Q1: $E(X)=\frac{\theta}{3}$ $E(X^2)=\frac{1}{3}$ $Var(X)=\frac{1}{3}-\frac{\theta^2}{9}$

$Var(3\bar{X})=9Var(X)/n=\frac{3-\theta^2}{n}$ when $n\to\infty$, $Var(3\bar{X})\to 0$

The $3\bar{X}$ is the unbiased estimator.

$MSE((3\bar{X}-\theta)^2)=Var(3\bar{X})=0$ when $n\to\infty$.

So, $3\bar{X}$ is the consistent of θ

Q2: $f(x|\theta)=\frac{1}{\sqrt{2\pi}\sqrt{\theta}}\cdot\exp[-\frac{(x-\theta)^2}{2\theta}]$

(a) $\ell(\theta|\underline{X})=\sum\log f(x_i|\theta)=-\frac{1}{2}n\log 2\pi-\frac{1}{2}\cdot n\log\theta-\frac{\sum(x_i-\theta)^2}{2\theta}$ equal with $\ell(\theta|\underline{X})=-n\log\theta-(\sum x_i^2\theta^{-1}-2\sum x_i+\theta$

$\frac{d\ell(\theta|X)}{d\theta}=\cancel{-\frac{n}{\theta}+\frac{\sum(x_i-\theta)}{\theta}+\frac{\sum(x_i-\theta)^2}{\theta^2}}-\frac{n}{\theta}+\frac{\sum x_i^2}{\theta^2}-1=0$

equal with $\theta^2+\theta-w=0$ where $w=\frac{\sum x_i^2}{n}$

$\hat{\theta}=\frac{-1+\sqrt{1+4w}}{2}$

(b). $I(\theta)=Var(\frac{d\log f(x|\theta)}{d\theta})=\frac{2n\theta+n}{2\theta^2}$

From the theory that $\sqrt{n}[\tau(\hat{\theta})-\tau(\theta)]\xrightarrow{d}N(0,\frac{(\tau'(\theta))^2}{I(\theta)})$, where $\hat{\theta}$ is the MLE of θ.

Let $\tau(\theta)=\theta$, we have:

$\sqrt{n}(\hat{\theta}-\theta)\xrightarrow{d}N(0,\frac{1}{I(\theta)})$

$\frac{1}{I(\theta)}$ is the approximate variance of $\hat{\theta}$, it is $\frac{2\theta^2}{2n\theta+n}$.

Q3: (a) $E(\frac{\sum x_iY_i}{\sum x_i^2})=E(\frac{\sum x_i(\beta x_i+\varepsilon_i)}{\sum x_i^2})=\beta+E(\frac{\sum x_i\varepsilon_i}{\sum x_i^2})$ From the theory that

$E(\frac{X}{Y})\approx\frac{E(X)}{E(Y)}$ and $Var(\frac{X}{Y})\approx\frac{1}{[E(Y)]^2}\cdot Var(X)+\frac{[E(X)]^2}{[E(Y)]^4}\cdot Var(Y)-2\frac{E(X)}{[E(Y)]^3}Cov(X,Y)$

So, $E(\frac{\sum x_i\varepsilon_i}{\sum(x_i^2)})\approx\frac{\sum E(x_i)\cdot E(\varepsilon_i)}{\sum E(x_i^2)}=0$, So, $E(\frac{\sum x_iY_i}{\sum x_i^2})\approx\beta$

$Var(\frac{\sum x_iY_i}{\sum x_i^2})=Var(\frac{\sum x_i\varepsilon_i}{\sum x_i^2})\approx\frac{\sum Var(\varepsilon_ix_i)}{\sum Var(x_i^2)}$

$Var(\varepsilon_ix_i)=E(\varepsilon_i^2x_i^2)-[E(x_i\varepsilon_i)]^2$

$=\sigma^2(\tau^2+\mu^2)$

By using the moment Generate function, we can get $E(X^4)$ and then get $Var(x_i^2)$

$Var(x_i^2)=2\tau^2(2\mu^2+\tau^2)$

So $Var(\frac{\sum x_iY_i}{\sum x_i^2})\approx\cancel{\frac{\sum Var(\varepsilon_ix_i)}{\sum Var(x_i^2)}}$

