PROBLEMS

7.1. A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then she wins twice, and if tails, then one-half of the value that appears on the die. Determine her expected winnings.

7.2. The game of Clue involves 6 suspects, 6 weapons, and 9 rooms. One of each is randomly chosen and the object of the game is to guess the chosen three.

(a) How many solutions are possible?
In one version of the game, the selection is made and then each of the players is randomly given three of the remaining cards. Let S, W, and R be, respectively, the numbers of suspects, weapons, and rooms in the set of three cards given to a specified player. Also, let X denote the number of solutions that are possible after that player observes his or her three cards.

(b) Express X in terms of S, W, and R.

(c) Find $E[X]$.

7.3. Gambles are independent, and each one results in the player being equally likely to win or lose 1 unit. Let W denote the net winnings of a gambler whose strategy is to stop gambling immediately after his first win. Find

(a) $P\{W > 0\}$

(b) $P\{W < 0\}$

(c) $E[W]$

7.4. If X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 1/y, & \text{if } 0 < y < 1,\ 0 < x < y \\ 0, & \text{otherwise} \end{cases}$$

find

(a) $E[XY]$

(b) $E[X]$

(c) $E[Y]$

7.5. The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are $(0, 0)$, to the point (x, y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.

7.6. A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.

7.7. Suppose that A and B each randomly and independently choose 3 of 10 objects. Find the expected number of objects

(a) chosen by both A and B;

(b) not chosen by either A or B;

(c) chosen by exactly one of A and B.

7.8. N people arrive separately to a professional dinner. Upon arrival, each person looks to see if he or she has any friends among those present. That person then sits either at the table of a friend or at an unoccupied table if none of those present is a friend. Assuming that each of the $\binom{N}{2}$ pairs of people is, independently, a pair of friends with probability p, find the expected number of occupied tables.
Hint: Let X_i equal 1 or 0, depending on whether the ith arrival sits at a previously unoccupied table.

7.9. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find

(a) the expected number of urns that are empty;

(b) the probability that none of the urns is empty.

7.10. Consider 3 trials, each having the same probability of success. Let X denote the total number of successes in these trials. If $E[X] = 1.8$, what is

(a) the largest possible value of $P\{X = 3\}$?

(b) the smallest possible value of $P\{X = 3\}$?

In both cases, construct a probability scenario that results in $P\{X = 3\}$ having the stated value.
Hint: For part (b), you might start by letting U be a uniform random variable on (0, 1) and then defining the trials in terms of the value of U.

7.11. Consider n independent flips of a coin having probability p of landing on heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if $n = 5$ and the outcome is $HHTHT$, then there are 3 changeovers. Find the expected number of changeovers.
Hint: Express the number of changeovers as the sum of $n - 1$ Bernoulli random variables.

7.12. A group of n men and n women is lined up at random.

(a) Find the expected number of men who have a woman next to them.

(b) Repeat part (a), but now assuming that the group is randomly seated at a round table.

7.13. A set of 1000 cards numbered 1 through 1000 is randomly distributed among 1000 people with each receiving one card. Compute the expected number of cards that are given to people whose age matches the number on the card.

7.14. An urn has m black balls. At each stage, a black ball is removed and a new ball that is black with probability p and white with probability $1 - p$ is put in its place. Find the expected number of stages needed until there are no more black balls in the urn.

NOTE: The preceding has possible applications to understanding the AIDS disease. Part of the body's immune system consists of a certain class of cells known as T-cells. There are 2 types of T-cells, called CD4 and CD8. Now, while the total number of T-cells in AIDS sufferers is (at least in the early stages of the disease) the same as that in healthy individuals, it has recently been discovered that the mix of CD4 and CD8 T-cells is different. Roughly 60 percent of the T-cells of a healthy person are of the CD4 type, whereas the percentage of the T-cells that are of CD4 type appears to decrease continually in AIDS sufferers. A recent model proposes that the HIV virus (the virus that causes AIDS) attacks CD4 cells and that the body's mechanism for replacing killed T-cells does not differentiate between whether the killed T-cell was CD4 or CD8. Instead, it just produces a new T-cell that is CD4 with probability .6 and CD8 with probability .4. However, although this would seem to be a very efficient way of replacing killed T-cells when each one killed is equally likely to be any of the body's T-cells (and thus has probability .6 of being CD4), it has dangerous consequences when facing a virus that targets only the CD4 T-cells.

7.15. In Example 2h, say that i and j, $i \neq j$, form a matched pair if i chooses the hat belonging to j and j chooses the hat belonging to i. Find the expected number of matched pairs.

7.16. Let Z be a standard normal random variable, and, for a fixed x, set

$$X = \begin{cases} Z & \text{if } Z > x \\ 0 & \text{otherwise} \end{cases}$$

Show that $E[X] = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

7.17. A deck of n cards numbered 1 through n is thoroughly shuffled so that all possible $n!$ orderings can be assumed to be equally likely. Suppose you are to make n guesses sequentially, where the ith one is a guess of the card in position i. Let N denote the number of correct guesses.

(a) If you are not given any information about your earlier guesses show that, for any strategy, $E[N] = 1$.

(b) Suppose that after each guess you are shown the card that was in the position in question. What do you think is the best strategy? Show that, under this strategy,

$$E[N] = \frac{1}{n} + \frac{1}{n-1} + \cdots + 1$$
$$\approx \int_1^n \frac{1}{x}\,dx = \log n$$

(c) Suppose that you are told after each guess whether you are right or wrong. In this case, it can be shown that the strategy which maximizes $E[N]$ is one that keeps on guessing the same card until you are told you are correct and then changes to a new card. For this strategy, show that

$$E[N] = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$
$$\approx e - 1$$

Hint: For all parts, express N as the sum of indicator (that is, Bernoulli) random variables.

7.18. Cards from an ordinary deck of 52 playing cards are turned face up one at a time. If the 1st card is an ace, or the 2nd a deuce, or the 3rd a three, or ..., or the 13th a king, or the 14 an ace, and so on, we say that a match occurs. Note that we do not require that the $(13n + 1)$th card be any particular ace for a match to occur but only that it be an ace. Compute the expected number of matches that occur.

7.19. A certain region is inhabited by r distinct types of a certain species of insect. Each insect caught will, independently of the types of the previous catches, be of type i with probability

$$P_i, i = 1, \ldots, r \qquad \sum_1^r P_i = 1$$

(a) Compute the mean number of insects that are caught before the first type 1 catch.

(b) Compute the mean number of types of insects that are caught before the first type 1 catch.

7.20. In an urn containing n balls, the ith ball has weight $W(i), i = 1, \ldots, n$. The balls are removed without replacement, one at a time, according to the following rule: At each selection, the probability that a given ball in the urn is chosen is equal to its weight divided by the sum of the weights remaining in the urn. For instance, if at some time $i_1, \ldots, i_r$ is the set of balls remaining in the urn, then the next selection will be i_j with probability $W(i_j)\Big/\sum_{k=1}^{r} W(i_k)$, $j = 1, \ldots, r$. Compute the expected number of balls that are withdrawn before ball number 1 is removed.

