Homework 3 (Due 2/7)

Question 1 For jointly distributed random vectors $\vec{X}$ and $\vec{Y}$, show that

$$\mathrm{Cov}(\boldsymbol{C}\vec{X}, \boldsymbol{D}\vec{Y}) = \boldsymbol{C}\mathrm{Cov}(\vec{X}, \vec{Y})\boldsymbol{D}^\top.$$

Question 2 Let $\vec{X} \in \mathbb{R}^p$ and $\vec{Y} \in \mathbb{R}^q$ be independent random vectors. Show that $\mathrm{Cov}(\vec{X}, \vec{Y}) = \mathbf{0}_{p\times q}$.

Question 3 For mutually independent random vectors $\vec{X}_1, \ldots, \vec{X}_n \in \mathbb{R}^p$, show that

$$\mathrm{Cov}(a_1\vec{X}_1 + \ldots + a_n\vec{X}_n + \vec{c}) = a_1^2\mathrm{Cov}(\vec{X}_1) + \ldots + a_n^2\mathrm{Cov}(\vec{X}_n).$$

Question 4 Let $\vec{X} \sim \mathcal{N}_p(\vec{\mu}, \mathbf{\Sigma})$. Let

$$\mathbf{\Sigma} = \sum_{j=1}^{p} \lambda_j \vec{v}_j \vec{v}_j^\top$$

be the spectral decomposition. Let $Y_j = \vec{v}_j^\top \vec{X}$ for all $j = 1, \ldots, p$. Show that $Y_1, \ldots, Y_p$ are mutually independent.

Question 5 Let $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \sim \mathcal{N}_4(\vec{0}, \mathbf{\Sigma})$, where

$$\mathbf{\Sigma} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) Find a, b, c and d, such that

$$\mathrm{Cov}(X_1, X_3 - (aX_1 + bX_2)) = \mathrm{Cov}(X_2, X_3 - (aX_1 + bX_2)) = 0,$$

and

$$\mathrm{Cov}(X_1, X_4 - (cX_1 + dX_2)) = \mathrm{Cov}(X_2, X_4 - (cX_1 + dX_2)) = 0.$$

(b) For any x_1, x_2, give the conditional distribution of $\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}$ given $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Question 6 Let

$$\vec{X}_1, \ldots, \vec{X}_{20}$$

be a random sample from $\mathcal{N}_2(\vec{\mu}, \mathbf{\Sigma})$, where $\vec{\mu} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{\Sigma} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$. Denote the sample mean and sample covariance by $\overline{\vec{X}}$ and $\boldsymbol{S}$, respectively. Derive the distributions of $\overline{\vec{X}}$ and $\left(\overline{\vec{X}} - \vec{\mu}\right)^\top \boldsymbol{S}^{-1} \left(\overline{\vec{X}} - \vec{\mu}\right)$, respectively.