Stat 200A Homework 2

Note, the "Problems" and "Theoretical Exercises" are listed in separate sections at the end of the chapter.

1. Chapter 2, Problem 52

   Solution. (a) There are $\binom{10}{8}$ ways to select 8 pairs from 10 and for each pair we can choose either the left or the right shoe (but not both). Thus, there are $\binom{10}{8}2^8$ outcomes such that no complete pair is selected. Then the probability is this divided by the number of selections of 8 from 20, so the probability is
$$\frac{\binom{10}{8}2^8}{\binom{20}{8}} = 0.09145035.$$
   (b) There are $\binom{10}{1}$ ways of choosing the pair that is complete. Then from the remaining 9 pairs we can select $\binom{9}{6}$ pairs that will be uncompleted, then 2^6 choices of either left or right shoe. The probability is
$$\frac{\binom{10}{1}\binom{9}{6}2^6}{\binom{20}{8}} = 0.4267683.$$

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2. Chapter 2, Theoretical Exercise 11

   Solution. First notice that $P(E \cup F) \leq 1$ by the first axiom of probability, and
$$P(E) + P(F) - P(EF) = P(E \cup F) \leq 1.$$
   Rearranging terms we get
$$P(EF) \geq P(E) + P(F) - 1.$$
   Applying this to the numbers gives us that particular answer.

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3. Chapter 2, Theoretical Exercise 20

   Solution. Suppose that all of the outcomes had probability $a > 0$. Because S is infinite then there exists an sequence of k different outcomes $\omega_1, \omega_2, \ldots, \omega_k$ where $k > 1/a$, thus
$$P(S) = \sum_{i=1}^{k} P(\{\omega_i\}) = \sum_{i=1}^{k} a = ka > 1.$$
   This violates the third axiom of probability. Now suppose that all outcomes have 0 probability. Because we S is countable then we can order the outcomes, $\{x_i\}_{i=1}^{\infty}$ and
$$P(S) = \sum_{i=1}^{\infty} P(\{\omega_i\}) = 0.$$
   which violates the 2nd axiom. Hence, a countably infinite sample space cannot have equal probabilities. But, it is possible for them to have positive probabilities, for example,
$$P(\{\omega_i\}) = \frac{1}{2^i}$$

for $i = 1, 2, \ldots$ is a valid probability.

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4. Chapter 3, Problem 15

Solution. Let F be the event in which the woman is a smoker, and E be the event in which ectopic pregnancy develops. Then we have that

$$\frac{P(E|F)}{P(E|F^C)} = 2.$$

We also know that $P(F) = .32$ then we want to know what is $P(F|E)$?

We have that $P(F^C) = 1 - P(F) = .68$. By Bayes rule,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)} = \frac{[P(E|F)/P(E|F^C)]P(F)}{[P(E|F)/P(E|F^C)]P(F) + P(F^C)}$$

$$= \frac{2(.32)}{2(.32) + .68} \approx .485$$

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5. Chapter 3, Problem 46 (Hint: The function $f(x) = x^2$ is convex.)

Solution. The intuitive solution is the following: Suppose without loss of generality that $p_m > p_f$ (if it was the opposite then we could just reverse the role of male and female), and we do not know the gender of the person. If we knew what the gender was then we would say that there will be an accident with probability p_m or p_f, but we do not, all we know is that the probability of being male is α. But when we see that there is an accident in the first year, that changes our probability that the person is male (it increases it). Then the individual is more risky because they are more likely to be male. Hence, the probability of an accident in the following year is increased.

The mathematical solution is the following: The function $f(x) = x^2$ is convex so that $(\alpha p_m + (1 - \alpha)p_f)^2 \leq \alpha p_m^2 + (1 - \alpha)p_f^2$. Let M be the event that the person is male. Then we have that $P(A_1)P(A_2|A_1) = P(A_1A_2) = P(A_1A_2|M)P(M) + P(A_1A_2|M^C)P(M^C) = \alpha p_m^2 + (1 - \alpha)p_f^2 \geq (\alpha p_m + (1 - \alpha)p_f)^2 = (P(A_1|M)P(M) + P(A_1|M^C)P(M^C))^2 = P(A_1)^2$. Hence, $P(A_2|A_1) \geq P(A_1)$.

