is said to be a *geometric* random variable with parameter p. Such a random variable represents the trial number of the first success when each trial is independently a success with probability p. Its mean and variance are given by

$$E[X] = \frac{1}{p} \quad \mathrm{Var}(X) = \frac{1-p}{p^2}$$

The random variable X whose probability mass function is given by

$$p(i) = \binom{i-1}{r-1} p^r (1-p)^{i-r} \quad i \geq r$$

is said to be a *negative binomial* random variable with parameters r and p. Such a random variable represents the trial number of the rth success when each trial is independently a success with probability p. Its mean and variance are given by

$$E[X] = \frac{r}{p} \quad \mathrm{Var}(X) = \frac{r(1-p)}{p^2}$$

A *hypergeometric* random variable X with parameters n, N, and m represents the number of white balls selected when n balls are randomly chosen from an urn that contains N balls of which m are white. The probability mass function of this random variable is given by

$$p(i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \quad i = 0, \ldots, m$$

With $p = m/N$, its mean and variance are

$$E[X] = np \quad \mathrm{Var}(X) = \frac{N-n}{N-1} np(1-p)$$

An important property of the expected value is that the expected value of a sum of random variables is equal to the sum of their expected values. That is,

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

PROBLEMS

4.1. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

4.2. Two fair dice are rolled. Let X equal the product of the 2 dice. Compute $P\{X = i\}$ for $i = 1, \ldots, 36$.

4.3. Three dice are rolled. By assuming that each of the $6^3 = 216$ possible outcomes is equally likely, find the probabilities attached to the possible values that X can take on, where X is the sum of the 3 dice.

4.4. Five men and 5 women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman. (For instance, $X = 1$

if the top-ranked person is female.) Find P{X = i}, $i = 1, 2, 3, \ldots, 8, 9, 10$.

4.5. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

4.6. In Problem 5, for $n = 3$, if the coin is assumed fair, what are the probabilities associated with the values that X can take on?

4.7. Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:

(a) the maximum value to appear in the two rolls;
(b) the minimum value to appear in the two rolls;
(c) the sum of the two rolls;
(d) the value of the first roll minus the value of the second roll?

4.8. If the die in Problem 7 is assumed fair, calculate the probabilities associated with the random variables in parts (a) through (d).

4.9. Repeat Example 1b when the balls are selected with replacement.

4.10. In Example 1d, compute the conditional probability that we win i dollars, given that we win something; compute it for $i = 1, 2, 3$.

4.11. **(a)** An integer N is to be selected at random from $\{1, 2, \ldots, (10)^3\}$ in the sense that each integer has the same probability of being selected. What is the probability that N will be divisible by 3? by 5? by 7? by 15? by 105? How would your answer change if $(10)^3$ is replaced by $(10)^k$ as k became larger and larger?

(b) An important function in number theory—one whose properties can be shown to be related to what is probably the most important unsolved problem of mathematics, the Riemann hypothesis—is the Möbius function $\mu(n)$, defined for all positive integral values n as follows: Factor n into its prime factors. If there is a repeated prime factor, as in $12 = 2 \cdot 2 \cdot 3$ or $49 = 7 \cdot 7$, then $\mu(n)$ is defined to equal 0. Now let N be chosen at random from $\{1, 2, \ldots (10)^k\}$, where k is large. Determine $P\{\mu(N) = 0\}$ as $k \to \infty$.

Hint: To compute $P\{\mu(N) \neq 0\}$, use the identity

$$\prod_{i=1}^{\infty} \frac{P_i^2 - 1}{P_i^2} = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{24}{25}\right)\left(\frac{48}{49}\right)\cdots = \frac{6}{\pi^2}$$

where P_i is the ith-smallest prime. (The number 1 is not a prime.)

4.12. In the game of Two-Finger Morra, 2 players show 1 or 2 fingers and simultaneously guess the number of fingers their opponent will show. If only one of the players guesses correctly, he wins an amount (in dollars) equal to the sum of the fingers shown by him and his opponent. If both players guess correctly or if neither guesses correctly, then no money is exchanged. Consider a specified player, and denote by X the amount of money he wins in a single game of Two-Finger Morra.

(a) If each player acts independently of the other, and if each player makes his choice of the number of fingers he will hold up and the number he will guess that his opponent will hold up in such a way that each of the 4 possibilities is equally likely, what are the possible values of X and what are their associated probabilities?

(b) Suppose that each player acts independently of the other. If each player decides to hold up the same number of fingers that he guesses his opponent will hold up, and if each player is equally likely to hold up 1 or 2 fingers, what are the possible values of X and their associated probabilities?

4.13. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.

4.14. Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P\{X = i\}, i = 0, 1, 2, 3, 4$.

4.15. The National Basketball Association (NBA) draft lottery involves the 11 teams that had the worst won–lost records during the year. A total of 66 balls are placed in an urn. Each of these balls is inscribed with the name of a team: Eleven have the name of the team with the worst record, 10 have the name of the team with the second-worst record, 9 have the name of the team with the third-worst record, and so on (with 1 ball having the name of the team with the 11th-worst record). A ball is then chosen at random, and the team whose name is on the ball is given the first pick in the draft of players about to enter the league. Another ball is then chosen, and if it "belongs" to a team different from the one that received the first draft pick, then the team to which it belongs receives the second draft pick. (If the ball belongs to the team receiving the first pick, then it is discarded and another one is chosen; this continues until the ball of another team is chosen.) Finally, another ball is chosen, and the team named on the ball (provided that it is different from the previous two teams) receives the third draft pick. The remaining draft picks 4 through 11 are then awarded to the 8 teams that did not "win the lottery," in inverse order of their won–lost records. For instance, if the team with the worst record did not receive any of the 3 lottery picks, then that team would receive the fourth draft pick. Let X denote the draft pick of the team with the worst record. Find the probability mass function of X.

