Page 1 Note 2.

Methods of finding tests:

- Likelihood ratio tests
- Union-intersection / Intersection-union tests
- Bayes test

1. Likelihood ratio tests:

Unrestricted likelihood maximum

$$\sup_{\theta \in \Theta} L(\theta | \mathbf{x})$$

Null-restricted likelihood maximum

$$\sup_{\theta \in \Theta_0} L(\theta | \mathbf{x})$$

The likelihood ratio test statistic for

$$\begin{cases} H_0: \theta \in \Theta_0 \\ H_1: \theta \in \Theta_1 \end{cases}$$

is

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta | \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta | \mathbf{x})} \begin{matrix} \leftarrow \text{null-restricted} \\ \leftarrow \text{unrestricted} \end{matrix}$$

Likelihood ratio test: H_0 is rejected iff $\lambda(\mathbf{x}) < c$ for some $0 \le c < 1$.

Usually, we find two mles:

$$\hat{\theta}_0 = \underset{\theta \in \Theta_0}{\operatorname{argmax}} \; L(\theta | \underset{\sim}{x}) \qquad \text{and}$$

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmax}} \; L(\theta | \underset{\sim}{x}).$$

Then $\lambda(\underset{\sim}{x}) = \dfrac{L(\hat{\theta}_0 | \underset{\sim}{x})}{L(\hat{\theta} | \underset{\sim}{x})}$.

Example 1. $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 is known

Find the LR statistic for

$$\begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta \neq \theta_0 \end{cases}.$$

Here $\Theta_0 = \{\theta_0\}$. $\Theta_1 = (-\infty, \theta_0) \cup (\theta_0, +\infty)$.

Recall that

$$L(\theta | \underset{\sim}{x}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \theta)^2\right\}$$

Null-restricted mle:

$$\hat{\theta}_0 = \underset{\theta \in \Theta_0}{\operatorname{argmax}} \; L(\theta | \underset{\sim}{x}) = \theta_0$$

Unrestricted mle:

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmax}} \; L(\theta | \underset{\sim}{x}) = \bar{x}.$$

Then

$$\lambda(\underset{\sim}{x}) = \frac{L(\hat{\theta}_0 \mid \underset{\sim}{x})}{L(\hat{\theta} \mid \underset{\sim}{x})}$$

$$= \frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\theta_0)^2\right\}}{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\bar{x})^2\right\}}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^n (x_i-\theta_0)^2 - \sum_{i=1}^n (x_i-\bar{x})^2\right]\right\}$$

It's straightforward to verify

$$\sum_{i=1}^n (x_i-\theta_0)^2 = \sum_{i=1}^n (x_i-\bar{x})^2 + n(\bar{x}-\theta_0)^2.$$

Then

$$\lambda(\underset{\sim}{x}) = \exp\left\{-\frac{1}{2\sigma^2} n(\bar{x}-\theta_0)^2\right\}$$

The rejection region:

$$\{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) \le c\} \Longleftrightarrow \left\{\underset{\sim}{x} : |\bar{x}-\theta_0| \ge \sqrt{\frac{-2\sigma^2 \log c}{n}}\right\}$$

$$\Longleftrightarrow \left\{\underset{\sim}{x} : \sqrt{n}\left|\frac{\bar{x}-\theta_0}{\sigma}\right| \ge \sqrt{-2\log c}\right\}.$$

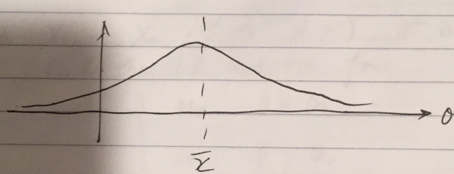
Example 2: Under the same setup. Find the LR statistic for

$$\begin{cases} H_0: \theta \le \theta_0 \\ H_1: \theta > \theta_0 \end{cases}$$

Here $\Theta_0 = (-\infty, \theta_0]$

$\Theta_1 = (\theta_0, \infty)$

Recall the graph of $L(\theta | \mathbf{x})$.



