PROBLEMS

6.1. Two fair dice are rolled. Find the joint probability mass function of X and Y when

(a) X is the largest value obtained on any die and Y is the sum of the values;

(b) X is the value on the first die and Y is the larger of the two values;

(c) X is the smallest and Y is the largest value obtained on the dice.

6.2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2;

(b) X_1, X_2, X_3.

6.3. In Problem 2, suppose that the white balls are numbered, and let Y_i equal 1 if the ith white ball is selected and 0 otherwise. Find the joint probability mass function of

(a) Y_1, Y_2;

(b) Y_1, Y_2, Y_3.

6.4. Repeat Problem 2 when the ball selected is replaced in the urn before the next selection.

6.5. Repeat Problem 3a when the ball selected is replaced in the urn before the next selection.

6.6. A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time, until the defective ones are identified. Denote by N_1 the number of tests made until the first defective is identified and by N_2 the number of additional tests until the second defective is identified. Find the joint probability mass function of N_1 and N_2.

6.7. Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first two successes. Find the joint mass function of X_1 and X_2.

6.8. The joint probability density function of X and Y is given by

$$f(x, y) = c(y^2 - x^2)e^{-y} \qquad -y \le x \le y,\ 0 < y < \infty$$

(a) Find c.

(b) Find the marginal densities of X and Y.

(c) Find $E[X]$.

6.9. The joint probability density function of X and Y is given by

$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ 0 < y < 2$$

(a) Verify that this is indeed a joint density function.

(b) Compute the density function of X.

(c) Find $P\{X > Y\}$.

(d) Find $P\{Y > \frac{1}{2} | X < \frac{1}{2}\}$.

(e) Find $E[X]$.

(f) Find $E[Y]$.

6.10. The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)} \qquad 0 \le x < \infty, 0 \le y < \infty$$

Find (a) $P\{X < Y\}$ and (b) $P\{X < a\}$.

6.11. A television store owner figures that 45 percent of the customers entering his store will purchase an ordinary television set, 15 percent will purchase a plasma television set, and 40 percent will just be browsing. If 5 customers enter his store on a given day, what is the probability that he will sell exactly 2 ordinary sets and 1 plasma set on that day?

6.12. The number of people that enter a drugstore in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drugstore, given that 10 women entered in that hour. What assumptions have you made?

6.13. A man and a woman agree to meet at a certain location about 12:30 P.M. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1 P.M., find the probability that the first to arrive waits no longer than 5 minutes. What is the probability that the man arrives first?

6.14. An ambulance travels back and forth at a constant speed along a road of length L. At a certain moment of time, an accident occurs at a point uniformly distributed on the road. [That is, the distance of the point from one of the fixed ends of the road is uniformly distributed over $(0, L)$.] Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, and assuming independence of the variables, compute the distribution of the distance of the ambulance from the accident.

6.15. The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if, for some constant c, its joint density is

$$f(x, y) = \begin{cases} c & \text{if}(x, y) \in R \\ 0 & \text{otherwise} \end{cases}$$

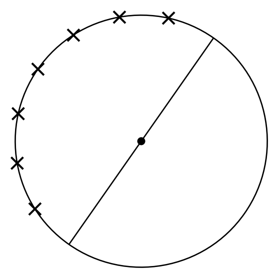
(a) Show that $1/c =$ area of region R.

Suppose that (X, Y) is uniformly distributed over the square centered at (0, 0) and with sides of length 2.

(b) Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.

(c) What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \le 1\}$.

6.16. Suppose that n points are independently chosen at random on the circumference of a circle, and we want the probability that they all lie in some semicircle. That is, we want the probability that there is a line passing through the center of the circle such that all the points are on one side of that line, as shown in the following diagram:



Let $P_1, \ldots, P_n$ denote the n points. Let A denote the event that all the points are contained in some semicircle, and let A_i be the event that all the points lie in the semicircle beginning at the point P_i and going clockwise for $180°$, $i = 1, \ldots, n$.

(a) Express A in terms of the A_i.

(b) Are the A_i mutually exclusive?

(c) Find $P(A)$.

6.17. Three points X_1, X_2, X_3 are selected at random on a line L. What is the probability that X_2 lies between X_1 and X_3?

6.18. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$.] Find the probability that the distance between the two points is greater than $L/3$.

6.19. Show that $f(x, y) = 1/x$, $0 < y < x < 1$, is a joint density function. Assuming that f is the joint density function of X, Y, find

(a) the marginal density of Y;

(b) the marginal density of X;

(c) $E[X]$;

(c) $E[Y]$.

