A random variable is said to have a *beta* distribution with parameters (a, b) if its probability density function is equal to

$$f(x) = \frac{1}{B(a,b)}x^{a-1}(1 - x)^{b-1} \quad 0 \le x \le 1$$

and is equal to 0 otherwise. The constant $B(a, b)$ is given by

$$B(a,b) = \int_0^1 x^{a-1}(1 - x)^{b-1}\,dx$$

The mean and variance of such a random variable are, respectively,

$$E[X] = \frac{a}{a + b} \qquad \text{Var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$$

PROBLEMS

5.1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?
(b) What is the cumulative distribution function of X?

5.2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

what is the probability that the system functions for at least 5 months?

5.3. Consider the function

$$f(x) = \begin{cases} C(2x - x^3) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

Could f be a probability density function? If so, determine C. Repeat if $f(x)$ were given by

$$f(x) = \begin{cases} C(2x - x^2) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

5.4. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \dfrac{10}{x^2} & x > 10 \\ 0 & x \le 10 \end{cases}$$

(a) Find $P\{X > 20\}$.
(b) What is the cumulative distribution function of X?
(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

5.5. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1 - x)^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

what must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01?

5.6. Compute $E[X]$ if X has a density function given by

(a) $f(x) = \begin{cases} \dfrac{1}{4}xe^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$;

(b) $f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$;

(c) $f(x) = \begin{cases} \dfrac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$.

5.7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

5.8. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \qquad x \ge 0$$

Compute the expected lifetime of such a tube.

5.9. Consider Example 4b of Chapter 4, but now suppose that the seasonal demand is a continuous random variable having probability density function f. Show that the optimal amount to stock is the value s^* that satisfies

$$F(s^*) = \frac{b}{b + \ell}$$

where b is net profit per unit sale, ℓ is the net loss per unit unsold, and F is the cumulative distribution function of the seasonal demand.

5.10. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

5.11. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

5.12. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0, 100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

5.13. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

5.14. Let X be a uniform (0, 1) random variable. Compute $E[X^n]$ by using Proposition 2.1, and then check the result by using the definition of expectation.

5.15. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $P\{X > 5\}$;

(b) $P\{4 < X < 16\}$;

(c) $P\{X < 8\}$;

(d) $P\{X < 20\}$;

(e) $P\{X > 16\}$.

5.16. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

5.17. A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.

5.18. Suppose that X is a normal random variable with mean 5. If $P\{X > 9\} = .2$, approximately what is $\text{Var}(X)$?

5.19. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P\{X > c\} = .10$.

5.20. If 65 percent of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain

(a) at least 50 who are in favor of the proposition;

(b) between 60 and 70 inclusive who are in favor;

(c) fewer than 75 in favor.

5.21. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 feet, 2 inches tall? What percentage of men in the 6-footer club are over 6 feet, 5 inches?

5.22. The width of a slot of a duralumin forging is (in inches) normally distributed with $\mu = .9000$ and $\sigma = .0030$. The specification limits were given as $.9000 \pm .0050$.

(a) What percentage of forgings will be defective?

(b) What is the maximum allowable value of σ that will permit no more than 1 in 100 defectives when the widths are normally distributed with $\mu = .9000$ and σ?

5.23. One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that the number 5 will appear less than 150 times.

5.24. The lifetimes of interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \times 10^6$ hours and $\sigma = 3 \times 10^5$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than 1.8×10^6?

5.25. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

5.26. Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins, but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas if it lands on heads less than 525 times, then we shall conclude that it is a fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

5.27. In 10,000 independent tosses of a coin, the coin landed on heads 5800 times. Is it reasonable to assume that the coin is not fair? Explain.

5.28. Twelve percent of the population is left handed. Approximate the probability that there are at least 20 left-handers in a school of 200 students. State your assumptions.

5.29. A model for the movement of a stock supposes that if the present price of the stock is s, then, after one period, it will be either us with probability p or ds with probability $1 - p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30 percent after the next 1000 periods if $u = 1.012, d = 0.990$, and $p = .52$.

5.30. An image is partitioned into two regions, one white and the other black. A reading taken from a randomly chosen point in the white section will give a reading that is normally distributed with $\mu = 4$ and $\sigma^2 = 4$, whereas one taken from a randomly chosen point in the black region will have a normally distributed reading with parameters (6, 9). A point is randomly chosen on the image and has a reading of 5. If the fraction of the image that is black is α, for what value of α would the probability of making an error be the same, regardless of whether one concluded that the point was in the black region or in the white region?

