200B HW#4 solution

January 31, 2020

- Instructions for Exercises 7.7.2- 7.7.16: In each of these exercises, assume that the random variables $X_1, \dots, X_n$ form a random sample of size n from the distribution specified in that exercise, and show that the statistic T specified in the exercise is a sufficient statistic for the parameter.

7.7.2

The geometric distribution with parameter p, which is unknown $(0 < p < 1)$; $T = \sum_{i=1}^n X_i$.

Solution: The p.m.f. for the geometric distribution is $f(x|p) = p(1-p)^x, x = 0, 1, \dots,$ which leads to the joint p.m.f.

$$f(\mathbf{x}|p) = \prod_{i=1}^n p(1-p)^{x_i}\mathbf{1}_{\{x_i \in \{0,1,\dots\}\}} = p^n(1-p)^{\sum_{i=1}^n x_i}\mathbf{1}_{\{\mathbf{x}\in\{0,1,\dots\}\}} = p^n(1-p)^T\mathbf{1}_{\{\mathbf{x}\in\{0,1,\dots\}\}}.$$

By the factorization theorem with $u(x) = \mathbf{1}_{\{\mathbf{x}\in\{0,1,\dots\}\}}$ and $v(T,p) = p^n(1-p)^T$, T is a sufficient statistic for p.

7.7.10

The uniform distribution on the interval $[a, b]$, where the value of b is known and the value of a is unknown $(a < b)$; $T = \min\ \{X_1, \dots, X_n\}$.

Solution: The p.d.f. for this uniform distribution is

$$f(x|a) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise,} \end{cases}$$

which leads to the joint p.d.f.

$$f_n(\mathbf{x}|a) = \begin{cases} \frac{1}{(b-a)^n} & \text{if } a \leq x_i \leq b \quad \forall i = 1, \dots n \\ 0 & \text{otherwise.} \end{cases}$$

This can be re-written as

$$f_n(\mathbf{x}|a) = \frac{1}{(b-a)^n} \mathbf{1}_{\{T \in [a,\infty)\}} \mathbf{1}_{\{\max\{x_1,\dots,x_n\} \in (-\infty,b]\}},$$

where $1_{\{\}}$ denotes the indicator function (note that b is known). By the factorization theorem with $u(x) = \mathbf{1}_{\{\max\{x_1,\dots,x_n\} \in (-\infty,b]\}}$ and $v(T,a) = \frac{1}{(b-a)^n} \mathbf{1}_{\{T \in [a,\infty)\}}$, T is a sufficient statistic for a.

7.7.12

Suppose that a random sample $X_1, \dots, X_n$ is drawn from the Pareto distribution with parameters x_0 and α. (See Exercise 16 in Sec. 5.7.)

(a) If x_0 is known and α is unknown

(b) If x_0 is unknown and α is known

Solution: The joint p.d.f. function is

$$f(\mathbf{x}|\alpha, x_0) = \frac{\alpha^n x_0^{\alpha n}}{[\prod_{i=1}^n x_i]^{\alpha+1}},$$

for all $x_i \geq x_0$, which can be expressed as

$$\frac{\alpha^n x_0^{\alpha n}}{[\prod_{i=1}^n x_i]^{\alpha+1}} \mathbf{1}_{\{\min\{x_1,\dots,x_n\} \in [x_0,\infty)\}}.$$

a) If x_0 is known, write

$$\left[\frac{\alpha^n x_0^{\alpha n}}{[\prod_{i=1}^n x_i]^{\alpha+1}} \right] \times \mathbf{1}_{\{\min\{x_1,\dots,x_n\} \in [x_0,\infty)\}}.$$

Let $T = \prod_{i=1}^n x_i$. By the factorization theorem with $u(x) = \mathbf{1}_{\{\min\{x_1,\dots,x_n\} \in [x_0,\infty)\}}$ and $v(T,\alpha) = \left[\frac{\alpha^n x_0^{\alpha n}}{T^{\alpha+1}} \right]$, T is sufficient for α.

b) If α is known, write

$$\left[\frac{\alpha^n}{[\prod_{i=1}^n x_i]^{\alpha+1}}\right] \times [x_0^{\alpha n}\mathbf{1}_{\{\min\{x_1,\dots,x_n\}\in[x_0,\infty)\}}].$$

Let $T = \min\{X_1, \dots, X_n\}$. By the factorization theorem with $u(x) = \left[\frac{\alpha^n}{[\prod_{i=1}^n x_i]^{\alpha+1}}\right]$ and $v(T, x_0) = [x_0^{\alpha n}\mathbf{1}_{\{T\in[x_0,\infty)\}}]$, T is sufficient for x_0.

7.7.16

Let θ be a parameter with parameter space Ω equal to an interval of real numbers (possibly unbounded). Let $\mathbf{X}$ have p.d.f. or p.f. $f_n(\mathbf{x}|\theta)$ conditional on θ. Let $T = r(\mathbf{X})$ be a statistic. Assume that T is sufficient. Prove that, for every possible prior p.d.f. for θ, the posterior p.d.f. of θ given $\mathbf{X} = \mathbf{x}$ depends on x only through $r(\mathbf{x})$.

Solution: Let $\xi(\theta)$ be a prior p.d.f. for θ. The posterior p.d.f. of θ is, according to Bayes' Theorem,

$$\xi(\theta|\mathbf{x}) = \frac{f_n(\mathbf{x}|\theta)\xi(\theta)}{\int_\Omega f_n(\mathbf{x}|\theta)\xi(\theta)d\theta} = \frac{u(\mathbf{x})v[r(\mathbf{x}),\theta]\xi(\theta)}{\int_\Omega u(\mathbf{x})v[r(\mathbf{x}),\theta]\xi(\theta)d\theta} = \frac{v[r(\mathbf{x}),\theta]\xi(\theta)}{\int v[r(\mathbf{x}),\theta]\xi(\theta)d\theta},$$

where the second equality uses the factorization theorem. One can see that this last expression depends on $\mathbf{x}$ only through $r(\mathbf{x})$. Notice that $u(\mathbf{x})$ is constant with respect to the integral in the denominator, and therefore cancels with the numerator.