6.20. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} xe^{-(x+y)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? If, instead, $f(x, y)$ were given by

$$f(x, y) = \begin{cases} 2 & 0 < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

would X and Y be independent?

6.21. Let

$$f(x, y) = 24xy \quad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x + y \le 1$$

and let it equal 0 otherwise.

(a) Show that $f(x, y)$ is a joint probability density function.

(b) Find $E[X]$.

(c) Find $E[Y]$.

6.22. The joint density function of X and Y is

$$f(x, y) = \begin{cases} x + y & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?

(b) Find the density function of X.

(c) Find $P\{X + Y < 1\}$.

6.23. The random variables X and Y have joint density function

$$f(x, y) = 12xy(1 - x) \quad 0 < x < 1,\ 0 < y < 1$$

and equal to 0 otherwise.

(a) Are X and Y independent?

(b) Find $E[X]$.

(c) Find $E[Y]$.

(d) Find $\mathrm{Var}(X)$.

(e) Find $\mathrm{Var}(Y)$.

6.24. Consider independent trials, each of which results in outcome $i, i = 0, 1, \ldots, k$, with probability $p_i, \sum_{i=0}^{k} p_i = 1$. Let N denote the number of trials needed to obtain an outcome that is not equal to 0, and let X be that outcome.

(a) Find $P\{N = n\}, n \ge 1$.

(b) Find $P\{X = j\}, j = 1, \ldots, k$.

(c) Show that $P\{N = n, X = j\} = P\{N = n\}P\{X = j\}$.

(d) Is it intuitive to you that N is independent of X?

(e) Is it intuitive to you that X is independent of N?

6.25. Suppose that 10^6 people arrive at a service station at times that are independent random variables, each of which is uniformly distributed over $(0, 10^6)$. Let N denote the number that arrive in the first hour. Find an approximation for $P\{N = i\}$.

6.26. Suppose that A, B, C, are independent random variables, each being uniformly distributed over $(0, 1)$.
- **(a)** What is the joint cumulative distribution function of A, B, C?
- **(b)** What is the probability that all of the roots of the equation $Ax^2 + Bx + C = 0$ are real?

6.27. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Also compute $P\{X_1 < X_2\}$.

6.28. The time that it takes to service a car is an exponential random variable with rate 1.
- **(a)** If A. J. brings his car in at time 0 and M. J. brings her car in at time t, what is the probability that M. J.'s car is ready before A. J.'s car? (Assume that service times are independent and service begins upon arrival of the car.)
- **(b)** If both cars are brought in at time 0, with work starting on M. J.'s car only when A. J.'s car has been completely serviced, what is the probability that M. J.'s car is ready before time 2?

6.29. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that
- **(a)** the total gross sales over the next 2 weeks exceeds \$5000;
- **(b)** weekly sales exceed \$2000 in at least 2 of the next 3 weeks?

What independence assumptions have you made?

6.30. Jill's bowling scores are approximately normally distributed with mean 170 and standard deviation 20, while Jack's scores are approximately normally distributed with mean 160 and standard deviation 15. If Jack and Jill each bowl one game, then assuming that their scores are independent random variables, approximate the probability that
- **(a)** Jack's score is higher;
- **(b)** the total of their scores is above 350.

6.31. According to the U.S. National Center for Health Statistics, 25.2 percent of males and 23.6 percent of females never eat breakfast. Suppose that random samples of 200 men and 200 women are chosen. Approximate the probability that
- **(a)** at least 110 of these 400 people never eat breakfast;
- **(b)** the number of the women who never eat breakfast is at least as large as the number of the men who never eat breakfast.

6.32. The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that an article of 10 pages contains (a) 0 and (b) 2 or more typographical errors? Explain your reasoning!

6.33. The monthly worldwide average number of airplane crashes of commercial airlines is 2.2. What is the probability that there will be
- **(a)** more than 2 such accidents in the next month?
- **(b)** more than 4 such accidents in the next 2 months?
- **(c)** more than 5 such accidents in the next 3 months?

Explain your reasoning!

6.34. Jay has two jobs to do, one after the other. Each attempt at job i takes one hour and is successful with probability p_i. If $p_1 = .3$ and $p_2 = .4$, what is the probability that it will take Jay more than 12 hours to be successful on both jobs?

6.35. In Problem 4, calculate the conditional probability mass function of X_1 given that
- **(a)** $X_2 = 1$;
- **(b)** $X_2 = 0$.

