4.3-1 : $T(n) = T(n-1) + n$

Assume $T(n) \le c(n+1)^2$ for $n < k$

$$T(k) = T(k-1) + k$$
$$\le ck^2 + k \le c(k+1)^2$$
$$\Rightarrow ck^2 + k \le ck^2 + 2ck + c$$
$$\Rightarrow k \le 2ck + c$$

it is correct for any value $c \ge 1$ for all $k \ge 1$

4.3-2. $T(n) = T(\lceil \frac{n}{2} \rceil) + 1$ is $O(\lg n)$

when $n=1 \Rightarrow T(1) = T(1) + 1 \Rightarrow T(1) = 0 \Rightarrow 0 \le C \cdot \lg 1$ for any C

when $K \geq 2$, Assume $T(n) \le C\lg n$ for all $n < K$,

Prove $T(K) \le C\lg K$ when $n = K$

$$T(K) = T(\lceil \tfrac{K}{2} \rceil) + 1$$
$$\le C \cdot \lg \lceil \tfrac{K}{2} \rceil + 1$$
$$\le C\lg(\tfrac{K}{2} + 1) + 1 \le C\lg K$$
$$\Rightarrow 1 \le C \lg \frac{2K}{K+2}, \quad \frac{2K}{K+2} \geq 1 \text{ for } K \geq 2.$$
$$\Rightarrow \frac{1}{\lg \frac{2K}{K+2}} \le C$$

suppose we have a function $f(x) = \frac{2K}{K+2}$

$$\frac{df(x)}{dx} = \frac{4}{(K+2)^2} > 0, \text{ so. } f(x) \uparrow \text{ when } x \uparrow.$$

So, the min value is $K=2 \Rightarrow f(2) = \frac{2 \cdot 2}{2+2} = 1$

$$\Rightarrow \text{max value of } \frac{1}{\lg \frac{2K}{K+2}} = 1$$

$$\Rightarrow 1 \le C.$$

4.3-7: $T(n) = 4T(\frac{n}{3}) + n$

Assume $T(n) \leq C \cdot n^{\log_3 4}$ for some $c > 0$ and $n < k$.

Prove $T(n) \leq cn^{\log_3 4}$ when $n = k$.

$$T(k) = 4T(\tfrac{k}{3}) + k \leq 4 \cdot c(\tfrac{k}{3})^{\log_3 4} + k$$
$$= 4 \cdot c \cdot k^{\log_3 4} \cdot \tfrac{1}{4} + k$$
$$\Rightarrow c \cdot k^{\log_3 4} + k < c \cdot k^{\log_3 4}$$
$$\Rightarrow k < 0, \text{ it is impossible.}$$

Assume that exist $\varepsilon > 0$ and $\varepsilon < \log_3 4$ and $T(n) \leq C \cdot (n^{\log_3 4} - n^{\varepsilon})$

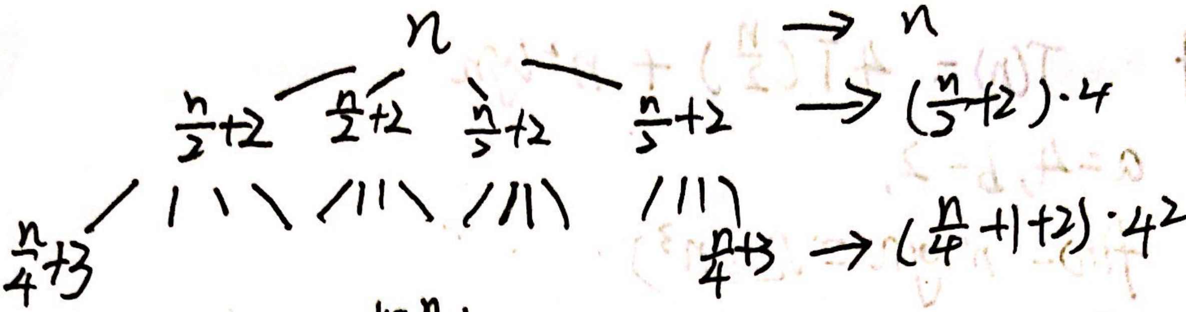
$$T(n) = 4T(\tfrac{n}{3}) + n$$
$$\leq 4 \cdot C \cdot [(\tfrac{k}{3})^{\log_3 4} - (\tfrac{k}{3})^{\varepsilon}] + k$$
$$= 4 \cdot C \cdot \tfrac{1}{4} \cdot k^{\log_3 4} - 4 \cdot c \cdot (\tfrac{k}{3})^{\varepsilon} + k \leq ck^{\log_3 4} - ck^{\varepsilon}$$
$$\Rightarrow -\tfrac{4}{3^{\varepsilon}} \cdot c \cdot k^{\varepsilon} + k \leq -ck^{\varepsilon}$$

