Homework 5 (Due 3/4)

Question 1 Suppose we are given the data

$z_1:$	2	2	2	0	-1	-2	-3
$z_2:$	1	-2	1	0	1	-2	1
$y:$	1	0	1	0	-1	0	-1

We aim at fitting the linear model $Y_i = \beta_0 + \beta_1 z_{i1} + \beta_2 z_{i2} + \epsilon_i, \quad i = 1, 2, \ldots, 7.$
(1) Find the least square estimate $\hat{\vec{\beta}}$;
(2) Find the R^2 statistic;
(3) Find $\hat{\sigma}^2$ and $\widehat{\text{Cov}}(\vec{\beta})$;
(4) Find a 95% confidence interval for β_1;
(5) Find 95% simultaneous confidence intervals for β_0, β_1 and β_2 based on the confidence region;
(6) Find 95% simultaneous confidence intervals for β_0, β_1 and β_2 based on Bonferroni correction;
(7) Test $H_0 : \beta_1 = \beta_2 = 0$ at the level of $\alpha = 0.05$;
(8) Find a 95% confidence interval for the mean response $\mathbb{E}(Y_0) = \beta_0 + \beta_1 \bar{z}_1 + \beta_2 \bar{z}_2$, where $\bar{z}_j$ is the sample mean of z_j, $j = 1, 2$.
(9) Find a 95% prediction interval for a new response Y_0 corresponding to $(\bar{z}_1, \bar{z}_2)$.

Question 2 Let $\boldsymbol{Z}$ be a $n \times (r+1)$ design matrix, and $\boldsymbol{H} = \boldsymbol{Z}(\boldsymbol{Z}^\top \boldsymbol{Z})^{-1}\boldsymbol{Z}^\top$ be the hat matrix. Show that
(1) Both $\boldsymbol{H}$ and $\boldsymbol{I} - \boldsymbol{H}$ are symmetric;
(2) $\boldsymbol{H}^2 = \boldsymbol{H}$, $(\boldsymbol{I} - \boldsymbol{H})^2 = \boldsymbol{I} - \boldsymbol{H}$, $\boldsymbol{H}(\boldsymbol{I} - \boldsymbol{H}) = \boldsymbol{0}$;
(3) All eigenvalues of $\boldsymbol{H}$ and $\boldsymbol{I} - \boldsymbol{H}$ are either 1 or 0;
(4) Both $\boldsymbol{H}$ and $\boldsymbol{I} - \boldsymbol{H}$ are positive semidefinite;
(5) $\boldsymbol{H}\boldsymbol{Z} = \boldsymbol{Z}$ and $(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z} = \boldsymbol{0}$.

Question 3 The design matrix $\boldsymbol{Z}$ is partitioned as

$$\boldsymbol{Z} = \left[\begin{array}{cccc|ccc} 1 & z_{11} & \ldots & z_{1q} & z_{1,q+1} & \ldots & z_{1,r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_{n1} & \ldots & z_{nq} & z_{n,q+1} & \ldots & z_{n,r} \end{array}\right] := [\boldsymbol{Z}_{(1)}, \boldsymbol{Z}_{(2)}],$$

Let $\boldsymbol{H} = \boldsymbol{Z}(\boldsymbol{Z}^\top \boldsymbol{Z})^{-1}\boldsymbol{Z}^\top$ and $\boldsymbol{H}_{(red)} = \boldsymbol{Z}_{(1)}(\boldsymbol{Z}_{(1)}^\top \boldsymbol{Z}_{(1)})^{-1}\boldsymbol{Z}_{(1)}^\top$ be the hat matrices for the full and reduced models, respectively. Show that

$$\boldsymbol{H}\boldsymbol{H}_{(red)} = \boldsymbol{H}_{(red)}\boldsymbol{H} = \boldsymbol{H}_{(red)}.$$

Question 4 Consider the classical linear regression model

$$\vec{Y_i} = \beta_0 + \beta_1 z_{i1} + \ldots + \beta_r z_{ir} + \epsilon_i, \quad i = 1, \ldots, n.$$

or in the matrix form

$$\boldsymbol{Y} = \boldsymbol{Z}\vec{\beta} + \boldsymbol{\epsilon}, \tag{0.1}$$

where the $n \times (r+1)$ design matrix $\boldsymbol{Z}$ is of rank $r+1$. Let $\boldsymbol{C}$ be a $r \times r$ invertible matrix, which determines the transformation of the explanatory variables

$$\begin{bmatrix} w_{i1} \\ \vdots \\ w_{ir} \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} z_{i1} \\ \vdots \\ z_{ir} \end{bmatrix}, \quad i = 1, \ldots, n.$$

We then consider the linear regression model

$$Y_i = \gamma_0 + \gamma_1 w_{i1} + \ldots + \gamma_r w_{ir} + \epsilon_i, \quad i = 1, \ldots, n,$$

or equivalently,

$$\boldsymbol{Y} = \boldsymbol{W}\vec{\gamma} + \boldsymbol{\epsilon}. \tag{0.2}$$

Let the least square estimate of (0.1) be $\hat{\vec{\beta}}$ and that of (0.2) be $\hat{\vec{\gamma}}$.
(1) Find the relationship between $\boldsymbol{W}$ and $\boldsymbol{Z}$ through

$$\widetilde{\boldsymbol{C}} = \begin{bmatrix} 1 & \vec{0}^\top \\ \vec{0} & \boldsymbol{C} \end{bmatrix}.$$

(2) Compare the fitted residuals $\hat{\vec{\epsilon}}_z$ and $\hat{\vec{\epsilon}}_w$ for the two linear models.
(3) Compare the R-square statistics R_z^2 and R_w^2 for the two linear models.
(4) Compare the F-test statistics for

$$H_0 : \beta_1 = \ldots = \beta_r = 0$$

and

$$H_0 : \gamma_1 = \ldots = \gamma_r = 0.$$

(5) Consider a new observation of the explanatory variates

$$\vec{z}_0^\top = [1, z_{01}, \ldots, z_{0r}],$$

and correspondingly

$$\vec{w}_0^\top = [1, w_{01}, \ldots, w_{0r}],$$

where

$$\begin{bmatrix} w_{01} \\ \vdots \\ w_{0r} \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} z_{01} \\ \vdots \\ z_{0r} \end{bmatrix}.$$

Compare the prediction intervals for Y_0 based on $\hat{\vec{\beta}}$ and $\hat{\vec{\gamma}}$, respectively.

Question 5 Fit a multiple linear regression to the dataset in Table 7.1 on Page 372 by implementing steps (1) - (9) in Question 1.