

## **Body Fat: Compare Models**

Va	ariat	les	in M	odel	$\hat{eta}_1$		,	$\hat{eta}_2$		<b>s</b> {β̂	}	s	$\hat{\beta}_2$ }	1	MSE	
M	odel	1: 2	<b>Χ</b> 1		0.85	72		-		0.12	88		-		7.95	
	odel		_		-		-	3565		-		-	100		6.3	
M	odel	3: 2	$X_1, X$	2	0.22	24	0.6	5594	.	0.30	34	0.2	2912		6.47	
М	odel	4: 2	$X_1, X$	$_{2},X_{3}$	4.33	34	-2	.857		3.01	6	2.	582		6.15	

- The regression coefficient for X<sub>1</sub> (X<sub>2</sub>)
   depending on which other X variables are included in the model.
- The standard errors of the fitted regression coefficients are becoming when more X variables are included into the model.
- MSE tends to as additional X variables are added into the model.











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М	odel	4: 2	$X_1, X$	$_{2}$ , $X_{2}$	4.33	34	-2	.857		3.01	6	2.	582		6.15	

- The regression coefficient for  $X_1$  ( $X_2$ ) varies drastically depending on which other X variables are included in the model.
- The standard errors of the fitted regression coefficients are becoming inflated when more X variables are included into the model.
- MSE tends to decrease as additional X variables are added into the model.











- $SSR(X_1) = 352.27, SSR(X_1|X_2) = 3.47.$
- The reason why  $SSR(X_1|X_2)$  is so small compared to  $SSR(X_1)$  is that  $X_1$  and  $X_2$  are with each other and with the response variable Y.
  - When  $X_2$  is already in the model, the marginal contribution from  $X_1$  in explaining Y is since  $X_2$  contains much of the information as  $X_1$  in terms of explaining Y.

What would happen if  $X_1$  and  $X_2$  were not correlated with Y, but were highly correlated among themselves?

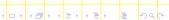
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- From the general linear test perspective, each T-test is a test, testing whether the of an X variable is significant given
  - X variables being included in the model.
- The three tests of the marginal effects of  $X_1$ ,  $X_2$ ,  $X_3$  together are to testing whether there is a regression relation between Y and  $(X_1, X_2, X_3)$ .
- The reduced model for each individual test contains
   X variables and thus may lead to non-significant results due
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- On the other hand, the reduced model for testing regression relation contains
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In Model 4, none of the three X variables is statistically significant by the T-tests. However, the F-test for regression relation is highly significant. Is there a paradox?

- From the general linear test perspective, each T-test is a marginal test, testing whether the marginal effect of an X variable is significant given all other X variables being included in the model.
- The three tests of the marginal effects of  $X_1$ ,  $X_2$ ,  $X_3$  together are not equivalent to testing whether there is a regression relation between Y and  $(X_1, X_2, X_3)$ .
- The reduced model for each individual test contains all other X variables and thus may lead to non-significant results due to multicollinearity.
- On the other hand, the reduced model for testing regression relation contains no X variable.



#### Effects of Multicollinearity: Summary

- With multicollinearity, the estimated regression coefficients tend to have sampling variability (i.e., standard errors). This leads to:
  - confidence intervals.
  - It's possible that of the regression coefficients is statistically significant, but at the same time there is a regression relation between the response variable and the entire set of X variables.
- Multicollinearity does not prevent us from getting a of the data.



## Effects of Multicollinearity: Summary

- With multicollinearity, the estimated regression coefficients tend to have large sampling variability (i.e., large standard errors). This leads to:
  - Wide confidence intervals.
  - It's possible that none of the regression coefficients is statistically significant, but at the same time there is a significant regression relation between the response variable and the entire set of X variables.
- Multicollinearity does not prevent us from getting a good fit of the data.



## Interpretation of Regression Coefficients and ESS

#### In the presence of multicollinearity:

- The regression coefficient of an X variable which other X variables are also in the model.
- Therefore, a regression coefficient reflect any inherent effect of the corresponding X variable on the response variable, but only a given whatever other X variables are also in the model.
- Similarly, there is sum of squares that can be ascribed to any one X variable.
  - The reduction in the total variation in Y ascribed to an X variable must be interpreted as a given other X variables also included in the model.

