Practice Midterm I

STA 200B

University of California, Davis

Exam Rules: This exam is closed book and closed notes. You may bring one page of notes, double-sided. Use of calculators, cell phones or any other electronic or communication devices is not allowed. You must show all of your work to receive credit. You will have 50 minutes to complete the exam.

Note: You do not need to show that the second derivative is negative when deriving MLEs. If needed, you may use that for the Beta(α, β) distribution we have $EX = \alpha/(\alpha+\beta)$, $\quad \text{var}(X) = \alpha\beta/[(\alpha+\beta)^2(\alpha+\beta+1)]$ and for the Gamma(α, β) distribution $EX = \alpha/\beta$, $\quad \text{var}(X) = \alpha/\beta^2$.

Name : ____________________

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1. Suppose that the number of defects in a 1200-foot roll of magnetic recording tape has a Poisson distribution for which the value of the mean (θ) is unknown.

   a) Suppose five rolls of this tape are selected at random. Determine the joint distribution $f(x_1, \ldots, x_5|\theta)$ of the five rolls.

   The p.m.f. of this Poisson distribution is

   $$f(x|\theta) = \frac{\theta^x}{x!}e^{-\theta}.$$

   So the joint p.m.f. is

   $$f(x_1, \ldots, f_5|\theta) = \prod_{i=1}^{5} f(x_i|\theta) = \prod_{i=1}^{5} \frac{\theta^{x_i}}{x_i!}e^{-\theta} = \frac{\theta^{\sum_{i=1}^5 x_i}}{\prod_{i=1}^5 (x_i!)}e^{-5\theta}.$$

   b) Suppose the number of defects found on the rolls are $2, 2, 6, 0$ and 3. If the prior distribution of θ is the gamma distribution with parameters $\alpha = 3$ and $\beta = 1$, find the posterior distribution of θ.

   The posterior p.d.f. of θ is

   $$\xi(\theta|\mathbf{x}) \propto f(x_1, \ldots, x_5|\theta)\xi(\theta) \propto \theta^{\sum_{i=1}^5 x_i} e^{-5\theta}\theta^{\alpha-1}e^{-\beta\theta} = \theta^{\sum_{i=1}^5 x_i + \alpha - 1}e^{-(5+\beta)\theta}$$

   It follows that the posterior distribution is a gamma distribution with parameters $\sum_{i=1}^5 x_i + \alpha = (2+2+6+0+3)+3 = 16$ and $5+\beta = 5+1 = 6$.

   c) Find the Bayes estimator with respect to the squared error loss function.

   The Bayes estimator is the mean of the posterior gamma distribution, which is 16/6=2.667.

d) Write an equation from which the Bayes estimator with respect to the absolute error loss function can be calculated.

The Bayes estimator $\hat{m}$ with respect to the absolute error loss function is the median of posterior distribution. Then it satisfies $\int_0^{\hat{m}} \frac{6^{16}}{\Gamma(16)} x^{15} e^{-6x} dx = \frac{1}{2}$.

e) Find a method of moments estimator for θ^2.

Since θ is the mean of the sample, a method of moments estimator of θ is $\hat{\theta} = \bar{X} = \frac{\sum_{i=1}^n X_i}{n}$. Then by invariance property $\bar{X}^2$ is a method of moments estimator of θ^2.

2. Suppose that $X_1, \ldots, X_n$ form a random sample where each of the X_i is the number of successes in N i.i.d. Bernoulli trials with parameter θ, i.e., θ is the probability of success.

a) Derive the MLE for θ.

$$\tilde{L}(\theta) = f(\mathbf{x}|\theta) = \prod_{i=1}^n \binom{N}{X_i} \theta^{X_i}(1-\theta)^{N-X_i} = (\prod_{i=1}^n \binom{N}{X_i})\theta^{\sum_{i=1}^n X_i}(1-\theta)^{nN-\sum_{i=1}^n X_i},$$

$$L(\theta) = \sum_{i=1}^n \log \frac{N!}{X_i!(N-X_i)!} + \sum_{i=1}^n X_i \log(\theta) + (nN - \sum_{i=1}^n X_i)\log(1-\theta),$$

$$\frac{dL(\theta)}{d\theta} = \frac{\sum_{i=1}^n X_i}{\theta} - \frac{nN - \sum_{i=1}^n X_i}{1-\theta}.$$

So the MLE of θ is $\frac{\sum_{i=1}^n X_i}{nN}$. (Note the requirement to check the second derivative, but it was stated that this does not need to be done specifically for this test).

b) Derive the MLE for EX_1^2.

$EX_1^2 = \mathrm{var}(X_1) + (EX_1)^2 = N\theta(1-\theta) + N^2\theta^2$ and by the invariance property of MLEs we can plug in the MLE for θ. Writing $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ this leads to the MLE of EX_1^2, which is $\bar{X}(1-\frac{\bar{X}}{N})+\bar{X}^2$.

c) Obtain a method of moments estimator for $\mathrm{var}(X_1)$.

$E(X_1) = N\theta$ and $\mathrm{var}(X_1) = N\theta(1-\theta)$. So $\mathrm{var}(X_1) = E(X_1)(1-\frac{E(X_1)}{N})$. We then need a method of moments estimator for $E(X_1)$ for which the entire sample needs to be used. The first sample moment is $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ and therefore a method of moments estimator of $\mathrm{var}(X_1)$ is $\bar{X}(1-\frac{\bar{X}}{N})$.