4.16. In Problem 15, let team number 1 be the team with the worst record, let team number 2 be the team with the second-worst record, and so on. Let Y_i denote the team that gets draft pick number i. (Thus, $Y_1 = 3$ if the first ball chosen belongs to team number 3.) Find the probability mass function of (a) Y_1, (b) Y_2, and (c) Y_3.

4.17. Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \dfrac{b}{4} & 0 \le b < 1 \\ \dfrac{1}{2} + \dfrac{b-1}{4} & 1 \le b < 2 \\ \dfrac{11}{12} & 2 \le b < 3 \\ 1 & 3 \le b \end{cases}$$

(a) Find $P\{X = i\}, i = 1, 2, 3$.
(b) Find $P\{\frac{1}{2} < X < \frac{3}{2}\}$.

4.18. Four independent flips of a fair coin are made. Let X denote the number of heads obtained. Plot the probability mass function of the random variable $X - 2$.

4.19. If the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \dfrac{1}{2} & 0 \le b < 1 \\ \dfrac{3}{5} & 1 \le b < 2 \\ \dfrac{4}{5} & 2 \le b < 3 \\ \dfrac{9}{10} & 3 \le b < 3.5 \\ 1 & b \ge 3.5 \end{cases}$$

calculate the probability mass function of X.

4.20. A gambling book recommends the following "winning strategy" for the game of roulette: Bet \$1 on red. If red appears (which has probability $\frac{18}{38}$), then take the \$1 profit and quit. If red does not appear and you lose this bet (which has probability $\frac{20}{38}$ of occurring), make additional \$1 bets on red on each of the next two spins of the roulette wheel and then quit. Let X denote your winnings when you quit.
(a) Find $P\{X > 0\}$.
(b) Are you convinced that the strategy is indeed a "winning" strategy? Explain your answer!
(c) Find $E[X]$.

4.21. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.
(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?
(b) Compute $E[X]$ and $E[Y]$.

4.22. Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when (a) $i = 2$ and (b) $i = 3$. Also, show in both cases that this number is maximized when $p = \frac{1}{2}$.

4.23. You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely.
(a) If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?
(b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?

4.24. A and B play the following game: A writes down either number 1 or number 2, and B must guess which one. If the number that A has written down is i and B has guessed correctly, B receives i units from A. If B makes a wrong guess, B pays $\frac{3}{4}$ unit to A. If B randomizes his decision by guessing 1 with probability p and 2 with probability $1 - p$, determine his expected gain if (a) A has written down number 1 and (b) A has written down number 2.

What value of p maximizes the minimum possible value of B's expected gain, and what is this maximin value? (Note that B's expected gain depends not only on p, but also on what A does.)

Consider now player A. Suppose that she also randomizes her decision, writing down number 1 with probability q. What is A's expected loss if (c) B chooses number 1 and (d) B chooses number 2?

What value of q minimizes A's maximum expected loss? Show that the minimum of A's maximum expected loss is equal to the maximum of B's minimum expected gain. This result, known as the minimax theorem, was first established in generality by the mathematician John von Neumann and is the fundamental result in the mathematical discipline known as the theory of games. The common value is called the value of the game to player B.

4.25. Two coins are to be flipped. The first coin will land on heads with probability .6, the second with probability .7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.
(a) Find $P\{X = 1\}$.
(b) Determine $E[X]$.

4.26. One of the numbers 1 through 10 is randomly chosen. You are to try to guess the number chosen by asking questions with "yes–no" answers. Compute the expected number of questions you will need to ask in each of the following two cases:
(a) Your ith question is to be "Is it i?" $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
(b) With each question you try to eliminate one-half of the remaining numbers, as nearly as possible.

4.27. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of A?

4.28. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

4.29. There are two possible causes for a breakdown of a machine. To check the first possibility would cost C_1 dollars, and, if that were the cause of the breakdown, the trouble could be repaired at a cost of R_1 dollars. Similarly, there are costs C_2 and R_2 associated with the second possibility. Let p and $1 - p$ denote, respectively, the probabilities that the breakdown is caused by the first and second possibilities. Under what conditions on $p, C_i, R_i, i = 1, 2$, should we check the first possible cause of breakdown and then the second, as opposed to reversing the checking order, so as to minimize the expected cost involved in returning the machine to working order?

Note: If the first check is negative, we must still check the other possibility.

4.30. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let X denote the player's winnings. Show that $E[X] = +\infty$. This problem is known as the St. Petersburg paradox.
(a) Would you be willing to pay $1 million to play this game once?
(b) Would you be willing to pay $1 million for each game if you could play for as long as you liked and only had to settle up when you stopped playing?

4.31. Each night different meteorologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\begin{array}{ll} 1 - (1 - p)^2 & \text{if it does rain} \\ 1 - p^2 & \text{if it does not rain} \end{array}$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of weather. Suppose now that a given meteorologist is aware of our scoring mechanism and wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^*, what value of p should he or she assert so as to maximize the expected score?

4.32. To determine whether they have a certain disease, 100 people are to have their blood tested. However, rather than testing each individual separately, it has been decided first to place the people into groups of 10. The blood samples of the 10 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 10 people, whereas if the test is positive, each of the 10 people will also be individually tested and, in all, 11 tests will be made on this group. Assume that the probability that a person has the disease is .1 for all people, independently of each other, and compute the expected number of tests necessary for each group. (Note that we are assuming that the pooled test will be positive if at least one person in the pool has the disease.)

4.33. A newsboy purchases papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. If his daily demand is a binomial random variable with $n = 10, p = \frac{1}{3}$, approximately how many papers should he purchase so as to maximize his expected profit?

4.34. In Example 4b, suppose that the department store incurs an additional cost of c for each unit of unmet demand. (This type of cost is often referred to as a goodwill cost because the store loses the goodwill of those customers whose demands it cannot meet.) Compute the expected profit when the store stocks s units, and determine the value of s that maximizes the expected profit.

4.35. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win $-\$1.00$. (That is, you lose \$1.00.) Calculate
(a) the expected value of the amount you win;
(b) the variance of the amount you win.

4.36. Consider Problem 22 with $i = 2$. Find the variance of the number of games played, and show that this number is maximized when $p = \frac{1}{2}$.

