# Stat 206: Linear Models

## Lecture 11

Nov. 4, 2019

# General Linear Tests

$\mathcal{I}$ and $\mathcal{J}$ are two non-overlapping index sets.

- **Full model**: Contain both $X_{\mathcal{I}}$ and $X_{\mathcal{J}}$.
- **Reduced model**: Contain only $X_{\mathcal{I}}$.
- Test whether $X_{\mathcal{J}}$ may be dropped out of the full model:

$$H_0 : \beta_j = 0, \ \text{ for } \textbf{all } j \in \mathcal{J}$$

vs.

$$H_a : \text{some } \beta_j : j \in \mathcal{J} \text{ are nonzero.}$$

Basic idea: Compare under the full model with $SSE$ under the reduced model by an F ratio:

- Under $H_0$ (i.e., the model):

$$F^* \sim_{H_0} \qquad .$$

- Reject $H_0$ at level $\alpha$ if the observed $F^*$ .

# F-test for Regression Relation

- Full model with $X_1, \cdots, X_{p-1}$:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_{p-1} X_{i,p-1} + \epsilon_i, \quad i = 1, \cdots n.$$

- Reduced model with no $X$ variable:

$$Y_i = \beta_0 + \epsilon_i, \quad i = 1, \cdots, n.$$

  So $SSE(R) =$ ,and $df_R =$ .

- $SSE(R) - SSE(F) =$ , and $df_R - df_F =$ .
- F ratio

$$F^* = \qquad .$$

# Test whether a Single $\beta_k = 0$

Body fat: Test for the model with all three predictors whether the midarm circumference ($X_3$) can be dropped.

- Full model: $SSE(F) = 98.40$ with d.f. 16.

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Null and alternative hypotheses:

$$H_0 : \qquad\qquad vs. \quad H_a : \qquad\qquad .$$

- Reduced model: $SSE(R) =$ with d.f.

- $F^* =$ .
- Pvalue= . So we
  $X_3$ from the full model.

# Equivalence between F-test and T-test

- Test whether $X_k$ can be dropped from a regression model with $p-1$ $X$ variables:
$$H_0 : \beta_k = 0 \;\; vs. \;\; H_a : \beta_k \neq 0.$$
- T-test:
$$T^* = \frac{\hat{\beta}_k}{s\{\hat{\beta}_k\}} \underset{H_0}{\sim} t_{(n-p)},$$
where $\hat{\beta}_k$ is the LS estimator of $\beta_k$ and $s\{\hat{\beta}_k\}$ is its standard error under the full model. Reject $H_0$ when $|T^*| > t(1-\alpha/2; n-p)$.
- $F^* = (T^*)^2$ and $F(1-\alpha; 1, n-p) = (t(1-\alpha/2; n-p))^2$. So for this test, F-test and T-test are equivalent.

*Notes: for one one-sided alternatives, we still need the T-tests.*

## Test whether Several $\beta_k = 0$

Body fat: Test whether both thigh circumference ($X_2$) and midarm circumference ($X_3$) can be dropped from the model with all three predictors.

- Full model: $SSE(F) = 98.40$ with d.f. 16.

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Null and alternative hypotheses:

$$H_0 : \qquad\qquad vs. \quad H_a :$$

- Reduced model: $SSE(R) =$ with d.f. .

- $F^* =$ .
- Pvalue= . The result is at $\alpha = 0.05$.

# Standardization

Different $X$ variables often have different units which could make their values vastly different.

- Regression coefficients are not comparable.
- Elements of $\mathbf{X}'\mathbf{X}$ could differ substantially in order of magnitude, causing numerical instability.
- A regression model can be reparametrized into a standardized regression model through centering and rescaling.
- This process is called **standardization**, a.k.a. **correlation transformation**.

