Principal Component Analysis

1 Principal components: concepts and calculation

We would like to explain the variance-covariance structure of a set of variables by a few linear combinations of these variables.

Let $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix}$ be a random vector with population mean $\vec{\mu}$ and population covariance matrix $\mathbf{\Sigma}$. Denote the spectral decomposition of $\mathbf{\Sigma}$ as

$$\mathbf{\Sigma} = \lambda_1 \vec{v}_1 \vec{v}_1^\top + \ldots + \lambda_p \vec{v}_p \vec{v}_p^\top,$$

where $\lambda_1 \geq \ldots \geq \lambda_p > 0$. Each eigenvector is represented as

$$\vec{v}_k = [v_{k1}, \ldots, v_{kp}]^\top.$$

1.1 The first principal component

Consider a linear combination of the variates by $\vec{a} = \begin{bmatrix} a_1 & \ldots & a_p \end{bmatrix}^\top$:

$$Y_1 = \vec{a}^\top \vec{X} = a_1 X_1 + a_2 X_2 + \ldots + a_p X_p.$$

In order to explain the variance-covariance of $\vec{X}$ as much as possible, we want to maximize the variance of Y_1. At the same time, in order to fix the scale, we impose the constraint $\|\vec{a}\| = 1$. Then the first principal component for $\vec{X}$ is defined by the following optimization:

$$\begin{aligned} \max \quad & \mathrm{Var}(Y_1) = \mathrm{Var}(\vec{a}^\top \vec{X}) = \vec{a}^\top \mathbf{\Sigma} \vec{a} \\ s.t. \quad & \|\vec{a}\|^2 = 1 \end{aligned}$$

The Lagrangian for the above optimization is

$$f(\vec{a}; \lambda) = \vec{a}^\top \mathbf{\Sigma} \vec{a} - \lambda(\vec{a}^\top \vec{a} - 1).$$

By setting the gradient to be equal to zero, we get

$$\nabla_{\vec{a}} f(\vec{a}; \lambda) = 2\mathbf{\Sigma}\vec{a} - 2\lambda\vec{a} = \vec{0},$$

that is

$$\mathbf{\Sigma}\vec{a} = \lambda\vec{a},$$

This implies that $\vec{a}$ must be a unit eigenvector, and λ is the corresponding eigenvalue. Notice what we aim to maximize is

$$\mathrm{Var}(Y_1) = \mathrm{Var}(\vec{a}^\top \vec{X}) = \vec{a}^\top \mathbf{\Sigma} \vec{a} = \lambda \vec{a}^\top \vec{a} = \lambda.$$

Since $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$, in order to maximize $\mathrm{Var}(Y_1)$, we must have

$$\begin{cases} \vec{a} = \vec{v}_1 \\ \mathrm{Var}(Y_1) = \lambda_1. \end{cases}$$

To sum up, we have the following result:

Proposition 1. *The first principal component is*

$$Y_1 = \vec{v}_1^\top \vec{X} = v_{11} X_1 + v_{12} X_2 + \ldots v_{1p} X_p,$$

where $\vec{v}_1$ is the eigenvector corresponding to the leading eigenvalue λ_1. Moreover, $Var(Y_1) = \lambda_1$.

1.2 The second principal component

In order to define the second principal component, we also look for some linear combination of the variates

$$Y_2 = \vec{a}^\top \vec{X} = a_1 X_1 + a_2 X_2 + \ldots + a_p X_p,$$

such that its variance is as large as possible. However, here we have two constraints: First, we need to impose $\|\vec{a}\|^2 = 1$ in order to fix the scale; Second, we require that Y_2 explains the variance-covariance of $\vec{X}$ that has not been explained by Y_1, which amounts to $\mathrm{cov}(Y_2, Y_1) = 0$. Notice that

$$\mathrm{cov}(Y_2, Y_1) = \mathrm{cov}(\vec{a}^\top \vec{X}, \vec{v}_1^\top \vec{X}) = \vec{a}^\top \mathrm{Cov}(\vec{X}) \vec{v}_1 = \vec{a}^\top \mathbf{\Sigma} \vec{v}_1 = \lambda_1 \vec{a}^\top \vec{v}_1,$$

so the constraint $\mathrm{cov}(Y_2, Y_1) = 0$ is equivalent to $\vec{a}^\top \vec{v}_1 = 0$. Consequently, the second principal component is defined through the following optimization

$$\begin{aligned} \max \quad & \mathrm{Var}(Y_2) = \mathrm{Var}(\vec{a}^\top \vec{X}) = \vec{a}^\top \mathbf{\Sigma} \vec{a} \\ s.t. \quad & \vec{a}^\top \vec{a} = 1, \\ & \vec{a}^\top \vec{v}_1 = 0. \end{aligned}$$

