Homework 1 (Due on 1/17)

Question 1 Let

$$\boldsymbol{A} = \begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_k \end{bmatrix} \in \mathbb{R}^{n\times k} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \vec{b}_1^\top \\ \vdots \\ \vec{b}_k^\top \end{bmatrix} \in \mathbb{R}^{k\times p}.$$

Show that

$$\boldsymbol{AB} = \vec{a}_1\vec{b}_1^\top + \vec{a}_2\vec{b}_2^\top + \ldots + \vec{a}_k\vec{b}_k^\top.$$

Question 2 Let

$$\boldsymbol{C} = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad \boldsymbol{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{bmatrix} \in \mathbb{R}^{p\times p}$$

and

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix} \in \mathbb{R}^{n\times p}.$$

Calculate $\boldsymbol{CAD}$.

Question 3 Let $\vec{q}_1, \ldots, \vec{q}_k \in \mathbb{R}^k$ be k unit and pairwise perpendicular vectors. Show that

$$\vec{q}_1\vec{q}_1^\top + \vec{q}_2\vec{q}_2^\top + \ldots + \vec{q}_k\vec{q}_k^\top = \boldsymbol{I}.$$

Question 4 Let $\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ -3 & 5 \\ 5 & -3 \\ -4 & -4 \end{bmatrix}$.

(a) Calculate $\boldsymbol{A}^\top\boldsymbol{A}$ and find its spectral decomposition.

(b) Find $(\boldsymbol{A}^\top\boldsymbol{A})^{-1}$ and $(\boldsymbol{A}^\top\boldsymbol{A})^{-\frac{1}{2}}$.

Question 5

(a) Let $\boldsymbol{S}$, $\boldsymbol{D}$ and $\boldsymbol{C}$ be $k\times k$ invertible matrices. Moreover, let $\vec{x}$ and $\vec{y}$ be k-dimensional vectors. Show the following equality

$$(\boldsymbol{D}\vec{x})^\top(\boldsymbol{CSD}^\top)^{-1}(\boldsymbol{C}\vec{y}) = \vec{x}^\top\boldsymbol{S}^{-1}\vec{y}.$$

(b) Set

$$\boldsymbol{S} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Calculate $(\boldsymbol{C}\vec{x})^\top(\boldsymbol{CSC}^\top)^{-1}(\boldsymbol{C}\vec{x})$ and $\vec{x}^\top\boldsymbol{S}^{-1}\vec{x}$. Does your answer contradict the claim in part (a)? Explain.