Homework 4 Solution

Question 1

Proof.

$$\mathbb{P}_{\text{null}}\left(\max_{1\le j\le p}\left|\frac{(\overline{X}_{1j}-\overline{X}_{2j})-\delta_{0j}}{S_{\text{pooled},j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right|\ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right)$$

$$=\mathbb{P}_{\text{null}}\left(\bigcup_{j=1}^{p}\left\{\left|\frac{(\overline{X}_{1j}-\overline{X}_{2j})-\delta_{0j}}{S_{\text{pooled},j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right|\ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right\}\right)$$

$$\le\sum_{j=1}^{p}\mathbb{P}_{\text{null}}\left(\left\{\left|\frac{(\overline{X}_{1j}-\overline{X}_{2j})-\delta_{0j}}{S_{\text{pooled},j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right|\ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right\}\right)$$

We know that under the null hypothesis, for any $j=1,\dots,p$, $\frac{(\overline{X}_{1j}-\overline{X}_{2j})-\delta_{0j}}{S_{\text{pooled},j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ follows $t_{n_1+n_2-2}$ distribution. Therefore,

$$\mathbb{P}_{\text{null}}\left(\left\{\left|\frac{(\overline{X}_{1j}-\overline{X}_{2j})-\delta_{0j}}{S_{\text{pooled},j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right|\ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right\}\right)=\frac{\alpha}{p},\quad j=1,\dots,p.$$

Then we can finish the proof as follows

$$\mathbb{P}_{\text{null}}\left(\max_{1\le j\le p}\left|\frac{(\overline{X}_{1j}-\overline{X}_{2j})-\delta_{0j}}{S_{\text{pooled},j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right|\ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right)\le\sum_{j=1}^{p}\frac{\alpha}{p}=\alpha.$$

□

Question 2

1. We have the pooled sample covariance matrix

$$\boldsymbol{S}_{\text{pooled}}=\frac{n_1-1}{n_1+n_2-2}S_1+\frac{n_2-1}{n_1+n_2-2}S_2=\frac{1}{2}\begin{bmatrix}16&8\\8&16\end{bmatrix}+\frac{1}{2}\begin{bmatrix}16&8\\8&16\end{bmatrix}=\begin{bmatrix}16&8\\8&16\end{bmatrix}$$

Moreover, we have $\bar{\vec{x}}_1-\bar{\vec{x}}_2=\begin{bmatrix}0\\-4\end{bmatrix}$. Then

$$T^2=\left(\bar{\vec{x}}_1-\bar{\vec{x}}_2\right)^\top\left(\left(\frac{1}{n_1}+\frac{1}{n_2}\right)S_{\text{pooled}}\right)^{-1}\left(\bar{\vec{x}}_1-\bar{\vec{x}}_2\right)=\frac{4}{3}$$

In contrast, the critical value is

$$\frac{(n_1+n_2-2)\,p}{n_1+n_2-1-p}F_{p,n_1+n_2-1-p}(\alpha)=\frac{68}{33}F_{2,33}(0.05)=6.7689$$

Here $T^2<\frac{(n_1+n_2-2)p}{n_1+n_2-1-p}F_{p,n_1+n_2-1-p}(\alpha)$. Therefore, we do not reject H_0 at the level of $\alpha=0.05$.

2. The $(1-\alpha)$ confidence ellipse of $\vec{\mu}_1 - \vec{\mu}_2$ is

$$
\begin{aligned}
&((\bar{\vec{x}}_1 - \bar{\vec{x}}_2) - (\vec{\mu}_1 - \vec{\mu}_2))^\top ((\frac{1}{n_1} + \frac{1}{n_2}) S_{\text{pooled}})^{-1} ((\bar{\vec{x}}_1 - \bar{\vec{x}}_2) - (\vec{\mu}_1 - \vec{\mu}_2)) \\
&\leq \frac{(n_1+n_2-2)p}{n_1+n_2-1-p} F_{p, n_1+n_2-1-p}(\alpha)
\end{aligned}
$$

The spectral decomposition of $((\frac{1}{n_1} + \frac{1}{n_2}) S_{\text{pooled}})$ is

$$
((\frac{1}{n_1} + \frac{1}{n_2}) S_{\text{pooled}}) = \frac{8}{3} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}^\top + \frac{8}{9} \begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}^\top
$$

Also, $\bar{\vec{x}}_1 - \bar{\vec{x}}_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, $c = \sqrt{\frac{(n_1+n_2-2)p}{n_1+n_2-1-p} F_{p, n_1+n_2-1-p}(\alpha)} = 2.6017$.