5.31. **(a)** A fire station is to be located along a road of length $A, A < \infty$. If fires occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to

$$\text{minimize } E[|X - a|]$$

when X is uniformly distributed over $(0, A)$.

(b) Now suppose that the road is of infinite length—stretching from point 0 outward to ∞. If the distance of a fire from point 0 is exponentially distributed with rate λ, where should the fire station now be located? That is, we want to minimize $E[|X - a|]$, where X is now exponential with rate λ.

5.32. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is

(a) the probability that a repair time exceeds 2 hours?

(b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

5.33. The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?

5.34. Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter $\frac{1}{20}$. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed, but rather is (in thousands of miles) uniformly distributed over (0, 40).

5.35. The lung cancer hazard rate $\lambda(t)$ of a t-year-old male smoker is such that

$$\lambda(t) = .027 + .00025(t - 40)^2 \qquad t \geq 40$$

Assuming that a 40-year-old male smoker survives all other hazards, what is the probability that he survives to (a) age 50 and (b) age 60 without contracting lung cancer?

5.36. Suppose that the life distribution of an item has the hazard rate function $\lambda(t) = t^3, t > 0$. What is the probability that
- **(a)** the item survives to age 2?
- **(b)** the item's lifetime is between .4 and 1.4?
- **(c)** a 1-year-old item will survive to age 2?

5.37. If X is uniformly distributed over $(-1, 1)$, find
- **(a)** $P\{|X| > \frac{1}{2}\}$;
- **(b)** the density function of the random variable $|X|$.

5.38. If Y is uniformly distributed over (0, 5), what is the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real?

5.39. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.

5.40. If X is uniformly distributed over (0, 1), find the density function of $Y = e^X$.

5.41. Find the distribution of $R = A \sin\theta$, where A is a fixed constant and θ is uniformly distributed on $(-\pi/2, \pi/2)$. Such a random variable R arises in the theory of ballistics. If a projectile is fired from the origin at an angle α from the earth with a speed ν, then the point R at which it returns to the earth can be expressed as $R = (\nu^2/g)\sin 2\alpha$, where g is the gravitational constant, equal to 980 centimeters per second squared.

THEORETICAL EXERCISES

5.1. The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by

$$f(x) = \begin{cases} ax^2 e^{-bx^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $b = m/2kT$ and k, T, and m denote, respectively, Boltzmann's constant, the absolute temperature of the gas, and the mass of the molecule. Evaluate a in terms of b.

5.2. Show that

$$E[Y] = \int_0^\infty P\{Y > y\}\,dy - \int_0^\infty P\{Y < -y\}\,dy$$

Hint: Show that

$$\int_0^\infty P\{Y < -y\}\,dy = -\int_{-\infty}^0 x f_Y(x)\,dx$$

$$\int_0^\infty P\{Y > y\}\,dy = \int_0^\infty x f_Y(x)\,dx$$

5.3. Show that if X has density function f, then

$$E[g(X)] = \int_{-\infty}^\infty g(x)f(x)\,dx$$

Hint: Using Theoretical Exercise 2, start with

$$E[g(X)] = \int_0^\infty P\{g(X) > y\}\,dy - \int_0^\infty P\{g(X) < -y\}\,dy$$

and then proceed as in the proof given in the text when $g(X) \geq 0$.

5.4. Prove Corollary 2.1.

5.5. Use the result that, for a nonnegative random variable Y,

$$E[Y] = \int_0^\infty P\{Y > t\}\,dt$$

to show that, for a nonnegative random variable X,

$$E[X^n] = \int_0^\infty n x^{n-1} P\{X > x\}\,dx$$

Hint: Start with

$$E[X^n] = \int_0^\infty P\{X^n > t\}\,dt$$

and make the change of variables $t = x^n$.

5.6. Define a collection of events $E_a, 0 < a < 1$, having the property that $P(E_a) = 1$ for all a but $P\left(\bigcap_a E_a\right) = 0$.
Hint: Let X be uniform over (0, 1) and define each E_a in terms of X.

5.7. The standard deviation of X, denoted $SD(X)$, is given by

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$$

Find $SD(aX + b)$ if X has variance σ^2.

5.8. Let X be a random variable that takes on values between 0 and c. That is, $P\{0 \leq X \leq c\} = 1$. Show that

$$\mathrm{Var}(X) \leq \frac{c^2}{4}$$

Hint: One approach is to first argue that

$$E[X^2] \leq cE[X]$$

and then use this inequality to show that

$$\mathrm{Var}(X) \le c^2[\alpha(1 - \alpha)] \quad \text{where } \alpha = \frac{E[X]}{c}$$

5.9. Show that Z is a standard normal random variable, then, for $x > 0$,
(a) $P\{Z > x\} = P\{Z < -x\}$;
(b) $P\{|Z| > x\} = 2P\{Z > x\}$;
(c) $P\{|Z| < x\} = 2P\{Z < x\} - 1$.