7.21. For a group of 100 people, compute

(a) the expected number of days of the year that are birthdays of exactly 3 people:

(b) the expected number of distinct birthdays.

7.22. How many times would you expect to roll a fair die before all 6 sides appeared at least once?

7.23. Urn 1 contains 5 white and 6 black balls, while urn 2 contains 8 white and 10 black balls. Two balls are randomly selected from urn 1 and are put into urn 2. If 3 balls are then randomly selected from urn 2, compute the expected number of white balls in the trio.
Hint: Let $X_i = 1$ if the ith white ball initially in urn 1 is one of the three selected, and let $X_i = 0$ otherwise. Similarly, let $Y_i = 1$ if the ith white ball from urn 2 is one of the three selected, and let $Y_i = 0$ otherwise. The number of white balls in the trio can now be written as $\sum_1^5 X_i + \sum_1^8 Y_i$.

7.24. A bottle initially contains m large pills and n small pills. Each day, a patient randomly chooses one of the pills. If a small pill is chosen, then that pill is eaten. If a large pill is chosen, then the pill is broken in two; one part is returned to the bottle (and is now considered a small pill) and the other part is then eaten.
(a) Let X denote the number of small pills in the bottle after the last large pill has been chosen and its smaller half returned. Find $E[X]$.
Hint: Define $n + m$ indicator variables, one for each of the small pills initially present and one for each of the m small pills created when a large one is split in two. Now use the argument of Example 2m.
(b) Let Y denote the day on which the last large pill is chosen. Find $E[Y]$.
Hint: What is the relationship between X and Y?

7.25. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that

$$X_1 \geq X_2 \geq \cdots \geq X_{N-1} < X_N$$

That is, N is the point at which the sequence stops decreasing. Show that $E[N] = e$.
Hint: First find $P\{N \geq n\}$.

7.26. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables having uniform distributions over (0, 1), find
(a) $E[\max(X_1, \ldots, X_n)]$;
(b) $E[\min(X_1, \ldots, X_n)]$.

***7.27.** If 101 items are distributed among 10 boxes, then at least one of the boxes must contain more than 10 items. Use the probabilistic method to prove this result.

***7.28.** The k-of-r-out-of-n circular reliability system, $k \leq r \leq n$, consists of n components that are arranged in a circular fashion. Each component is either functional or failed, and the system functions if there is no block of r consecutive components of which at least k are failed. Show that there is no way to arrange 47 components, 8 of which are failed, to make a functional 3-of-12-out-of-47 circular system.

***7.29.** There are 4 different types of coupons, the first 2 of which compose one group and the second 2 another group. Each new coupon obtained is type i with probability p_i, where $p_1 = p_2 = 1/8, p_3 = p_4 = 3/8$. Find the expected number of coupons that one must obtain to have at least one of
(a) all 4 types;
(b) all the types of the first group;
(c) all the types of the second group;
(d) all the types of either group.

7.30. If X and Y are independent and identically distributed with mean μ and variance σ^2, find

$$E[(X - Y)^2]$$

7.31. In Problem 6, calculate the variance of the sum of the rolls.

7.32. In Problem 9, compute the variance of the number of empty urns.

7.33. If $E[X] = 1$ and $\text{Var}(X) = 5$, find
(a) $E[(2 + X)^2]$;
(b) $\text{Var}(4 + 3X)$.

7.34. If 10 married couples are randomly seated at a round table, compute (a) the expected number and (b) the variance of the number of wives who are seated next to their husbands.

7.35. Cards from an ordinary deck are turned face up one at a time. Compute the expected number of cards that need to be turned face up in order to obtain
(a) 2 aces;
(b) 5 spades;
(c) all 13 hearts.

7.36. Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die. Compute $\text{Cov}(X, Y)$.

7.37. A die is rolled twice. Let X equal the sum of the outcomes, and let Y equal the first outcome minus the second. Compute $\text{Cov}(X, Y)$.

7.38. The random variables X and Y have a joint density function given by

$$f(x, y) = \begin{cases} 2e^{-2x}/x & 0 \leq x < \infty, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Compute $\text{Cov}(X, Y)$.

7.39. Let $X_1, \ldots$ be independent with common mean μ and common variance σ^2, and set $Y_n = X_n + X_{n+1} + X_{n+2}$. For $j \geq 0$, find $\text{Cov}(Y_n, Y_{n+j})$.

7.40. The joint density function of X and Y is given by

$$f(x,y) = \frac{1}{y}e^{-(y+x/y)}, \quad x > 0, y > 0$$

Find $E[X]$, $E[Y]$, and show that $\text{Cov}(X,Y) = 1$.

7.41. A pond contains 100 fish, of which 30 are carp. If 20 fish are caught, what are the mean and variance of the number of carp among the 20? What assumptions are you making?

7.42. A group of 20 people consisting of 10 men and 10 women is randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of a man and a woman. Now suppose the 20 people consist of 10 married couples. Compute the mean and variance of the number of married couples that are paired together.

7.43. Let $X_1, X_2, \ldots, X_n$ be independent random variables having an unknown continuous distribution function F, and let $Y_1, Y_2, \ldots, Y_m$ be independent random variables having an unknown continuous distribution function G. Now order those $n + m$ variables, and let

$$I_i = \begin{cases} 1 & \text{if the } i\text{th smallest of the } n + m \\ & \text{variables is from the } X \text{ sample} \\ 0 & \text{otherwise} \end{cases}$$

The random variable $R = \sum_{i=1}^{n+m} iI_i$ is the sum of the ranks of the X sample and is the basis of a standard statistical procedure (called the Wilcoxon sum-of-ranks test) for testing whether F and G are identical distributions. This test accepts the hypothesis that $F = G$ when R is neither too large nor too small. Assuming that the hypothesis of equality is in fact correct, compute the mean and variance of R.

Hint: Use the results of Example 3e.

7.44. Between two distinct methods for manufacturing certain goods, the quality of goods produced by method i is a continuous random variable having distribution $F_i, i = 1, 2$. Suppose that n goods are produced by method 1 and m by method 2. Rank the $n + m$ goods according to quality, and let

$$X_j = \begin{cases} 1 & \text{if the } j\text{th best was produced from} \\ & \text{method 1} \\ 2 & \text{otherwise} \end{cases}$$

For the vector $X_1, X_2, \ldots, X_{n+m}$, which consists of n 1's and m 2's, let R denote the number of runs of 1. For instance, if $n = 5, m = 2$, and $X = 1, 2, 1, 1, 1, 1, 2$, then $R = 2$. If $F_1 = F_2$ (that is, if the two methods produce identically distributed goods), what are the mean and variance of R?

7.45. If X_1, X_2, X_3, and X_4 are (pairwise) uncorrelated random variables, each having mean 0 and variance 1, compute the correlations of
(a) $X_1 + X_2$ and $X_2 + X_3$;
(b) $X_1 + X_2$ and $X_3 + X_4$.

7.46. Consider the following dice game, as played at a certain gambling casino: Players 1 and 2 roll a pair of dice in turn. The bank then rolls the dice to determine the outcome according to the following rule: Player $i, i = 1, 2$, wins if his roll is strictly greater than the bank's. For $i = 1, 2$, let

$$I_i = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0 & \text{otherwise} \end{cases}$$

and show that I_1 and I_2 are positively correlated. Explain why this result was to be expected.