Therefore, the ellipse has center $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$, with axes of directions $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $\begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, and half axis lengths $c\sqrt{\lambda_1} = 4.2486$ and $c\sqrt{\lambda_2} = 2.4529$.

3. The simultaneous confidence intervals for $\mu_{1j} - \mu_{2j}$, $j = 1, \dots, p$ based on T^2 are

$$
\begin{aligned}
&(\bar{x}_{1j} - \bar{x}_{2j}) - s_{\text{pooled},j} \sqrt{(\frac{1}{n_1} + \frac{1}{n_2}) \frac{(n_1+n_2-2)p}{n_1+n_2-1-p} F_{p, n_1+n_2-1-p}(\alpha)} \\
&\leq \mu_{1j} - \mu_{2j} \\
&\leq (\bar{x}_{1j} - \bar{x}_{2j}) + s_{\text{pooled},j} \sqrt{(\frac{1}{n_1} + \frac{1}{n_2}) \frac{(n_1+n_2-2)p}{n_1+n_2-1-p} F_{p, n_1+n_2-1-p}(\alpha)}
\end{aligned}
$$

We have $\bar{x}_{11} - \bar{x}_{21} = 0$, $s_{\text{pooled},1} = \sqrt{(\boldsymbol{S}_{\text{pooled}})_{11}} = 4$, $\bar{x}_{12} - \bar{x}_{22} = -4$,$s_{\text{pooled},2} = \sqrt{(\boldsymbol{S}_{\text{pooled}})_{22}} = 4$. The resulting simultaneous confidence intervals are

$$
-3.4690 \leq \mu_{11} - \mu_{21} \leq 3.4690, \quad -7.4690 \leq \mu_{12} - \mu_{22} \leq -0.5310
$$

4. The formulas for simultaneous $1-\alpha$ confidence intervals for $\mu_{1j} - \mu_{2j}$, $j = 1, \dots, p$ with Bonferroni correction are

$$
\begin{aligned}
&(\bar{x}_{1j} - \bar{x}_{2j}) - s_{\text{pooled},j} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} t_{n_1+n_2-2}(\frac{\alpha}{2p}) \\
&\leq \mu_{1j} - \mu_{2j} \\
&\leq (\bar{x}_{1j} - \bar{x}_{2j}) + s_{\text{pooled},j} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} t_{n_1+n_2-2}(\frac{\alpha}{2p})
\end{aligned}
$$

We have $t_{n_1+n_2-2}(\frac{\alpha}{2p}) = t_{34}(0.0125) = 2.345$. The resulting simultaneous confidence intervals are

$$
-3.1267 \leq \mu_{11} - \mu_{21} \leq 3.1267, \quad -7.1267 \leq \mu_{12} - \mu_{22} \leq -0.8733
$$

Question 3 Rearrange the equalities in the null hypothesis then we can see that it is equivalent to

$$H_0 : \begin{bmatrix} \mu_{12} - \mu_{11} \\ \mu_{13} - \mu_{12} \end{bmatrix} = \begin{bmatrix} \mu_{22} - \mu_{21} \\ \mu_{23} - \mu_{22} \end{bmatrix},$$

which can be written as $H_0 : \boldsymbol{C}\vec{\mu}_1 = \boldsymbol{C}\vec{\mu}_2$ where $\boldsymbol{C} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Conduct the linear transformations $\vec{y}_{lj} = \boldsymbol{C}\vec{x}_{lj}$ for all $l = 1, 2$ and $j = 1, \ldots, n$. Then we have

$$\bar{\vec{y}}_1 = \boldsymbol{C}\bar{\vec{x}}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{S}_{y,1} = \boldsymbol{C}\boldsymbol{S}_1\boldsymbol{C}^\top = \begin{bmatrix} 16 & -8 \\ -8 & 16 \end{bmatrix},$$

$$\bar{\vec{y}}_2 = \boldsymbol{C}\bar{\vec{x}}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \boldsymbol{S}_{y,2} = \boldsymbol{C}\boldsymbol{S}_2\boldsymbol{C}^\top = \begin{bmatrix} 16 & -8 \\ -8 & 16 \end{bmatrix}.$$

