Note 3

Page 1 Union-intersection Tests

① Constructing tests for more complicated null hypothesis from simpler null hypothesis

② Initial tests + follow-ups: familywise error control

Suppose the null hypothesis can be expressed as an intersection

$$\Theta_0 = \bigcap_{\gamma \in \Gamma} \Theta_\gamma .$$

where Γ is an arbitrary index set.

For any $\gamma \in \Gamma$, suppose we have hypothesis

$$H_{0\gamma}: \theta \in \Theta_\gamma \quad \text{v.s.} \quad H_{1\gamma}: \theta \in \Theta_\gamma^c$$

with corresponding rejection regions

$$\{x: T_\gamma(x) \in R_\gamma\}.$$

Then, the rejection region of the union-intersection test for

$$H_0: \theta \in \Theta_0 \quad \text{v.s.} \quad H_1: \theta \in \Theta_0^c$$

is given by

$$\bigcup_{\gamma \in \Gamma} \{x: T_\gamma(x) \in R_\gamma\}$$

In fact, Θ_0 holds $\Longleftrightarrow$ All Θ_γ hold, so

Θ_0 is rejected $\Longleftrightarrow$ One of $H_{0\gamma}$ is rejected.

Page 1: One particular special case is

$$\{\underset{\sim}{x} : T_\gamma(\underset{\sim}{x}) \in R_\gamma\} = \{\underset{\sim}{x} : T_\gamma(\underset{\sim}{x}) > c\} \quad \text{for the same } c$$

Then $$\bigcup_{\gamma \in \Gamma} \{\underset{\sim}{x} : T_\gamma(\underset{\sim}{x}) \in R_\gamma\} = \{\underset{\sim}{x} : \sup_{\gamma \in \Gamma} T_\gamma(\underset{\sim}{x}) > c\}$$

Example 1: Normal union-intersection test:

$$X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$H_0 : \mu = \mu_0 \quad \text{v.s.} \quad H_1 : \mu \neq \mu_0.$$

Write H_0: $\{\mu : \mu \leq \mu_0\} \cap \{\mu : \mu \geq \mu_0\}$

H_{0L}: $\mu \leq \mu_0$ v.s. H_{1L}: $\mu > \mu_0$.

The LRT to reject H_{0L} is

$$\frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}} > t. \quad (t > 0).$$

H_{0U}: $\mu \geq \mu_0$ v.s. H_{1U}: $\mu < \mu_0$

The LRT to reject H_{0U} is

$$\frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}} < -t$$

Then, the union-intersection test for H_0 v.s. H_1 is

$$\{\underset{\sim}{x} : \frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}} > t\} \cup \{\underset{\sim}{x} : \frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}} < -t\}$$

$$= \{\underset{\sim}{x} : \left|\frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}\right| > t\}.$$

i.e., the two-sided t-test.

Page 3 Example 2. Multivariate two-sample test

$$\vec{X}_{11}, \cdots, \vec{X}_{1n_1} \overset{iid}{\sim} N_p(\vec{\mu}_1, \Sigma). \quad \bar{\vec{x}}_1, S_1$$
$$\vec{X}_{21}, \cdots, \vec{X}_{2n_2} \overset{iid}{\sim} N_p(\vec{\mu}_2, \Sigma). \quad \bar{\vec{x}}_2, S_2.$$

Pooled sample covariance matrix $S_{pooled} = \frac{n_1-1}{n_1+n_2-2} S_1 + \frac{n_2-1}{n_1+n_2-2} S_2$

Test: $H_0: \vec{\mu}_1 = \vec{\mu}_2$ v.s. $H_1: \vec{\mu}_1 \neq \vec{\mu}_2$.

For any $\underset{(p\times 1)}{\vec{a}} \in \mathbb{R}^p$, $\vec{a} \neq \vec{0}$, define the hypothesis

$$H_{0\vec{a}}: \vec{a}^T\vec{\mu}_1 = \vec{a}^T\vec{\mu}_2 \quad \text{v.s.} \quad H_{1\vec{a}}: \vec{a}^T\vec{\mu}_1 \neq \vec{a}_2^T\vec{\mu}_2$$

Notice that this follows the union-intersection framework, since

$$\{\vec{\mu}_1 = \vec{\mu}_2\} \Longleftrightarrow \{\vec{a}^T\vec{\mu}_1 = \vec{a}^T\vec{\mu}_2 \text{ for all } \vec{a} \neq \vec{0}\}$$

For each $\vec{a}$: $H_{0\vec{a}}$ v.s. $H_{1\vec{a}}$.

Sample 1: $\vec{a}^T\vec{X}_{11}, \cdots, \vec{a}^T\vec{X}_{1n_1} \overset{iid}{\sim} N(\vec{a}^T\vec{\mu}_1, \vec{a}^T\Sigma\vec{a})$

sample mean $\vec{a}^T\bar{\vec{x}}_1$, sample variance $\vec{a}^T S_1 \vec{a}$.

Sample 2: $\vec{a}^T\vec{X}_{21}, \cdots, \vec{a}^T\vec{X}_{2n_2} \overset{iid}{\sim} N(\vec{a}^T\vec{\mu}_2, \vec{a}^T\Sigma\vec{a})$

sample mean $\vec{a}^T\bar{\vec{x}}_2$, sample variance $\vec{a}^T S_2 \vec{a}$.