# Correlation Transformation

Define transformed variables:

$$X_{ik}^{*} = \frac{1}{\sqrt{n-1}}\left(\frac{X_{ik}-\overline{X}_k}{s_{X_k}}\right), \quad k = 1, \cdots, p-1,$$

where

$$\overline{X}_k = \frac{1}{n}\sum_{i=1}^{n} X_{ik}, \quad s_{X_k} = \sqrt{\frac{\sum_{i=1}^{n}(X_{ik}-\overline{X}_k)^2}{n-1}}, \quad (k = 1, \cdots, p-1).$$

are sample means and sample standard deviations, respectively.

- The sample means of the transformed variables are
  .
- The sample standard deviations of the transformed variables are .
- So all variables are and are
  .
- Correlation transformation the pairwise (sample) correlations among the $X$ variables, the (sample) correlations between the $X$ variables and the response variable.

# Standardized Regression Model

Rewrite the regression model in terms of standardized variables:

$$Y_i = \beta_0^* + \beta_1^* X_{i1}^* + \beta_2^* X_{i2}^* + \cdots + \beta_{p-1}^* X_{i,p-1}^* + \epsilon_i, \quad i = 1, \cdots n,$$

where

$$\beta_k^* = \qquad\qquad (k = 1, \cdots, p-1), \quad \beta_0^* =$$

is a "reparametrization" of the original model.

# Design Matrix of Standardized Model

$$\underset{n\times p}{\mathbf{X}^*} = \begin{bmatrix} 1 & X_{11}^* & \cdots & X_{1,p-1}^* \\ 1 & X_{21}^* & \cdots & X_{2,p-1}^* \\ \vdots & \vdots & \dots & \vdots \\ 1 & X_{n1}^* & \cdots & X_{n,p-1}^* \end{bmatrix}.$$

$$\underset{p\times p}{\mathbf{X}^{*\prime}\mathbf{X}^*} = \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & 1 & r_{12} & \cdots & r_{1,p-1} \\ 0 & r_{21} & 1 & \cdots & r_{2,p-1} \\ 0 & \vdots & \dots & \vdots & \\ 0 & r_{p-1,1} & r_{p-1,2} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} n & \mathbf{0}^T \\ \mathbf{0} & \underset{(p-1)\times(p-1)}{\mathbf{r}_{XX}} \end{bmatrix},$$

where $\mathbf{r}_{XX}$ is the sample correlation matrix of the $X$ variables.

# Correlation Matrix

- Its $(k, l)$-element $r_{kl}$ is the sample correlation coefficient between $X_k, X_l$:
- All its elements are numbers
.
- Its diagonal elements are , since the correlation of a variable with itself is .
- Correlation matrix is a matrix:

# $\mathbf{X'Y}$ Matrix of Standardized Model

$$\underset{p\times 1}{\mathbf{X}^{*\prime}\mathbf{Y}} = \begin{bmatrix} n\bar{Y} \\ \sqrt{n-1}s_Y r_{Y1} \\ \sqrt{n-1}s_Y r_{Y2} \\ \vdots \\ \sqrt{n-1}s_Y r_{Y,p-1} \end{bmatrix} = \sqrt{n-1}s_Y \begin{bmatrix} \frac{n}{\sqrt{n-1}s_Y}\bar{Y} \\ \underset{(p-1)\times 1}{\mathbf{r}_{XY}} \end{bmatrix}$$

where $r_{Yk}$ is the sample correlation coefficient between $Y$ and $X_k$:

$$r_{Yk} = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(X_{ik} - \overline{X}_k)(Y_i - \overline{Y})}{s_{X_k} s_Y}, \quad k = 1, \cdots p-1.$$

# LS Fit of Standardized Model

$$\underset{p\times 1}{\hat{\boldsymbol{\beta}}^*} = \begin{bmatrix} \hat{\beta}_0^* \\ \hat{\beta}_1^* \\ \hat{\beta}_2^* \\ \vdots \\ \hat{\beta}_{p-1}^* \end{bmatrix} = \begin{bmatrix} \bar{Y} \\ \underset{(p-1)\times 1}{\sqrt{n-1}\, s_Y \mathbf{r}_{XX}^{-1} \mathbf{r}_{XY}.} \end{bmatrix}$$