The pooled sample covariance of the new sample is

$$\boldsymbol{S}_{y,pooled} = \frac{n_1 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_{y,1} + \frac{n_2 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_{y,2} = \frac{1}{2}\begin{bmatrix} 16 & -8 \\ -8 & 16 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 16 & -8 \\ -8 & 16 \end{bmatrix} = \begin{bmatrix} 16 & -8 \\ -8 & 16 \end{bmatrix}.$$

1. The test statistic for $H_0 : \boldsymbol{C}\vec{\mu}_1 = \boldsymbol{C}\vec{\mu}_2$ is

$$T^2 = \left(\bar{\vec{y}}_1 - \bar{\vec{y}}_2\right)^\top \left(\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{y,pooled}\right)^{-1} \left(\bar{\vec{y}}_1 - \bar{\vec{y}}_2\right) = 9$$

   The critical values is

$$\frac{(n_1 + n_2 - 2)\, p}{n_1 + n_2 - 1 - p} F_{p,n_1+n_2-1-p}(\alpha) = \frac{2(18 + 18 - 2)}{18 + 18 - 1 - 2} F_{2,18+18-1-2}(0.05) = 6.7689$$

   So the T^2 statistic is greater than the critical value. We reject H_0 at the level of $\alpha = 0.05$.

2. The $(1 - \alpha)$ confidence ellipse of $\vec{d}_1 - \vec{d}_2$ is

$$((\bar{\vec{y}}_1 - \bar{\vec{y}}_2) - (\vec{d}_1 - \vec{d}_2))^\top ((\frac{1}{n_1} + \frac{1}{n_2})S_{\text{y,pooled}})^{-1}((\bar{\vec{y}}_1 - \bar{\vec{y}}_2) - (\vec{d}_1 - \vec{d}_2))$$
$$\leq \frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - 1 - p} F_{p,n_1+n_2-1-p}(\alpha)$$

   The spectral decomposition of $((\frac{1}{n_1} + \frac{1}{n_2})S_{\text{y,pooled}})$ is

$$((\frac{1}{n_1} + \frac{1}{n_2})S_{\text{y,pooled}}) = \frac{8}{3}\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}^\top + \frac{8}{9}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}^\top$$

   Also, $\bar{\vec{y}}_1 - \bar{\vec{y}}_2 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $c = \sqrt{\frac{(n_1+n_2-2)p}{n_1+n_2-1-p} F_{p,n_1+n_2-1-p}(\alpha)} = 2.6017$.

   Therefore, the ellipse has center $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$, with axes of directions $\begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, and half axis lengths $c\sqrt{\lambda_1} = 4.2486$ and $c\sqrt{\lambda_2} = 2.4529$.

3. The simultaneous confidence intervals for $d_{1j} - d_{2j}$, $j = 1, \ldots, p$ based on T^2 are

$$(\bar{y}_{1j} - \bar{y}_{2j}) - s_{\text{y,pooled},j}\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})\frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - 1 - p} F_{p,n_1+n_2-1-p}(\alpha)}$$
$$\leq d_{1j} - d_{2j}$$
$$\leq (\bar{y}_{1j} - \bar{y}_{2j}) + s_{\text{y,pooled},j}\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})\frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - 1 - p} F_{p,n_1+n_2-1-p}(\alpha)}$$

We have $\bar{y}_{11} - \bar{y}_{21} = -2$, $s_{\text{y,pooled},1} = \sqrt{(\boldsymbol{S}_{\text{y,pooled}})_{11}} = 4$, $\bar{y}_{12} - \bar{y}_{22} = -2$,$s_{\text{y,pooled},2} = \sqrt{(\boldsymbol{S}_{\text{y,pooled}})_{22}} = 4$. The resulting simultaneous confidence intervals are

$$-5.4690 \leq d_{11} - d_{21} \leq 1.4690, \quad -5.4690 \leq d_{12} - d_{22} \leq 1.4690$$

