutative triangle as follows:

$$\begin{tikzcd} A_1 \otimes_{A_0} R_{s_i,s_j} \arrow[r] \arrow[d] & (L_i)_{s_j} \\ (L_j)_{s_i} \arrow[ur, dashed, "\simeq"'] & \end{tikzcd}$$

In other words, the following triangle commutes:

$$\begin{tikzcd} (L_i)_{s_j s_k} \arrow[r, "\alpha_{ij}"] \arrow[d, "\alpha_{ik}"'] & (L_j)_{s_i s_k} \\ (L_k)_{s_i s_j} \arrow[ur, "\alpha_{jk}"'] & \end{tikzcd}$$

There are surjections from $A_1 \otimes_{A_0} R_{s_i s_j k}$ to all of them. This implies that the triangle commutes and there is a module L over R with isomorphism $\beta_i \colon L_{s_i} \simeq L_i$, which implies that $L \in \mathrm{Line}_R$. Similarly, using Zariski descent for morphisms, we get a map $\varphi\colon A \otimes_{A_0} R \to \mathrm{Sym}_R(L)$ such that $\varphi_{s_i} = \varphi_i$. Therefore, the map φ is surjective and $(L, \varphi) \in X(R)$. For uniqueness: let (L', φ') satisfy the same property, then there exists a unique morphism such that the following diagram commutes:

$$\begin{tikzcd} (L_i)_{s_j s_k} \arrow[r, "\alpha_{ij}"] \arrow[d, "\alpha_{ik}"'] & (L_j)_{s_i s_k} \\ (L_k)_{s_i s_j} \arrow[ur, "\alpha_{jk}"'] & \end{tikzcd}$$

which follows from the Zariski descent for morphism. Let $X = \mathrm{Proj}(A)$, suppose that I is finite. Since filtered colimits preserves finite limits in Set, the Zariski descent for $\mathrm{Proj}_d(A_0)$ implies that Zariski descent for $\mathrm{Proj}(A)$. If I is infinite, then this can be reduced to the finite case as before. $\square$

Lemma 3.62. *Let A be an $\mathbf{N}$-graded ring, let $d \geq 1$, and let $f \in A_d$. Let $\mathrm{D}(f) \subseteq \mathrm{Proj}_d(A)$ be the subfunctor of pairs (L, φ) such that $\varphi(f)$ generates L (which is open by Proposition 3.51).*

(a) There is a canonical isomorphism $\mathrm{D}(f) \simeq \mathrm{Spec}(A_{(f)})$.

(b) For every $n \geq 1$, the map $\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A)$ induces an isomorphism $\mathrm{D}(f) \xrightarrow{\sim} \mathrm{D}(f^n)$. Moreover, the square

$$\begin{tikzcd} \mathrm{D}(f) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{D}(f^n) \arrow[d, hook] \\ \mathrm{Proj}_d(A) \arrow[r] & \mathrm{Proj}_{nd}(A) \end{tikzcd}$$

is cartesian.

(c) Let $f_1, \ldots, f_k \in A$ be homogeneous elements of positive degrees. Under the isomorphism in (b), we have $\mathrm{D}(f_1, \ldots, f_k) = \mathrm{D}(f_1) \cap \ldots \cap \mathrm{D}(f_k)$ inside $\mathrm{Proj}_d(A)$ for all sufficiently divisible d.

Theorem 3.63 (Structure of Proj)**.** *Let A be an $\mathbf{N}$-graded ring and let $d \geq 1$.*

(a) The map $\mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ is an open immersion.

(b) Suppose that A is generated as an A_0-algebra by homogeneous elements whose degrees divide d. Then the map $\mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ is an isomorphism. In particular, if A is generated by $A_{\leq 1}$, then $\mathrm{Proj}_1(A) \xrightarrow{\sim} \mathrm{Proj}(A)$.

Corollary 3.64. *Let A be a finitely generated $\mathbf{N}$-graded k-algebra such that $k \to A_0$ is surjective. Then $\mathrm{Proj}(A)$ is a projective k-scheme.*

Remark 3.65. The functor Proj extends the functor Spec as follows: there is a commutative triangle (up to natural isomorphism)

$$\begin{tikzcd} \mathrm{CAlg} \arrow[r, "(-)[x]"] \arrow[dr, "\mathrm{Spec}"'] & \mathrm{CAlg}^{\mathbf{N},\mathrm{es}} \arrow[d, "\mathrm{Proj}"] \\ & \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \end{tikzcd}$$

where the horizontal functor sends R to the $\mathbf{N}$-graded ring $R[x]$. The follows from Theorem 3.63 as

$$\mathrm{Proj}(R[x]) \simeq \mathrm{Proj}_1(R[x]) = \mathbf{P}^0_R = \mathrm{Spec}(R).$$

Definition 3.66 (Irrelevant ideal)**.** Let A be an $\mathbf{N}$-graded ring. The irrelevant ideal of A is the homogeneous ideal $A_+ = \bigoplus_{d\geq 1} A_d$.

Remark 3.67 (Spec vs. Proj). The relation between $\mathbf{A}^I$ and $\P^I$ from before extends to a relation between $\mathrm{Spec}(A)$ and $\mathrm{Proj}(A)$. Namely, let $\mathrm{D}(A_+) \subseteq \mathrm{Spec}(A)$ be the nonvanishin locus of the irrelevant ideal. Then there is a zigzag

$$\mathrm{Spec}(A) \hookleftarrow \mathrm{D}(A_+) \to \mathrm{Proj}(A)$$

over $\mathrm{Spec}(A_0)$, natural in $A \in \mathrm{CAlg}^{\mathbf{N},\mathrm{es}}$, where the second map is defined as follows. Since any family generating the unit ideal contains a finite generating subfamily, we have

$$\mathrm{D}(A_+) = \operatorname*{colim}_{d\in\mathbf{N}^{\mathrm{div}}_{>0}} \mathrm{D}(A_d).$$

The map $\mathrm{D}(A_+) \to \mathrm{Proj}(A)$ is then the colimit of the maps

$$\mathrm{D}(A_d) \to \mathrm{Proj}_d(A), \quad (\varphi\colon A \to R) \mapsto (\bigoplus_{n\in\mathbf{N}} \varphi|_{A_{nd}}\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(R)).$$

There is a canonical action of the affine group scheme $\mathbf{G}_m$ opn $\mathrm{Spec}(A)$, which restricts to each $\mathrm{D}(A_d)$: for any $\varphi\colon A \to R$ and $\lambda \in R^\times$, we define $\lambda\varphi\colon A \to R$ by

$$(\lambda\varphi)(a) = \lambda^d\varphi(a), \quad \text{for all } a \in A_d.$$

Each map $\mathrm{D}(A_d) \to \mathrm{Proj}_d(A)$ is then $\mathbf{G}_m$-invariant, and we have the following generalization of PProspostiino 3.24: the induced map $\mathrm{D}(A_1)/\mathbf{G}_m \to \mathrm{Proj}_1(A)$ is a monomorphism, whose image consits exactly of the trivial quotient lines; in particular, it is a bijection on local rings and principal ideal domains. This does not holds for arbitrary $d \in \mathbf{N}$, but it will turn out that $\mathrm{Proj}_d(A)$ is still the quotient $\mathrm{D}(A_d)/\mathbf{G}_m$ in the category of *schemes*.

Definition 3.68 (Vanishing and nonvanishing loci)**.** Let A be an $\mathbf{N}$-graded ring and let $F \subset A$ be a homogeneous subset.

(a) The vanishing locus of F is the subfunctor $\mathrm{V}(F) \subseteq \mathrm{Proj}(A)$, which is the union over $d \in \mathbf{N}^{\mathrm{div}}_0$ of the subfunctors $\mathrm{V}_d(F) \subseteq \mathrm{Proj}_d(A)$ given by

$$\mathrm{V}_d(F)(R) = \{\varphi\colon A^{(d)\otimes_{A_0}} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid \varphi(f) \text{ for all } f \in (F)^{(d)}\}.$$

(b) The nonvanishing locus of F is the subfunctor $\mathrm{D}(F) \subseteq \mathrm{Proj}(A)$, which is the union over $d \in \mathbf{N}^{\mathrm{div}}_{>0}$ of the subfunctors $\mathrm{D}_d(F) \subseteq \mathrm{Proj}_d(A)$ given by

$$\mathrm{D}_d(F)(R) = \{\varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid \varphi((F)_{nd}) \text{ generates } L^{\otimes n} \text{ for some } n \in N\}$$

Remark 3.69. One can easily check that V and D satisfies exactly the same formal properties as in the affine case (Proposition 2.59). Note also that $\mathrm{V}(F)$ depends only on (F) and $\mathrm{D}(F)$ only on $\sqrt{(F)}$. If A is generated by $A_{\leq 1}$, we have the following simpler description as subfunctors of $\mathrm{Proj}_1(A) \simeq \mathrm{Proj}(A)$:

$$\mathrm{V}(F)(R) = \{\varphi\colon A \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid \varphi(f) = 0 \text{ for all } f \in F\}$$
$$\mathrm{D}(F)(R) = \{\varphi\colon A \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid (\varphi(F)) \text{ contains } L^{\otimes n} \text{ for some } n \in \mathbf{N}\}.$$

Remark 3.70. Let A be an $\mathbf{N}$-graded ring.

(a) For any homogeneous subset $F \subseteq A$, there is a canonical isomorphism $\mathrm{Proj}(A/(F)) \xrightarrow{\sim} \mathrm{V}(F)$.
(b) Let $f \in A_d$ with $d \geq 1$. Then the nonvanishing locus $\mathrm{D}(f)$ of Definition 3.68 is contained in $\mathrm{Proj}_d(A)$ and matches the one from Lemma 3.62, by part (b) of the lemma. By part (a) of the lemma, there is a canonical isomorphism $\mathrm{Proj}(A_f) \xrightarrow{\sim} \mathrm{D}(f)$.
(c) If $f \in A_0$, there is a canonical isomorphism $\mathrm{Proj}(A_f) \xrightarrow{\sim} \mathrm{D}(f)$.

Proposition 3.71. *Let A be an $\mathbf{N}$-graded ring and let $F \subseteq A$ be a homogeneous subset. then*

(a) $\mathrm{V}(F)$ *is a closed subfunctor of* $\mathrm{Proj}(A)$.
(b) $\mathrm{D}(F)$ *is an open subfunctor of* $\mathrm{Proj}(A)$.

Example 3.72 (The Veronese embedding). Let k be a ring, M a k-module, and $d \in \mathbf{N}$. The *Veronese embedding of degree* d is the map of algebraic k-functors

$$\rho_d\colon \mathbf{P}(M) \to \mathbf{P}(\mathrm{Sym}_k^d(M))$$

sending a quotient line $M \otimes_k R \twoheadrightarrow L$ to its d-th symmetric power $\mathrm{Sym}_k^d(M) \otimes_k R \twoheadrightarrow \mathrm{Sym}_R^d(L) = L^{\otimes d}$. If $M = k^{(I)}$, the Veronese embedding of degree d is the map

$$\rho_d\colon \mathbf{P}_k^I \to \mathbf{P}_k^{\mathrm{Sym}^d(I)},$$

where $\mathrm{Sym}^d(I)$ is the quotient of I^d be the action of the symmetric group Σ_d. If $M = k^{n+1}$, this becomes the map

$$\rho_d\colon \mathbf{P}_k^n \to \mathbf{P}^{\binom{n+d}{n}-1},$$

which is given in coordinates by the formula

$$[a_0 : \ldots : a_n] \mapsto [\text{all degree } d \text{ monomials in the } a_i\text{'s}].$$

We claim that ρ_d is a closed immersion if $d \geq 1$. Indeed, ρ_d is Proj of a surjective degree d map of graded rings. Explicitly, it is the composite

$$\mathrm{Proj}_1(\mathrm{Sym}_k(M)) \to \mathrm{Proj}_d(\mathrm{Sym}_k(M)) \to \mathrm{Proj}_1(\mathrm{Sym}_k(\mathrm{Sym}_k^d(M))),$$

where the first map sends (L, φ) to $(L^{\otimes d, \varphi^{(d)}})$ and the second map is Proj_1 of the surjective map

$$\mathrm{Sym}_k(\mathrm{Sym}_k^d(M)) \twoheadleftarrow \mathrm{Sym}_k(M)^{(d)}.$$

If $d \geq 1$, the first map is an isomorphism by Theorem 3.63, and the second map is a closed immersion by Proposition 3.71.

Analogously to quasi-affine scheme, we define quasi-projective schemes as nonvanishing loci in projective scheme:

Definition 3.73 (Quasi-projective scheme)**.** Let k be a ring. An algebraic k-functor X is a *quasi-projective k-scheme* if there exists an $\mathbf{N}$-graded k-algebra A, generated by a finite subset of A_1, and a *finite* homogeneous subset $F \subseteq A$ such that $X \simeq \mathrm{D}(F) \subseteq \mathrm{Proj}(A)$. We denote by $\mathrm{QProj}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ the full subcategory spanned by the quasi-projective k-schemes.

We will see in ?? that every quasi-affine k-scheme of finite type is quasi-projective. Thus, under this finiteness assumption, quasi-projective k-schemes subsume all types of k-schemes discussed so far.

3.7. **Saturation.** The goal of this section is to classify closed and open subfunctors of $\mathrm{Proj}(A)$. This is more subtle than the analogous result for $\mathrm{Spec}(A)$ (Proposition 2.65), since there can be different homogeneous ideals in A with the same vanishing locus in $\mathrm{Proj}(A)$:

Example 3.74. The homogeneous ideal (x) and (x^2, xy) in $\mathbf{Z}[x, y]$ have the same vanishing locus in $\mathbf{P}^1 = \mathrm{Proj}(\mathbf{Z}[x, y])$. Since $(x^2, xy) \subseteq (x)$, we have $\mathrm{V}(x) \subseteq \mathrm{V}(x^2, xy)$. For the converse, consider an arbitrary quotient line $(a, b)\colon R^2 \twoheadrightarrow L$ in $\P^1(R)$ such that $a^2 = 0$ and $ab = 0$ in $L^{\otimes 2}$. Since $\mathrm{Sym}^2_R(R^2) \to L^{\otimes 2}$ is surjective, $L^{\otimes 2}$ is generated by (a^2, ab, b^2), hence by b^2 alone. Since L is invertible, this implies that b generates L. From $ab = 0$, we then deduce that $a = 0$, as desired.

Definition 3.75 (Saturation)**.** Let A be an $\mathbf{N}$-graded ring and let $H \subset A$ be a homogeneous ideal. The *saturation* of H is the homogeneous ideal

$$H^{\mathrm{sat}} = \{x \in H \mid \text{for all } f \in A_1, \text{ there exists } n \in N \text{ such that } f^n x \in H\}.$$

We say that H is *saturated*, if $H^{\mathrm{sat}} = H$.

Remark 3.76. Saturated ideals are determined by their *tail*: if $H \subseteq A$ is a homogeneous ideal and $D \subset \mathbf{N}$ is an infinite subset, then $H^{\mathrm{sat}} = (\bigoplus_{d \in D} H_d)^{\mathrm{sat}}$.

Example 3.77.
- (a) In $\mathbf{Z}[x, y]$, we have $(x^2, xy) = (x)$.
- (b) If $\mathfrak{p} \subseteq A$ is a homogeneous prime ideal not containing A_1, then $\mathfrak{p}$ is saturated.
- (c) In any polynomial ring $k[x_i \mid i \in I]$, (0) is saturated.

We briefly consider a more general context for this definiton:

Definition 3.78 (I-nilpotent, I-local module)**.** Let R be a ring, $I \subseteq R$ a subset, and M an R-module. An element $x \in M$ is called I*-nilpotent* if, for every $f \in I$, there exists $n \in \mathbf{N}$ such that $f^n x = 0$. The I-nilpotent elements of M form a submodule $\Gamma_I M \subseteq M$. The module M is called:
- (a) I*-nilpotent* if $\Gamma_I M = M$;
- (b) I*-local* if every R-linear map $h\colon P \to Q$ with I-nilpotent kernel and cokernel induces a bijection
$$h^*\colon \mathrm{Map}(Q, M) \xrightarrow{\sim} \mathrm{Map}(P, M).$$
We denote the corresponding full subcategories of Mod_R by $\mathrm{Mod}_R^{I-\mathrm{nil}}$ and $\mathrm{Mod}_R^{I-\mathrm{loc}}$.

Note that these conditions depend only on the radical ideal generated by I. The categories $\mathrm{Mod}^{I-\mathrm{nil}}$ and $\mathrm{Mod}_R^{I-\mathrm{loc}}$ are abelian and fit in a short exact sequence of Grothendieck abelian categories

$$\begin{tikzcd} \mathrm{Mod}_R^{I-\mathrm{loc}} \arrow[r, shift left, hook] & \mathrm{Mod}_R \arrow[l, shift left, "\Gamma_I"] \arrow[r, shift left, "\mathrm{L}_I"] & \mathrm{Mod}_R^{I-\mathrm{loc}} \arrow[l, shift left, hook'] \end{tikzcd}$$

where all functors are exact. In particular, an R-module M is I-nilpotent if and only if $\mathrm{L}_I M = 0$, and $\Gamma_I M$ is the kernel of the unit map $M \to \mathrm{L}_I M$.

Remark 3.79. An R-module M is I-nilpotent if and only if $M_f = 0$ for all $f \in I$. Hence, an R-linear map $M \to N$ induces an isomorphism $\mathrm{L}_I M \xrightarrow{\sim} \mathrm{L}_I N$ if and only if it induces isomorphisms $M_f \xrightarrow{\sim} N_f$ for all $f \in I$. In particular, for any $f \in R$, $\{f\}$-local is a synonymous with f-periodic, and $\mathrm{L}_{\{f\}} M = M_f$.

Remark 3.80. We will give later a geometric interpretation of $\mathrm{Mod}_R^{I-\mathrm{loc}}$ when I is finite: it can be identified with the category of quasi-coherent modules on the open subscheme $\mathrm{D}(I) \subseteq \mathrm{Spec}(R)$.

Example 3.81.
- (a) Let L be a line over R and let $s\colon R \to L$. Then an R-module is s-periodic if and only if it is $\mathrm{im}(s^\vee)$-local.
- (b) Let A be an $\mathbf{N}$-graded ring. The saturation of a homogeneous ideal $H \subseteq A$ is the kernel of $A \to \mathrm{L}_{A_1}(A/H)$. In particular, $(0)^{\mathrm{sat}} = \Gamma_{A_1} A$.

Proposition 3.82 (Conservativity of Proj)**.** *Let A and B be $\mathbf{N}$-graded rings and let $A \to B$ be an eventually surjective map. The following conditions are equivalent:*

(a) $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ *is an isomorphism.*
(b) For all $d \geq 1$, $\mathrm{L}_{A_d} \to \mathrm{L}_{A_d} B^{(d)}$.

Corollary 3.83 (Functoriality projective Nullstellensatz)**.** *Sending a homogeneous subset $F \subseteq k[x_i \mid i \in I]$ to its vanishing locus $\mathrm{V}(F) \subseteq \mathbf{P}^I_k$ induces an order-resersing bijection*

$$\mathrm{V}\colon \{\textit{saturated homogeneous ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbf{P}^I_k\}.$$

For simplicity, we state the next result for $\mathbf{N}$-graded rings that are generated in degrees ≤ 1.

Proposition 3.84 (Classification of closed and open subfunctors of projective schemes)**.** *Let A be an $\mathbf{N}$-graded ring generated by $A_{\leq 1}$.*

(a) The construction $F \mapsto \mathrm{V}(F)$ induces an order-reversing injection

$$\{\textit{saturated homogeneous ideals in } A\} \hookrightarrow \{\textit{closed subfunctors of } \mathrm{Proj}(A)\},$$

which is bijective if A is finitely generated as an A_0-algebra.

(b) The construction $F \mapsto \mathrm{D}(F)$ induces an order-preserving bijection

$$\{\textit{radical homogeneous ideals in } A_+\} \xrightarrow{\sim} \{\textit{open subfunctors of } \mathrm{Proj}(A)\}.$$

If $A \to B$ is surjective, these are further equivalent to: for all $d \leq 1$, $\ker(A \to B)^{(d)}$ is A_d-nilpotent. If A is generated by $A_{\leq 1}$, it suffices to consider $d = 1$ in (b) and (c).