5.10. Let $f(x)$ denote the probability density function of a normal random variable with mean μ and variance σ^2. Show that $\mu - \sigma$ and $\mu + \sigma$ are points of inflection of this function. That is, show that $f''(x) = 0$ when $x = \mu - \sigma$ or $x = \mu + \sigma$.

5.11. Let Z be a standard normal random variable Z, and let g be a differentiable function with derivative g'.
(a) Show that $E[g'(Z)] = E[Zg(Z)]$
(b) Show that $E[Z^{n+1}] = nE[Z^{n-1}]$
(c) Find $E[Z^4]$.

5.12. Use the identity of Theoretical Exercise 5 to derive $E[X^2]$ when X is an exponential random variable with parameter λ.

5.13. The median of a continuous random variable having distribution function F is that value m such that $F(m) = \frac{1}{2}$. That is, a random variable is just as likely to be larger than its median as it is to be smaller. Find the median of X if X is
(a) uniformly distributed over (a, b);
(b) normal with parameters μ, σ^2;
(c) exponential with rate λ.

5.14. The mode of a continuous random variable having density f is the value of x for which $f(x)$ attains its maximum. Compute the mode of X in cases (a), (b), and (c) of Theoretical Exercise 5.13.

5.15. If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is exponential with parameter λ/c.

5.16. Compute the hazard rate function of X when X is uniformly distributed over $(0, a)$.

5.17. If X has hazard rate function $\lambda_X(t)$, compute the hazard rate function of aX where a is a positive constant.

5.18. Verify that the gamma density function integrates to 1.

5.19. If X is an exponential random variable with mean $1/\lambda$, show that

$$E[X^k] = \frac{k!}{\lambda^k} \qquad k = 1, 2, \ldots$$

Hint: Make use of the gamma density function to evaluate the preceding.

5.20. Verify that

$$\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$$

when X is a gamma random variable with parameters α and λ.

5.21. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
Hint: $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}\,dx$. Make the change of variables $y = \sqrt{2x}$ and then relate the resulting expression to the normal distribution.

5.22. Compute the hazard rate function of a gamma random variable with parameters (α, λ) and show it is increasing when $\alpha \ge 1$ and decreasing when $\alpha \le 1$.

5.23. Compute the hazard rate function of a Weibull random variable and show it is increasing when $\beta \ge 1$ and decreasing when $\beta \le 1$.

5.24. Show that a plot of $\log(\log(1 - F(x))^{-1})$ against $\log x$ will be a straight line with slope β when $F(\cdot)$ is a Weibull distribution function. Show also that approximately 63.2 percent of all observations from such a distribution will be less than α. Assume that $v = 0$.

5.25. Let

$$Y = \left(\frac{X - \nu}{\alpha}\right)^\beta$$

Show that if X is a Weibull random variable with parameters ν, α, and β, then Y is an exponential random variable with parameter $\lambda = 1$ and vice versa.

5.26. If X is a beta random variable with parameters a and b, show that

$$E[X] = \frac{a}{a + b}$$
$$\mathrm{Var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$$

5.27. If X is uniformly distributed over (a, b), what random variable, having a linear relation with X, is uniformly distributed over $(0, 1)$?

5.28. Consider the beta distribution with parameters (a, b). Show that
(a) when $a > 1$ and $b > 1$, the density is unimodal (that is, it has a unique mode) with mode equal to $(a - 1)/(a + b - 2)$;
(b) when $a \le 1, b \le 1$, and $a + b < 2$, the density is either unimodal with mode at 0 or 1 or U-shaped with modes at both 0 and 1;
(c) when $a = 1 = b$, all points in $[0, 1]$ are modes.

5.29. Let X be a continuous random variable having cumulative distribution function F. Define the random variable Y by $Y = F(X)$. Show that Y is uniformly distributed over $(0, 1)$.

5.30. Let X have probability density f_X. Find the probability density function of the random variable Y defined by $Y = aX + b$.

5.31. Find the probability density function of $Y = e^X$ when X is normally distributed with parameters μ and σ^2. The random variable Y is said to have a *lognormal distribution* (since $\log Y$ has a normal distribution) with parameters μ and σ^2.

