Homework 2 Solution

Question 1 By solving the matrix product, we have

$$\boldsymbol{A} = \lambda_1 \vec{v}_1 \vec{v}_1^\top + ... + \lambda_k \vec{v}_k \vec{v}_k^\top$$

Since $\boldsymbol{P}$ is an orthogonal matrix, all $\vec{v}_i$'s are unit and pairwise perpendicular. That is to say, for all i, $\vec{v}_i^\top \vec{v}_i = 1$ and for all $i \neq j$, $\vec{v}_i^\top \vec{v}_j = 0$.
Therefore, for all i,

$$\begin{aligned} \boldsymbol{A}\vec{v}_i &= \lambda_1 \vec{v}_1 \vec{v}_1^\top \vec{v}_i + ... + \lambda_k \vec{v}_k \vec{v}_k^\top \vec{v}_i \\ &= \lambda_j \vec{v}_i \end{aligned}$$

which means λ_i's are eigenvalues for $\boldsymbol{A}$, and $\vec{v}_i$'s are eigenvectors for $\boldsymbol{A}$.
By matrix product,

$$\begin{aligned} \boldsymbol{AB} &= \boldsymbol{P\Lambda P}^\top \boldsymbol{P\Gamma P}^\top \\ &= \boldsymbol{P\Lambda\Gamma P}^\top \\ &= \lambda_1\gamma_1 \vec{v}_1 \vec{v}_1^\top + ... + \lambda_k\gamma_k \vec{v}_k \vec{v}_k^\top \end{aligned}$$

Similarly, $\lambda_i\gamma_i$'s are eigenvalues for $\boldsymbol{AB}$, and $\vec{v}_i$'s are eigenvectors for $\boldsymbol{AB}$.

Question 2 Let $\boldsymbol{X} = \begin{bmatrix} \vec{x}_1^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix}$, $\boldsymbol{Y} = \begin{bmatrix} \vec{y}_1^\top \\ \vdots \\ \vec{y}_n^\top \end{bmatrix}$. Since for all i, $\vec{y}_i = \boldsymbol{C}\vec{x}_i$, we have

$$\boldsymbol{Y} = \boldsymbol{XC}^\top$$

$$
\begin{aligned}
\boldsymbol{S}_{\boldsymbol{Y},\boldsymbol{X}} &= \frac{1}{n-1}\begin{bmatrix} \sum_{i=1}^{n}(y_{i1}-\bar{y_1})(x_{i1}-\bar{x_1}) & \sum_{i=1}^{n}(y_{i1}-\bar{y_1})(x_{i2}-\bar{x_2}) & \dots & \sum_{i=1}^{n}(y_{i1}-\bar{y_1})(x_{ip}-\bar{x_p}) \\ \sum_{i=1}^{n}(y_{i2}-\bar{y_2})(x_{i1}-\bar{x_1}) & \sum_{i=1}^{n}(y_{i2}-\bar{y_2})(x_{i2}-\bar{x_2}) & \dots & \sum_{i=1}^{n}(y_{i2}-\bar{y_2})(x_{ip}-\bar{x_p}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}(y_{iq}-\bar{y_q})(x_{i1}-\bar{x_1}) & \sum_{i=1}^{n}(y_{iq}-\bar{y_q})(x_{i2}-\bar{x_2}) & \cdots & \sum_{i=1}^{n}(y_{iq}-\bar{y_q})(x_{ip}-\bar{x_p}) \end{bmatrix} \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\begin{bmatrix} y_{i1}-\bar{y_1} \\ y_{i2}-\bar{y_2} \\ \vdots \\ y_{iq}-\bar{y_q} \end{bmatrix}\begin{bmatrix} x_{i1}-\bar{x_1} & x_{i2}-\bar{x_2} & \dots & x_{ip}-\bar{x_p} \end{bmatrix} \\
&= \frac{1}{n-1}\sum_{i=1}^{n}(\vec{y}_i-\bar{\vec{y}})(\vec{x}_i-\bar{\vec{x}})^\top \\
&= \frac{1}{n-1}\begin{bmatrix} (\vec{y}_1-\bar{\vec{y}}) & \dots & (\vec{y}_n-\bar{\vec{y}}) \end{bmatrix}\begin{bmatrix} (\vec{x}_1-\bar{\vec{x}})^\top \\ \vdots \\ (\vec{x}_n-\bar{\vec{x}})^\top \end{bmatrix} \\
&= \frac{1}{n-1}(\boldsymbol{Y}-\vec{1}_n\bar{\vec{y}}^\top)^\top(\boldsymbol{X}-\vec{1}_n\bar{\vec{x}}^\top) \\
&= \frac{1}{n-1}(\boldsymbol{X}\boldsymbol{C}^\top-\vec{1}_n(\boldsymbol{C}\bar{\vec{x}})^\top)^\top(\boldsymbol{X}-\vec{1}_n\bar{\vec{x}}^\top) \\
&= \frac{1}{n-1}\boldsymbol{C}(\boldsymbol{X}-\vec{1}_n\bar{\vec{x}}^\top)^\top(\boldsymbol{X}-\vec{1}_n\bar{\vec{x}}^\top) \\
&= \boldsymbol{C}\boldsymbol{S}_{\boldsymbol{X}}
\end{aligned}
$$