Remark 3.85. Proposition 3.84 generalizes to all $\mathbf{N}$-graded ring A as follows: (b) remains unchanged, while in (a) we must replace the left-hand side with the set of homogeneous ideals H having the following property: H contains every homogeneous element $x \in A$ that satisfies $(x)_d \subseteq H^{(d),\mathrm{sat}}$ for all $d \geq 1$.

3.8. **Projective closure.** Recall from Projecctive completions of finite affine spaces that any affine space $\mathbf{A}^I$ admits a canonical open embedding $\mathbf{A}^I \hookrightarrow \mathbf{P}^{I \sqcup \{0\}}$, which identifies $\mathbf{A}^I$ with the nonvanishing locus of x_0 in $\mathbf{P}^{I \sqcup \{0\}}$. The goal of this section is to compare vanishing and nonvanishing loci in $\mathbf{A}^I$ and $\mathbf{P}^{I \sqcup \{0\}}$.

Proposition 3.86. *Let I be a set not containing 0, let $F \subseteq k[x_0, x_i \mid i \in I]$ be a homogeneous subset, and let $F_0 \subseteq k[x_i \mid i \in I]$ be obtained from F by setting $x_0 = 1$. Consider the vanishing loci $\mathrm{V}(F) \subseteq \mathbf{P}^{I \sqcup \{0\}}$ and $\mathrm{V}(F_0) \subseteq \mathbf{A}^I_k$ and the vanishing loci $\mathrm{D}(F) \subseteq \mathbf{P}^{I \sqcup \{0\}}$ and $\mathrm{F}_0 \subseteq \mathbf{A}^I_k$. Then*

$$\mathrm{V}(F) \cap \mathbf{A}^I_k = \mathrm{V}(F_0) \quad \textit{and} \quad \mathrm{D}(F) \cap \mathbf{A}^I_k = \mathrm{D}(F_0).$$

Next, we want to describe the projective closure of affine vanishing loci.

Definition 3.87 (Closure of subfunctor)**.** Let X be an algebraic functor and $Y \subseteq X$ a subfunctor. The *closure* of Y in X is the smallest closed subfunctor of X containing Y. This is well-defined as arbitraty intersections of closed subfunctors are closed (by Proposition 2.59(ii)).

Definition 3.88 (Homogenization of polynomials)**.** Let k be a ring and I a set (not containing 0).

(a) Let $f \in k[x_i \mid i \in I]$. The *homogenization* of f is the homogeneous polynomial

$$f^{\mathrm{h}} = x_0^d f((\frac{x_i}{x_0})_{i \in I}) \in k[x_0, x_i \mid i \in I],$$

where d is the maximal degree of a monomial with nonzero coefficient in f, or equivalently the smallest integer for which the above expression is a polynomial.

(b) Let $F \subseteq k[x_i \mid i \in I]$ be an ideal. The *homogenization* of F is the homogeneous ideal

$$F^{\mathrm{h}} = (f^{\mathrm{h}} \mid f \in F) \subseteq k[x_0, x_i \mid i \in I].$$

Remark 3.89. (a) We can recover any polynomial or ideal from its homogenization by setting $x_0 = 1$ is in F. It follows that homogenization preserves radical and prime ideals.
(b) We have $(f^{\mathrm{h}}) = (f)^{\mathrm{h}}$, but this does not generalize to ideals with two or more generators. For example, if F is the ideal in $k[x_1, x_2]$ generated by $f = x_1 + x_2^2$ and $g = x_2$, then $F^{\mathrm{h}} = (x_1, x_2)$, but the ideal generated by $f^{\mathrm{h}} = x_0x_1 + x_2^2$ and g^{h} does not contain x_1.

Corollary 3.90 (Projective closure of vanishing loci)**.** *Let I be a finite set and $F \subseteq k[x_i \mid i \in I]$ an ideal with vanishing locus $\mathrm{V}(F) \subseteq \mathbf{A}_k^I$. Then $\mathrm{V}(F^{\mathrm{h}}) \subseteq \mathbf{P}^{I \sqcup \{0\}}$ is the closure of $\mathrm{V}(F)$.*

Corollary 3.91 (Quasi-affine schemes of finite type are quasi-projective)**.** *Let k be a ring, A a k-algebra of finite type, and $F \subseteq A$ a finite subset. Then the subfunctor $\mathrm{D}(F) \subseteq \mathrm{Spec}(A)$ is a quasi-projective k-scheme.*

Definition 3.92 (Finite scheme)**.** Let k be a ring. An algebraic k-functor X is finite k-scheme if there exists a finite k-algebra A such that $X \simeq \mathrm{Spec}(A)$.

Proposition 3.93 (Finite schemes are projective)**.** *Let $X \subseteq \mathbf{A}_k^I$ be a closed subfunctor such that every function on X is integral over k, i.e. satisfies a monic polynomial equation over k. Then X is closed in $\mathbf{P}_k^{I \sqcup \{0\}}$. In particular, every finite k-scheme is projective.*

Remark 3.94. Conversely, if an affine k-scheme is a projective, then it finite; we will prove this later using the notion of properness. This means that any closed subscheme of $\mathbf{A}_k^n$, if not finite, must "go to infinity" and not remain closed in $\mathbf{P}_k^n$. This is in stark contrast to the analogous situation in differential geometry, where there exists positive dimensional closed submanifold of $\mathbf{R}^n$ that are "away from infinity", i.e., remain closed in $\mathbf{RP}^n$.

3.9. **Example of projective schemes.**

Example 3.95 (Weighted projective spaces). Let I be a set and $w\colon I \to \mathbf{N}_{>0}$ a map. Then we can equip the polynomial k-algebra $k[x_i \mid i \in I]$ with the $\mathbf{N}$-graded structure in which x_i is homogeneous of degree $w(i)$. The resulting algebraic k-functor $\mathbf{P}_k^w = \mathrm{Proj}(k[x_i \mid i \in I])$ is called the *weighted projective space* with weights w. It is a projective k-scheme if I is a finite, by Corollary 3.64.

Example 3.96 (The Segre embedding). Let k be a ring and let M and N be k-modules. The *Segre embedding* is the map of algebraic k-functors

$$\varsigma\colon \mathbf{P}_k^I \times \mathbf{P}_k^J \to \mathbf{P}_k^{I \times J}.$$

If $M = k^{n+1}$ and $N = k^{n+1}$, this becomes the map

$$\varsigma\colon \mathbf{P}_k^m \times \mathbf{P}_k^n \to \mathbf{P}_k^{mn+m+n},$$

which is given in coordinates by the formula

$$([a_0 : \ldots : a_m], [b_0 : \ldots : b_n]) \mapsto [\text{all products } a_ib_j].$$

More generally, we have a Segre embedding

$$\varsigma\colon \prod_{i=1}^n \mathbf{P}(M_i) \to \mathbf{P}(\bigotimes_{i=1}^n M_i)$$

for any finite family of k-modules $M_1, \ldots, M_n$. We claim that the Segre embedding is a closed immersion. Given surjective maps $M' \twoheadrightarrow M$ and $N' \twoheadrightarrow N$, we have a commutative square

$$\begin{tikzcd}
\mathbf{P}(M) \times \mathbf{P}(N) \arrow[r, "\varsigma"] \arrow[d, hook] & \mathbf{P}(M \otimes_k N) \arrow[d, hook] \\
\mathbf{P}(M') \times \mathbf{P}(N') \arrow[r, "\varsigma"] & \mathbf{P}(M' \otimes_k N')
\end{tikzcd}$$

where the vertical maps are closed immersions. Hence if the bottom horizontal map is closed immersion, so is the top horizontal map. We may therefore assume that M and N have bases $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$, so that $M\otimes_k N$ has basis $(e_i\otimes f_j)_{i,j}$. For any map $\operatorname{Spec}(R)\to \mathbf{P}(M\otimes_k N)$, the pullback $(\mathbf{P}(M)\times\mathbf{P}(N))\times_{\mathbf{P}(M\otimes_k N)}\operatorname{Spec}(R)$ satisfies Zariksi descent, by Proposition 3.61 and Corollary 2.72. Using Proposition 2.79, we see that it is enough to show that each restriction

$$\varsigma_{ij}\colon \varsigma^{-1}(\mathrm{D}(e_i\otimes f_j))\mapsto \mathrm{D}(e_i\otimes f_j)$$

of ς is a closed immersion. The open subfunctor $\mathrm{D}(e_i\otimes f_j)\subseteq \mathbf{P}(M\otimes_k N)$ is an affine space on the set $(I\times J)\setminus\{(i,j)\}$. Its preimage $\varsigma^{-1}(\mathrm{D}(e_i\otimes f_j))\subseteq \mathbf{P}(M)\times\mathbf{P}(N)$ is the subfunctor $\mathrm{D}(e_i)\times\mathrm{D}(f_j)$, which is an affine space on the set $(I\setminus\{i\})\sqcup(J\setminus\{j\})$. By inspection, the map ς_{ij} is then Spec of the map of polynomial rings

$$k[z_{kl}\mid (k,l)\in (I\times J)\setminus\{i,j\}]\to k[x_k,y_l\mid k\in I\setminus\{i\}, l\in J\setminus\{j\}],\quad z_{kl}\mapsto x_ky_l,$$

where $x_i=y_j=1$. Since $z_{kj}\mapsto x_k$ and $z_{il}\mapsto y_l$, this map is surjective, so that $\varsigma_{i,j}$ is a closed immersion as desired.

Corollary 3.97. *Let X and Y be projective k-schemes. Then $X\times Y$ is a projective k-scheme.*

Remark 3.98. The Veronese embedding of Example 3.72 is determined by the Segre embedding as follows: for any k-module M and $d\geq 1$, there is a commutative square

$$\begin{tikzcd} \mathbf{P}(M) \arrow[r, "\rho_d"] \arrow[d, hook, "\Delta"'] & \mathbf{P}(\mathrm{Sym}^d_K(M)) \arrow[d, hook] \\ \mathbf{P}(M)^{\times d} \arrow[r, "\varsigma"] & \mathbf{P}(M^{\otimes d}) \end{tikzcd}$$

where the vertical maps are the inclusions of the Σ_d-fixed points. This gives another proof that ρ_d is a closed immersion.

Next we consider a generalization of projective spaces where we replace quotient lines with quotient spaces of arbitrary rank.

Definition 3.99 (Grassmannian)**.** Let k be a ring, M a k-module, and $n\in\mathbf{N}$. The rank n *Grassmannian* of M is the algebraic k-functor $\mathrm{Gr}_n(M)$ given by

$$\mathrm{Gr}_n(M)(R)=\{\text{quotient spaces of } M\otimes_k R \text{ of constant rank } n\}.$$

Remark 3.100. By definition, $\mathrm{Gr}_1(M)=\mathbf{P}(M)$. If M is a vector space of constant rank r, then duality induces isomorphism $\mathrm{Gr}_n(M)\simeq \mathrm{Gr}_{r-n}(M^\vee)$(Corollary 3.22).

Example 3.101 (The Plücker embedding)*.* Let M be a k-module and let $n\in\mathbf{N}$. The *Plücker embedding* is the map of algebraic k-functors

$$\varpi\colon\ \mathrm{Gr}_n(M)\to\mathbf{P}(\Lambda^n_k(M))$$

that sends a quotient space of rank n to its n-th exterior power, which is a quotient line. We claim that the Plücker embedding is a closed immersion. For any surjective map $M'\twoheadrightarrow M$, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Gr}_n(M) \arrow[r, "\varpi"] \arrow[d, hook] & \mathbf{P}(\Lambda^n_k(M)) \arrow[d, hook] \\ \mathrm{Gr}_n(M') \arrow[r, "\varpi"] & \mathbf{P}(\Lambda^n_k(M')) \end{tikzcd}$$

where the vertical maps are closed immersions. We can therefore assume that M has a basis $(e_i)_{i\in I}$, so that $\Lambda^n_k(M)$ has an induced basis $(e_J)_J$ (defined up to signs) indexed by the set $\binom{I}{n}$ of n-element subsets of I. Zariski descent for modules implies, as in Proposition 3.61, that $\mathrm{Gr}_n(M)$ satisfies Zariski descent. By Proposition 2.79, it thus suffices to show that each map $\varpi_J\colon \varpi^{-1}(\mathrm{D}(e_J)) \to \mathrm{D}(e_J)$ is a closed immersion. Then open subfunctor $\mathrm{D}(e_J) \subseteq \mathbf{P}(\Lambda^n_k)(M)$ is an affine space on the set $\binom{I}{n} \setminus \{J\}$. The preimage $\varpi^{-1}(\mathrm{D}(e_J)) \subseteq \mathrm{Gr}_n(M)$ is the subfunctor consisting og quotient spaces $R^{(I)} \twoheadrightarrow V$ such that the composite $R^{(J)} \hookrightarrow R^{(I)} \twoheadrightarrow V$ is an isomorphism, which is an affine space on the set $(I \setminus J) \times I$. By inspection, the map ϖ_J is $\mathbf{A}(-)$ of the map of free k-modules given by

$$e_K \mapsto \begin{cases} e_{(i,j)} & \text{if } K = (J \setminus \{j\} \cup \{j\}), \\ 0 & \text{else} \end{cases}$$

As this map is evidently surjective, ϖ_J is a closed immersion, as desired. If M is of finite type, it follows that the Grassmannian $\mathrm{Gr}_n(M)$ is projective k-scheme.

Definition 3.102 (Flag schemes)**.** Let M be a k-module and let $n = (n_1, \ldots, n_s)$ be an increasing sequence of natural numbers.

(a) A *flag of type* n on M is a sequence of quotient modules

$$M \twoheadrightarrow F_s \twoheadrightarrow \ldots \twoheadrightarrow F_1$$

where each F_i is a vector space of constant rank n_i.

(b) The type n *flag scheme* of M is the algebraic k-functor $\mathrm{Flag}_n(M)$ given by

$$\mathrm{Flag}_n(M)(R) = \{\text{flags of type } n \text{ on } M \otimes_k R\}.$$

Remark 3.103. By definition, $\mathrm{Flag}_{(n)}(M) = \mathrm{Gr}_n(M)$. If M is a vector space of constant rank r, then duality induces isomorphisms $\mathrm{Flag}_{(n_1,\ldots,n_s)}(M) \simeq \mathrm{Flag}_{(r-n_s,\ldots,r-n_1)}(M^\vee)$.

Example 3.104 (Projectivity of flag schemes)*.* Sending a flag to its components define a monomorphism

$$\mathrm{Flag}_n(M) \hookrightarrow \prod_{i=1}^{s} \mathrm{Gr}_{n_i}(M).$$

We claim that it is a closed immersion, and hence that $\mathrm{Flag}_n(M)$ is a projective k-scheme if M is of finite type. Indeed, for any R-point $x\colon \operatorname{Spec}(R) \to \prod_{i=1}^s \mathrm{Gr}_{n_i}(M)$ classifying quotient spaces $F_1, \ldots, F_s$ of $M \otimes_k R$, the preimage of $\mathrm{Flag}_n(M)$ by x is exactly the joint vanishing locus of the R-linear maps

$$\ker(M \otimes_k R \twoheadrightarrow F_{i+1}) \hookrightarrow M \otimes_k R \twoheadrightarrow F_i$$

for $1 \le i \le s-1$, which is a closed subfunctor of $\operatorname{Spec}(R)$ by Proposition 3.51.

3.10. **The projective Nullstellensatz.** Hilbert's Nullstellensatz (2.88) implies an analogous result for solutions of homogeneous polynomial equations in projective space. We record this classical result here, although it is mostly of historical interest.

For a ring k, define the maps

$$\{\text{homogeneous subsets of } k[x_0, \ldots, x_n]\} \underset{\mathrm{I}}{\overset{\mathrm{V}}{\rightleftarrows}} \{\text{subsets of } \mathbf{P}^n(k)\}$$

as follows:

$$\begin{aligned} (V)(F) &= \{x \in \mathbf{P}^n(k) \mid f(x) = 0 \text{ for all } f \in F\} \\ \mathrm{I}(X) &= \{f \in k[x_0, \ldots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in X\} \end{aligned}$$

Note that both maps are order-reversing and that $F \subseteq \mathrm{I}(\mathrm{V}(F))$ and $X \subseteq \mathrm{V}(\mathrm{I}(X))$ (in other words, this is an adjunction between posets). Note also that $\mathrm{I}(X)$ is always saturated homogeneous ideal in $k[x_0, \ldots, x_n]$, which is radical if k is reduced. Call a subset $X \subseteq \mathbf{P}^n(k)$ *algebraic* if it lies in the image of V, or equivalently if $X = \mathrm{V}(\mathrm{I}(X))$.

Proposition 3.105 (Projective Nullstellensatz). *Let k be an algebraically closed field and let $n \in \mathbf{N}$. For any homogeneous subset $F \subseteq k[x_0, \dots, x_n]$, we have*

$$\mathrm{I}(\mathrm{V}(F)) = \sqrt{(F)}^{\mathrm{sat}}.$$

Consequently, the maps V *and* I *define a one-to-one correspondence*

$$\{\textit{saturated radical homogeneous ideals in } k[x_0, \dots, x_n]\} \underset{\mathrm{I}}{\overset{\mathrm{V}}{\rightleftarrows}} \{\textit{algebraic subsets of } \mathbf{P}^n(k)\}$$

Corollary 3.106. *Let k be a field and let $n \in \mathbf{N}$. Then there exists an order reversing bijection*

$$\{\textit{saturated radical homogeneous ideals in } k[x_0, \dots, x_n]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbf{P}^n \colon \mathrm{Field}_k^{\mathrm{fin}} \to \mathrm{Set}\}.$$

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Corollary 3.107. *Let R be a ring, L a line, $s \in L$. Then* $\mathrm{D} = \mathrm{Spec}(R_s)$*, where $D(s)$ is the epimorphisms $R \xrightarrow{s}$.*

In summary, if we have a d-homogeneous polynomial $f \in \mathbf{Z}[x_1, \dots, x_n]_d$, this gives us an open subfunctor $\mathrm{D}(f) \subseteq \mathbf{P}^n$. For all map $x \colon \mathrm{Spec}(R) \to \mathbf{P}^n$, we have $x^{-1}(\mathrm{D}(f)) = \mathrm{D}(s)$, we have an epimorphism $a \colon R^{n+1} \twoheadrightarrow L$, where $s = f(a) \in L^{\otimes d}$. The inclusion $\mathrm{D}(f) \hookrightarrow \mathrm{Spec}(R)$ is *relatively affine*.

3.11. **Improved Zariski descent.** Let $(f_i)_{i \in I}$ in R generates the unit ideal, then the line L over R, $(s_i)_{i \in I}$ is a generating system. Then for all R-modules M, we have equalizer

$$M \to \prod_{i \in I} M_{s_i} \rightrightarrows \prod_{i,j \in I} M_{s_i, s_j}$$

There are two ways to prove this: Key input: given $s_1, \dots s_n \in L$ and $d \geq 1$, the elements $s_1, \dots, s_n$ generates L is equivalent to $s_1^d, \dots s_n^d$ generate $L^{\otimes d}$: for $R^n \to L$, choose $(f_1 \dots, f_r) = R$ such that $L_{f_i} = R_{f_i}$ for all $i \in I$. Second: $R^n \to L$ is a surjection is equivalent to for all $i \in I$, the map $R_{f_i} \to L_{f_i}$ is surjective. The map $R^n \to L^{\otimes d}$ is surjective if and only if for all $i \in I$, the map $R^n_{f_i} \to L^{\otimes d}_{f_i}$ is surjective. Then reduce to the special case.

3.12. **The projective functor.**

Notation 3.108. We denote A a $\mathbf{N}$-graded ring, $d \in \mathbf{N}$. We adopt the following notation:

(a) The ring $A^{(d)}$ is the $\mathbf{N}$-graded ring $\bigoplus_{n \in \mathbf{N}} A_{nd}$ such that $(A^{(d)})_n = A_{nd}$.
(b) The ring $A_+ = \bigoplus_{d \geq 1} A_d$ is called the *irrelevant ideal*.
(c) Let $f \in A$ be a homogeneous element. Then A_f is a graded ring; we denote $A_{(f)} = (A_f)_{(0)}$.

