PS 3, Problem 3, 5, 7, 11, 15, 17, 24, 25

Quiz 3: all material related to PS 3.

(& possible SD of r.v).

Problem 3.

$p(A) = 0.3$

$p(B) = 0.4$.

$p(A \text{ and } B) = p(A \cap B) = 0.15$.

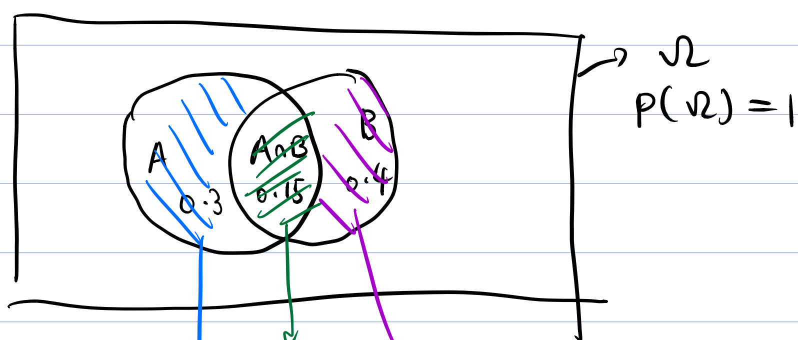
$p(A \text{ or } B)$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) \quad (1)$$

$$= 0.3 + 0.4 - 0.15$$

$$= 0.55.$$

Venn diagram



0.15 $\quad 0.15 \quad$ $0.25 = (0.4 - 0.15)$.

$= p(B \setminus A)$. "difference"

$= (0.3 - 0.15)$

$= p(A \setminus B)$

"difference" $= \{x : x \in A, x \notin B\}$.

$$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A) \qquad (2)$$
$$= 0.15 + 0.15 + 0.25$$
$$= 0.55.$$

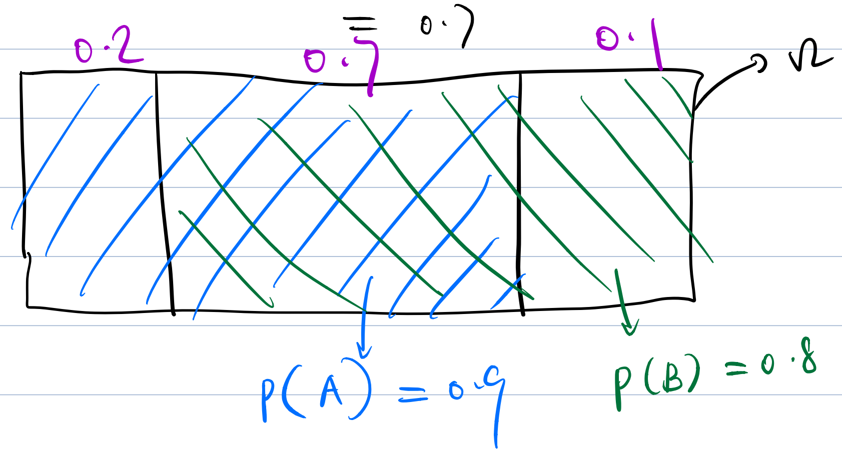
(1) and (2) are equivalent.

$$P(A) + P(B) - P(A \cap B)$$
$$= \left(P(A \setminus B) + \cancel{P(A \cap B)} \right) + \left(P(B \setminus A) + P(A \cap B) \right)$$
$$\cancel{- P(A \cap B)}$$
$$= P(A \setminus B) + P(A \cap B) + P(B \setminus A).$$

Problem 5. $P(A) = 0.9$ $P(B) = 0.8$.

(i) $P(A \cap B)$??

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \qquad (1) \checkmark$$
$$= 0.9 + 0.8 - P(A \cup B)$$
$$\geq 0.9 + 0.8 - P(\Omega)$$
$$= 0.9 + 0.8 - 1$$
$$= 0.7$$



In this case, we really have. $A \cup B = \Omega$

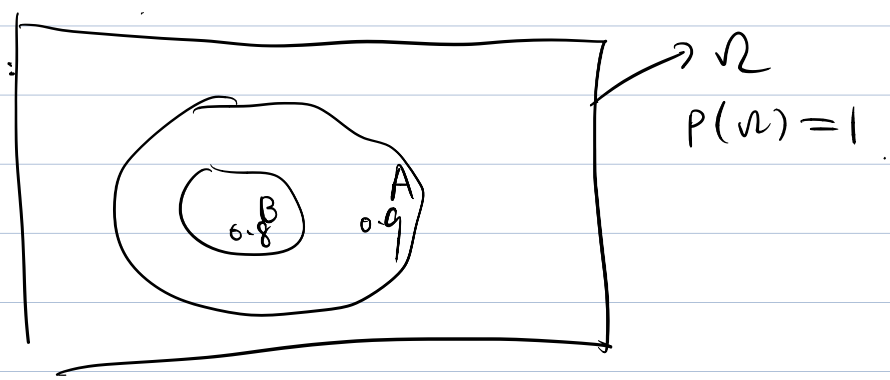
$\Rightarrow$ "=" in the inequality holds

$\Rightarrow$ 0.7 is the minimum value of $p(A \cap B)$.

If there are other possible values of $p(A \cap B)$, then we could claim that it is not determined.

- $p(A \cap B) \leq p(B) = 0.8$, because $A \cap B \subseteq B$.

Special case:



$B \subseteq A$, $B \cap A = B$.

$\Rightarrow p(B \cap A) = p(B) = 0.8$.

$0.7 \leq p(A \cap B) \leq 0.8$, both ended could be attained.

