STA 200A: Homework 3 Solutions

Note, the "Problems" and "Theoretical Exercises" are listed in separate sections at the end of the chapter.

The problem numbers are based on the **9th edition**. (A copy of these problems is available on the course webpage under the folder 'book problems'.)

1. Three cooks A,B,C, bake a special cake and with probabilities $0.02, 0.03, 0.05$ respectively the cake fails to rise. In the restaurant where they work, A bakes this cake 50% of the time, B bakes it 30%, and C bakes it 20%. What proportion of the failures are caused by A?

Solution: Let F indicate failure. We are interested in $P(A|F)$ which can be calculated with the Bayes rule,

$$P(A|F) = \frac{P(F|A)P(A)}{P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C)} = \frac{.02(.5)}{.02(.5) + .03(.3) + .05(.2)} \approx .345.$$

2. 3.T11

Solution: At least one head is the complement of all tails, which has probability $(1-p)^n$. The probability of at least one head being less 1/2 happens if and only if the probability of all tails is a most 1/2. Hence,

$$(1-p)^n < \frac{1}{2} \quad \text{iff} \quad n > -\frac{\log 2}{\log(1-p)}$$

is required.

3. 3.P66

Solution: (a) Let A be the event that 1 and 2 work. Let B be the event that 3 and 4 work. Finally, let C be the event that 5 works. Then the probability that the circuit works is

$$P((A \cup B) \cap C) \underbrace{=}_{\text{since } A \cup B \text{ and } C \text{ are indep.}} P(A \cup B) \cdot P(C) = \Big(P(A) + P(B) - P(A \cap B)\Big)P(C)$$

Note that

$$P(A) = p_1 p_2$$
$$P(B) = p_3 p_4$$
$$P(A \cap B) = p_1 p_2 p_3 p_4.$$
$$P(C) = p_5.$$

Hence, the circuit works with probability

$$(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5.$$

(b) Let $E_1 = \{1 \text{ and } 4 \text{ close}\}$, $E_2 = \{1,\ 3,\ 5 \text{ all close}\}$, $E_3 = \{2,\ 5 \text{ close}\}$, $E_4 = \{2,\ 3,\ 4 \text{ close}\}$.

$$\begin{aligned}
P(E_1 \cup E_2 \cup E_3 \cup E_4) =& P(E_1) + P(E_2) + P(E_3) + P(E_4) - P(E_1E_2) - P(E_1E_3) - P(E_1E_4) \\
& - P(E_2E_3) - P(E_2E_4) + P(E_3E_4) + P(E_1E_2E_3) + P(E_1E_2E_4) \\
& + P(E_1E_3E_4) + P(E_2E_3E_4) - P(E_1E_2E_3E_4) \\
=& P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4 \\
& - P_1P_2P_3P_5 - P_2P_3P_4P_5 - 2P_1P_2P_3P_4P_5 + 3P_1P_2P_3P_4P_5.
\end{aligned} \tag{1}$$

4. 4.P21

Solution:

(a) $E[X]$ is going to be larger, because the more populous buses are more likely for X, while for Y the buses are equally likely.

(b)

$$E[X] = 40\frac{40}{148} + 33\frac{33}{148} + 25\frac{25}{148} + 50\frac{50}{148} \approx 39.3$$

while

$$E[Y] = 40\frac{1}{4} + 33\frac{1}{4} + 25\frac{1}{4} + 50\frac{1}{4} = 37.$$

5. 4.P26

Solution:

(a) The number of questions required is uniformly distributed over $1, \ldots, 10$ which has expectation of $\sum_{i=1}^{10} \frac{i}{10} = 11(10)/(2(10)) = 11/2$.

(b) Question 1: is it greater than 5 or not? Question 2: of the remaining, is it greater than the first 2? Question 3: of the remaining is it the first? Question 4: if it is not determined then of the remaining is it the first? Then we can write the number of questions needed for each positions,

$$(3, 3, 3, 4, 4, 3, 3, 3, 4, 4).$$

Each has equal probability, so the expectation is $(3(6) + 4(4))/10 = 3.4$.

6. 4.T13

Solution: The distribution is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

and if we want to maximize this with respect to p then this is the same as maximizing

$$p^k (1-p)^{n-k}.$$

Moreover, this is the same as maximizing

$$\log(p^k(1-p)^{n-k}) = k \log p + (n-k) \log(1-p).$$

We can take the derivative of this to obtain

$$k\frac{1}{p} - (n-k)\frac{1}{1-p}$$

and we can find an optimum by setting this to be 0.

$$k\left(\frac{1}{p} + \frac{1}{1-p}\right) = n\frac{1}{1-p}$$

$$\frac{k}{n}\frac{1}{p(1-p)} = \frac{1}{1-p}$$

$$p = \frac{k}{n}$$

So the maximum likelihood estimate is $\hat{p} = X/n$ (and it is random!).