The resulting Lagrangian is thus

$$f(\vec{a}; \lambda, \gamma) = \vec{a}^\top \mathbf{\Sigma} \vec{a} - \lambda(\|\vec{a}\|^2 - 1) - \gamma \vec{a}^\top \vec{v}_1.$$

Again, by setting the gradient to be equal to zero, we get

$$\nabla_{\vec{a}} f(\vec{a}; \lambda, \gamma) = 2\mathbf{\Sigma}\vec{a} - 2\lambda\vec{a} - \gamma\vec{v}_1 = \vec{0}.$$

Taking the inner product of both sides with $\vec{v}_1$, we have

$$\vec{v}_1^\top (2\mathbf{\Sigma}\vec{a} - 2\lambda\vec{a} - \gamma\vec{v}_1) = 0.$$

The first term

$$\vec{v}_1^\top \mathbf{\Sigma} \vec{a} = \vec{a}^\top \mathbf{\Sigma} \vec{v}_1 = \vec{a}^\top (\lambda_1 \vec{v}_1) = \lambda_1 \vec{a}^\top \vec{v}_1 = 0.$$

By the constraint, the second term is $\vec{v}_1^\top \vec{a} = 0$. As a result, $\gamma \|\vec{v}_1\|^2 = \gamma = 0$. Then the stationary condition is still in the form of

$$\mathbf{\Sigma} \vec{a} = \lambda \vec{a}.$$

This implies that $\vec{a}$ must be a unit eigenvector while λ is the corresponding eigenvalue. Note that we actually want to maximize

$$\mathrm{Var}(Y_2) = \vec{a}^\top \mathbf{\Sigma} \vec{a} = \lambda \|\vec{a}\|^2 = \lambda.$$

Since $\vec{a}^\top \vec{v}_1 = 0$, the best we can get is

$$\begin{cases} \vec{a} = \vec{v}_2 \\ \mathrm{Var}(Y_2) = \lambda_2. \end{cases}$$

To sum up, we have the following result:

Proposition 2. *The second principal component is*

$$Y_2 = \vec{v}_2^\top \vec{X} = v_{21} X_1 + v_{22} X_2 + \ldots v_{2p} X_p,$$

where $\vec{v}_2$ is the eigenvector corresponding to the second largest eigenvalue λ_2. Moreover, $Var(Y_2) = \lambda_2$.

1.3 General concepts on principal components

In general, the k-th principal component can be defined iteratively through the following procedure. Given the existing principal components $Y_1, \ldots, Y_{k-1}$, we look for some linear combination of the variates

$$Y_k = \vec{a}^\top \vec{X} = a_1 X_1 + a_2 X_2 + \ldots + a_p X_p,$$

such that its variance is as large as possible. Still, we need to impose $\|\vec{a}\|^2 = 1$ in order to fix the scale, and require that Y_k to explain the variance-covariance of $\vec{X}$ that has not been explained by $Y_1, \ldots, Y_{k-1}$, which amounts to

$$\mathrm{cov}(Y_k, Y_1) = \mathrm{cov}(Y_k, Y_2) = \ldots = \mathrm{cov}(Y_k, Y_{k-1}) = 0.$$

Consequently, the k-th principal component is defined through the following optimization

$$\begin{aligned} \max \quad & \mathrm{Var}(Y_k) = \mathrm{Var}(\vec{a}^\top \vec{X}) = \vec{a}^\top \mathbf{\Sigma} \vec{a} \\ s.t. \quad & \vec{a}^\top \vec{a} = 1, \\ & \vec{a}^\top \vec{v}_1 = 0, \\ & \quad \vdots \\ & \vec{a}^\top \vec{v}_{k-1} = 0. \end{aligned}$$

In general, we have the following result

Proposition 3. *The k-th principal component is*

$$Y_k = \vec{v}_k^\top \vec{X} = v_{k1}X_1 + v_{k2}X_2 + \dots v_{kp}X_p,$$

where $\vec{v}_k$ is the eigenvector corresponding to the k-th largest eigenvalue λ_k. Moreover, $Var(Y_k) = \lambda_k$.

The coefficients $v_{k1}, \dots, v_{kp}$ are referred to as loadings on the random variables X_1, ..., X_p for the k-th principal component Y_k.

