(c) Show that, under H_0, $2T$ has a chi squared distribution, and find the number of degrees of freedom. (*Hint:* Obtain the joint distribution of the $n-1$ nontrivial terms $X_i/(\min_i X_i)$ conditional on $\min_i X_i$. Put these $n-1$ terms together, and notice that the distribution of T given $\min_i X_i$ does not depend on $\min_i X_i$, so it is the unconditional distribution of T.)

8.6 Suppose that we have two independent random samples: $X_1,\ldots,X_n$ are exponential(θ), and $Y_1,\ldots,Y_m$ are exponential(μ).

(a) Find the LRT of $H_0\colon \theta=\mu$ versus $H_1\colon \theta\neq\mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T=\frac{\Sigma X_i}{\Sigma X_i+\Sigma Y_i}.$$

(c) Find the distribution of T when H_0 is true.

8.7 We have already seen the usefulness of the LRT in dealing with problems with nuisance parameters. We now look at some other nuisance parameter problems.

(a) Find the LRT of

$$H_0\colon \theta\le 0 \qquad \text{versus} \qquad H_1\colon \theta>0$$

based on a sample $X_1,\ldots,X_n$ from a population with probability density function $f(x|\theta,\lambda)=\frac{1}{\lambda}e^{-(x-\theta)/\lambda}I_{[\theta,\infty)}(x)$, where both θ and λ are unknown.

(b) We have previously seen that the exponential pdf is a special case of a gamma pdf. Generalizing in another way, the exponential pdf can be considered as a special case of the Weibull(γ,β). The Weibull pdf, which reduces to the exponential if $\gamma=1$, is very important in modeling reliability of systems. Suppose that $X_1,\ldots,X_n$ is a random sample from a Weibull population with both γ and β unknown. Find the LRT of $H_0\colon \gamma=1$ versus $H_1\colon \gamma\neq 1$.

8.8 A special case of a normal family is one in which the mean and the variance are related, the $\mathrm{n}(\theta,a\theta)$ family. If we are interested in testing this relationship, regardless of the value of θ, we are again faced with a nuisance parameter problem.

(a) Find the LRT of $H_0\colon a=1$ versus $H_1\colon a\neq 1$ based on a sample $X_1,\ldots,X_n$ from a $\mathrm{n}(\theta,a\theta)$ family, where θ is unknown.

(b) A similar question can be asked about a related family, the $\mathrm{n}(\theta,a\theta^2)$ family. Thus, if $X_1,\ldots,X_n$ are iid $\mathrm{n}(\theta,a\theta^2)$, where θ is unknown, find the LRT of $H_0: a=1$ versus $H_1\colon a\neq 1$.

8.9 Stefanski (1996) establishes the arithmetic-geometric-harmonic mean inequality (see Example 4.7.8 and Miscellanea 4.9.2) using a proof based on likelihood ratio tests. Suppose that $Y_1,\ldots,Y_n$ are independent with pdfs $\lambda_i e^{-\lambda_i y_i}$, and we want to test H_0: $\lambda_1=\cdots=\lambda_n$ vs. H_1: λ_i are not all equal.

(a) Show that the LRT statistic is given by $(\bar{Y})^{-n}/(\prod_i Y_i)^{-1}$ and hence deduce the arithmetic-geometric mean inequality.

(b) Make the transformation $X_i=1/Y_i$, and show that the LRT statistic based on $X_1,\ldots,X_n$ is given by $[n/\sum_i(1/X_i)]^n/\prod_i X_i$ and hence deduce the geometric-harmonic mean inequality.

8.10 Let $X_1,\ldots,X_n$ be iid Poisson(λ), and let λ have a gamma(α,β) distribution, the conjugate family for the Poisson. In Exercise 7.24 the posterior distribution of λ was found, including the posterior mean and variance. Now consider a Bayesian test of $H_0\colon \lambda\le\lambda_0$ versus $H_1\colon \lambda>\lambda_0$.