$\frac{1}{\mu_Y^2}\cdot Var(X)$ where $Y=\sum x_i^2$, $X=\sum x_i\cdot\varepsilon_i$

$=\frac{n\sigma^2(\mu^2+\tau^2)}{[n(\mu^2+\tau^2)]^2}=\frac{\sigma^2}{n(\mu^2+\tau^2)}$

(b) $L = \frac{\sum Y_i}{\sum X_i} = \beta + \frac{\sum \varepsilon_i}{\sum X_i}$

$E(L) = \beta + E\left(\frac{\sum \varepsilon_i}{\sum X_i}\right) \approx \beta + \frac{\sum E(\varepsilon_i)}{\sum E(X_i)} = \beta$

$Var(L) = Var\left(\frac{\sum \varepsilon_i}{\sum X_i}\right) \approx \frac{1}{(\sum E(X_i))^2} \cdot Var(\sum \varepsilon_i) + \frac{(\sum E(\varepsilon_i))^2}{(\sum E(X_i))^4} \cdot Var(\sum X_i)$

$= \frac{n\sigma^2}{n^2\mu^2} + 0$

$= \frac{\sigma^2}{n\mu^2}$

(c) $L = \frac{\sum(\frac{Y_i}{X_i})}{n} = \beta + \frac{1}{n} \cdot \sum \frac{\varepsilon_i}{X_i}$

$E(L) = \beta + E(\frac{\varepsilon_i}{X_i}) = \beta$

$var(L) = Var(\frac{1}{n} \cdot \sum \frac{\varepsilon_i}{X_i}) = \frac{1}{n} \cdot Var(\sum \frac{\varepsilon_i}{X_i}) \approx \frac{1}{n} \cdot Var(\frac{\varepsilon_i}{X_i}) = \frac{\sigma^2}{n\mu^2}$

Q4: $Var(T_n) = var(\frac{\sqrt{n}}{\bar{X}_n}) = n^3 Var(\frac{1}{B})$ where $B = \sum X_i \sim N(\mu, n\sigma^2)$

$Var(\frac{1}{B}) = E((\frac{1}{B})^2) - [E(\frac{1}{B})]^2$ $E(\frac{1}{B})$ is a fixed value, we should calculate $E((\frac{1}{B})^2)$

$E((\frac{1}{B})^2) > \sqrt{\frac{1}{2\pi\sigma^2}} \int_0^1 \frac{1}{B^2} \exp\{-(B-\mu)^2/2\sigma^2\} dB > \sqrt{\frac{1}{2\pi\sigma^2}} K \cdot \int_0^1 \frac{1}{B^2} dB = \infty.$

where $K = \max_{0<B<1} \exp\{-(B-\mu)^2/2\sigma^2\}$

So, $Var(\frac{1}{B}) = \infty$

(b) If we delete the interval $(-\delta, \delta)$ in the sample space, this means that $\frac{1}{\bar{X}_n}$ has the probablity to 0, So, we can use the formular in Q3:

$Var(T_n) = Var(\frac{\sqrt{n}}{\bar{X}_n}) \approx \frac{n}{\mu^4} var(\bar{X}_n)$

$= \frac{\sigma^2}{\mu^4} < 0$

(c) $P\{-\delta < \sum X_i < \delta\}$

$= P\{-\delta - \mu < \sum X_i - \mu < \delta - \mu\}$

$= P\{\frac{(-\delta-\mu)\sqrt{n}}{\sigma} < \frac{(\sum X_i - \mu)\sqrt{n}}{\sigma} < \frac{(\delta-\mu)\sqrt{n}}{\sigma}\}$ suppose $\sigma > 0$

$< P\{Z < \frac{\sqrt{n}(\delta-\mu)}{\sigma}\}$ $Z \sim N(0,1)$

when $n \to \infty$, the probability that $P\{Z < \frac{\sqrt{n}(\delta-\mu)}{\sigma}\} = 0$

Q5: By the asymptotic efficient of MLEs Theorem,

Let $\hat{p}$ denote the MLE of p, $\tau(p) = p(1-p)$, we have:

$$\sqrt{n}[\tau(\hat{p}) - \tau(p)] \xrightarrow{d} N(0, \frac{(\tau'(p))^2}{I(\theta)})$$

