Stat 206: Linear Models

Lecture 10

October 30, 2019

Prediction of a New Observation

- $Y_{h(new)} = \mathbf{X}_h'\beta + \epsilon_h$: with the observations Y_is.
- Predicted value: $\widehat{Y}_h :=$.

  $\sigma^2(pred_h) :=$.

- Standard error for prediction:

  $s(pred_h) =$.

- $(1 - \alpha)$-prediction interval for $Y_{h(new)}$:

Multiple Regression: Example

$n = 30$ cases, response variable Y and three predictor variables X_1, X_2, X_3.

case	Y	X1	X2	X3
1	3.01	1.06	0.86	-1.23
2	-3.40	-0.30	-0.08	-0.48
3	2.74	1.05	0.22	-0.40
...	...	...	...	...
30	-1.42	2.12	-0.8	-0.62

Example: Model 2

Nonadditive model with interaction between X_1 and X_2:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i1} X_{i2} + \epsilon_i, \quad i = 1, \cdots, 30.$$

($p = 5$)

```
Call:
lm(formula = Y ~ X1 + X2 + X3 + X1:X2, data = data)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.8832     0.2153   4.103  0.00038 ***
X1             1.5946     0.2421   6.587 6.69e-07 ***
X2             1.7091     0.2605   6.560 7.16e-07 ***
X3             2.1266     0.2687   7.916 2.85e-08 ***
X1:X2          1.0076     0.2467   4.084  0.00040 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.026 on 25 degrees of freedom
Multiple R-squared: 0.933,      Adjusted R-squared: 0.9223
F-statistic: 87.04 on 4 and 25 DF,  p-value: 2.681e-14
```

Model 3

Predict a new observation when $X_1 = 0.8, X_2 = 0.5, X_3 = -1$ under Model 2.

- Standard error for prediction:

$$s(pred) = \qquad\qquad\qquad\qquad\qquad\qquad .$$

- A 99%-prediction interval for Y_{hnew}:

$$1.290 \pm 2.787 \times 1.1098 = [-1.803, 4.383].$$

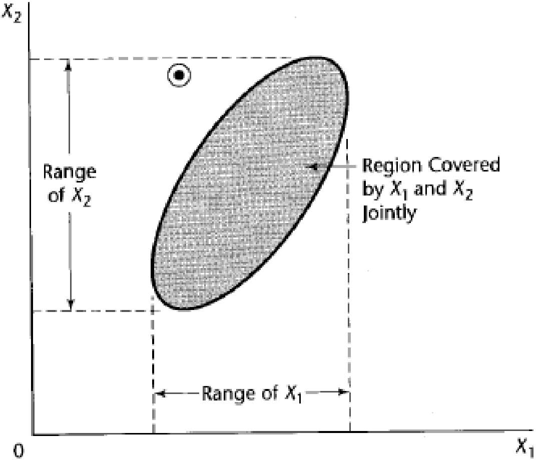
- R codes.

```
> newX=data.frame(X1=0.8, X2=0.5, X3=-1)
> predict.lm(fit2, newX, interval="confidence",
+ level=0.99, se.fit=TRUE)

> predict.lm(fit2, newX, interval="prediction",
+ level=0.99, se.fit=TRUE)
```

Hidden Extrapolations

- Recall that extrapolation occurs when predicting the response variable for values of the X variable(s)
  of the original data.
- It's possible that, the fitted model
  when extended outside the range of the observations.
- With more than one X variables, the levels of
  define the region of the observations. One can not merely look at the ranges of each X variable.
- With two X variables, we can look at their scatter plot.
- Procedure to identify hidden extrapolation for more than two X variables will be discussed later.


X_2
Range of X_2
Region Covered by X_1 and X_2 Jointly
Range of X_1
0
X_1

Extra Sum of Squares

$\mathcal{I}$ and $\mathcal{J}$ are two **non-overlapping** index sets.

- **Extra sum of squares (ESS)**:

$$SSR(X_{\mathcal{J}}|X_{\mathcal{I}}) := \qquad\qquad\qquad\qquad\qquad\qquad .$$

- It indicates the

- Degrees of freedom: $d.f.(SSR(X_{\mathcal{J}}|X_{\mathcal{I}})) = \qquad\qquad\qquad .$
- Mean squares: $MSR(X_{\mathcal{J}}|X_{\mathcal{I}}) := \qquad\qquad\qquad .$

Notations.

- $\mathcal{I}$: an index set; $X_{\mathcal{I}} := \{X_i : i \in \mathcal{I}\}$.
  - E.g. $\mathcal{I} = \{2, 3\}$, $X_{\mathcal{I}} = \{X_2, X_3\}$.
- $SSE(X_{\mathcal{I}})$ and $SSR(X_{\mathcal{I}})$ denote the error sum of squares and regression sum of squares, respectively, under the regression model with $X_{\mathcal{I}} := \{X_i : i \in \mathcal{I}\}$ being the X variables.
  - E.g., $SSE(X_2, X_3)$ is the error sum of squares of the model with X_2 and X_3.

Some properties of ESS.