4.37. Find $\text{Var}(X)$ and $\text{Var}(Y)$ for X and Y as given in Problem 21.

4.38. If $E[X] = 1$ and $\text{Var}(X) = 5$, find
(a) $E[(2 + X)^2]$;
(b) $\text{Var}(4 + 3X)$.

4.39. A ball is drawn from an urn containing 3 white and 3 black balls. After the ball is drawn, it is replaced and another ball is drawn. This process goes on indefinitely. What is the probability that, of the first 4 balls drawn, exactly 2 are white?

4.40. On a multiple-choice exam with 3 possible answers for each of the 5 questions, what is the probability that a student will get 4 or more correct answers just by guessing?

4.41. A man claims to have extrasensory perception. As a test, a fair coin is flipped 10 times and the man is asked to predict the outcome in advance. He gets 7 out of 10 correct. What is the probability that he would have done at least this well if he had no ESP?

4.42. Suppose that, in flight, airplane engines will fail with probability $1 - p$, independently from engine to engine. If an airplane needs a majority of its engines operative to complete a successful flight, for what values of p is a 5-engine plane preferable to a 3-engine plane?

4.43. A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability .2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded? What independence assumptions are you making?

4.44. A satellite system consists of n components and functions on any given day if at least k of the n components function on that day. On a rainy day each of the components independently functions with probability p_1, whereas on a dry day they each independently function with probability p_2. If the probability of rain tomorrow is α, what is the probability that the satellite system will function?

4.45. A student is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. He figures that if he has an on day, then each of his examiners will pass him, independently of each other, with probability .8, whereas if he has an off day, this probability will be reduced to .4. Suppose that the student will pass the examination if a majority of the examiners pass him. If the student feels that he is twice as likely to have an off day as he is to have an on day, should he request an examination with 3 examiners or with 5 examiners?

4.46. Suppose that it takes at least 9 votes from a 12-member jury to convict a defendant. Suppose also that the probability that a juror votes a guilty person innocent is .2, whereas the probability that the juror votes an innocent person guilty is .1. If each juror acts independently and if 65 percent of the defendants are guilty, find the probability that the jury renders a correct decision. What percentage of defendants is convicted?

4.47. In some military courts, 9 judges are appointed. However, both the prosecution and the defense attorneys are entitled to a peremptory challenge of any judge, in which case that judge is removed from the case and is not replaced. A defendant is declared guilty if the majority of judges cast votes of guilty, and he or she is declared innocent otherwise. Suppose that when the defendant is, in fact, guilty, each judge will (independently) vote guilty with probability .7, whereas when the defendant is, in fact, innocent, this probability drops to .3.
(a) What is the probability that a guilty defendant is declared guilty when there are (i) 9, (ii) 8, and (iii) 7 judges?
(b) Repeat part (a) for an innocent defendant.
(c) If the prosecution attorney does not exercise the right to a peremptory challenge of a judge, and if the defense is limited to at most two such challenges, how many challenges should the defense attorney make if he or she is 60 percent certain that the client is guilty?

4.48. It is known that diskettes produced by a certain company will be defective with probability .01, independently of each other. The company sells the diskettes in packages of size 10 and offers a money-back guarantee that at most 1 of the 10 diskettes in the package will be defective. The guarantee is that the customer can return the entire package of diskettes if he or she finds more than one defective diskette in it. If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them?

4.49. When coin 1 is flipped, it lands on heads with probability .4; when coin 2 is flipped, it lands on heads with probability .7. One of these coins is randomly chosen and flipped 10 times.

(a) What is the probability that the coin lands on heads on exactly 7 of the 10 flips?

(b) Given that the first of these ten flips lands heads, what is the conditional probability that exactly 7 of the 10 flips land on heads?

4.50. Suppose that a biased coin that lands on heads with probability p is flipped 10 times. Given that a total of 6 heads results, find the conditional probability that the first 3 outcomes are

(a) h, t, t (meaning that the first flip results in heads, the second in tails, and the third in tails);

(b) t, h, t.

4.51. The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that the next page you read contains (a) 0 and (b) 2 or more typographical errors? Explain your reasoning!

4.52. The monthly worldwide average number of airplane crashes of commercial airlines is 3.5. What is the probability that there will be

(a) at least 2 such accidents in the next month;

(b) at most 1 accident in the next month?

Explain your reasoning!

4.53. Approximately 80,000 marriages took place in the state of New York last year. Estimate the probability that, for at least one of these couples,

(a) both partners were born on April 30;

(b) both partners celebrated their birthday on the same day of the year.

State your assumptions.

4.54. Suppose that the average number of cars abandoned weekly on a certain highway is 2.2. Approximate the probability that there will be

(a) no abandoned cars in the next week;

(b) at least 2 abandoned cars in the next week.

4.55. A certain typing agency employs 2 typists. The average number of errors per article is 3 when typed by the first typist and 4.2 when typed by the second. If your article is equally likely to be typed by either typist, approximate the probability that it will have no errors.

4.56. How many people are needed so that the probability that at least one of them has the same birthday as you is greater than $\frac{1}{2}$?

4.57. Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

(a) Find the probability that 3 or more accidents occur today.

(b) Repeat part (a) under the assumption that at least 1 accident occurs today.

4.58. Compare the Poisson approximation with the correct binomial probability for the following cases:

(a) $P\{X = 2\}$ when $n = 8, p = .1$;

(b) $P\{X = 9\}$ when $n = 10, p = .95$;

(c) $P\{X = 0\}$ when $n = 10, p = .1$;

(d) $P\{X = 4\}$ when $n = 9, p = .2$.

4.59. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $\frac{1}{100}$, what is the (approximate) probability that you will win a prize

(a) at least once?

(b) exactly once?

(c) at least twice?

4.60. The number of times that a person contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 5$. Suppose that a new wonder drug (based on large quantities of vitamin C) has just been marketed that reduces the Poisson parameter to $\lambda = 3$ for 75 percent of the population. For the other 25 percent of the population, the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 2 colds in that time, how likely is it that the drug is beneficial for him or her?