Question 3

(a)

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{s_{X,Y}}{s_{X,X}} = \frac{2.5}{6.952381} = 0.359589 \\
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1\bar{x} = -0.1541096
\end{aligned}
$$

(b) Let $\boldsymbol{Z} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 0 & 0 \\ -1 & -1 \\ -2 & -1 \\ -3 & -1 \end{bmatrix}$

Then

$$
\begin{aligned}
\bar{\vec{z}} &= \frac{1}{n}\boldsymbol{Z}^\top\vec{1}_n \\
&= \begin{bmatrix} \frac{3}{7} \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{S}_Z &= \frac{1}{n-1}(\boldsymbol{Z} - \vec{1_n}\bar{\vec{z}}^\top)^\top(\boldsymbol{Z} - \vec{1_n}\bar{\vec{z}}^\top) \\
&= \begin{bmatrix} 6.952381 & 2.5 \\ 2.5 & 1 \end{bmatrix}
\end{aligned}
$$

(c) Let $\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}$. Then $\boldsymbol{U} = \begin{bmatrix} X \\ Y - \alpha X \end{bmatrix} = \boldsymbol{C}\begin{bmatrix} X \\ Y \end{bmatrix}$.

We have

$$
\begin{aligned}
\boldsymbol{S}_U &= \boldsymbol{C}\boldsymbol{S}_Z\boldsymbol{C}^\top \\
&= \begin{bmatrix} 6.952381 & 2.5 - 6.952381\alpha \\ 2.5 - 6.952381\alpha & 1 - 2 \times 2.5\alpha + 6.952381\alpha^2 \end{bmatrix}
\end{aligned}
$$

When $2.5 - 6.952381\alpha = 0$, the sample covariance of $Y - \alpha X$ and X is 0. So $\alpha = \frac{2.5}{6.952381} = 0.359589$.

Question 4

$$\hat{\beta}_1 = \frac{s_{X,Y}}{s_{X,X}}$$

Let $\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}$. Then $\boldsymbol{U} = \begin{bmatrix} X \\ Y - \alpha X \end{bmatrix} = \boldsymbol{C}\begin{bmatrix} X \\ Y \end{bmatrix}$.

We have

$$
\begin{aligned}
\boldsymbol{S}_U &= \boldsymbol{C}\boldsymbol{S}_Z\boldsymbol{C}^\top \\
&= \begin{bmatrix} s_{X,X} & S_{X,Y} - S_{X,X}\alpha \\ S_{X,Y} - S_{X,X}\alpha & S_{Y,Y} - 2S_{X,Y}\alpha + S_{X,X}\alpha^2 \end{bmatrix}
\end{aligned}
$$

When $S_{X,Y} - S_{X,X}\alpha$, the sample covariance of $Y - \alpha X$ and X is 0. So $\alpha = \frac{S_{X,Y}}{S_{X,X}} = \hat{\beta}_1$.

Question 5 First let's look at the sample covariance between Y_j and Y_k.

$$
\begin{aligned}
s^y_{jk} &= \frac{1}{n-1}\sum_{i=1}^{n}(y_{ij} - \bar{y_j})(y_{ik} - \bar{y_k}) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}((c_j x_{ij} + d_j) - (c_j\bar{x}_j + d_j))((c_k x_{ik} + d_k) - (c_k\bar{x}_k + d_k)) \\
&= \frac{1}{n-1}\sum_{i=1}^{n} c_j(x_{ij} - \bar{x}_j)c_k(x_{ik} - \bar{x}_k) \\
&= c_j c_k \frac{1}{n-1}\sum_{i=1}^{n}(x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k) \\
&= c_j c_k s^x_{jk}
\end{aligned}
$$

