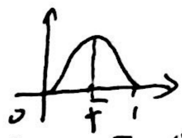
$Q_1$: $\theta_0 \in (-\infty, \theta_0]$ $\quad f(x|\theta) = \theta^x (1-\theta)^{1-x}$, $L(\theta|\underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^{\sum Y_i}(1-\theta)^{n-\sum Y_i}$

$\theta_1 \in (\theta_0, +\infty)$

the unrestricted MLE: $\hat{\theta} = \frac{\sum Y_i}{n} = \bar{Y}$

null-restricted MLE: $\hat{\theta}_0 = \begin{cases} \theta_0 & \text{if } \theta_0 \le \bar{Y} \\ \bar{Y} & \text{if } \theta_0 \ge \bar{Y} \end{cases}$




we only consider the ratio under $\theta_0 \le \bar{Y}$. Because if $\theta_0 > \bar{Y}$, the LR is 1

$$\lambda(\underline{Y}) = \frac{L(\hat{\theta}_0|\underline{Y})}{L(\hat{\theta}|\underline{Y})} = \frac{\theta_0^{\sum Y_i}(1-\theta_0)^{n-\sum Y_i}}{(\bar{Y})^{\sum Y_i}(1-\bar{Y})^{n-\sum Y_i}} < c \quad \text{for } \theta_0 \le \bar{Y}$$

under the $\lambda(\underline{Y}) < c$, we can get $\sum Y_i > a$ for $\sum Y_i > n\theta_0$.

So, this can convert: $\sum Y_i > \max\{a, \theta_0 \cdot n\} \Rightarrow \sum Y_i > b$.

$Q_2$: $X_1 \cdots X_n \overset{iid}{\sim} f(x|\theta) = \begin{cases} \theta e^{-\theta x} & x \ge 0 \\ 0 & x < 0 \end{cases}$ $\quad Y_1 \cdots Y_m \overset{iid}{\sim} f(y|\mu) = \begin{cases} \mu e^{-\mu y} & y \ge 0 \\ 0 & y < 0. \end{cases}$

(1)

$$L(\theta, \mu|\underline{Y}, \underline{X}) = \theta^n \cdot \mu^m \cdot e^{-\theta \sum X_i} \cdot e^{-\mu \cdot \sum Y_j}$$

the unrestricted MLE: $\hat{\theta} = \frac{n}{\sum X_i}$ $\quad \hat{\mu} = \frac{m}{\sum Y_i}$

under the null $H_0: \theta = \mu$, we can get the MLE under null hypothesis:

$$P = \theta = \mu \Rightarrow L(P_0|\underline{X}, \underline{Y}) = P^n e^{-P\sum X_i} \cdot e^{-P\sum Y_i} \cdot P^m$$

$$\hat{P} = \frac{m+n}{\sum X_i + \sum Y_i}$$

$$\text{the } \lambda(\underline{X}, \underline{Y}) = \frac{L(\hat{P}|\underline{X}, \underline{Y})}{L(\hat{\theta}, \hat{\mu}|\underline{X}, \underline{Y})} = \frac{(\hat{P})^{m+n} \cdot e^{-(m+n)}}{(\frac{n}{\sum X_i})^n \cdot e^{-n} e^{-m} (\frac{m}{\sum Y_i})^m} = \frac{(\hat{P})^{m+n}}{(\hat{\theta})^n \cdot (\hat{\mu})^m} < c.$$

(2) $\frac{(\hat{P})^{m+n}}{(\hat{\theta})^n (\hat{\mu})^m} < c \Rightarrow (m+n)\log\frac{m+n}{\sum X_i + \sum Y_i} - n\log\frac{n}{\sum X_i} - m\log\frac{m}{\sum Y_i} < \log c$.

$$\Rightarrow m\log\left(\frac{m+n}{\sum X_i + \sum Y_i} \cdot \frac{\sum Y_i}{m}\right) + n\log\left(\frac{m+n}{\sum X_i + \sum Y_i} \cdot \frac{\sum X_i}{n}\right) < \log c.$$

$$\Rightarrow m\log\left(\frac{\sum Y_i}{\sum X_i + \sum Y_i}\right) + m\log\left(\frac{m+n}{m}\right) + n\log\left(\frac{\sum X_i}{\sum X_i + \sum Y_i}\right) + n\log\frac{m+n}{n} < \log c$$

$$\Rightarrow m\log(1-T) + n\log(T) < \log c - (m+n)\log\left(\frac{m+n}{n}\right), \quad T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

(3) under the $H_0$ is true, $P = \theta = \mu$.