4.61. The probability of being dealt a full house in a hand of poker is approximately .0014. Find an approximation for the probability that, in 1000 hands of poker, you will be dealt at least 2 full houses.

4.62. Consider n independent trials, each of which results in one of the outcomes $1, \ldots, k$ with respective probabilities $p_1, \ldots, p_k$, $\sum_{i=1}^{k} p_i = 1$. Show that if all the p_i are small, then the probability that no trial outcome occurs more than once is approximately equal to $\exp(-n(n-1)\sum_i p_i^2/2)$.

4.63. People enter a gambling casino at a rate of 1 every 2 minutes.

(a) What is the probability that no one enters between 12:00 and 12:05?

(b) What is the probability that at least 4 people enter the casino during that time?

4.64. The suicide rate in a certain state is 1 suicide per 100,000 inhabitants per month.

(a) Find the probability that, in a city of 400,000 inhabitants within this state, there will be 8 or more suicides in a given month.

(b) What is the probability that there will be at least 2 months during the year that will have 8 or more suicides?

(c) Counting the present month as month number 1, what is the probability that the first month to have 8 or more suicides will be month number $i, i \geq 1$?

What assumptions are you making?

4.65. Each of 500 soldiers in an army company independently has a certain disease with probability $1/10^3$. This disease will show up in a blood test, and to facilitate matters, blood samples from all 500 soldiers are pooled and tested.

(a) What is the (approximate) probability that the blood test will be positive (that is, at least one person has the disease)?

Suppose now that the blood test yields a positive result.

(b) What is the probability, under this circumstance, that more than one person has the disease?

One of the 500 people is Jones, who knows that he has the disease.

(c) What does Jones think is the probability that more than one person has the disease?

Because the pooled test was positive, the authorities have decided to test each individual separately. The first $i - 1$ of these tests were negative, and the ith one—which was on Jones—was positive.

(d) Given the preceding, scenario, what is the probability, as a function of i, that any of the remaining people have the disease?

4.66. A total of $2n$ people, consisting of n married couples, are randomly seated (all possible orderings being equally likely) at a round table. Let C_i denote the event that the members of couple i are seated next to each other, $i = 1, \ldots, n$.

(a) Find $P(C_i)$.

(b) For $j \neq i$, find $P(C_j|C_i)$.

(c) Approximate the probability, for n large, that there are no married couples who are seated next to each other.

4.67. Repeat the preceding problem when the seating is random but subject to the constraint that the men and women alternate.

4.68. In response to an attack of 10 missiles, 500 antiballistic missiles are launched. The missile targets of the antiballistic missiles are independent, and each antiballstic missile is equally likely to go towards any of the target missiles. If each antiballistic missile independently hits its target with probability .1, use the Poisson paradigm to approximate the probability that all missiles are hit.

4.69. A fair coin is flipped 10 times. Find the probability that there is a string of 4 consecutive heads by

(a) using the formula derived in the text;

(b) using the recursive equations derived in the text.

(c) Compare your answer with that given by the Poisson approximation.

4.70. At time 0, a coin that comes up heads with probability p is flipped and falls to the ground. Suppose it lands on heads. At times chosen according to a Poisson process with rate λ, the coin is picked up and flipped. (Between these times the coin remains on the ground.) What is the probability that the coin is on its head side at time t? *Hint* What would be the conditional probability if there were no additional flips by time t, and what would it be if there were additional flips by time t?

4.71. Consider a roulette wheel consisting of 38 numbers 1 through 36, 0, and double 0. If Smith always bets that the outcome will be one of the numbers 1 through 12, what is the probability that

(a) Smith will lose his first 5 bets;

(b) his first win will occur on his fourth bet?

4.72. Two athletic teams play a series of games; the first team to win 4 games is declared the overall winner. Suppose that one of the teams is stronger than the other and wins each game with probability .6, independently of the outcomes of the other games. Find the probability, for $i = 4, 5, 6, 7$, that the stronger team wins the series in exactly i games. Compare the probability that the stronger team wins with the probability that it would win a 2-out-of-3 series.

4.73. Suppose in Problem 72 that the two teams are evenly matched and each has probability $\frac{1}{2}$ of winning each game. Find the expected number of games played.

4.74. An interviewer is given a list of people she can interview. If the interviewer needs to interview 5 people, and if each person (independently) agrees to be interviewed with probability $\frac{2}{3}$, what is the probability that her list of people will enable her to obtain her necessary number of interviews if the list consists of (a) 5 people and (b) 8 people? For part (b), what is the probability that the interviewer will speak to exactly (c) 6 people and (d) 7 people on the list?

4.75. A fair coin is continually flipped until heads appears for the 10th time. Let X denote the number of tails that occur. Compute the probability mass function of X.

4.76. Solve the Banach match problem (Example 8e) when the left-hand matchbox originally contained

N_1 matches and the right-hand box contained N_2 matches.

4.77. In the Banach matchbox problem, find the probability that, at the moment when the first box is emptied (as opposed to being found empty), the other box contains exactly k matches.

4.78. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly n selections?

4.79. Suppose that a batch of 100 items contains 6 that are defective and 94 that are not defective. If X is the number of defective items in a randomly drawn sample of 10 items from the batch, find (a) $P\{X = 0\}$ and (b) $P\{X > 2\}$.

4.80. A game popular in Nevada gambling casinos is Keno, which is played as follows: Twenty numbers are selected at random by the casino from the set of numbers 1 through 80. A player can select from 1 to 15 numbers; a win occurs if some fraction of the player's chosen subset matches any of the 20 numbers drawn by the house. The payoff is a function of the number of elements in the player's selection and the number of matches. For instance, if the player selects only 1 number, then he or she wins if this number is among the set of 20, and the payoff is \$2.2 won for every dollar bet. (As the player's probability of winning in this case is $\frac{1}{4}$, it is clear that the "fair" payoff should be \$3 won for every \$1 bet.) When the player selects 2 numbers, a payoff (of odds) of \$12 won for every \$1 bet is made when both numbers are among the 20,

(a) What would be the fair payoff in this case?
Let $P_{n,\,k}$ denote the probability that exactly k of the n numbers chosen by the player are among the 20 selected by the house.