Similarly, we have $s^y_{jj} = c_j^2 s^x_{jj}$. Therefore,

$$
\begin{aligned}
r^y_{jk} &= \frac{s^y_{jk}}{\sqrt{s^y_{jj}s^y_{kk}}} \\
&= \frac{c_j c_k s^x_{jk}}{\sqrt{c_j^2 s^x_{jj} c_k^2 s^x_{kk}}} \\
&= \frac{s^x_{jk}}{\sqrt{s^x_{jj}s^x_{kk}}} \\
&= r^x_{jk}
\end{aligned}
$$

Question 6

$$
\vec{Y} = \begin{bmatrix} \frac{1}{2}(X_1+X_2) \\ \frac{1}{2}(X_2+X_3) \\ \frac{1}{2}(X_3+X_4) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = C\vec{X}
$$

Therefore,

$$
\begin{aligned}
\boldsymbol{S}_Y &= C\boldsymbol{S}_X C^\top \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} \\ 1 & \frac{3}{2} & 1 \\ \frac{1}{4} & 1 & \frac{3}{2} \end{bmatrix}
\end{aligned}
$$

Question 7

(a) From Question 3,

$$
\boldsymbol{S} = \begin{bmatrix} 6.952381 & 2.5 \\ 2.5 & 1 \end{bmatrix}
$$

The spectral decomposition of $\boldsymbol{S}$ is given by

$$
\boldsymbol{S} = \begin{bmatrix} -0.9396023 & 0.3422683 \\ -0.3422683 & -0.9396023 \end{bmatrix} \begin{bmatrix} 7.863054 & 0 \\ 0 & 0.08932673 \end{bmatrix} \begin{bmatrix} -0.9396023 & -0.3422683 \\ 0.3422683 & -0.9396023 \end{bmatrix}
$$

From the property of spectral decomposition, the spectral decomposition of $\boldsymbol{S}^{-1}$ is given by

$$
\boldsymbol{S}^{-1} = \begin{bmatrix} -0.9396023 & 0.3422683 \\ -0.3422683 & -0.9396023 \end{bmatrix} \begin{bmatrix} \frac{1}{7.863054} & 0 \\ 0 & \frac{1}{0.08932673} \end{bmatrix} \begin{bmatrix} -0.9396023 & -0.3422683 \\ 0.3422683 & -0.9396023 \end{bmatrix}
$$

(b) From Question 3, $\bar{x} = \begin{bmatrix} \frac{3}{7} \\ 0 \end{bmatrix}$

$$[u_1, u_2] = \begin{bmatrix} -0.9396023 & 0.3422683 \\ -0.3422683 & -0.9396023 \end{bmatrix}$$

$$\lambda_1 = 7.863054$$
$$\lambda_2 = 0.08932673$$

$$c^2 = 4$$
$$c = 2$$

In all, this ellipse has center $\begin{bmatrix} \frac{3}{7} \\ 0 \end{bmatrix}$, the axes are of direction $u_1 = \begin{bmatrix} -0.9396023 \\ -0.3422683 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0.3422683 \\ -0.9396023 \end{bmatrix}$, with corresponding half axis length $c\sqrt{\lambda_1} = 5.6082276$ and $c\sqrt{\lambda_2} = 0.5977516$

(c)

$$\begin{aligned} \boldsymbol{R} &= \begin{bmatrix} 1 & 0.9481417 \\ 0.9481417 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.7071068 & -0.7071068 \\ 0.7071068 & 0.7071068 \end{bmatrix} \begin{bmatrix} 1.948142 & 0.00000000 \\ 0.000000 & 0.05185834 \end{bmatrix} \begin{bmatrix} 0.7071068 & 0.7071068 \\ -0.7071068 & 0.7071068 \end{bmatrix} \end{aligned}$$

$$\boldsymbol{R}^{-1} = \begin{bmatrix} 0.7071068 & -0.7071068 \\ 0.7071068 & 0.7071068 \end{bmatrix} \begin{bmatrix} \frac{1}{1.948142} & 0.00000000 \\ 0.000000 & \frac{1}{0.05185834} \end{bmatrix} \begin{bmatrix} 0.7071068 & 0.7071068 \\ -0.7071068 & 0.7071068 \end{bmatrix}$$

(d)

$$[u_1, u_2] = \begin{bmatrix} 0.7071068 & -0.7071068 \\ 0.7071068 & 0.7071068 \end{bmatrix}$$

$$\lambda_1 = 1.948142$$
$$\lambda_2 = 0.05185834$$

$$c^2 = 4$$
$$c = 2$$

In all, this ellipse has center $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The axes are of direction $u_1 = \begin{bmatrix} 0.7071068 \\ 0.7071068 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -0.7071068 \\ 0.7071068 \end{bmatrix}$, with corresponding half axis length $c\sqrt{\lambda_1} = 2.7915169$ and $c\sqrt{\lambda_2} = 0.4554485$