# Interpretation of Regression Coefficients and ESS

#### In the presence of multicollinearity:

- The regression coefficient of an X variable depends on which other X variables are also in the model.
- Therefore, a regression coefficient does **not** reflect any inherent effect of the corresponding X variable on the response variable, but only a marginal effect given whatever other X variables are also in the model.
- Similarly, there is **no** unique sum of squares that can be ascribed to any one X variable.
  - The reduction in the total variation in Y ascribed to an X variable must be interpreted as a margin reduction given other X variables also included in the model.

# Quantify Multicollinearity: Variance Inflation Factor

Under the standardized model:

$$\sigma^{2}(\hat{oldsymbol{eta}}^{*})=$$

- The kth diagonal element of the inverse correlation matrix  $\mathbf{r}_{XX}^{-1}$  is called the **variance inflation factor (VIF)** for  $\hat{\beta}_k^*$ , denoted by  $VIF_k$ .
- The variance of the estimated regression coefficient  $\hat{\beta}_k^*$ :

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• The variance of the estimated regression coefficient  $\hat{\beta}_k$  in the original model:

## Quantify Multicollinearity: Variance Inflation Factor

Under the standardized model:

$$\sigma^{2}(\hat{\boldsymbol{\beta}}^{*}) = \sigma^{2}\begin{bmatrix} \frac{1}{n} & \mathbf{0}^{T} \\ \mathbf{0} & \mathbf{r}_{XX}^{-1} \end{bmatrix}$$

- The kth diagonal element of the inverse correlation matrix  $\mathbf{r}_{xx}^{-1}$ is called the variance inflation factor (VIF) for  $\hat{\beta}_{k}^{*}$ , denoted by  $VIF_k$ .
- The variance of the estimated regression coefficient  $\hat{\beta}_{i}^{*}$ :

$$\sigma^2(\hat{\beta}_k^*) = VIF_k\sigma^2, \quad k = 1, \cdots, p-1.$$

The variance of the estimated regression coefficient  $\hat{\beta}_k$  in the original model:

$$\sigma^{2}(\hat{\beta}_{k}) = VIF_{k} \times \frac{\sigma^{2}}{\sum_{i=1}^{n} (X_{ik} - \bar{X}_{k})^{2}}, \quad k = 1, \dots, p-1.$$

#### It can be shown that

$$VIF_k = \frac{1}{1 - R^2} (\geq 1), \quad k = 1, \dots, p - 1,$$

where  $R_k^2$  is the coefficient of multiple determination when  $X_k$  is regressed on the rest of X variables  $\{X_j: 1 \le j \ne k \le p-1\}$ .

- If  $X_k$  is uncorrelated with the rest of the X variables, then  $R_k^2 =$  and  $VIF_k =$
- If  $R_k^2 > 0$ , then  $VIF_k$ , indicating an variance for  $\hat{\beta}_k^*$  (eqv.  $\hat{\beta}_k$ ) due to the between  $X_k$  and the other X variables.
- If  $X_k$  has a perfect linear association with the rest of the X variables, then  $R_k^2 = , VIF_k =$  and so the variance of  $\hat{\beta}_k^*$  (eqv.  $\hat{\beta}_k$ ) is
- In practice,  $\max_k VIF_k > 10$  is often taken as an indication that multicollinearity is high.



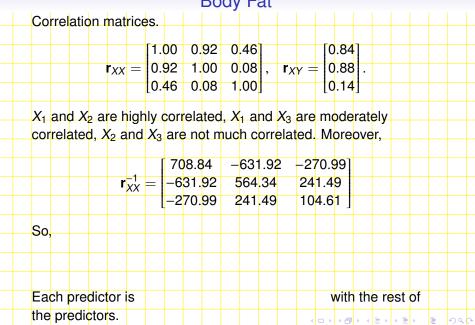
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where  $R_k^2$  is the coefficient of multiple determination when  $X_k$  is regressed on the rest of X variables  $\{X_j: 1 \le j \ne k \le p-1\}$ .