(b) Compute $P_{n,\,k}$

(c) The most typical wager at Keno consists of selecting 10 numbers. For such a bet the casino pays off as shown in the following table. Compute the expected payoff:

Keno Payoffs in 10 Number Bets

Number of matches	Dollars won for each \$1 bet
0–4	-1
5	1
6	17
7	179
8	1,299
9	2,599
10	24,999

4.81. In Example 8i, what percentage of i defective lots does the purchaser reject? Find it for $i = 1, 4$. Given that a lot is rejected, what is the conditional probability that it contained 4 defective components?

4.82. A purchaser of transistors buys them in lots of 20. It is his policy to randomly inspect 4 components from a lot and to accept the lot only if all 4 are nondefective. If each component in a lot is, independently, defective with probability .1, what proportion of lots is rejected?

4.83. There are three highways in the county. The number of daily accidents that occur on these highways are Poisson random variables with respective parameters .3, .5, and .7. Find the expected number of accidents that will happen on any of these highways today.

4.84. Suppose that 10 balls are put into 5 boxes, with each ball independently being put in box i with probability p_i, $\sum_{i=1}^{5} p_i = 1$.

(a) Find the expected number of boxes that do not have any balls.

(b) Find the expected number of boxes that have exactly 1 ball.

4.85. There are k types of coupons. Independently of the types of previously collected coupons, each new coupon collected is of type i with probability p_i, $\sum_{i=1}^{k} p_i = 1$. If n coupons are collected, find the expected number of distinct types that appear in this set. (That is, find the expected number of types of coupons that appear at least once in the set of n coupons.)

THEORETICAL EXERCISES

4.1. There are N distinct types of coupons, and each time one is obtained it will, independently of past choices, be of type i with probability $P_i, i = 1, \ldots, N$. Let T denote the number one need select to obtain at least one of each type. Compute $P\{T = n\}$.
Hint: Use an argument similar to the one used in Example 1e.

4.2. If X has distribution function F, what is the distribution function of e^X?

4.3. If X has distribution function F, what is the distribution function of the random variable $\alpha X + \beta$, where α and β are constants, $\alpha \neq 0$?

4.4. For a nonnegative integer-valued random variable N, show that

$$E[N] = \sum_{i=1}^{\infty} P\{N \geq i\}$$

Hint: $\sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} P\{N = k\}$. Now interchange the order of summation.

4.5. For a nonnegative integer-valued random variable N, show that

$$\sum_{i=0}^{\infty} iP\{N > i\} = \frac{1}{2}(E[N^2] - E[N])$$

Hint: $\sum_{i=0}^{\infty} iP\{N > i\} = \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P\{N = k\}$. Now interchange the order of summation.

4.6. Let X be such that

$$P\{X = 1\} = p = 1 - P\{X = -1\}$$

Find $c \neq 1$ such that $E[c^X] = 1$.

4.7. Let X be a random variable having expected value μ and variance σ^2. Find the expected value and variance of

$$Y = \frac{X - \mu}{\sigma}$$

4.8. Find $\text{Var}(X)$ if

$$P(X = a) = p = 1 - P(X = b)$$

4.9. Show how the derivation of the binomial probabilities

$$P\{X = i\} = \binom{n}{i} p^i (1 - p)^{n-i}, \quad i = 0, \ldots, n$$

leads to a proof of the binomial theorem

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

when x and y are nonnegative.
Hint: Let $p = \frac{x}{x+y}$.

4.10. Let X be a binomial random variable with parameters n and p. Show that

$$E\left[\frac{1}{X + 1}\right] = \frac{1 - (1 - p)^{n+1}}{(n + 1)p}$$

4.11. Consider n independent sequential trials, each of which is successful with probability p. If there is a total of k successes, show that each of the $n!/[k!(n - k)!]$ possible arrangements of the k successes and $n - k$ failures is equally likely.

4.12. There are n components lined up in a linear arrangement. Suppose that each component independently functions with probability p. What is the probability that no 2 neighboring components are both nonfunctional?
Hint: Condition on the number of defective components and use the results of Example 4c of Chapter 1.

4.13. Let X be a binomial random variable with parameters (n, p). What value of p maximizes $P\{X = k\}, k = 0, 1, \ldots, n$? This is an example of a statistical method used to estimate p when a binomial (n, p) random variable is observed to equal k. If we assume that n is known, then we estimate p by choosing that value of p which maximizes $P\{X = k\}$. This is known as the *method of maximum likelihood estimation*.

4.14. A family has n children with probability $\alpha p^n, n \geq 1$, where $\alpha \leq (1 - p)/p$.

(a) What proportion of families has no children?

(b) If each child is equally likely to be a boy or a girl (independently of each other), what proportion of families consists of k boys (and any number of girls)?

4.15. Suppose that n independent tosses of a coin having probability p of coming up heads are made. Show that the probability that an even number of heads results is $\frac{1}{2}[1 + (q - p)^n]$, where $q = 1 - p$. Do this by proving and then utilizing the identity

$$\sum_{i=0}^{[n/2]} \binom{n}{2i} p^{2i} q^{n-2i} = \frac{1}{2}\left[(p + q)^n + (q - p)^n\right]$$

where $[n/2]$ is the largest integer less than or equal to $n/2$. Compare this exercise with Theoretical Exercise 3.5 of Chapter 3.

4.16. Let X be a Poisson random variable with parameter λ. Show that $P\{X = i\}$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ.
Hint: Consider $P\{X = i\}/P\{X = i - 1\}$.

4.17. Let X be a Poisson random variable with parameter λ.

(a) Show that

$$P\{X \text{ is even}\} = \frac{1}{2}\left[1 + e^{-2\lambda}\right]$$

by using the result of Theoretical Exercise 15 and the relationship between Poisson and binomial random variables.