That is $\tau(\hat{p})$ is a consistent and asymptotic efficient estimator of $\tau(p)$

(b). $\sqrt{n}(\hat{p} - p) \xrightarrow{d} N(0, \frac{1}{I(\theta)})$, $\sigma^2 = \frac{1}{I(\theta)}$

By the theorem $n(g(\hat{p}) - g(p)) \xrightarrow{d} \sigma^2 \frac{g''(p)}{2} \cdot \chi_1^2$ where $g'(p) = 0 \Rightarrow p = \frac{1}{2}$

suppose $g(p) = p(1-p)$, we can have

$$n(\hat{p}(1-\hat{p}) - \frac{1}{4}) \xrightarrow{d} \frac{1}{I(\theta)} \cdot (-1) \cdot \chi_1^2$$

$$\hat{p}(1-\hat{p}) \xrightarrow{d} \frac{-1}{nI(\theta)} \chi_1^2 + \frac{1}{4}$$

$\log f(x|\theta) = x \log p + (1-x) \log(1-p)$

$\frac{\partial \log f(x|\theta)}{\partial \theta} = \frac{x}{p} - \frac{1}{1-p}(1-x)$, $Var(\frac{\partial \log f(x|\theta)}{\partial \theta}) = Var(\frac{x-p}{p(1-p)}) = \frac{1}{p^2(1-p)^2} \cdot Var(x)$

$= \frac{1}{p(1-p)} = I(\theta)$

$$\hat{p}(1-\hat{p}) \xrightarrow{d} \frac{-p(1-p)}{n} \chi_1^2 + \frac{1}{4}, \quad \hat{p}(1-\hat{p}) \xrightarrow{d} -\frac{1}{4n}\chi_1^2 + \frac{1}{4}$$

(c) $Var(\hat{p}(1-\hat{p}))$

$E(\hat{p}) = p$

$E(\hat{p}^2) = \frac{p + (n-1)p^2}{n}$

$E(\hat{p}^3) = \frac{p + 3(n-1)p^2 + (n-1)(n-2)p^3}{n^2}$

$E(\hat{p}^4) = \frac{p + 7(n-1)p^2 + 6(n-1)(n-2)p^3 + (n-1)(n-2)(n-3)p^4}{n^3}$ those formulas are all from websites

$Var(\hat{p}(1-\hat{p})) = Var(\hat{p}^2) + Var(\hat{p}) - 2cov(\hat{p}, \hat{p}^2)$

$= E(\hat{p}^2) - E^2(\hat{p}) + E(\hat{p}^4) - 2E(\hat{p}^3) + 2E(\hat{p})E(\hat{p}^2)$

$= \frac{(n^2 - 2n + 1)p + (-5n^2 + 12n - 7)p^2 + (8n^2 - 20n + 12)p^3 + (n-1)(6-4n)p^4}{n^3}$

from above formula, we can know that:

若 $p \neq \frac{1}{2}$, $nVar(\hat{p}(1-\hat{p})) \longrightarrow p(1-p)(1-2p)^2$, which is consistent with (a)

若 $p = \frac{1}{2}$, $Var(\hat{p}(1-\hat{p})) = \frac{1}{8}$, which is constant

So, it is the reason for the failure of approximations any clearer.

Q7: (a) $f(x|p) = p^x \cancel{\ } (1-p)^{1-x}$, $f_n(\underline{x}|p) = p^{\sum x_i}(1-p)^{n-\sum x_i}$, $\log f_n(\underline{x}|p) = \sum x_i \log p + (n-\sum x_i)\log(1-p)$

The MLE $\cancel{\ }$ $\hat{p} = \frac{\sum x_i}{n}$, $\cancel{\ }$ with $p = p_0$, we have:

$$\lambda(\underline{x}) = \frac{p_0^{\sum x_i}(1-p_0)^{n-\sum x_i}}{\hat{p}^{\sum x_i}(1-\hat{p})^{n-\sum x_i}} \quad \hat{p} = \frac{\sum x_i}{n}$$

$$-2\log \lambda(\underline{x}) = -2\left[\sum x_i \log\frac{p_0}{\hat{p}} + (n-\sum x_i)\log\frac{1-p_0}{1-\hat{p}}\right]$$

(b) $T = -2\log(\lambda(\underline{x})) = -2\left[\sum x_i \log\frac{p_0}{\hat{p}} + (n-\sum x_i)\log\frac{1-p_0}{1-\hat{p}}\right] \xrightarrow{d} \chi_1^2$

We would reject H_0 at level α if $T > \chi^2_{1,\alpha}$.

