STA 200A: Homework 8

Note: Below the notation 3.T11 means Chapter 3, Theoretical Exercise 11. Similarly, the notation 4.P21 means Chapter 4, Problem 21.

1. 7.P4

Solution:

(a)

$$E[XY] = \int_0^1 \int_0^y xy\frac{1}{y}dxdy = \int_0^1 \int_0^y xdxdy = \int_0^1 \frac{y^2}{2}dy = \frac{1}{6}$$

(b)

$$E[X] = \int_0^1 \int_0^y \frac{x}{y}dxdy = \int_0^1 \frac{y}{2}dy = \frac{1}{4}$$

(c)

$$E[Y] = \int_0^1 \int_0^y dxdy = \int_0^1 ydy = \frac{1}{2}$$

2. 7.P19

Solution:

(a) This is a Geometric random variable minus 1 (because the number of insects before the first type 1 is the first type 1 time minus 1), so the mean is $\frac{1}{P_1} - 1$. It is not a big deal if you misinterpreted this to be $1/P_1$.

(b) This one is more difficult. I condition on the first time for a type 1, call this T_1 which is Geometric($1/P_1$). Let T_j be the first time type j appears, which conditional on T_1 is geometric $1/Q_i$ for $Q_i = P_i/(\sum_{j\neq 1} P_j)$ before T_j. Then the expected number of types is

$$E\sum_j 1\{T_j < T_1\} = \sum_j (1 - P\{T_j \geq T_1\}) = \sum_{j\neq 1}(1 - \sum_n P\{T_j \geq n | T_1 = n\}P\{T_1 = n\})$$

$$\sum_{n=1}^{\infty} P\{T_j \geq n | T_1 = n\}P\{T_1 = n\} = \sum_{n=1}^{\infty}(1-Q_j)^{n-1}(1-P_1)^{n-1}P_1 = P_1\frac{1}{1-(1-Q_j)(1-P_1)}$$

$$= \frac{P_1}{P_1 + Q_j - P_1 Q_j}.$$

So the expectation is

$$E\sum_j 1\{T_j < T_1\} = \sum_{j\neq 1}\left(1 - \frac{P_1}{P_1 + Q_j - P_1 Q_j}\right).$$

3. Let X and Y be continuous random variables with a joint density function $f_{X,Y}(x,y)$, and let $Z = Y/X$. Show that the density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f_{X,Y}(x, xz) dx.$$

Solution:

Let us next consider the quotient of two continuous random variables. The derivation is very similar to that for the sum of such variables, given previously: We first find the cdf and then differentiate to find the density. Suppose that X and Y are continuous with joint density function f and that $Z = Y/X$. Then $F_Z(z) = P(Z \leq z)$ is the probability of the set of (x, y) such that $y/x \leq z$. If $x > 0$, this is the set $y \leq xz$; if $x < 0$, it is the set $y \geq xz$. Thus,

$$F_Z(z) = \int_{-\infty}^{0} \int_{xz}^{\infty} f(x, y)\, dy\, dx + \int_{0}^{\infty} \int_{-\infty}^{xz} f(x, y)\, dy\, dx$$

To remove the dependence of the inner integrals on x, we make the change of variables $y = xv$ in the inner integrals and obtain

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{0} \int_{z}^{-\infty} x f(x, xv)\, dv\, dx + \int_{0}^{\infty} \int_{-\infty}^{z} x f(x, xv)\, dv\, dx \\ &= \int_{-\infty}^{0} \int_{-\infty}^{z} (-x) f(x, xv)\, dv\, dx + \int_{0}^{\infty} \int_{-\infty}^{z} x f(x, xv)\, dv\, dx \\ &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} |x| f(x, xv)\, dx\, dv \end{aligned}$$

Finally, differentiating (again under an assumption of continuity), we find

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f(x, xz)\, dx$$

4. 7.T1

Solution: Let $f(a) = E[(X-a)^2] = E[X^2] - 2aE[X] + a^2$. Then set $f'(a) = -2E[X] + 2a = 0$, we get $a = E[X]$. Also $f''(a) = 2 > 0$, so we know the minimizer is $a = E[X]$.

5. 7.T2

Solution: We follow the hint. Let $g(a) = E[|X-a|] = \int_{-\infty}^{\infty} |x-a| f(x)dx = -\int_{-\infty}^{a}(x-a)f(x)dx + \int_{a}^{\infty}(x-a)f(x)dx$. Then set $g'(a) = \int_{-\infty}^{a} f(x)dx - \int_{a}^{\infty} f(x)dx = 0$. Solve for a we have $a = F^{-1}(1/2)$. Also $g''(a) = 2f(a) > 0$.

6. 7.P26

Solution: Please fill this in. Note that the cdf for $X_{(n)}$ is $F_{X_{(n)}}(x) = x^n$. Take derivate to get the pdf $f_{X_{(n)}}(x) = nx^{n-1}$. So $E[X_{(n)}] = \int_0^1 xnx^{n-1}dx = \int_0^1 nx^n dx = \frac{n}{n+1}$

Similarly, the cdf of $X_{(1)}$ is $F_{X_{(1)}}(x) = 1-(1-x)^n$, and this gives the density $f_{X_{(1)}}(x) = n(1-x)^{n-1}$. So $E[X_{(1)}] = \int_0^1 xn(1-x)^{n-1}dx = \int_0^1 (1-x)^n dx = \frac{1}{n+1}$.

Note that if you recognize $X_{(n)}$ has the same distribution as $1 - X_{(1)}$, the second part of the calculation can be obtained for free.

7. 7.P38

Solution:

$$E[XY] = \int_0^\infty \int_0^x 2xy\frac{e^{-2x}}{x}dydx = \int_0^\infty \int_0^x 2ye^{-2x}dydx = \frac{1}{2}\int_0^\infty x^2 2e^{-2x}dx$$

but this is the one half of the second moment of an exponential with rate 2 (which is 1/2), which is

$$E[XY] = \frac{1}{4}.$$

$$E[X] = \int_0^\infty \int_0^x 2x\frac{e^{-2x}}{x}dydx = \int_0^\infty \int_0^x 2e^{-2x}dydx = \int_0^\infty x2e^{-2x}dx = \frac{1}{2}.$$

$$E[Y] = \int_0^\infty \int_0^x 2y\frac{e^{-2x}}{x}dydx = \int_0^\infty xe^{-2x}dx = \frac{1}{2}\int_0^\infty x2e^{-2x}dx = \frac{1}{4}.$$

Finally, this means that

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$$

8. 7.P48

Solution: (a) $X \sim Geometric(1/6)$. So $E[X] = 1/(1/6) = 6$.

(b) If $Y = 1$, then $X \neq 1$. Consider $Z = X - 1$ given $Y = 1$, is again a geometric distribution with $p = 1/6$. So $E[X|Y = 1] = 6 + 1 = 7$.