6.36. In Problem 3, calculate the conditional probability mass function of Y_1 given that
- **(a)** $Y_2 = 1$;
- **(b)** $Y_2 = 0$.

6.37. In Problem 5, calculate the conditional probability mass function of Y_1 given that
- **(a)** $Y_2 = 1$;
- **(b)** $Y_2 = 0$.

6.38. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1, \ldots, X\}$. Call this second number Y.
- **(a)** Find the joint mass function of X and Y.
- **(b)** Find the conditional mass function of X given that $Y = i$. Do it for $i = 1, 2, 3, 4, 5$.
- **(c)** Are X and Y independent? Why?

6.39. Two dice are rolled. Let X and Y denote, respectively, the largest and smallest values obtained. Compute the conditional mass function of Y given $X = i$, for $i = 1, 2, \ldots, 6$. Are X and Y independent? Why?

6.40. The joint probability mass function of X and Y is given by

$$p(1,1) = \frac{1}{8} \qquad p(1,2) = \frac{1}{4}$$
$$p(2,1) = \frac{1}{8} \qquad p(2,2) = \frac{1}{2}$$

- **(a)** Compute the conditional mass function of X given $Y = i, i = 1, 2$.
- **(b)** Are X and Y independent?

(c) Compute $P\{XY \leq 3\}, P\{X + Y > 2\}, P\{X/Y > 1\}$.

6.41. The joint density function of X and Y is given by

$$f(x, y) = xe^{-x(y+1)} \quad x > 0, \ y > 0$$

(a) Find the conditional density of X, given $Y = y$, and that of Y, given $X = x$.
(b) Find the density function of $Z = XY$.

6.42. The joint density of X and Y is

$$f(x, y) = c(x^2 - y^2)e^{-x} \quad 0 \leq x < \infty, \ -x \leq y \leq x$$

Find the conditional distribution of Y, given $X = x$.

6.43. An insurance company supposes that each person has an accident parameter and that the yearly number of accidents of someone whose accident parameter is λ is Poisson distributed with mean λ. They also suppose that the parameter value of a newly insured person can be assumed to be the value of a gamma random variable with parameters s and α. If a newly insured person has n accidents in her first year, find the conditional density of her accident parameter. Also, determine the expected number of accidents that she will have in the following year.

6.44. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over (0, 1), compute the probability that the largest of the three is greater than the sum of the other two.

6.45. A complex machine is able to operate effectively as long as at least 3 of its 5 motors are functioning. If each motor independently functions for a random amount of time with density function $f(x) = xe^{-x}, x > 0$, compute the density function of the length of time that the machine functions.

6.46. If 3 trucks break down at points randomly distributed on a road of length L, find the probability that no 2 of the trucks are within a distance d of each other when $d \leq L/2$.

6.47. Consider a sample of size 5 from a uniform distribution over (0, 1). Compute the probability that the median is in the interval $\left(\frac{1}{4}, \frac{3}{4}\right)$.

6.48. If X_1, X_2, X_3, X_4, X_5 are independent and identically distributed exponential random variables with the parameter λ, compute
(a) $P\{\min(X_1, \ldots, X_5) \leq a\}$;
(b) $P\{\max(X_1, \ldots, X_5) \leq a\}$.

6.49. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a set of n independent uniform (0,1) random variables. Find the conditional distribution of $X_{(n)}$ given that $X_{(1)} = s_1, X_{(2)} = s_2, \ldots, X_{(n-1)} = s_{n-1}$.

6.50. Let Z_1 and Z_2 be independent standard normal random variables. Show that X, Y has a bivariate normal distribution when $X = Z_1, \ Y = Z_1 + Z_2$.

6.51. Derive the distribution of the range of a sample of size 2 from a distribution having density function $f(x) = 2x, \ 0 < x < 1$.

6.52. Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is

$$f(x, y) = \frac{1}{\pi} \quad x^2 + y^2 \leq 1$$

Find the joint density function of the polar coordinates $R = (X^2 + Y^2)^{1/2}$ and $\Theta = \tan^{-1} Y/X$.

6.53. If X and Y are independent random variables both uniformly distributed over (0, 1), find the joint density function of $R = \sqrt{X^2 + Y^2}, \Theta = \tan^{-1} Y/X$.

6.54. If U is uniform on $(0, 2\pi)$ and Z, independent of U, is exponential with rate 1, show directly (without using the results of Example 7b) that X and Y defined by

$$X = \sqrt{2Z} \cos U$$
$$Y = \sqrt{2Z} \sin U$$

are independent standard normal random variables.