- $SSR(X_{\mathcal{J}}|X_{\mathcal{I}})$.
- Usually $SSR(X_{\mathcal{J}}|X_{\mathcal{I}})$ $SSR(X_{\mathcal{I}}|X_{\mathcal{J}})$.
- ESS is also the marginal of the regression sum of squares, i.e.,

$$SSR(X_{\mathcal{J}}|X_{\mathcal{I}}) = \qquad .$$

Body Fat

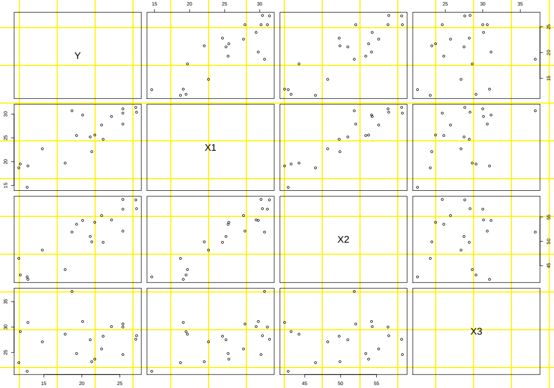
A researcher measured the amount of body fat (Y) of 20 healthy females 25 to 34 years old, together with three (potential) predictor variables, triceps skinfolds thickness (X_1), thigh circumference (X_2), and midarm circumference (X_3). The amount of body fat was obtained by a cumbersome and expensive procedure requiring immersion of the person in water. Thus it would be helpful if a regression model with some or all of these predictors could provide reliable estimates of body fat as these predictors are easy to measure.

A snapshot of the data.

case	X1	X2	X3	Y
Triceps	Thigh	MidArm	BodyFat	
1	19.5	43.1	29.1	11.9
2	24.7	49.8	28.2	22.8
3	30.7	51.9	37.0	18.7
4	29.8	54.3	31.1	20.1
5	19.1	42.2	30.9	12.9
6	25.6	53.9	23.7	21.7
...	...	...	...	...

First check the variable type, distribution, etc.,of each variable.

Scatter plot matrix: Check pairwise relationship of these variables.



Do you see any particular patterns?

Correlation matrix.

	X1	X2	X3	Y
X1	1.0000000	0.9238425	0.4577772	0.8432654
X2	0.9238425	1.0000000	0.0846675	0.8780896
X3	0.4577772	0.0846675	1.0000000	0.1424440
Y	0.8432654	0.8780896	0.1424440	1.0000000

X_1 and X_2 are correlated, X_1 and X_3 are correlated, X_2 and X_3 are correlated.

Consider the following 4 models.

- Model 1: regression of Y on X_1

$$Y_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Model 2: regression of Y on X_2

$$Y_i = \beta_0 + \beta_2 X_{i2} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Model 3: regression of Y on X_1 and X_2

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Model 4: regression of Y on X_1, X_2 and X_3.

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i, \quad i = 1, \cdots, 20.$$

Boy Fat: Model 1

```
> summary(fit1)

Call:
lm(formula = Y ~ X1, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)  -1.4961     3.3192  -0.451    0.658
X1            0.8572     0.1288   6.656 3.02e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.82 on 18 degrees of freedom
Multiple R-squared: 0.7111,     Adjusted R-squared: 0.695
F-statistic:  44.3 on 1 and 18 DF,  p-value: 3.024e-06

> anova(fit1)
Analysis of Variance Table

Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1          1 352.27  352.27  44.305 3.024e-06 ***
Residuals 18 143.12    7.95
```

Boy Fat: Model 2

```
> summary(fit2)

Call:
lm(formula = Y ~ X2, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -23.6345     5.6574  -4.178 0.000566 ***
X2            0.8565     0.1100   7.786  3.6e-07 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.51 on 18 degrees of freedom
Multiple R-squared: 0.771,      Adjusted R-squared: 0.7583
F-statistic: 60.62 on 1 and 18 DF,  p-value: 3.6e-07

> anova(fit2)
Analysis of Variance Table

Response: Y
Df Sum Sq Mean Sq F value  Pr(>F)
X2         1 381.97  381.97  60.617 3.6e-07 ***
Residuals 18 113.42    6.30
```

Boy Fat: Model 3

```
> summary(fit3)

Call:
lm(formula = Y ~ X1 + X2, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -19.1742     8.3606  -2.293   0.0348 *
X1            0.2224     0.3034   0.733   0.4737
X2            0.6594     0.2912   2.265   0.0369 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.543 on 17 degrees of freedom
Multiple R-squared: 0.7781,     Adjusted R-squared: 0.7519
F-statistic:  29.8 on 2 and 17 DF,  p-value: 2.774e-06

> anova(fit3)
Analysis of Variance Table

Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1          1 352.27  352.27 54.4661 1.075e-06 ***
X2          1  33.17   33.17  5.1284    0.0369 *
Residuals 17 109.95    6.47
```