- These are called *fitted standardized regression coefficients*.
- Relationships with the LS estimators of the original model:

$$\begin{aligned} \hat{\beta}_k &= \frac{1}{\sqrt{n-1}\, s_{X_k}} \hat{\beta}_k^*, \quad k = 1, \cdots, p-1 \\ \hat{\beta}_0 &= \overline{Y} - \hat{\beta}_1 \overline{X}_1 - \cdots - \hat{\beta}_{p-1} \overline{X}_{p-1}. \end{aligned}$$

*Do fitted values, residuals and sums of squares change due to standardization of the X variables?*

# Body Fat

Sample means and sample standard deviations ($n = 20$):

$$\overline{Y} = 20.20, \quad \overline{X}_1 = 25.30, \quad \overline{X}_2 = 51.17, \quad \overline{X}_3 = 27.62;$$

$$s_Y = 5.11, \quad s_{X_1} = 5.02, \quad s_{X_2} = 5.23, \quad s_{X_3} = 3.65.$$

Correlation matrices:

$$\mathbf{r}_{XX} = \begin{bmatrix} 1.00 & 0.92 & 0.46 \\ 0.92 & 1.00 & 0.08 \\ 0.46 & 0.08 & 1.00 \end{bmatrix}, \quad \mathbf{r}_{XY} = \begin{bmatrix} 0.84 \\ 0.88 \\ 0.14 \end{bmatrix}.$$

Least-squares estimators of the standardized model:

$$\hat{\beta}_0^* = \overline{Y} = 20.20, \quad \begin{bmatrix} \hat{\beta}_1^* \\ \hat{\beta}_2^* \\ \hat{\beta}_3^* \end{bmatrix} = \sqrt{n-1}\, s_Y \mathbf{r}_{XX}^{-1} \mathbf{r}_{XY} = 27.5 \times \begin{bmatrix} 4.26 \\ -2.93 \\ -1.56 \end{bmatrix}.$$

Least-squares estimators of the original model:

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 4.33 \\ -2.86 \\ -2.18 \end{bmatrix} = \begin{bmatrix} \frac{5.11}{5.02} \times 4.26 \\ \frac{5.11}{5.23} \times (-2.93) \\ \frac{5.11}{3.65} \times (-1.56) \end{bmatrix}.$$

# Multicollinearity

Multicollinearity refers to the situation when the $X$ variables are among themselves.

- This term is often reserved for the situation when the inter-correlation/collinearity among the $X$ variables is
  .
- $X$ variables being nearly collinear means

- To understand the effects of multicollinearity, we consider two extreme situations:
    (i) When the $X$ variables are not correlated with each other at all
    (ii) When they are perfectly intercorrelated.
- In practice, it is usually somewhere in between (i) and (ii).

# Uncorrelated $X$ Variables

- $\mathbf{r}_{XX} =$
- Fitted standardized regression coefficients:

$$\hat{\beta}_k^* = \qquad\qquad\qquad, \quad k = 1, \cdots, p-1$$

are the between the response variable $Y$ and individual $X$ variables.

- Variance-covariance matrix:

$$\sigma^2\{\begin{bmatrix} \hat{\beta}_0^* \\ \hat{\beta}_1^* \\ \hat{\beta}_2^* \\ \vdots \\ \hat{\beta}_{p-1}^* \end{bmatrix}\} =$$

- So the LS estimators of the standardized model are . *How about the LS estimators of the original model?*

When the $X$ variables are uncorrelated, the effect of an $X$ variable
other $X$ variables in the model.

- The LS fitted regression coefficient of an $X$ variable is
  by which other (uncorrelated) $X$ variables are in the model.
- The LS fitted regression coefficients of the $X$ variables are
  with each other.
- The contribution of an $X$ variable in reducing the error sum of squares is the other (uncorrelated) $X$ variables in the model, i.e.