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6. Chapter 3, Theoretical Exercise 6,

Solution.

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1 - P((E_1 \cup E_2 \cup \ldots \cup E_n)^C) = 1 - P(E_1^C E_2^C \cdots E_n^C)$$

$$= 1 - \prod_{i=1}^{n} P(E_i^C) = 1 - \prod_{i=1}^{n}(1 - P(E_i))$$

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7. Chapter 3, Theoretical Exercise 16 This problem introduces a simple meteorological model, more complicated versions of which have been proposed in the meteorological literature. Consider a sequence of days and let R_i denote the event that it rains on day i. Suppose that $P(R_i|R_{i-1}) = \alpha$ and $P(R_i^c|R_{i-1}^c) = \beta$. Suppose further that only today's weather is relevant to predicting tomorrow's; that is, $P(R_i|R_{i-1} \cap R_{i-2} \cap ... \cap R_0) = P(R_i|R_{i-1})$.

$$P(R_i|R_{i-1}) = \alpha, \qquad P(R_i^c|R_{i-1}) = 1 - \alpha$$
$$P(R_i^c|R_{i-1}^c) = \beta, \qquad P(R_i|R_{i-1}^c) = 1 - \beta$$

(a) If the probability of rain today is p, what is the probability of rain tomorrow? $(P(R_i) = p)$

$$P(R_{i+1}) = P(R_{i+1}|R_i)P(R_i) + P(R_{i+1}|R_i^c)P(R_i^c) = \alpha p + (1-\beta)(1-p)$$

(b) What is the probability of rain the day after tomorrow?

$$\begin{aligned} P(R_{i+2}) &= P(R_{i+2}|R_{i+1})P(R_{i+1}) + P(R_{i+2}|R_{i+1}^c)P(R_{i+1}^c) \\ &= \alpha[\alpha p + (1-\beta)(1-p)] + (1-\beta)[1 - (\alpha p + (1-\beta)(1-p))] \end{aligned}$$

(c) What is the probability of rain n days from now? What happens as $n \to \infty$? Let $P(R_0)$ denote the probability of raining today, i.e., $P(R_0) = p$, and the first part of the question is asking to find the probability of raining n days from now, i.e., $P(R_n)$.

$$\begin{aligned} P(R_n) &= P(R_n|R_{n-1})P(R_{n-1}) + P(R_n|R_{n-1}^c)P(R_{n-1}^c) \\ &= \alpha P(R_{n-1}) + (1-\beta)[1 - P(R_{n-1})] \end{aligned}$$

Denoting $P(R_n) = a_n$, and $b = 1 - \beta$, we have

$$a_n = \alpha a_{n-1} + b[1 - a_{n-1}] = b + (\alpha - b)a_{n-1}$$

Further denoting, $c = \alpha - b$, will yield the following recursion

$$a_n = b + ca_{n-1}, \quad \text{with} \quad a_0 = P(R_0) = p$$

now the goal is to solve this recursion,i.e.,

$$\begin{aligned} a_1 &= b + ca_0 = b + cp \\ a_2 &= b + ca_1 = b + c(b + cp) = b + bc + c^2 p \\ a_3 &= b + ca_3 = b + c(b + bc + c^2 p) = b + bc + bc^2 + c^3 p \\ &\vdots \\ a_n &= b + bc + bc^2 + ... + bc^{n-1} + c^n p \end{aligned}$$

Here we will use the following property of finite sequences:

$$\sum_{j=0}^{n} x^j = \frac{1 - x^{n+1}}{1 - x}$$

i.e.,

$$a_n = b(1 + c + c^2 + ... + c^{n-1}) + c^n p = b\frac{1 - c^n}{1 - c} + c^n p$$

hence

$$P(R_n) = b\frac{1 - c^n}{1 - c} + c^n p, \qquad b = 1 - \beta, \quad c = \alpha - b = \alpha + \beta - 1$$

noting that $c^n \to 0$ as $n \to \infty$, we will have

$$P(R_n) \to \frac{b}{1 - c} = \frac{1 - \beta}{2 - \beta - \alpha}, \quad \text{as} \quad n \to \infty$$

Reading Assignment. Please read Chapters 2 and 3 of the text.