- If  $X_k$  is uncorrelated with the rest of the X variables, then  $R_k^2 = 0$  and  $VIF_k = 1$  (no inflation).
- If  $R_k^2 > 0$ , then  $VIF_k > 1$ , indicating an inflated variance for  $\hat{\beta}_k^*$  (eqv.  $\hat{\beta}_k$ ) due to the intercorrelation between  $X_k$  and the other X variables.
- If  $X_k$  has a perfect linear association with the rest of the X variables, then  $R_k^2 = 1$ ,  $VIF_k = \infty$  and so the variance of  $\hat{\beta}_k^*$  (eqv.  $\hat{\beta}_k$ ) is infinity (ill-defined).
- In practice,  $\max_k VIF_k > 10$  is often taken as an indication that multicollinearity is high.





$$\mathbf{r}_{XX} = \begin{bmatrix} 1.00 & 0.92 & 0.46 \\ 0.92 & 1.00 & 0.08 \\ 0.46 & 0.08 & 1.00 \end{bmatrix}, \quad \mathbf{r}_{XY} = \begin{bmatrix} 0.84 \\ 0.88 \\ 0.14 \end{bmatrix}.$$

 $X_1$  and  $X_2$  are highly correlated,  $X_1$  and  $X_3$  are moderately correlated,  $X_2$  and  $X_3$  are not much correlated. Moreover,

$$\mathbf{r}_{XX}^{-1} = \begin{bmatrix} 708.84 & -631.92 & -270.99 \\ -631.92 & 564.34 & 241.49 \\ -270.99 & 241.49 & 104.61 \end{bmatrix}$$

So, 
$$R_1^2 = 0.9986$$
,  $R_2^2 = 0.9982$ ,  $R_3^2 = 0.9904$ .

Each predictor is highly intercorrelated with the rest of the predictors.



#### Coefficient of Partial Determination

It measures the marginal contribution in proportional reduction in SSE by adding one X variable into a model.

Definition.

$$\begin{array}{c} R_{Y,j|1,\cdots,j-1,j+1,\cdots,p-1}^2 \\ SSE(X_1,\cdots,X_{j-1},X_{j+1},\cdots,X_{p-1}) - SSE(X_1,\cdots,X_{p-1}) \\ \vdots \\ SSE(X_1,\cdots,X_{j-1},X_{j+1},\cdots,X_{p-1}) \\ \end{array} \\ = \begin{array}{c} SSR(X_j|X_1,\cdots,X_{j-1},X_{j+1},\cdots,X_{p-1}) \\ SSE(X_1,\cdots,X_{j-1},X_{j+1},\cdots,X_{p-1}) \end{array}$$

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- Coefficients of partial determination are in between 0 and 1.
- For example,  $R_{Y,1|2}^2 = \frac{SSR(X_1|X_2)}{SSE(X_2)}$  is the proportional reduction in SSE by including  $X_1$  into the model with  $X_2$ .

From R outputs, we can obtain a number of coefficients of partial determination. E.g.:

$$R_{Y,1|2}^2 = \frac{SSE(X_2) - SSE(X_1, X_2)}{SSE(X_2)} = \frac{113.42 - 109.95}{113.43} = 3.1\%.$$

$$R_{Y,3|12}^2 =$$

 $R_{Y,2|1}^2 =$ 

• When  $X_2$  is added to the model containing  $X_1$ , SSE is reduced by ; When  $X_1$  is added to the model containing  $X_2$ , SSE is reduced by ; When  $X_3$  is added to the model containing  $X_1$ ,  $X_2$ , SSE is reduced by

From R outputs, we can obtain a number of coefficients of partial determination. E.g.:

$$R_{Y,2|1}^2 = \frac{SSE(X_1) - SSE(X_1, X_2)}{SSE(X_1)} = \frac{143.12 - 109.95}{143.12} = 23.2\%.$$

$$R_{Y,1|2}^{2} = \frac{SSE(X_{2}) - SSE(X_{1}, X_{2})}{SSE(X_{2})} = \frac{113.42 - 109.95}{113.43} = 3.1\%.$$

$$R_{Y,3|12}^{2} = \frac{SSR(X_{3}|X_{1}, X_{2})}{SSE(X_{1}, X_{2})} = \frac{11.55}{109.95} = 10.5\%.$$

 When X<sub>2</sub> is added to the model containing X<sub>1</sub>, SSE is reduced by 23.2%; When  $X_1$  is added to the model containing  $X_2$ , SSE is reduced by 3.1%; When  $X_3$  is added to the model containing  $X_1, X_2$ , SSE is reduced by 10.5%.