# Crew Productivity

A study on the effect of work crew size ($X_1$) and level of bonus pay ($X_2$) on productivity ($Y$).

| case | X1<br>crew-size | X2<br>bonus-pay | Y<br>productivity |
|---|---|---|---|
| 1 | 4 | 2 | 42 |
| 2 | 4 | 2 | 39 |
| 3 | 4 | 3 | 48 |
| 4 | 4 | 3 | 51 |
| 5 | 6 | 2 | 49 |
| 6 | 6 | 2 | 53 |
| 7 | 6 | 3 | 61 |
| 8 | 6 | 3 | 60 |

Pairwise correlation matrix.

| | X1 | X2 | Y |
|---|---|---|---|
| X1 | 1.00 | 0.00 | 0.74 |
| X2 | 0.00 | 1.00 | 0.64 |
| Y | 0.74 | 0.64 | 1.00 |

$X_1$ and $X_2$ are uncorrelated.

# Crew Productivity: Model 1

```
Call:
lm(formula = Y ~ X1, data = data)

Residuals:
   Min     1Q Median     3Q    Max 
-6.750 -3.750  0.125  4.500  6.000 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept)   23.500     10.111   2.324   0.0591 .
X1             5.375      1.983   2.711   0.0351 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 5.609 on 6 degrees of freedom
Multiple R-squared: 0.5505,     Adjusted R-squared: 0.4755 
F-statistic: 7.347 on 1 and 6 DF,  p-value: 0.03508

> anova(fit1)
Analysis of Variance Table

Response: Y
          Df Sum Sq Mean Sq F value  Pr(>F)  
X1         1 231.12 231.125   7.347 0.03508 *
Residuals  6 188.75  31.458
```

# Crew Productivity: Model 2

```
Call:
lm(formula = Y ~ X2, data = data)

Residuals:
   Min      1Q Median      3Q     Max 
-7.000 -4.688 -0.250  5.250  7.250 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept)   27.250     11.608   2.348   0.0572 .
X2             9.250      4.553   2.032   0.0885 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1 

Residual standard error: 6.439 on 6 degrees of freedom
Multiple R-squared: 0.4076,     Adjusted R-squared: 0.3088 
F-statistic: 4.128 on 1 and 6 DF,  p-value: 0.08846 

> anova(fit2)
Analysis of Variance Table

Response: Y
          Df Sum Sq Mean Sq F value  Pr(>F)  
X2         1 171.12 171.125  4.1276 0.08846 .
Residuals  6 248.75  41.458                  
```

# Crew Productivity: Model 3

```
Call:
lm(formula = Y ~ X1 + X2, data = data)

Residuals:
     1      2      3      4      5      6      7      8
 1.625 -1.375 -1.625  1.375 -2.125  1.875  0.625 -0.375

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.3750     4.7405   0.079 0.940016
X1            5.3750     0.6638   8.097 0.000466 ***
X2            9.2500     1.3276   6.968 0.000937 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.877 on 5 degrees of freedom
Multiple R-squared: 0.958,      Adjusted R-squared: 0.9412
F-statistic: 57.06 on 2 and 5 DF,  p-value: 0.000361

> anova(fit3)
Analysis of Variance Table

Response: Y
          Df  Sum Sq Mean Sq F value    Pr(>F)
X1         1 231.125 231.125  65.567 0.0004657 ***
X2         1 171.125 171.125  48.546 0.0009366 ***
Residuals  5  17.625   3.525
```

# Perfectly Correlated $X$ variables

A set of $X$ variables is said to be *collinear* if one or several of them may be expressed as a linear combination of the other $X$ variables (including $\mathbf{1}_n$).