Null-restricted mle:

$$\hat{\theta}_0 = \underset{\theta \le \theta_0}{\operatorname{argmax}}\ L(\theta | \mathbf{x}) = \begin{cases} \theta_0 & \text{if } \theta_0 < \bar{x} \\ \bar{x} & \text{if } \theta_0 \ge \bar{x} \end{cases}$$

Unrestricted mle:

$\hat{\theta} = \bar{x}$.

Then $\lambda(\underset{\sim}{x}) = \dfrac{L(\hat{\theta}_0 | \underset{\sim}{x})}{L(\hat{\theta} | \underset{\sim}{x})} = \begin{cases} 1 & \text{if } \bar{x} \le \theta_0 \\ \exp\left(-\frac{1}{2\sigma^2} n(\bar{x}-\theta_0)^2\right) & \text{if } \bar{x} > \theta_0 \end{cases}$

For $0 < c \le 1$, the rejection region becomes

$$\{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) \le c\} = \left\{\underset{\sim}{x} : \sqrt{n}\left|\frac{\bar{x}-\theta_0}{\sigma}\right| \ge \sqrt{-2\log c},\ \bar{x} > \theta_0\right\}$$

$$= \left\{\underset{\sim}{x} : \sqrt{n}\left(\frac{\bar{x}-\theta_0}{\sigma}\right) \ge \sqrt{-2\log c}\right\}$$

Example 3: $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 is unknown.
Find the LR statistic for

$$\begin{cases} H_0 : \theta \le \theta_0 \\ H_1 : \theta > \theta_0 \end{cases}$$

Note that we now have two parameters, θ and σ^2

$$\Theta_0 = \{(\theta, \sigma^2) : \theta \le \theta_0\}$$
$$\Theta_1 = \{(\theta, \sigma^2) : \theta > \theta_0\}$$

The likelihood function is

$$L(\theta, \sigma^2 | \underset{\sim}{x}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\theta)^2\right\}.$$

For the unrestricted mle,

It's well known that

$$\hat{\theta} = \bar{x}, \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

$$L(\hat{\theta}, \hat{\sigma}^2 \mid \mathbf{x}) = \left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\hat{\sigma}^2} n\hat{\sigma}^2\right\}$$

$$= \left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp\left\{-\frac{n}{2}\right\}$$

Under the null restriction $\theta \le \theta_0$. Notice that we can first obtain

$$\hat{\theta}_{res} = \begin{cases} \theta_0 & \text{if } \theta_0 < \bar{x} \\ \bar{x} & \text{if } \theta_0 \ge \bar{x} \end{cases}.$$

Then

$$\hat{\sigma}^2_{res} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\theta}_{res})^2 = \begin{cases} \frac{1}{n}\sum_{i=1}^{n}(x_i - \theta_0)^2 := \hat{\sigma}_0^2 & \text{if } \theta_0 < \bar{x} \\ \hat{\sigma}^2 & \text{if } \theta_0 \ge \bar{x}. \end{cases}$$

$$\Rightarrow L(\hat{\theta}_{res}, \hat{\sigma}^2_{res} \mid \mathbf{x}) = \begin{cases} \left(\frac{1}{\hat{\sigma}_0\sqrt{2\pi}}\right)^n \exp\left(-\frac{n}{2}\right) & \text{if } \theta_0 < \bar{x} \\ \left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp\left(-\frac{n}{2}\right) & \text{if } \theta_0 \ge \bar{x} \end{cases}$$

Then

$$\lambda(\mathbf{x}) = \frac{L(\hat{\theta}_{res}, \hat{\sigma}^2_{res} \mid \mathbf{x})}{L(\hat{\theta}, \hat{\sigma}^2 \mid \mathbf{x})} = \begin{cases} \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^n & \text{if } \theta_0 < \bar{x} \\ 1 & \text{if } \theta_0 \ge \bar{x}. \end{cases}$$

Then for any $0 < c < 1$, H_0 is rejected iff

$$\lambda(\underset{\sim}{x}) < c \iff \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} < c \quad \text{and} \quad \bar{x} > \theta_0$$

