Homework 4 (Due 2/21)

Question 1 We have introduced in class the Bonferroni-corrected two-sample test: $H_0 : \vec{\mu}_1 - \vec{\mu}_2 = \vec{\delta}_0$ is rejected if

$$\max_{1 \le j \le p} \left| \frac{(\overline{x}_{1j} - \overline{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right| \ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right).$$

Prove the following Type I error control:

$$\mathbb{P}_{null}\left(\max_{1 \le j \le p} \left| \frac{(\overline{X}_{1j} - \overline{X}_{2j}) - \delta_{0j}}{S_{pooled,j}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right| \ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right) \right) \le \alpha.$$

Question 2 Consider a sample of size $n_1 = 18$ from $\mathcal{N}_2(\vec{\mu}_1, \mathbf{\Sigma}_1)$ and a sample of size $n_2 = 18$ from $\mathcal{N}_2(\vec{\mu}_2, \mathbf{\Sigma}_2)$. Assume $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$. The summary statistics for these two samples are

$$\bar{\vec{x}}_1 = \begin{bmatrix} 85 \\ 83 \end{bmatrix}, \quad \boldsymbol{S}_1 = \begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix}, \quad \bar{\vec{x}}_2 = \begin{bmatrix} 85 \\ 87 \end{bmatrix}, \quad \boldsymbol{S}_2 = \begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix}.$$

1. Test $H_0 : \vec{\mu}_1 = \vec{\mu}_2$ at the level of $\alpha = .05$ with Hotelling's T^2;
2. Plot 95% confidence region for $\vec{\mu}_1 - \vec{\mu}_2$;
3. Find 95% simultaneous confidence intervals for $\mu_{1j} - \mu_{2j},\ j = 1, 2$ with T^2;
4. Find $\ge$ 95% simultaneous confidence intervals for $\mu_{1j} - \mu_{2j},\ j = 1, 2$ with Bonferroni correction.

Question 3 Consider two independent samples from 3-variate multivariate normal populations:

$$\text{Population 1 with } \vec{\mu}_1 = \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \end{bmatrix}: \text{ sample size } n_1 = 18,\ \bar{\vec{x}}_1 = \begin{bmatrix} 80 \\ 80 \\ 80 \end{bmatrix},\ \boldsymbol{S}_1 = \begin{bmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{bmatrix};$$

$$\text{Population 2 with } \vec{\mu}_2 = \begin{bmatrix} \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix}: \text{ sample size } n_2 = 18,\ \bar{\vec{x}}_2 = \begin{bmatrix} 78 \\ 80 \\ 82 \end{bmatrix},\ \boldsymbol{S}_2 = \begin{bmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{bmatrix}.$$

Furthermore, assume the population covariance matrices of the two populations are the same. Denote

$$\vec{d}_1 = \begin{bmatrix} d_{11} \\ d_{12} \end{bmatrix} = \begin{bmatrix} \mu_{12} - \mu_{11} \\ \mu_{13} - \mu_{12} \end{bmatrix}$$

and

$$\vec{d}_2 = \begin{bmatrix} d_{21} \\ d_{22} \end{bmatrix} = \begin{bmatrix} \mu_{22} - \mu_{21} \\ \mu_{23} - \mu_{22} \end{bmatrix}.$$

1. Test $H_0 : \vec{d}_1 = \vec{d}_2$ at the level of $\alpha = .05$ with Hotelling's T^2;
2. Plot 95% confidence region for $\vec{d}_1 - \vec{d}_2$;
3. Find 95% simultaneous confidence intervals for $d_{1j} - d_{2j},\ j = 1, 2$ with T^2;
4. Find $\ge$ 95% simultaneous confidence intervals for $d_{1j} - d_{2j},\ j = 1, 2$ with Bonferroni correction.

Question 4 Read Example 6.4 on Page 289, redo the analysis, and do Exercise 6.7 on Page 338.