STA 200A: Homework 4

Note: Below the notation 3.T11 means Chapter 3, Theoretical Exercise 11. Similarly, the notation 4.P21 means Chapter 4, Problem 21.

1. Let X be a Poisson rv with parameter λ. Calculate $E\left[\frac{1}{X+1}\right]$.

Solution:

$$\begin{aligned}
E\left[\frac{1}{X+1}\right] &= \sum_{x=0}^{\infty} \frac{1}{1+X} \frac{e^{-\lambda}\lambda^X}{X!} \\
&= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^X}{(X+1)!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^X}{(X+1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{X+1}}{(X+1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \left(\sum_{x=0}^{\infty} \frac{\lambda^X}{X!} - 1 \right) \\
&= \frac{e^{-\lambda}}{\lambda} (e^{\lambda} - 1) \\
&= \frac{1 - e^{-\lambda}}{\lambda}
\end{aligned}$$

2. 4.T10

Solution: We need the following fact:

$$\left(\frac{n+1}{k+1}\right)\binom{n}{k} = \binom{n+1}{k+1}.$$

Now, if $X \sim \text{Binomial}(n,\, p)$,

$$\begin{aligned}
E\Big[\frac{1}{X+1}\Big] &= \sum_{k=0}^{n} \frac{1}{k+1} P\{X = k\} \\
&= \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{1}{p(n+1)} \sum_{k=0}^{n} \binom{n+1}{k+1} p^{k+1} (1-p)^{n+1-(k+1)} \\
&= \frac{1}{p(n+1)} \sum_{k=1}^{n+1} \binom{n+1}{k} p^{k} (1-p)^{n+1-k} \\
&= \frac{1}{p(n+1)} \Big[\sum_{k=0}^{n+1} \binom{n}{k} p^{k} (1-p)^{n+1-k} - (1-p)^{n+1} \Big] \\
&= \frac{1-(1-p)^{n+1}}{p(n+1)}
\end{aligned}$$

3. There are n distinct items arranged in a random order. (All orderings are equally likely). The items are searched sequentially until a desired item is found. What is the expected number of items searched?

Solution: Each item is equally likely to be requested so probability is $1/n$ to be requested, i.e., $f_X(k) = P(X = k) = p(k) = \frac{1}{n}$, where X is the number of items searched until item is found.

$$E[X] = \sum_x x f_X(x) = \sum_{k=1}^{n} k p(k) = \sum_{k=1}^{n} k \frac{1}{n} = \frac{1}{n} \sum_{k=1}^{n} = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

4. The university administration assures a mathematician that he has only 1 chance in 10,000 of being trapped in a much-maligned elevator in the mathematics building. If he goes to work 5 days a week, 52 weeks a year, for 10 years, and always rides the elevator up to his office when he first arrives, what is the probability that he will never be trapped? That he will be trapped once? Twice? Assume that the outcomes on all the days are mutually independent (a dubious assumption in practice).

   Finally, compute the same probabilities by approximating the Binomial distribution with Poisson distribution, and see how close the numbers are.

Solution:

Note that we have $5 \times 52 \times 10 = 2,600$ days in total (total number of trials) with the probability of being trapped $1/10,000$. Denote X as the number of being trapped in the elevator and since the outcomes on all the days are mutually independent, $X \sim$Binomial$(2600, 1/10000)$. Hence we have

$$
\begin{aligned}
P(X=0) &= \binom{2600}{0}\left(\frac{1}{10000}\right)^{0}\left(1-\frac{1}{10000}\right)^{2600-0} = .77\\
P(X=1) &= \binom{2600}{1}\left(\frac{1}{10000}\right)^{1}\left(1-\frac{1}{10000}\right)^{2600-1} = .20\\
P(X=2) &= \binom{2600}{2}\left(\frac{1}{10000}\right)^{2}\left(1-\frac{1}{10000}\right)^{2600-2} = .02
\end{aligned}
$$

The numbers for the Poisson approximation with $\lambda = np = 0.26$ are

$$
\begin{aligned}
P(X=0) &\approx 0.771\\
P(X=1) &\approx 0.200\\
P(X=1) &\approx 0.026
\end{aligned}
$$

5. An urn contains balls numbered 1 to N. Let X be the largest number drawn in n drawings when random sampling with replacement is used. (The event $X \leq k$ means that each of n numbers drawn is less than or equal to k.)

