Q1: $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, 1)$, $H_0: \mu = \mu_0$. $H_1: \mu \neq \mu_0$: H_0 is rejected If $|\bar{X}_n - \mu_0| \geq C$.

$T = |\bar{X}_n - \mu_0|$ $\alpha = \sup_{\mu = \mu_0} \pi(\theta|\delta) = \pi(\mu_0|\delta) = Pr(T \geq c | \mu_0)$

$\Rightarrow \alpha = Pr(|\bar{X}_n - \mu_0| \geq C | \theta = \mu_0)$

$= Pr(\bar{X}_n - \mu_0 \geq C | \theta = \mu_0) + Pr(\bar{X}_n - \mu_0 \leq -C | \theta = \mu_0)$

$= 2 Pr(\bar{X}_n - \mu_0 \geq C) = 0.05$. Because $\sqrt{n}(\bar{X} - \mu_0) \sim N(0,1)$

$\Rightarrow Pr(Z \geq 5C) = 0.025 \Rightarrow C = \frac{\Phi(0.025)}{5}$

Q2: $f(x|\theta) = \begin{cases} 1 & \theta - \frac{1}{2} \leq x \leq \theta + \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$, $x - \frac{1}{2} \leq \theta \leq x + \frac{1}{2}$, If $x - \frac{1}{2} \geq 4$, we should reject the H_0

So, the test is If $X \geq 3.5$, we reject H_0 and $\pi(\theta|\delta) = 0$ for $\theta \leq 3$ and $\pi(\theta|\delta) = 1$ for $\theta \geq 4$.

Q3: $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, 1)$, $H_0: \mu \geq \mu_0$ $H_1: \mu \leq \mu_0$, reject H_0 $T \geq C$, LRT reject H_0: $\sqrt{n}(\bar{X} - \mu_0) <$

$\pi(\mu|\delta_c) = Pr(T \geq C | \mu)$ Let's $T = -\sqrt{n}(\bar{X} - \mu_0)$

$= \cancel{Pr(-\sqrt{n}(\bar{X} - \mu) - \mu\sqrt{n} + \ldots)}$

$= Pr(-\sqrt{n}(\bar{X} - \mu_0) \geq C | \mu)$

$= Pr(\sqrt{n}(\bar{X} - \mu_0) \leq -C) = Pr(\sqrt{n}(\bar{X} - \mu) + \mu\sqrt{n} - \mu_0\sqrt{n} \leq -C)$

$= Pr(Z \leq (\mu_0 - \mu)\sqrt{n} - C)$

$= \Phi((\mu_0 - \mu)\sqrt{n} - C)$ is an decresing function of μ.

Q4: P-value $= \sup_{\theta \in \Omega_0} Pr(Z \geq z | \theta)$, $Z = \sqrt{n}\frac{\bar{X} - \mu_0}{\sigma}$, $H_0: \mu \leq \mu_0$ $H_1: \mu > \mu_0$, rej $Z > c$

$= \sup_{\theta \in \Omega_0} Pr(\sqrt{n}\frac{\bar{X} - \mu_0}{\sigma} \geq z)$, $\theta = \mu$. $\Omega_0: \mu \leq \mu_0$

$\cancel{\sup_{\mu \in \Omega_0} 1 - \Phi(z)}$ $= \sup_{\mu \in \Omega_0} Pr(\sqrt{n}\frac{\bar{X} - \mu}{\sigma} > z + \frac{\sqrt{n}\mu_0}{\sigma} - \frac{\sqrt{n}\mu}{\sigma})$

$= 1 - \min_{\mu \in \Omega_0} Pr(\sqrt{n}\frac{\bar{X} - \mu}{\sigma} \leq z + \frac{\sqrt{n}\mu_0}{\sigma} - \frac{\sqrt{n}\mu}{\sigma})$ $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N$

$= 1 - \Phi(z + \frac{\sqrt{n}\mu_0}{\sigma} - \frac{\sqrt{n}\mu_0}{\sigma}) = 1 - \Phi(z)$

Q5: (1) $\pi(\theta|\delta_c) = Pr(X \geq c | \theta)$

$= \int_c^{+\infty} \frac{1}{\pi[1 + (x - \theta)^2]} dx = \frac{1}{\pi}(1 - \arctan(C - \theta))$, so, it is an incresing function of

(2) $\cancel{\frac{1}{\pi}(1 - \arctan(c - \theta))}$ $\alpha = \sup_{\theta \in \Omega_0} \pi(\theta|\delta_c) = \sup_{\theta \in \Omega_0} \frac{1}{\pi}(1 - \arctan(C - \theta)) = 0.05$. when $\theta = \theta_0$ find max.

$= \frac{1}{\pi}(1 - \arctan(C - \theta_0)) = 0.05$

$\Rightarrow C = \tan(1 - 0.05\pi) + \theta_0$.