- The design matrix $\underset{n\times p}{\mathbf{X}}$ is . So the matrix $\underset{p\times p}{\mathbf{X}'\mathbf{X}}$ is .
- LS estimators are because the least-squares equation
$$\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{Y}$$
has solutions.
- This means that there exist vectors $\mathbf{b}$ that minimize the least squares criterion:

$$Q(\mathbf{b}) = \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_{i1} - \cdots - b_{p-1} X_{i,p-1})^2 .$$

- If $X$ variables are perfectly correlated, then there exists a nonzero vector $\underset{p\times 1}{\mathbf{c}}$ such that

$$\underset{n\times p}{\mathbf{X}}\underset{p\times 1}{\mathbf{c}} = \mathbf{0}_n.$$

- If $\mathbf{b}$ is a solution to the least-squares equation, i.e.,

$$\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{Y},$$

  then $\mathbf{b} + k\mathbf{c}$ is also a solution where $k \in \mathbb{R}$ is an arbitrary scalar since

$$\begin{aligned} \mathbf{X}'\mathbf{X}(\mathbf{b} + k\mathbf{c}) &= \mathbf{X}'\mathbf{X}\mathbf{b} + k\mathbf{X}'\mathbf{X}\mathbf{c} \\ &= \mathbf{X}'\mathbf{Y} + k\mathbf{X}'\mathbf{0}_n = \mathbf{X}'\mathbf{Y}. \end{aligned}$$

- Similarly, if $\mathbf{b}$ minimizes the least-squares criterion function $Q(\cdot)$, then $\mathbf{b} + k\mathbf{c}$ also minimizes $Q(\cdot)$ since

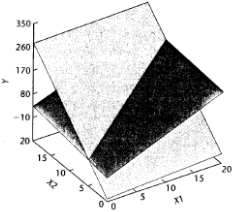
$$\begin{aligned} Q(\mathbf{b}) &= (\mathbf{Y} - \mathbf{X}\mathbf{b})'(\mathbf{Y} - \mathbf{X}\mathbf{b}) \\ &= (\mathbf{Y} - \mathbf{X}(\mathbf{b} + k\mathbf{c}))'(\mathbf{Y} - \mathbf{X}(\mathbf{b} + k\mathbf{c})) = Q(\mathbf{b} + k\mathbf{c}). \end{aligned}$$

# Example

| case | X1 | X2 | Y |
|---|---|---|---|
| 1 | 2 | 6 | 24 |
| 2 | 8 | 9 | 82 |
| 3 | 6 | 8 | 66 |
| 4 | 10 | 10 | 98 |

- $X$ variables (including the column of 1) are perfectly correlated since $X_2 = 5 + 0.5X_1$.
- There are infinitely many response functions that fit this data equally "best" (with $SSE = 17.14$).

**FIGURE 7.2** Two Response Planes That Intersect when $X_2 = 5 + .5X_1$.



- The two response surfaces in the figure are completely different, but they have the same $y$ values on $X_2 = 5 + 0.5X_1$: $y = 7.14 + 9.29X_1$.
- Actually, any response surface that passes the intersecting line will fit the data equally well as these two, e.g.,

$$\widehat{Y} = 7.14 + 9.29X_1, \quad \widehat{Y} = -85.71 + 18.57X_2.$$

*Can you think about some others?*

```
Call:
lm(formula = Y ~ X1, data = data)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   7.1429     3.5341   2.021  0.18066
X1            9.2857     0.4949  18.764  0.00283 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.928 on 2 degrees of freedom
Multiple R-squared: 0.9944,     Adjusted R-squared: 0.9915
F-statistic: 352.1 on 1 and 2 DF,  p-value: 0.002828

Call:
lm(formula = Y ~ X2, data = data)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -85.7143     8.2956  -10.33  0.00924 **
X2           18.5714     0.9897   18.76  0.00283 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.928 on 2 degrees of freedom
Multiple R-squared: 0.9944,     Adjusted R-squared: 0.9915
F-statistic: 352.1 on 1 and 2 DF,  p-value: 0.002828
```

```
Call:
lm(formula = Y ~ X1 + X2, data = data)

Residuals:
      1       2       3       4
-1.7143  0.5714  3.1429 -2.0000

Coefficients: (1 not defined because of singularities)
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   7.1429     3.5341   2.021  0.18066
X1            9.2857     0.4949  18.764  0.00283 **
X2                NA         NA      NA       NA
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.928 on 2 degrees of freedom
Multiple R-squared: 0.9944,     Adjusted R-squared: 0.9915
F-statistic: 352.1 on 1 and 2 DF,  p-value: 0.002828
```

Here, R discards $X_2$ and fits a model only using $X_1$.