1.4 Verification of covariance structures of the PCs

Denote

$$\boldsymbol{V} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_p \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & \dots & v_{p1} \\ v_{12} & v_{22} & \dots & v_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1p} & v_{2p} & \dots & v_{pp} \end{bmatrix}, \tag{1.1}$$

where the row and column indices require attention. Recall that $\boldsymbol{\Sigma} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top$, where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{bmatrix}.$$

The random vector of population principal components can thus be written as

$$\vec{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} = \begin{bmatrix} \vec{v}_1^\top \vec{X} \\ \vdots \\ \vec{v}_p^\top \vec{X} \end{bmatrix} = \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_p^\top \end{bmatrix} \vec{X} = \boldsymbol{V}^\top \vec{X}.$$

The linear relationship $\vec{Y} = \boldsymbol{V}^\top \vec{X}$ gives

$$\mathrm{Cov}(\vec{Y}) = \boldsymbol{V}^\top \mathrm{Cov}(\vec{X}) \boldsymbol{V} = \boldsymbol{V}^\top \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top \boldsymbol{V} = \boldsymbol{\Lambda},$$

which implies

$$\mathrm{Var}(Y_k) = \lambda_k, \text{ for } k = 1, \dots, p.$$

and

$$\mathrm{Cov}(Y_j, Y_k) = 0 \text{ for } j \neq k.$$

1.5 Standardization

In certain applications, it is common to standardize the original variates $X_1, \dots, X_p$ into

$$Z_1 = \frac{X_1 - \mu_1}{\sqrt{\sigma_{11}}}, \quad Z_2 = \frac{X_2 - \mu_2}{\sqrt{\sigma_{22}}}, \dots, Z_p = \frac{X_p - \mu_p}{\sqrt{\sigma_{pp}}}.$$

Then it is straightforward to get

$$\mathrm{Cov}(\vec{Z}) = \begin{bmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1p} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \dots & \rho_{pp} \end{bmatrix} = \mathrm{Corr}(\vec{X})$$

where

$$\rho_{jk} = \frac{\sigma_{jk}}{\sqrt{\sigma_{jj}\sigma_{kk}}}.$$

If we still represent the spectral decomposition for the covariance of the standardized variables as

$$\mathrm{Cov}(\vec{Z}) = \lambda_1 \vec{v}_1 \vec{v}_1^\top + \ldots + \lambda_p \vec{v}_p \vec{v}_p^\top$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$. The principal components of $Z_1, \ldots, Z_p$ are

$$Y_k = \vec{v}_k^\top \vec{Z} = v_{k1} Z_1 + \ldots v_{kp} Z_p.$$

2 Basic Principal Component Analysis

2.1 Contribution of variables to the determination of PCs

One standard method to compare the contributions of different variables to the determination of a particular PC is through the formula:

$$Y_k = \vec{v}_k^\top \vec{X} = v_{k1} X_1 + v_{k2} X_2 + \ldots v_{kp} X_p.$$

In other words, we compare the contributions of $X_1, \ldots, X_p$ to the determination of Y_k based on the loadings $v_{k1}, \ldots, v_{kp}$.

Here we introduce the second method: Compare the contributions of $X_1, \ldots, X_p$ to the determination of Y_k based on the correlations $\mathrm{Corr}(X_1, Y_k), \ldots, \mathrm{Corr}(X_p, Y_k)$. Recall that we have $\vec{Y} = \boldsymbol{V}^\top \vec{X}$, where $\boldsymbol{V}$ is defined in (1.1). Then,

$$\begin{aligned} \mathrm{Cov}(\vec{Y}, \vec{X}) &= \mathrm{Cov}(\boldsymbol{V}^\top \vec{X}, \vec{X}) = \boldsymbol{V}^\top \mathrm{Cov}(\vec{X}) = \boldsymbol{V}^\top \boldsymbol{\Sigma} = \boldsymbol{V}^\top \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^\top = \boldsymbol{\Lambda} \boldsymbol{V}^\top \\ &= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1p} \\ v_{21} & v_{22} & \dots & v_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ v_{p1} & v_{p2} & \dots & v_{pp} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 v_{11} & \lambda_1 v_{12} & \dots & \lambda_1 v_{1p} \\ \lambda_2 v_{21} & \lambda_2 v_{22} & \dots & \lambda_2 v_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_p v_{p1} & \lambda_p v_{p2} & \dots & \lambda_p v_{pp} \end{bmatrix}, \end{aligned}$$

The covariance between the k-th principal component and the j-th variate is

$$\mathrm{Cov}(Y_k, X_j) = \lambda_k v_{kj}.$$

We further have

$$\text{Corr}(Y_k, X_j) = \frac{\text{Cov}(Y_k, X_j)}{\sqrt{\text{Var}(Y_k)\text{Var}(X_j)}} = \frac{\lambda_k v_{kj}}{\sqrt{\lambda_k \sigma_{jj}}} = v_{kj}\sqrt{\frac{\lambda_k}{\sigma_{jj}}}.$$

This gives the second method to compare the contributions of X_j's to the determination of Y_k through the correlation coefficients $v_{kj}\sqrt{\frac{\lambda_k}{\sigma_{jj}}}$ for $j = 1, \ldots, p$.