6.55. X and Y have joint density function

$$f(x, y) = \frac{1}{x^2 y^2} \quad x \geq 1, \ y \geq 1$$

(a) Compute the joint density function of $U = XY, V = X/Y$.
(b) What are the marginal densities?

6.56. If X and Y are independent and identically distributed uniform random variables on (0, 1), compute the joint density of
(a) $U = X + Y, V = X/Y$;
(b) $U = X, V = X/Y$;
(c) $U = X + Y, V = X/(X + Y)$.

6.57. Repeat Problem 6.56 when X and Y are independent exponential random variables, each with parameter $\lambda = 1$.

6.58. If X_1 and X_2 are independent exponential random variables, each having parameter λ, find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$.

6.59. If X, Y, and Z are independent random variables having identical density functions $f(x) = e^{-x}, 0 < x < \infty$, derive the joint distribution of $U = X + Y, \ V = X + Z, W = Y + Z$.

6.60. In Example 8b, let $Y_{k+1} = n + 1 - \sum_{i=1}^{k} Y_i$. Show that $Y_1, \ldots, Y_k, Y_{k+1}$ are exchangeable. Note that Y_{k+1} is the number of balls one must observe to obtain a special ball if one considers the balls in their reverse order of withdrawal.

6.61. Consider an urn containing n balls numbered $1, \ldots, n$, and suppose that k of them are randomly withdrawn. Let X_i equal 1 if ball number i is removed and let X_i be 0 otherwise. Show that $X_1, \ldots, X_n$ are exchangeable.

THEORETICAL EXERCISES

6.1. Verify Equation (1.2).

6.2. Suppose that the number of events occurring in a given time period is a Poisson random variable with parameter λ. If each event is classified as a type i event with probability p_i, $i = 1, \ldots, n$, $\sum p_i = 1$, independently of other events, show that the numbers of type i events that occur, $i = 1, \ldots, n$, are independent Poisson random variables with respective parameters λp_i, $i = 1, \ldots, n$.

6.3. Suggest a procedure for using Buffon's needle problem to estimate π. Surprisingly enough, this was once a common method of evaluating π.

6.4. Solve Buffon's needle problem when $L > D$.
ANSWER: $\dfrac{2L}{\pi D}(1 - \sin\theta) + 2\theta/\pi$, where $\cos\theta = D/L$.

6.5. If X and Y are independent continuous positive random variables, express the density function of (a) $Z = X/Y$ and (b) $Z = XY$ in terms of the density functions of X and Y. Evaluate the density functions in the special case where X and Y are both exponential random variables.

6.6. If X and Y are jointly continuous with joint density function $f_{X,Y}(x, y)$, show that $X + Y$ is continuous with density function

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_{X,Y}(x, t - x)\, dx$$

6.7. **(a)** If X has a gamma distribution with parameters (t, λ), what is the distribution of cX, $c > 0$?
(b) Show that

$$\frac{1}{2\lambda}\chi^2_{2n}$$

has a gamma distribution with parameters n, λ when n is a positive integer and χ^2_{2n} is a chi-squared random variable with $2n$ degrees of freedom.

6.8. Let X and Y be independent continuous random variables with respective hazard rate functions $\lambda_X(t)$ and $\lambda_Y(t)$, and set $W = \min(X, Y)$.
(a) Determine the distribution function of W in terms of those of X and Y.
(b) Show that $\lambda_W(t)$, the hazard rate function of W, is given by

$$\lambda_W(t) = \lambda_X(t) + \lambda_Y(t)$$

6.9. Let $X_1, \ldots, X_n$ be independent exponential random variables having a common parameter λ. Determine the distribution of $\min(X_1, \ldots, X_n)$.

6.10. The lifetimes of batteries are independent exponential random variables, each having parameter λ. A flashlight needs 2 batteries to work. If one has a flashlight and a stockpile of n batteries, what is the distribution of time that the flashlight can operate?

6.11. Let X_1, X_2, X_3, X_4, X_5 be independent continuous random variables having a common distribution function F and density function f, and set

$$I = P\{X_1 < X_2 < X_3 < X_4 < X_5\}$$

(a) Show that I does not depend on F.
Hint: Write I as a five-dimensional integral and make the change of variables $u_i = F(x_i)$, $i = 1, \ldots, 5$.
(b) Evaluate I.
(c) Give an intuitive explanation for your answer to (b).