   Show that when N is large that
$$E[X] \approx \frac{n}{n+1}N.$$

Solution: Note that $P(X \leq k) = (k/N)^n$. Hence

$$P(X=k) = P(X \leq k) - P(X \leq k-1) = (k^n - (k-1)^n)N.$$

Therefore

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{N} kP(X=k) && (1)\\
&= \frac{1}{N^n}\sum_{k=1}^{N} k\cdot(k^n-(k-1)^n) && (2)\\
&= \frac{1}{N^n}\sum_{k=1}^{N} k^{n+1} - k(k-1)^n && (3)\\
&= \frac{1}{N^n}\sum_{k=1}^{N} k^{n+1} - (k-1+1)(k-1)^n && (4)\\
&= \frac{1}{N^n}\sum_{k=1}^{N} k^{n+1} - (k-1+1)(k-1)^n && (5)\\
&= \frac{1}{N^n}\sum_{k=1}^{N} k^{n+1} - (k-1)^{n+1} - (k-1)^n && (6)
\end{aligned}
$$

Note that the sum $\sum k^{n+1} - (k-1)^{n+1}$ is telescoping and so is equal to $N^{n+1} - 0$. Hence,

$$E[X] = \frac{1}{N^n}\left(N^{n+1} - \sum_{k=1}^{n}(k-1)^n\right)$$

But, using an integral approximation,

$$\sum_{k=1}^{N}(k-1)^n = \sum_{k=0}^{N-1} k^n \approx \int_0^{N-1} x^n dx = \frac{(N-1)^{n+1}}{n+1}.$$

Combining everything,

$$E[X] \approx \frac{1}{N^n}\left(N^{n+1} - \frac{1}{n+1}(N-1)^{n+1}\right) \approx \frac{n}{n+1}N.$$

6. Let X be a binomial rv with parameters (n,p). Let t be a fixed real number. Derive a formula for the expectation $E[e^{tX}]$.

Solution: The expectation is equal to $(pe^t + (1-p))^n$. To see this we begin with

$$E[e^{tX}] = \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x} = \sum_{x=0}^{n}\binom{n}{x}(pe^t)^x(1-p)^{n-x}.$$

Using the binomial theorem we have

$$\sum_{x=0}^{n}\binom{n}{x}u^x v^{n-x} = (u+v)^n.$$

Hence, letting $u = pe^t$ and $v = 1-p$, we reach the claimed answer.

7. Let X be a binomial rv with parameters (n,p). Show that $E[X] = np$ by directly evaluating the sum $\sum_{x=0}^{n} xP(X=x)$.

Solution: Note that $x\binom{n}{x} = n\binom{n-1}{x-1}$. Hence

$$\begin{aligned}
E[X] &= \sum_{x=0}^{n} n\binom{n-1}{x-1}p^x(1-p)^{n-x} && (7)\\
&= \sum_{x=0}^{n-1} n\binom{n-1}{y}p^{y+1}(1-p)^{n-(y+1)} && \text{where we define } y = x-1. \quad (8)\\
&= np\sum_{x=0}^{n-1}\binom{n-1}{y}p^y(1-p)^{n-(y+1)} && (9)\\
&= np\sum_{z=0}^{n-1}P(Z=z) && \text{where } Z \text{ is defined to be binomial with params } (n-1,p). \quad (10)\\
&= np\cdot 1 && \text{since all values of a pmf must sum to 1.} \quad (11)
\end{aligned}$$

8. 4.P23

Solution: (a) Let assume one buys x of the commodity at the start of the week, then in cash one has $C = 1000 - 2x$. Here we have x ounces of our commodity with $0 \leq x \leq 500$. Then at the end of the week our total value is given by

$$V = 1000 - 2x + Yx,$$

where Y is the random variable representing the cost per ounce of the commodity. We desire to maximize $E[V]$. We have

$$\begin{aligned}
E[V] &= 1000 - 2x + x\sum_{i=2}^{2} y_i p(y_i) \\
&= 1000 - 2x = x\Big(1(\frac{1}{2}) + 4(\frac{1}{2})\Big) \\
&= 1000 + \frac{x}{2}
\end{aligned}$$

Since this is an increasing linear function of x, to maximize our expected amount of money, we should buy as much as possible. Thus let $x = 500$ i.e. buy all that one can.

(b) We desire to maximize the expected amount of the commodity that one posses. Now by purchasing x at the beginning of the week, one is then left with $1000 - 2x$ cash to buy more at the end of the week. The amount of the commodity A, that we have at the end of the week is given by

$$A = x + \frac{1000 - 2x}{Y}$$

where Y is the random variable denoting the cost per ounce of our commodity at the end of the week. Then the expected value of A is then given by

$$\begin{aligned}
E[A] &= x + \sum_{i=1}^{2}\Big(\frac{1000 - 2x}{y_i}\Big)p(y_i) \\
&= x + \Big(\frac{1000 - 2x}{1}\Big)\Big(\frac{1}{2}\Big) + \Big(\frac{1000 - 2x}{1}\Big)\Big(\frac{1}{2}\Big) \\
&= 625 - \frac{x}{4}.
\end{aligned}$$

Which is linear and decreases with increasing x. Thus we should pick $x = 0$ i.e. buy none of the commodities now and buy it all at the end of the week.