(b) Verify the formula in part (a) directly by making use of the expansion of $e^{-\lambda} + e^{\lambda}$.

4.18. Let X be a Poisson random variable with parameter λ. What value of λ maximizes $P\{X = k\}, k \geq 0$?

4.19. Show that X is a Poisson random variable with parameter λ, then

$$E[X^n] = \lambda E[(X + 1)^{n-1}]$$

Now use this result to compute $E[X^3]$.

4.20. Consider n coins, each of which independently comes up heads with probability p. Suppose that n is large and p is small, and let $\lambda = np$. Suppose that all n coins are tossed; if at least one comes up heads, the experiment ends; if not, we again toss all n coins, and so on. That is, we stop the first time that at least one of the n coins come up heads. Let X denote the total number of heads that appear. Which of the following reasonings concerned with approximating $P\{X = 1\}$ is correct (in all cases, Y is a Poisson random variable with parameter λ)?

(a) Because the total number of heads that occur when all n coins are rolled is approximately a Poisson random variable with parameter λ,

$$P\{X = 1\} \approx P\{Y = 1\} = \lambda e^{-\lambda}$$

(b) Because the total number of heads that occur when all n coins are rolled is approximately a Poisson random variable with parameter λ, and because we stop only when this number is positive,

$$P\{X = 1\} \approx P\{Y = 1 | Y > 0\} = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

(c) Because at least one coin comes up heads, X will equal 1 if none of the other $n - 1$ coins come up heads. Because the number of heads resulting from these $n - 1$ coins is approximately Poisson with mean $(n - 1)p \approx \lambda$,

$$P\{X = 1\} \approx P\{Y = 0\} = e^{-\lambda}$$

4.21. From a set of n randomly chosen people, let E_{ij} denote the event that persons i and j have the same birthday. Assume that each person is equally likely to have any of the 365 days of the year as his or her birthday. Find

(a) $P(E_{3,4}|E_{1,2})$;

(b) $P(E_{1,3}|E_{1,2})$;

(c) $P(E_{2,3}|E_{1,2} \cap E_{1,3})$.

What can you conclude from your answers to parts (a)–(c) about the independence of the $\binom{n}{2}$ events E_{ij}?

4.22. An urn contains $2n$ balls, of which 2 are numbered 1, 2 are numbered 2, ..., and 2 are numbered n. Balls are successively withdrawn 2 at a time without replacement. Let T denote the first selection in which the balls withdrawn have the same number (and let it equal infinity if none of the pairs withdrawn has the same number). We want to show that, for $0 < \alpha < 1$,

$$\lim_n P\{T > \alpha n\} = e^{-\alpha/2}$$

To verify the preceding formula, let M_k denote the number of pairs withdrawn in the first k selections, $k = 1, \ldots, n$.

(a) Argue that when n is large, M_k can be regarded as the number of successes in k (approximately) independent trials.

(b) Approximate $P\{M_k = 0\}$ when n is large.

(c) Write the event $\{T > \alpha n\}$ in terms of the value of one of the variables M_k.

(d) Verify the limiting probability given for $P\{T > \alpha n\}$.

4.23. Consider a random collection of n individuals. In approximating the probability that no 3 of these individuals share the same birthday, a better Poisson approximation than that obtained in the text (at least for values of n between 80 and 90) is obtained by letting E_i be the event that there are at least 3 birthdays on day i, $i = 1, \ldots, 365$.

(a) Find $P(E_i)$.

(b) Give an approximation for the probability that no 3 individuals share the same birthday.

(c) Evaluate the preceding when $n = 88$ (which can be shown to be the smallest value of n for which the probability exceeds .5).

4.24. Here is another way to obtain a set of recursive equations for determining P_n, the probability that there is a string of k consecutive heads in a sequence of n flips of a fair coin that comes up heads with probability p:

(a) Argue that, for $k < n$, there will be a string of k consecutive heads if either

1. there is a string of k consecutive heads within the first $n - 1$ flips, or
2. there is no string of k consecutive heads within the first $n - k - 1$ flips, flip $n - k$ is a tail, and flips $n - k + 1, \ldots, n$ are all heads.

(b) Using the preceding, relate P_n to P_{n-1}. Starting with $P_k = p^k$, the recursion can be used to obtain P_{k+1}, then P_{k+1}, and so on, up to P_n.

4.25. Suppose that the number of events that occur in a specified time is a Poisson random variable with parameter λ. If each event is counted with probability p, independently of every other event, show that the number of events that are counted is a Poisson random variable with parameter λp. Also, give an intuitive argument as to why this should be so. As an application of the preceding result, suppose that the number of distinct uranium deposits in a given area is a Poisson random variable with parameter $\lambda = 10$. If, in a fixed period of time, each deposit is discovered independently with probability $\frac{1}{50}$, find the probability that (a) exactly 1, (b) at least 1, and (c) at most 1 deposit is discovered during that time.

4.26. Prove

$$\sum_{i=0}^{n} e^{-\lambda}\frac{\lambda^i}{i!} = \frac{1}{n!}\int_{\lambda}^{\infty} e^{-x}x^n dx$$

Hint: Use integration by parts.

4.27. If X is a geometric random variable, show analytically that

$$P\{X = n + k|X > n\} = P\{X = k\}$$

Using the interpretation of a geometric random variable, give a verbal argument as to why the preceding equation is true.

4.28. Let X be a negative binomial random variable with parameters r and p, and let Y be a binomial random variable with parameters n and p. Show that

$$P\{X > n\} = P\{Y < r\}$$

Hint: Either one could attempt an analytical proof of the preceding equation, which is equivalent to proving the identity

$$\sum_{i=n+1}^{\infty}\binom{i-1}{r-1}p^r(1-p)^{i-r} = \sum_{i=0}^{r-1}\binom{n}{i} \times p^i(1-p)^{n-i}$$

or one could attempt a proof that uses the probabilistic interpretation of these random variables. That is, in the latter case, start by considering a sequence of independent trials having a common probability p of success. Then try to express the events $\{X > n\}$ and $\{Y < r\}$ in terms of the outcomes of this sequence.