Construction 3.109. Let A be a $\mathbf{N}$-graded ring. We will define $\mathrm{Proj}(A) \in \mathrm{Fun}(\mathrm{CAlg}_{A_0}, \mathrm{Set})$: let $R \in \mathrm{CAlg}_{A_0}$, then

$\mathrm{Proj}_1(A)(R) = \{(L, \varphi) \mid L$ quotient line of $A_1 \otimes_{A_0} R$, φgraded R-algebra map $A \otimes_{A_0} R \to \mathrm{Sym}_R(L)$ such that $\varphi_1$is th

Equivalently, it is the set of quotient R-algebras $A \otimes_{A_0} R \twoheadrightarrow R$ with B_1 a line over B and $\mathrm{Sym}_R(B_1) \simeq B$. Let $d \in \mathbf{N}$, we define $\mathrm{Proj}_d(A) = \mathrm{Proj}_1(A^{(d)})$. For each $n \in \mathbf{N}$, there is a map $\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A)$, given by $(L, \varphi) \mapsto (L^{\otimes n}, \varphi^{(n)})$. In other words, it is given by

$$A^{(d)} \otimes A_0 R \twoheadrightarrow \mathrm{Sym}_R(L) \mapsto \varphi^{(n)} \cdot A^{(dn)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L^{\otimes n})$$

which gives us a functor $\mathbf{N}^{\div} \to \mathrm{Fun}(\mathrm{CAlg}_{A_0}, \mathrm{Set})$ by $d \mapsto \mathrm{Proj}_d(A)$. We define $\mathrm{Proj}(A) = \mathrm{colim}_{d \in \mathbf{N}^{\div}_{\neq 0}}(\mathrm{Proj}_d(A)) \to \mathrm{Spec}(A_0)$.

Remark 3.110. (a) Let $d \geq 1$, then $d\mathbf{N}^{\div}_{\neq 0} \subseteq \mathbf{N}^{\div}_{\neq 0}$ is a cofinal subset. This implies that $\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)})$.

(b) Functoriality in A: let $A \xrightarrow{\alpha} B$ is map, when do we have $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$? If α is surjective, then we get $\mathrm{Proj}(\alpha)$ by precomposition with α. In fact, we say that $A \to B$ is *eventually surjective* if for every $d \geq 1$ and $b \in B_d$, there exists $n \geq 0$ such that $A_{nd} \otimes A_0 \to B_{nd}$ with $b^n \in B_{nd}$.
(c) Even better" if you have a degree map $A \to B$, i.e. a graed map $A \to B^{(d)}$, which is eventually surjective, then $\mathrm{Proj}(B) \simeq \mathrm{Proj}(B^{(d)}) \to \mathrm{Proj}(A)$.

Theorem 3.111. *Let A be a $\mathbf{N}$-graded ring and $d \geq 1$.*

(a) The map $\mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ is an open immersion.
(b) Suppose A is generated as an A_0-algebra by homogeneous elements with degrees dividing d. Then $\mathrm{Proj}_d(A) \simeq \mathrm{Proj}(A)$.

Corollary 3.112. *If A is finitely generated over A_0. Then $\mathrm{Proj}(A)$ is a projective A_0-scheme.*

Proof. If $d >> 0$, then $A^{(d)}$ is generated by A_d as A_0-objects. □

Example 3.113 (Veronese embeddings). Fix a set I and $d >> 1$. Let $I(d) = I^d/\Sigma_d$. There is a map

$$k[z_{i_1,\dots,i_d} \mid i_1, \dots, i_d \in I(d)] \twoheadrightarrow k[x_i \mid i \in I], \quad z_{i_1,\dots,i_d} \mapsto x_{i_1} \dots x_{i_d}$$

This gives us $\mathrm{hSol}_\Sigma \subseteq \mathbf{P}_k^{I(d)}$ and $\mathrm{hSol}_\Sigma \simeq \mathrm{Proj}(k[x_i \mid i \in I]^{(d)}) \simeq \mathrm{Proj}(k[x_i \mid i \in I]) = \mathbf{P}_k^I$. This gives us an embedding $\mathbf{P}_k^I \hookrightarrow \mathbf{P}_k^{I(d)}$ called the *degreed Veronese embedding.* Let $|I| = 2, d = 2$ and $|I(d) = 2$, then we have an embedding $\mathbf{P}_k^1 \hookrightarrow \mathbf{P}_k^2$, where $\mathbf{P}^1 \simeq V(z_{01}^2 - z_{00}z_{11}) \subseteq \mathbf{P}^2$. This is not easy to prove. Let $d = 3$ and $|I(d)| = 4$. Then we have $\mathbf{P}^1 \simeq V(z_{000}z_{111} - z_{001}z_{011}, z_{001}^2 - z_{000}z_{011}, z_{011}^2 - z_{001}z_{111})$. In projective coordinates $\mathbf{P}^1 \hookrightarrow \mathbf{P}^2$, we have $(a, b) \colon R^2 \twoheadrightarrow L \mapsto (a^2, ab, b^2) \colon R^2(2) \twoheadrightarrow L^{\otimes}$.

4. Quasi-coherent modules

4.1. Limits of categories.

Definition 4.1 (Category-valued functor)**.** Let $\mathcal{J}$ be a category. A *functor*

$$F\colon \mathcal{J} \to \mathrm{Cat}_\infty$$

consists of

(a) for every object $I \in \mathcal{J}$, a category $F(I)$;
(b) for every map $f\colon I \to J$ in $\mathcal{J}$, a functor $F(f)\colon F(I) \to F(J)$;
(c) for every object $I \in \mathcal{J}$, a natural isomorphism $\eta_I\colon \mathrm{id}_{F(I)} \simeq F(\mathrm{id}_I)$;
(d) for every pair of maps $f\colon I \to J$ and $g\colon J \to K$ in $\mathcal{J}$, a natural isomorphism $\mu_{f,g}\colon F(g)\circ F(f) \simeq F(g\circ f)$;

satisfying the following conditions:

(a) for every map $f\colon I \to J$, the following diagrams commute:

$$\begin{tikzcd} F(f) \arrow[r, "\eta_I"] \arrow[dr, "\mathrm{id}"'] & F(f)\circ F(\mathrm{id}_I) \arrow[d, "\mu_{\mathrm{id}_I,f}"] \\ & F(f), \end{tikzcd} \qquad \begin{tikzcd} F(f) \arrow[r, "\mu_J"] \arrow[dr, "\mathrm{id}"'] & F(\mathrm{id}_J)\circ F(f) \arrow[d, "\mu_{f,\mathrm{id}_J}"] \\ & F(f); \end{tikzcd}$$

(b) for every triple of maps $f\colon I \to J$, $g\colon J \to K$ and $h\colon K \to L$ in $\mathcal{J}$, the following square commutes:

$$\begin{tikzcd} F(h)\circ F(g)\circ F(f) \arrow[r, "\mu_{f,g}"] \arrow[d, "\mu_{g,h}"'] & F(h)\circ F(g\circ f) \arrow[d, "\mu_{g\circ f,h}"] \\ F(h\circ g)\circ F(f) \arrow[r, "\mu_{f,h\circ g}"'] & F(h\circ g\circ f) \end{tikzcd}$$

Example 4.2 (Self-indexing functor). Let $\mathcal{C}$ be a category with pullbacks. There is then a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad X \mapsto \mathcal{C}_{/X},$$

called the *self-indexing functor* of $\mathcal{C}$, sending a map $f\colon X \to Y$ to the pullback functor

$$f^*\colon \mathcal{C}_{/X} \to \mathcal{C}_{/Y}, \quad U \mapsto U \times_X Y.$$

The natural transformation $\eta_X\colon \mathrm{id}_{\mathcal{C}_{/X}} \simeq \mathrm{id}_X^*$ and $\mu_{f,g}\colon f^*\circ g^* \simeq (g\circ f)^*$ are given by the canonical isomorphisms $U \simeq U\times_X X$ and $(U\times_X Y)\times_Y Z$, induced by the universal property of pullbacks.

Example 4.3 (Categories of modules). The assignment $R \mapsto \mathrm{Mod}_R$ is functorial in three ways:

(a) There is a functor

$$\mathrm{Mod}^*\colon \mathrm{CAlg} \to \mathrm{Cat}_\infty, \quad R \mapsto \mathrm{Mod}_R,$$

sending a ring map $f\colon R \to S$ to the extension of scalars functor

$$f^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto M\otimes_R S.$$

The natural isomorphisms $\eta_R\colon \mathrm{id}_{\mathrm{Mod}_R} \simeq \mathrm{id}_R^*$ and $\mu_{f,g}\colon g^*\circ f^* \simeq (g\circ f)^*$ are given by the canonical isomorphisms $M \simeq M\otimes_R R$ and $(M\otimes_R S)\otimes_S T \simeq M\otimes_R T$, induced by the universal property of scalar extension.

(b) There is a functor

$$\mathrm{Mod}_*\colon \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad R \mapsto \mathrm{Mod}_R,$$

sending a ring map $f\colon R \to S$ to the restriction of scalars functor

$$f_*\colon \mathrm{Mod}_S \to \mathrm{Mod}_R, \quad M \mapsto M.$$

In this case, the natural isomorphisms μ_R and $\eta_{f,g}$ are take to be the identity.

(c) Finally there is a functor

$$\mathrm{Mod}^!\colon \mathrm{CAlg} \to \mathrm{Cat}_\infty, \quad R \mapsto \mathrm{Mod}_R,$$

sending a ring map $f\colon R \to S$ to the coextension of scalars functor

$$f^!\colon \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(S, M).$$

As in (a), the natural isomorphisms μ_R and $\eta_{f,g}$ are induced by the universal property of scalar coextension.

Example 4.4 (Categories of algebras). The assignment $R \mapsto \mathrm{CAlg}_R$ is both covariant and contravariant in R as in (a) and (b) of Example 4.3. There is however no analogue of (c) of Example 4.3 for algebras.

Example 4.5 (Subcategories of modules). Any property of modules that is preserved by extension of scalars defines a subfunctor of $\mathrm{Mod}^*\colon \mathrm{CAlg} \to \mathrm{Cat}_\infty$. This holds for the following properties, and any combination thereof: finitely generated, finitely presented, flat, projective, free, vector space, line, etc.

Example 4.6 (Posets of ideals). If we view posets as categories, then a poset-valued functor $\mathcal{C} \to \mathrm{Pos}$ defines a category-valued functor (where η_I and $\mu_{f,g}$ are identities). For example, we have the following functors $\mathrm{CAlg} \to \mathrm{Pos}$:

(a) The poset Id_R of ideals in R is functorial in R: a map $R \to S$ induces $\mathrm{Id}_R \to \mathrm{Id}_S, \quad I \mapsto IS$.
(b) The poset Rad_R of radical ideals in R is functorial in R: a map $R \to S$ induces $\mathrm{Rad}_R \to \mathrm{Rad}_S \quad, I \mapsto \sqrt{IS}$.

Notation 4.7. Let $\mathcal{J}$ be a category. The *left cone* $\mathcal{J}^{\triangleleft}$ on $\mathcal{J}$ is the category obtained from $\mathcal{J}$ by adjoining a new object $-\infty$ which is strictly initial, i.e., such that $\mathrm{Map}(-\infty, I) = *$ for all $I \in \mathcal{J}^{\triangleleft}$ and $\mathrm{Map}(I, -\infty) = \varnothing$ for all $i \in \mathcal{J}$. Dually, the *right cone* $J^{\triangleright}$ is obtained from $\mathcal{J}$ by adding a strictly final object ∞.

Definition 4.8 (Limits of categories)**.** Let $F\colon \mathcal{J} \to \mathrm{Cat}_\infty$ be a functor. The *limit* of F, denoted by $\lim F$ or $\lim_{I \in \mathcal{J}} F(I)$, is the following category:

- An object of $\lim F$ consists of objects $x_I \in F(I)$ for all $I \in \mathcal{J}$ and isomorphisms

$$\alpha_f\colon F(f)(x_I) \xrightarrow{\sim} x_J$$

for all $f\colon I \to J$ in $\mathcal{J}$ satisfying the *cocycle conditions*:

(a) for every $I \in \mathcal{J}$, the following triangle commutes

$$\begin{tikzcd} x_I \arrow[r, "\mu_I(x_I)"] \arrow[dr, "\mathrm{id}"'] & F(\mathrm{id}_I)(x_I) \arrow[d, "\alpha_{\mathrm{id}_I}"] \\ & x_I. \end{tikzcd}$$

(b) for every pair of morphisms $f\colon I \to J$ and $g\colon J \to K$ in $\mathcal{J}$, the following square commutes:

$$\begin{tikzcd} F(g)(F(f)(x_I)) \arrow[r, "F(g)(\alpha_f)"] \arrow[d, "\mu_{f,g}(x_I)"'] & F(g)(x_J) \arrow[d, "\alpha_g"] \\ F(g \circ f)(x_I) \arrow[r, "\alpha_{g \circ f}"'] & x_K. \end{tikzcd}$$

- A morphsim in $\lim F$ from (x,α) to (y,β) consists of morphisms $\varphi_I\colon x_I \to y_I$ for all $I \in \mathcal{J}$, such that for each map $f\colon I \to J$ in $\mathcal{J}$, the following square commutes:

$$\begin{tikzcd} F(f)(x_I) \arrow[r, "F(f)(\varphi_I)"] \arrow[d, "\alpha_I"'] & F(f)(y_I) \arrow[d, "\beta_f"] \\ x_J \arrow[r, "\varphi_J"'] & y_J. \end{tikzcd}$$

- Identities and composition are defined pointwise.

By construction, there is an extension $\bar{F}\colon \mathcal{J}^{\triangleleft} \to \mathrm{Cat}_\infty$ of F with $\bar{F}(-\infty) = \lim F$. If $\bar{F}\colon \mathcal{J}^{\triangleleft} \to \mathrm{Cat}_\infty$ is any extension of F, then $\bar{F}$ induces a functor $\bar{F}(-\infty) \to \lim F$, and we say that $\bar{F}$ is a limit diagram if that functor is an equivalence.

Remark 4.9 (Colimits of categories). Once limits have been defined, we can also define colimits: given $F\colon \mathcal{J} \to \mathrm{Cat}_\infty$, an extension $\bar{F}\colon \mathcal{J}^{\triangleleft} \to \mathrm{Cat}_\infty$ is called a *colimit diagram* if, for every category $\mathcal{E}$, the functor

$$Fun(\bar{F}(-), \mathcal{E})\colon (\mathcal{J}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}_\infty$$

is a limit diagram. One can show that colimits of categories always exists, although they are difficult to describe explicitly in general.

Example 4.10 (Pullbacks of categories). Given functors $\mathcal{C} \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{D}$, the pullback $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is the category whose objects are triples (x, y, α) with $x \in \mathcal{C}$, $y \in \mathcal{D}$ and $\alpha\colon f(x) \xrightarrow{\sim} g(y)$ in $\mathcal{E}$.

Example 4.11. Let $\mathcal{C}$ be a category and consider the diagram $\mathcal{C} \rightrightarrows \mathcal{C}$, where both functors are the identity. The limit of this diagram is equivalent to the category $\mathrm{Fun}(B\mathbf{Z}, \mathcal{C})$ of $\mathbf{Z}$-equivariant objects of $\mathcal{C}$. In fact, an object in the limit is a pair (x, α), where $x \in \mathcal{C}$ and $\alpha\colon x \simeq x$. An automorphism $\alpha \in \mathrm{Aut}_{\mathcal{C}}(x)$ corresponds precisely to an $\mathbf{Z}$-action on x.

Example 4.12 (Zariski descent for modules). Limits of categories allow us to reformulate Theorem 2.70 more succinctly: it is exactly the statement that, if $(f_i)_{i\in I}$ generates the unit ideal in R, then the diagram of categories

$$\mathrm{Mod}_R \to \prod_{i\in I} \mathrm{Mod}_{R_{f_i}} \rightrightarrows \prod_{i,j\in I} \mathrm{Mod}_{R_{f_i,f_j}} \substack{\to\\\to\\\to} \prod_{i,j,k\in I} \mathrm{Mod}_{R_{f_i,f_j,f_k}}.$$

obtained by restricting the functor Mod^* from Example 4.3 is a limit diagram. In the case $I = \{1, 2\}$, this is equivalent to the simpler statement that the square of categories

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[r] \arrow[d] & \mathrm{Mod}_{R_{f_1}} \arrow[d] \\ \mathrm{Mod}_{R_{f_2}} \arrow[r] & \mathrm{Mod}_{R_{f_1 f_2}} \end{tikzcd}$$

is cartesian. Moreover, these results hold for any subfunctor of Mod^* defined by a Zariski-local property (as in Example 4.5), such as $R \mapsto \mathrm{Vect}_R$ and $R \mapsto \mathrm{Line}_R$.

4.2. **Quasi-coherence.**

Definition 4.13 (Quasi-coherent objects)**.** Let $F\colon \mathrm{CAlg} \to \mathrm{Cat}_\infty$ be a functor. For any algebraic functor X, we define the category $F(X)$ of quasi-coherent F-objects over X, or simply *F-objects* over X, as the limit

$$F(X) = \lim_{\mathrm{Spec}(R)\to X} F(R),$$

where the indexing category is the category of elements of X. For example:

(a) The category of *quasi-coherent modules* over X is $\mathrm{Mod}_X = \lim_{\mathrm{Spec}(R)\to X} \mathrm{Mod}_R$.

(b) The category of quasi-coherent vector spaces over X is $\mathrm{Vect}_X = \lim_{\mathrm{Spec}(R)\to X} \mathrm{Vect}_R$. These are called *vector bundles* over X.
(c) The category of *quasi-coherent lines over* X is $\mathrm{Line}_X = \lim_{\mathrm{Spec}(R)\to X} \mathrm{Line}_R$. These are called *line bundles over* X.
(d) The category of *quasi-coherent algebras* over X is $\mathrm{CAlg}_X = \lim_{\mathrm{Spec}(R)\to X} \mathrm{CAlg}_R$.
(e) The poset of *quasi-coherent ideals* over X is $\mathrm{Id}_X = \lim_{\mathrm{Spec}(R)\to X} \mathrm{Id}_| R$.
(f) The poset of *quas-coherent radical ideals* over X is $\mathrm{Rad}_X = \lim_{\mathrm{Spec}(R)\to X} \mathrm{Rad}_R$.

Remark 4.14 (Functoriality of quasi-coherent objects). (a) The assignment $X \mapsto F(X)$ has a structure of functor $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{Cat}_\infty$. In particular, for every map $f : Y \to X$, we have a *base change* or *pullback* functor

$$f^* : F(X) \to F(Y),$$

given by precomposition with the induced functor $\mathrm{El}(Y) \to \mathrm{El}(X)$. The right adjoint to f^*, if it exists, is called the *pushforward* functor and denoted by f_*.
(b) For $X = \mathrm{Spec}(R)$, evaluation at id_R induces an equivalence of categories $F(X) \xrightarrow{\sim} F(R)$. In other words, the construction of Definition 4.2 is an extension of F along the Yoneda embedding $\mathrm{CAlg} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}}$ (in fact, it is the *right Kan extension*). For example, a quasi-coherent module (algebra, ideal, etc.) over $\mathrm{Spec}(R)$ is the same as an R-module (algebra, ideal, etc.).
(c) There is an obvious notion of a *natural transformation* between category valued functors. Any natural transformation $\alpha : F \to G$ between functors $\mathrm{CAlg} \to \mathrm{Cat}_\infty$ extends automatically to a natural transformation between their quasi-coherent extensions $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{Cat}_\infty$. In particular, for any algebraic functor X, there is an induced functor $\alpha_X : F(X) \to G(X)$.