**When $X$ variables are perfectly correlated, we may still get a fit of the data.**

- The least-squares fitted values $\widehat{\mathbf{Y}}$ is and is the

  of the response vector $\mathbf{Y}$ to the linear subspace of $\mathbb{R}^n$ generated by the columns of the design matrix $\mathbf{X}$ (the column space).
- However, the regression coefficients are

  .

# Body Fat: Compare Models

| Variables in Model | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $s\{\hat{\beta}_1\}$ | $s\{\hat{\beta}_2\}$ | MSE |
|---|---|---|---|---|---|
| Model 1: $X_1$ | 0.8572 | - | 0.1288 | - | 7.95 |
| Model 2: $X_2$ | - | 0.8565 | - | 0.1100 | 6.3 |
| Model 3: $X_1, X_2$ | 0.2224 | 0.6594 | 0.3034 | 0.2912 | 6.47 |
| Model 4: $X_1, X_2, X_3$ | 4.334 | -2.857 | 3.016 | 2.582 | 6.15 |

- The regression coefficient for $X_1$ ($X_2$) depending on which other $X$ variables are included in the model.
- The standard errors of the fitted regression coefficients are becoming when more $X$ variables are included into the model.
- MSE tends to as additional $X$ variables are added into the model.

R outputs

- $SSR(X_1) = 352.27$, $SSR(X_1|X_2) = 3.47$.
- The reason why $SSR(X_1|X_2)$ is so small compared to $SSR(X_1)$ is that $X_1$ and $X_2$ are with each other and with the response variable $Y$.
  - When $X_2$ is already in the model, the marginal contribution from $X_1$ in explaining $Y$ is since $X_2$ contains much of the information as $X_1$ in terms of explaining $Y$.

*What would happen if $X_1$ and $X_2$ were not correlated with $Y$, but were highly correlated among themselves?*

In Model 4, none of the three $X$ variables is statistically significant by the T-tests. However, the F-test for regression relation is highly significant. Is there a paradox?

- From the general linear test perspective, each T-test is a test, testing whether the of an $X$ variable is significant given $X$ variables being included in the model.
- The three tests of the marginal effects of $X_1, X_2, X_3$ together are to testing whether there is a regression relation between $Y$ and $(X_1, X_2, X_3)$.
- The reduced model for each individual test contains $X$ variables and thus may lead to non-significant results due to .
- On the other hand, the reduced model for testing regression relation contains $X$ variable.

Model 4

# Effects of Multicollinearity: Summary

- With multicollinearity, the estimated regression coefficients tend to have sampling variability (i.e., standard errors). This leads to:
  - confidence intervals.
  - It's possible that of the regression coefficients is statistically significant, but at the same time there is a regression relation between the response variable and the entire set of $X$ variables.
- Multicollinearity does not prevent us from getting a of the data.

◂ prediction

# Interpretation of Regression Coefficients and ESS

In the presence of multicollinearity:

- The regression coefficient of an $X$ variable
  which other $X$ variables are also in the model.
- Therefore, a regression coefficient reflect any inherent effect of the corresponding $X$ variable on the response variable, but only a
  given whatever other $X$ variables are also in the model.
- Similarly, there is sum of squares that can be ascribed to any one $X$ variable.
  - The reduction in the total variation in $Y$ ascribed to an $X$ variable must be interpreted as a
    given other $X$ variables also included in the model.