When the variables are standardized from X_j to Z_j, we have and

$$\text{Corr}(Y_k, Z_j) = v_{kj}\sqrt{\frac{\lambda_k}{\rho_{jj}}} = v_{kj}\sqrt{\lambda_k}.$$

This implies that for a fixed k, the loadings and correlation coefficients between the k-th PC Y_k and $Z_1, \ldots, Z_p$ are proportional. Therefore, there is no difference in comparing of the contributions of variables to the determination of Y_k based on either loadings or correlations.

2.2 Selecting the number of PCs

Recall that the spectral decomposition of the population covariance is

$$\boldsymbol{\Sigma} = \lambda_1 \vec{v}_1 \vec{v}_1^\top + \lambda_2 \vec{v}_2 \vec{v}_2^\top + \ldots + \lambda_p \vec{v}_p \vec{v}_p^\top.$$

Denote

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \ldots & \sigma_{pp} \end{bmatrix} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top.$$

The trace formula gives

$$\text{trace}(\boldsymbol{\Sigma}) = \text{trace}(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top) = \text{trace}(\boldsymbol{\Lambda}\boldsymbol{V}^\top\boldsymbol{V}) = \text{trace}(\boldsymbol{\Lambda}),$$

which is equivalent to

$$\sigma_{11} + \sigma_{22} + \ldots + \sigma_{pp} = \lambda_1 + \lambda_2 + \ldots + \lambda_p.$$

Since $\text{Var}(Y_k) = \lambda_k$ for $k = 1, \ldots, p$ and $\text{Var}(X_j) = \sigma_{jj}$, the trace formula gives

$$\text{Var}(X_1) + \ldots + \text{Var}(X_p) = \text{Var}(Y_1) + \ldots + \text{Var}(Y_p)$$

Definition 4. *The proportion of the total variance due to the first k principal components is defined as*

$$\frac{Var(Y_1) + \ldots + Var(Y_k)}{Var(Y_1) + \ldots + Var(Y_p)} = \frac{\lambda_1 + \ldots + \lambda_k}{\lambda_1 + \ldots + \lambda_p} = \frac{\lambda_1 + \ldots + \lambda_k}{\sigma_{11} + \ldots + \sigma_{pp}}.$$

Example: If

$$\frac{\lambda_1 + \lambda_2 + \lambda_3}{\sigma_{11} + \sigma_{22} + \ldots + \sigma_{pp}} > 90\%,$$

then we can replace $X_1, \ldots, X_p$ with Y_1, Y_2 and Y_3 without much loss of information.

3 Sample PCA

3.1 Summary of results

Let

$$\vec{x}_1, \ldots, \vec{x}_n$$

be a sample with sample mean $\overline{\vec{x}}$ and sample covariance $\boldsymbol{S}$. By considering the spectral decomposition of the sample covariance

$$\boldsymbol{S} = \hat{\lambda}_1 \vec{u}_1 \vec{u}_1^\top + \ldots + \hat{\lambda}_p \vec{u}_p \vec{u}_p^\top,$$

where $\vec{u}_k = [u_{k1}, \ldots, u_{kp}]^\top$, we have the following results about sample PCs in the analogy to population PCs:

- The k-the sample PC is defined as

  $$\widehat{Y}_k = u_{k1} X_1 + u_{k2} X_2 + \ldots + u_{kp} X_p.$$

  The coefficients $u_{k1}, \ldots, u_{kp}$ are referred to as loadings for the k-th sample principal component $\widehat{Y}_k$. In particular, the i-th observation of the k-th sample principal component as

  $$\hat{y}_{ik} = \vec{u}_k^\top \vec{x}_i = u_{k1} x_{i1} + u_{k2} x_{i2} + \ldots + u_{kp} x_{ip}.$$

- The sample variance of $\widehat{Y}_k$ is $\hat{\lambda}_k$, and for $k \neq j$, the sample covariance between $\widehat{Y}_k$ and $\widehat{Y}_j$ is 0.