Example 4.15 (Quasi-coherent modules). Let us spell out explicitly the definition of a quasi-coherent module. A quasi-coherent module M over an algebraic functor X consists of the following data:
(a) for every $x : \mathrm{Spec}(R) \to X$, an R-module $M(x)$;
(b) for every commutative triangle

$$\begin{tikzcd} \mathrm{Spec}(S) \arrow[r, "y"] \arrow[d, "\varphi"'] & X \\ \mathrm{Spec}(R) \arrow[ur, "x"'] & \end{tikzcd}$$

an S-linear isomorphism $\varphi^*(M(x)) \simeq M(Y)$;

such that the isomorphism in (b) satisfy the cocyle conditions of Definition 4.8. Under the equivalence $\mathrm{Mod}_{\mathrm{Spec}(R)} \simeq \mathrm{Mod}_R$ of Remark 4.14, the R-module $M(x)$ in (a) is identified with the pullback $x^*(M)$, and the isomorphism in (b) is an instance of the natural isomorphism $\mu_{\mathrm{Spec}(\varphi),x} : \mathrm{Spec}(\varphi)^* \circ x^* \simeq y^*$, which is part of the functor $\mathrm{Mod} : \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ of Remark 4.14.

Remark 4.16 (Pushforward of quasi-coherent modules). Let $f : Y \to X$ be a map of algebraic functors. For quasi-coherent modules and algebras, the pushforward functor f_* always exists, moduluo "size issues" as in Remark 2.53. That is, if M is a quasi-coherent module over Y, then $f_*(M)$ is potentially a large quasi-coherent module over X. This does not happen if X and Y are accessible, hence does not happen in practice.

Remark 4.17. Let $F : \mathrm{CAlg} \to \mathrm{Cat}_\infty$ be a functor. Any properties or constructions within the categories $F(R)$ that are compatible with base change automatically extend to the categories of quasi-coherent F-objects. For example:
(a) (Properties of modules) A quasi-coherent module can be of *finite type*, of *finite presentation*, *flat*, *projective*, *free*, etc.

(b) (Properties of module maps) A morphism of quasi-coherent modules can be *zero*, *surjective*, *universally injective*, etc. On the other hand, injectivity does not a priori make sense, as this property is not preserved by base change.
(c) (Sums of module maps) Two morphisms $f, g\colon M \to N$ in Mod_X have a sum $f + g\colon M \to N$. This defines an abelian group structure on $\mathrm{Map}_{\mathrm{Mod}_X}(M, N)$.
(d) (Colimits of modules) The category of quasi-coherent modules over X has a symmetric monoidal structure with $(M \otimes N)(x) = M(x) \otimes N(x)$. For any $d \in \mathbf{N}$, there are symmetric, exterior, and divided power functors Sym^d, Λ^d, $\Gamma^d\colon \mathrm{Mod}_X \to \mathrm{Mod}_X$.
(e) (Operations with ideals) Quasi-coherent ideals $I, J \in \mathrm{Id}_X$ have a sum $I + J$ and a product IJ, defined by $(I + J)(x) = I(x) + J(x)$ and $(IJ)(x) = I(x)J(x)$.
(f) (Radical of an ideal) Anhy quasi-coherent ideal $I \in \mathrm{Id}_X$ has a *radical* $\sqrt{I} \in \mathrm{Rad}_X$ defined by $(\sqrt{I})(x) = \sqrt{I(x)}$.
(g) (Properties of algebras) A quasi-coherent algebra can be of *finite type*, of *finite presentation*, *finite*, *free*, etc. A quasi-coherent $\mathbf{N}$-graded algebra can be *generated* by A_1, *finitely generated as an A_0-algebra*, etc.
(h) (Properties of algebra maps) A morphism of quasi-coherent algebras can be *surjective*, *a localization*, have a *nilpotent kernel*, etc. A morphism of quasi-coherent $\mathbf{N}$-graded algebras can also be *essentially surjective*.
(i) (Underlying module and symmetric algebra) There is a forgetful functor $\mathrm{CAlg}_X \to \mathrm{Mod}_X$ with a left adjoint Sym, such that the Underlying quasi-coherent module of $\mathrm{Sym}(M)$ is $\bigoplus_{d\in\mathbf{N}} \mathrm{Sym}^d(M)$.
(j) (Kernel of a quasi-coherent algebra) There is a *kernel* functor $\mathrm{CAlg}_X \to \mathrm{Id}_X$, which extends the functors $\mathrm{CAlg}_R \to \mathrm{Id}_R$ sending a ring map $R \to A$ to its kernel.
(k) etc.

Warning 4.18. Beware that the forgetful functors $\mathrm{Rad}_R \hookrightarrow \mathrm{Id}_R \to \mathrm{Mod}_R$ are *not* compatible with base change, and so do not extend to arbitrary algebraic functors X (although they will when X is a scheme): a quasi-coherent radical ideal over X need not have an underlying quasi-coherent ideal, and a quasi-coherent ideal over X need not have an underlying quasi-coherent module.

Notation 4.19 (The quasi-coherent algebra of functions)**.** Let X be an algebraic functor. We denote $\mathcal{O}(X) \in \mathrm{CAlg}_X$ the quasi-coherent algebra given by $\mathcal{O}_X(x\colon \mathrm{Spec}(R) \to X) = \mathcal{O}(\mathrm{Spec}(R)) = R$. Its underlying quasi-coherent module is the unit of the tensor product in Mod_X, since R is the unit in Mod_R.

Definition 4.20. Let X be an algebraic functor and M a quasi-coherent module over X. We denote by $M(X)$ or $\Gamma(X, M)$ the limit $\lim_{x\in\mathrm{El}(X)} M(x)$, which is a module over the ring of functions $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$. Elements of $M(X)$ are called *global sections of M*.

4.3. Classification of closed and open subfunctors.

Proposition 4.21 (Classification of closed and open subfunctors)**.** *Let X be an algebraic functor.*
(a) There is an isomorphism of posets

$$\mathrm{V}\colon \{\textit{quasi-coherent ideal over } X^{\mathrm{op}}\} \xrightarrow{\sim} \{\textit{closed subfunctors of } X\}.$$

(b) There is an isomoprhism of posets

$$\mathrm{D}\colon \{\textit{quasi-coherent radical ideals over } X\} \xrightarrow{\sim} \{\textit{open subfunctors of } X\}$$

Definition 4.22 (Open complement)**.** Let $Z \subseteq X$ be closed subfunctor defined by the quasi-coherent ideal I. The *open complement* $X \setminus Z$ of Z in X is the open subfunctor defined by the quasi-coherent radical ideal $\sqrt{I}$.

Example 4.23. Let I be a set.

(a) The punctured affine space $\mathbf{A}^I \setminus \{0\}$ (redExample 2.57) is the open complement of 0 in $\mathbf{A}^I$.
(b) In $\mathbf{P}^{I\sqcup\{0\}}$, the affine space $\mathrm{D}(x_0) \simeq \mathbf{A}^I$ is the open complement of the hyperplane at infinity $\mathrm{V}(x_0) \simeq \mathbf{P}^I$.

Warning 4.24. Different closed subfunctors can have the same open complement, since different ideals can have the same radical (e.g., (x) and (x^2) in $\mathbf{Z}[x]$). For this reason, there is no notion of *closed image* of f, also called the *scheme-theoretic image*, is the smallest closed subfunctor of X through which f factors. By Proposition 4.21, the closed image of f is $\mathrm{V}(I) \subseteq X$, where the quasi-coherent ideal I is the kernel of the quasi-coherent algebra $f_*(\mathcal{O}_Y)$ over X.

Example 4.25 (Closed image). Any map of ring $\varphi\colon A \to B$ admits a canonical factroization $A \twoheadrightarrow A/\ker(\varphi) \hookrightarrow B$, so that $\mathrm{V}(\ker(\varphi)) \subseteq \mathrm{Spec}(A)$ is the smallest closed subfunctor through which $\mathrm{Spec}(\varphi)$ factors. This factroization generalizes to algebraic functors as follows. Let $f\colon X \to Y$ be a map of algebraic functors. The *closed image* of f, also called the *scheme-theoretical image*, is the smallest closed subfunctor of X through which f factors. By Proposition 4.21, the closed image of f is $\mathrm{V}(I) \subseteq X$, where the quasi-coherent ideal I is the kernel of the quasi-coherent algebra $f_*(\mathcal{O}_Y)$ over X.

Remark 4.26 (Loci associated with maps of quasi-coherent modules). The loci from Definition 3.48 have a direct generalization to any algebraic functor X: a map $f\colon M \to N$ in Mod_X defines a subfunctors $\mathrm{V}(f)$, $\mathrm{Epi}(f)$, $\mathrm{Mono}(f)$, and $\mathrm{Iso}(f)$ on X. These are characterized by the property that their preimage along any R-point $x\colon \mathrm{Spec}(R) \to X$ is the corresponding locus for the R-linear map $f(x)\colon M(x) \to N(x)$. Proposition 3.51 then immediately generalizes to this setting: if N is a vector bundle, then $\mathrm{V}(f)$ is a closed subfunctor and $\mathrm{Epi}(f)$ is an open subfunctor of X, and if both M and N are vector bundles, then also $\mathrm{Mono}(f)$ and $\mathrm{Iso}(f)$ are open subfunctors of X.

Call a subfunctor *clopen*, if it is both closed and open. Recall that $\mathrm{Idem}(R)$ is the set of idempotent elements in a ring R.

Proposition 4.27 (Classification of clopen subfunctors)**.** *For any algebraic functor X, there is a bijection*

$$\mathrm{Idem}(\mathcal{O}(X)) \xrightarrow{\sim} \{\textit{clopen subfunctors of } X\}, \quad e \mapsto \mathrm{V}(e) = \mathrm{D}(1-e).$$

Remark 4.28. By Example 2.30 and Remark 2.53, there is a bijection

$$\mathrm{Map}(X, \mathrm{Spec}(\mathbf{Z} \times \mathbf{Z})) \xrightarrow{\sim} \mathrm{Idem}(\mathcal{O}(X)), \quad f \mapsto f^*(1,0).$$

Thus, clopen subfunctors are also in bijection with maps to $\mathrm{Spec}(\mathbf{Z} \times \mathbf{Z})$, which is the coproduct $* \sqcup *$ in the category of affine schemes. This is analogous to the following (much more obvious) topological statement: if T is a topological space, there is a bijection between clopen subsets of T and continuous maps $T \to * \sqcup *$.

4.4. **Relative** Spec **and** Proj**.** The constructions Spec and Proj are compatible with base change in the following sense. If A is an R-algebra (resp. an $\mathbf{N}$-graded R-algebra) and $R \to S$ is a ring map, then the following squares are cartesian:

$$\begin{array}{ccc} \mathrm{Spec}(A \otimes_R S) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R), \end{array} \qquad \begin{array}{ccc} \mathrm{Proj}(A \otimes_R S) & \longrightarrow & \mathrm{Proj}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R). \end{array}$$

This allows us to extend Spec and Proj to quasi-coherent algebras:

Definition 4.29 (Relative Spec and Proj)**.** Let X be an algebraic functor.

(a) Let A be a quasi-coherent algebra over X. The algebraic functor $\mathrm{Spec}(A)$ over X is defined by

$$\mathrm{Spec}(A)(R) = \{(x,a) \mid x \in X(R) \text{ and } a \text{ is a section of } \mathrm{Spec}(A(x)) \to \mathrm{Spec}(R)\}.$$

This defines a functor $\mathrm{Spec}\colon \mathrm{CAlg}_X^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/X}$.

(b) Let A be a quasi-coherent $\mathbf{N}$-graded algebra over X. The algebraic functor $\mathrm{Proj}(A)$ over X is defined by

$$\mathrm{Proj}(A)(R) = \{(x,a) \mid x \in X(R) \text{ and } a \text{ is a section of } \mathrm{Proj}(A(x)) \to \mathrm{Spec}(R)\}.$$

This defines a functor $\mathrm{Proj}\colon (\mathrm{CAlg}_X^{\mathbf{N},\mathrm{es}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/X}$.

Definition 4.30 (Relative affine and projective spaces)**.** Let X be an algebraic functor and let M be a quasi-coherent module over X.

(a) The *affine space* $\mathbf{A}(M)$ over X is $\mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbf{A}(M)(R) = \{(x,a) \mid x \in X(R) \text{ and } a\colon M(x) \to R \text{ is an } R\text{-linear map}\}.$$

(b) The *punctured affine space* $\mathbf{A}(M) \setminus \{0\}$ over X is $\mathrm{D}(M) \subseteq \mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$(\mathbf{A}(M) \setminus \{0\})(R) = \{(x,a) \mid x \in X(R) \text{ and } a\colon M(x) \twoheadrightarrow R \text{ is a surjective } R\text{-linear map}\}.$$

(c) The *projective space* $\mathbf{P}(M)$ over X is $\mathrm{Proj}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbf{P}(M)(R) = \{(x,a) \mid x \in X(R) \text{ and } a\colon M(x) \twoheadrightarrow L \text{ is a quotient } R\text{-line}\}.$$

Remark 4.31 (Morphisms into Spec and Proj). Using **??**, we obtain the following descriptions of maps into $\mathrm{Spec}(A)$ or $\mathrm{Proj}(A)$. Let X and Y be algebraic functors.

(a) For any $A \in \mathrm{CAlg}_X$, a map $Y \to \mathrm{Spec}(A)$ consists of a map $f\colon Y \to X$ and a map $f^*(A) \to \mathcal{O}(Y)$ in CAlg_Y (cf. Remark 2.53).

(b) Let A be a quasi-coherent $\mathbf{N}$-graded algebra over X generated by A_1. Then a map $Y \to \mathrm{Proj}(A)$ consists of a map $f\colon Y \to X$ and a quotient line bundle $f^*(A_1) \twoheadrightarrow L$ in Mod_Y such that the induced map $\mathrm{Sym}(f^*(A_1)) \twoheadrightarrow \mathrm{Sym}(L)$ factors through $f^*(A)$.

The relative Spec and Proj functors allows us to relativize the notions of affine, quasi-affine, projective, and quasi-projective schemes:

Definition 4.32 (Affine and quasi-affine morphisms)**.** Let $f\colon Y \to X$ ve a map of algebraic functors.

(a) We say that f is *affine*, or that Y is *affine over* X, if there exists $A \in \mathrm{CAlg}_X$ and an isomorphism $Y \simeq \mathrm{Spec}(A)$ over X. We denote by Aff_X the category of algebraic functors affine over X.

(b) We say that f is *quasi-affine*, or that Y is *quasi-affine over* X, if there exists $V \in \mathrm{Aff}_X$, $I \in \mathrm{Rad}_V^{\mathrm{ft}}$, and an isomorphism $Y \simeq \mathrm{D}(I)$ over X. We denote by QAff_X the category of algebraic functors quasi-affine over X.

Definition 4.33 (Projective and quasi-projective morphism)**.** Let $f\colon Y \to X$ be a map of algebraic functors.

(a) We say that f is *projective*, or that Y is projective over X, if there exists $A \in \mathrm{CAlg}_X^{\mathbf{N},\mathrm{es}}$, generated by A_1 an finitely generated, and an isomorphism $Y \simeq \mathrm{Proj}(A)$ over X. We denote by Proj_X the category of algebraic functors projective over X.

(b) We say that f is *quasi-projective*, or that Y is *quasi-projective over* X, if there exists $V \in \mathrm{Proj}_X$, $I \in \mathrm{Rad}_V^{\mathrm{ft}}$, and an isomorphism $Y \simeq \mathrm{D}(I)$ over X. We denote by QProj_X the category of algebraic functors quasi-projective over X.

Affineness turns out to be a *quasi-coherent property* in the sense that the functor $X \mapsto \mathrm{Aff}_X$ is right Kan extended from affine schemes:

Proposition 4.34 (Characterization of affine morphisms). *Let X be an algebraic functor.*

(a) A map $Y \to X$ is affine if and only if, for every ring R and every map $\mathrm{Spec}(R) \to X$, the pullback $Y \times_X \mathrm{Spec}(R)$ is affine.
(b) The functor Spec *defines an equivalence of categories*

$$\mathrm{Spec}\colon \mathrm{CAlg}_X^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Aff}_X,$$

whose inverse sends $f\colon Y \to X$ to $f_*(\mathcal{O}_Y)$.

Warning 4.35. The analogue of **??** 4.34 (a) does not hold for quasi-affine, projective, and quasi-projective morphisms.

Example 4.36 (Closed immersions as affine morphisms). By **??** 4.34, any closed immersion $Z \hookrightarrow X$ is affine, and $Z \simeq \mathrm{Spec}(i_*(\mathcal{O}_Z))$. Moreover, an affine map $f\colon Y \to X$ is a closed immersion if and only if the induced map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is surjective.

4.5. Modules over quasi-projective schemes.

Definition 4.37 (Zariski-local epimorphism). A map of algebraic functors $Y \to X$ is called *Zariski-local epimorphism*, if for every $\mathrm{Spec}(R) \to X$, there exists $f_1, \dots, f_n \in R$ generating the unit ideal such that each composite $\mathrm{Spec}(R_{f_i}) \hookrightarrow \mathrm{Spec}(R) \to X$ lifts to Y.

Example 4.38.
(a) Any epimorphism is a Zariski-local epimorphism. In partiuclar, any map with a section, such as the projection $\mathbf{A}^n \times X \to X$, is a Zariski-local epimorphism.
(b) Let $(F_i)_{i\in I}$ be a family of subsets of $\mathcal{O}(X)$ whose union generates the unit ideal. Then the map $\coprod_{i\in I} \mathrm{D}(F_i) \to X$ is a Zariski-local epimorphism.
(c) Let L be a line bundle over X. A family $(s_i)_{i\in I}$ of global sections of L is called *generating*, if for every $x \in \mathrm{El}(X)$, the family $(s_i(x))_{i\in I}$ generates the line $L(x)$. In this case, the map $\coprod_{i\in I} \mathrm{D}(s_i) \to X$ is a Zariski-local epimorphism.
(d) The map $\mathbf{A}^I \setminus \{0\} \to \mathbf{P}^I$ is a Zariski-local epimorphism, by Proposition 3.28. More generally, for any $\mathbf{N}$-graded ring A, the map $\mathrm{D}(A_1) \to \mathrm{Proj}_1(A)$ is a Zariski-local epimorphism (see Remark 3.67).

Proposition 4.39 (Zariski descent for quasi-coherent modules). *Let $(Y_i \to X)_{i\in I}$ be family of maps of algebraic functors such that $\coprod_{i\in I} Y_i \to X$ is a Zariski-local epimorphism. Then the following diagram of categories is a limit diagram:*

$$\mathrm{Mod}_X \to \prod_{i\in I} \mathrm{Mod}_{Y_i} \rightrightarrows \prod_{i,j\in I} \mathrm{Mod}_{Y_i \times_X Y_j} \substack{\to\\\to\\\to} \prod_{i,j,k\in I} \mathrm{Mod}_{Y_i \times_X Y_j \times_X Y_k}.$$

In particular, the functor $\mathrm{Mod}_X \to \prod_{i\in I} \mathrm{Mod}_{Y_i}$ is conservative.

Remark 4.40. Proposition 4.39 is a formal consequence of the Zariski descent property of Mod^* from Example 4.12: for any functor $F\colon \mathrm{CAlg} \to \mathrm{Cat}_\infty$ that satisfies the latter property, its quasi-coherent extension $X \mapsto F(X)$ also satisfies Proposition 4.39. For example, there are analogous statements for quasi-coherent algebras, ideals, and radical ideals.

Corollary 4.41 (Zariski descent for functions). *Let $(Y_i \to X)_{i\in I}$ be a family of maps of algebraic functors such that $\coprod_{i\in I} Y_i \to X$ is a Zariski-local epimorphism. Then the following diagram of rings is an equalizer:*

$$\mathcal{O}(X) \to \prod_{i\in I} \mathcal{O}(Y_i) \rightrightarrows \prod_{i,j\in I} \mathcal{O}(Y_i \times_X Y_j).$$

For the following examples, recall the notion of I-local module and the associated localization functor $\mathrm{L}_I\colon \mathrm{Mod}_A \to \mathrm{Mod}_A^{\mathrm{I-loc}}$ (see Definition 3.78).

Example 4.42 (Modules over nonvanishing loci). Let A be a ring and $I \subseteq A$ a subset. Then the map $\coprod_{f\in I} \operatorname{Spec}(A_f) \to \mathrm{D}(I)$ is a Zariski-local epimorphism. By Proposition 4.39, there is a limit diagram

$$\mathrm{Mod}_{\mathrm{D}(I)} \to \prod_{f\in I} \mathrm{Mod}_{A_f} \rightrightarrows \prod_{f,g\in I} \mathrm{Mod}_{A_{fg}} \overset{\to}{\underset{\to}{\to}} \prod_{f,g,h\in I} \mathrm{Mod}_{A_{fgh}}\,.$$

In particular, the pullback functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\mathrm{D}(I)}$ factors through the localization functor L_I.