6.12. Show that the jointly continuous (discrete) random variables $X_1, \ldots, X_n$ are independent if and only if their joint probability density (mass) function $f(x_1, \ldots, x_n)$ can be written as

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} g_i(x_i)$$

for nonnegative functions $g_i(x)$, $i = 1, \ldots, n$.

6.13. In Example 5c we computed the conditional density of a success probability for a sequence of trials when the first $n + m$ trials resulted in n successes. Would the conditional density change if we specified which n of these trials resulted in successes?

6.14. Suppose that X and Y are independent geometric random variables with the same parameter p.

(a) Without any computations, what do you think is the value of

$$P\{X = i|X + Y = n\}?$$

Hint: Imagine that you continually flip a coin having probability p of coming up heads. If the second head occurs on the nth flip, what is the probability mass function of the time of the first head?

(b) Verify your conjecture in part (a).

6.15. Consider a sequence of independent trials, with each trial being a success with probability p. Given that the kth success occurs on trial n, show that all possible outcomes of the first $n - 1$ trials that consist of $k - 1$ successes and $n - k$ failures are equally likely.

6.16. If X and Y are independent binomial random variables with identical parameters n and p, show analytically that the conditional distribution of X given that $X + Y = m$ is the hypergeometric distribution. Also, give a second argument that yields the same result without any computations.
Hint: Suppose that $2n$ coins are flipped. Let X denote the number of heads in the first n flips and Y the number in the second n flips. Argue that given a total of m heads, the number of heads in the first n flips has the same distribution as the number of white balls selected when a sample of size m is chosen from n white and n black balls.

6.17. Suppose that $X_i, i = 1, 2, 3$ are independent Poisson random variables with respective means $\lambda_i, i = 1, 2, 3$. Let $X = X_1 + X_2$ and $Y = X_2 + X_3$. The random vector X, Y is said to have a bivariate Poisson distribution. Find its joint probability mass function. That is, find $P\{X = n, Y = m\}$.

6.18. Suppose X and Y are both integer-valued random variables. Let

$$p(i|j) = P(X = i|Y = j)$$

and

$$q(j|i) = P(Y = j|X = i)$$

Show that

$$P(X = i, Y = j) = \frac{p(i|j)}{\sum_i \frac{p(i|j)}{q(j|i)}}$$

6.19. Let X_1, X_2, X_3 be independent and identically distributed continuous random variables. Compute

(a) $P\{X_1 > X_2|X_1 > X_3\}$;

(b) $P\{X_1 > X_2|X_1 < X_3\}$;

(c) $P\{X_1 > X_2|X_2 > X_3\}$;

(d) $P\{X_1 > X_2|X_2 < X_3\}$.

6.20. Let U denote a random variable uniformly distributed over (0, 1). Compute the conditional distribution of U given that

(a) $U > a$;

(b) $U < a$;

where $0 < a < 1$.

6.21. Suppose that W, the amount of moisture in the air on a given day, is a gamma random variable with parameters (t, β). That is, its density is $f(w) = \beta e^{-\beta w}(\beta w)^{t-1}/\Gamma(t), w > 0$. Suppose also that given that $W = w$, the number of accidents during that day—call it N—has a Poisson distribution with mean w. Show that the conditional distribution of W given that $N = n$ is the gamma distribution with parameters $(t + n, \beta + 1)$.

6.22. Let W be a gamma random variable with parameters (t, β), and suppose that conditional on $W = w, X_1, X_2, \ldots, X_n$ are independent exponential random variables with rate w. Show that the conditional distribution of W given that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ is gamma with parameters $\left(t + n, \beta + \sum_{i=1}^{n} x_i\right)$.

6.23. A rectangular array of mn numbers arranged in n rows, each consisting of m columns, is said to contain a *saddlepoint* if there is a number that is both the minimum of its row and the maximum of its column. For instance, in the array

$$\begin{array}{rrr} 1 & 3 & 2 \\ 0 & -2 & 6 \\ .5 & 12 & 3 \end{array}$$

the number 1 in the first row, first column is a saddlepoint. The existence of a saddlepoint is of significance in the theory of games. Consider a rectangular array of numbers as described previously and suppose that there are two individuals—A and B—that are playing the following game: A is to choose one of the numbers $1, 2, \ldots, n$ and B one of the numbers $1, 2, \ldots, m$. These choices are announced simultaneously, and if A chose i and B chose j, then A wins from B the amount specified by the number in the ith row, jth column of the array. Now suppose that the array contains a saddlepoint—say the number in the row r and column k—call this number x_{rk}. Now if player A chooses row r, then that player can guarantee herself a win of at least x_{rk} (since x_{rk} is the minimum number in the row r). On the other hand, if player B chooses column k, then he can guarantee that he will lose no more than x_{rk} (since x_{rk} is the maximum number in the column k). Hence, as A has a way of playing that guarantees her a win of x_{rk} and as B has a way of playing that guarantees he will lose no more than x_{rk}, it seems reasonable to take these two strategies as being optimal and declare that the value of the game to player A is x_{rk}.

