Homework 3 Solution

Question 1 From the definition of population cross-covariance matrix, we have

$$\begin{aligned}
Cov(\vec{X},\vec{Y}) &= \begin{bmatrix} E((X_1-E(X_1))(Y_1-E(Y_1))) & \dots & E((X_1-E(X_1))(Y_q-E(Y_q))) \\ E((X_2-E(X_2))(Y_1-E(Y_1))) & \dots & E((X_2-E(X_2))(Y_q-E(Y_q))) \\ \vdots & \ddots & \vdots \\ E((X_p-E(X_p))(Y_1-E(Y_1))) & \cdots & E((X_p-E(X_p))(Y_q-E(Y_q))) \end{bmatrix} \\
&= E\left[(\vec{X}-E(\vec{X}))(\vec{Y}-E(\vec{Y}))^\top\right]
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
Cov(\boldsymbol{C}\vec{X},\boldsymbol{D}\vec{Y}) &= E\left[(\boldsymbol{C}\vec{X}-E(\boldsymbol{C}\vec{X}))(\boldsymbol{D}\vec{Y}-E(\boldsymbol{D}\vec{Y}))^\top\right] \\
&= \boldsymbol{C}Cov(\vec{X},\vec{Y})\boldsymbol{D}^\top
\end{aligned}$$

Question 2 As $\vec{X}$ and $\vec{Y}$ are independent, we know that X_i and Y_j are independent for all i,j. Therefore, $Cov(X_i,Y_j)=0$. Based on the definition of population cross-covariance matrix, every element in $Cov(\vec{X},\vec{Y})$ is equal to 0. Therefore, $Cov(\vec{X},\vec{Y})=\mathbf{0}_{p\times q}$

Question 3 By question 2, we know that $Cov(\vec{X}_i,\vec{X}_j)=\mathbf{0}_{p\times p}$, for all $i\neq j$. Furthermore, for constant vector $\vec{c}$, we have $Cov(\vec{X}_i,\vec{c})=\mathbf{0}_{p\times p}$.
Therefore, we have

$$\begin{aligned}
Cov(a_1\vec{X}_1+a_2\vec{X}_2+...+a_n\vec{X}_n+\vec{c}) &= \sum_{i=1}^{n} Cov(a_i\vec{X}_i,a_i\vec{X}_i) + \sum_{i\neq j} Cov(a_i\vec{X}_i,a_j\vec{X}_j) + \sum_{i=1}^{n} Cov(a_i\vec{X}_i,\vec{c}) \\
&= \sum_{i=1}^{n} Cov(a_i\vec{X}_i,a_i\vec{X}_i) \\
&= \sum_{i=1}^{n} E((a_i\vec{X}_i-E(a_i\vec{X}_i))(a_i\vec{X}_i-E(a_i\vec{X}_i))^\top) \\
&= \sum_{i=1}^{n} a_i^2 Cov(\vec{X}_i) \\
&= a_1^2Cov(\vec{X}_1)+a_2^2Cov(\vec{X}_2)+...+a_n^2Cov(\vec{X}_n)
\end{aligned}$$

Question 4 Let

$$\vec{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ . \\ Y_p \end{bmatrix} = \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ . \\ \vec{v}_p^\top \end{bmatrix} \vec{X}$$

Then,

$$
\begin{aligned}
Cov(\vec{Y}) &= \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ . \\ \vec{v}_p^\top \end{bmatrix} Cov(\vec{X}) \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & ... & \vec{v}_p \end{bmatrix} \\
&= \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ . \\ \vec{v}_p^\top \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & ... & \vec{v}_p \end{bmatrix} \begin{bmatrix} \lambda_1 & . & . & 0 \\ . & . & & . \\ . & & . & . \\ 0 & . & . & \lambda_p \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ . \\ \vec{v}_p^\top \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & ... & \vec{v}_p \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & . & . & 0 \\ . & . & & . \\ . & & . & . \\ 0 & . & . & \lambda_p \end{bmatrix}
\end{aligned}
$$

Therefore, $\vec{Y}$ follows a multivariate Normal distribution, with covariance matrix being diagonal. So Y_i and Y_j are independent for $i \neq j$.