7.47. Consider a graph having n vertices labeled $1, 2, \ldots, n$, and suppose that, between each of the $\binom{n}{2}$ pairs of distinct vertices, an edge is independently present with probability p. The degree of vertex i, designated as D_i, is the number of edges that have vertex i as one of their vertices.
(a) What is the distribution of D_i?
(b) Find $\rho(D_i, D_j)$, the correlation between D_i and D_j.

7.48. A fair die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a 6 and a 5. Find
(a) $E[X]$;
(b) $E[X|Y = 1]$;
(c) $E[X|Y = 5]$.

7.49. There are two misshapen coins in a box; their probabilities for landing on heads when they are flipped are, respectively, .4 and .7. One of the coins is to be randomly chosen and flipped 10 times. Given that two of the first three flips landed on heads, what is the conditional expected number of heads in the 10 flips?

7.50. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y}, \quad 0 < x < \infty, \quad 0 < y < \infty$$

Compute $E[X^2|Y = y]$.

7.51. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty$$

Compute $E[X^3|Y = y]$.

7.52. A population is made up of r disjoint subgroups. Let p_i denote the proportion of the population that is in subgroup $i, i = 1, \ldots, r$. If the average weight of the members of subgroup i is $w_i, i = 1, \ldots, r$, what is the average weight of the members of the population?

7.53. A prisoner is trapped in a cell containing 3 doors. The first door leads to a tunnel that returns him to his cell after 2 days' travel. The second leads to a tunnel that returns him to his cell after 4 days' travel. The third door leads to freedom after 1 day of travel. If it is assumed that the prisoner will always select doors 1, 2, and 3 with respective probabilities .5, .3, and .2, what is the expected number of days until the prisoner reaches freedom?

7.54. Consider the following dice game: A pair of dice is rolled. If the sum is 7, then the game ends and you win 0. If the sum is not 7, then you have the option of either stopping the game and receiving an amount equal to that sum or starting over again. For each value of $i, i = 2, \ldots, 12$, find your expected return if you employ the strategy of stopping the first time that a value at least as large as i appears. What value of i leads to the largest expected return?
Hint: Let X_i denote the return when you use the critical value i. To compute $E[X_i]$, condition on the initial sum.

7.55. Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability .6, compute the expected number of ducks that are hit. Assume that the number of ducks in a flock is a Poisson random variable with mean 6.

7.56. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.

7.57. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 2.5. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

7.58. A coin having probability p of coming up heads is continually flipped until both heads and tails have appeared. Find
(a) the expected number of flips;
(b) the probability that the last flip lands on heads.

7.59. There are $n + 1$ participants in a game. Each person independently is a winner with probability p. The winners share a total prize of 1 unit. (For instance, if 4 people win, then each of them receives $\frac{1}{4}$, whereas if there are no winners, then none of the participants receive anything.) Let A denote a specified one of the players, and let X denote the amount that is received by A.
(a) Compute the expected total prize shared by the players.
(b) Argue that $E[X] = \dfrac{1 - (1 - p)^{n+1}}{n + 1}$.
(c) Compute $E[X]$ by conditioning on whether A is a winner, and conclude that

$$E[(1 + B)^{-1}] = \frac{1 - (1 - p)^{n+1}}{(n + 1)p}$$

when B is a binomial random variable with parameters n and p.

7.60. Each of $m + 2$ players pays 1 unit to a kitty in order to play the following game: A fair coin is to be flipped successively n times, where n is an odd number, and the successive outcomes are noted. Before the n flips, each player writes down a prediction of the outcomes. For instance, if $n = 3$, then a player might write down (H, H, T), which means that he or she predicts that the first flip will land on heads, the second on heads, and the third on tails. After the coins are flipped, the players count their total number of correct predictions. Thus, if the actual outcomes are all heads, then the player who wrote (H, H, T) would have 2 correct predictions. The total kitty of $m + 2$ is then evenly split up among those players having the largest number of correct predictions.

Since each of the coin flips is equally likely to land on either heads or tails, m of the players have decided to make their predictions in a totally random fashion. Specifically, they will each flip one of their own fair coins n times and then use the result as their prediction. However, the final 2 of the players have formed a syndicate and will use the following strategy: One of them will make predictions in the same random fashion as the other m players, but the other one will then predict exactly the opposite of the first. That is, when the randomizing member of the syndicate predicts an H, the other member predicts a T. For instance, if the randomizing member of the syndicate predicts (H, H, T), then the other one predicts (T, T, H).
(a) Argue that exactly one of the syndicate members will have more than $n/2$ correct predictions. (Remember, n is odd.)
(b) Let X denote the number of the m nonsyndicate players that have more than $n/2$ correct predictions. What is the distribution of X?

(c) With X as defined in part (b), argue that

$$E[\text{payoff to the syndicate}] = (m + 2) \times E\left[\frac{1}{X + 1}\right]$$

(d) Use part (c) of Problem 59 to conclude that

$$E[\text{payoff to the syndicate}] = \frac{2(m + 2)}{m + 1} \times \left[1 - \left(\frac{1}{2}\right)^{m+1}\right]$$

and explicitly compute this number when $m = 1, 2$, and 3. Because it can be shown that

$$\frac{2(m + 2)}{m + 1}\left[1 - \left(\frac{1}{2}\right)^{m+1}\right] > 2$$

it follows that the syndicate's strategy always gives it a positive expected profit.

7.61. Let $X_1, \ldots$ be independent random variables with the common distribution function F, and suppose they are independent of N, a geometric random variable with parameter p. Let $M = \max(X_1, \ldots, X_N)$.
(a) Find $P\{M \le x\}$ by conditioning on N.
(b) Find $P\{M \le x | N = 1\}$.
(c) Find $P\{M \le x | N > 1\}$.
(d) Use (b) and (c) to rederive the probability you found in (a).

7.62. Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0, 1) random variables. In Example 5i we showed that, for $0 \le x \le 1, E[N(x)] = e^x$, where

$$N(x) = \min\left\{n : \sum_{i=1}^{n} U_i > x\right\}$$

This problem gives another approach to establishing that result.
(a) Show by induction on n that, for $0 < x \le 1$ and all $n \ge 0$,

$$P\{N(x) \ge n + 1\} = \frac{x^n}{n!}$$

Hint: First condition on U_1 and then use the induction hypothesis.
Use part (a) to conclude that

$$E[N(x)] = e^x$$

7.63. An urn contains 30 balls, of which 10 are red and 8 are blue. From this urn, 12 balls are randomly withdrawn. Let X denote the number of red and Y the number of blue balls that are withdrawn. Find $\text{Cov}(X, Y)$
(a) by defining appropriate indicator (that is, Bernoulli) random variables

$$X_i, Y_j \text{ such that } X = \sum_{i=1}^{10} X_i, Y = \sum_{j=1}^{8} Y_j$$

(b) by conditioning (on either X or Y) to determine $E[XY]$.