(3) P-value $= \sup_{\theta \in \Omega_0} Pr(T \geq t | \theta)$, $T = X$, $t = x$

$= \sup_{\theta \in \Omega_0} Pr(X \geq x | \theta)$

$= \sup_{\theta \in \Omega_0} \frac{1}{\pi}(1 - \arctan(x - \theta)) = \frac{1}{\pi}(1 - \arctan(x - \theta_0))$

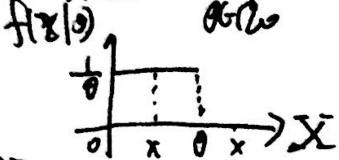
Q6: $f(x|\theta) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \\ 0 & \text{otherwise.} \end{cases}$ $H_0: \theta \le 1$ $H_1: \theta > 1$, reject H_0 if $X \ge c$.

Pvalue $= \sup_{\theta \in \Theta_0} \Pr(X \ge x)$

If $x < 0$, Pvalue $= \sup_{\theta \in \Theta_0} \Pr(X \ge x) = 1$ $\quad \theta \le 1$

If $0 \le x \le 1$ Pvalue $= \sup_{\theta \in \Theta_0} \Pr(X \ge x) = 1 - \min_{\theta \le 1} \frac{x}{\theta} = 1 - x$

If $x > 1$ Pvalue $= \sup_{\theta \in \Theta_0} \Pr(X \ge x) = 0.$



Q7: (a) $T = \sum_{i=1}^{10} X_i$ is the sufficient statistic of Bernoulli distribution.

$g(t|p) = \text{Binomial}(10, p) = \binom{10}{t} \cdot p^t (1-p)^{10-t}$.

$\frac{g(t|p=\frac{1}{4})}{g(t|p=\frac{1}{2})} > k \Rightarrow (\frac{1}{2})^t \cdot (\frac{3}{2})^{10-t} > k \Rightarrow t\log\frac{1}{2} + (10-t)\log\frac{3}{2} > k \Rightarrow -\frac{1}{2}t > c_0$
$\Rightarrow t < c_1$, this is the UMP α test, reject H_0 if $t < c_1$

$\alpha = \sup_{\theta \in \Theta_0} \Pr(T < c|\theta) \Rightarrow 0.0547 = \Pr(T < c|\theta = \frac{1}{2})$
$\Rightarrow c = 3.$

So, reject H_0 if $t < 3$, this is the UMP test. The power of this test is:

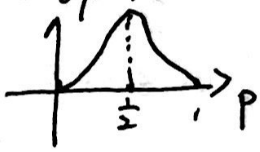
$P(T < 3|\theta = \frac{1}{2}) = 0.0547$ $\quad P(T < 3|\theta = \frac{1}{4}) = 0.526.$

(b) $T = \sum_{i=1}^{10} X_i \ge 6$, H_0 $p \le \frac{1}{2}$ $\quad H_1$ $p > \frac{1}{2}$

$\alpha = \sup_{\theta \in \Theta_0} P_r(T \ge 6|\theta)$

$= \sup_{p \le \frac{1}{2}} \binom{10}{6} p^6 (1-p)^4 + \binom{10}{7} p^7 (1-p)^3 + \binom{10}{8} p^8 (1-p)^2 + \binom{10}{9} \cdot p^9 (1-p) + \binom{10}{10} p^{10}$

when $p = \frac{1}{2}$, this function will be maximized.



$= 0.377$

$\pi(\theta|\delta) = \sum_{k=6}^{10} \binom{10}{k} p^k (1-p)^{10-k}$

(c) $\alpha = \sup_{p=\frac{1}{2}} \Pr(T < c|p) = \sum_{k=0}^{c-1} \binom{10}{k} \cdot (\frac{1}{2})^k (\frac{1}{2})^{10-k}$, choose c from 1 to 11, we can get different α level.

Q8: $\alpha = \sup_{\theta \in \Theta_0} \Pr(X > \frac{1}{2}|\theta) = \sup_{\theta \le 1} \int_{\frac{1}{2}}^{1} \frac{1}{B(\theta, 1)} \cdot x^{\theta - 1} dx = \sup_{\theta \le 1} \frac{1}{B(\theta, 1)} \left(1^\theta - (\frac{1}{2})^\theta\right)$, when $\theta = 1$, get

max, $\alpha = \frac{1}{B(1,1)} \cdot \frac{1}{2} = \frac{1}{2}$

(b) $f(x|\theta_1 = 2) = \frac{1}{B(2,1)} x$ $\quad \frac{f(x|\theta_1 = 2)}{f(x|\theta_0 = 1)} = \frac{x}{B(2,1)} > k \Rightarrow x > c$

$f(x|\theta_0 = 1) = 1$

reject H_0 when $x > c$, $\alpha = \sup_{\theta = 1} \Pr(X > c|\theta) = \int_c^1 \frac{1}{B(1,1)} \cdot dx = 1 - c = \alpha$

$\Rightarrow c = 1 - \alpha.$

So, it is the UMP test of α level when $x > 1 - \alpha$.