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Q1: $f(x|\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$

(a)

$L(\theta|\underline{x}) = (\frac{1}{\theta})^n e^{-\frac{\Sigma x_i}{\theta}}$

$\ell(\theta|\underline{x}) = -n\log\theta - \frac{\Sigma x_i}{\theta}$

$\psi(\theta|\underline{x}) = -\frac{n}{\theta} - \Sigma x_i (-1)\theta^{-2}$

$= -\frac{n}{\theta} + \Sigma x_i \theta^{-2}$

$\psi(\theta|\underline{x}) = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0$

$+n\theta = \Sigma x_i$

MLE $\rightarrow$ $\hat{\theta}_n = \bar{x}$

$\psi(\theta|\underline{x}) = -n\theta^{-1} + \Sigma x_i \theta^{-2}$

$\psi'(\theta|\underline{x}) = +n\theta^{-2} - 2\Sigma x_i \cdot \theta^{-3}$

So. $\hat{I}_n(\hat{\theta}_n) = -n\hat{\theta}_n^{-2} + 2\Sigma x_i \cdot \hat{\theta}_n^{-3}$

By the MLE therom, we have:

$$\sqrt{\hat{I}_n(\hat{\theta}_n)}\,(\hat{\theta}_n - \theta) \xrightarrow{d} N(0,1)$$

So, The $1-\alpha$ wald's confidence interval for θ is

$$\hat{\theta}_n - Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{\hat{I}_n(\hat{\theta}_n)}} \le \theta \le \hat{\theta}_n + Z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{\hat{I}_n(\hat{\theta}_n)}}$$

where $\sqrt{\hat{I}_n(\hat{\theta}_n)} = \sqrt{-n\cdot\hat{\theta}_n^{-2} + 2\Sigma x_i \hat{\theta}_n^{-3}}$

(b) The score test statistic is

$$\frac{\psi(\theta)}{\sqrt{I_n(\theta_0)}} \xrightarrow{d} N(0,1)$$

$I(\theta) = Var(\frac{\partial \log f(x|\theta)}{\partial\theta}) = Var(\frac{\partial -\log\theta - \frac{x}{\theta}}{\partial\theta})$

$= Var(-\frac{1}{\theta} + x\theta^{-2})$

$= Var(\theta^{-2}\cdot X)$

$= \frac{1}{\theta^4} Var(X)$ $\quad Var(X) = \frac{1}{(\frac{1}{\theta})^2} = \theta^2$

$= \frac{1}{\theta^2}$

$I_n(\theta) = n I(\theta) = \frac{n}{\theta^2}$

$\psi(\theta) = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2}$

So, The CI for θ is

$$\{|\frac{\psi(\theta)}{\sqrt{I_n(\theta)}}| < Z_{\frac{\alpha}{2}}\}.$$

$$= \{|\frac{\hat{\theta}_n - \theta}{n\cdot\theta^2} \cdot \frac{1}{\sqrt{\frac{n}{\theta^2}}}| < Z_{\frac{\alpha}{2}}\}$$

$$= \{|\frac{\hat{\theta}_n - \theta}{n\cdot\theta^2} \cdot \frac{\theta}{\sqrt{n}}| \le Z_{\frac{\alpha}{2}}\}$$

$$\Rightarrow \{|\frac{\hat{\theta}_n - \theta}{n^{\frac{1}{2}}\cdot\theta}| < Z_{\frac{\alpha}{2}}\}$$

It is not same with wald CI.

Q2 (a) $f(x|\theta) = \sqrt{\frac{\theta}{2\pi}} \cdot e^{\frac{-\theta x^2}{2}}$ it is the $N(0, \frac{1}{\theta})$

$L(\theta|\underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) = (\frac{\theta}{2\pi})^{\frac{n}{2}} \cdot \exp\{-\frac{\theta}{2}\Sigma x_i^2\}$

$\ell(\theta|\underline{x}) = \frac{n}{2}\log\theta - \frac{n}{2}\log 2\pi - \frac{\theta}{2}\Sigma x_i^2$

$\psi(\theta|\underline{x}) = \frac{n}{2}\cdot\frac{1}{\theta} - \frac{1}{2}\Sigma x_i^2$

