Problems : 1, 6, 7, 8, 13, 15, 16

Problem 1:

For X:

x	$p(x)$
0	1/4
1	1/2
$\sqrt{2}$	1/4

$\xrightarrow{Y = X^2}$

For Y:

y	$p(y)$
0	1/4
1	1/2
2	1/4

$$
\begin{aligned}
P(Y=2) &= P(X^2=2) = P(X = \pm\sqrt{2}) \\
&= P(X=\sqrt{2}) \\
&= 1/4.
\end{aligned}
$$

(other values are similar).

Conclusion:

$$
Y = \begin{cases} 0, & \text{w/ prob. } 1/4 \\ 1, & \text{w/ prob. } 1/2 \\ 2, & \text{w/ prob. } 1/4 \end{cases}
$$

or, using the probability mass function (p.m.f)

(pmf are used for discrete r.v).

$$
p_Y(y) = \begin{cases} \frac{1}{4}, & y=0; \\ \frac{1}{2}, & y=1; \\ \frac{1}{4}, & y=2. \\ 0, & \text{otherwise}. \end{cases}
$$

Problem 6.

$EX = 1$, $var(x) = 2.5$

$EY = ?$, $Y = (2-x)^2$.

- $E[A+B] = E[A] + E[B]$
- $E[a \cdot A] = a \cdot E[A]$

$$
\begin{aligned}
E[Y] &= E[(2-x)^2] \\
&= E[4 - 4x + x^2] \\
&= E[4] - E[4x] + E[x^2] \\
&= 4 - 4 \cdot E[x] + E[x^2]
\end{aligned}
$$

$E[x] = 1$,

$var(x) = E[x^2] - (E[x])^2$

$$
\begin{aligned}
\Rightarrow E[x^2] &= var(x) + (E[x])^2 \\
&= 2.5 + (1)^2 \\
&= 3.5
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= 4 - 4 \cdot (1) + (3.5) \\
&= 3.5
\end{aligned}
$$

Problem 7.

X_1 = the change of price in the first hour;
X_2 = 〃 second hour;
X_3 = 〃 third hour.

$E[X_1] = 1200$, $\quad var(X_1) = 100$,
$E[X_2] = -2000$, $\quad Var(X_2) = 200$,
$E[X_3] = 10$, $\quad var(X_3) = 3$.

$$E[X_1+X_2+X_3] = E[X_1] + E[X_2] + E[X_3]$$
$$= 1200 - 2000 + 10$$
$$= -790$$

do not need independence. holds at anytime.

(different from answer key)

$Var(X_1+X_2+X_3) = ?$

In the case where X_1, X_2, X_3 are independent, we have

$$Var(X_1+X_2+X_3) = var(X_1) + var(X_2) + var(X_3) \quad (*)$$

For the case where they are dependent, it involves covariance.
It is not simply determined by variances. (Not fixed)
$\Rightarrow$ No enough information given.

Problem 8.

What is binomial distribution? # of success in multiple independent trials.

Binomial (n, p) — n: # of trials, p: success prob.

$p = P(\text{a single promotion is from marketing}) = \frac{1}{4}$

$n = 6$

pmf: $p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n.$

$$P(X=4) = \binom{6}{4}(0.25)^4(0.75)^2 = 0.03296$$

$$P(X \le 4) = \sum_{x=0}^{4} \binom{6}{x}(0.25)^x(0.75)^{6-x}$$

$$= 0.99536$$

R code:

```
dbinom(4, 6, 0.25)            # 0.03295898
sum(dbinom(0:4, 6, 0.25))     # 0.9953613
```

Problem 13.

x	$p(x)$
2	1/2
4	1/4
8	1/8
16	1/16
$\vdots$	$\vdots$

Verify that this is a pmf.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

or:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

For discrete X:

$$E[X] = \sum_x x \cdot p(x)$$

$$= \sum_{n=1}^{\infty} 2^n \cdot p(2^n)$$

$$= \sum_{n=1}^{\infty} 2^n \cdot \left(\frac{1}{2^n}\right)$$

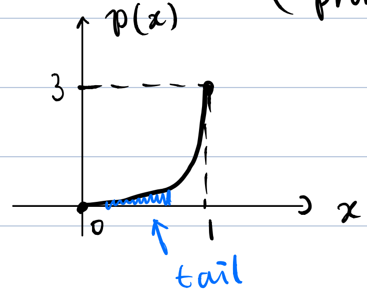
$$= \sum_{n=1}^{\infty} 1.$$

$$= \infty.$$

The expected winnings from this game is ∞.

- A r.v. could have $+\infty$ expectation, even though it is always finite. (It always takes finite values)

Problem 15 $\quad p_X(x)$, pdf of continuous r.v.
(prob. density function)



(i) Shape? (left-skewed, right-skewed, symmetric).

What is left-skewed? It has long left tail.

✓

(ii) CDF of a continuous r.v. is the integral of its p.d.f

$$F_X(x) = \int_{-\infty}^{x} p_X(t)\,dt$$

$$= \int_{0}^{x} (3t^2)\,dt$$

$$= t^3 \Big|_0^x = x^3, \quad \forall\, 0 \le x \le 1.$$

$$F_X(x) = P(X \le x)$$

$$\Rightarrow \quad F_X(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}$$

(iii) $\quad P(X > 0.75) = 1 - P(X \le 0.75)$

$$= 1 - F_X(0.75)$$

$$= 1 - \left(\tfrac{3}{4}\right)^3 = 37/64.$$

Problem 16.

$$F_X(x) = \sqrt{x}, \quad 0 < x < 1.$$

$$\begin{aligned} P(X > 0.5) &= 1 - P(X \leq 0.5) \\ &= 1 - F_X(0.5) \\ &= 1 - \sqrt{0.5} \\ &= 0.29 \end{aligned}$$

Supplementary Material:

Problem 2: $sd(aX+b) = sd(aX) = a \cdot Sd(X)$

$\Rightarrow sd(2+X) = sd(X) = 4$

Problem 3: $Var(aX+b) = Var(aX) = a^2 Var(X)$

$\Rightarrow Var(3X+12) = 3^2 \cdot Var(X)$

$= 3^2 \cdot 10$

$= 90$

Problem 4: $E[4X-2] = 4 \cdot E[X] - 2$

$= 4 \cdot 2 - 2$

$= 6$

Problem 11: Let X be the number of instruments that you double your value.

$\Rightarrow X \sim \text{Binomial}(63, \frac{1}{2})$ $(n=63,\ p=\frac{1}{2})$

Your return value is $10X - 5 \cdot 63 = 10X - 315$

$E[10X-315] = 10 \cdot E[X] - 315$

$= 10 \cdot 63 \cdot \frac{1}{2} - 5 \cdot 63$

$= 0$

$sd(10X - 315) = 10 \cdot sd(X)$

$= 10 \cdot \sqrt{63 \cdot \frac{1}{2} \cdot (1 - \frac{1}{2})}$

$= 39.686$