Example 4.43 (Modules over Proj). Let A be an $\mathbf{N}$-graded ring.

(a) Let $I \subseteq A_+$ be a homogeneous subset such that $A_+ \subseteq \sqrt{(I)}$. Then the map $\coprod_{f\in I} \operatorname{Spec}(A_{(f)}) \to \operatorname{Proj}(A)$ is a Zariski-local epimorphism. Proposition 4.39 gives the limit diagram

$$\mathrm{Mod}_{\mathrm{Proj}(A)} \to \prod_{f\in I} \mathrm{Mod}_{A_{(f)}} \rightrightarrows \prod_{f,g\in I} \mathrm{Mod}_{A(fg)} \overset{\to}{\underset{\to}{\to}} \prod_{f,g,h\in I} \mathrm{Mod}_{A(fgh)}\,.$$

(b) Consider $\mathrm{D}(A_1) \subseteq \operatorname{Spec}(A)$. By Remark 3.67, the canonical map $\mathrm{D}(A_1) \to \operatorname{Proj}_1(A)$ is a Zariski-local epimorphism, and it induces a monomorphism $\mathrm{D}(A_1)/\mathbf{G}_m \hookrightarrow \operatorname{Proj}_1(A)$. Hence, the n-fold fiber product of $\mathrm{D}(A_1)$ over $\operatorname{Proj}_1(A)$ is the same as n-fold fiber product over $\mathrm{D}(A_1)/\mathbf{G}_m$, which is $\mathrm{D}(A_1) \times \mathbf{G}_m^{n-1}$, since the action of $\mathbf{G}_m$ on $\mathrm{D}(A_1)$ is free. By **??**, we get a limit diagram of the form

$$\mathrm{Mod}_{\mathrm{Proj}_1(A)} \to \mathrm{Mod}_{\mathrm{D}(A_1)} \rightrightarrows \mathrm{Mod}_{\mathrm{D}(A_1)\times\mathbf{G}_m} \overset{\to}{\underset{\to}{\to}} \mathrm{Mod}_{\mathrm{D}(A_1)\times\mathbf{G}_m\times\mathbf{G}_m}\,.$$

Example 4.44 (Modules over $\mathbf{P}^1$). In case $\mathbf{P}^1_k = \operatorname{Proj}(k[x,y])$, Example 4.43 (a) applies with the two element set $I = \{x,y\}$. We have $k[x,y]_{(x)} = k[u]$ with $u = y/x$, $k[x,y]_{(y)} = k[v]$ with $v = x/y$, and $k[x,y]_{(xy)} = k[t^{\pm 1}]$ with $t = y/x$. Hence, the limit diagram may be rewritten as a cartesian square

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbf{P}^1_k} \arrow[r] \arrow[d] & \mathrm{Mod}_{k[u]} \arrow[d, "u\mapsto t"] \\
\mathrm{Mod}_{k[v]} \arrow[r, "v\mapsto t^{-1}"] & \mathrm{Mod}_{k[t^{\pm1}]}\,.
\end{tikzcd}$$

A quasi-coherent module over $\mathbf{P}^1_k$ is therefore a triple (M,N,α), where M is a $k[u]$-module, N is a $k[v]$-module, and $\varphi\colon M[u^{-1}] \xrightarrow{\sim} N[v^{-1}]$ is an isomorphism to $k[t^{\pm1}]$-modules (i.e., an isomorphism of k-module such that the action of u on $M[u^{-1}]$ is *inverse* to the action of v on $N[v^{-1}]$).

Example 4.45 (Modules over the affine line with doubled origin). Let X be the affine line with doubled origin over k, which is the algebraic k-functor defined by

$$X(R) = \{(f,e) \mid f \in R, e \in R/(f),\ \text{and } e^2 = e\}.$$

Let $X_0 \subseteq X$ and $X_1 \subseteq X$ be the loci where $e = 0$ and $e = 1$, respectively. One can check that:

- X_0 and X_1 are open subfunctors of X such that $X_0 \sqcup X_1 \to X$ is a Zariski-local epimorphism;
- The map $X \to \mathbf{A}^1_k,\ (f,e) \mapsto f$, restricts to isomorphism

$$X_0 \xrightarrow{\sim} \mathbf{A}^1_k, \quad X_1 \xrightarrow{\sim} \mathbf{A}^1_k, \quad \text{and} \quad X_0 \cap X_1 \xrightarrow{\sim} \mathbf{G}_{m,k}.$$

By Proposition 4.39, we obtain a cartesian square

$$\begin{tikzcd}
\mathrm{Mod}_X \arrow[r] \arrow[d] & \mathrm{Mod}_{k[x]} \arrow[d, "x\mapsto x"] \\
\mathrm{Mod}_{k[x]} \arrow[r, "x\mapsto x"] & \mathrm{Mod}_{k[x^{\pm1}]}\,.
\end{tikzcd}$$

A quasi-coherent module over X is therefore a triple (M,N,α), where M is a $k[x]$-module, N is a $k[x]$-module, and $\alpha\colon M[x^{-1}] \xrightarrow{\sim} N[x^{-1}]$ is an isomorphism of $k[x^{\pm1}]$-modules (or equivalently of $k[x]$-modules).

Example 4.46 (Modules over Grassmannian). Let M be a k-module and let $n \in \mathbf{N}$. Let $\mathrm{St}_n(M)$ be the algebraic k-functor given by

$$\mathrm{St}_n(M)(R) = \{\text{surjective } R\text{-linear maps } M \otimes_k R \to R^n\},$$

called the *Stiefel scheme* of M. By Proposition 3.51, this is an open subfunctor of the affine space $\mathbf{A}(M^n)$. There is an obvious map $\mathrm{St}_n(M) \to \mathrm{Gr}_n(M)$, which ia a Zariski-local epimorphism, since vector spaces are Zariski-locally free. Moreover, there is a free action of the affine group scheme GL_n on St_n, such that $\mathrm{St}_n(M) \to \mathrm{Gr}_n(M)$ is GL_n-invariant and induces a monomorphism $\mathrm{St}_n(M)/\mathrm{GL}_n \hookrightarrow \mathrm{Gr}_n(M)$; in the case $n = 1$, this recovers the knoen monomorphism $(\mathbf{A}(M) \setminus \{0\})/\mathbf{G}_m \hookrightarrow ¶(M)$. As in Example 4.43, we then obtain a limit diagram of the form

$$\mathrm{Mod}_{\mathrm{Gr}_n(M)} \to \mathrm{Mod}_{\mathrm{St}_n(M)} \rightrightarrows \mathrm{Mod}_{\mathrm{St}_n(M)\times \mathrm{GL}_n} \overset{\to}{\underset{\to}{\to}} \mathrm{Mod}_{\mathrm{St}_n(M)\times \mathrm{GL}_n \times \mathrm{GL}_n},$$

where each term can be further expanded using Example 4.42.

Using Example 4.42, we can identify quasi-coherent modules over quasi-affine schemes:

Theorem 4.47 (Quasi-coherent modules over quasi-affine schemes). *Let A be a ring and let $I \subseteq A$ be a finite subset with nonvanishing locus $\mathrm{D}(I) \subseteq \mathrm{Spec}(A)$. Then the pullback functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\mathrm{D}(I)}$ induces an equivalence of categories*

$$\mathrm{Mod}_A^{\mathrm{I-loc}} \xrightarrow{\sim} .$$

Corollary 4.48 (Affine completion of quasi-affine schemes). *Let X be a quasi-affine scheme. Then the canonical map $X \to \mathrm{Spec}(\mathcal{O}(X))$ is an open immersion defined by a finitely generated radical ideal.*

Construction 4.49 (Quasi-coherent modules associated with a $\mathbf{Z}$-graded module). Let A be an $\mathbf{N}$-graded ring. We define a functor

$$\{\mathbf{Z}\text{-graded } A\text{-module}\} \to \mathrm{Mod}_{\mathrm{Proj}(A)}, \quad M \mapsto \tilde{M},$$

as follows. Let M be a $\mathbf{Z}$-graded A-module and let $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$ classify a quotient $\mathbf{N}$-graded algebra $\varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L)$ for some $d \geq 1$ (if A is generated by $A_{\leq 1}$, we can take $d = 1$). Let $\tilde{\varphi}$ be the induced map of $\mathbf{Z}$-graded rings $A^{(d)} \to \bigoplus_{n\in\mathbf{Z}} L^{\otimes n}$. We set

$$\tilde{M}(x) = \tilde{\varphi}^*(M^{(d)})_0 \in \mathrm{Mod}_R .$$

Note that for every $d \geq 1$ and $f \in A_d$, the pullback functor $\mathrm{Mod}_{\mathrm{Proj}(A)} \to \mathrm{Mod}_{A(f)}$ sends $\tilde{M}$ to $M_{(f)}$. In particular, by Example 4.43 (a), the functor $M \mapsto \tilde{M}$ factors through the localization functor L_{A_+}.

Using either (a) or (b) of **??**, we can make the following computation:

Theorem 4.50 (Quasi-coherent modules over quasi-projective schemes). *Let A be an $\mathbf{N}$-graded ring generated by $A_{\leq 1}$ and $I \subseteq A_1$ a finite subset with nonvanishing locus $\mathrm{D}(I) \subseteq \mathrm{Proj}(A)$. Then the functor $M \mapsto \tilde{M}$ induces an equivalence*

$$\{I\text{-local } \mathbf{Z}\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{D}(I)} .$$

In particular, if A is a finitely generated A_0-algebra, then

$$\{A_1\text{-local } Z\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{Proj}(A)}$$

4.6. **Serre twists.** Let M be a $\mathbf{Z}$-graded module over a $\mathbf{Z}$-graded ring A and let $d \in \mathbf{Z}$. Recall from Notation 3.36 that $M(d)$ denotes the $\mathbf{Z}$-graded A-module with $M(d)_n = M_{n+d}$.

Definition 4.51 (Serre twists)**.** Let A be an $\mathbf{N}$-graded ring and let $d \in \mathbf{Z}$. We denote by $\mathcal{O}(d)$ the quasi-coherent module over $\mathrm{Proj}(A)$ associated with the $\mathbf{Z}$-graded A-module $A(d)$ (see Construction 4.49). Given $M \in \mathrm{Mod}_{\mathrm{Proj}(A)}$, the d-th *Serre twists* of M is the quasi-coherent module over $\mathrm{Proj}(A)$ given by

$$M(d) = M \otimes \mathcal{O}(d).$$

Remark 4.52. If A is generated by $A_{\leq 1}$, then an R-point $x \in \mathrm{Proj}(A)(R)$ is a quotient line $A_1 \otimes_{A_0} R \twoheadrightarrow L$ satisfying some condition. Unraveling the definition, we see that $\mathcal{O}(d)(x) = L^{\otimes d}$.

Proposition 4.53 (Properties of Serre twists)**.** *Let A be an $\mathbf{N}$-graded ring generated by $A_{\leq}$.*

(a) For any $d \in \mathbf{Z}$, $\mathcal{O}(d)$ is a line bundle over $\mathrm{Proj}(A)$.

(b) For any $\mathbf{Z}$-graded A-module M and $d \in \mathbf{Z}$, there is an isomorphism $\tilde{M}(d) \xrightarrow{\sim} \widetilde{M(d)}$. In particular, for all $d, e \in \mathbf{Z}$, there are isomorphisms

$$\mathcal{O}(d) \otimes \mathcal{O}(e) \xrightarrow{\sim} \mathcal{O}(d+e) \quad \textit{and} \quad \mathcal{O}(1)^{\otimes d} \xrightarrow{\sim} \mathcal{O}(d),$$

so that $\bigoplus_{d \in \mathbf{Z}} \mathcal{O}(d)$ has a structure of quasi-coherent $\mathbf{Z}$-graded algebra over $\mathrm{Proj}(A)$.

(c) There is a canonical map of $\mathbf{Z}$-graded rings

$$\mathrm{L}_{A_1} A \to \bigoplus_{d \in \mathbf{Z}} \Gamma(\mathrm{Proj}(A), \mathcal{O}(d)),$$

and, for any $\mathbf{Z}$-graded A-module M, a canonical map of $\mathbf{Z}$-graded A-modules

$$\mathrm{L}_{A_1} M \to \bigoplus_{d \in \mathbf{Z}} \Gamma(\mathrm{Proj}(A), \tilde{M}(d)),$$

which are isomorphisms if A is finitely generated as an A_0-algebra.

Remark 4.54. If A is an arbitrary $\mathbf{N}$-graded ring and M a $\mathbf{Z}$-graded A-module, there are still canonical maps

$$\tilde{M}(d) \to \widetilde{M(d)} \quad \text{and} \quad (\mathrm{L}_{A_+} M)_0 \to \Gamma(\mathrm{Proj}(A), \tilde{M})$$

but they need not be isomorphisms. If A is generated over A_0 by homogeneous elements whose degrees divide d, then $\mathcal{O}$ is a line bundle over $\mathrm{Proj}(A)$ and the first map is an isomorphism.

Definition 4.55 (Tautological line bundle)**.** Let A be an $\mathbf{N}$-graded ring generated by $A_{\leq 1}$. The line bundle $\mathcal{O}(1)$ over $\mathrm{Proj}(A)$ is called the *tautological line bundle*.

Example 4.56. If $A = k[x_0, \ldots, x_n]$, then

$$\mathrm{L}_{A_1} A = \begin{cases} k[x_0^{\pm 1}] & n = 0 \\ A & n \geq 1 \end{cases}$$

(this is exactly the computation of $\mathcal{O}(\mathbf{A}_k^{n+1} \setminus \{0\})$, see Proposition 2.77). Hence, $\Gamma(\mathbf{P}_k^n, \mathcal{O}(d)) \simeq k[x_0, \ldots, x_n]_d$ for all $d \geq 0$ and $\Gamma(\mathbf{P}_k^n, \mathcal{O}(d)) = 0$ for all $d < 0$ and $n \geq 1$. For $d = 0$, this says that $\mathcal{O}(\mathbf{P}_k^n) = k$, i.e., that every function on projective n-space is constant.

Example 4.57. Under the description of $\mathrm{Mod}_{\mathbf{P}_k^1}$ from Example 4.44, the quasi-coherent module $\mathcal{O}(d)$ over $\mathbf{P}_k^1$ is the triple $(k[u], k[v], \alpha)$, where $\alpha \colon k[x^{\pm 1}] \xrightarrow{sim} k[v^{\pm 1}]$ is the $k[t^{\pm 1}]$-linear isomorphism sending 1 to v^d (hence u^n to v^{d-n}).

Example 4.58 (Tautological vector bundles over Grassmannian). Let k be a ring, M a k-module, and $n \in \mathbf{N}$. The *Tautological vector bundle* $\mathcal{T}$ on $\mathrm{Gr}_n(M)$ is the rank n vector bundle such that, for any R-point $x\colon \operatorname{Spec}(R) \to \mathrm{Gr}_n(M)$ classifying a quotient space $M \otimes_k R \twoheadrightarrow V$, $\mathcal{T}(x) = V$. For $n = 1$, this recovers the tautological line bundle $\mathcal{O}(1)$ on $\P(M) = \operatorname{Proj}(\operatorname{Sym}_k(M))$. If $\varpi\colon \mathrm{Gr}_n(M) \hookrightarrow \mathbf{P}(\Lambda^n_k(M))$ is the Plücker embedding (Example 3.101), then $\varpi^*(\mathcal{O}(1)) \simeq \Lambda^n(\mathcal{T})$ in $\mathrm{Vect}_{\mathrm{Gr}_n(M)}$.

Remark 4.59. Let A be an $\mathbf{N}$-graded ring generated by $A_{\leq 1}$. If $f \in A$ is homogeneous of degree d, it defines by Proposition 4.53 a global section of $\mathcal{O}(d)$, or equivalently a map $\mathcal{O}(-d) \to \mathcal{O}$ in $\mathrm{Mod}_{\operatorname{Proj}(A)}$. Hence, any homogeneous subset $F \subseteq A$ induces a map $\bigoplus_{f \in F} \mathcal{O}(-d_f) \to \mathcal{O}$, where f has degree d_f. One can check that the loci $\mathrm{V}(F)$ and $\mathrm{D}(F)$ in Proj defined in Definition 3.69 corresponds to the line bundle $\mathcal{O}(d)$ on $\operatorname{Proj}(A)$. In fact, by Proposition 4.53 (c), the data of this line bundle is exactly what allows us to recover (the A_1-localization of) A from the algebraic functor $\operatorname{Proj}(A)$.

We can make this more precise as follows. Let $\mathrm{CAlg}^{\mathbf{N},1} \subseteq \mathrm{CAlg}^{\mathbf{N},\mathrm{es}}$ be the full subcategory spanned by the $\mathbf{N}$-graded rings generated in degrees ≤ 1. Let LineBdl be the category of pairs (X, L), where X is an algebraic functor and $L \in \mathrm{Line}_X$, whose morphisms $(X', L') \to (X, L)$ are pairs (f, λ) with $f\colon X' \to X$ and $\lambda\colon L' \xrightarrow{\sim} f^*(L)$. Then there is a functor

$$(\operatorname{Proj}(-), \mathcal{O}(1))\colon (\mathrm{CAlg}^{\mathbf{N},1})^{\mathrm{op}} \to \mathrm{LineBdl},$$

which is fully faithful on the subcategory of $\mathbf{N}$-graded rings of the form $(\mathrm{L}_{A_1} A)_{\geq 0}$, where A is finitely generated over A_0.

5. Locales and topological spaces

The starting point of *pointless topology* is the observation that most topological spaces (including for example all Hausdorff spaces) are determined by their posets of open subsets. Since posets are ubiquitous in mathematics, this leads to the appearance of topology in unexpected places. In this chapter, we will see that the poset of open subfunctors of any algebraic functor Z is isomorphic to the poset of open subsets of an associated topoogical space $|X|$, which we will describe explicitly when $X = \mathrm{Spec}(A)$ or $\mathrm{Proj}(A)$.

Notation 5.1. Let P be a poset. Given a family $(x_i)_{i\in I}$ in P, we denote by $\bigvee_{i\in I} x_i$ its supremum (least upper bound) and by $\bigwedge_{i\in I} x_i$ its infimum (greatest lower bound), if they exist.

Remark 5.2.

(a) If we view a poset as a category, suprema are colimits and infima are limits.
(b) The supremum of the empty family (i.e., the colimit of the empty diagram) is the smallest element (i.e., the initial object). Dually, the infimum of the empty family is the largest element.
(c) A poset admits all suprema if and only if it admits all infima. Indeed, the infimum of a family is the supremum of all elements below the family.

Definition 5.3 (Locale)**.** A *locale* is a poset $\mathcal{O}$ satisfying the following condition

(a) (Completeness) $\mathcal{O}$ admits all suprema (hence all infima).
(b) (Distributivity) For any $u \in \mathcal{O}$ and any family $(v_i)_{i\in I}$,

$$u \wedge (\bigvee_{i\in I} v_i) = \bigvee_{i\in I} (u \wedge v_i).$$

A morphism of locales $f\colon \mathcal{O}' \to \mathcal{O}$ is a morphism of posets $f^*\colon \mathcal{O} \to \mathcal{O}'$ (in the other direction!) that preserves colimits and finite limits. We denote by Loc the category of locales.

Example 5.4 (Totally ordered sets). Any totally ordered set with suprema is a locale. For example, the posets $\mathbf{N} \cup \{+\infty\}$ and $\mathbf{R} \cup \{\pm\infty\}$ are locales.

Example 5.5 (The locale of a topological space). Let T be a topological space. The poset $\mathrm{Open}(T)$ of open subsets of T is a locale:

- Given a family of open subsets $(U_i)_{i\in I}$, its supremum is the union $\bigcup_{i\in I} U_i$ and its infimum is the interior of the intersection $\bigcap_{i\in I} U_i$.
- The distributivity law follows from the set-theoretic distributive law and the fact that *finite* intersections of open subsets are open.

