STA 200A: Homework 5

Note: Below the notation 3.T11 means Chapter 3, Theoretical Exercise 11. Similarly, the notation 4.P21 means Chapter 4, Problem 21.

1. Let $X \sim \text{Gamma}(\alpha, \lambda)$. Find $E[1/X]$.

Solution:

Suppose $\alpha > 1$. Then,

$$
\begin{aligned}
E\Big[\frac{1}{X}\Big] &= \int_0^\infty \frac{1}{x}\frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}dx \\
&= \frac{\lambda^2}{\Gamma(\alpha)}\int_0^\infty (\lambda x)^{\alpha-2}e^{-\lambda x}dx \quad \text{(next consider the change of variable } y = \lambda x,\ dy = \lambda dx\text{)} \\
&= \frac{\lambda^2}{\Gamma(\alpha)}\frac{1}{\lambda}\underbrace{\int_0^\infty y^{\alpha-2}e^{-y}dy}_{=\Gamma(\alpha-1)} \\
&= \frac{\lambda}{\Gamma(\alpha)}\Gamma(\alpha-1) \\
&= \frac{\lambda}{\alpha-1} \quad \text{provided } \alpha > 1, \quad \text{since } \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1).
\end{aligned}
$$

2. Let $X \sim \text{Uniform}[0,1]$, and put $Y = \sqrt{X}$. Find $E[Y]$.

Solution:

We first find the density of Y then take expectation.
Let $g(t) = \sqrt{t}$, which is monotone in $[0,1]$. Then $Y = g(X)$. For $X \in [0,1]$, $Y \in [0,1]$.

$$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) = 1 \cdot 2y = 2y.$$

Now,

$$E[Y] = \int_0^1 y \cdot 2y = \int_0^1 2y^2 = \frac{2}{3}.$$

Another way is to consider $E[g(X)]$ to do integration with respect to X without finding the distribution of Y.

3. Let X be an Exponential(λ) rv. Derive a formula for $E[X^2]$.

Solution: Answer: $E[X^2] = 2/\lambda^2$.

Let's use the result from Theoretical Problem 5.5. For an exponential random variable X, $P(X \geq x) = 1 - P(X \leq x) = e^{-\lambda x}$

So

$$E[X^n] = \int_0^\infty n x^{n-1} e^{-\lambda x} dx.$$

Now set $n = 2$, we have

$$\begin{aligned} E[X^2] &= \int_0^\infty 2x e^{-\lambda x} dx \\ &= 2\Big[\left.\frac{x e^{-\lambda x}}{-\lambda}\right|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x} dx \Big] \\ &= \frac{2}{\lambda^2} \end{aligned}$$

4. Find the density function of $Y = e^Z$, where $Z \sim \mathcal{N}(\mu, \sigma^2)$. This is called the log-normal density, since $\log Y$ is normally distributed.

Solution: Again we will first find the cdf of Y interms of the cdf of Z (since we know the distribution of Z), then differentiate it to find the pdf, i.e.,

$$F_Y(y) = P(Y \leq y) = P(e^Z \leq y) = P(Z \leq \ln y) = F_Z(\ln y)$$

then the pdf can be expressed as

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_Z(\ln y)}{dy} = \frac{1}{y} f_Z(\ln y)$$

hence we have

$$f_Y(y) = \frac{1}{y}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}[\ln y - \mu]^2}$$

5. Let $f(x) = (1 + \alpha x)/2$ for $-1 \leq x \leq 1$ and $f(x) = 0$ otherwise, where $-1 \leq \alpha \leq 1$. Show that f is a density and find the corresponding cdf. Find the upper quartile of the distribution in terms of α.

Solution: If f is a density then $\int_{-1}^1 f(x)dx = 1$, i.e.,

$$\int_{-1}^1 (1 + \alpha x)/2 = \frac{1}{2}\left[x + \frac{\alpha x^2}{2}\right]_{x=-1}^{x=1} = \frac{1}{2}[1 + \alpha/2 - (-1 + \alpha/2)] = \frac{1}{2}2 = 1$$

then the cdf can be computed by:

$$F(x) = \int_{-1}^x f(t)dt = \frac{1}{2}\int_{-1}^x (1 + \alpha t)dt = \frac{1}{2}\left[t + \frac{\alpha t^2}{2}\right]_{t=-1}^{t=x} = \frac{2x + \alpha x^2 - \alpha + 2}{4}$$

Let $x_{3/4}$ be the upper quartile. It can be obtained by solving the following equation for x,

$$F(x_{3/4}) = .75.$$

6. Suppose X has the density function $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ otherwise.
   (a) Find c.
   (b) Find the cdf $F_X(x)$.
   (c) What is $P(0.1 \le X < 0.5)$?

Solution:

(a) In order to find c, we need to integrate the density (in the given domain) to 1 and solve for c, i.e.,

$$\int_0^1 cx^2 = 1 \quad \Rightarrow \quad \left[\frac{cx^3}{3}\right]_0^1 = 1 \quad \Rightarrow \quad \frac{c}{3} = 1 \quad \Rightarrow \quad c = 3$$

hence the density function is

$$f_X(x) = \begin{cases} 3x^2, & x \in [0,1] \\ 0, & \text{o'wise} \end{cases}$$

(b) and the cdf can be obtained as $F_X(x) = \int_0^x f_X(t)dt$, i.e.,

$$F_X(x) = 3\int_0^x t^2 dt = 3\left[\frac{t^3}{3}\right]_{t=0}^{t=x} = x^3, \quad \forall x \in [0,1]$$

hence the cdf can be explicitly express as follows:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^3, & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

(c) Note that for constants c_1 and c_2, $P(c_1 \le X \le c_2) = F_X(c_2) - F_X(c_1)$, hence we have

$$P(0.1 \le X \le 0.5) = F_X(0.5) - F_X(0.1) = (0.5)^3 - (0.1)^3 = 0.124$$

7. If $X \sim \mathcal{N}(0, \sigma^2)$, find the density of $Y = |X|$.

Solution: Let F_X and F_Y denote the cdf of random variables X and Y, respectively with the corresponding pdfs f_X and f_Y. Then we can find the cdf of Y as follows:

$$F_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y) = F_X(y) - F_X(-y)$$

Note that the pdf can be found by differentiating the corresponding cdf, i.e.,

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}[F_X(y) - F_X(-y)] = f_X(y) + f_X(-y)$$

Since X has mean zero ($X \sim \mathcal{N}(0, \sigma^2)$), we have $f_X(y) = f_X(-y)$, which yields $f_Y(y) = 2f_X(y)$, hence

$$f_Y(y) = \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}, \quad y > 0$$

8. Let $Z \sim N(0,1)$ be a standard normal rv. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function, with derivative g'. Show that the following identity holds.

$$E[g'(Z)] = E[Zg(Z)].$$

(You may assume both of these expectations exist.)

Solution:

Let f be the pdf of a standard normal random variable. It's easy to check that $f'(x) = -xf(x)$. Now,

$$\begin{aligned}
E[g'(Z)] &= \int g'(x)f(x)dx \\
&= \int f(x)dg(x) \\
&= f(x)g(x)|_{-\infty}^{\infty} - \int g(x)f'(x)dx \\
&= 0 + \int g(x)xf(x)dx \\
&= E[Zg(Z)]
\end{aligned}$$

9. Chapter 5, Theoretical Exercise 11 (just parts b and c)

Solution:

(b). Let $g(z) = z^n$, and apply part (a). The result is immediate.

(c). It can be done by using the result from part (b). $E[Z^4] = 3E[Z^2] = 3$.