7.64. Type i light bulbs function for a random amount of time having mean μ_i and standard deviation $\sigma_i, i = 1, 2$. A light bulb randomly chosen from a bin of bulbs is a type 1 bulb with probability p and a type 2 bulb with probability $1 - p$. Let X denote the lifetime of this bulb. Find
(a) $E[X]$;
(b) $\text{Var}(X)$.

7.65. The number of winter storms in a good year is a Poisson random variable with mean 3, whereas the number in a bad year is a Poisson random variable with mean 5. If next year will be a good year with probability .4 or a bad year with probability .6, find the expected value and variance of the number of storms that will occur.

7.66. In Example 5c, compute the variance of the length of time until the miner reaches safety.

7.67. Consider a gambler who, at each gamble, either wins or loses her bet with respective probabilities p and $1 - p$. A popular gambling system known as the Kelley strategy is to always bet the fraction $2p - 1$ of your current fortune when $p > \frac{1}{2}$. Compute the expected fortune after n gambles of a gambler who starts with x units and employs the Kelley strategy.

7.68. The number of accidents that a person has in a given year is a Poisson random variable with mean λ. However, suppose that the value of λ changes from person to person, being equal to 2 for 60 percent of the population and 3 for the other 40 percent. If a person is chosen at random, what is the probability that he will have (a) 0 accidents and (b) exactly 3 accidents in a certain year? What is the conditional probability that he will have 3 accidents in a given year, given that he had no accidents the preceding year?

7.69. Repeat Problem 68 when the proportion of the population having a value of λ less than x is equal to $1 - e^{-x}$.

7.70. Consider an urn containing a large number of coins, and suppose that each of the coins has some probability p of turning up heads when it is flipped. However, this value of p varies from coin to coin. Suppose that the composition of the urn is such that if a coin is selected at random from it, then the p-value of the coin can be regarded as being the value of a random variable that is uniformly distributed over [0, 1]. If a coin is selected at random from the urn and flipped twice, compute the probability that

(a) the first flip results in a head;

(b) both flips result in heads.

7.71. In Problem 70, suppose that the coin is tossed n times. Let X denote the number of heads that occur. Show that

$$P\{X = i\} = \frac{1}{n+1} \qquad i = 0, 1, \ldots, n$$

Hint: Make use of the fact that

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

when a and b are positive integers.

7.72. Suppose that in Problem 70 we continue to flip the coin until a head appears. Let N denote the number of flips needed. Find

(a) $P\{N \geq i\}, i \geq 0$;

(b) $P\{N = i\}$;

(c) $E[N]$.

7.73. In Example 6b, let S denote the signal sent and R the signal received.

(a) Compute $E[R]$.

(b) Compute $\text{Var}(R)$.

(c) Is R normally distributed?

(d) Compute $\text{Cov}(R, S)$.

7.74. In Example 6c, suppose that X is uniformly distributed over (0, 1). If the discretized regions are determined by $a_0 = 0, a_1 = \frac{1}{2}$, and $a_2 = 1$, calculate the optimal quantizer Y and compute $E[(X - Y)^2]$.

7.75. The moment generating function of X is given by $M_X(t) = \exp\{2e^t - 2\}$ and that of Y by $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$. If X and Y are independent, what are

(a) $P\{X + Y = 2\}$?

(b) $P\{XY = 0\}$?

(c) $E[XY]$?

7.76. Let X be the value of the first die and Y the sum of the values when two dice are rolled. Compute the joint moment generating function of X and Y.

7.77. The joint density of X and Y is given by

$$f(x, y) = \frac{1}{\sqrt{2\pi}} e^{-y} e^{-(x-y)^2/2} \qquad 0 < y < \infty, \quad -\infty < x < \infty$$

(a) Compute the joint moment generating function of X and Y.

(b) Compute the individual moment generating functions.

7.78. Two envelopes, each containing a check, are placed in front of you. You are to choose one of the envelopes, open it, and see the amount of the check. At this point, either you can accept that amount or you can exchange it for the check in the unopened envelope. What should you do? Is it possible to devise a strategy that does better than just accepting the first envelope?

Let A and $B, A < B$, denote the (unknown) amounts of the checks, and note that the strategy that randomly selects an envelope and always accepts its check has an expected return of $(A + B)/2$. Consider the following strategy: Let $F(\cdot)$ be any strictly increasing (that is, continuous) distribution function. Choose an envelope randomly and open it. If the discovered check has the value x, then accept it with probability $F(x)$ and exchange it with probability $1 - F(x)$.

(a) Show that if you employ the latter strategy, then your expected return is greater than $(A + B)/2$.
Hint: Condition on whether the first envelope has the value A or B.
Now consider the strategy that fixes a value x and then accepts the first check if its value is greater than x and exchanges it otherwise.

(b) Show that, for any x, the expected return under the x-strategy is always at least $(A + B)/2$ and that it is strictly larger than $(A + B)/2$ if x lies between A and B.

(c) Let X be a continuous random variable on the whole line, and consider the following strategy: Generate the value of X, and if $X = x$, then employ the x-strategy of part (b). Show that the expected return under this strategy is greater than $(A + B)/2$.

7.79. Successive weekly sales, in units of one thousand dollars, have a bivariate normal distribution with common mean 40, common standard deviation 6, and correlation .6.

(a) Find the probability that the total of the next 2 weeks' sales exceeds 90.

(b) If the correlation were .2 rather than .6, do you think that this would increase or decrease the answer to (a)? Explain your reasoning.

(c) Repeat (a) when the correlation is .2.

THEORETICAL EXERCISES

7.1. Show that $E[(X - a)^2]$ is minimized at $a = E[X]$.

7.2. Suppose that X is a continuous random variable with density function f. Show that $E[|X - a|]$ is minimized when a is equal to the median of F.
Hint: Write

$$E[|X - a|] = \int |x - a| f(x)\, dx$$

Now break up the integral into the regions where $x < a$ and where $x > a$, and differentiate.

7.3. Prove Proposition 2.1 when
(a) X and Y have a joint probability mass function;
(b) X and Y have a joint probability density function and $g(x, y) \geq 0$ for all x, y.

7.4. Let X be a random variable having finite expectation μ and variance σ^2, and let $g(\cdot)$ be a twice differentiable function. Show that

$$E[g(X)] \approx g(\mu) + \frac{g''(\mu)}{2}\sigma^2$$

Hint: Expand $g(\cdot)$ in a Taylor series about μ. Use the first three terms and ignore the remainder.

7.5. Let $A_1, A_2, \ldots, A_n$ be arbitrary events, and define $C_k = \{\text{at least } k \text{ of the } A_i \text{ occur}\}$. Show that

$$\sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P(A_k)$$

Hint: Let X denote the number of the A_i that occur. Show that both sides of the preceding equation are equal to $E[X]$.

7.6. In the text, we noted that

$$E\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} E[X_i]$$

when the X_i are all nonnegative random variables. Since an integral is a limit of sums, one might expect that

$$E\left[\int_0^{\infty} X(t)dt\right] = \int_0^{\infty} E[X(t)]\, dt$$

whenever $X(t), 0 \leq t < \infty$, are all nonnegative random variables; and this result is indeed true. Use it to give another proof of the result that, for a nonnegative random variable X,

$$E[X] = \int_0^{\infty} P\{X > t\}\, dt$$

Hint: Define, for each nonnegative t, the random variable $X(t)$ by

$$X(t) = \begin{cases} 1 & \text{if } t < X \\ 0 & \text{if } t \geq X \end{cases}$$

Now relate $\int_0^{\infty} X(t)dt$ to X.