4.29. For a hypergeometric random variable, determine

$$P\{X = k + 1\}/P\{X = k\}$$

4.30. Balls numbered 1 through N are in an urn. Suppose that $n, n \le N$, of them are randomly selected without replacement. Let Y denote the largest number selected.

(a) Find the probability mass function of Y.

(b) Derive an expression for $E[Y]$ and then use Fermat's combinatorial identity (see Theoretical Exercise 11 of Chapter 1) to simplify the expression.

4.31. A jar contains $m + n$ chips, numbered $1, 2, \ldots, n + m$. A set of size n is drawn. If we let X denote the number of chips drawn having numbers that exceed each of the numbers of those remaining, compute the probability mass function of X.

4.32. A jar contains n chips. Suppose that a boy successively draws a chip from the jar, each time replacing the one drawn before drawing another. The process continues until the boy draws a chip that he has previously drawn. Let X denote the number of draws, and compute its probability mass function.

4.33. Show that Equation (8.6) follows from Equation (8.5).

4.34. From a set of n elements, a nonempty subset is chosen at random in the sense that all of the nonempty subsets are equally likely to be selected. Let X denote the number of elements in the chosen subset. Using the identities given in Theoretical Exercise 12 of Chapter 1, show that

$$E[X] = \frac{n}{2 - \left(\frac{1}{2}\right)^{n-1}}$$

$$\text{Var}(X) = \frac{n \cdot 2^{2n-2} - n(n+1)2^{n-2}}{(2^n - 1)^2}$$

Show also that, for n large,

$$\text{Var}(X) \backsim \frac{n}{4}$$

in the sense that the ratio $\text{Var}(X)$ to $n/4$ approaches 1 as n approaches ∞. Compare this formula with the limiting form of $\text{Var}(Y)$ when $P\{Y = i\} = 1/n, i = 1, \ldots, n$.

4.35. An urn initially contains one red and one blue ball. At each stage, a ball is randomly chosen and then replaced along with another of the same color. Let X denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then X is equal to 2.

(a) Find $P\{X > i\}, i \ge 1$.

(b) Show that, with probability 1, a blue ball is eventually chosen. (That is, show that $P\{X < \infty\} = 1$.)

(c) Find $E[X]$.

4.36. Suppose the possible values of X are $\{x_i\}$, the possible values of Y are $\{y_j\}$, and the possible values of $X + Y$ are $\{z_k\}$. Let A_k denote the set of all pairs of indices (i,j) such that $x_i + y_j = z_k$; that is, $A_k = \{(i,j) : x_i + y_j = z_k\}$.

(a) Argue that

$$P\{X + Y = z_k\} = \sum_{(i,j)\in A_k} P\{X = x_i, Y = y_j\}$$

(b) Show that

$$E[X + Y] = \sum_k \sum_{(i,j)\in A_k} (x_i + y_j)P\{X = x_i, Y = y_j\}$$

(c) Using the formula from part (b), argue that

$$E[X + Y] = \sum_i \sum_j (x_i + y_j)P\{X = x_i, Y = y_j\}$$

(d) Show that

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j),$$
$$P(Y = y_j) = \sum_i P\{X = x_i, Y = y_j\}$$

(e) Prove that

$$E[X + Y] = E[X] + E[Y]$$

SELF-TEST PROBLEMS AND EXERCISES

4.1. Suppose that the random variable X is equal to the number of hits obtained by a certain baseball player in his next 3 at bats. If $P\{X = 1\} = .3, P\{X = 2\} = .2$, and $P\{X = 0\} = 3P\{X = 3\}$, find $E[X]$.

4.2. Suppose that X takes on one of the values 0, 1, and 2. If for some constant $c, P\{X = i\} = cP\{X = i - 1\}, i = 1, 2$, find $E[X]$.

4.3. A coin that, when flipped, comes up heads with probability p is flipped until either heads or tails has occurred twice. Find the expected number of flips.

4.4. A certain community is composed of m families, n_i of which have i children, $\sum_{i=1}^{r} n_i = m$. If one of the families is randomly chosen, let X denote the number of children in that family. If one of the $\sum_{i=1}^{r} in_i$ children is randomly chosen, let Y denote the total number of children in the family of that child. Show that $E[Y] \geq E[X]$.

4.5. Suppose that $P\{X = 0\} = 1 - P\{X = 1\}$. If $E[X] = 3\text{Var}(X)$, find $P\{X = 0\}$.

4.6. There are 2 coins in a bin. When one of them is flipped, it lands on heads with probability .6, and when the other is flipped, it lands on heads with probability .3. One of these coins is to be randomly chosen and then flipped. Without knowing which coin is chosen, you can bet any amount up to 10 dollars, and you then either win that amount if the coin comes up heads or lose it if it comes up tails. Suppose, however, that an insider is willing to sell you, for an amount C, the information as to which coin was selected. What is your expected payoff if you buy this information? Note that if you buy it and then bet x, you will end up either winning $x - C$ or $-x - C$ (that is, losing $x + C$ in the latter case). Also, for what values of C does it pay to purchase the information?

4.7. A philanthropist writes a positive number x on a piece of red paper, shows the paper to an impartial observer, and then turns it face down on the table. The observer then flips a fair coin. If it shows heads, she writes the value $2x$ and, if tails, the value $x/2$, on a piece of blue paper, which she then turns face down on the table. Without knowing either the value x or the result of the coin flip, you have the option of turning over either the red or the blue piece of paper. After doing so and observing the number written on that paper, you may elect to receive as a reward either that amount or the (unknown) amount written on the other piece of paper. For instance, if you elect to turn over the blue paper and observe the value 100, then you can elect either to accept 100 as your reward or to take the amount (either 200 or 50) on the red paper. Suppose that you would like your expected reward to be large.

(a) Argue that there is no reason to turn over the red paper first, because if you do so, then no matter what value you observe, it is always better to switch to the blue paper.