when $\varepsilon = 1$

$$\Rightarrow -\tfrac{4}{3}ck + k \leq -ck$$
$$\Rightarrow k \leq (\tfrac{4}{3} - 1)\, ck$$
$$\Rightarrow 1 \leq \tfrac{1}{3}c$$
$$\Rightarrow 3 \leq c$$

this means that there must exsit $c \geq 3$ that can make $T(k) \leq C[k^{\log_3 4} - k]$ right.

A.4-3



$$T(n) = n + \left(\frac{n}{2}+2\right)\cdot 4 + \sum_{i=2}^{\log_2 n - 1}\left(\frac{n}{2^i} + \frac{1}{2^{i-2}} + 2\right)\cdot 4^i + \underbrace{\Theta(n^2)}_{4^{\log_2 n} = n^2}$$

$$= n + 2n + 8 + (n+4)[(n+1)-4] + \frac{2}{3}(2n^2 - 16)$$

$$= O(n^2)$$

Assume $T(n) \le c(n^2 - dn)$ when $n<k$

$$T(k) = 4T\left(\frac{k}{2}+2\right) + n$$

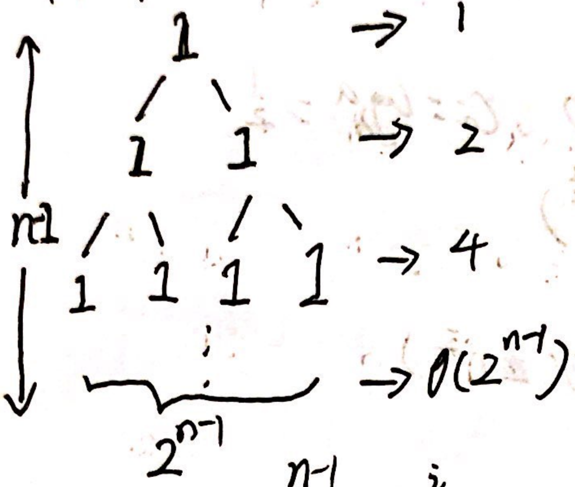
$$\le 4\left(\left(\frac{k}{2}+2\right)^2 - d\cdot\left(\frac{k}{2}+2\right)\right) + k$$

$$\Rightarrow c(n^2 - dn) - (cd - 8c - 1)n - (cd-2)\cdot 8c \le c(n^2-dn)$$

$\Rightarrow$ ~~it is correct when $n=k$, $T(n) \le c(n^2-dn)$~~

$T(k) \le c(n^2 - dn)$ when $n=k$.

4.4-4:



$n-k=1$

$k=n-1$

$$T(n) = \sum_{i=0}^{n-1} 2^i$$

$$= 2^n - 1$$

$$= O(2^n)$$

Assume $T(n) \le c\cdot 2^n + dn$ when $n<k$

Prove $T(k) \le c\cdot 2^k + dk$ when $n=k$

$$T(k) = 2T(k-1)+1$$

[illegible]

$$\le 2[c\cdot 2^{k-1} + d(k-1)] + 1 \le c\cdot 2^k + dk$$

$$2dk - 2d + 1 \le dk$$

$$dk \le 2d - 1$$

$$d \le \frac{-1}{k-2}$$

So, we can have

$$d \le -1$$

So, there exist $c>0$ that can make

$T(n) \le c\cdot 2^n + dn$ correct.