If the nm numbers in the rectangular array described are independently chosen from an arbitrary continuous distribution, what is the probability that the resulting array will contain a saddle-point?

6.24. If X is exponential with rate λ, find $P\{[X] = n, X - [X] \leq x\}$, where $[x]$ is defined as the largest integer less than or equal to x. Can you conclude that $[X]$ and $X - [X]$ are independent?

6.25. Suppose that $F(x)$ is a cumulative distribution function. Show that (a) $F^n(x)$ and (b) $1 - [1 - F(x)]^n$ are also cumulative distribution functions when n is a positive integer.
Hint: Let $X_1, \ldots, X_n$ be independent random variables having the common distribution function F. Define random variables Y and Z in terms of the X_i so that $P\{Y \leq x\} = F^n(x)$ and $P\{Z \leq x\} = 1 - [1 - F(x)]^n$.

6.26. Show that if n people are distributed at random along a road L miles long, then the probability that no 2 people are less than a distance D miles apart is when $D \leq L/(n - 1)$, $[1 - (n - 1)D/L]^n$. What if $D > L/(n - 1)$?

6.27. Establish Equation (6.2) by differentiating Equation (6.4).

6.28. Show that the median of a sample of size $2n + 1$ from a uniform distribution on (0, 1) has a beta distribution with parameters $(n + 1, n + 1)$.

6.29. Verify Equation (6.6), which gives the joint density of $X_{(i)}$ and $X_{(j)}$.

6.30. Compute the density of the range of a sample of size n from a continuous distribution having density function f.

6.31. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the ordered values of n independent uniform (0, 1) random variables. Prove that for $1 \leq k \leq n + 1$,

$$P\{X_{(k)} - X_{(k-1)} > t\} = (1 - t)^n$$

where $X_{(0)} \equiv 0, X_{(n+1)} \equiv t$.

6.32. Let $X_1, \ldots, X_n$ be a set of independent and identically distributed continuous random variables having distribution function F, and let $X_{(i)}, i = 1, \ldots, n$ denote their ordered values. If X, independent of the $X_i, i = 1, \ldots, n$, also has distribution F, determine
(a) $P\{X > X_{(n)}\}$;
(b) $P\{X > X_{(1)}\}$;
(c) $P\{X_{(i)} < X < X_{(j)}\}, 1 \leq i < j \leq n$.

6.33. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having distribution function F and density f. The quantity $M \equiv [X_{(1)} + X_{(n)}]/2$, defined to be the average of the smallest and largest values in $X_1, \ldots, X_n$, is called the *midrange* of the sequence. Show that its distribution function is

$$F_M(m) = n\int_{-\infty}^{m} [F(2m - x) - F(x)]^{n-1} f(x)\,dx$$

6.34. Let $X_1, \ldots, X_n$ be independent uniform (0, 1) random variables. Let $R = X_{(n)} - X_{(1)}$ denote the range and $M = [X_{(n)} + X_{(1)}]/2$ the midrange of $X_1, \ldots, X_n$. Compute the joint density function of R and M.

6.35. If X and Y are independent standard normal random variables, determine the joint density function of

$$U = X \quad V = \frac{X}{Y}$$

Then use your result to show that X/Y has a Cauchy distribution.

SELF-TEST PROBLEMS AND EXERCISES

6.1. Each throw of an unfair die lands on each of the odd numbers 1, 3, 5 with probability C and on each of the even numbers with probability $2C$.
(a) Find C.
(b) Suppose that the die is tossed. Let X equal 1 if the result is an even number, and let it be 0 otherwise. Also, let Y equal 1 if the result is a number greater than three and let it be 0 otherwise. Find the joint probability mass function of X and Y. Suppose now that 12 independent tosses of the die are made.
(c) Find the probability that each of the six outcomes occurs exactly twice.
(d) Find the probability that 4 of the outcomes are either one or two, 4 are either three or four, and 4 are either five or six.
(e) Find the probability that at least 8 of the tosses land on even numbers.