$X_i \overset{iid}{\sim} Exp(P)$ $\quad Y_i \overset{iid}{\sim} Exp(P)$ $\Rightarrow$ $2PX_i \sim Exp(\frac{1}{2}) = Gamma(1, \frac{1}{2}) = \chi^2_2$

$2P \cdot \sum X_i \sim Gamma(m, \frac{1}{2}) = \chi^2_{2m}$

$2P \cdot (\sum X_i + \sum Y_i) \sim Gamma(m+n, \frac{1}{2}) = \chi^2_{2(m+n)}$

$$\text{So, } T = \frac{2P\sum X_i}{2P(\sum X_i + \sum Y_i)} = \frac{\chi^2_{2m}}{\chi^2_{2(m+n)}} = F_{2m, 2(m+n)}$$

So, $T$ is the $F$ distribution.

Q3: one-way layout:

$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ $\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$, $\sum \alpha_i = 0$, $i = 1 \cdots I$, $j = 1 \cdots J$

Test: $H_0: \alpha_1 = \cdots = \alpha_I = 0$, $H_1$: not all $\alpha_i$ are zero.

$\lambda(\vec{Y}) = \left(\frac{\hat\sigma}{\hat\sigma_0}\right)^{IJ}$ $\hat\sigma^2$ is the MLE of $\hat\sigma$ under full model
$\hat\sigma_0$ is the MLE of $\sigma^2$ under reduced model

$$\sum_i \sum_j (y_{ij} - \bar{y}_{..})^2 = J \cdot \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_i \sum_j (y_{ij} - \bar{y}_{i.})^2$$

$$SS_T^2 = SS_{Factor}^2 + SS_E^2.$$

under the full model: $\hat\sigma^2 = \frac{1}{IJ} \cdot SSE$. under the reduced model: $\hat\sigma_0^2 = \frac{1}{IJ}(SS_F^2 + SS_E^2)$

$$\lambda(\vec{Y}) = \frac{SSE^2}{SS_F^2 + SS_E^2} < C^{\frac{2}{IJ}}$$

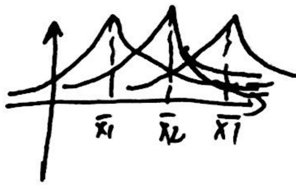
Q4: $X_{11} \cdots X_{1n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2)$ $X_{21} \cdots X_{2n_2} \overset{iid}{\sim} N(\mu_2, \sigma^2)$

$$L(\mu_1, \mu_2, \sigma \mid \underline{X_1}, \underline{X_2}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n_1+n_2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\left(\sum^{n_1}(X_{i1} - \mu_1)^2 + \sum^{n_2}(X_{i2} - \mu_2)^2\right)\right\}$$

The MLE of $\mu_1, \mu_2, \sigma^2$ under unrestricted is $\hat\mu_1 = \bar{X}_1$ $\hat\mu_2 = \bar{X}_2$, $\hat\sigma^2 = \frac{\sum(X_{i1} - \bar{X}_1)^2 + \sum(X_{i2} - \bar{X}_2)^2}{n_1 + n_2}$

Under the $H_0$ hypothesis: $\mu_1 \le \mu_2$ we can get:

$$\hat\theta_0 = \begin{cases} \hat\mu_1 = \bar{X}_1, \hat\mu_2 = \bar{X}_2, \hat\sigma_0^2 = \frac{\sum(X_{i1} - \bar{X}_1)^2 + \sum(X_{i2} - \bar{X}_2)^2}{n_1 + n_2} & \text{if } \bar{X}_1 \le \bar{X}_2 \\ \hat\mu_1 = \bar{X}_2, \hat\mu_2 = \bar{X}_2, \hat\sigma_0^2 = \frac{\sum(X_{i1} - \bar{X}_2)^2 + \sum(X_{i2} - \bar{X}_2)^2}{n_1 + n_2} & \text{if } \bar{X}_1 > \bar{X}_2 \end{cases}$$




$$\lambda(\underline{X_1}, \underline{X_2}) = \frac{L(\hat\theta_0 \mid \underline{X_1} \underline{X_2})}{L(\hat\theta \mid \underline{X_1} \underline{X_2})} = \left(\frac{\hat\sigma}{\hat\sigma_0}\right)^{n_1+n_2} < C.$$

