STA 200A: Extra Practice Problems for Studying

Note: These problems are only a source of extra practice, and will not be collected for a grade. Some of these problems are more challenging than typical homework problems, and they should not be viewed as a "sample exam".

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1. If all poker hands are equally likely, what is the probability of getting a full house? (Note that getting a full house means getting 3 of a kind, and 2 of another kind.)

Solution: How many possible hands? Answer: $|\Omega| = \binom{52}{5}$.

There are 4 suits (types), and 13 distinct cards within each suit.

To get a full house, you must get 2 of a kind (suit 1), and 3 of a kind (suit 2)

How many choices for the first suit: Ans: 13
How many choices for the second suit? Ans: 12

How many ways to choose 2 cards from the first suit? Ans: $\binom{4}{2}$.

How many ways to choose 3 cards from the second suit: Ans: $\binom{4}{3}$.

Consequently,

$$P(A) = \frac{13 \cdot 12 \cdot \binom{4}{2} \cdot \binom{4}{3}}{\binom{52}{5}}.$$

2. Suppose a coin with head probability p is flipped n times. (All flips are independent.) Let X be the number of consecutive heads observed. (In other words, X is the number of times a head is followed by another head.) For instance, if the sequence HTHHH occurs, then $X = 2$.

Derive a formula for $E[X]$ in terms of p.

Solution: Let A_i be the event that a head occurs on both the flip i and $(i+1)$ where i ranges $1, \ldots, (n-1)$. Also, let the indicator random variable 1_{A_i} be equal to 1 if A_i occurs, and be 0 otherwise. Then,

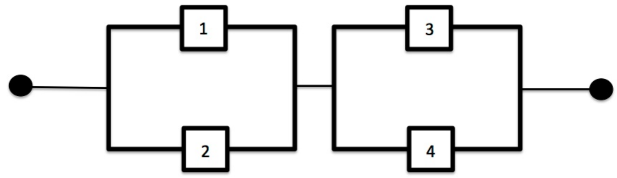
$$X = \sum_{i=1}^{n-1} 1_{A_i},$$

and furthermore

$$E[X] = \sum_{i=1}^{n-1} E[1_{A_i}] = \sum_{i=1}^{n-1} P(A_i) = \sum_{i=1}^{n-1} p^2 = (n-1)p^2,$$

where we have used the fact that $P(A_i) = p^2$.

3. Let A_i be the event that the ith component in the circuit below works. Assume the events A_1, A_2, A_3, A_4 are independent. If each component works with probability p, what is the probability that current can flow across the circuit?



Solution: Let W be the event that the circuit works. Then,

$$W = (A_1 \cup A_2) \cap (A_3 \cup A_4).$$

Since all the components are independent, it follows that the events $C = A_1 \cup A_2$ and $D = A_3 \cup A_4$ are independent. Hence,

$$P(W) = P(C \cap D) = P(C)P(D).$$

Now, we also have a rule for probability of a union, which is that

$$P(C) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = 2p - p^2.$$

$$P(D) = P(A_3 \cup A_4) = P(A_3) + P(A_4) - P(A_4 \cap A_4) = 2p - p^2.$$

So, altogether,

$$P(W) = (2p - p^2)(2p - p^2).$$

Does this make sense in the limit that $p \to 0$ or $p \to 1$? Yes.

4. Consider the joint density function $f_{X,Y}(x,y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{if } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$ Derive a formula for $E[X|Y = 1]$ in terms of λ.

Solution:

Note that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^y \lambda^3 x e^{-\lambda y} dx = \frac{\lambda^3}{2} y^2 e^{-\lambda y}.$$

Hence, $f_{X|Y}(x|y) = f_{X,Y}(x,y) \Big/ f_Y(y) = \frac{\lambda^3 x e^{-\lambda y}}{(\lambda^3/2) y^2 e^{-\lambda y}}$ (for $0 \le x \le y$, and 0 otherwise). So,

$$\begin{aligned} E[X|Y = 1] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|1)dx && (1) \\ &= \int_0^1 x \frac{\lambda^3 x e^{-\lambda}}{(\lambda^3/2)e^{-\lambda}} dx && (2) \\ &= 2\int_0^1 x^2 dx && (3) \\ &= 2/3. && (4) \end{aligned}$$

5. Consider a two stage experiment. In the first stage, n fair dice are rolled. In the second stage, the dice that came up 6 in the first stage are rolled again. Let X be the number of dice that come up 6 in the second stage. Calculate $E[X]$ and $\text{var}(X)$ in terms of n.

Solution:
Expectation. Let Y denote the number of dice that come up 6 in the first stage. Then, the tower property says

$$E[X] = E[E[X|Y]] = E[(1/6)Y] = (1/6)E[Y] = (1/6)(1/6)n = n/36.$$

Note that in the second step above, we are using the fact that if we condition on a fixed value $Y = y$, then X has a Binomial distribution based on y trials and success probability $1/6$.

Variance.
Next we use the law of total variance,

$$\text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y]) \qquad (*)$$

Let's handle the second term on the right side of (*). Recall that $E[X|Y] = (1/6)Y$, and by itself, Y is Binomial$(n, 1/6)$.

Hence $\text{var}(E[X|Y]) = \text{var}((1/6)Y) = (1/6)^2\,\text{var}(Y) = (1/6)^2 n \cdot (1/6)(1-1/6) = \frac{5}{6^4}n$.