Then the sample variance is

$$\frac{n_1-1}{n_1+n_2-2}\vec{a}^T S_1 \vec{a} + \frac{n_2-1}{n_1+n_2-2}\vec{a}^T S_2 \vec{a} = \vec{a}^T S_{pooled}\vec{a}.$$

Page 4 Then, the rejection region for the two-sample t-test is

$$\left\{ \underset{\sim}{x} : \frac{(\vec{a}^T \overline{\vec{x}_1} - \vec{a}^T \overline{\vec{x}_2})^2}{(\frac{1}{n_1} + \frac{1}{n_2}) \vec{a}^T S_{pooled} \vec{a}} > c \right\}$$

The union-intersection test for H_0 has the rejection region

$$\bigcup_{\vec{a} \neq \vec{0}} \left\{ \underset{\sim}{x} : \frac{(\vec{a}^T \overline{\vec{x}_1} - \vec{a}^T \overline{\vec{x}_2})^2}{(\frac{1}{n_1} + \frac{1}{n_2}) \vec{a}^T S_{pooled} \vec{a}} > c \right\}$$

$$= \left\{ \underset{\sim}{x} : \max_{\vec{a} \neq \vec{0}} \frac{|\vec{a}^T(\overline{\vec{x}_1} - \overline{\vec{x}_2})|^2}{(\frac{1}{n_1} + \frac{1}{n_2}) \vec{a}^T S_{pooled} \vec{a}} > c \right\}$$

By extended Cauchy-Schwarz,

$$|\vec{a}^T(\overline{\vec{x}_1} - \overline{\vec{x}_2})|^2 \le (\vec{a}^T S_{pooled} \vec{a})(\overline{\vec{x}_1} - \overline{\vec{x}_2})^T S_{pooled}^{-1} (\overline{\vec{x}_1} - \overline{\vec{x}_2}),$$

we have

$$\frac{|\vec{a}^T(\overline{\vec{x}_1} - \overline{\vec{x}_2})|^2}{(\frac{1}{n_1} + \frac{1}{n_2}) \vec{a}^T S_{pooled} \vec{a}} \le \frac{(\vec{a}^T S_{pooled} \vec{a})(\overline{\vec{x}_1} - \overline{\vec{x}_2})^T S_{pooled}^{-1} (\overline{\vec{x}_1} - \overline{\vec{x}_2})}{(\frac{1}{n_1} + \frac{1}{n_2}) \vec{a}^T S_{pooled} \vec{a}}$$

$$= (\overline{\vec{x}_1} - \overline{\vec{x}_2})^T \left((\frac{1}{n_1} + \frac{1}{n_2}) S_{pooled} \right)^{-1} (\overline{\vec{x}_1} - \overline{\vec{x}_2})$$

The equality can be reached (by which $\vec{a}$?)

Page 5 Then

$$\max_{\vec{a} \neq \vec{0}} \frac{|\vec{a}^T(\overline{\vec{x}_1} - \overline{\vec{x}_2})|^2}{(\frac{1}{n_1}+\frac{1}{n_2})\vec{a}^T S_{pooled}\,\vec{a}} = (\overline{\vec{x}_1} - \overline{\vec{x}_2})^T\left((\tfrac{1}{n_1}+\tfrac{1}{n_2})S_{pooled}\right)^{-1}(\overline{\vec{x}_1} - \overline{\vec{x}_2}).$$

The union-intersection test is thus

$$(\overline{\vec{x}_1} - \overline{\vec{x}_2})^T\left((\tfrac{1}{n_1}+\tfrac{1}{n_2})S_{pooled}\right)^{-1}(\overline{\vec{x}_1} - \overline{\vec{x}_2}) > c,$$

which is the same as Hotelling's T^2.

Bayes Tests

$X \sim f(x|\theta)$ pdf/pmf of x given θ

$\theta \sim \pi(\theta)$ prior distribution

Posterior distribution

$$\theta \sim \pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{\int f(x|\theta)\pi(\theta)\,d\theta}.$$

Test $H_0: \theta \in \Theta_0$ v.s. $H_1: \theta \in \Theta_1$.

Bayes test: H_0 is rejected if the posterior ~~distribution~~ probability

$$\mathbb{P}(\theta \in \Theta_0 | x) < c.$$

Example 1 (Normal Bayes Test)

$$X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$$

$$\pi(\theta) = N(\mu, \tau^2)$$

$$H_0: \theta \le \theta_0 \quad \text{v.s.} \quad H_1: \theta > \theta_0$$

Page 6 Posterior

$$\pi(\theta|x) = \mathcal{N}\left(\frac{n\tau^2\bar{x} + \sigma^2\mu}{n\tau^2+\sigma^2}, \frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}\right)$$

$$\Longrightarrow \mathbb{P}(H_0 \text{ is true}|x) = \mathbb{P}(\theta \le \theta_0|x) = F(\theta_0|x).$$

For simplicity, let's reject H_0 if

$$\mathbb{P}(H_0 \text{ is true}|x) = F(\theta_0|x) \le \frac{1}{2}$$

$$\Longleftrightarrow \theta_0 \le \text{posterior mean} = \frac{n\tau^2\bar{x} + \sigma^2\mu}{n\tau^2+\sigma^2}$$

$$\Longleftrightarrow \bar{x} \geq \theta_0 + \frac{\sigma^2}{n\tau^2}(\theta_0 - \mu).$$

How to interpret the result?

$\bar{x}$ = "observed" mean

θ_0 = hypothetical mean

μ = prior mean