$$\begin{cases} \hat\sigma^2 = \frac{\sum(X_{i1} - \bar{X}_1)^2 + \sum(X_{i2} - \bar{X}_2)^2}{n_1 + n_2} \\ \hat\sigma_0^2 = \frac{\sum(X_{i1} - \bar{X}_2)^2 + \sum(X_{i2} - \bar{X}_2)^2}{n_1 + n_2} = \hat\sigma^2 + \frac{n_1(\bar{X}_1 - \bar{X}_2)^2}{n_1 + n_2} \end{cases}$$

$$\lambda(\underline{X_1}, \underline{X_2}) = \frac{\hat\sigma^2}{\hat\sigma^2 + \frac{n_1(\bar{X}_1 - \bar{X}_2)^2}{n_1 + n_2}} < C^{\frac{2}{n_1+n_2}}$$

$$\Rightarrow \hat\sigma^2 < \frac{C^{\frac{2}{n_1+n_2}}}{1 - C^{\frac{2}{n_1+n_2}}} \cdot \frac{n_1(\bar{X}_1 - \bar{X}_2)^2}{n_1 + n_2}. \Rightarrow \frac{1 - C^{\frac{2}{n_1+n_2}}}{C^{\frac{2}{n_1+n_2}} \cdot n_1} < \frac{(\bar{X}_1 - \bar{X}_2)^2}{(n_1+n_2) \cdot \frac{\sum(X_{i1} - \bar{X}_1)^2 + \sum(X_{i2} - \bar{X}_2)^2}{n_1 + n_2}}$$

$$\Rightarrow \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\sum(X_{i1} - \bar{X}_1)^2 + \sum(X_{i2} - \bar{X}_2)^2}} > b \Rightarrow \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \cdot \hat\sigma} \sim t > \frac{b\sqrt{(n_1+n_2)}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = a.$$

So, it is equivalent two sample one sided t- test:

Q5: $\vec{X_1} \cdots \vec{X_n} \sim N_p(\vec{\mu}, \Sigma)$, $\bar{\vec{X}}$ is the mean, $S_0$ is the sample variance.

$$f(\vec{X}) = \frac{1}{(2\pi)^{\frac{p}{2}} \cdot |\Sigma|^{\frac{1}{2}}} \cdot \exp\{\tfrac{1}{2}(\vec{X}-\vec{\mu})^T \cdot \Sigma^{-1} \cdot (\vec{X}-\vec{\mu})\}$$, $|\Sigma|$ is the determinant of matrix $\Sigma$.

$$L(\vec{\mu}, \Sigma \mid \underline{X}) = \left(\frac{1}{2\pi^{p/2}} \cdot \frac{1}{|\Sigma|^{\frac{1}{2}}}\right)^n \cdot \exp\{\tfrac{1}{2} \cdot \sum_{i=1}^{n} (\vec{X_i}-\vec{\mu})^T \cdot \Sigma^{-1} (\vec{X_i}-\vec{\mu})\}$$

The MLE of unrestricted region is: $\hat{\mu} = \bar{\vec{X}}$, $\hat{\Sigma} = S$ $\quad S = \frac{1}{n} \cdot \Sigma(\vec{X_i} - \bar{\vec{X}})(\vec{X_i} - \bar{\vec{X}})^T$

The MLE of under null restricted is $\hat{\vec{\mu}_0} = \vec{\mu_0}$, $\hat{\Sigma}_0 = S_0$. $S_0 = \frac{1}{n} \Sigma(\vec{X_i} - \vec{\mu_0}) \cdot (\vec{X_i} - \vec{\mu_0})^T$

$$\lambda(\underline{X}) = \frac{L(\hat{\vec{\mu}_0}, \hat{\Sigma}_0 \mid \underline{X})}{L(\hat{\vec{\mu}}, \hat{\Sigma} \mid \underline{X})} = \left(\frac{|S|^{\frac{1}{2}}}{|S_0|^{\frac{1}{2}}}\right)^n < C \Rightarrow \frac{|S|}{|S_0|} < C^{\frac{2}{n}}$$

$$S_0 = \frac{1}{n} \Sigma(\vec{X_i} - \bar{\vec{X}} + \bar{\vec{X}} - \vec{\mu_0})(\vec{X_i} - \bar{\vec{X}} + \bar{\vec{X}} - \vec{\mu_0})^T = S + (\bar{\vec{X}} - \vec{\mu_0}) \times (\bar{\vec{X}} - \vec{\mu_0})^T$$
$$= S + A$$