So, the MLE of θ is: $\frac{n}{2}\frac{1}{\theta} - \frac{1}{2}\Sigma x_i^2 = 0$

$$\hat{\theta}_n = \frac{n}{\Sigma x_i^2}$$

$\psi'(\theta|\underline{x}) = -\frac{n}{2}\cdot\theta^{-2}$

$\hat{I}_n(\hat{\theta}_n) = -\psi'(\hat{\theta}_n) = \frac{n}{2\hat{\theta}_n^2}$

By the MLE theorm:

$$\sqrt{\hat{I}_n(\hat{\theta}_n)}\,(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0,1)$$

So, the $1-\alpha$ CI is

$$\hat{\theta}_n - Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{\hat{I}_n(\hat{\theta}_n)}} \le \theta \le \hat{\theta}_n + Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{\hat{I}_n(\hat{\theta}_n)}}$$

$$\hat{\theta}_n - Z_{\frac{\alpha}{2}}\cdot\frac{\sqrt{2}\,\hat{\theta}_n}{\sqrt{n}} \le \theta \le \hat{\theta}_n + Z_{\frac{\alpha}{2}}\sqrt{\frac{2}{n}}\cdot\hat{\theta}_n$$

(b) $\psi(\theta) = \frac{n}{2\theta} - \frac{1}{2}\Sigma x_i^2$

$\psi'(\theta) = -\frac{n}{2}\cdot\theta^{-2}$

$I_n(\theta) = -E(\psi'(\theta))$

$= -E(-\frac{n}{2}\theta^{-2})$

$= \frac{n}{2\theta^2}$

The score test statistic is

$$\frac{\psi(\theta_0)}{\sqrt{I_n(\theta_0)}} = \frac{\frac{n}{2\theta} - \frac{1}{2}\Sigma x_i^2}{\sqrt{\frac{n}{2}}\cdot\frac{1}{\theta}} = \frac{n - \Sigma x_i^2\cdot\theta}{2\theta}\cdot\frac{\sqrt{2}\cdot\theta}{\sqrt{n}} = \frac{n - \Sigma x_i^2\cdot\theta}{\sqrt{2n}} = \frac{\hat{\theta}_n - \theta}{\frac{\sqrt{2n}}{\Sigma x_i^2}}$$

So, the CI of score test interval is

$$\left\{\left|\frac{\hat{\theta}_n - \theta}{\frac{\sqrt{2n}}{\Sigma x_i^2}}\right| < Z_{\frac{\alpha}{2}}\right\} = \left\{\left|\frac{\hat{\theta}_n - \theta}{\sqrt{\frac{2}{n}}\cdot\hat{\theta}_n}\right| < Z_{\frac{\alpha}{2}}\right\}.$$

It is the same with Wald CI.

Q3:

$P(X=0) = p$

$P(X=1) = p - p^2$

$P(X=2) = (1-p)^2$

$\cancel{f(x|p) = p^{(1-x)}(1-p)^x}$

we use n_0, n_1, n_2 as the Count of $0, 1, 2$.

$\cancel{f(x|p) = p^{n_0}(p-p^2)^{n_1}(1-p)^{2n_2}}$

$$L(p|\underline{x}) = p^{n_0}(p-p^2)^{n_1}(1-p)^{2n_2}$$

$$\ell(p|\underline{x}) = n_0 \log p + n_1 \log(p-p^2) + 2n_2 \log(1-p)$$

$$\psi(p|\underline{x}) = \frac{n_0}{p} + \frac{n_1}{p(1-p)}\cdot(1-2p) + 2n_2 \cdot \frac{1}{1-p}(-1)$$

$$= \frac{n_0}{p} + \frac{n_1(1-2p)}{p(1-p)} - \frac{2n_2}{1-p}$$

$$\psi(p|\underline{x}) = 0 \Rightarrow n_0(1-p) + n_1(1-2p) - 2n_2 \cdot p = 0$$

$$\hat{p} = \frac{n_0+n_1}{n_0+2n_1+2n_2}$$

$\cancel{\text{So}}$ $\psi'(p|\underline{x}) = -(n_0+n_1)p^{-2} + (2n_2-n_1)(1-p)^{-2}$

$$\hat{I}_n(\hat{p}) = -\psi(\hat{p}|\underline{x}) = -(n_0+n_1)\hat{p}^{-2} + (2n_2-n_1)(1-\hat{p})^{-2}$$

By the theorem of MLE

$$\cancel{\sqrt{n}(\hat{p}-p)}\ \sqrt{\hat{I}_n(\hat{p})}\,(\hat{p}-p) \xrightarrow{d} N(0,1)$$

So, The CI of wald test is

$$\hat{p} - Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{\hat{I}_n(\hat{p})}} \le p \le \hat{p} + Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{\hat{I}_n(\hat{p})}}$$

Q4: (1) reject H_{0i}, $P_i < \frac{0.5}{10} = 0.05$.