6.2. The joint probability mass function of the random variables X, Y, Z is

$$p(1,2,3) = p(2,1,1) = p(2,2,1) = p(2,3,2) = \frac{1}{4}$$

Find (a) $E[XYZ]$, and (b) $E[XY + XZ + YZ]$.

6.3. The joint density of X and Y is given by

$$f(x,y) = C(y - x)e^{-y} \quad -y < x < y, \quad 0 < y < \infty$$

(a) Find C.
(b) Find the density function of X.
(c) Find the density function of Y.
(d) Find $E[X]$.
(e) Find $E[Y]$.

6.4. Let $r = r_1 + \ldots + r_k$, where all r_i are positive integers. Argue that if $X_1, \ldots, X_r$ has a multinomial distribution, then so does $Y_1, \ldots, Y_k$ where, with $r_0 = 0$,

$$Y_i = \sum_{j=r_{i-1}+1}^{r_{i-1}+r_i} X_j, \quad i \le k$$

That is, Y_1 is the sum of the first r_1 of the $X's$, Y_2 is the sum of the next r_2, and so on.

6.5. Suppose that X, Y, and Z are independent random variables that are each equally likely to be either 1 or 2. Find the probability mass function of (a) XYZ, (b) $XY + XZ + YZ$, and (c) $X^2 + YZ$.

6.6. Let X and Y be continuous random variables with joint density function

$$f(x, y) = \begin{cases} \dfrac{x}{5} + cy & 0 < x < 1,\ 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.
(a) What is the value of c?
(b) Are X and Y independent?
(c) Find $P\{X + Y > 3\}$.

6.7. The joint density function of X and Y is

$$f(x, y) = \begin{cases} xy & 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?
(b) Find the density function of X.
(c) Find the density function of Y.
(d) Find the joint distribution function.
(e) Find $E[Y]$.
(f) Find $P\{X + Y < 1\}$.

6.8. Consider two components and three types of shocks. A type 1 shock causes component 1 to fail, a type 2 shock causes component 2 to fail, and a type 3 shock causes both components 1 and 2 to fail. The times until shocks 1, 2, and 3 occur are independent exponential random variables with respective rates λ_1, λ_2, and λ_3. Let X_i denote the time at which component i fails, $i = 1, 2$. The random variables X_1, X_2 are said to have a joint bivariate exponential distribution. Find $P\{X_1 > s, X_2 > t\}$.

6.9. Consider a directory of classified advertisements that consists of m pages, where m is very large. Suppose that the number of advertisements per page varies and that your only method of finding out how many advertisements there are on a specified page is to count them. In addition, suppose that there are too many pages for it to be feasible to make a complete count of the total number of advertisements and that your objective is to choose a directory advertisement in such a way that each of them has an equal chance of being selected.

(a) If you randomly choose a page and then randomly choose an advertisement from that page, would that satisfy your objective? Why or why not?

Let $n(i)$ denote the number of advertisements on page $i, i = 1, \ldots, m$, and suppose that whereas these quantities are unknown, we can assume that they are all less than or equal to some specified value n. Consider the following algorithm for choosing an advertisement.

Step 1. Choose a page at random. Suppose it is page X. Determine $n(X)$ by counting the number of advertisements on page X.
Step 2. "Accept" page X with probability $n(X)/n$. If page X is accepted, go to step 3. Otherwise, return to step 1.
Step 3. Randomly choose one of the advertisements on page X.

Call each pass of the algorithm through step 1 an iteration. For instance, if the first randomly chosen page is rejected and the second accepted, than we would have needed 2 iterations of the algorithm to obtain an advertisement.

(b) What is the probability that a single iteration of the algorithm results in the acceptance of an advertisement on page i?
(c) What is the probability that a single iteration of the algorithm results in the acceptance of an advertisement?
(d) What is the probability that the algorithm goes through k iterations, accepting the jth advertisement on page i on the final iteration?
(e) What is the probability that the jth advertisement on page i is the advertisement obtained from the algorithm?
(f) What is the expected number of iterations taken by the algorithm?

6.10. The "random" parts of the algorithm in Self-Test Problem 8 can be written in terms of the generated values of a sequence of independent uniform (0, 1) random variables, known as random numbers. With $[x]$ defined as the largest integer less than or equal to x, the first step can be written as follows:

Step 1. Generate a uniform (0, 1) random variable U. Let $X = [mU] + 1$, and determine the value of $n(X)$.

