STA200C HW2

1. The null hypothesis H_0 is simple. Therefore, the size α of the test is

$$\alpha = \Pr(\text{Rejecting } H_0 | \mu = \mu_0).$$

When $\mu = \mu_0$, the random variable $Z = n^{1/2}(\overline{X}_n - \mu_0)$ will have the standard normal distribution. Hence, since $n = 25$,

$$\alpha = \Pr(|\overline{X}_n - \mu_0| \geq c) = \Pr(|Z| \geq 5c) = 2[1 - \Phi(5c)].$$

where $\Phi(\cdot)$ is the c.d.f. of standard normal distribution. Thus, $\alpha = 0.05$ if and only if $\Phi(5c) = 0.975$. It is found from a table of the standard normal distribution that $5c = 1.96$ and $c = 0.392$.

2. Let δ be the test procedure that rejects the H_0 when $X > 3 + \frac{1}{2}$. Then

$$\begin{aligned} \pi(\theta|\delta) &= 0 \quad \text{for } \theta \leq 3 \\ \pi(\theta|\delta) &= 1 \quad \text{for } \theta \geq 4 \end{aligned}$$

If H_0 is true, then $X \in [3 - \frac{1}{2}, 3 + \frac{1}{2}]$ will surely be smaller than $3 + \frac{1}{5}$. If H_1 is true, then $X \in [4 - \frac{1}{2}, 4 + \frac{1}{2}]$ will surely be greater than $3 + \frac{1}{5}$. Therefore, the test procedure which rejects H_0 if and only if $X > 3 + \frac{1}{2}$ will have probability 0 of leading to a wrong decision, no matter what the true value of θ is.

3. Let

$$\begin{aligned} H_0 &: \mu \geq \mu_0 \\ H_1 &: \mu < \mu_0 \end{aligned}$$

Let δ be the test that rejects H_0 if $T := \overline{X}_n < c$ for some critical value c.

$$\begin{aligned} \pi(\mu|\delta) &= P(\overline{X}_n < c|\mu) \\ &= P(\sqrt{n}(\overline{X}_n - \mu) < \sqrt{n}(c - \mu)|\mu) \\ &= 1 - \Phi(\sqrt{n}(c - \mu)) \end{aligned}$$

where $\Phi(\cdot)$ is the c.d.f. of standard normal distribution. As $\Phi(\cdot)$ is an increasing function, $\pi(\mu|\delta)$ is decreasing in μ.

4. $X_1, \ldots, X_n \sim N(\mu, 1)$ i.i.d. The hypothesis is

$$\begin{aligned} H_0 &: \mu \leq \mu_0 \\ H_1 &: \mu > \mu_0 \end{aligned}$$

Let

$$Z = \sqrt{n}\frac{\overline{X}_n - \mu_0}{\sigma} = \sqrt{n}(\overline{X}_n - \mu_0)$$

By theorem 8.3.27 in Casella, we can construct the p-value as:

$$\begin{aligned} p\text{-value} &= P(Z \geq z|\mu_0) \\ &= 1 - \Phi(\sqrt{n}(\overline{x}_n - \mu_0)) \\ &= \Phi(\sqrt{n}(\mu_0 - \overline{x}_n)) \end{aligned}$$

where Φ is the c.d.f. of standard normal distribution.

5. (a) The power function of δ_c is

$$\begin{aligned} \pi(\theta|\delta_c) &= P(X \geq c|\theta) \\ &= \int_c^\infty \frac{dx}{\pi(1 + (x - \theta)^2} \\ &= \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan(c - \theta)\right) \end{aligned}$$

Since arctan is an increasing function and $c - \theta$ is a decreasing function of θ, the power functions is increasing in θ.