If $f\colon T \to S$ is a map of topological spaces, then $f^{-1}\colon \mathrm{Open}(S) \to \mathrm{Open}(T)$ is a morphism of posets that preserves colimits and finite limits. Hence, it is a morphism of locales $\mathrm{Open}(T) \to \mathrm{Open}(S)$. This defines a functor

$$\mathrm{Open}\colon \mathrm{Top} \to \mathrm{Loc}\,.$$

Remark 5.6. Locales are also called *frames* or *complete Heyting algebras*. However, these other names come with different notions of morphisms, hence describe different categories.

Proposition 5.7 (Limits and colimits of locales)**.** *The category* Loc *admits limits and colimits, and the inclusion* $\mathrm{Loc} \hookrightarrow \mathrm{Pos}^{\mathrm{op}}$ *preserves colimits. The initial object of* Loc *is* $0 = \mathrm{Open}(\varnothing) = \{\varnothing\}$ *and the final object is* $1 = \mathrm{Open}(*) = \{\varnothing \to *\}$.

Definition 5.8 (Points of a locale)**.** Let $\mathcal{O}$ be a locale. We define a topological space $\mathrm{Pt}(\mathcal{O})$ as follows:

- The elements of $\mathrm{Pt}(\mathcal{O})$ are the *points* of $\mathcal{O}$, i.e. the morphisms of locales $1 \to \mathcal{O}$.
- The open sets of $\mathrm{Pt}(\mathcal{O})$ are the subsets $\mathrm{Pt}(u) = \{p\colon 1 \to \mathcal{O} \mid p^*(u) = *\}$ for all $u \in \mathcal{O}$.

Given a morphism of locales $f\colon \mathcal{O}' \to \mathcal{O}$, the induced map $\mathrm{Pt}(\mathcal{O}') \to \mathrm{Pt}(\mathcal{O})$ is continous, since the preimage of $\mathrm{Pt}(u)$ is $\mathrm{Pt}(f^*(u))$. Hence, we have a functor

$$\mathrm{Pt}\colon \mathrm{Loc} \to \mathrm{Top}\,.$$

Remark 5.9 (Explicit description of points). Let P be a poset. A map of *sets* $f\colon P \to \{\varnothing \to *\}$ is determined by the subset $F = f^{-1}(*) \subseteq P$. It is a map of posets if and only if:

(a) F is upward-closed.

If this is the case and P has finite limits, then f preserves finite limits if and only if

(b) F is nonempty and $x \wedge y \in F$ whenver $x, y \in F$.

If moreover P is complete, then f preserves colimits if and only if:

(c) whenever $\bigvee_{i\in I} x_i \in F$, there existss $i \in I$ with $x_i \in F$.

Subsets $F \subset P$ satisfying (a) and (b) are called *filters* on P, and they are called *completely prime* if they also satisfy (c). Thus, we can identify the points of a locale $\mathcal{O}$ with the completely prime filters on $\mathcal{O}$.

Proposition 5.10 (The adjunction between topological spaces and locales)**.** *The functor* $\mathrm{Open}\colon \mathrm{Top} \to \mathrm{Loc}$ *is left adjoint to the functor* $\mathrm{Pt}\colon \mathrm{Loc} \to \mathrm{Top}$*. This adjunction is idempotent, i.e., it restricts to an equivalence between the essentail images of both functors.*

Definition 5.11 (Sober spaces and spatial locales)**.**

(a) A topological space T is *sober*, if it lies in the essential image of $\mathrm{Pt}\colon \mathrm{Loc} \to \mathrm{Top}$ or equivalently if the unit map $T \to \mathrm{Pt}(\mathrm{Open}(T))$ is an isomorphism.

(b) A locale $\mathcal{O}$ is *spatial* if it lies in the essential image of $\mathrm{Open}\colon \mathrm{Top} \to \mathrm{Loc}$, or equivalently if the counit map $\mathrm{Open}(\mathrm{Pt}(\mathcal{O})) \to \mathcal{O}$ is an isomorphism.

Remark 5.12. By **??**, the adjunction $\mathrm{Open} \dashv \mathrm{Pt}$ restricts to an equivalence between the category $\mathrm{Top}^{\mathrm{sob}}$ of sober topological spaces and the category of $\mathrm{Loc}^{\mathrm{spa}}$ of spatial locales. Moreover, the inclusion $\mathrm{Top}^{\mathrm{sob}} \hookrightarrow \mathrm{Top}$ has a left adjoint given by $\mathrm{Pt} \circ \mathrm{Open}$, called *soberification*, and the inclusion $\mathrm{Loc}^{\mathrm{spa}} \hookrightarrow \mathrm{Loc}$ has a right adjoint given by $\mathrm{Open} \circ \mathrm{Pt}$ called *spatialization*. In partiuclar, limits of sober spaces are sober and colimits of spatial locales are spatial.

Proposition 5.13 (Characterization of sober spaces)**.** *Let T be a topological space. Then the points of the locale* $\mathrm{Open}(T)$ *can be identified with the irreducible closed subsets of T, and the unit map* $T \to \mathrm{Pt}(\mathrm{Open}(T))$ *sends t to the closure of $\{t\}$. Hence, a topological space T is sober if and only if the map*

$$\{\textit{points of } T\} \to \{\textit{irreducible closed subsets of } T\}, \quad t \mapsto \overline{\{t\}},$$

is a bijection, i.e., if and only if every irreducible closed subsets of T has a unique generic point.

Remark 5.14.

(a) Every Hausdorff space is sober, as its irreducible subsets are singletons.

(b) Being sober is a local property: if T admits an open covering by sober spaces, then T is sober.

5.1. **Locales of radical ideals.** Let R be a ring. Recall that the poset of open subfunctors of $\mathrm{Spec}(R)$ is isomorphic to the poset Rad_R of radical ideals in R (Proposition 2.65). We claim that is is a locale. The supremum of a family of radical ideals $(K_i)_{i\in I}$ is the radical of $\sum_{i\in I} K_i$. The distributivity law reads

$$K \cap \sqrt{\sum_{i\in I} L_i} = \sqrt{\sum_{i\in I} (K \cap L_I)},$$

and it follows from the following facts: we have $K \cap L = \sqrt{KL}$ for any radical ideals K and L, and the *product* of ideals distributes over sums. By contrast, the poset Id_R of all ideals is usually not a locale, since the intersection of ideals does not distribute over sums.

For any ring map $R \to S$, the base change map $\mathrm{Id}_R \to \mathrm{Id}_S$, $I \mapsto IS$, preserves products and sums of ideals. It follows that the base change map $\mathrm{Rad}_R \to \mathrm{Rad}_S$, $I \mapsto \sqrt{IS}$, preserves finite intersections and suprema, hence is a morphism of locales $\mathrm{Rad}_S \to \mathrm{Rad}_R$.

We now computes the points of the locale Rad_R.

Definition 5.15 (Prime spectrum)**.** Let R be a ring. The *prime spectrum* of R is the topological space $\mathrm{Prim}(R)$ defined as follows:

- The elements of $\mathrm{Prim}(R)$ are the prime ideals of R.
- The open sets of $\mathrm{Prim}(R)$ are the subsets $\mathrm{Prim}(I) = \{\mathfrak{p} \mid I \not\subseteq \mathfrak{p}\}$ for all $I \subseteq R$.

Since prime ideals are radical, $\mathrm{Prim}(I)$ depends only on the radical ideal $\sqrt{(I)}$ generated by I.

Proposition 5.16 (The locale of a ring)**.** *Let R be a ring.*

(a) There is a homeomorphism

$$\mathrm{Prim}(R) \xrightarrow{\sim} \mathrm{Pt}(\mathrm{Rad}_R), \quad \mathfrak{p} \mapsto \{I \in \mathrm{Rad}_R \mid I \not\subseteq \mathfrak{p}\},$$

under which the open subset $\mathrm{Pt}(I)$ corresponds to the open subset $\mathrm{Prim}(I)$.

(b) The locale Rad_R is spatial, i.e., the map from (a) induces an isomorphism $\mathrm{Open}(\mathrm{Prim}(R)) \xrightarrow{\sim} \mathrm{Rad}_R$.

(c) The topological space $\mathrm{Prim}(R)$ is spectral, i.e.e, it is sober and its quasi-compact open subsets form a basis of the topology that is closed under finite intersections.

Remark 5.17. One can show that, conversely, every spectral space is homeomorphic to $\mathrm{Prim}(R)$ for some ring R (which is however far from unique).

Let now A be an $\mathbf{N}$-graded ring, and let hRad_A be the poset of saturated radical homogeneous ideals in A. By Proposition 3.84 (and Remark 3.85), the poset hRad_A is isomorphic to the poset of open subfunctors of $\mathrm{Proj}(A)$. This poset is also a locale: the distributivity law follows from the one in Rad_A and the fact that saturation preserves finite intersections. We now compute the points of hRad_A.

Definition 5.18 (Homogeneous prime spectrum)**.** Let A be an $\mathbf{N}$-graded ring. The *homogeneous prime spectrum* of A is the topological space $\mathrm{hPrim}(A)$ defined as follows:

- The elements of $\mathrm{hPrim}(A)$ are the saturated homogeneous prime ideals of A, or equivalently the homogeneous prime ideals of A that do not conatain A_+ (see (*d*) of Remark 3.76).
- The open sets of $\mathrm{hPrim}(R)$ are the subsets $\mathrm{hPrim}(I) = \{\mathfrak{p} \mid I \not\subseteq \mathfrak{p}\}$ for all homogeneous subsets $I \subseteq A$.

Note that $\mathrm{hPrim}(I)$ depends only on the saturated radical ideal $\sqrt{(I)}^{\mathrm{sat}}$ generated by I.

Proposition 5.19 (The locale of an $\mathbf{N}$-graded ring)**.** *Let A be an $\mathbf{N}$-graded ring.*

(a) There is a homeomorphism

$$\mathrm{hPrim}(A) \xrightarrow{\sim} \mathrm{Pt}(\mathrm{hRad}_A), \quad \mathfrak{p} \mapsto \{I \in \mathrm{hRad}_A \mid I \not\subseteq \mathfrak{p}\},$$

under which the open subset $\mathrm{Pt}(I)$ corresponds to the open subset $\mathrm{hPrim}(I)$.

(b) The locale hRad_A is spatial, i.e., the map from (a) induces an isomorphism $\mathrm{Open}(\mathrm{hPrim}(A)) \xrightarrow{\sim} \mathrm{hRad}_A$.

(c) The topological space $\mathrm{hPrim}(A)$ is sober and its quasi-compact open subsets form a basis of the topology that is closed under binary intersections. If A is finitely generated as an A_0-algebra, then $\mathrm{hPrim}(A)$ is spectral.

5.2. **The topological space of an algebraic functor.** LKet $\mathrm{Open}(X)$ denote the poset of open subfunctors of an algebraic functor X. Since the preimage of an open subfunctor is open, we have a functor

$$\mathrm{Open}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Pos}^{\mathrm{op}}.$$

By Proposition 4.21, $\mathrm{Open}(X)$ is isomorphic to the poset Rad_X of quasi-coherent radical ideals over X, so that there is an isomorphism of posets

$$\mathrm{Open}(X) \xrightarrow{\sim} \lim_{\mathrm{Spec}(R)\to X} \mathrm{Open}(\mathrm{Spec}(R)).$$

In Section 5.1, we saw that the restriction of $\mathrm{Open}(X)$ to the subcategory $\mathrm{Aff} \subseteq \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$ of affine schemes lands in the subcategory of locales. Combining these facts with Proposition 5.7, we immediately deduce the following result:

Proposition 5.20. *The functor* $\mathrm{Open}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Pos}^{\mathrm{op}}$ *lands in the subcategory* $\mathrm{Loc} \subseteq \mathrm{Pos}^{\mathrm{op}}$*, and the induced functor*

$$\mathrm{Open}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Loc}$$

is colimit preserving.

Since colimits of spatial locales are spatial (Remark 5.12), we deduce:

Corollary 5.21. *For any algebraic functor X, the locale* $\mathrm{Open}(X)$ *is spatial.*

Remark 5.22. The final object in $\mathrm{Open}(X)$ is $\mathrm{D}(1) = X$ and the initial object is $\mathrm{D}(0) = \varnothing_X$, where $\varnothing_X(0) = X(0)$ and $\varnothing_X(R) = \varnothing$ for all $R \neq 0$. For any $U, V \in \mathrm{Open}(X)$, we have $U \wedge V = U \cap V$.

We now define an explicit topological space $|X|$ whose locale of open sets is $\mathrm{Open}(X)$. If $X = \mathrm{Spec}(R)$, the prime spectrum $\mathrm{Prim}(R)$ is the unique sober space with this property, by Proposition 5.16. On the other hand, by Remark 3.5, there is a bijection

$$\{\text{connected components of } \mathrm{Field}_R\} \xrightarrow{\sim} \mathrm{Prim}(R), \quad (\varphi\colon R \to k) \mapsto \ker(\varphi),$$

with inverse given by $\mathfrak{p} \mapsto \kappa(\mathfrak{p})$.

Construction 5.23 (Underlying space of an algebraic functor)**.** Let X be an algebraic functor and let $\mathrm{Field}_X \subseteq \mathrm{El}(X)^{\mathrm{op}}$ be the full subcategory spanned by the field-valued points of X. We define a topological space $|X|$ as follows:

- The points of $|X|$ are the connected components of the category Field_X.
- A subset $U \subseteq |X|$ is open if and only if, for every ring R and every R-point $x\colon \mathrm{Spec}(R) \to X$, the preimage $|x|^{-1}(U) \subseteq \mathrm{Prim}(R)$ is open.

Proposition 5.24. *The functor*

$$\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Top}, \quad X \mapsto |X|,$$

is colimit preserving. Hence, for any algebraic functor X, there is a natural isomorphism of posets

$$\mathrm{Open}(X) \xrightarrow{\sim} \mathrm{Open}(|X|), \quad U \mapsto |U|.$$

Remark 5.25. (a) Since every field admits an algebraic closure field; such a k-point is sometimes called a *geometry point* of X, although the exact menaning of "geometric point" varies from sources to sources.
(b) If X is affine, then $|X|$ is sober. However, as colimits of sober spaces need not to be sober, $|X|$ is not sober in general. Its soberification is the space $\mathrm{Pt}(\mathrm{Open}(X))$.
(c) If $X = \mathrm{Proj}(A)$, then $|X|$ is sober as it is covered by the open subspaces $|\mathrm{Spec}(A_{(f)})|$, which are sober. Hence, the canonical map $|X| \to \mathrm{Pt}(\mathrm{Open}(X)) \simeq \mathrm{hPrim}(A)$ is homeomorphism.

Remark 5.26 (Points in algebraic geometry). We now have introduced two distinct notions of "points" of an algebraic functor X:

(a) If R is any ring, an *R-point* or *R-valued point* of X is an element of $X(R)$.
(b) A *point* of X usually refers to a point of the topological space $|X|$, which is an equivalence class of field-valued points of X.

For fixed R, the R-points of X only form a set $X(R)$, but for varying R they form a category, namely the category of elements $\mathrm{El}(X)$. On the other hand, the point of X in the sense of (b) form a topological space $|X|$. These two notions of points are related by the zigzag of functors

$$\mathrm{El}(X)^{\mathrm{op}} \hookleftarrow \mathrm{Field}_X \twoheadleftarrow |X|.$$

Lemma 5.27 (Underlying space of pullbacks)**.** *Let $Y \to X \leftarrow Z$ be maps of algebraic functors. Then the canonical map*

$$|Y \times_X Z| \to |Y| \times_{|X|} |Z|$$

is surjective. It is bijective if $Y \to X$ is a monomorphism.

Proposition 5.28 (Underlying space of immersions)**.** *Let X be an algebraic functor and let $Y \hookrightarrow X$ be an open immersion, a closed immersion, or an immersion. Then the induced map $|Y| \to |X|$ is an open embedding, a closed embedding, or a locally closed embedding, respectively.*

Remark 5.29 (Closed subfunctors vs. closed subspaces). By Proposition 5.28, there is a map of posets

$$\mathrm{Closed}(X) \to \mathrm{Closed}(|X|), \quad Z \mapsto |Z|.$$

In contrast to Proposition 5.24, this map to almost never bijective, but it is surjective if $X = \mathrm{Spec}(A)$ or if $X = \mathrm{Proj}(A)$: a preimage of a closed subset of $|X|$ is given by the vanishing locus of the radical ideal in A corresponding to the open complement. In general, if $Z \subseteq X$ is a closed subfunctor with open complement U (Definition 4.22), then $|Z| = |X| \setminus |U|$.

Definition 5.30 (Topological properties of algebraic functors)**.** Let P be a property of topological spaces, such as *connected*, *locally connected*, *irreducible*, or *discrete*. We say that an algebraic functor X has property P if $|X|$ has property P[5].

Example 5.31. If $X = \mathrm{Spec}(R)$, so that $|X|$ is the prime spectrum $\mathrm{Prim}(R)$, we have the following standard results from commutative algebra:

emark 5.43
(a) X is connected if and only if R has exactlyl two idempotent element (see below for a generalization of this statement to arbitrary algebraic functors).
(b) X is irreducible if and only if the nilradical $\sqrt{0} \subseteq R$ is prime.
(c) X is discrete if and only if R is artinian.

Definition 5.32 (Topological properties of maps of algebraic functors)**.** Let P be a property of maps of topological spaces, such as *surjective*, *injective*, *bijective*, *dominant*, *submersive*, *open*, *closed*, or *homeomorphism*[6] A map of algebraic functors $f\colon Y \to X$ is said to have property P if the map $|f|\colon |Y| \to |X|$ has property P. We say that f is *universally* P if for every $X' \to X$, the base change $Y \times_X X' \to X'$ has property P.

Proposition 5.33. **??** *Let $f\colon Y \to X$ be a map of algebraic functors.*

(a) Let P be any of the following properties: surjective, injective, bijective, dominant, submersive, open, closed, or homeomorphism. Then f is universally P if and only if, for every ring R and every $\mathrm{Spec}(R) \to X$, the base change $Y \times_X \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ has property P.
(b) f is universally surjective if and only if it is surjective.

[5]Provided this does not conflict with any other definitions. For example, an affine scheme X is called *noetherian* if the ring $\mathcal{O}(X)$ is noetherian, which is strictly stronger than $|X|$ being noetherian; the latter property is then called *topologically noetherian*

[6]A continuous map $f\colon T \to S$ is *dominant* if $f(T)$ is dense in S, and it is *submersive*, if it is surjective and S had the quotient topology.

(c) If f is a monomorphism, then it is universally injective.
(d) If f is a Zariski-local epimorphism, then it is universally submersive and in particular universally surjective.

Example 5.34. With the exception of surjectivity, the properties listed in **??** are not automatically universal, even for maps between affine schemes:

(a) The map $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$ is a homeomorphism, but it is not universally injective: its pullback along itself is $\mathrm{Spec}(\mathbf{C} \times \mathbf{C}) \to \mathrm{Spec}(\mathbf{C})$, which induces the map of topological spaces $* \sqcup * \to *$.
(b) If k is a field, then $|\mathrm{Spec}(k)| = *$ and hence any map $X \to \mathrm{Spec}(k)$ is closed. The map $\mathbf{A}^1_k \to \mathrm{Spec}(k)$ is however not universally closed: its pullback along itself is the projection $\mathbf{A}^2_k \to \mathbf{A}^1_k$, which is not closed as it restricts to an isomorphism $\mathrm{V}(xy-1) \xrightarrow{\sim} \mathbf{A}^1_k \setminus \{0\}$.
(c) An example of an open morphism that is not universally open is

$$f\colon \mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x,y])/(y^2 - x^3 - x^2), \quad x \mapsto t^2 - 1, \quad y \mapsto t(t^2-1),$$

where k is a field of characteristic not 2, which is the resolution of singularities of the affine nodal cubic. The map $|f|$ is the quotient map $|\mathbf{A}^1_k| \twoheadrightarrow |\mathbf{A}^1_k|/(-1 \sim 1)$, which is open but whose pullback along itself is not open. By Lemma 5.27, this implies that the pullback of f along itself is not open.

Corollary 5.35 (Zariski codescent for the underlying space)**.** *Let $(Y_i \to X)_{i\in I}$ be a family of maps of algebraic functors such that $\coprod_{i\in I} Y_i \to X$ is Zariski-local epimorphism. Then the diagram*

$$\coprod_{i,j\in I} |Y_i \times_X Y_j| \rightrightarrows \coprod_{i\in I} |Y_i| \to |X|$$

is a coequalizer diagram of topological spaces.