(b) Let y be a fixed nonnegative value, and consider the following strategy: Turn over the blue paper, and if its value is at least y, then accept that amount. If it is less than y, then switch to the red paper. Let $R_y(x)$ denote the reward obtained if the philanthropist writes the amount x and you employ this strategy. Find $E[R_y(x)]$. Note that $E[R_0(x)]$ is the expected reward if the philanthropist writes the amount x when you employ the strategy of always choosing the blue paper.

4.8. Let $B(n, p)$ represent a binomial random variable with parameters n and p. Argue that

$$P\{B(n,p) \le i\} = 1 - P\{B(n,1-p) \le n-i-1\}$$

Hint: The number of successes less than or equal to i is equivalent to what statement about the number of failures?

4.9. If X is a binomial random variable with expected value 6 and variance 2.4, find $P\{X = 5\}$.

4.10. An urn contains n balls numbered 1 through n. If you withdraw m balls randomly in sequence, each time replacing the ball selected previously, find $P\{X = k\}, k = 1, \dots, m$, where X is the maximum of the m chosen numbers.
Hint: First find $P\{X \le k\}$.

4.11. Teams A and B play a series of games, with the first team to win 3 games being declared the winner of the series. Suppose that team A independently wins each game with probability p. Find the conditional probability that team A wins
(a) the series given that it wins the first game;
(b) the first game given that it wins the series.

4.12. A local soccer team has 5 more games left to play. If it wins its game this weekend, then it will play its final 4 games in the upper bracket of its league, and if it loses, then it will play its final games in the lower bracket. If it plays in the upper bracket, then it will independently win each of its games in this bracket with probability .4, and if it plays in the lower bracket, then it will independently win each of its games with probability .7. If the probability that the team wins its game this weekend is .5, what is the probability that it wins at least 3 of its final 4 games?

4.13. Each of the members of a 7-judge panel independently makes a correct decision with probability .7. If the panel's decision is made by majority rule, what is the probability that the panel makes the correct decision? Given that 4 of the judges agreed, what is the probability that the panel made the correct decision?

4.14. On average, 5.2 hurricanes hit a certain region in a year. What is the probability that there will be 3 or fewer hurricanes hitting this year?

4.15. The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter λ. However, such a random variable can be observed only if it is positive, since if it is 0 then we cannot know that such an insect was on the leaf. If we let Y denote the observed number of eggs, then

$$P\{Y = i\} = P\{X = i | X > 0\}$$

where X is Poisson with parameter λ. Find $E[Y]$.

4.16. Each of n boys and n girls, independently and randomly, chooses a member of the other sex. If a boy and girl choose each other, they become a couple. Number the girls, and let G_i be the event that girl number i is part of a couple. Let $P_0 = 1 - P(\cup_{i=1}^{n} G_i)$ be the probability that no couples are formed.
(a) What is $P(G_i)$?
(b) What is $P(G_i | G_j)$?
(c) When n is large, approximate P_0.
(d) When n is large, approximate P_k, the probability that exactly k couples are formed.
(e) Use the inclusion–exclusion identity to evaluate P_0.

4.17. A total of $2n$ people, consisting of n married couples, are randomly divided into n pairs. Arbitrarily number the women, and let W_i denote the event that woman i is paired with her husband.
(a) Find $P(W_i)$.
(b) For $i \ne j$, find $P(W_i | W_j)$.
(c) When n is large, approximate the probability that no wife is paired with her husband.
(d) If each pairing must consist of a man and a woman, what does the problem reduce to?

4.18. A casino patron will continue to make \$5 bets on red in roulette until she has won 4 of these bets.
(a) What is the probability that she places a total of 9 bets?
(b) What is her expected winnings when she stops?
Remark: On each bet, she will either win \$5 with probability $\frac{18}{38}$ or lose \$5 with probability $\frac{20}{38}$.

4.19. When three friends go for coffee, they decide who will pay the check by each flipping a coin and then letting the "odd person" pay. If all three flips produce the same result (so that there is no odd person), then they make a second round of flips, and they continue to do so until there is an odd person. What is the probability that
(a) exactly 3 rounds of flips are made?
(b) more than 4 rounds are needed?

4.20. Show that if X is a geometric random variable with parameter p, then

$$E[1/X] = \frac{-p\log(p)}{1-p}$$

Hint: You will need to evaluate an expression of the form $\sum_{i=1}^{\infty} a^i/i$. To do so, write $a^i/i = \int_0^a x^{i-1}dx$, and then interchange the sum and the integral.

4.21. Suppose that

$$P\{X = a\} = p, \quad P\{X = b\} = 1 - p$$

(a) Show that $\frac{X-b}{a-b}$ is a Bernoulli random variable.

(b) Find Var(X).

4.22. Each game you play is a win with probability p. You plan to play 5 games, but if you win the fifth game, then you will keep on playing until you lose.

(a) Find the expected number of games that you play.

(b) Find the expected number of games that you lose.

4.23. Balls are randomly withdrawn, one at a time without replacement, from an urn that initially has N white and M black balls. Find the probability that n white balls are drawn before m black balls, $n \leq N, m \leq M$.

4.24. Ten balls are to be distributed among 5 urns, with each ball going into urn i with probability p_i, $\sum_{i=1}^{5} p_i = 1$. Let X_i denote the number of balls that go into urn i. Assume that events corresponding to the locations of different balls are independent.

(a) What type of random variable is X_i? Be as specific as possible.

(b) For $i \neq j$, what type of random variable is $X_i + X_j$?

(c) Find $P\{X_1 + X_2 + X_3 = 7\}$.

4.25. For the match problem (Example 5m in Chapter 2), find

(a) the expected number of matches.

(b) the variance of the number of matches.

4.26. Let α be the probability that a geometric random variable X with parameter p is an even number.

(a) Find α by using the identity $\alpha = \sum_{i=1}^{\infty} P\{X = 2i\}$.

(b) Find α by conditioning on whether $X = 1$ or $X > 1$.