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4.5-1:

a. $a=2, b=4, \log_b a = \log_4 2 = \frac{1}{2}$

$f(n) = 1 = O(1) = O(n^0)$, $c=0$, $c_i = \log_b a = \frac{1}{2}$

$c < c_i \Rightarrow T(n) = \Theta(n^{\frac{1}{2}})$

b. $a=2, b=4, \log_b a = \frac{1}{2}$

$f(n) = \sqrt{n} = \Theta(n^{\log_b a}) = \Theta(n^{\frac{1}{2}})$

$\Rightarrow T(n) = n^{\frac{1}{2}} \lg n$

c. $a=2, b=4, \log_b a = \frac{1}{2}$

$f(n) = n = \Omega(n)$

$a \cdot f(\frac{n}{b}) = 2 \cdot \frac{n}{4} = \frac{n}{2} \le k \cdot f(n) = k \cdot n$

for some constant $k<1$ and large n.

So, $T(n) = \Theta(n)$

d. $a=2, b=4. \log_b a = \frac{1}{2}$

$f(n) = n^2 = \Omega(n^2)$

$a \cdot f(\frac{n}{b}) = \frac{n}{2} \le k \cdot f(n) = k \cdot n$ for some constant $k<1$ and large n

$T(n) = \Theta(n^2)$

4.5-4 $T(n) = 4T(\frac{n}{2}) + n^2 \lg n$

$a=4, b=2,$

$f(n) = n^2 \lg n = \Omega(n^3)$

$a \cdot f(\frac{n}{b}) = 4(\frac{n}{2})^2 \lg\frac{n}{2} = n^2 \lg n - n^2 \lg 2$

$k f(n) = k n^2 \lg n$

$af(\frac{n}{b}) \le k f(n) \Rightarrow n^2 \lg n - n^2 \lg 2 \le k n^2 \lg n$ for $k<1$

$$1 - \frac{\lg 2}{\lg n} \le k.$$

we can't find a constant k that make $1-\frac{\lg 2}{\lg n} \le k < 1$. so we can't use master theorm.

$n^2 \lg n \rightarrow n^2 \lg n$

$(\frac{n}{2})^2 \lg\frac{n}{2} \rightarrow 4 \cdot (\frac{n}{2})^2 \lg\frac{n}{2}$ $\quad \lg_2 n \quad (\frac{1}{2})^k n = 1$

$k = \lg_2 n$

$(\frac{n}{4})^2 \lg\frac{n}{4} \rightarrow 16 \cdot (\frac{n}{4})^2 \lg\frac{n}{4}$

$n^2 = 4^{\lg_2 n}$

$$T(n) = \sum_{i=0}^{\lg_2 n - 1} (2^2)^i \cdot \frac{n^2}{2^{2i}} \lg\frac{n}{2^i} + \theta(n^2)$$

$$= n^2 \sum_{i=0}^{\lg_2 n - 1} (\lg n - \lg 2^i) + \theta(n^2)$$

$$\le n^2 \sum_{i=0}^{\lg_2 n} \lg n + \theta(n^2)$$

$$= n^2 \lg n \cdot \lg n + \theta(n^2)$$

$$= O(n^2 \lg^2 n)$$

4-2:

a. Passed by Pointer:

$$T(n) = T(\tfrac{n}{2}) + C$$

$a = 1$, $b = 2$, $\log_2 1 = 0$

$f(n) = \Theta(n^0 \log^0 n) = \Theta(1)$

so, $T(n) = \Theta(\log n)$

Passed by copy:

$$T(n) = T(\tfrac{n}{2}) + \cancel{\;} cN$$
$$= 2cN + T(\tfrac{n}{4})$$
$$= 3cN + T(\tfrac{n}{8})$$
$$= \sum_{i=0}^{\log N - 1} cN$$
$$= cN\log n = \Theta(n\log n)$$

