Note 4

Page 1 Types of errors and power functions

		Decision	
		Accept H_0	Reject H_0
truth	H_0	No error	Type I error
	H_1	Type II error	No error

Probability of error

$\theta \in \Theta_0$, Probability of Type I error / false alarm

$$\mathbb{P}_\theta \text{ (reject } H_0)$$

$\theta \in \Theta_1$. Probability of Type II error / missed detection

$$\mathbb{P}_\theta \text{ (not reject } H_0) = 1 - \mathbb{P}_\theta \text{ (reject } H_0).$$

Def: The power function of a test with rejection region R is the function of $\theta \in \Theta$, defined as

$$\beta(\theta) = \mathbb{P}_\theta (X \in R).$$

Then $\theta \in \Theta_0$: probability of Type I error $= \mathbb{P}_\theta(\text{reject } H_0) = \beta(\theta)$

$\theta \in \Theta_1$: probability of Type II error $= 1 - \mathbb{P}_\theta(\text{reject } H_0) = 1 - \beta(\theta)$

We wish that
$$\begin{cases} \beta(\theta) \approx 0 & \text{if } \theta \in \Theta_0 \\ \beta(\theta) \approx 1 & \text{if } \theta \in \Theta_1 \end{cases}$$

Page 2 Example: Normal power function

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known. An LRT reject $H_0: \theta \le \theta_0$ v.s. $H_1: \theta > \theta_0$ if

$$\frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} > c$$

where c is some threshold.

Power function

$$\beta(\theta) = \mathbb{P}_\theta\left(\frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} > c\right)$$

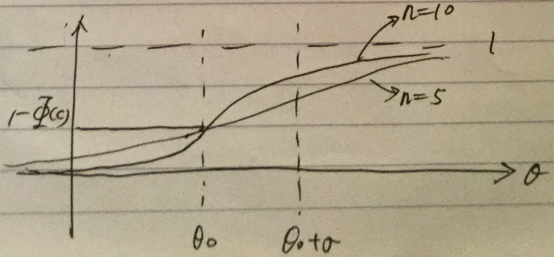
$$= \mathbb{P}_\theta\left(\frac{\bar{x} - \theta}{\sigma/\sqrt{n}} > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

$$= \mathbb{P}_\theta\left(Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \qquad Z \sim N(0,1)$$

$$= 1 - \mathbb{P}_\theta\left(Z \le c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

$$= 1 - \Phi\left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

Then
$$\begin{cases} \beta(\theta) \uparrow \\ \lim_{\theta \to -\infty} \beta(\theta) = 0 \\ \lim_{\theta \to \infty} \beta(\theta) = 1 \end{cases}$$



Suppose we need maximum type I error to be 0.1, and the maximum Type II error to be 0.2 if $\theta \geq \theta_0 + \sigma$.
How to choose c and n?

Maximum Type I error

$$\max_{\theta \in \Theta_0} \beta(\theta) = \max_{\theta \in \theta_0} \beta(\theta) = \beta(\theta_0) = 1 - \Phi(c) = 0.1$$

$$\Rightarrow \quad c = 1.28$$

Maximum Type II error

$$\max_{\theta \geq \theta_0 + \sigma} 1 - \beta(\theta) = 1 - \min_{\theta \geq \theta_0 + \sigma} \beta(\theta) = 1 - \beta(\theta_0 + \sigma)$$

$$= \Phi(c - \sqrt{n}) = 0.2.$$

$$\Rightarrow \quad n = 4.49.$$

Since n is an integer, we choose $n = 5$.

Size and level of tests.

Usually one cannot minimize the Type I and Type II errors at the same time. How to find a reasonable trade-off?

Solution: Place a bound on the Type I error, and try to minimize the Type II error.

Page 4 Def (Size of a test): For $0 \le \alpha \le 1$, a test with power function $\beta(\theta)$ is a size α test, if

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha.$$

Def (Level of a test): For $0 \le \alpha \le 1$, a test with power function $\beta(\theta)$ is a level α test, if its size is less than or equal to α, i.e.,

$$\sup_{\theta \in \Theta_0} \beta(\theta) \le \alpha.$$

Size of LRT: Find a critical value c, such that

$$\sup_{\theta \in \Theta_0} \mathbb{P}_\theta(\lambda(x) < c) = \alpha.$$

How to determine c:

1) Specific sampling distribution
2) Large sample theory

Example : $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, 1)$

$H_0: \theta = \theta_0$, $H_1: \theta \ne \theta_0$.

LRT: H_0 is rejected if $\sqrt{n}\,|\bar{x} - \theta_0| \ge t$.