7. 4.T17

Solution:
(a) Since X is a Poisson random variable the probability mass function for X is given by

$$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}.$$

To help solve this problem it is helpful to recall that a binomial random variable with parameters (n, p) can be approximated by a Poisson random variable with $\lambda =$ np, and that this approximation improves as $n \to \infty$. To begin then, let E denote the event that X is even. Then to evaluate the expression $P(E)$ we will use the fact that a binomial random variable can be approximated by a Poisson random variable. When we consider X to be a binomial random variable we have from theoretical Exercise 15 in this chapter that

$$P(E) = \frac{1}{2}(1 + (q-p)^n).$$

Using the Poisson approximation to the binomial we will have that $p = \lambda/n$ and $q = 1-p = 1-\lambda/n$, so the above expression becomes

$$P(E) = \frac{1}{2}(1 + (1 - \frac{2\lambda}{n})^n).$$

Taking n to infinity (as required to make the binomial approximation by the Poisson distribution exact) and remembering that

$$\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$$

the probability PE above goes to $\frac{1}{2}(1 + e^{-2\lambda})$ as we were to show.

(b) To directly evaluate this probability consider the summation representation of the requested probability. When we look at this it looks like the Taylor expansion of $\cos(\lambda)$ but without the required alternating $(-1)^i$ factor. This observation might trigger the recollection that the above series is in fact the Taylor expansion of the $\cosh(\lambda)$ function. This can be seen from the definition of the cosh function.

8. 4.T25

Solution: We can solve this problem by conditioning on the number of true events (from the original Poisson random variable N) that occur. We begin by letting M be the number of events counted by our filtered Poisson random variable. Then we want to show that M is another Poisson random variable with parameter λp. To do so consider the probability that M has counted j filtered events, by conditioning on the number of observed events from the original Poisson random variable. We find

$$P(M = j) = \sum_{n=0}^{\infty} P(M = j|N = n)(\frac{e^{-\lambda}\lambda^n}{n!})$$

The conditional probability in this sum can be computed using the acceptance rule defined above. For if we have n original events the number of derived events is a binomial random variable with parameters (n, p). Specifically then we have

$$P(M = j|N = n) = \binom{n}{j} p^j (1-p)^{n-j} \quad \text{if} \quad j \leq n \qquad (\text{else } = 0)$$

Putting this result into the original expression for $PM = j$ we find that

$$P(M = j) = \sum_{n=j}^{\infty} \binom{n}{j} p^j (1-p)^{n-j} (\frac{e^{-\lambda}\lambda^n}{n!})$$

To evaluate this we note that $\binom{n}{j}\frac{1}{n!} = \frac{1}{j!(n-j)!}$, so that the above simplifies as following

$$P(M = j) = e^{-p\lambda}\frac{(p\lambda)^j}{j!}$$

, from which we can see M is a Poisson random variable with parameter λ, p as claimed.

9. Two boys play basketball in the following way. They take turns shooting and stop when a basket is made. Player A goes first and has probability p_1 of making a basket on any throw. Player B, who shoots second, has probability p_2 of making a basket. The outcomes of the successive trials are assumed to be independent.

   (a) Find the frequency function for the total number of attempts.

   Solution: Let X denote the number of attempts to end the game. Here consider the iterative approach, i.e.,

$$\begin{aligned}
P(X=1) &= p_1 \\
P(X=2) &= (1-p_1)p_2 \\
P(X=3) &= (1-p_1)(1-p_2)p_1 \\
P(X=4) &= (1-p_1)(1-p_2)(1-p_1)p_2 \\
&\vdots \\
P(X=2n) &= (1-p_1)^n(1-p_2)^{n-1}p_2 \\
P(X=2n+1) &= (1-p_1)^n(1-p_2)^n p_1
\end{aligned}$$

hence in general for any $k \in \mathbb{Z}^+$, we have

$$P(X=k) = \left(\begin{array}{ll} (1-p_1)^{\frac{k}{2}}(1-p_2)^{\frac{k}{2}-1}p_2 & \text{: k is even} \\ (1-p_1)^{\frac{k-1}{2}}(1-p_2)^{\frac{k-1}{2}}p_1 & \text{: k is odd} \end{array} \right.$$

(b) What is the probability that player A wins?

Solution:

$$\begin{aligned}
P(\text{player A wins}) &= P(X=1)+P(X=3)+P(X=5)+... \\
&= \sum_{n=0}^{\infty} P(X=2n+1) \\
&= \sum_{n=0}^{\infty}(1-p_1)^n(1-p_2)^n p_1 = p_1\sum_{n=0}^{\infty}(1-p_1)^n(1-p_2)^n \\
&= \frac{p_1}{1-(1-p_1)(1-p_2)}
\end{aligned}$$

the last equality $\sum_{n=0}^{\infty}(1-p_1)^n(1-p_2)^n = 1/[1-(1-p_1)(1-p_2)]$ holds from the geometric series, i.e., $\sum_{r=0}^{\infty} x^r = 1/(1-x)$, $\forall x \in (0,1)$.