5.32. Let X and Y be independent random variables that are both equally likely to be either $1, 2, \ldots, (10)^N$, where N is very large. Let D denote the greatest common divisor of X and Y, and let $Q_k = P\{D = k\}$.

(a) Give a heuristic argument that $Q_k = \frac{1}{k^2}Q_1$.
Hint: Note that in order for D to equal k, k must divide both X and Y and also X/k, and Y/k must be relatively prime. (That is, X/k, and Y/k must have a greatest common divisor equal to 1.)

(b) Use part (a) to show that

$$Q_1 = P\{X \text{ and } Y \text{ are relatively prime}\}$$
$$= \frac{1}{\sum_{k=1}^{\infty} 1/k^2}$$

It is a well-known identity that $\sum_{1}^{\infty} 1/k^2 = \pi^2/6$, so $Q_1 = 6/\pi^2$. (In number theory, this is known as the Legendre theorem.)

(c) Now argue that

$$Q_1 = \prod_{i=1}^{\infty} \left(\frac{P_i^2 - 1}{P_i^2} \right)$$

where P_i is the ith-smallest prime greater than 1.
Hint: X and Y will be relatively prime if they have no common prime factors. Hence, from part (b), we see that

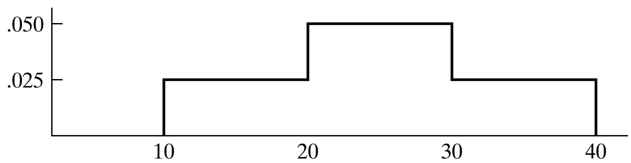
$$\prod_{i=1}^{\infty} \left(\frac{P_i^2 - 1}{P_i^2} \right) = \frac{6}{\pi^2}$$

which was noted without explanation in Problem 11 of Chapter 4. (The relationship between this problem and Problem 11 of Chapter 4 is that X and Y are relatively prime if XY has no multiple prime factors.)

5.33. Prove Theorem 7.1 when $g(x)$ is a decreasing function.

SELF-TEST PROBLEMS AND EXERCISES

5.1. The number of minutes of playing time of a certain high school basketball player in a randomly chosen game is a random variable whose probability density function is given in the following figure:



Find the probability that the player plays

(a) over 15 minutes;
(b) between 20 and 35 minutes;
(c) less than 30 minutes;
(d) more than 36 minutes.

5.2. For some constant c, the random variable X has the probability density function

$$f(x) = \begin{cases} cx^n & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) c and (b) $P\{X > x\}, 0 < x < 1$.

5.3. For some constant c, the random variable X has the probability density function

$$f(x) = \begin{cases} cx^4 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) $E[X]$ and (b) $\text{Var}(X)$.

5.4. The random variable X has the probability density function

$$f(x) = \begin{cases} ax + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = .6$, find (a) $P\{X < \frac{1}{2}\}$ and (b) $\text{Var}(X)$.

5.5. The random variable X is said to be a discrete uniform random variable on the integers $1, 2, \ldots, n$ if

$$P\{X = i\} = \frac{1}{n} \quad i = 1, 2, \ldots, n$$

For any nonnegative real number x, let $\text{Int}(x)$ (sometimes written as $[x]$) be the largest integer that is less than or equal to x. Show that U is a uniform random variable on (0, 1), fain

$X = \text{Int}(nU) + 1$ is a discrete uniform random variable on $1, \ldots, n$.

5.6. Your company must make a sealed bid for a construction project. If you succeed in winning the contract (by having the lowest bid), then you plan to pay another firm 100 thousand dollars to do the work. If you believe that the minimum bid (in thousands of dollars) of the other participating companies can be modeled as the value of a random variable that is uniformly distributed on (70, 140), how much should you bid to maximize your expected profit?

5.7. To be a winner in a certain game, you must be successful in three successive rounds. The game depends on the value of U, a uniform random variable on (0, 1). If $U > .1$, then you are successful in round 1; if $U > .2$, then you are successful in round 2; and if $U > .3$, then you are successful in round 3.

(a) Find the probability that you are successful in round 1.
(b) Find the conditional probability that you are successful in round 2 given that you were successful in round 1.
(c) Find the conditional probability that you are successful in round 3 given that you were successful in rounds 1 and 2.
(d) Find the probability that you are a winner.

5.8. A randomly chosen IQ test taker obtains a score that is approximately a normal random variable with mean 100 and standard deviation 15. What is the probability that the score of such a person is (a) above 125; (b) between 90 and 110?

