Homework 4 (Due on 6/1 Monday)

Question 1 Familywise error rate can be extended to the k-FWER in that

$$k\text{-FWER} = \mathbb{P}(V \geq k),$$

i.e., the probability of at least k false rejections. Consider the modified Bonferroni procedure: $H_{0,i}$ is rejected iff $p_i < k\alpha/n$. Show that this modified Bonferroni's method controls FWER at level α.

Question 2 Let the order statistics of the n p-values be

$$p_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(n)},$$

with corresponding hypotheses

$$H_{0,(1)}, H_{0,(2)}, \ldots, H_{0,(n)}.$$

Also, define

$$\alpha_i = \begin{cases} \frac{k\alpha}{n} & i \leq k, \\ \frac{k\alpha}{n+k-i} & i > k. \end{cases}$$

Consider the modified Holm's procedure

- **Step 1**: If $p_{(1)} \leq \alpha_1$, reject $H_{0,(1)}$ and go to Step 2; otherwise, accept $H_{0,(1)}$,..., $H_{0,(n)}$ and stop;
- ....
- **Step i**: If $p_{(i)} \leq \alpha_i$, reject $H_{0,(i)}$ and go to the next step; otherwise, accept $H_{0,(i)}$,..., $H_{0,(n)}$ and stop;
- ...
- **Step n**: If $p_{(n)} \leq \alpha_n$, reject $H_{0,(n)}$; otherwise, accept $H_{0,(n)}$ and stop.

Show that Holm's procedure controls k-FWER at level α.

Question 3 Let $U_1, \ldots, U_n$ be i.i.d. uniform random variables on $[0,1]$. Let the ordered statistics be

$$U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}.$$

By the result of FDR control with BH procedure, show that

$$\mathbb{P}\left(U_{(i)} \geq \frac{i\alpha}{n}, \quad i = 1, \ldots, n\right) \geq 1 - \alpha.$$

Question 4 In the proof of FDR control by BH procedure, we have defined the events

$$C_r^{(1)} = \left\{p_{(1)}^{(1)}, \ldots, p_{(r-1)}^{(1)} \leq \frac{qr}{n},\ p_{(r)}^{(i)} > \frac{q(r+1)}{n}, \ldots, p_{(n-1)}^{(i)} > q\right\}$$

for $r = 1, \ldots, n$, where $p_{(1)}^{(1)}, \ldots, p_{(n-1)}^{(1)}$ are the order statistics of $p_2, \ldots, p_n$. Show that

- $C_1^{(1)}$, $C_2^{(1)}$, ... $C_n^{(1)}$ are mutually disjoint;
- $C_1^{(1)}$, $C_1^{(1)} \cup C_2^{(1)}$, ... , $C_1^{(1)} \cup \ldots \cup C_n^{(1)}$ are a sequence of increasing sets in $\boldsymbol{p} = [p_1, \ldots, p_n]^\top$.