4. The formulas for simultaneous $1-\alpha$ confidence intervals for $d_{1j} - d_{2j}$, $j = 1, \ldots, p$ with Bonferroni correction are

$$\begin{aligned}
&(\bar{y}_{1j} - \bar{y}_{2j}) - s_{\text{y,pooled},j}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}t_{n_1+n_2-2}(\frac{\alpha}{2p}) \\
&\leq d_{1j} - d_{2j} \\
&\leq (\bar{y}_{1j} - \bar{y}_{2j}) + s_{\text{y,pooled},j}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}t_{n_1+n_2-2}(\frac{\alpha}{2p})
\end{aligned}$$

We have $t_{n_1+n_2-2}(\frac{\alpha}{2p}) = t_{34}(0.0125) = 2.345$. The resulting simultaneous confidence intervals are

$$-5.1267 \leq d_{11} - d_{21} \leq 1.1267, \quad -5.1267 \leq d_{12} - d_{22} \leq 1.1267$$

Question

$$S_{\text{pooled}} = \begin{bmatrix} 10963.7 & 21505.5 \\ 21505.5 & 63661.3 \end{bmatrix}$$

$$\bar{\vec{x}}_1 - \bar{\vec{x}}_2 = \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix}$$

Then

$$T^2 = \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)^\top \left(\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right)^{-1} \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right) = 16.0662$$

The critical values is

$$\frac{(n_1 + n_2 - 2)\,p}{n_1 + n_2 - 1 - p}F_{p,n_1+n_2-1-p}(\alpha) = \frac{2(45+55-2)}{45+55-1-2}F_{2,45+55-1-2}(0.05) = 6.26$$

So the T^2 statistic is greater than the critical value. We reject H_0 at the level of $\alpha = 0.05$. Suppose the coefficient vector that is most responsible for rejection is $\vec{a}$. Then the linear combinations have sample means $\vec{a}^\top \bar{\vec{x}}_1$ and $\vec{a}^\top \bar{\vec{x}}_2$, and sample covariances $\vec{a}^\top S_1 \vec{a}$ and $\vec{a}^\top S_2 \vec{a}$. Therefore, the pooled standard error $s_{\vec{a},\text{pooled}}$ satisfies

$$\begin{aligned}
s^2_{\vec{a},\text{pooled}} &= \frac{n_1 - 1}{n_1 + n_2 - 2}\vec{a}^\top S_1 \vec{a} + \frac{n_2 - 1}{n_1 + n_2 - 2}\vec{a}^\top S_2 \vec{a} \\
&= \vec{a}^\top (\frac{n_1 - 1}{n_1 + n_2 - 2}S_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}S_2)\vec{a} \\
&= \vec{a}^\top (S_{\text{pooled}})\vec{a}
\end{aligned}$$

To test $H_0 : \vec{a}^\top(\vec{\mu}_1 - \vec{\mu}_2) = 0$, we can form the following t-statistic

$$t_{\vec{a},\alpha} = \frac{\vec{a}^\top(\bar{\vec{x}}_1 - \bar{\vec{x}}_2)}{s_{\vec{a},\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

To get the most responsible $\vec{a}$, we need to find the $\vec{a}$ such that its corresponding $t_{\vec{a},\alpha}$ has the largest absolute value. So we need to find $\vec{a}$ that maximizes $t^2_{\vec{a},\alpha}$.

$$t^2_{\vec{a},\alpha} = \frac{(\vec{a}^\top(\bar{\vec{x}}_1 - \bar{\vec{x}}_2))^2}{\vec{a}^\top((\frac{1}{n_1} + \frac{1}{n_2})S_{\text{pooled}})\vec{a}}$$

According to the maximization lemma,

$$t^2_{\vec{a},\alpha} \leq (\bar{\vec{x}}_1 - \bar{\vec{x}}_2)^\top \left(\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right)^{-1} (\bar{\vec{x}}_1 - \bar{\vec{x}}_2) = T^2$$

and the equity holds if $\vec{a} = c((\frac{1}{n_1} + \frac{1}{n_2})S_{\text{pooled}})^{-1}(\bar{\vec{x}}_1 - \bar{\vec{x}}_2) = c(S_{\text{pooled}})^{-1}(\bar{\vec{x}}_1 - \bar{\vec{x}}_2)$. Here c can be any nonzero scalar.
So when $\vec{a} = c\begin{bmatrix}0.0421\\0.0641\end{bmatrix}$, $t^2_{\vec{a},\alpha}$ is maximized.