Now let's handle the first term on the right side of (*). Again, when we condition on $Y = y$, the variable X is Binomial based on y trials and success probability $1/6$, so

$$\text{var}(X|Y) = Y \cdot (1/6)(1-1/6) = \frac{5}{36}Y.$$

In turn, we get

$$E[\text{var}(X|Y)] = E[(5/36)Y] = (5/36) \cdot (1/6)n,$$

since Y is Binomial$(n, 1/6)$.

Putting the pieces together gives

$$\text{var}(X) = (\frac{5}{6^3} + \frac{5}{6^4})n.$$

6. Suppose X, Y, and Z are independent Uniform[0,1] variables. Calculate the numerical value of the probability $P(X^2 \geq YZ)$.

Solution:

$$\begin{aligned}
P(X^2 \geq YZ) &= 1 - P(X < \sqrt{YZ}) \quad \text{since } Y, Z \geq 0 && (5)\\
&= 1 - E[1\{X < \sqrt{YZ}\}] && (6)\\
&= 1 - E[E[1\{X < \sqrt{YZ}\}|YZ]] && (7)\\
&= 1 - E[\sqrt{Y}\sqrt{Z}] \quad \text{since } X \text{ is Uniform[0,1], we have } P(X < x) = x. && (8)\\
&= 1 - E[\sqrt{Y}]E[\sqrt{Z}] && (9)\\
& && (10)
\end{aligned}$$

Now $E[\sqrt{Y}] = \int_0^1 y^{1/2}dy = \frac{2}{3}$, and $E[\sqrt{Z}]$, is the same. Hence,

$$P(X^2 \geq YZ) = 1 - (2/3)^2 = 5/9.$$

7. Suppose that you have a stick of length 1 meter. If it is randomly broken into two pieces, what is the expected length of the smaller piece? (By randomly broken, we mean that the "break point" is uniformly distributed in the interval [0,1].)

Solution: If we let U ∼Uniform[0,1] denote the location of the break point, then the length L of the smaller piece is $L = \min(U, 1-U)$. So, we want to calculate $E[L]$. This is

$$\begin{aligned} E[L] &= \int_0^1 \min(u, 1-u)du && (11) \\ &= \int_0^{1/2} udu + \int_{1/2}^1 (1-u)du && (12) \\ &= (1/8) + (1/8) && (13) \\ &= 1/4 && (14) \end{aligned}$$

(This is smaller than you might have expected.)

8. Let U be a Uniform[0,1] random variable. Also, suppose that conditionally on $U = u$, the random variable X has a Binomial(n, u) distribution (corresponding to success probability u). Calculate the mgf of X and use this to show that X is equally likely to take the values $0, 1, \ldots, n$.

Solution: Note that if Y is a rv uniformly distributed on $0, \ldots, n$ then

$$M_Y(t) = \frac{1}{n+1}\Big(1 + e^t + e^{2t} + \cdots + e^{nt}\Big).$$

We want to show that $M_X(t) = M_Y(t)$. To begin, we use the tower property,

$$\begin{aligned} M_X(t) &= E[e^{tX}] && (15) \\ &= E[E[e^{tX}|U]]. && (16) \end{aligned}$$

Conditionally on U, the variable $X = V_1, \ldots, V_n$ is a sum of n independent coin flips each with head probability U, and so

$$\begin{aligned}
M_X(t) &= E[E[e^{t(V_1+\cdots V_n)}|U]] && (17)\\
&= E\Big[E[e^{tV_1}|U]\cdots E[e^{tV_n}|U]\Big] && (18)\\
&= E\Big[\Big(Ue^t+(1-U)\Big)^n\Big] && (19)\\
&= \int_0^1 (ue^t+(1-u))^n du && (20)\\
&= \frac{1}{e^t-1}\int_1^{e^t} w^n dw \qquad \text{change of variable } w := ue^t+1-u && (21)\\
&= \frac{1}{e^t-1}\frac{e^{t(n+1)}-1}{n+1} && (22)\\
&= \frac{1}{n+1}\Big(1+e^t+e^{2t}+\cdots+e^{nt}\Big). && (23)\\
& && (24)
\end{aligned}$$

This last step is algebraic. The final expression is the mgf of a variable that takes the values $0,\ldots,n$ each with probability $1/(n+1)$.

9. Suppose that the pdf of a rv X is given by $f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, where x is any real number. Consider the random variable $Y = 1/X$. Let y be a fixed real number, and derive a formula for the pdf $f_Y(y)$ as a function of y.

Solution:

Let's look at the cdf.

$$F_Y(y) = P(Y \le y) = P((1/X) \le y) \tag{25}$$

For simplicity, let's suppose y is positive to start with.

Note that

$$\begin{aligned}
\{(1/X) \le y\} &= \Big(\{(1/X) \le y\} \cap \{X < 0\}\Big) \cup \Big(\{(1/X) \le y\} \cap \{X \ge 0\}\Big) && (26)\\
&= \{X < 0\} \cup \{X \ge 1/y\} && (27)
\end{aligned}$$

where the possibility $X = 0$ doesn't matter since X has a continuous distribution. Also note that the two sets on the right are disjoint. Hence,

$$F_Y(y) = P(X \le 0) + \Big(1 - F(1/y)\Big)$$

Now, let's differentiate with respect to y.

$$\begin{aligned}
f_Y(y) &= f_X(1/y)\cdot(y^{-2}) && (28)\\
&= \frac{1}{y^2}\frac{1}{\pi}\frac{1}{1+(1/y)^2} && (29)\\
&= \frac{1}{\pi}\frac{1}{1+y^2}. && (30)
\end{aligned}$$

If we had let $y < 0$ at the beginning, the calculation could have been done in a similar way, and we would have gotten the same answer.

Hence, it turns out that Y has the same distribution as X.