$\det(S_0) = \det(S+A) \geq \det(S) + \det(A)$ under $S$ and $A$ are positive semidefinite matrices.

so, $\lambda(\underline{X}) < C \Rightarrow \frac{|S|}{|S_0|} < C^{\frac{2}{n}} \Rightarrow \frac{|S|}{|S_0|} \leq \frac{|S|}{|S|+|A|} < C^{\frac{2}{n}} \Rightarrow \frac{1-C^{\frac{2}{n}}}{C^{\frac{2}{n}}} < \frac{|A|}{|S|}$

$\Rightarrow b < |A| \cdot |S^{-1}| \Rightarrow b < |A S^{-1}|$, By cauchy-schwarz, we can get $|AS^{-1}| \leq |n(\bar{\vec{X}} - \vec{\mu_0}) \cdot S^{-1} \cdot (\bar{\vec{X}} - \vec{\mu_0})^T|$, so, we have:

$$|n(\bar{\vec{X}} - \vec{\mu_0}) S^{-1} (\bar{\vec{X}} - \vec{\mu_0})^T| > b$$
$$\Rightarrow (\bar{\vec{X}} - \vec{\mu_0})^T (\tfrac{1}{n} \cdot S)^{-1} (\bar{\vec{X}} - \vec{\mu_0}) > d.$$

Which is the same as $T^2$ statistic.

Q6: $H_0: \vec{\mu} = \vec{\mu_0}$ $\quad H_1: \vec{\mu} \neq \vec{\mu_0}$ $\quad H_{0\vec{a}}: \vec{a}^T \vec{\mu} = \vec{a}^T \cdot \vec{\mu_0}$, $\quad H_{1\vec{a}}: \vec{a}^T \vec{\mu} \neq \vec{a}^T \cdot \vec{\mu_0}$

For any $\vec{a} \neq \vec{0} \in \mathbb{R}^p$, $H_{0\vec{a}}$:

$$\vec{a}^T \cdot \vec{X_1}, \cdots \vec{a}^T \cdot \vec{X_n} \overset{iid}{\sim} N(\vec{a}^T \cdot \vec{\mu}, \vec{a}^T \Sigma \vec{a})$$

The reject region of one sample t-test:

$$\{\underline{X}: |\sqrt{n} \frac{\vec{a}^T \cdot \bar{\vec{X}} - \vec{a}^T \cdot \vec{\mu_0}}{\sqrt{\vec{a}^T \cdot S \cdot \vec{a}}}| > C\}$$

The union-intersection for $H_0$ is the region:

$$\bigcup_{\vec{a} \neq \vec{0}} \{\underline{X}: |\sqrt{n} \cdot \frac{\vec{a}^T \cdot \bar{\vec{X}} - \vec{a}^T \cdot \vec{\mu_0}}{\sqrt{\vec{a}^T \cdot S \cdot \vec{a}}}| > C\} = \max_{\vec{a} \neq \vec{0}} \frac{(\vec{a}^T \cdot \bar{\vec{X}} - \vec{a}^T \cdot \vec{\mu_0})^2}{(\vec{a}^T \cdot S \vec{a}) \cdot \frac{1}{n}} > C^2$$

under the extened cauchy-schwarz inequality:

$$(\vec{a}^T \cdot (\bar{\vec{X}} - \vec{\mu_0}))^2 \leq (\vec{a}^T \cdot S \cdot \vec{a}) \cdot (\bar{\vec{X}} - \vec{\mu_0})^T \cdot S^{-1} \cdot (\bar{\vec{X}} - \vec{\mu_0})$$

$$\text{so. } \frac{(\vec{a}^T (\bar{\vec{X}} - \vec{\mu_0}))^2}{(\vec{a}^T \cdot S \cdot \vec{a}) \frac{1}{n}} \leq \frac{(\bar{\vec{X}} - \vec{\mu_0})^T \cdot S^{-1} \cdot (\bar{\vec{X}} - \vec{\mu_0})}{\frac{1}{n}}$$
$$= (\bar{\vec{X}} - \vec{\mu_0})^T \cdot (\tfrac{1}{n} \cdot S)^{-1} (\bar{\vec{X}} - \vec{\mu_0}) > C^2$$

$$\Rightarrow \{\underline{X}: (\bar{\vec{X}} - \vec{\mu_0})^T \cdot (\tfrac{1}{n} \cdot S)^{-1} \cdot (\bar{\vec{X}} - \vec{\mu_0}) > C^2\}.$$

it is equivalent to the $T^2$ statistic.