$P_1 = 0.1 > 0.05$. $P_2 = 0.3 > 0.05$, $P_3 = 0.5 > 0.05$ $P_4 = 0.7 > 0.05$ $P_5 = 0.9 > 0.05$.

So. $V = 0$

$P_6 = 0.01 < 0.05$ $P_7 = 0.03 < 0.05$, $P_8 = 0.05 = 0.05$ $P_9 = 0.07 > 0.05$ $P_{10} = 0.09 > 0.05$

$R = 2$

(2) $P_{(1)} = 0.01 < \frac{0.5}{10} = 0.05$ So, $R = 4$, $V = 0$

$P_{(2)} = 0.03 < \frac{0.5}{9}$

$P_{(3)} = 0.05 < \frac{0.5}{8} = 0.0625$.

$P_{(4)} = 0.07 < \frac{0.5}{7} = 0.0714$

$P_{(5)} = 0.09 > \frac{0.5}{6}$ Stop

(3) $P_{(1)} \le P_{(2)} \cdots \le P_{(10)}$ So, ~~reject~~ $R = 6$, $V = 1$

$P_{(1)} = 0.01 \le \frac{1}{10} \cdot 0.2$

$P_{(2)} = 0.03 \le \frac{2}{10} \cdot 0.2$

$P_{(3)} = 0.05 \le \frac{3}{10} \times 0.2$

$P_{(4)} = 0.07 \le \frac{4}{10} \times 0.2$

$P_{(5)} = 0.09 \le \frac{5}{10} \times 0.2$

$P_{(6)} = 0.1 \le \frac{6}{10} \times 0.2$

$P_{(7)} = 0.3 > \frac{7}{10} \times 0.2$

~~$P_{(8)} = 0.5$~~

Q5: $\cancel{\quad}$

$$\frac{V}{R+1} = \frac{V}{1}\cdot I\{R=0\} + \frac{V}{2}\cdot I\{R=1\} + \cdots + \frac{V}{n+1}\cdot I\{R=n\}.$$

So
$$E\left(\frac{V}{R+1}\right) = E\left(\sum_{r=0}^{n} \frac{V}{r+1}\cdot I\{R=r\}\right)$$
$$= \sum_{r=0}^{n} \frac{1}{r+1}\cdot E\left(V\cdot I\{R=r\}\right)$$
$$= \sum_{r=0}^{n} \frac{1}{r+1}\cdot E\left(\sum_{i=1}^{n_0} I\{P_i \le \tfrac{qr}{n}\}\cdot I\{R=r\}\right)$$
$$= \sum_{r=0}^{n} \frac{1}{r+1} \sum_{i=1}^{n_0} E\left(I\{P_i \le \tfrac{qr}{n}, R=r\}\right)$$
$$= \sum_{r=0}^{n} \frac{1}{r+1} \sum_{i=1}^{n_0} P\{P_i \le \tfrac{qr}{n}, R=r\}$$

Because $P\{P_i \le \frac{qr}{n}, R=r\} = P\{P_j \le \frac{qr}{n}, R=r\}$ for $i \ne j$

$$= \sum_{r=0}^{n} \frac{1}{r+1}\, n_0 \cdot P\{P_1 \le \tfrac{qr}{n}, R=r\}$$
$$= \sum_{r=0}^{n} \frac{n_0}{r+1} P\left(\{P_1 \le \tfrac{qr}{n}\} \cap C_r^{(1)}\right)$$
$$\le \sum_{r=0}^{n} \frac{n_0}{r+1}\cdot \frac{qr}{n}\cdot P\left(C_r^{(1)}\right)$$
$$= \cancel{\frac{n_0 qr}{n(r+1)}}\ \frac{n_0 qr}{(r+1)n} \sum_{r=0}^{n} P\left(C_r^{(1)}\right) = \frac{n_0 qr}{(r+1)n} \le q.$$