7.7. We say that X is *stochastically larger* than Y, written $X \geq_{\text{st}} Y$, if, for all t,

$$P\{X > t\} \geq P\{Y > t\}$$

Show that if $X \geq_{\text{st}} Y$, then $E[X] \geq E[Y]$ when
(a) X and Y are nonnegative random variables;
(b) X and Y are arbitrary random variables.
Hint: Write X as

$$X = X^+ - X^-$$

where

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0 \end{cases}, \quad X^- = \begin{cases} 0 & \text{if } X \geq 0 \\ -X & \text{if } X < 0 \end{cases}$$

Similarly, represent Y as $Y^+ - Y^-$. Then make use of part (a).

7.8. Show that X is stochastically larger than Y if and only if

$$E[f(X)] \geq E[f(Y)]$$

for all increasing functions f.
Hint: Show that $X \geq_{\text{st}} Y$, then $E[f(X)] \geq E[f(Y)]$ by showing that $f(X) \geq_{\text{st}} f(Y)$ and then using Theoretical Exercise 7.7. To show that if $E[f(X)] \geq E[f(Y)]$ for all increasing functions f, then $P\{X > t\} \geq P\{Y > t\}$, define an appropriate increasing function f.

7.9. A coin having probability p of landing on heads is flipped n times. Compute the expected number of runs of heads of size 1, of size 2, and of size $k, 1 \leq k \leq n$.

7.10. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed positive random variables. For $k \leq n$, find

$$E\left[\frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{n} X_i}\right]$$

7.11. Consider n independent trials, each resulting in any one of r possible outcomes with probabilities $P_1, P_2, \ldots, P_r$. Let X denote the number of outcomes that never occur in any of the trials. Find

$E[X]$ and show that, among all probability vectors $P_1, \ldots, P_r, E[X]$ is minimized when $P_i = 1/r, i = 1, \ldots, r$.

7.12. Let $X_1, X_2, \ldots$ be a sequence of independent random variables having the probability mass function

$$P\{X_n = 0\} = P\{X_n = 2\} = 1/2, \quad n \geq 1$$

The random variable $X = \sum_{n=1}^{\infty} X_n/3^n$ is said to have the *Cantor distribution*. Find $E[X]$ and $\text{Var}(X)$.

7.13. Let $X_1, \ldots, X_n$ be independent and identically distributed continuous random variables. We say that a record value occurs at time $j, j \leq n$, if $X_j \geq X_i$ for all $1 \leq i \leq j$. Show that

(a) $E[\text{number of record values}] = \sum_{j=1}^{n} 1/j$;

(b) $\text{Var}(\text{number of record values}) = \sum_{j=1}^{n} (j-1)/j^2$.

7.14. For Example 2i, show that the variance of the number of coupons needed to amass a full set is equal to

$$\sum_{i=1}^{N-1} \frac{iN}{(N-i)^2}$$

When N is large, this can be shown to be approximately equal (in the sense that their ratio approaches 1 as $N \to \infty$) to $N^2\pi^2/6$.

7.15. Consider n independent trials, the ith of which results in a success with probability P_i.

(a) Compute the expected number of successes in the n trials—call it μ.

(b) For a fixed value of μ, what choice of $P_1, \ldots, P_n$ maximizes the variance of the number of successes?

(c) What choice minimizes the variance?

***7.16.** Suppose that each of the elements of $S = \{1, 2, \ldots, n\}$ is to be colored either red or blue. Show that if $A_1, \ldots, A_r$ are subsets of S, there is a way of doing the coloring so that at most $\sum_{i=1}^{r} (1/2)^{|A_i|-1}$ of these subsets have all their elements the same color (where $|A|$ denotes the number of elements in the set A).

7.17. Suppose that X_1 and X_2 are independent random variables having a common mean μ. Suppose also that $\text{Var}(X_1) = \sigma_1^2$ and $\text{Var}(X_2) = \sigma_2^2$. The value of μ is unknown, and it is proposed that μ be estimated by a weighted average of X_1 and X_2. That is, $\lambda X_1 + (1 - \lambda)X_2$ will be used as an estimate of μ for some appropriate value of λ. Which value of λ yields the estimate having the lowest possible variance? Explain why it is desirable to use this value of λ.

7.18. In Example 4f, we showed that the covariance of the multinomial random variables N_i and N_j is equal to $-mP_iP_j$ by expressing N_i and N_j as the sum of indicator variables. We could also have obtained that result by using the formula

$$\text{Var}(N_i + N_j) = \text{Var}(N_i) + \text{Var}(N_j) + 2\,\text{Cov}(N_i, N_j)$$

(a) What is the distribution of $N_i + N_j$?

(b) Use the preceding identity to show that $\text{Cov}(N_i, N_j) = -mP_iP_j$.

7.19. Show that X and Y are identically distributed and not necessarily independent, then

$$\text{Cov}(X + Y, X - Y) = 0$$

7.20. *The Conditional Covariance Formula*. The conditional covariance of X and Y, given Z, is defined by

$$\text{Cov}(X, Y|Z) \equiv E[(X - E[X|Z])(Y - E[Y|Z])|Z]$$

(a) Show that

$$\text{Cov}(X, Y|Z) = E[XY|Z] - E[X|Z]E[Y|Z]$$

(b) Prove the conditional covariance formula

$$\text{Cov}(X, Y) = E[\text{Cov}(X, Y|Z)] + \text{Cov}(E[X|Z], E[Y|Z])$$

(c) Set $X = Y$ in part (b) and obtain the conditional variance formula.

7.21. Let $X_{(i)}, i = 1, \ldots, n$, denote the order statistics from a set of n uniform (0, 1) random variables, and note that the density function of $X_{(i)}$ is given by

$$f(x) = \frac{n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i} \quad 0 < x < 1$$

(a) Compute $\text{Var}(X_{(i)}), i = 1, \ldots, n$.

(b) Which value of i minimizes, and which value maximizes, $\text{Var}(X_{(i)})$?

7.22. Show that $Y = a + bX$, then

$$\rho(X, Y) = \begin{cases} +1 & \text{if } b > 0 \\ -1 & \text{if } b < 0 \end{cases}$$

7.23. Show that Z is a standard normal random variable and if Y is defined by $Y = a + bZ + cZ^2$, then

$$\rho(Y, Z) = \frac{b}{\sqrt{b^2 + 2c^2}}$$

7.24. Prove the Cauchy–Schwarz inequality, namely,

$$(E[XY])^2 \le E[X^2]E[Y^2]$$

Hint: Unless $Y = -tX$ for some constant, in which case the inequality holds with equality, if follows that, for all t,

$$0 < E[(tX + Y)^2] = E[X^2]t^2 + 2E[XY]t + E[Y^2]$$

Hence, the roots of the quadratic equation

$$E[X^2]t^2 + 2E[XY]t + E[Y^2] = 0$$

must be imaginary, which implies that the discriminant of this quadratic equation must be negative.

