Q1: $P(V \geq k) \leq \frac{\cancel{E(V)}E(V)}{k}$, This is by markov's inequality.

$$\frac{E(V)}{k} = \frac{E(\sum_{i=1}^{n_0} 1\{V_i=1\})}{k} = \frac{\sum_{i=1}^{n_0} E[1\{V_i=1\}]}{k} = \frac{\sum_{i=1}^{n_0} P(V_i=1)}{k}$$, where $\begin{cases} V_i=1 & \text{if } H_{0,i} \\ V_i=0 & \text{if } H_{0,i} \text{ Acc} \end{cases}$

$$= \frac{\sum_{i=1}^{n_0} P_i}{k} \leq \sum_{i=1}^{n_0} \frac{k\alpha}{nk} \cancel{\quad} = n_0\frac{\alpha}{n} \leq \alpha$$

Q2: suppose that P_{i_k} is the kth minimum p-value amoung true nulls, obviously, we have $i_k \leq n-n_0+k \Rightarrow n_0 \leq n-i_k+k$.

Holm's procedure commits k false rejection only if:

$$P_{(1)} \leq \alpha_1 \quad \cdots \quad P_{i_k} \leq \alpha_k = \begin{cases} \frac{k\alpha}{n} & \text{if } i_k \leq k \\ \frac{k\alpha}{n+k-i_k} & \text{if } i_k > k \end{cases}$$

$$\left.\begin{array}{l} \text{if } i_k \leq k, \ \alpha_k = \frac{k\alpha}{n} \leq \frac{k\alpha}{n_0} \\ \text{if } i_k > k, \ \alpha_k = \frac{k\alpha}{n+k-i_k} \leq \frac{k\alpha}{n_0} \end{array}\right] \Rightarrow P_{i_k} \leq \alpha_k \leq \frac{k\alpha}{n_0}.$$

$$P(V \geq k) \leq \frac{E(V)}{k} = \frac{\sum_{i=1}^{n_0} P_i}{k} \leq \sum_{i=1}^{n_0} \frac{k\alpha}{n_0 \cdot k} = \alpha$$

Q3: $P_{(1)} \leq P_{(2)} \leq \cdots \leq P_{(n)}$, suppose an event $T_i = \{P_{(i)} \geq \frac{i\alpha}{n}\}$, $U_{(1)} = P_{(1)}, \cdots, U_{(n)} = P_{(n)}$

$$\cancel{\quad} P(P_{(i)} \geq \frac{i\alpha}{n}, i=1\cdots n) = P(\bigcap_{i=1}^{n} T_i)$$

$$1 - P(\bigcap_{i=1}^{n} T_i) = P([\bigcap_{i=1}^{n} T_i]^c) = P(\bigcup_{i=1}^{n} T_i^c) = P(P_{(i)} < \frac{i\alpha}{n}, i=1\cdots n)$$

Now, we should prove that $P(P_{(i)} < \frac{i\alpha}{n}, i=1\cdots n) \leq \alpha$

From the BH(α) procedure, The event $\{P_{(i)} < \frac{i\alpha}{n}, i=1\cdots n\}$ ~~[illegible]~~ means that n is the largest index for which $P_{(i)} \leq \frac{i}{n}q$, so, this event is equalvilent with ~~[illegible]~~ the event: $\{R=n\}$

$$= \{P_1 \leq \cancel{\quad} \frac{\alpha n}{n} = \alpha, R=n\} = \{P_1 \leq \alpha\} \cap C_n^{(1)}$$

So, $P(P_{(i)} < \frac{i\alpha}{n}, i=1\cdots n) = P(\{P_1 \leq \alpha\} \cap C_n^{(1)})$

$= P(P_1 \leq \alpha) \cdot P(C_n^{(1)})$ because P_1 independent with $C_n^{(1)}$

$= \alpha \cdot P(C_n^{(1)}) \leq \alpha$

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Q4 (1) $C_r^{(1)} = \{P_{(1)}^{(1)} \cdots P_{(r-1)}^{(1)} \le \frac{qr}{n}, P_{(r)}^{(1)} > \frac{q(r+1)}{n}, \dots, P_{(n-1)}^{(1)} > q\}$

$= \{P_{(r-1)}^{(1)} \le \frac{qr}{n}, P_{(s)}^{(1)} \geq \frac{q(s+1)}{n}, \forall s > r-1\}$

$= \{R = r\}$.

$C_i^{(1)} \cap C_j^{(1)} = \{R=i\} \cap \{R=j\} = \emptyset$ for $i \neq j$.

So, $C_1^{(1)}, \dots, C_n^{(1)}$ are mutually disjoint.

(2) $C_1^{(1)} = \{P_{(1)}^{(1)} > \frac{2q}{n}, P_{(2)}^{(1)} > \frac{3q}{n}, \dots, P_{(n-1)}^{(1)} > q\}$,

For vector $\vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_{(1)}^{(1)} \\ \vdots \\ p_{(n-1)}^{(1)} \end{bmatrix} = \begin{bmatrix} p_1 \\ t_1 \\ \vdots \\ t_{n-1} \end{bmatrix}$, $\vec{p} \in C_1$

obviously for any $\vec{y} = \vec{p} + \varepsilon$, $\varepsilon > 0$, $\vec{y} = \begin{bmatrix} p_1 + \varepsilon \\ p_{(1)}^{(1)} + \varepsilon \\ \vdots \\ p_{(n-1)}^{(1)} + \varepsilon \end{bmatrix}$

$\vec{y}$ is still belong to $C_1^{(1)}$, $\vec{y} \in C_1^{(1)}$, So, $C_1^{(1)}$ is a increasing set

$C_2^{(1)} = \{P_{(1)}^{(1)} \le \frac{2q}{n}, P_{(2)}^{(1)} > \frac{3q}{n}, \dots P_{(n-1)}^{(1)} > q\}$

$C_1^{(1)} \cup C_2^{(1)} = \{P_{(2)}^{(1)} > \frac{3q}{n} \cdots P_{(n-1)}^{(1)} > q\} \cup \{P_{(1)}^{(1)} = \mathbb{R}^1\}$. this means $P_{(1)}^{(1)}$ can be any value.

for $\vec{p} \in C_1^{(1)} \cup C_2^{(1)}$, $\vec{y} = \vec{p} + \varepsilon$, $\varepsilon > 0$, $\vec{y} = \begin{bmatrix} R + \varepsilon \\ P_{(1)}^{(1)} + \varepsilon \\ \vdots \\ P_{(n-1)}^{(1)} + \varepsilon \end{bmatrix}$

obviously, $\vec{y}$ is still belong to $C_1^{(1)} \cup C_2^{(1)}$.

$C_1^{(1)} \cup C_2^{(1)} \cup \cdots \cup C_k^{(1)} = \{P_{(1)}^{(1)} > \frac{(k+1)q}{n} \cdots P_{(n-1)}^{(1)} > q\} \cup \{P_{(i)}^{(1)} = \mathbb{R}^1, \text{for } i = 1 \cdots k\}$.

it is also a increasing set.