Note that

$$n\hat{\sigma}_0^2 = \sum_{i=1}^{n} (x_i - \theta_0)^2$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \theta_0)^2$$

$$= n\hat{\sigma}^2 + n(\bar{x} - \theta_0)^2$$

$$\Longrightarrow \hat{\sigma}_0^2 = \hat{\sigma}^2 + (\bar{x} - \theta_0)^2$$

Then

$$\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2} = \frac{\hat{\sigma}^2}{\hat{\sigma}^2 + (\bar{x} - \theta_0)^2} < c^{\frac{2}{n}}$$

$$\iff \hat{\sigma}^2 < \frac{c^{\frac{2}{n}}}{1 - c^{\frac{2}{n}}} (\bar{x} - \theta_0)^2.$$

Along with $\bar{x} > \theta_0$, we get the rejection region

$$\bar{x} - \theta_0 > \hat{\sigma}\sqrt{\frac{1 - c^{\frac{2}{n}}}{c^{\frac{2}{n}}}}.$$

Notice that $\log b \approx b - 1$ when $b \approx 1$.

Then $\sqrt{\frac{1 - c^{\frac{2}{n}}}{c^{\frac{2}{n}}}} = \sqrt{\frac{1}{c^{\frac{2}{n}}} - 1} \approx \sqrt{\log \frac{1}{c^{\frac{2}{n}}}} = \sqrt{-\frac{2}{n}\log c}$

Then when n is large, the rejection region is approximately

$$\sqrt{n}\left(\frac{\bar{x} - \theta_0}{\hat{\sigma}}\right) > \sqrt{-2\log c}.$$

Example 4 : Shift family of exponential distribution

$X_1 \cdots X_n$ iid exponential with pdf

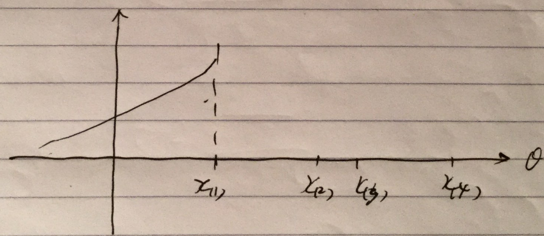
$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

Find the LR statistic for $\begin{cases} H_0 : & \theta \leq \theta_0 \\ H_1 : & \theta > \theta_0 \end{cases}$

Recall that in this case

$$L(\theta|\underline{x}) = \begin{cases} e^{-\sum_{i=1}^{n}(x_i-\theta)} & \theta \leq x_{(1)} \\ 0 & \theta > x_{(1)} \end{cases}$$

where $x_{(1)} = \min_i x_i$. The graph is



For this testing problem, $\Theta_0 = (-\infty, \theta_0]$
$\Theta_1 = (\theta_0, \infty)$.

The unrestricted mle : $\hat{\theta} = x_{(1)}$

The null-restricted mle : $\hat{\theta}_0 = \begin{cases} \theta_0 & \text{if } \theta_0 \leq x_{(1)} \\ x_{(1)} & \text{if } \theta_0 > x_{(1)} \end{cases}$

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~~Then the LR test statistic is~~

~~$\lambda(X) =$~~

The unrestricted maximum likelihood

$$L(\hat{\theta}|X) = L(x_{(1)}|X) = e^{-\sum_{i=1}^{n}(x_i - x_{(1)})}$$

The null-restricted maximum likelihood

$$L(\hat{\theta}_0|X) = \begin{cases} e^{-\sum_{i=1}^{n}(x_i - \theta_0)} & \text{if } \theta_0 \le x_{(1)} \\ e^{-\sum_{i=1}^{n}(x_i - \bar{x})} & \text{if } \theta_0 > x_{(1)} \end{cases}$$

Then the LR test statistic is

$$\lambda(X) = \frac{L(\hat{\theta}_0|X)}{L(\hat{\theta}|X)} = \begin{cases} e^{n(\theta_0 - x_{(1)})} & \text{if } x_{(1)} \ge \theta_0 \\ 1 & \text{if } x_{(1)} < \theta_0 \end{cases}$$

Then for any $0 < c < 1$, the rejection region is

$$\{\lambda(X) < c\} \Longleftrightarrow \{X : e^{n(\theta_0 - x_{(1)})} < c\}$$

$$\Longleftrightarrow \{X : x_{(1)} - \theta_0 > -\tfrac{1}{n}\log c\}$$

Note that in these examples, likelihood ratio is based on the sufficient statistic. In fact, by the factorization theorem,

$$f(\mathbf{x}|\theta) = g(T(\mathbf{x})|\theta)\, h(\mathbf{x}). \quad \text{then}$$

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta|\mathbf{x})}{\sup_{\Theta} L(\theta|\mathbf{x})} = \frac{\sup_{\Theta_0} g(T(\mathbf{x})|\theta)}{\sup_{\Theta} g(T(\mathbf{x})|\theta)} := \lambda^*(T(\mathbf{x})).$$

This implies that if the sufficient statistic is known, we might be able to find the LR test more simply.