(b) To make the size of the test 0.05, we need to solve

$$\begin{aligned} 0.05 &= \pi(\theta = \theta_0|\delta_c) \\ 0.05 &= \frac{1}{\pi}\left(\frac{1}{\pi} - \arctan(c - \theta_0)\right) \end{aligned}$$

for c. We get

$$c = \theta_0 + \tan(0.45\pi) = \theta_0 + 6.314$$

(c) The p-value when $X = x$ is observed is, theorem 8.3.27 in Casella,

$$P(X \geq x|\theta = \theta_0) = \frac{1}{\pi}\left(\frac{1}{\pi} - \arctan(x - \theta_0)\right)$$

6. The p-value when $X = x$ is observed is the size of the test that rejects H_0 when $X \geq x$, namely

$$P(X \geq x|\theta = 1) = \begin{cases} 0 & \text{if } x \geq 1, \\ 1 - x & \text{if } 0 < x < 1 \end{cases}$$

7. (a) From Corollary 8.3.13 we can base the test on $\sum_i X_i$, the sufficient statistic. Let $Y = \sum_i X_i \sim \text{binomial}(10, p)$ and let $f(y|p)$ denote the pmf of Y. By Corollary 8.3.13, a test that rejects if $f(y|1/4)/f(y|1/2) > k$ is UMP of its size.

For $p_2 > p_2$,

$$\frac{f(y|p_2)}{f(y|p_1)} = \frac{\binom{n}{y} p_2^y (1-p_2)^{n-y}}{\binom{n}{y} p_1^y (1-p_1)^{n-y}} = \left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^y \left(\frac{1-p_2}{1-p_1}\right)^n$$

Both $p_2/p_1 > 1$ and $(1-p_1)/(1-p_2) > 1$. Thus the ratio is increasing in y. The ratio $f(y|1/2)/f(y|1/4)$ is increasing in y.

So the ratio $f(y|1/4)/f(y|1/2)$ is decreasing in y, and rejecting for large value of the ratio is equivalent to rejecting for small values of y.

To get $\alpha = .0547$, we must find c such that $P(Y \le c|p = 1/2) = .0547$. Trying values $c = 0, 1, ...,$ we find that for $c = 2$, $P(Y \le 2|p = 1/2) = .0547$. So the test that rejects if $Y \le 2$ is the UMP size $\alpha = .0547$ test.

The power of the test is $P(Y \le 2|p = 1/4) \approx .526$.

(b) The size of the test is $P(Y \ge 6|p = 1/2) \approx .377$. The power function is $\beta(\theta) = \sum_{k=6}^{10} \binom{10}{k} \theta^k (1-\theta)^{10-k}$.

(c) There is a nonrandomized UMP test for all α levels corresponding to the probabilities

$$P(Y \le i|p = 1/2) = \sum_{k=i}^{10} \binom{10}{k}$$

where i is an integer.

α can have any of the values $0, \frac{1}{1024}, \frac{11}{1024}, \frac{56}{1024}, \frac{176}{1024}, \frac{386}{1024}, \frac{638}{1024}, \frac{848}{1024}, \frac{968}{1024}, \frac{1013}{1024}, \frac{1023}{1024}, 1$.

8. (a) The test is: Reject H_0 if $X > 1/2$. So the power function is

$$\begin{aligned}
\beta(\theta) &= P_\theta(X > 1/2) \\
&= \int_{1/2}^{1} \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1}(x-1)^{1-1} dx \\
&= \theta \frac{1}{\theta} x^\theta \Big|_{1/2}^{1} \\
&= 1 - \frac{1}{2^\theta}
\end{aligned}$$

The size is $\sup_{\theta \in H_0} \beta(\theta) = \sup_{\theta \le 1}(1 - 1/2^\theta) = 1 - 1/2 = 1/2$.

(b) By the Neyman-Pearson Lemma, the most powerful test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ is given by: Reject H_0 if $f(x|2)/f(x|1) > k$ for some $k \ge 0$. Substituting the beta pdf gives

$$\frac{f(x|2)}{f(x|1)} = \frac{\frac{1}{\beta(2,1} x^{2-1}(1-x)^{1-1}}{\frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1}} = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} x = 2x.$$

Thus, the MP test is: Reject H_0 if $X > k/2$. We now use the α level to determine k. We have

$$\begin{aligned}
\alpha &= \sup_{\theta \in \Theta_0} \beta(\theta) \\
&= \beta(1) \\
&= \int_{k/2}^{1} f_X(x|1)dx \\
&= \int_{k/2}^{1} x^{1-1}(1-x)^{1-1}dx \\
&= 1 - \frac{k}{2}.
\end{aligned}$$

Thus $1 - k/2 = \alpha$, so the most powerful α level test is reject H_0 if $X > 1 - \alpha$.