(a) Explain why the above is equivalent to step 1 of Problem 8.

Hint: What is the probability mass function of X?

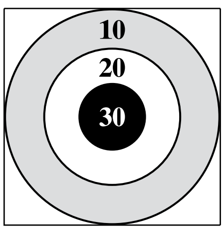
(b) Write the remaining steps of the algorithm in a similar style.

6.11. Let $X_1, X_2, \ldots$ be a sequence of independent uniform (0, 1) random variables. For a fixed constant c, define the random variable N by

$$N = \min\{n : X_n > c\}$$

Is N independent of X_N? That is, does knowing the value of the first random variable that is greater than c affect the probability distribution of when this random variable occurs? Give an intuitive explanation for your answer.

6.12. The accompanying dartboard is a square whose sides are of length 6:



The three circles are all centered at the center of the board and are of radii 1, 2, and 3, respectively. Darts landing within the circle of radius 1 score 30 points, those landing outside this circle, but within the circle of radius 2, are worth 20 points, and those landing outside the circle of radius 2, but within the circle of radius 3, are worth 10 points. Darts that do not land within the circle of radius 3 do not score any points. Assuming that each dart that you throw will, independently of what occurred on your previous throws, land on a point uniformly distributed in the square, find the probabilities of the accompanying events:

(a) You score 20 on a throw of the dart.

(b) You score at least 20 on a throw of the dart.

(c) You score 0 on a throw of the dart.

(d) The expected value of your score on a throw of the dart.

(e) Both of your first two throws score at least 10.

(f) Your total score after two throws is 30.

6.13. A model proposed for NBA basketball supposes that when two teams with roughly the same record play each other, the number of points scored in a quarter by the home team minus the number scored by the visiting team is approximately a normal random variable with mean 1.5 and variance 6. In addition, the model supposes that the point differentials for the four quarters are independent. Assume that this model is correct.

(a) What is the probability that the home team wins?

(b) What is the conditional probability that the home team wins, given that it is behind by 5 points at halftime?

(c) What is the conditional probability that the home team wins, given that it is ahead by 5 points at the end of the first quarter?

6.14. Let N be a geometric random variable with parameter p. Suppose that the conditional distribution of X given that $N = n$ is the gamma distribution with parameters n and λ. Find the conditional probability mass function of N given that $X = x$.

6.15. Let X and Y be independent uniform (0, 1) random variables.

(a) Find the joint density of $U = X, V = X + Y$.

(b) Use the result obtained in part (a) to compute the density function of V.

6.16. You and three other people are to place bids for an object, with the high bid winning. If you win, you plan to sell the object immediately for 10 thousand dollars. How much should you bid to maximize your expected profit if you believe that the bids of the others can be regarded as being independent and uniformly distributed between 7 and 11 thousand dollars?

6.17. Find the probability that $X_1, X_2, \ldots, X_n$ is a permutation of $1, 2, \ldots, n$, when $X_1, X_2, \ldots, X_n$ are independent and

(a) each is equally likely to be any of the values $1, \ldots, n$;

(b) each has the probability mass function $P\{X_i = j\} = p_j, j = 1, \ldots, n$.

6.18. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent random vectors, with each vector being a random ordering of k ones and $n - k$ zeroes. That is, their joint probability mass functions are

$$P\{X_1 = i_1, \ldots, X_n = i_n\} = P\{Y_1 = i_1, \ldots, Y_n = i_n\} = \frac{1}{\binom{n}{k}}, \quad i_j = 0, 1, \sum_{j=1}^{n} i_j = k$$

Let

$$N = \sum_{i=1}^{n} |X_i - Y_i|$$

denote the number of coordinates at which the two vectors have different values. Also, let M denote the number of values of i for which $X_i = 1, Y_i = 0$.

(a) Relate N to M.
(b) What is the distribution of M?
(c) Find $E[N]$.
(d) Find $\text{Var}(N)$.

***6.19.** Let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal random variables, and let

$$S_j = \sum_{i=1}^{j} Z_i$$

(a) What is the conditional distribution of S_n given that $S_k = y$, for $k = 1, \ldots, n$?
(b) Show that, for $1 \le k \le n$, the conditional distribution of S_k given that $S_n = x$ is normal with mean xk/n and variance $k(n - k)/n$.

6.20. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables. Find

(a) $P\{X_6 > X_1 | X_1 = \max(X_1, \ldots, X_5)\}$
(b) $P\{X_6 > X_2 | X_1 = \max(X_1, \ldots, X_5)\}$