7.25. Show that if X and Y are independent, then

$$E[X|Y = y] = E[X] \quad \text{for all } y$$

(a) in the discrete case;
(b) in the continuous case.

7.26. Prove that $E[g(X)Y|X] = g(X)E[Y|X]$.

7.27. Prove that if $E[Y|X = x] = E[Y]$ for all x, then X and Y are uncorrelated; give a counterexample to show that the converse is not true.
Hint: Prove and use the fact that $E[XY] = E[XE[Y|X]]$.

7.28. Show that $\text{Cov}(X, E[Y|X]) = \text{Cov}(X, Y)$.

7.29. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables. Find

$$E[X_1|X_1 + \cdots + X_n = x]$$

7.30. Consider Example 4f, which is concerned with the multinomial distribution. Use conditional expectation to compute $E[N_iN_j]$, and then use this to verify the formula for $\text{Cov}(N_i, N_j)$ given in Example 4f.

7.31. An urn initially contains b black and w white balls. At each stage, we add r black balls and then withdraw, at random, r balls from the $b + w + r$ balls in the urn. Show that

$$E[\text{number of white balls after stage } t] = \left(\frac{b + w}{b + w + r}\right)^t w$$

7.32. For an event A, let I_A equal 1 if A occurs and let it equal 0 if A does not occur. For a random variable X, show that

$$E[X|A] = \frac{E[XI_A]}{P(A)}$$

7.33. A coin that lands on heads with probability p is continually flipped. Compute the expected number of flips that are made until a string of r heads in a row is obtained.
Hint: Condition on the time of the first occurrence of tails to obtain the equation

$$E[X] = (1 - p)\sum_{i=1}^{r} p^{i-1}(i + E[X]) + (1 - p)\sum_{i=r+1}^{\infty} p^{i-1}r$$

Simplify and solve for $E[X]$.

7.34. For another approach to Theoretical Exercise 33, let T_r denote the number of flips required to obtain a run of r consecutive heads.
(a) Determine $E[T_r|T_{r-1}]$.
(b) Determine $E[T_r]$ in terms of $E[T_{r-1}]$.
(c) What is $E[T_1]$?
(d) What is $E[T_r]$?

7.35. The probability generating function of the discrete nonnegative integer valued random variable X having probability mass function p_j, $j \ge 0$, is defined by

$$\phi(s) = E[s^X] = \sum_{j=0}^{\infty} p_j s^j$$

Let Y be a geometric random variable with parameter $p = 1 - s$, where $0 < s < 1$. Suppose that Y is independent of X, and show that

$$\phi(s) = P\{X < Y\}$$

7.36. One ball at a time is randomly selected from an urn containing a white and b black balls until all of the remaining balls are of the same color. Let $M_{a,b}$ denote the expected number of balls left in the urn when the experiment ends. Compute a recursive formula for $M_{a,b}$ and solve when $a = 3$ and $b = 5$.

7.37. An urn contains a white and b black balls. After a ball is drawn, it is returned to the urn if it is white; but if it is black, it is replaced by a white ball from another urn. Let M_n denote the expected number of white balls in the urn after the foregoing operation has been repeated n times.
(a) Derive the recursive equation

$$M_{n+1} = \left(1 - \frac{1}{a + b}\right)M_n + 1$$

(b) Use part (a) to prove that

$$M_n = a + b - b\left(1 - \frac{1}{a + b}\right)^n$$

(c) What is the probability that the $(n + 1)$st ball drawn is white?

7.38. The best linear predictor of Y with respect to X_1 and X_2 is equal to $a + bX_1 + cX_2$, where a, b, and c are chosen to minimize

$$E[(Y - (a + bX_1 + cX_2))^2]$$

Determine a, b, and c.

7.39. The best quadratic predictor of Y with respect to X is $a + bX + cX^2$, where a, b, and c are chosen to minimize $E[(Y - (a + bX + cX^2))^2]$. Determine a, b, and c.

7.40. Use the conditional variance formula to determine the variance of a geometric random variable X having parameter p.

7.41. Let X be a normal random variable with parameters $\mu = 0$ and $\sigma^2 = 1$, and let I, independent of X, be such that $P\{I = 1\} = \frac{1}{2} = P\{I = 0\}$. Now define Y by

$$Y = \begin{cases} X & \text{if } I = 1 \\ -X & \text{if } I = 0 \end{cases}$$

In words, Y is equally likely to equal either X or $-X$.

(a) Are X and Y independent?
(b) Are I and Y independent?
(c) Show that Y is normal with mean 0 and variance 1.
(d) Show that $\text{Cov}(X, Y) = 0$.

7.42. It follows from Proposition 6.1 and the fact that the best linear predictor of Y with respect to X is $\mu_y + \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x)$ that if

$$E[Y|X] = a + bX$$

then

$$a = \mu_y - \rho\frac{\sigma_y}{\sigma_x}\mu_x \qquad b = \rho\frac{\sigma_y}{\sigma_x}$$

(Why?) Verify this directly.

7.43. Show that, for random variables X and Z,

$$E[(X - Y)^2] = E[X^2] - E[Y^2]$$

where

$$Y = E[X|Z]$$

7.44. Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that, by the end of its lifetime, each individual will have produced j new offspring with probability $P_j, j \ge 0$, independently of the number produced by any other individual. The number of individuals initially present, denoted by X_0, is called the size of the zeroth generation. All offspring of the zeroth generation constitute the first generation, and their number is denoted by X_1. In general, let X_n denote the size of the nth generation. Let $\mu = \sum_{j=0}^{\infty} jP_j$ and $\sigma^2 = \sum_{j=0}^{\infty}(j - \mu)^2 P_j$ denote, respectively, the mean and the variance of the number of offspring produced by a single individual. Suppose that $X_0 = 1$—that is, initially there is a single individual in the population.

(a) Show that

$$E[X_n] = \mu E[X_{n-1}]$$

(b) Use part (a) to conclude that

$$E[X_n] = \mu^n$$

(c) Show that

$$\text{Var}(X_n) = \sigma^2\mu^{n-1} + \mu^2\text{Var}(X_{n-1})$$

(d) Use part (c) to conclude that

$$\text{Var}(X_n) = \begin{cases} \sigma^2\mu^{n-1}\left(\dfrac{\mu^n - 1}{\mu - 1}\right) & \text{if } \mu \ne 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

The model just described is known as a *branching process*, and an important question for a population that evolves along such lines is the probability that the population will eventually die out. Let π denote this probability when the population starts with a single individual. That is,

$$\pi = P\{\text{population eventually dies out}|X_0 = 1\}$$

(e) Argue that π satisfies

$$\pi = \sum_{j=0}^{\infty} P_j \pi^j$$

Hint: Condition on the number of offspring of the initial member of the population.

7.45. Verify the formula for the moment generating function of a uniform random variable that is given in Table 7.7. Also, differentiate to verify the formulas for the mean and variance.