#### Interpretation of Coefficient of Partial Determination

- $SSR(X_i|X_1,\cdots,X_{i-1},X_{i+1},\cdots,X_{p-1})$  is the SSR when regressing the residuals  $e(Y|X_{-(i)}) = Y - \hat{Y}(X_{-(i)})$  to the residuals  $e(X_i|X_{-(i)}) = X_i - \hat{X}_i(X_{-(i)})$ , where  $X_{-(j)} = \{X_i : 1 \le i \ne j \le p\}.$
- So  $R_{Y,i|1,...|i-1|i+1,...,p-1}^2$  is the

between the two sets of residuals obtained by regressing Y and  $X_i$  to the rest of variables  $X_{-(i)}$ , respectively.

• So  $R_{Y,j|1,...,j-1,j+1,...,p-1}^2$  measures the linear association between Y and  $X_j$  after have been adjusted for.

#### Interpretation of Coefficient of Partial Determination

- $SSR(X_j|X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_{p-1})$  is the SSR when regressing the residuals  $e(Y|X_{-(j)}) = Y \hat{Y}(X_{-(j)})$  to the residuals  $e(X_j|X_{-(j)}) = X_j \hat{X}_j(X_{-(j)})$ , where  $X_{-(j)} = \{X_l : 1 \le l \ne j \le p\}$ . (Discussed in the Lab session)
- So  $R^2_{Y,j|1,...,j-1,j+1,...,p-1}$  is the coefficient of simple determination (i.e., the squared correlation coefficient) between the two sets of residuals obtained by regressing Y and  $X_i$  to the rest of variables  $X_{-(i)}$ , respectively.
- So  $R^2_{Y,j|1,\dots,j-1,j+1,\dots,p-1}$  measures the linear association between Y and  $X_j$  after the linear effects of  $X_{-(j)}$  have been adjusted for.

Example.  $R_{V,1|2}^2$ .

- Regress Y on  $X_2$ :  $e_i(Y|X_2) = Y_i \widehat{Y}_i(X_2)$ ,  $i = 1, \dots n$ .
- Regress  $X_1$  on  $X_2$ :  $e_i(X_1|X_2) = X_{i1} \hat{X}_{i1}(X_2)$ ,  $i = 1, \dots, n$ .
- $R_{Y1|2}^2$  equals to the coefficient of simple determination between  $e_i(Y|X_2)$  and  $e_i(X_1|X_2)$ .
- It measures the linear association between Y and X<sub>1</sub> after the linear effects of X<sub>2</sub> have been adjusted for.

#### Partial Correlations

The **signed** square-root of a coefficient of partial determination is called a partial correlation.

- The sign is the same as the sign of the corresponding fitted regression coefficient.
- Partial correlation is the between the
- Partial correlations can be used to find the "best" X variable to be added next for inclusion in the regression model.

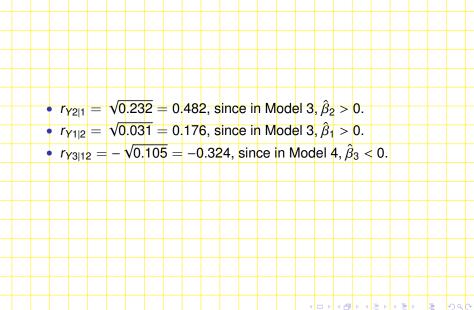
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	•	r	/1 2	=										
	•	r	/3 1:	$_{2} =$										

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# LS Fitted Regression Coefficients as Partial Coefficients

The LS fitted regression coefficients  $\hat{\beta}$  are indeed partial coefficients.

- Consider p-1 X variables in the model. Let  $\hat{\beta}_i$  be the LS fitted regression coefficient for Xi.
- Then  $\hat{\beta}_i$  equals to the LS fitted regression coefficient when regressing the residuals  $e(Y|X_{-(i)}) = Y - \hat{Y}(X_{-(i)})$  to the residuals  $e(X_i|X_{-(i)}) = X_i - \hat{X}_i(X_{-(i)})$ , where  $X_{-(i)} = \{X_l : 1 \le l \ne j \le p\}.$

Confirm this numerically with some of homework data sets.