Definition 5.36 (Residue field)**.** Leet X be an algebraic functor and let $x \in |X|$. Let $\mathrm{Field}_{(X,x)}$ be the full subcategory of Field_X spanned by the field-valued points hitting x. If the category $\mathrm{Field}_{(X,x)}$ has an initial object, we call it the *residue field* of X at x and denote it by $\kappa(x)$.

Let $f\colon Y \to X$ be a map of algebraic functors, let $y \in |Y|$ and let $x = f(y) \in |X|$. If both residue fields $\kappa(y)$ and $\kappa(x)$ exist, then the universal property of $\kappa(x)$ gives a map of fields $\kappa(x) \to \kappa(y)$, called the *residual field extension* of f at y.

Proposition 5.37 (Existence of residue fields)**.** *Let X be an algebraic functor, let $Y \hookrightarrow X$ be an immersion, and let $y \in |Y|$ be a point with image $x \in |X|$. Then the functor $\mathrm{Field}_{(Y,y)} \to \mathrm{Field}_{(X,x)}$ is an equivalence. In particular, the residue field of X at x exists if and only if the residue field of Y at y exists, in which case the residual field extension $\kappa(x) \to \kappa(y)$ is an isomorphism.*

Example 5.38. (a) Let $X = \mathrm{Spec}(R)$ and let $x \in |X|$ be given by the prime ideal $\mathfrak{p} \subseteq R$. Then the residue field $\kappa(x)$ exists and is the usual residue field $\kappa(\mathfrak{p})$, by Remark 3.5.
(b) Let $X = \mathrm{Proj}(A)$ and let $x \in |X|$ be given by the saturated homogeneous prime ideal $\mathfrak{p} \subseteq A$. By Remark 3.76, there exists a homogeneous element $f \in A_+$ with $f \notin \mathfrak{p}$. Then $\mathfrak{p}_{(f)}$ is a prime ideal in $A_{(f)}$ and the open immersion $\mathrm{Spec}(A_{(f)}) \hookrightarrow \mathrm{Proj}(A)$ sends $\mathfrak{p}_{(f)}$ to $\mathfrak{p}$. By Proposition 5.37 and (a), the residue field $\kappa(x)$ exists and is given by $\kappa(\mathfrak{p}_{(f)}) \simeq \kappa(\mathfrak{p})_0$.

5.3. **Open coverings.** The notion of *open covering* of a topological space T depends only on the locale $\mathrm{Open}(T)$: a family of open subsets $(U_i \subseteq T)_{i\in I}$ is an open covering if and only if its supremum in $\mathrm{Open}(T)$ equals T. In the locale of an algebraic functor, there are several useful characterization of this property:

Proposition 5.39. *Let X be an algebraic functor, let $(U_i \subseteq X)_{i\in I}$ be a family of open subfunctors, and let $K_i \in \mathrm{Rad}_X$ be the quasi-coherent radical ideal of U_i. The following conditions are equivalent:*

(a) $X = \bigvee_{i\in I} U_i$ *in the poset* $\mathrm{Open}(X)$.
(b) $\mathcal{O}(X) = \bigvee_{i\in I} K_i$ *in the poset* Rad_X.
(c) For every ring R *and every* $x\colon \mathrm{Spec}(R) \to X$, $\bigcup_{i\in I} K_i(x)$ *generates the unit ideal in* R.
(d) For every local ring R, $X(R) = \bigcup_{i\in I} U_i(R)$.
(e) For every field k, $X(k) = \bigcup_{i\in I} U_i(k)$.
(f) The induced map $\coprod_{i\in I} U_i \to X$ *is a Zariski-local epimorphism (Definition 4.37).*
(g) The induced map $\coprod_{i\in I} U_i \to X$ *is surjective (Definition 5.32).*

Definition 5.40 (Open covering)**.** A family of open subfunctors $(U_i \subseteq X)_{i\in I}$ is called an *open covering* of X if it satisfies the equivalent conditions of Proposition 5.39

Remark 5.41. By Proposition 5.39, open coverings of an algebraic functor X are equivalent to open coverings of its underlying space $|X|$.

Example 5.42 (Open coverings of affine and projective schemes)*.*
(a) Let R be a ring and $(F_i)_{i\in I}$ a family of subsets of R. Then $(\mathrm{D}(F_i) \subseteq \mathrm{Spec}(R))_{i\in I}$ is an open covering of $\mathrm{Spec}(R)$ if and only if $\bigcup_{i\in I} F_i$ generates the unit ideal of R, by Proposition 2.65. In general, the subfunctor $\mathrm{D}(F_i)$ form an open covering of $\mathrm{D}(\bigcup_{i\in I} F_i)$.
(b) Let R be a ring, L a line over R, and $(s_i)_{i\in I}$ a family of elements of L. Then $(D(s_i) \subseteq \mathrm{Spec}(R))_{i\in I}$ is an open covering of $\mathrm{Spec}(R)$ if and only if $(s_i)_{i\in I}$ generates L.
(c) Let A be an $\mathbf{N}$-graded ring and $(F_i)_{i\in I}$ a family of homogeneous subsets of R. Then $(\mathrm{D}(F_i) \subseteq \mathrm{Proj}(A))_{i\in I}$ is an open covering of $\mathrm{Proj}(A)$ if and only if the saturated radical ideal generated by $\bigcup_{i\in I} F_i$ is the unit ideal of A, by Proposition 3.84. For example, the affine open subschemes $\mathrm{D}(f) \simeq \mathrm{Spec}(A_{(f)})$ with $f \in A_+$ homogeneous form an open covering of $\mathrm{Proj}(A)$. In general, the subfunctors $\mathrm{D}(F_i)$ form an open covering of $\mathrm{D}(\bigcup_{i\in I} F_i)$.
(d) Let k be a ring, M a k-module, and $n \in \mathbf{N}$. For every $\alpha \in M^n$, let $U(\alpha) \subseteq \mathrm{Gr}_n(M)$ be the subfunctor consisting of the quotient spaces $\varphi\colon M \otimes_k R \twoheadrightarrow V$ such that $\varphi \circ \alpha\colon R^n \to V$ is an isomorphism, which is open by Proposition 3.51. If $(\alpha_i)_{i\in I}$ is a family in M^n, then $(U(\alpha_i) \subseteq \mathrm{Gr}_n(M))_{i\in I}$ is an open covering of $\mathrm{Gr}_n(M)$ if and only if the image of $(\alpha_i)_{i\in I}$ in $\Lambda^n_k M$ is a generating family (Example 3.101).

Remark 5.43 (Clopen subfunctors vs. clopen subspaces)*.* For any algebraic functor X, the map

$$\mathrm{Clopen}(X) \to \mathrm{Clopen}(|X|), \quad U \mapsto |U|,$$

is bijective. Indeed, let $U \subseteq X$ be an open subfucntor such that $|U|$ is clopen in $|X|$, and let $V \subseteq X$ be the open subfunctor such that $|X| = |U| \sqcup |V|$. Then U and V form an open covering of X such that $U \cap V = \varnothing_X$, so that the map $\mathcal{O}(X) \to \mathcal{O}(U) \times \mathcal{O}(V)$ is an isomorphism by Corollary 4.41. We therefore find an idempotent function $e \in \mathcal{O}(X)$ such that $U \subseteq \mathrm{D}(e)$ and $\mathrm{V} \subseteq \mathrm{D}(1-e)$. Since $U \vee V = X$ and $\mathrm{D}(e) \wedge \mathrm{D}(1-e) = \varnothing_X$ in $\mathrm{Open}(X)$, the distributivity law implies that $U = \mathrm{D}(e)$. As e is idempotent, we have $\mathrm{D}(e) = \mathrm{V}(1-e)$, so that U is closed, as desired. In particular, by Proposition 4.27, X is connected (in the sense of Definition 5.32) if and only if the ring $\mathcal{O}(X)$ has exactly two idempotent elements.

Proposition 5.44 (Closure properties of open coverings)**.**
- *(Refinement) A family of open subfunctors* $(U_i \subseteq X)_{i\in I}$ *is an open covering if it is refined by an open covering* $(V_j \subseteq X)_{j\in J}$*, i.e., if there exists* $\alpha\colon J \to I$ *such that* $V_j \subseteq U_{\alpha(j)}$ *for all* $j \in J$.
- *(Composition) If* $(U_i \subseteq X)_{i\in I}$ *and* $(V_{ij} \subseteq U_i)_{j\in J_i}$ *are open coverings, then* $(V_{ij} \subseteq X)_{(i,j)\in\coprod_{i\in I} J_i}$ *is an open covering.*
- *(Intersection) If* $(U_i \subseteq X)_{i\in I}$ *and* $(V_j \subseteq X)_{j\in J}$ *are open coverings.*
- *(Base change) If* $(U_i \subseteq X)_{i\in I}$ *is an open covering and* $f\colon Y \to X$ *is a map of algebraic functors, then* $(f^{-1}(U_i) \subseteq Y)_{i\in I}$ *is an open covering.*

6. Sheaves

In this chapter, we introduce the formalism of Grothendieck topologies and sheaves. A Grothendieck topology τ on a category $\mathcal{C}$ determines (and is determined by) a subcategory $\mathrm{Sh}_\tau(\mathcal{C})$ of the presheaf category $\mathrm{P}(\mathcal{C})$, whose objects are called τ-sheaves on $\mathcal{C}$. These are presheaves on $\mathcal{C}$ satisfying a certain local-to-global condition with respect to the topology τ, called τ-descent.

In algebraic geometry, we typically use this formalism in the following situations:

(a) $\mathcal{C}$ is the category of open subsets of a topological space T. This was the original notion of sheaf discovered by Leray in the 1940s, before Grothendieck's categorical generalization in the 1950s.

(b) $\mathcal{C} = \mathrm{CAlg}_k^{\mathrm{op}}$ is the opposite of the category of k-algebras, or equivalently the category Aff_k of affine k-schemes, on which there is a plethora of commonly used Grothendieck's topologies. The main ones are the Zariski, Nisnevich, étale, fppf ("fidèlement plat de présentation finie"), and fpqc ("fidèlement plat quasi-compac") topologies.

(c) The topologies in (b) extend to various enlargements of Aff_k, such as the category Sch_k of k-schemes, which we will introduce later, and even the whole category $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ of algebraic k-functors.

All the "Zariski descent" statements established so far are examples of descent statements for the so-called Zariski topology on $\mathrm{CAlg}^{\mathrm{op}}$ (e.g., Corollary 2.72 and Proposition 3.61) or on $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ (e.g., Corollary 4.41).

6.1. **Sieves and descent.**

Definition 6.1 (Sieve)**.** Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$. A *sieve* on X is a subpresheaf of the representable presheaf $\text{よ}(X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

Remark 6.2 (Sieves as subcategories)*.* A *left closed subcategory* of a category $\mathcal{C}$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that, if $X \in \mathcal{D}$ and $Y \to X$ is any morphism in $\mathcal{C}$, then $Y \in \mathcal{D}$. If R is a sieve on an object $X \in \mathcal{C}$, is category of element $\mathrm{El}(R)$ is a left closed subcategory of $\mathrm{El}(\text{よ}(X)) = \mathcal{C}_{/X}$. This defines an order-preserving bijection

$$\mathrm{El}\colon \{\text{sieves on } X\} \xrightarrow{\sim} \{\text{left closed subcategories of } \mathcal{C}_{/X}\}.$$

In practice, we think of sieves on X either as subfunctors of $\text{よ}(X)$ or as left closed subcategories of $\mathcal{C}_{/X}$, depending on the situation.

Notation 6.3 (Pullback of sieves)**.** Let R be a sieve on X and let $f\colon X \to Y$ be a map. We denote by $f^*(R)$ the pullback $R \times_{\text{よ}(X)} \text{よ}(Y)$, which is a sieve on Y. Concretely, $f^*(R)$ consists of all maps to Y whose composition with f belongs to R.

Definition 6.4 (Generated sieve)**.** Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$. The *sieve on X generated by a family of maps* $(Y_i \to X)_{i\in I}$ is the image of the map $\coprod_{i\in I} \text{よ}(Y_i) \to \text{よ}(X)$, i.e., the sieve consisting of all maps to X that factor through Y_i for some i.

Definition 6.5 (Descent)**.** Let $\mathcal{C}$ be a category, $F \in \mathrm{P}(\mathcal{C})$ a presheaf on $\mathcal{C}$ and a sieve on $X \in \mathcal{C}$. We say that F *satisfies descent* along R if the inclusion $R \subseteq \text{よ}(X)$ induces an isomorphism

$$F(X) = \mathrm{Map}(\text{よ}(X), F) \xrightarrow{\sim} \mathrm{Map}(R, F).$$

Example 6.6 (The empty sieve)*.* For any $X \in \mathcal{C}$, the empty subpresheaf of $\text{よ}(X)$ is a sieve on X, called the *empty sieve* on X. Since the empty presheaf is an initial object in $\mathrm{P}(\mathcal{C})$, a presheaf F satifies descent along this sieve if and only if $F(X)$ is a one-point set.

Example 6.7 (Sieve generated by two subojects)*.* Let $X \in \mathcal{C}$, let $U, V \subseteq X$ be a pair of subobjects of X, and let R be the sieve on X generated by U and V. If the intersection $U \cap V$ exists, then R is the pushout $よ(U) \sqcup_{よ(U\cap V)} よ(V)$ in $\mathrm{P}(\mathcal{C})$. Hence, a presheaf F on $\mathcal{C}$ satisfies descent along R if and only if the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

is cartesian.

Lemma 6.8 (Presentation of sieves)**.** *Let R be a sieve on X and let $(Y_i \to X)_{i\in I}$ be a family of maps generating R. The the maps $よ(Y_i) \to R$ induce an isomorphism of presheaves*

$$\operatorname{colim}(\coprod_{i,j\in I} よ(Y_i) \times_{よ(X)よ(Y_j) \rightrightarrows \coprod_{i\in I}} よ(Y_i)) \xrightarrow{\sim} R.$$

W: Todo

Proof. □

Proposition 6.9 (Characterization of descent)**.** *Let R be a sieve on $X \in \mathcal{C}$ and let $(Y_i \to X)_{i\in I}$ be a family of maps generating R. For a presheaf $F \in \mathrm{P}(\mathcal{C})$, the following conditions are equivalent:*

(a) F satisfies descent along R.
(b) The canonical map

$$F(X) \to \lim_{Y\in \mathrm{El}(R)} F(Y)$$

is an isomorphism.
(c) The diagram

$$F(X) \to \prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} \mathrm{Map}(よ(Y)_i \times_{よ(X)} よ(Y_j), F)$$

is an equalizer

Remark 6.10. If in Proposition 6.9 the pullbacks $Y_i \times_X Y_j$ exists in $\mathcal{C}$, we can rewrite Condition (c) as follows:

c' The diagram

$$F(X) \to \prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} F(Y_i \times_X Y_j)$$

is an equalizer.

Example 6.11 (Monogenic sieve). Let R be the sieve on $X \in \mathcal{C}$ generated by a single map $Y \to X$. If the pullback $Y \times_X Y$ exists in $\mathcal{C}$, then a presheaf $F \in \mathrm{P}(\mathcal{C})$ satisfies descent along R if and only if the following diagram is an equalizer

$$F(X) \to F(Y) \rightrightarrows F(Y \times_X Y).$$

Remark 6.12 (Descent for categories). We can defin for a presheaf of categories $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ (in the sense of Definition 4.1) via Condition (b) of Proposition 6.9. Namely, if R is a sieve on $X \in \mathcal{C}$, we say that F *satisfies descent along* R if the canonical functor

$$F(X) \to \lim_{Y\in \mathrm{El}(R)} F(Y)$$

is an equivalence of categories. It turns out that this is *not* equivalent to Condition (b) of Proposition 6.9. Instead, assuming that the relevant fiber products exist in $\mathcal{C}$ for simplicity, it is equivalent to the following condition c" The diagram of categories

$$F(X) \to \prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} F(Y_i \times_X Y_j) \substack{\to\\\to\\\to} \prod_{i,j,k} \prod_{i,j,k\in I} F(Y_i \times_X Y_j \times_X Y_k)$$

is a limit diagram.

This explains why the descent condition takes this more complicated form for presheaves of categories. Note that (c'') is equivalent to (c') if F is the presheaf of posets.

6.2. **Grothendieck topologies.**

Definition 6.13 (Grothendieck topologies, site)**.** A *Grothendieck topology* τ on a category $\mathcal{C}$, or *topology* for short, assigns to each object $X \in \mathcal{C}$ a collection of sieves on X, called τ*-covering sieves*, subject to the following conditions:

(a) If R is a τ-covering sieve on X and $f: Y \to X$ is any map, then the pullback $f^*(R)$ is a τ-covering sieve on Y.
(b) Let S be a sieve on X and let R be a τ-covering sieve on X. If for all $f: Y \to X$ in R, $f^*(S)$ is τ-covering, then S itself is τ-covering.
(c) The maximal sieve $よ(X)$ on any X is τ-covering.

A category equipped with a Grothendieck topology is called a *site*

Remark 6.14. The following further properties of a Grothendieck topology τ are consequences of the axioms $(a) - (c)$:

d If R is a τ-covering sieve on X and S is any sieve on X containing R, then S is a τ-covering.
e Any finite intersection of τ-covering sieves on X is τ-covering.

Definition 6.15 (Sheaf)**.** Let $\mathcal{C}$ be a category and τ a topology on $\mathcal{C}$. A presheaf $F \in \mathrm{P}(C)$ is called a τ*-sheaf* if it satisfies descent along every τ-covering sieve. We denote by $\mathrm{Sh}_\tau(\mathcal{C}) \subseteq \mathrm{P}(\mathcal{C})$ the full subcategory of τ-sheaves.

Remark 6.16 (Posets of topologies). Let τ and ρ be topologies on $\mathcal{C}$, we say that τ is *coarser* than ρ and ρ is *finer* than τ, and we write $\tau \leq \rho$, if every τ-covering sieve is a ρ-covering sieve. The collection of topologies on $\mathcal{C}$ is a poset under the relation $\leq$. By definition, any intersection of topologies on $\mathcal{C}$ is again a topology. The poset of topologies on $\mathcal{C}$ therefore admits all infima, and hence also all suprema. In particular, any collection σ of sieves on objects of $\mathcal{C}$ generates a topology $\bar{\sigma}$, which is the coarsest topology on $\mathcal{C}$ containing σ.

Example 6.17. On any category $\mathcal{C}$, we can consider the following topologies:

(a) The *discrete topology* is the finest topology: all sieves are covering. The final presheaf $*$ is the only sheaf in the discret topology.
(b) The *indiscrete topology* if the coarsest topology: only maximal sieves are covering. All presheaves are sheaves in the indiscrete topology

Example 6.18. Locales and topological spaces Let $\mathcal{O}$ be a locale, viewed as a category. If we define a sieve on $u \in \mathcal{O}$ to be a covering if its supremum is u, we obtain a topology on $\mathcal{O}$, called the *canonical topology*. Indeed, Conditions (a) and (b) follow from the distributivity law, an Condition (c) is clear. In particular, if T is a topological space, this defines the canonical topology on the category $\mathrm{Open}(T)$.

Example 6.19 (Weiss topologies). Let T be a topological space and κ cardinal. The κ*-Weiss topology* on $\mathrm{Open}(T)$ is defined as follows: a sieve R on U is covering if any subset of U of size $< \kappa$ contained in some element R. This recovers the canonical topology for $\kappa = 2$ (Example 6.18), the discrete topology for $\kappa = 0$ and the indiscrete topology for $\kappa > |T|$ (Example 6.17). For $\kappa = 1$, the covering sieves are exactly the nonempty sieves, and the corresponding sheaves are the constant presheaves.

Warning 6.20. The notion of Grothendieck topology is *not* a direct generalization of the notion of topology on a set. Example 6.18 explains the relation between the two notions. Historically, Grothendieck named the new concepty "topology" because it replaced topological spaces in the context of sheaf theory.