7.46. For a standard normal random variable Z, let $\mu_n = E[Z^n]$. Show that

$$\mu_n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ \dfrac{(2j)!}{2^j j!} & \text{when } n = 2j \end{cases}$$

Hint: Start by expanding the moment generating function of Z into a Taylor series about 0 to obtain

$$E[e^{tZ}] = e^{t^2/2} = \sum_{j=0}^{\infty} \frac{(t^2/2)^j}{j!}$$

7.47. Let X be a normal random variable with mean μ and variance σ^2. Use the results of Theoretical Exercise 46 to show that

$$E[X^n] = \sum_{j=0}^{[n/2]} \frac{\binom{n}{2j} \mu^{n-2j}\sigma^{2j}(2j)!}{2^j j!}$$

In the preceding equation, $[n/2]$ is the largest integer less than or equal to $n/2$. Check your answer by letting $n = 1$ and $n = 2$.

7.48. If $Y = aX + b$, where a and b are constants, express the moment generating function of Y in terms of the moment generating function of X.

7.49. The positive random variable X is said to be a *lognormal* random variable with parameters μ and σ^2 if $\log(X)$ is a normal random variable with mean μ and variance σ^2. Use the normal moment generating function to find the mean and variance of a lognormal random variable.

7.50. Let X have moment generating function $M(t)$, and define $\Psi(t) = \log M(t)$. Show that

$$\Psi''(t)|_{t=0} = \mathrm{Var}(X)$$

7.51. Use Table 7.2 to determine the distribution of $\sum_{i=1}^{n} X_i$ when $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables, each having mean $1/\lambda$.

7.52. Show how to compute $\mathrm{Cov}(X, Y)$ from the joint moment generating function of X and Y.

7.53. Suppose that $X_1, \ldots, X_n$ have a multivariate normal distribution. Show that $X_1, \ldots, X_n$ are independent random variables if and only if

$$\mathrm{Cov}(X_i, X_j) = 0 \quad \text{when } i \neq j$$

7.54. If Z is a standard normal random variable, what is $\mathrm{Cov}(Z, Z^2)$?

7.55. Suppose that Y is a normal random variable with mean μ and variance σ^2, and suppose also that the conditional distribution of X, given that $Y = y$, is normal with mean y and variance 1.

(a) Argue that the joint distribution of X, Y is the same as that of $Y + Z, Y$ when Z is a standard normal random variable that is independent of Y.

(b) Use the result of part (a) to argue that X, Y has a bivariate normal distribution.

(c) Find $E[X]$, $\mathrm{Var}(X)$, and $\mathrm{Corr}(X, Y)$.

(d) Find $E[Y|X = x]$.

(e) What is the conditional distribution of Y given that $X = x$?

SELF-TEST PROBLEMS AND EXERCISES

7.1. Consider a list of m names, where the same name may appear more than once on the list. Let $n(i)$, $i = 1, \ldots, m$, denote the number of times that the name in position i appears on the list, and let d denote the number of distinct names on the list.

(a) Express d in terms of the variables $m, n(i), i = 1, \ldots, m$. Let U be a uniform (0, 1) random variable, and let $X = [mU] + 1$.

(b) What is the probability mass function of X?

(c) Argue that $E[m/n(X)] = d$.

7.2. An urn has n white and m black balls that are removed one at a time in a randomly chosen order. Find the expected number of instances in which a white ball is immediately followed by a black one.

7.3. Twenty individuals consisting of 10 married couples are to be seated at 5 different tables, with 4 people at each table.

(a) If the seating is done "at random," what is the expected number of married couples that are seated at the same table?

(b) If 2 men and 2 women are randomly chosen to be seated at each table, what is the expected number of married couples that are seated at the same table?

7.4. If a die is to be rolled until all sides have appeared at least once, find the expected number of times that outcome 1 appears.

7.5. A deck of $2n$ cards consists of n red and n black cards. The cards are shuffled and then turned over one at a time. Suppose that each time a red card is turned over, we win 1 unit if more red cards than black cards have been turned over by that time. (For instance, if $n = 2$ and the result is r b r b, then we would win a total of 2 units.) Find the expected amount that we win.

7.6. Let $A_1, A_2, \ldots, A_n$ be events, and let N denote the number of them that occur. Also, let $I = 1$ if all of these events occur, and let it be 0 otherwise. Prove Bonferroni's inequality, namely,

$$P(A_1 \cdots A_n) \geq \sum_{i=1}^{n} P(A_i) - (n - 1)$$

Hint: Argue first that $N \leq n - 1 + I$.

7.7. Let X be the smallest value obtained when k numbers are randomly chosen from the set $1, \ldots, n$. Find $E[X]$ by interpreting X as a negative hypergeometric random variable.

7.8. An arriving plane carries r families. A total of n_j of these families have checked in a total of j pieces of luggage, $\sum_j n_j = r$. Suppose that when the plane lands, the $N = \sum_j jn_j$ pieces of luggage come out of the plane in a random order. As soon as a family collects all of its luggage, it immediately departs the airport. If the Sanchez family checked in j pieces of luggage, find the expected number of families that depart after they do.

***7.9.** Nineteen items on the rim of a circle of radius 1 are to be chosen. Show that, for any choice of these points, there will be an arc of (arc) length 1 that contains at least 4 of them.

7.10. Let X be a Poisson random variable with mean λ. Show that if λ is not too small, then

$$\text{Var}(\sqrt{X}) \approx .25$$

Hint: Use the result of Theoretical Exercise 4 to approximate $E[\sqrt{X}]$.

7.11. Suppose in Self-Test Problem 3 that the 20 people are to be seated at seven tables, three of which have 4 seats and four of which have 2 seats. If the people are randomly seated, find the expected value of the number of married couples that are seated at the same table.

7.12. Individuals 1 through $n, n > 1$, are to be recruited into a firm in the following manner: Individual 1 starts the firm and recruits individual 2. Individuals 1 and 2 will then compete to recruit individual 3. Once individual 3 is recruited, individuals 1, 2, and 3 will compete to recruit individual 4, and so on. Suppose that when individuals $1, 2, \ldots, i$ compete to recruit individual $i + 1$, each of them is equally likely to be the successful recruiter.

(a) Find the expected number of the individuals $1, \ldots, n$ who did not recruit anyone else.

(b) Derive an expression for the variance of the number of individuals who did not recruit anyone else, and evaluate it for $n = 5$.

7.13. The nine players on a basketball team consist of 2 centers, 3 forwards, and 4 backcourt players. If the players are paired up at random into three groups of size 3 each, find (a) the expected value and (b) the variance of the number of triplets consisting of one of each type of player.

7.14. A deck of 52 cards is shuffled and a bridge hand of 13 cards is dealt out. Let X and Y denote, respectively, the number of aces and the number of spades in the hand.

(a) Show that X and Y are uncorrelated.

(b) Are they independent?

7.15. Each coin in a bin has a value attached to it. Each time that a coin with value p is flipped, it lands on heads with probability p. When a coin is randomly chosen from the bin, its value is uniformly distributed on (0, 1). Suppose that after the coin is chosen, but before it is flipped, you must predict whether it will land on heads or on tails. You will win 1 if you are correct and will lose 1 otherwise.

(a) What is your expected gain if you are not told the value of the coin?