Passed by subrange

$$T(n) = T(\tfrac{n}{2}) + cn$$

$f(n) = cn = \Omega(n)$

$a \cdot f(\tfrac{n}{b}) = \frac{c}{2}n \le k f(n) = kcn$ for $k < 1$

So, $T(n) = \Theta(n)$

Scanned with CamScanner

4-2, b.

passed by pointer: $T(n) = 2T(\frac{n}{2}) + cn$

$a=2, b=2$, $\log_2 a = 1$

$f(n) = \theta(n) = \theta(n^1 \lg^0 n)$

$\Rightarrow T(n) = \theta(n\lg n)$

passed by copy:

$T(n) = 2T(\frac{n}{2}) + 2N + cn$

$= 2[2T(\frac{n}{4}) + 2N + c\frac{n}{2}) + 2N + cn$

$= 4T(\frac{n}{4}) + 2c(\frac{n}{2}) + cn + 4N$

$= \sum_{i=0}^{\lg n - 1} cn + N \sum_{i=0}^{\lg n-1} 2^i$

$= cn\lg n + nN - N = \theta(nN) = \theta(n^2)$

passed by subarray:

$T(n) = 2T(\frac{n}{2}) + 2 \cdot \frac{n}{2} + cn$

$= 2T(\frac{n}{2}) + (c+1)n$

$f(n) = \theta(n) \Rightarrow T(n) = \theta(n\lg n)$

9. (a) ~~[illegible]~~ ~~[illegible]~~

~~[illegible]~~ 235 789

146 | 1 < 2 one time
↓ | 1 < 7 two time.

Times: 2 1

46 235 789

2 < 4 one time

Times: 2 1 2 | 2 < 7 two time

46 35 789

3 < 4 one time

Times: 2 1 2 3 | 3 < 7 two time.

46 5 789.

4 < 5

Times: 2 1 2 3 4 | 4 < 7

6 5 789.

5 < 6

Times 2 1 2 3 4 5 | 5 < 7

6 789.

Times 1 1 2 3 4 5 6 $\boxed{789}$ | 6 < 7

Total times : ~~[illegible]~~ $2 \times 5 + 1 = 11 < \frac{5}{3} \times 9$ when $n = 9$.

Find the smallest value from given 3 values. it needs 2 times compare.

The worst case:

~~[illegible]~~

As Example of $n = 9$

147 , 258 , 369

it needs $\frac{2}{3} \times n \times 2$ times to sort on $\frac{2}{3}n$ values.

7 , 8 , 9

The final $\frac{1}{3}n$ only needs $\frac{1}{3}n$ compares.

So, $\frac{2}{3}n \times 2 + \frac{1}{3}n = \frac{5}{3}n$ is the worst case.

Scanned with CamScanner

(b) $T(n) = 3T(\frac{n}{3}) + \frac{5}{3}n$

$f(n) = \frac{5}{3}n = O(n)$

$a=3,\ b=3,\ \log_3 3 = 1$

$\Rightarrow T(n) = n\log n$

10. $T(n) = 2T(\frac{n}{2}) + \frac{n}{2}\cdot 2 + \sum_{i=0}^{\frac{n}{2}-1}\sum_{j=i+1}^{\frac{n}{2}-1} 1$

$= 2T(\frac{n}{2}) + O(n^2)$

$f(n) = \Omega(n^2)$, $a=2, b=2, \log_b a = 1$

$af(\frac{n}{b}) = 2\cdot(\frac{n}{2})^2 = \frac{1}{2}n^2 \le k\cdot n^2$ for constant $k<1$

so $T(n) = \Theta(n^2)$