Problem set 5

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Instructions: Here is our fifth set of practice problems.

Practice problems

(1) Suppose that X is a random variable with PDF given by

$$p_X(x) = \begin{cases} c(1-x^2) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of c?

answer: $c = 3/2$

(2) Suppose that f is a function taking positive values between 0 and 10. Furthermore,

$$\int_0^{10} f(t)\,\mathrm{d}t = 5\ .$$

For what factor a would the function $g(x) = a \cdot f(x)$ be a probability density function for a random variable with possible values between 0 and 10?

answer: $a = 1/5$

(3) Suppose that X is a random variable with PDF given by

$$p_X(x) = \begin{cases} \frac{3}{8}(2-\sqrt{x}) & \text{if } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

What is the CDF of X?

(4) What is the shape of the pdf in problem (3)?

answer: Right skewed

(5) For the pdf in problem (3), what is $P(X \leq 0.3)$? $P(X \geq 0.5)$?

answer: ≈ 0.1839; ≈ 0.7134

(6) For the pdf in problem (3), what is the mean? The variance?

answer: $6/5$; $148/175$

answer: $F_X(x) = \frac{3x}{4} - \frac{x^{3/2}}{4}$

(7) Suppose that a random variable with possible values in (0,1) has the CDF

$$F_X(x) = x^2$$

for $0 < x < 1$. What is the PDF for this random variable?

answer:

$$p_X(x) = \begin{cases} 2x & \textit{if } \; 0 < x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

(8) Suppose that X is a random variable with PDF given by

$$p_X(x) = \begin{cases} cx^2 & \text{if } \; 0 < x < 1 \\ 0 & \text{if otherwise} \end{cases}$$

what is $P(X \geq 0.5)$?

answer: $^7/_8$

(9) Suppose that X is a random variable with PDF given by

$$p_X(x) = \begin{cases} c\sin(x^3) & \text{if } \; 0 < x < 1 \\ 0 & \text{if otherwise} \end{cases}$$

what is $P(X = 0.25)$?

answer: 0

(10) Suppose that X is a random variable with PDF given by

$$p_X(x) = \begin{cases} cx^2 & \text{if } \; -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of the constant of integration c?

answer: $^3/_2$

(11) What is the shape of the pdf in problem (10)?

answer: bimodal and symmetric

(12) For the pdf in problem (10), find the CDF

answer: $F_X(x) = \frac{x^3}{2} + \frac{1}{2}$

(13) For the pdf in problem (10), find the quantile function

answer: $F_X^{-1}(p) = (2p-1)^{1/3}$

(14) For the pdf in problem (10), find the mean and the median

answer: 0; 0

(15) Suppose that X is a random variable with PDF given by

$$p_X(x) = \begin{cases} \frac{4}{3}x^{1/3} & \text{if } 0 < x < 1 \\ 0 & \text{if otherwise} \end{cases}$$

What is the CDF?

answer: $F_X(x) = x^{4/3}$

(16) For the pdf in problem (15), what is the 60^{th} percentile? The median?

answer: $0.6^{3/4}$; $0.5^{3/4}$

(17) For the pdf in problem (15), what is the IQR?

answer: $0.75^{3/4} - 0.25^{3/4}$

(18) For the pdf in problem (15), what is the mean? The variance?

answer: $4/7$; $18/245$

(19) The continuous uniform distribution is characterized by the fact that all values between endpoints a and b are equally likely. Suppose X is a continuous uniform random variable. The PDF of X is given by

$$f_X(x) = \begin{cases} \alpha & \text{if } a \leq x \leq b; \\ 0 & \text{otherwise.} \end{cases}$$

Solve for α, and find $P(X \leq c)$, where $a \leq c \leq b$.

answer: $\frac{1}{b-a}$; $\frac{c-a}{b-a}$

(20) A computer repairman records how long it takes him to fix each computer his customers bring to him. In general, the problems are quite easy for him to fix, but since he must spend some time turning the computer on and off, he cannot finish any job in less than 6 minutes (0.1 hours). After some thought, he figures out that the time (in hours) needed to repair a computer (X) is well-modeled by the following PDF.

$$f_X(x) = \frac{1}{10x^2},$$

where $x \geq 0.1$. A customer runs into his store 15 minutes (0.25 hours) before closing and begs the repairman to fix his computer. The repairman says that he will try his best, but if he can't finish by closing time, he'll have to continue working on it tomorrow. What is the probability the customer gets his computer back before the store closes?

answer: 0.6

(21) Suppose X is a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } 3 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Compute $E(X)$, $\text{var}(X)$, $E\left(\frac{1}{X}\right)$.

answer: 4.5*; 0.75 ;* $\frac{1}{3}\log 2 \approx 0.2310491$

(22) If $X \sim \mathcal{N}(2,4)$ then $Y = 2X - 1$ is also a normal random variable such that $Y \sim \mathcal{N}(\mu, \sigma^2)$. Find μ and σ^2.

answer: $\mu = 3$, $\sigma^2 = 16$.

(23) Let $X \sim \mathcal{N}(3,9)$. Find:

(a) $P(2 < X < 5)$
(b) $P(X > 0)$
(c) $P(|X - 3| > 6)$

answer: (a) .3779, (b) .8413, (c) .0456

(24) Let Z be a standard normal random variable.

(a) Find $P(Z > 0.90)$
(b) Find $P(Z < 0.90)$
(c) Find $P(0 \leq Z < 2.12)$
(d) Find $P(-1.5 \leq Z < 2.3)$
(e) Find $P(0 > Z > -1.5)$
(f) Find $P(Z \leq 1.53)$
(g) Find $P(Z > -0.49)$
(h) Find $P(0.35 < Z < 2.01)$
(i) Find $P(|Z| > 1.28)$
(j) Find $P(|Z| < 1.28)$

answer: (a) 0.1841, (b) 0.8159, (c) 0.483, (d) 0.9225, (e) 0.4332, (f) 0.937, (g) 0.688, (h) 0.341, (i) 0.2, (j)0.7994

(25) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the probability that X is less than σ away from μ.

answer: 0.68

(26) Let $X \sim \mathcal{N}(5, 100)$. Find

(a) $P(X > 10)$

(b) $P(-20 < X < 15)$

answer: 0.3085; 0.8351

(27) What is the IQR of the standard normal distribution?

answer: ≈ 1.3490

(28) Let X be a normal random variable such that $EX = 10$ and $\text{var}(X) = 36$, compute

(a) $P(X > 5)$

(b) $P(4 < X < 16)$

(c) $P(X < 8)$

(d) $P(X < 20)$

(e) $P(X > 16)$

answer: (a) .7977, (b) .6827, (c) .3695, (d) .9522, (e) .1587