(b) Suppose now that you are allowed to inspect the coin before it is flipped, with the result of your inspection being that you learn the value of the coin. As a function of p, the value of the coin, what prediction should you make?

(c) Under the conditions of part (b), what is your expected gain?

7.16. In Self-Test Problem 1, we showed how to use the value of a uniform (0, 1) random variable (commonly called a *random number*) to obtain the value of a random variable whose mean is equal to the expected number of distinct names on a list. However, its use required that one choose a random position and then determine the number of times that the name in that position appears on the list. Another approach, which can be more efficient when there is a large amount of replication of names, is as follows: As before, start by choosing the random variable X as in Problem 1. Now identify the name in position X, and then go through the list, starting at the beginning, until that name appears. Let I equal 0 if you encounter that name before getting to position X, and let I equal 1 if your first encounter with the name is at position X. Show that $E[mI] = d$.
Hint: Compute $E[I]$ by using conditional expectation.

7.17. A total of m items are to be sequentially distributed among n cells, with each item independently being put in cell j with probability p_j, $j = 1, \ldots, n$. Find the expected number of collisions that occur, where a collision occurs whenever an item is put into a nonempty cell.

7.18. Let X be the length of the initial run in a random ordering of n ones and m zeroes. That is, if the first k values are the same (either all ones or all zeroes), then $X \geq k$. Find $E[X]$.

7.19. There are n items in a box labeled H and m in a box labeled T. A coin that comes up heads with probability p and tails with probability $1 - p$ is flipped. Each time it comes up heads, an item is removed from the H box, and each time it comes up tails, an item is removed from the T box. (If a box is empty and its outcome occurs, then no items

are removed.) Find the expected number of coin flips needed for both boxes to become empty.
Hint: Condition on the number of heads in the first $n + m$ flips.

7.20. Let X be a nonnegative random variable having distribution function F. Show that if $\overline{F}(x) = 1 - F(x)$, then

$$E[X^n] = \int_0^\infty x^{n-1}\overline{F}(x)\,dx$$

Hint: Start with the identity

$$X^n = n\int_0^X x^{n-1}\,dx = n\int_0^\infty x^{n-1}I_X(x)\,dx$$

where

$$I_x(x) = \begin{cases} 1, & \text{if } x < X \\ 0, & \text{otherwise} \end{cases}$$

***7.21.** Let $a_1, \ldots, a_n$, not all equal to 0, be such that $\sum_{i=1}^n a_i = 0$. Show that there is a permutation $i_1, \ldots, i_n$ such that $\sum_{j=1}^n a_{i_j}a_{i_{j+1}} < 0$.
Hint: Use the probabilistic method. (It is interesting that there need not be a permutation whose sum of products of successive pairs is positive. For instance, if $n = 3$, $a_1 = a_2 = -1$, and $a_3 = 2$, there is no such permutation.)

7.22. Suppose that X_i, $i = 1, 2, 3$, are independent Poisson random variables with respective means λ_i, $i = 1, 2, 3$. Let $X = X_1 + X_2$ and $Y = X_2 + X_3$. The random vector X, Y is said to have a bivariate Poisson distribution.
(a) Find $E[X]$ and $E[Y]$.
(b) Find $\text{Cov}(X, Y)$.
(c) Find the joint probability mass function $P\{X = i, Y = j\}$.

7.23. Let (X_i, Y_i), $i = 1, \ldots$, be a sequence of independent and identically distributed random vectors. That is, X_1, Y_1 is independent of, and has the same distribution as X_2, Y_2, and so on. Although X_i and Y_i can be dependent, X_i and Y_j are independent when $i \neq j$. Let

$$\mu_x = E[X_i], \quad \mu_y = E[Y_i], \quad \sigma_x^2 = \text{Var}(X_i),$$
$$\sigma_y^2 = \text{Var}(Y_i), \quad \rho = \text{Corr}(X_i, Y_i)$$

Find $\text{Corr}(\sum_{i=1}^n X_i, \sum_{j=1}^n Y_j)$.

7.24. Three cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let X denote the number of aces chosen.
(a) Find $E[X|\text{the ace of spades is chosen}]$.
(b) Find $E[X|\text{at least one ace is chosen}]$.

7.25. Let Φ be the standard normal distribution function, and let X be a normal random variable with mean μ and variance 1. We want to find $E[\Phi(X)]$. To do so, let Z be a standard normal random variable that is independent of X, and let

$$I = \begin{cases} 1, & \text{if } Z < X \\ 0, & \text{if } Z \geq X \end{cases}$$

(a) Show that $E[I|X = x] = \Phi(x)$.
(b) Show that $E[\Phi(X)] = P\{Z < X\}$.
(c) Show that $E[\Phi(X)] = \Phi(\frac{\mu}{\sqrt{2}})$.
Hint: What is the distribution of $X - Z$?

The preceding comes up in statistics. Suppose you are about to observe the value of a random variable X that is normally distributed with an unknown mean μ and variance 1, and suppose that you want to test the hypothesis that the mean μ is greater than or equal to 0. Clearly you would want to reject this hypothesis if X is sufficiently small. If it results that $X = x$, then the *p-value* of the hypothesis that the mean is greater than or equal to 0 is defined to be the probability that X would be as small as x if μ were equal to 0 (its smallest possible value if the hypothesis were true). (A small p-value is taken as an indication that the hypothesis is probably false.) Because X has a standard normal distribution when $\mu = 0$, the p-value that results when $X = x$ is $\Phi(x)$. Therefore, the preceding shows that the expected p-value that results when the true mean is μ is $\Phi(\frac{\mu}{\sqrt{2}})$.

7.26. A coin that comes up heads with probability p is flipped until either a total of n heads or of m tails is amassed. Find the expected number of flips.
Hint: Imagine that one continues to flip even after the goal is attained. Let X denote the number of flips needed to obtain n heads, and let Y denote the number of flips needed to obtain m tails. Note that $\max(X, Y) + \min(X, Y) = X + Y$. Compute $E[\max(X, Y)]$ by conditioning on the number of heads in the first $n + m - 1$ flips.

7.27. A deck of n cards numbered *1* through n, initially in any arbitrary order, is shuffled in the following manner: At each stage, we randomly choose one of the cards and move it to the front of the deck, leaving the relative positions of the other cards unchanged. This procedure is continued until all but one of the cards has been chosen. At this point it follows by symmetry that all $n!$ possible orderings are equally likely. Find the expected number of stages that are required.

7.28. Suppose that a sequence of independent trials in which each trial is a success with probability p is

performed until either a success occurs or a total of n trials has been reached. Find the mean number of trials that are performed.
Hint: The computations are simplified if you use the identity that, for a nonnegative integer valued random variable X,

$$E[X] = \sum_{i=1}^{\infty} P\{X \geq i\}$$

7.29. Suppose that X and Y are both Bernoulli random variables. Show that X and Y are independent if and only if $\text{Cov}(X, Y) = 0$.

7.30. In the generalized match problem, there are n individuals of whom n_i wear hat size i, $\sum_{i=1}^{r} n_i = n$. There are also n hats, of which h_i are of size i, $\sum_{i=1}^{r} h_i = n$. If each individual randomly chooses a hat (without replacement), find the expected number who choose a hat that is their size.