Proposition 6.21 (The adjunction between topologies and subcategories)**.** *Let $\mathcal{C}$ be a category. The map of posets*

$$\{\textit{topologies on } \mathcal{C}\} \to \{\textit{subcategories of } \mathrm{P}(\mathcal{C})\}, \quad \tau \mapsto \mathrm{Sh}_\tau(\mathcal{C}),$$

has a left adjoint $\mathcal{E} \to \tau_{\mathcal{E}}$, i.e., there exists a finest topology $\tau_{\mathcal{E}}$ such that $\mathcal{E} \subseteq \mathrm{Sh}_{\tau_{\mathcal{E}}}(\mathcal{C})$. A sieve R on X is $\tau_{\mathcal{E}}$ covering if and only if, for every map $f: Y \to X$, every $F \in \mathcal{E}$ satisfies descent along $f^*(R)$.

Corollary 6.22. *Let $(\tau_i)_{i\in I}$ be a family of topologies on $\mathcal{C}$ and let $\tau = \bigvee_{i\in I} \tau_i$ be their supremum. Then*

$$\mathrm{Sh}_\tau(\mathcal{C}) = \bigcap_{i\in I} \mathrm{Sh}_{\tau_i}(\mathcal{C}).$$

Corollary 6.23. *Let $\mathcal{C}$ be a category and let σ be a collection of sieves on $\mathcal{C}$ that is closed under pullbacks (i.e, satisfies Condition (a) of Definition 6.13). Let $\bar(\sigma)$ be the topology generated by σ. Then a presheaf F on $\mathcal{C}$ is a $\bar{\sigma}$-sheaf if and only if it satifies descent along all sieves in σ.*

In practice, topologies are often defined by means of covering families rather than covering sieves:

Definition 6.24 (Pretopology)**.** A *pretopology* π on a category $\mathcal{C}$ assigns to each object $X \in \mathcal{C}$ a collection of families of $\mathcal{C}_{/X}$, called *π-covering families* or *π-covers*, subject to the following conditions:

(a) If $(U_i \to X)_{i\in I}$ is a π-covering family and $f: Y \to X$ is any map, then the pullbacks $U_i \times_X Y$ exist in $\mathcal{C}$ and the family $(U_i \times_X Y \to Y)_{i\in I}$ is π-covering.
(b) If $(U_i \to X)_{i\in I}$ and $(V_{ij} \to U_i)_{j\in J}$, are π-covering families, then $(V_{ij} \to X)_{i\in I, j\in J_i}$ is π-covering.
(c) The singleton family $(\mathrm{id}_X: X \to X)$ is π-covering.

The *topology associated with* a pretopology π is the coarsest topology for which π-coverings families generate covering sieves (which exists by Remark 6.16).

Example 6.25. Let τ be a topology on $\mathcal{C}$. If (and only if) $\mathcal{C}$ has pullbacks, then the τ-covering families form a pretopology on $\mathcal{C}$, whose associated topology is τ. For example, if X is a topological space or an algebraic functor, covering families for the canonical topology on $\mathrm{Open}(X)$ are precisely open coverings, which in particular form a pretopology on $\mathrm{Open}(X)$.

Proposition 6.26 (Pretopologies and descent)**.** *Let $\mathcal{C}$ be a category, let π be a pretopology on $\mathcal{C}$ and let τ be the topology associated with π.*

(a) A sieve is τ-covering if and only if it contains a π-covering family.
(b) A presheaf F is a τ-sheaf if and only if, for every π-covering family $(U_i \to X)_{i\in I}$, the following diagram is an equalizer:

$$F(X) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_X U_j).$$

Example 6.27 (The standard topology on Top). Open coverings form a pretopology on the category Top of all topological spaces (and many similar categories, like that of smooth manifolds). If τ is the assoicated topology, Proposition 6.26 implies the following:

(a) A sieve on X is τ-covering if and only if it contains an open covering of X. Hence, a family $(Y_i \to X)_{i\in I}$ is τ-covering if and only if it locally has sections, i.e., if and only if every point of X has a neighborhood that maps to some Y_i over X.

(b) A presheaf $F\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ is a τ-sheaf if and only if $F_{|\,\mathrm{Open}(T)}$ is a sheaf on T for each $T \in \mathrm{Top}$.

For example, for every $X \in \mathrm{Top}$, the presheaf $よ(X)$ of X-valued functions is a τ-sheaf (by Example 6.18).

Example 6.28 (The Zariski topology). There is a pretopology on $\mathrm{CAlg}^{\mathrm{op}}$ whose covers are the families $(R \to R_{f_i})_{i\in I}$ suchthat $(f_i)_{i\in I}$ generates the unit ideal in R. The assoicated topology is called the *Zariski topology*. By Proposition 6.26, we have the following:

(a) A sieve $S \subseteq \mathrm{Spec}(R)$ is Zariski-covering if and only if the set of $f \in R$ such that $R \to R_f$ belongs to S generates the unit ideal. Consequently, an arbitrary family $(R \to R_i)_{i\in I}$ is Zariski-covering if and only if $\coprod_{i\in I} \mathrm{Spec}(S_i) \to \mathrm{Spec}(R)$ is a Zariski-local epimorphism (Definition 4.37).
(b) An algebraic functor $X\colon \mathrm{CAlg} \to \mathrm{Set}$ is a Zariski sheaf if and only if, for any ring R and any family $(f_i)_{i\in I}$ generating the unit ideal in R, the following diagram is an equalizer:
$$X(R) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j}).$$

For example, any affine scheme is a Zariski sheaf by Corollary 2.72, as is $\mathrm{Proj}(A)$ for any $\mathbf{N}$-graded ring A by Proposition 3.61. Note that if we only consider families $(f_i)_{i\in I}$ where I is finite, we obtain another pretopology on $\mathrm{CAlg}^{\mathrm{op}}$, whose associated topology is *also* the Zariski topology. Hence, the condition in (b) is *equivalent* to the seemingly weaker condition where we require that I be finite. On the other hand, by Proposition 6.9, it is also equivalent to the following seemingly stronger condition: for every Zariski-covering family $(R \to R_i)_{i\in I}$, the diagram
$$X(R) \to \prod_{i\in I} X(R_i) \rightrightarrows \prod_{i,j\in I} X(R_i \otimes_R R_j)$$
is an equalizer.

Example 6.29 (The fpqc and fppf topologies). The definition of the Zariski topology on $\mathrm{CAlg}^{\mathrm{op}}$ is a special case of a more general construction. Let E be a class of ring maps, which contains isomorphism and is closed under composition and cobase change. Then there is a pretopology on $\mathrm{CAlg}^{\mathrm{op}}$ whose covers are the families $(R \to R_i)_{i\in I}$ such that:

- I is finite;
- $\coprod_{i\in I} \mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ is surjective;
- each map $R \to R_i$ belongs to E.

For example:

(a) If E is the class of localizations $R \to R_f$, this recovers the Zariski topology of Example 6.28.
(b) If E is the class of flat maps, the associated topology on $\mathrm{CAlg}^{\mathrm{op}}$ is called the *fpqc topology*.
(c) If E is the class of flat maps of finite presentation, the associated topology on $\mathrm{CAlg}^{\mathrm{op}}$ is called the *fppf topology*.

Since the localization maps $R \to R_f$ are flat and of finite presentation, we have the following relations between these topologies: $\mathrm{Zar} \subseteq \mathrm{fppf} \subseteq \mathrm{fpqc}$.

Definition 6.30 (Dense subcategory)**.** Let $(\mathcal{C}, \tau)$ be a site. A *dense subcategory* of $(\mathcal{C}, \tau)$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that eveery $\mathcal{D} \subseteq \mathcal{C}$ such that every $X \in \mathcal{C}$ admits a τ-cover $(U_i \to X)_{i\in I}$ with $U_i \in \mathcal{D}$.

Proposition 6.31 (The "comparison lemma")**.** *Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{D} \subseteq \mathcal{C}$ a dense subcategory. Let $\tau|_{\mathcal{D}}$ be the collection of sieves $R|_{\mathcal{D}} \subseteq よ_{\mathcal{D}}(X)$ for all $X \in \mathcal{D}$ and all τ-covering sieves $R \subseteq よ_{\mathcal{C}}(X)$. Then $\tau|_{\mathcal{D}}$ is a topology on $\mathcal{D}$ and the functor* $\mathrm{P}(\mathcal{C}) \to \mathrm{P}(\mathcal{D})$*,* $F \mapsto F|_{\mathcal{D}}$*, restricts to an equivalence of categories* $\mathrm{Sh}_\tau(\mathcal{C}) \xrightarrow{\sim} \mathrm{Sh}_{\tau|_{\mathcal{D}}}(\mathcal{D})$.

Example 6.32 (Comparison lemma for presheaves). Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Then $\mathcal{D}$ is dense with respect to the indiscrete topology ((*b*) of Example 6.17) if and only if every object of $\mathcal{C}$ is a retract of an object of $\mathcal{D}$. In this case, the comparison lemma says that $\mathrm{P}(\mathcal{C}) \xrightarrow{\sim} \mathrm{P}(\mathcal{D})$.

Example 6.33 (Basis of a topological space). Let T be a topological space and let $\mathcal{B} \subseteq \mathrm{Open}(T)$ be a basis of its topology. By definition, $\mathcal{B}$ is dense with respect to the canonical topology on $\mathrm{Open}(T)$, and the restriction of the canonical topology to $\mathcal{B}$ consists of sieves generated by open coverings in $\mathcal{B}$. Hence, the comparison lemma provides an equivalence $\mathrm{Sh}(T) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{B})$.

Example 6.34 (Sheaves on manifolds). Let Man_n be the category of n-dimensional topological manifolds, equipped with the standard topology (induced by the pretopology of open coverings). Since by definition every n-manifold admits an open covering by manifolds homeomorphic to $\mathbf{R}^n$, the full subcategory $\{\mathbf{R}^n\} \subseteq \mathrm{Man}_n$ spanned by $\mathbf{R}^n$ is dense. By comparison lemma, $\mathrm{Sh}(\mathrm{Man}_n) \simeq \mathrm{Sh}(\{\mathbf{R}^n\})$.

Example 6.35 (Sheaves on $\mathrm{Spec}(R)$). Let R be a ring. Consider the following two full subcategories of the locale $\mathrm{Open}(\mathrm{Spec}(R)) \simeq \mathrm{Open}(\mathrm{Prim}(R)) \simeq \mathrm{Rad}_R$:

(a) $\mathrm{Open}^{\mathrm{pr}}(\mathrm{Spec}(R))$ consists of open subfunctors of the form $\mathrm{D}(f)$ with $f \in R$ (called the *principal open subschemes* of $\mathrm{Spec}(R)$);
(b) $\mathrm{Open}^{\mathrm{aff}}(\mathrm{Spec}(R))$ consists of the *affine* open subfunctors of $\mathrm{Spec}(R)$.

We have $\mathrm{Open}^{\mathrm{pr}}(\mathrm{Spec}(R)) \subseteq \mathrm{Open}^{\mathrm{aff}}(\mathrm{Spec}(R))$, and both subcategories are dense in $\mathrm{Open}(\mathrm{Spec}(R))$, since $\mathrm{D}(I) = \bigvee_{f \in I} \mathrm{D}(f)$. By the comparison lemma, the three categories of sheaves are equivalent. In particular, every sheaf on $\mathrm{Open}^{\mathrm{pr}}(\mathrm{Spec}(R))$ *extends uniquely* to a sheaf on $\mathrm{Open}(\mathrm{Spec}(R))$.

Example 6.36 (Sheaves on presheaves). Let $\mathcal{C}$ be a small category, which we view as a full subcategory of $\mathrm{P}(\mathcal{C})$ via the Yoneda embedding. Any topology τ on $\mathcal{C}$ extends canonically to a topology $\hat{\tau}$ on $\mathrm{P}(\mathcal{C})$ as follows: a sieve R on a presheaf F is $\hat{\tau}$-covering if, for any $よ(X) \to F$, the sieve of all $Y \to X$ such that $よ(Y) \to よ(X) \to F$ in R is τ-covering. Since every object of $\mathrm{P}(\mathcal{C})$ is a quotient of a coproduct of representable presheaves, the comparison lemma applies to the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{P}(\mathcal{C})$, so that

$$\mathrm{Sh}_\tau(\mathcal{C}) \simeq \mathrm{Sh}_{\hat{\tau}}(\mathrm{P}(\mathcal{C})).$$

(The assumption that $\mathcal{C}$ is small can be removed either by considering $\widehat{\mathrm{Set}}$-valued sheaves or by replacing $\mathrm{P}(\mathcal{C})$ by the full subcategory of presheaves that are *small* colimits of representables, cf. Remark 2.52)

Remark 6.37. The comparison lemma remains true for sheaves of categories (as in Remark 6.12). Applied to $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$ and τ the Zariski topology, Example 6.36, Example 6.36 shows that a Zariski sheaf of categories on $\mathrm{CAlg}^{\mathrm{op}}$ extends uniquely to a sheaf of categories on $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$. By the proof of the comparison lemma, this extension is given by right Kan extension, i.e. by forming categories of quasi-coherent objects (Definition 4.13). Granted this, Example 6.36 justifies Remark 4.40.

6.3. Sheafification.

Theorem 6.38. *Let $\mathcal{C}$ be a small category and τ a topology on $\mathcal{C}$. Then the inclusion $\mathrm{Sh}_\tau(\mathcal{C}) \subseteq \mathrm{P}(\mathcal{C})$ admits a left adjoint*

$$\alpha_\tau \colon \mathrm{P}(\mathcal{C}) \to \mathrm{Sh}_\tau(\mathcal{C}),$$

called τ-sheafification. Moreover:

(a) α_τ is left exact, i.e., preserves finite limits.
(b) A sieve $R \subseteq よ(X)$ is τ-covering if and only if $\mathrm{a}_\tau(R) \xrightarrow{\sim} \mathrm{a}_\tau(よ(X))$.

Remark 6.39. The assumption that $\mathcal{C}$ is small can be replaced with the following weaker assumption: for every $X \in \mathcal{C}$, there exists a small collection J of τ-covering sieves on X such that every τ-covering sieve on X contains a sieve in J. This assumption holds for example for the Zariski and fppf topologies on $\mathrm{CAlg}^{\mathrm{op}}$, so that Zariski sheafification and fppf sheafification of algebraic functors make sense. However, it does not hold for the fpqc topology on $\mathrm{CAlg}^{\mathrm{op}}$, and indeed fpqc sheafification does *not* exists.

Corollary 6.40. *Let $(\mathcal{C}, \tau)$ be a small site. Then the category $\mathrm{Sh}_\tau(\mathcal{C})$ admits limits and colimits: limits are computed objectwise, while colimits are computed by sheafifying objectwise colimits.*

Corollary 6.41. *Let $\mathcal{C}$ be a small category. Then there is an isomorphism of posets*

$$\{\textit{topologies on } \mathcal{C}\} \xrightarrow{\simeq} \{\textit{full subcategories of } \mathrm{P}(\mathcal{C}) \textit{ with left exact left adjoint}\},$$

given by $\tau \mapsto \mathrm{Sh}_\tau(\mathcal{C})$.

Definition 6.42 (Local epimorphism, monomorphism, isomorphism)**.** Let $(\mathcal{C}, \tau)$ be a site, let $f\colon F \to G$ be a map in $\mathrm{P}(\mathcal{C})$, and let $\Delta_f\colon F \to F \times_G F$ be its diagonal.

(a) f is a *local epimorphism* if, for every $X \in \mathcal{C}$ and every $x \in G(X)$, the sieve on X consisting of all $u\colon Y \to X$ such that $u^*(x)$ is in the image of f is a τ-covering sieve.
(b) f is a *τ-local monomorphism*, if its diagonal Δ_f is a τ-local epimorphism.
(c) f is a *τ-local isomorphism* if it is both a τ-local epimorphism and a τ-local monomorphism.

Proposition 6.43. *Let $(\mathcal{C}, \tau)$ be a small site and let f be a monomorphism in $\mathrm{P}(\mathcal{C})$. Then $\mathrm{a}_\tau(f)$ is an epimorphism in $\mathrm{Sh}_\tau(\mathcal{C})$ if and only if f is a τ-local epimorphism. The same statement holds for monomorphisms and for isomorphisms.*

Corollary 6.44 (Epi-mono factorization)**.** *Let $(\mathcal{C}, \tau)$ be a small site. Every map $f\colon F \to G$ in $\mathrm{Sh}_\tau(\mathcal{C})$ factors uniquely (up to unique isomorphism) as*

$$F \xrightarrow{f_{\mathrm{epi}}} \mathrm{im}_\tau(f) \xrightarrow{f_{\mathrm{mono}}} G,$$

where f_{epi} is an epimorphism and f_{mono} is a monomorphism. This factorization is obtained by applying a_τ to the factorization $F \to \mathrm{im}(f) \to G$ in $\mathrm{P}(\mathcal{C})$. In particular, f_{epi} is the coequalizer in $\mathrm{Sh}_\tau(\mathcal{C})$ of the two projections $F \times_G F \rightrightarrows F$.

Warning 6.45. A map in $\mathrm{Sh}_\tau(\mathcal{C})$ is a monomorphism in $\mathrm{Sh}_\tau(\mathcal{C})$ if and only if it is a monomorphism in $\mathrm{P}(\mathcal{C})$ (since the inclusion $\mathrm{Sh}_\tau(\mathcal{C}) \hookrightarrow \mathrm{P}(\mathcal{C})$ preserves limits and hence monomrophisms). However, the corresponding statements for epimorphisms does not hold: epimorphisms in $\mathrm{Sh}_\tau(\mathcal{C})$ are usually not epimorphisms in $\mathrm{P}(\mathcal{C})$, i.e., they are not objectwise surjective. For example, if R is a ring and $(f, g) = R$, then $\mathrm{Spec}(R_f \times R_g) \to \mathrm{Spec}(R)$ is an epimorphism in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$, since it is a Zariski-local epimorphism, but it is not objectwise surjective.

In the case of sheaves on a topological space, we can rephrase the criterion of Proposition 6.43 in terms of *stalks*:

Definition 6.46 (Stalk)**.** Let T be a topological space, let $x \in T$ and let $F \in \mathrm{P}(\mathrm{Open}(T))$. The *stalk* of F at x is the set

$$F_x = \operatorname*{colim}_{x \in U} F(U),$$

where the colimit is indexed by the poset of open neighborhoods of x.

Remark 6.47. Since the colimit in the definition of the stalk F_x is a filtered colimit, the functor

$$\mathrm{P}(\mathrm{Open}(T)) \to \mathrm{Set}, \quad F \mapsto F_x,$$

preserves colimits and finite limits.

Proposition 6.48 (Stalkwise characterization of epimorphisms, monomorphisms, and isomorphisms)**.** *Let T be a topological space, let $f\colon F \to G$ be a map of $\mathrm{P}(\mathrm{Open}(T))$, and let $\mathrm{a}\colon \mathrm{P}(\mathrm{Open}(T) \to \mathrm{Sh}(T))$ be the sheafification functor. Then $\mathrm{a}(f)$ is an epimorphism (resp. a monomorphism; an isomorphism) in $\mathrm{Sh}(T)$ if and only if, for every $x \in T$, the induced map of stalks $f_x\colon F_x \to G_x$ is surjective (resp. injective, bijective).*

Corollary 6.49. *Let T be a topological space and let $F \in \mathrm{P}(\mathrm{Open}(T))$. Then the unit map $F \to \mathrm{a}(F)$ induced a bijection on all stalks.*

Appendix A. Recollection on commutative algebra

Proposition A.1. *Let R be a ring and V be an R-vector space. For any R-module N and P, there is a canonical isomorphism*

$$\operatorname{Hom}_R(M, N \otimes_R P) \simeq \operatorname{Hom}_R(M, N) \otimes_R P$$

Proof. By Proposition 3.2, there exists V' such that the functor $V' \otimes_R (-)$ is right adjoint to $V \otimes_R (-)$. Since right adjoint is essentially unique, there exists a canonical isomorphism $V' \otimes_R (-) \simeq \operatorname{Hom}_R(V, -)$. The claim then follows from

$$\operatorname{Hom}_R(V, N \otimes_R P) \simeq V' \otimes_R (N \otimes_R P) \simeq (V' \otimes_R N) \otimes_R P \simeq \operatorname{Hom}_R(M, N) \otimes_R P.$$

$\square$

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