Examples of LRT in linear models ~~and multivariate statistics~~

Example 1. Linear models.

$$Y_i = \beta_0 + \beta_1 z_{i1} + \beta_2 z_{i2} + \cdots + \beta_r z_{ir} + \varepsilon_i$$

$i = 1, \ldots, n$

where $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

In matrix form, $\vec{Y} = Z\vec{\beta} + \vec{\varepsilon}$, where

$$\vec{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & z_{11} & z_{12} & \cdots & z_{1r} \\ 1 & z_{21} & z_{22} & \cdots & z_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_{n1} & z_{n2} & \cdots & z_{nr} \end{bmatrix}$$

$$\vec{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_r \end{bmatrix}, \quad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Test: $\beta_{q+1} = \cdots = \beta_r = 0$.

Denote $\vec{z_i}^T = [1, z_{i1}, \ldots, z_{ir}]$. We have the distribution $Y_i \sim N(\vec{z_i}^T \vec{\beta}, \sigma^2)$, with

the pdf $f(y_i \mid \vec{\beta}, \sigma^2) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{1}{2\sigma^2}(y_i - \vec{z_i}^T \vec{\beta})^2\right]$

Then $f(\vec{y} \mid \vec{\beta}, \sigma^2) = \prod_{i=1}^{n} f(y_i \mid \vec{\beta}, \sigma^2)$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \vec{z}_i^T\vec{\beta})^2\right\}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\|\vec{y} - X\vec{\beta}\|^2\right\}$$

In other words

$$L(\vec{\beta}, \sigma^2 \mid \vec{y}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\|\vec{y} - Z\vec{\beta}\|^2\right\}$$

Unrestricted MLE:

$\hat{\vec{\beta}} = (Z^TZ)^{-1}Z^T\vec{y}$, $\hat{\vec{\varepsilon}} = \vec{y} - Z\hat{\vec{\beta}}$, $\hat{\sigma}^2 = \frac{1}{n}\|\hat{\vec{\varepsilon}}\|^2$

Then

$$L(\hat{\vec{\beta}}, \hat{\sigma}^2 \mid \vec{y}) = \left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\hat{\sigma}^2}\|\vec{y} - Z\hat{\vec{\beta}}\|^2\right\}$$

$$= \left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\hat{\sigma}^2} n\hat{\sigma}^2\right\}$$

$$= \left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp\left\{-\frac{n}{2}\right\}$$

Under the null model: $Z = [\overset{1+q}{Z_{(1)}} \mid \overset{r-q}{Z_{(2)}}]$

Under the null, $\beta_{q+1} = \cdots = \beta_r = 0$, similarly, we have

$$L(\hat{\vec{\beta}}_0, \hat{\sigma}_0^2 \mid \vec{y}) = \left(\frac{1}{\hat{\sigma}_0\sqrt{2\pi}}\right)^n \exp\left\{-\frac{n}{2}\right\},$$

where $\hat{\sigma}_0^2 = \frac{1}{n}\|\hat{\vec{\varepsilon}}_0\|^2 = \frac{1}{n}\|\vec{y} - Z_{(1)}(Z_{(1)}^T Z_{(1)})^{-1}Z_{(1)}^T\vec{y}\|^2$.