(ii) A and B mutually exclusive $\Leftrightarrow p(A \cap B) = 0$

$$\Leftrightarrow p(A) + p(B) = p(A \cup B)$$
$$= 0.9 + 0.8$$
$$= 1.7 > 1.$$

impossible. (the probability has value between 0 and 1, by definition).

$\Rightarrow$ A and B must not be mutually exclusive.

$p(A \cap B) \geq 0.7 > 0$

$\Rightarrow p(A \cap B) \neq 0.$ $\Rightarrow$ the same conclusion.

$$p(A)+p(B)-1 \leq \left\{ \begin{matrix} 0 \leq \\ \end{matrix} \right\} p(A \cap B) \leq \min\{p(A), p(B)\}$$

$$\max\{p(A), p(B)\} \leq p(A \cup B) \left\{ \begin{matrix} \leq p(A)+p(B) \\ \leq 1 \end{matrix} \right.$$

Problem 7.

$$\underline{p(A \text{ and } B)} = P(\text{total is twelve, first is six})$$
$$= p(\text{first is six, second is six}).$$
$$= p(\text{first is six}) \cdot p(\text{second is six}).$$
$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

(independence of two events / $\underline{r.v}$).

(1,1)	(1,2)	...	(1,6)
(2,1)	(2,2)	...	(2,6)
⋮	⋮	⋱	⋮
(6,1)	(6,2)	...	$\boxed{(6,6)}$

each "outcome" has probability $\frac{1}{36}$.

e.g. $p(\{(6,6)\}) = \frac{1}{36}$.

Problem 15.

$p(A) = 0.2$

$p(B) = 0.4$.

A implies B, A contains in B, $(A \subseteq B)$.

$A \subseteq B \Longleftrightarrow A \cap B = A$.

$$p(A \cap B) = p(A).$$

$\underbrace{A^c}_{\text{compliment}} = \Omega \setminus A$, $p(A^c) = p(\Omega) - p(A)$

$= 1 - p(A)$.

in fact, $B \setminus A$.

$= \{x : x \in B, x \notin A\}$

$= \{x : x \in B, x \in A^c\}$

$= A^c \cap B$.

$$p(B \mid A^c) = \frac{p(B \cap A^c)}{p(A^c)} = \frac{p(B \setminus A)}{1 - p(A)}$$

$$= \frac{p(B) - p(A)}{1 - p(A)} = \frac{0.4 - 0.2}{1 - 0.2}$$

$$= \frac{1}{4}.$$

$p(B) = p(B \setminus A) + p(B \cap A)$ (common)

$= p(B \setminus A) + p(A)$ $(A \subseteq B)$

$\Longrightarrow p(B \setminus A) = p(B) - p(A)$

Problem 17

Let $A = \{\text{the boy cries wolf}\}$

$B = \{\text{the wolf presents}\}$

By condition: $P(A|B) = 0.8$

$P(A|B^c) = 0.3$

$P(B) = 0.2$

we want to know $= P(B|A)$.

Bayes formula:

$$P(B|A) = \frac{P(A|B)\cdot P(B)}{P(A|B)\cdot P(B) + P(A|B^c)\cdot P(B^c)}$$

$$= \frac{0.8 \times 0.2}{0.8 \times 0.2 + 0.3 \times 0.8}$$

$$= \frac{0.16}{0.16 + 0.24} = 0.4$$

Problem 24:

$$1 = P(\Omega)$$
$$= P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2)$$
$$= 0.08 + 0.14 + 0.15 + 0.31 + P(X=2)$$
$$= 0.68 + P(X=2)$$
$$\Rightarrow P(x=2) = 0.32$$

Problem 25:

$E[X] = \sum_i i \cdot P(X=i)$, where i goes over all possible values of X.

In this problem:

$$E[X] = (-2) \cdot P(X=-2) + (-1) \cdot P(X=-1) + 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$$
$$= (-2) \cdot 0.08 + (-1) \cdot 0.14 + (0) \cdot 0.15 + 1 \cdot 0.31 + 2 \cdot 0.32$$
$$= -0.16 - 0.14 + 0.31 + 0.64$$
$$= 0.65$$

Bonus:

Problem 10.

$A = \{$ transaction 1 involves inside trading $\}$.

$B = \{$ transaction 2 involves inside trading $\}$

Then, $P(A) = P(B) = 0.47$

$P(A \cap B) = 0.22$

We want the value of $P(A \cup B)$.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.47 + 0.47 - 0.22 \\ &= 0.72 \end{aligned}$$

Problem 12.

(See the explanation in problem 7.)

$$\begin{aligned} P(A \cap B) &\geq P(A) + P(B) - 1 \\ &= 0.5 + 0.7 - 1 \\ &= 0.2 \end{aligned}$$

which is also the minimum value.

Problem 21.

We can denote two events:

$A = \{$ the grasped die is weighted $\}$

$B = \{$ the result is 5 $\}$

(why these two? Because they fit the conditions given by the problem)

By machnism of the dice:

$$p(B|A) = 1, \qquad P(B|A^c) = \frac{1}{6}.$$

By the number of dice: $P(A) = \frac{1}{5}$.

We want: $P(B)$ and $P(A|B)$.

(1) Total probability formula:

$$p(B) = p(B|A)\cdot p(A) + p(B|A^c)\cdot p(A^c)$$

$$= 1\cdot\frac{1}{5} + \frac{1}{6}\cdot\frac{4}{5} = \frac{1}{3}.$$

(2) Bayes formula, or definition:

$$p(A|B) = \frac{P(B|A)\cdot p(A)}{p(B)}$$

$$= \frac{1\cdot\frac{1}{5}}{\frac{1}{3}} = \frac{3}{5}$$