GLT

Boy Fat: Model 4

```
> summary(fit4)
Call:
lm(formula = Y ~ X1 + X2 + X3, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)  117.085     99.782   1.173    0.258
X1             4.334      3.016   1.437    0.170
X2            -2.857      2.582  -1.106    0.285
X3            -2.186      1.595  -1.370    0.190

Residual standard error: 2.48 on 16 degrees of freedom
Multiple R-squared: 0.8014,     Adjusted R-squared: 0.7641
F-statistic: 21.52 on 3 and 16 DF,  p-value: 7.343e-06

> anova(fit4)
Analysis of Variance Table

Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1          1 352.27  352.27 57.2768 1.131e-06 ***
X2          1  33.17   33.17  5.3931   0.03373 *
X3          1  11.55   11.55  1.8773   0.18956
Residuals 16  98.40    6.15
```

GLT

Body Fat: ESS

From the R outputs, we can derive a number of extra sums of squares. For example:

- $$SSR(X_2|X_1) = \qquad .$$
- $$SSR(X_1|X_2) = \qquad .$$
- Both extra sums of squares have degrees of freedom , so $MSR(X_2|X_1) =$ and $MSR(X_1|X_2) =$.
- The reduction of SSE by adding to a model with is much more than the reduction of SSE by adding to a model with .

- $$SSR(X_3|X_1, X_2) = \qquad .$$

  This extra sum of squares has degrees of freedom ,
  so $MSR(X_3|X_1, X_2) = \qquad .$

- $$SSR(X_2, X_3|X_1) = \qquad .$$

  This extra sums of squares has degrees of freedom ,
  so $MSR(X_2, X_3|X_1) = \qquad .$

Are there other ESS that can be derived from the R outputs?

Decomposition of SSR into ESS

For a model with multiple X variables, the regression sum of squares (SSR) can be expressed as the of several extra sums of squares.

- For example:

$$SSR(X_1, X_2) = \qquad .$$

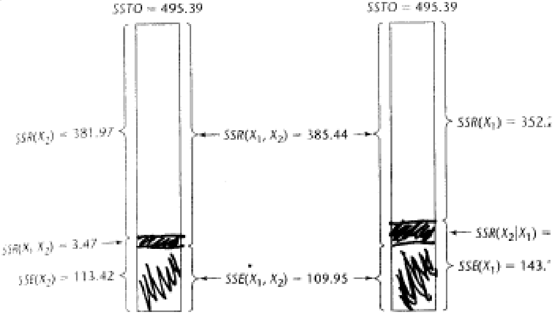
  $SSR(X_1)$ measures the contribution by in the model, whereas $SSR(X_2|X_1)$ measures the contribution when , given that X_1 is already in the model.

- However, such decomposition is usually not unique. For example,

$$SSR(X_1, X_2) = \qquad .$$

Chapter 7

FIGURE 7.1 Schematic Representation of Extra Sums of Squares—Body Fat Exampl



Adapted from Applied Linear Statistical Models by Kutner, Nachtsheim, Neter and Li

- More X variables, decompositions. For example, with three X variables:

$$
\begin{aligned}
SSR(X_1, X_2, X_3) &= SSR(X_1) + SSR(X_2|X_1) + SSR(X_3|X_1, X_2) \\
SSR(X_1, X_2, X_3) &= SSR(X_2) + SSR(X_1|X_2) + SSR(X_3|X_1, X_2) \\
SSR(X_1, X_2, X_3) &= SSR(X_1) + SSR(X_2, X_3|X_1), \quad \cdots, \cdots
\end{aligned}
$$

- Body Fat.
  - From Model 1, $SSR(X_1) = 352.27$; Also $SSR(X_2|X_1) = 33.17$ and $SSR(X_3|X_1, X_2) = 11.55$. So

$$SSR(X_1, X_2, X_3) =$$

  - From Model 2, $SSR(X_2) = 381.97$; Also $SSR(X_1|X_2) = 3.47$. So

$$SSR(X_1, X_2, X_3) =$$

Read *anova()* output

It provides decomposition of *SSR* into single d.f. ESS, of the *X* variables entering the model.

```
Call:
lm(formula = Y ~ X1 + X2 + X3, data = fat)
> anova(fit4)
Analysis of Variance Table
Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1          1 352.27  352.27 57.2768 1.131e-06 ***
X2          1  33.17   33.17  5.3931   0.03373 *
X3          1  11.55   11.55  1.8773   0.18956
Residuals 16  98.40    6.15
```

Source of Variation	SS	d.f.	MS
Regression			
Error			
Total			

For example: $SSR(X_2, X_3|X_1) =$

How to get $SSR(X_2|X_1, X_3)$ from the R output of Model 4? We need to enter the X variables in a different order, i.e.,

.

```
Call:
lm(formula = Y ~ X1 + X3 + X2, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)  117.085     99.782   1.173    0.258
X1             4.334      3.016   1.437    0.170
X3            -2.186      1.595  -1.370    0.190
X2            -2.857      2.582  -1.106    0.285

> anova(fit4.alt2)
Analysis of Variance Table
Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 352.27  352.27 57.2768 1.131e-06 ***
X3         1  37.19   37.19  6.0461   0.02571 *
X2         1   7.53    7.53  1.2242   0.28489
Residuals 16  98.40    6.15
```

Then we can get $SSR(X_2|X_1, X_3) =$.