- Instructions for Exercises 7.8.2-7.8.4: In each exercise, assume that the random variables $X_1, \dots, X_n$ form a random sample of size n from the distribution specified in the exercise, and show that the statistics T_1 and T_2 specified in the exercise are jointly sufficient statistics.

7.8.2

A beta distribution for which both parameters α and β are unknown $(\alpha > 0, \beta > 0)$; $T_1 = \sum_{i=1}^n X_i$ and $T_2 = \prod_{i=1}^n (1 - X_i)$.

Solution: The joint p.d.f. is

$$f_n(\mathbf{x}|\alpha,\beta) = \left[\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n [\prod_{i=1}^n x_i]^{\alpha-1}[\prod_{i=1}^n (1-x_i)]^{\beta-1}\right] \mathbf{1}_{\{\min(x_i)\in(0,\infty)\}}\mathbf{1}_{\{\max(x_i)\in(-\infty,1)\}}.$$

let $T_1 = \prod_{i=1}^n X_i$ and $T_2 = \prod_{i=1}^n (1-X_i)$. By the factorization theorem with $u(x) = \mathbf{1}_{\{\min(x_i)\in(0,\infty)\}}\mathbf{1}_{\{\max(x_i)\in(-\infty,1)\}}$ and $v((T_1,T_2),(\alpha,\beta)) = \left[\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n T_1^{\alpha-1}T_2^{\beta-1}\right]$, T_1 and T_2 are jointly sufficient for α and β.

7.8.4

The uniform distribution on the interval $[\theta, \theta+3]$, where the value of θ is unknown ($-\infty < \theta < \infty$); $T_1 = \min\{X_1,\ldots,X_n\}$ and $T_2 = \max\{X_1,\ldots,X_n\}$.

Solution: The joint p.d.f. is

$$f_n(\mathbf{x}|\theta) = \frac{1}{3^n}\mathbf{1}_{\{\min(x_i)\in[\theta,\infty)\}}\mathbf{1}_{\{\max(x_i)\in(-\infty,\theta+3]\}}.$$

Let $T_1 = \min(X_i)$ and $T_2 = \max(X_i)$. By the factorization theorem with $u(x) = 1$ and $v((T_1,T_2),\theta) = \frac{1}{3^n}\mathbf{1}_{T_1\in[\theta,\infty)}\mathbf{1}_{T_2\in(-\infty,\theta+3]}$, T_1 and T_2 are jointly sufficient for θ.

7.10.21

Suppose that a random sample $X_1,\ldots,X_n$ is to be taken from the normal distribution with unknown mean θ and variance 100, and the prior distribution of θ is the normal distribution with specified mean μ_0 and variance 25. Suppose that θ is to be estimated using the squared error loss function, and the sampling cost of each observation is 0.25 (in appropriate units). If the total cost of the estimation procedure is equal to the expected loss of the Bayes estimator plus the sampling cost $(0.25)n$, what is the sample size n for which the total cost will be a minimum?

Solution: By theorem 7.3.3, the posterior distribution of θ is a normal distribution with mean μ_1 and variance ν_1^2, where

$$\mu_1 = \frac{100\mu_0 + 25\bar{x}_n n}{100 + 25n}$$
$$\nu_1^2 = \frac{100 * 25}{100 + 25n} = \frac{100}{4 + n}.$$

With squared error loss, the expected loss of the Bayes estimator is

$$E((\mu_1 - \theta)^2 | x) = \nu_1^2 = \frac{100}{4 + n}.$$

So the cost function is

$$cost(n) = \frac{100}{4 + n} + 0.25n,$$

with derivatvie $\frac{dcost(n)}{dn} = 0.25 - \frac{100}{(4+n)^2}$ and second derivative is $\frac{dcost(n)^2}{d^2n} = \frac{2\times 100}{(4+n)^3} > 0$. So it is minimized at $\frac{dcost(n)}{dn} = 0$ which gives $n = 16$.

Additional Problem

(a) Show that the Gamma distributions form an exponential family (use the form as given in the textbook/lecture notes).

Solution: The p.d.f. of Gamma distribution is $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x + (\alpha-1)\log(x)}$. Let $\theta = (\alpha, \beta)$ then we can see it belongs to an exponential family with $a(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}$, $b(x) = 1$, $k = 2$, $c_1(\theta) = -\beta$, $d_1(x) = x$, $c_2(\theta) = \alpha - 1$ and $d_2(x) = \log(x)$.

(b) Use the exponential family representation to solve problem 7.7.14:
Suppose that $X_1, \ldots, X_n$ form a random sample from the gamma distribution specified in Exercise 6. Show that the statistic $T \sum_{i=1}^n \log(X_i)$ is a sufficient statistic for the parameter α.

Solution: Similar as in additional problem (a), we treat β as a constant and set $a(\alpha) = \frac{\beta^\alpha}{\Gamma(\alpha)}$, $b(x) = x^{\alpha-1}$, $k = 1$, $c_1(\alpha) = \alpha - 1$ and $d_1(x) = \log(x)$. It belongs to an exponential family and $T = \sum_{i=1}^n d_1(X_i) = \sum_{i=1}^n \log(X_i)$ is a sufficient statistic of α.