- The sample correlation between Y_k and X_j is $u_{kj} \sqrt{\frac{\hat{\lambda}_k}{s_{jj}}}$.

- The total sample covariances is

  $$s_{11} + s_{22} + \ldots + s_{pp} = \hat{\lambda}_1 + \hat{\lambda}_2 + \ldots + \hat{\lambda}_p,$$

  and the proportion of the total variance due to the first k sample principal components:

  $$\frac{\hat{\lambda}_1 + \ldots + \hat{\lambda}_k}{\hat{\lambda}_1 + \ldots + \hat{\lambda}_p} = \frac{\hat{\lambda}_1 + \ldots + \hat{\lambda}_k}{s_{11} + \ldots + s_{pp}}.$$

3.2 Reduction of number of columns in the dataset

Consider the spectral decomposition of the sample covariance in the matrix form:

$$\boldsymbol{S} = \hat{\lambda}_1 \vec{u}_1 \vec{u}_1^\top + \ldots + \hat{\lambda}_p \vec{u}_p \vec{u}_p^\top = \boldsymbol{U} \widehat{\boldsymbol{\Lambda}} \boldsymbol{U}^\top,$$

where

$$\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_p \geq 0,$$

$$\boldsymbol{U} = [\vec{u}_1, \ldots, \vec{u}_p] == \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{p1} \\ u_{12} & u_{22} & \ldots & u_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1p} & u_{2p} & \ldots & u_{pp} \end{bmatrix},$$

and

$$\widehat{\mathbf{\Lambda}} = \begin{bmatrix} \hat{\lambda}_1 & & \\ & \ddots & \\ & & \hat{\lambda}_k \end{bmatrix}$$

Then the i-th observation of all sample principal components is

$$\hat{\vec{y}}_i = \begin{bmatrix} \hat{y}_{i1} \\ \vdots \\ \hat{y}_{ip} \end{bmatrix} = \begin{bmatrix} u_{11}x_{i1} + u_{12}x_{i2} + \ldots + u_{1p}x_{ip} \\ \vdots \\ u_{p1}x_{i1} + u_{p2}x_{i2} + \ldots + u_{pp}x_{ip} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1p} \\ u_{21} & u_{22} & \ldots & u_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ u_{p1} & u_{p2} & \ldots & u_{pp} \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{ip} \end{bmatrix} = \boldsymbol{U}^\top \vec{x}_i.$$

Then, the data matrix of sample principal components is

$$\widehat{\boldsymbol{Y}} := \begin{bmatrix} \hat{\vec{y}}_1^\top \\ \hat{\vec{y}}_2^\top \\ \vdots \\ \hat{\vec{y}}_n^\top \end{bmatrix} = \begin{bmatrix} \vec{x}_1^\top \boldsymbol{U} \\ \vec{x}_2^\top \boldsymbol{U} \\ \vdots \\ \vec{x}_n^\top \boldsymbol{U} \end{bmatrix} = \begin{bmatrix} \vec{x}_1^\top \\ \vec{x}_2^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix} \boldsymbol{U} = \boldsymbol{X}\boldsymbol{U} = \boldsymbol{X}[\vec{u}_1, \ldots, \vec{u}_p].$$

In particular, if we only keep the observations of the first two sample PCs, we get

$$\begin{bmatrix} \hat{y}_{11} & \hat{y}_{12} \\ \hat{y}_{21} & \hat{y}_{22} \\ \vdots & \vdots \\ \hat{y}_{n1} & \hat{y}_{n2} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{np} \end{bmatrix} \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \\ \vdots & \vdots \\ u_{1p} & u_{2p} \end{bmatrix}$$

In practice, one is interested in plotting the PC scores of the observations for $\widehat{Y}_1$ and $\widehat{Y}_2$, i.e., the scatter plot of

$$\begin{bmatrix} \hat{y}_{11} \\ \hat{y}_{12} \end{bmatrix}, \begin{bmatrix} \hat{y}_{21} \\ \hat{y}_{22} \end{bmatrix}, \ldots, \begin{bmatrix} \hat{y}_{n1} \\ \hat{y}_{n2} \end{bmatrix}.$$

Meanwhile, each variable X_j should be also presented in the plot as the vector of loadings $\begin{bmatrix} u_{1j} \\ u_{2j} \end{bmatrix}$, which will be helpful for the interpretation of $\widehat{Y}_1$ and $\widehat{Y}_2$.

4 Data analysis and interpretation

See, e.g., Example 8.5 on page 451.