5.9. Suppose that the travel time from your home to your office is normally distributed with mean 40 minutes and standard deviation 7 minutes. If you want to be 95 percent certain that you will not be late for an office appointment at 1 P.M., what is the latest time that you should leave home?

5.10. The life of a certain type of automobile tire is normally distributed with mean 34,000 miles and standard deviation 4000 miles.

(a) What is the probability that such a tire lasts over 40,000 miles?
(b) What is the probability that it lasts between 30,000 and 35,000 miles?
(c) Given that it has survived 30,000 miles, what is the conditional probability that the tire survives another 10,000 miles?

5.11. The annual rainfall in Cleveland, Ohio is approximately a normal random variable with mean 40.2 inches and standard deviation 8.4 inches. What is the probability that

(a) next year's rainfall will exceed 44 inches?
(b) the yearly rainfalls in exactly 3 of the next 7 years will exceed 44 inches?

Assume that if A_i is the event that the rainfall exceeds 44 inches in year i (from now), then the events $A_i, i \geq 1$, are independent.

5.12. The following table uses 1992 data concerning the percentages of male and female full-time workers whose annual salaries fall into different ranges:

Earnings range	Percentage of females	Percentage of males
≤9999	8.6	4.4
10,000–19,999	38.0	21.1
20,000–24,999	19.4	15.8
25,000–49,999	29.2	41.5
≥50,000	4.8	17.2

Suppose that random samples of 200 male and 200 female full-time workers are chosen. Approximate the probability that

(a) at least 70 of the women earn \$25,000 or more;
(b) at most 60 percent of the men earn \$25,000 or more;
(c) at least three-fourths of the men and at least half the women earn \$20,000 or more.

5.13. At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he or she will still be with the teller after an additional 4 minutes?

5.14. Suppose that the cumulative distribution function of the random variable X is given by

$$F(x) = 1 - e^{-x^2} \quad x > 0$$

Evaluate (a) $P\{X > 2\}$; (b) $P\{1 < X < 3\}$; (c) the hazard rate function of F; (d) $E[X]$; (e) $\text{Var}(X)$. *Hint*: For parts (d) and (e), you might want to make use of the results of Theoretical Exercise 5.

5.15. The number of years that a washing machine functions is a random variable whose hazard rate function is given by

$$\lambda(t) = \begin{cases} .2 & 0 < t < 2 \\ .2 + .3(t - 2) & 2 \leq t < 5 \\ 1.1 & t > 5 \end{cases}$$

(a) What is the probability that the machine will still be working 6 years after being purchased?
(b) If it is still working 6 years after being purchased, what is the conditional probability that it will fail within the next 2 years?

5.16. A standard Cauchy random variable has density function

$$f(x) = \frac{1}{\pi(1 + x^2)} \quad -\infty < x < \infty$$

Show that X is a standard Cauchy random variable, then $1/X$ is also a standard Cauchy random variable.

5.17. A roulette wheel has 38 slots, numbered 0, 00, and 1 through 36. If you bet 1 on a specified number then you either win 35 if the roulette ball lands on that number or lose 1 if it does not. If you continually make such bets, approximate the probability that

(a) you are winning after 34 bets;
(b) you are winning after 1000 bets;
(c) you are winning after 100,000 bets.

Assume that each roll of the roulette ball is equally likely to land on any of the 38 numbers.

5.18. There are two types of batteries in a bin. When in use, type i batteries last (in hours) an exponentially distributed time with rate $\lambda_i, i = 1,2$. A battery that is randomly chosen from the bin will be a type i battery with probability $p_i, \sum_{i=1}^{2} p_i = 1$. If a randomly chosen battery is still operating after t hours of use, what is the probability that it will still be operating after an additional s hours?

5.19. Evidence concerning the guilt or innocence of a defendant in a criminal investigation can be summarized by the value of an exponential random variable X whose mean μ depends on whether the defendant is guilty. If innocent, $\mu = 1$; if guilty, $\mu = 2$. The deciding judge will rule the defendant guilty if $X > c$ for some suitably chosen value of c.

(a) If the judge wants to be 95 percent certain that an innocent man will not be convicted, what should be the value of c?
(b) Using the value of c found in part (a), what is the probability that a guilty defendant will be convicted?

5.20. For any real number y, define y^+ by

$$y^+ = \begin{matrix} y, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{matrix}$$

Let c be a constant.

(a) Show that

$$E[(Z - c)^+] = \frac{1}{\sqrt{2\pi}} e^{-c^2/2} - c(1 - \Phi(c))$$

when Z is a standard normal random variable.

(b) Find $E[(X - c)^+]$ when X is normal with mean μ and variance σ^2.