STA 200A: Homework 7 Solutions

Note: Below the notation 3.T11 means Chapter 3, Theoretical Exercise 11. Similarly, the notation 4.P21 means Chapter 4, Problem 21.

1. 6.P15

Solution:

(a) Let $|R|$ be the area of R.

$$\int_R f(x,y)dxdy = c|R| = 1$$

(b) Let $1(A)$ be 1 if A is true and 0 otherwise. The area is $2^2 = 4$ so density is

$$f(x,y) = \frac{1}{4}1\{|x|<1, |y|<1\}.$$

and hence,

$$f(x) = \frac{1}{2}1\{|x|<1\}, \quad f(y) = \frac{1}{2}1\{|y|<1\}$$

which implies

$$f(x,y) = f(x)f(y),$$

and they are indeed independent.

(c) The probability of $x^2 + y^2 < 1$ is the area of the unit circle divided by 4, $\pi/4$.

2. 6.P20

Solution: The density can be factorized by

$$f(x,y) = \left(xe^{-x}1\{x>0\}\right)\left(e^{-y}1\{y>0\}\right)$$

which means that they are independent. Also, if

$$f(x,y) = 21\{0<x<y, 0<y<1\}$$

then we have that $Y < 1/2$ implies that $X < 1/2$ with probability 1 while marginally $X > 1/2$ has a non-zero probability.

3. 6.P22

Solution:

(a) The density does not factorize so they are not independent.

(b)

$$\int_0^1 (x+y)dy = x + \frac{1}{2}$$

for $0 \le x \le 1$.

(c)

$$P\{X+Y<1\} = \int_0^1 \int_0^{1-x} (x+y)dydx = \int_0^1 (x(1-x) + \frac{(1-x)^2}{2})dx = \int_0^1 (\frac{1}{2}(-x^2+1))dx = \frac{1}{3}$$

4. 6.P44

Solution: We can compute this from the following: $\{X_{(3)} > X_{(1)} + X_{(2)}\} = \{X_3 > X_1 + X_2\} \cup \{X_1 > X_2 + X_3\} \cup \{X_2 > X_1 + X_3\}$ and each of these events is mutually exclusive and they have equal probability. The density of the sum of two independent uniforms is

$$f_{X_1+X_2}(z) = \int_0^1 1\{0 < z - x < 1\}dx = \int_0^1 1\{z-1 < x < z\}dx$$

$$= \int_{\max(0,z-1)}^{\min(z,1)} dx = \min(z,1) - \max(0, z-1), 0 < z < 2.$$

The probability of one of these events is

$$P\{X_3 > X_1 + X_2\} = \int_0^1 P\{X_3 > z\} f_{X_1+X_2}(z)dz = \int_0^1 (1-z)(\min(z,1) - \max(0,z-1))dz$$

$$= \int_0^1 (1-z)zdz = \frac{1}{2}z^2 - \frac{1}{3}z^3\Big|_0^1 = \frac{1}{6}$$

Multiply this by 3 you get 1/2.

5. 6.P48

Solution: (a) The event $\{\min(X_1, \ldots, X_5) \le a\} = \cup_{i=1}^5 \{X_i \le a\} = (\cap_{i=1}^5 \{X_i > a\})^C$, so

$$P\{\min(X_1, \ldots, X_5) \le a\} = 1 - \prod_{i=1}^5 P\{X_i > a\} = 1 - e^{-5\lambda a}.$$

(b) The event $\{\max(X_1, \ldots, X_5) \le a\} = \cap_{i=1}^5 \{X_i \le a\}$, so

$$P\{\max(X_1, \ldots, X_5) \le a\} = \prod_{i=1}^5 P\{X_i \le a\} = (1 - e^{-\lambda a})^5.$$

6. 6.T25

Solution: Consider the order statistic, $X_{(n)}$, then the probability $P\{X_{(n)} \leq x\} = P\{X_i \leq x, i = 1, \ldots, n\} = F(x)^n$, so it is a CDF. Also, $P\{X_{(1)} \leq x\} = 1 - P\{X_{(1)} > x\} = 1 - P\{X_i > x, i = 1, \ldots, n\} = 1 - (1 - F(x))^n$ so it is a CDF.

7. Let X and Y have the joint density function

$$f(x, y) = k(x - y), \qquad 0 \leq y \leq x \leq 1$$

and 0 elsewhere, where $k > 0$ is a constant.

(a) Sketch the region over which the density is positive and use it in determining limits of integration to answer the following questions.

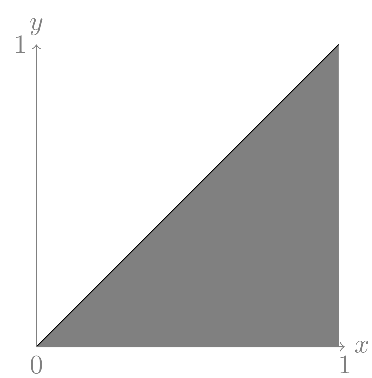
(b) Find k.

(c) Find the marginal densities of X and Y.

(d) Find the conditional densities of Y given X and X given Y.

Solution:

(a) Sketch the region over which the density is positive and use it in determining limits of integration to answer the following questions.



You can look at the shaded area in two ways:

$0 \leq x \leq 1$ and $0 \leq y \leq x$;
$0 \leq y \leq 1$ and $y \leq x \leq 1$.

(b) Find k.

We need $\int_0^1 \int_0^x k(x - y)dydx = 1$ (or $\int_0^1 \int_y^1 k(x - y)dxdy = 1$).

$$\begin{aligned}
\int_0^1 \int_0^x k(x - y)dydx &= \int_0^1 \left[xy - \frac{1}{2}y^2\right]_0^x dx \\
&= k \int_0^1 (x^2 - \frac{1}{2}x^2 - 0)dx \\
&= k \int_0^1 \frac{1}{2}x^2 dx \\
&= k \left[\frac{1}{6}x^3\right]_0^1 \\
&= \frac{1}{6}k
\end{aligned}$$

Therefore, $k = 6$.

(c) Find the marginal densities of X and Y.

$$\begin{aligned} f_X(x) &= \int_0^x 6(x-y)dy \qquad \text{for } 0 \le x \le 1 \\ &= 6\left[xy - \frac{1}{2}y^2\right]_0^x \\ &= 6(x^2 - \frac{1}{2}x^2 - 0) \\ &= 3x^2 \qquad \text{for } 0 \le x \le 1 \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_y^1 6(x-y)dx \qquad \text{for } 0 \le y \le 1 \\ &= 6\left[\frac{1}{2}x^2 - xy\right]_y^1 \\ &= 6(\frac{1}{2} - y - \frac{1}{2}y^2 + y^2) \\ &= 3 - 6y + 3y^2 \qquad \text{for } 0 \le y \le 1 \end{aligned}$$

(d) Find the conditional densities of Y given X and X given Y.

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \qquad \text{for } 0 \le y \le x \le 1 \\ &= \frac{6(x-y)}{3x^2} \\ &= \frac{2(x-y)}{x^2} \qquad \text{for } 0 \le y \le x \le 1 \end{aligned}$$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{for } 0 \le y \le x \le 1 \\ &= \frac{6(x-y)}{3y^2 - 6y + 3} \\ &= \frac{2(x-y)}{y^2 - 2y + 1} \qquad \text{for } 0 \le y \le x \le 1 \end{aligned}$$