Then

$$\lambda(\vec{y}) = \frac{L(\hat{\vec{\beta}}_0, \hat{\sigma}_0^2 \mid \vec{y})}{L(\hat{\vec{\beta}}, \hat{\sigma}^2 \mid \vec{y})} = \frac{\left(\frac{1}{\hat{\sigma}_0\sqrt{2\pi}}\right)^n \exp(-\frac{n}{2})}{\left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp(-\frac{n}{2})} = \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^n.$$

Then the rejection region: For some $0<c<1$.

$$\lambda(\vec{y}) < c \iff \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^n < c$$

$$\iff \frac{\hat{\sigma}_0^2 - \hat{\sigma}^2}{\hat{\sigma}^2} > \left(\frac{1}{c}\right)^{\frac{2}{n}} - 1$$

$$\iff \frac{1}{\hat{\sigma}^2}\left(\|\hat{\vec{\varepsilon}}_0\|^2 - \|\hat{\vec{\varepsilon}}\|^2\right) > n\left[\left(\frac{1}{c}\right)^{\frac{2}{n}} - 1\right]$$

This is the same as F-test.

Example 2: The Two-way Layout.

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad i=1,\cdots,I,\ j=1,\cdots,J$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0,\sigma^2), \quad \sum_{i=1}^{I}\alpha_i = 0, \quad \sum_{j=1}^{J}\beta_j = 0.$$

Test: $H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$, $\quad H_1: H_0$ is not true.

As a particular example of linear model,

$$\lambda(Y) = \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^{IJ}, \text{ where}$$

$\hat{\sigma}^2$ is the MLE of σ^2 under the full model.
$\hat{\sigma}_0^2$ is the MLE of σ^2 under the null model.
Denote

$$\overline{Y}_{i\cdot} = \frac{1}{J}\sum_{j=1}^{J} Y_{ij} \quad \text{for } i=1,\cdots,I$$

$$\overline{Y}_{\cdot j} = \frac{1}{I}\sum_{i=1}^{I} Y_{ij} \quad \text{for } j=1,\cdots,J$$

$$\overline{Y}_{\cdot\cdot} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij}.$$

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We also have the sum-of-squares decomposition

$$\sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{\cdot\cdot})^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{i\cdot}-\bar{Y}_{\cdot j}+\bar{Y}_{\cdot\cdot})^2$$

$$+ J\sum_{i=1}^{I}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2$$

$$+ I\sum_{j=1}^{J}(\bar{Y}_{\cdot j}-\bar{Y}_{\cdot\cdot})^2$$

denoted as $S^2_{Tot} = S^2_{Resid} + S^2_A + S^2_B$.

Under the full model

$$\hat{\sigma}^2 = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{i\cdot}-\bar{Y}_{\cdot j}+\bar{Y}_{\cdot\cdot})^2 = \frac{1}{IJ}S^2_{Resid}$$

Under the null model $\alpha_1 = \cdots = \alpha_I = 0$,

$$\hat{\sigma}_0^2 = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{\cdot j})^2$$

$$= \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{\cdot j}-\bar{Y}_{i\cdot}+\bar{Y}_{\cdot\cdot}+\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2$$

$$= \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\Big[(Y_{ij}-\bar{Y}_{\cdot j}-\bar{Y}_{i\cdot}+\bar{Y}_{\cdot\cdot})^2 + 2(Y_{ij}-\bar{Y}_{\cdot j}-\bar{Y}_{i\cdot}+\bar{Y}_{\cdot\cdot})(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot}) + (\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2\Big]$$

$$= \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{\cdot j}-\bar{Y}_{i\cdot}+\bar{Y}_{\cdot\cdot})^2$$

$$+ \frac{2}{IJ}\sum_{i=1}^{I}\Big[(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})\underbrace{\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{\cdot j}-\bar{Y}_{i\cdot}+\bar{Y}_{\cdot\cdot})}_{=0}\Big]$$

$$+ \frac{1}{I}\sum_{i=1}^{I}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2$$

$$= \frac{1}{IJ}S^2_{Resid} + \frac{1}{IJ}S^2_A$$

Then $$\lambda(Y) = \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^{IJ} = \left(\frac{S^2_{Resid}}{S^2_{Resid} + S^2_A}\right)^{\frac{IJ}{2}}$$

Then

$$\lambda(Y) < c \iff \frac{S^2_{Resid}}{S^2_{Resid} + S^2_A} < c^{\frac{2}{IJ}} \iff \frac{S^2_A}{S^2_{Resid}} > \frac{1}{c^{\frac{2}{IJ}}} - 1 .$$