If $\theta = \theta_0$, we have

$$\sqrt{n}(\bar{x} - \theta_0) \sim N(0, 1)$$

Then

$$\sup_{\theta \in \Theta_0} \mathbb{P}_\theta\left(\sqrt{n}\,|\bar{x} - \theta_0| \ge t\right)$$

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$$= \mathbb{P}_{\theta_0}\left(\sqrt{n}\,|\bar{X} - \theta_0| \geq t\right)$$
$$= 2(1 - \Phi(t)) = \alpha$$
$$\Longleftrightarrow \Phi(t) = 1 - \frac{\alpha}{2} \Longleftrightarrow t = z_{\frac{\alpha}{2}}.$$

Then H_0 is rejected at level α if

$$\sqrt{n}\,|\bar{x} - \theta_0| > z_{\frac{\alpha}{2}}.$$

Most powerful tests

Def: Let $\mathcal{C}$ be a class of tests for testing $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_0^c$. A test in $\mathcal{C}$ is called uniformly most powerful (UMP), if its power function $\beta(\theta)$ satisfies

$$\beta(\theta) \geq \tilde{\beta}(\theta) \quad \text{for all } \theta \in \Theta_0^c$$

and all $\tilde{\beta}(\theta)$ is the power function of a test in $\mathcal{C}$.

We often will consider $\mathcal{C} = \{\text{tests of level } \alpha\}$. In this case, we call the optimal test the UMP level α test.

Neyman-Pearson Lemma: Consider a simple test

$$H_0: \theta = \theta_0 \quad \text{vs} \quad H_1: \theta = \theta_1$$

Let the pdf/pmf corresponding to θ_i be

$$f(x \mid \theta_i), \quad i = 0, 1.$$

 Consider a test with the rejection region R satisfying

$$\left.\begin{array}{ll} x \in R & \text{if } f(x|\theta_1) > k f(x|\theta_0) \\ x \in R^c & \text{if } f(x|\theta_1) < k f(x|\theta_0) \end{array}\right\} \quad (1) \quad \text{for some } k$$

and

$$\mathbb{P}_{\theta_0}(X \in R) = \alpha. \qquad (2)$$

Then we have

a. (Sufficiency) Any test satisfying (1) and (2) is a most powerful level α test.

b. (Necessity) If a test exists satisfying (1) and (2) with $k > 0$, then any MP level α test satisfies (1), except perhaps on a set A satisfying

$$\mathbb{P}_{\theta_0}(X \in A) = \mathbb{P}_{\theta_1}(X \in A) = 0.$$

We only give the proof of sufficiency for the case of pdf.

Proof: Define $\phi(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{if } x \in R^c. \end{cases}$

Suppose the rejection region $\tilde{R}$ is a level α test. And define

$$\tilde{\phi}(x) = \begin{cases} 1 & \text{if } x \in \tilde{R} \\ 0 & \text{if } x \in \tilde{R}^c \end{cases}$$

Then $\mathbb{E}_{\theta_0} \tilde{\phi}(X) = \int \tilde{\phi}(x) f(x|\theta_0)\, d\mu(x)$ Lebesgue measure

$$= \mathbb{P}_{\theta_0}(X \in \tilde{R}) \le \alpha.$$

Page 7. Claim

$$\int (\phi(x) - \tilde{\phi}(x))(f(x|\theta_1) - k f(x|\theta_0))\, d\mu(x) \geq 0.$$

Case 1. $f(x|\theta_1) > k f(x|\theta_0)$

$\phi(x) = 1,\ \tilde{\phi}(x) \leq 1 \Rightarrow$ Integrand ≥ 0

Case 2. $f(x|\theta_1) < k f(x|\theta_0)$

$\phi(x) = 0,\ \tilde{\phi}(x) \geq 0 \Rightarrow$ Integrand ≥ 0.

Case 3. $f(x|\theta_1) = k f(x|\theta_0) \Rightarrow$ Integrand $= 0$.

Then $\mathbb{P}_{\theta_1}(X \in R) - \mathbb{P}_{\theta_1}(X \in \tilde{R})$

$$= \mathbb{E}_{\theta_1}(\phi(X) - \tilde{\phi}(X))$$

$$= \int (\phi(x) - \tilde{\phi}(x)) f(x|\theta_1)\, d\mu(x)$$

$$\geq k \int (\phi(x) - \tilde{\phi}(x)) f(x|\theta_0)\, d\mu(x)$$

$$= k \Big(\underbrace{\mathbb{P}_{\theta_0}(X \in R)}_{= \alpha} - \underbrace{\mathbb{P}_{\theta_0}(X \in \tilde{R})}_{\leq \alpha} \Big) \geq 0$$

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