Question 5

(a)

$$
\begin{aligned}
Cov(X_1, X_3 - (aX_1 + bX_2) &= Cov(X_1, X_3) - aCov(X_1, X_1) - bCov(X_1, X_2) \\
&= 0 - 2a - b \\
&= 0
\end{aligned}
$$

Similarly, $1 - a - 2b = 0$. So $a = -\frac{1}{3}$, $b = \frac{2}{3}$.

Similarly,

$$
\begin{aligned}
0 - 2c - d = 0 \\
0 - c - 2d = 0
\end{aligned}
$$

So $c = d = 0$.

(b) Let $\vec{Y} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 - (-\frac{1}{3}X_1 + \frac{2}{3}X_2) \\ X_4 \end{bmatrix}$. Then $\vec{Y}$ follows a multivariate Normal distribution. By part (a), $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $\begin{bmatrix} X_3 - (-\frac{1}{3}X_1 + \frac{2}{3}X_2) \\ X_4 \end{bmatrix}$ are independent.

Also we have

$$
\begin{aligned}
Cov(\vec{Y}) &= Cov(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{X}) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\top \\
&= \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\end{aligned}
$$

Because of independence, the distribution of $\begin{bmatrix} X_3 - (-\frac{1}{3}X_1 + \frac{2}{3}X_2) \\ X_4 \end{bmatrix}$ will not be affected by the value of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. So given $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} X_3 - (-\frac{1}{3}X_1 + \frac{2}{3}X_2) \\ X_4 \end{bmatrix} \sim \mathcal{N}_2(\vec{0}, \begin{bmatrix} \frac{4}{3} & 1 \\ 1 & 2 \end{bmatrix})$. Therefore, given $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$\begin{bmatrix} X_3 \\ X_4 \end{bmatrix} |(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) \sim_2 \mathcal{N}(\begin{bmatrix} (-\frac{1}{3}x_1 + \frac{2}{3}x_2) \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} & 1 \\ 1 & 2 \end{bmatrix})$$

Question 6

$$\bar{\vec{X}} = \frac{1}{n}\sum_{i=1}^{n} \vec{X}_i$$

is still Normally distributed. And

$$\begin{aligned} E(\bar{\vec{X}}) &= E(\frac{1}{n}\sum_{i=1}^{n} \vec{X}_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} E(\vec{X}_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} \vec{\mu} \\ &= \vec{\mu} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} Cov(\bar{\vec{X}}) &= Cov(\frac{1}{n}\sum_{i=1}^{n} \vec{X}_i) \\ &= Cov(\sum_{i=1}^{n} \frac{1}{n}\vec{X}_i) \\ &= \sum_{i=1}^{n} Cov(\frac{1}{n}\vec{X}_i) \\ &= \sum_{i=1}^{n} \frac{1}{n^2} Cov(\vec{X}_i) \\ &= \sum_{i=1}^{n} \frac{1}{n^2}\Sigma \\ &= \frac{1}{n}\Sigma \\ &= \frac{1}{20}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{4} \end{bmatrix} \end{aligned}$$

So,

$$\vec{\bar{X}} \sim \mathcal{N}_2(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{4} \end{bmatrix})$$

By property 6 in lecture note,

$$n(\vec{\bar{X}} - \vec{\mu})^\top \boldsymbol{S}^{-1}(\vec{\bar{X}} - \vec{\mu}) \sim \frac{(n-1)p}{n-p} F_{p,n-p}$$

Therefore,

$$\begin{aligned}(\vec{\bar{X}} - \vec{\mu})^\top \boldsymbol{S}^{-1}(\vec{\bar{X}} - \vec{\mu}) &\sim \frac{(n-1)p}{n(n-p)} F_{p,n-p} \\ &\sim \frac{38}{360} F_{2,18}\end{aligned}$$