3. Suppose that the proportion θ of defective items in large shipment is unknown and that the prior distribution of θ is the beta distribution with parameters 1 and 10. Assume in a random sample of 20 items one find that 1 item is defective.

   a) What is the expected value and variance of the prior distribution?

      The expected value equals $\frac{1}{1+10} = 0.091$.
      The variance of the prior distribution equals $\frac{1\cdot 10}{(1+10)^2(1+10+1)} = 0.007$.

   b) What is the posterior distribution?

      The prior p.d.f. of θ is $\xi(\theta) \propto \theta^{1-1}(1-\theta)^{10-1} = (1-\theta)^9$.
      Let $X_1, \ldots, X_{20}$ be the random samples. Then $X_i|\theta$ follows iid Bernoulli(θ) for $i = 1, \ldots, 20$. The likelihood function is $f(\mathbf{x}|\theta) = \theta(1-\theta)^{19}$.
      The posterior p.d.f. of θ is

      $$\xi(\theta|\mathbf{x}) \propto f(\mathbf{x}|\theta)\xi(\theta) \propto \theta(1-\theta)^{19}(1-\theta)^9 = \theta(1-\theta)^{28}.$$

      Thus, the posterior distribution of θ is the beta distribution with parameters 2 and 29.

   c) What is the Bayes estimator for θ if one uses the quadratic loss function?

      The Bayes estimator for θ is the mean of the posterior distribution, which is $\frac{2}{2+29} = 0.065$.

   d) Find the MLE for θ. Is it the same as the Bayes estimator?

      The log-likelihood function is

      $$L(\theta) = \log f(\mathbf{x}|\theta) = \log\theta + 19\log(1-\theta).$$

      Let $\dfrac{dL(\theta)}{d\theta} = \dfrac{1}{\theta} - \dfrac{19}{1-\theta} = 0$, we get the MLE $\hat{\theta} = 1/20 = 0.05$.
      Thus, the MLE is not the same as the Bayes estimator.

   e) Suppose that you change the sampling plan and will keep on sampling until you find 3 defective items. Let X be the number of non-defective items until this happens. Note the negative binomial distribution with parameters θ and k has the pmf $g(x|\theta) = \binom{x+k-1}{k-1}\theta^k(1-\theta)^x$.
      Derive the MLE and Bayes estimator again.

      X has the negative binomial distribution with parameters 3 and θ. So the likelihood function is

      $$g(x|\theta) = \binom{x+2}{2}\theta^3(1-\theta)^x$$

      The log-likelihood function is

      $$L_1(\theta) = \log g(x|\theta) = 3\log\theta + x\log(1-\theta) + \log\binom{x+2}{2}.$$

      Let $\dfrac{dL_1(\theta)}{d\theta} = \dfrac{3}{\theta} - \dfrac{x}{1-\theta} = 0$, we get the MLE is $3/(3+x)$.
      The posterior p.d.f. of θ is

      $$\xi(\theta|x) \propto g(x|\theta)\xi(\theta) \propto \theta^3(1-\theta)^x(1-\theta)^9 = \theta^3(1-\theta)^{x+9}$$

      It corresponds to the p.d.f. of the beta distribution with parameters 4 and $x+10$.
      Thus, the Bayes estimator is the mean of this distribution, which is $\frac{4}{4+(x+10)} = \frac{4}{x+14}$.

4. Let $X_1, \dots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x|\theta_1, \theta_2) = \frac{1}{\theta_2} e^{-(x-\theta_1)/\theta_2},$$

for $x \geq \theta_1$, $-\infty < \theta_1 < \infty$, and $\theta_2 > 0$.

a) Find jointly sufficient statistics (T_1, T_2) where θ_1 and θ_2 are both unknown.

$$f_n(\mathbf{x}|\theta_1) = \frac{1}{\theta_2^n} \exp\{-\frac{1}{\theta_2} \sum_{i=1}^{n} X_i + n\frac{\theta_1}{\theta_2}\} \mathbf{1}_{\{X_{(1)} \in [\theta_1, \infty)\}}.$$

So $(T_1, T_2) = (\sum_{i=1}^n X_i, X_{(1)})$ is jointly sufficient for (θ_1, θ_2) by factorization theorem with $u(\mathbf{x}) = 1$ and $v((T_1, T_2), (\theta_1, \theta_2)) = \frac{1}{\theta_2^n} \exp\{-\frac{1}{\theta_2} T_1 + n\frac{\theta_1}{\theta_2}\} \mathbf{1}_{\{T_2 \in [\theta_1, \infty)\}}$.

b) If θ_2 is known, find a sufficient statistic for θ_1.

$$f_n(\mathbf{x}|\theta_1) = \frac{1}{\theta_2^n} \exp\{-\frac{1}{\theta_2} \sum_{i=1}^{n} X_i\} \exp\{n\frac{\theta_1}{\theta_2}\} \mathbf{1}_{\{X_{(1)} \in [\theta_1, \infty)\}}.$$

So $T = X_{(1)}$ is sufficient for θ_1 by factorization theorem with $u(\mathbf{x}) = \frac{1}{\theta_2^n} \exp\{-\frac{1}{\theta_2} \sum_{i=1}^n X_i\}$ and $v(T, \theta_1) = \exp\{n\frac{\theta_1}{\theta_2}\} \mathbf{1}_{\{T \in [\theta_1, \infty)\}}$.