Q8. (a) The MLE of μ is $\hat{\mu} = \bar{x}_n$

So, the wald statistic is $W = \sqrt{\hat{I}_n(\hat{\mu})}\cdot(\hat{\mu}-\mu_0)$

$$\ell(\theta|\underline{x}) = \sum \log f(x_i|\theta) = n\log\frac{1}{\sqrt{2\pi}\sigma} - \frac{\sum(x_i-\mu)^2}{2\sigma^2}$$

$$\frac{\partial^2 \ell(\theta|\underline{x})}{\partial\theta^2} = -\frac{n}{\sigma^2} \quad \hat{I}_n(\hat{\mu}) = -\ell''(\theta|\underline{x}) = \frac{n}{\sigma^2}$$

So, $W = \sqrt{\frac{n}{\sigma^2}}(\hat{\mu}-\mu_0) = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}$

(b) By the theorem, $\sqrt{n}(\tau(\hat{\sigma}) - \tau(\sigma_0)) \xrightarrow{d} N\left(0, \frac{(\tau'(\theta))^2}{I(\theta)}\right)$, $\hat{\sigma}$ is the MLE of σ and $\tau(\theta) = \theta^2$, we can have:

$$\sqrt{n}(\hat{\sigma}^2 - \sigma_0^2) \xrightarrow{d} N\left(0, \frac{4\sigma^2}{I(\sigma)}\right)$$

$$\frac{\partial \log f(x|\sigma)}{\partial\sigma} = -\frac{1}{\sigma} + (x-\mu)^2\cdot\sigma^{-3}$$

$$I(\sigma) = Var\left(\frac{\partial \log f(x|\sigma)}{\partial\sigma}\right) = Var\left(\left(\frac{x-\mu}{\sigma}\right)^2\cdot\frac{1}{\sigma}\right) = \frac{1}{\sigma^2}\cdot Var\left(\left(\frac{x-\mu}{\sigma}\right)^2\right) = \frac{2}{\sigma^2}$$

So, $\sqrt{n}(\hat{\sigma}^2 - \sigma_0^2) \xrightarrow{d} N(0, 2\sigma^4)$, By ~~[illegible]~~ Slutsky's theorem

$$\frac{\hat{\sigma}^2 - \sigma_0^2}{\sqrt{2}\cdot\hat{\sigma}^2/\sqrt{n}} \xrightarrow{d} N(0,1)$$, By the δ-method,

The wald statistic is $W = \cancel{\ } \frac{\hat{\sigma} - \sigma_0}{\sqrt{\hat{\sigma}^2/2n}}$

Q9 (a) $\hat{I}_n(\hat{\mu}) = -\ell''(\theta|\underline{x}) = \frac{n}{\sigma^2}$, we have derived from Q8 (a).

$\psi(\hat{\mu}) = \frac{n}{\sigma^2}(\bar{X}-\mu)$

So, the score test is: $\frac{\frac{n}{\sigma^2}(\bar{X}-\mu)}{(n/\sigma^2)^{\frac{1}{2}}} = \sqrt{n}\cdot\frac{\bar{X}-\mu}{\sigma}$.

(b). $\psi(\sigma_0) = \frac{-n\cdot\sigma_0^2 + n\cdot\hat{\sigma}^2}{\sigma_0^3}$ $\hat{\sigma}$ is the MLE of σ

$I_n(\sigma_0) = \frac{2n}{\sigma_0^2}$

$$\frac{\psi(\sigma_0)}{\sqrt{I_n(\sigma_0)}} = \frac{n(\hat{\sigma}^2-\sigma_0^2)}{\sigma_0^3}\cdot\frac{\sigma_0}{\sqrt{2n}} = \frac{\sqrt{n}(\hat{\sigma}^2-\sigma_0^2)}{\sqrt{2}\cdot\sigma_0^2}$$