Problem set 6

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Instructions: Here is our sixth set of practice problems.

Practice problems

(1) Consider competing firms A and B. The firms will expand into X_A, X_B new markets, respectively, where $0 \leq X_A \leq 2$, $0 \leq X_B \leq 2$. Suppose that each firm has the same marginal distribution, given below. If the two firms expand independently, what is the joint probability distribution of their numbers of expansions? Would you say that it would be an unusual coincidence for them both to expand to 2 new markets?

x	$p(x)$
0	$1/2$
1	$1/3$
2	$1/6$

answer: Owing to their independence, we can obtain the joint distribution from the marginal distributions:

$p_{X_1,X_2}(x_1,x_2)$		$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$p_{X_1}(x_1)$
x_1	0	$1/4$	$1/6$	$1/12$	$1/2$
	1	$1/6$	$1/9$	$1/18$	$1/3$
	2	$1/12$	$1/18$	$1/36$	$1/6$
$p_{X_2}(x_2)$		$1/2$	$1/3$	$1/6$	

From this we have $P(X_A = 2, X_B = 2) = 1/36$, so yes, this would be fairly unusual.

(2) Suppose that you will roll two fair six-sided dice. Let U be the first result, and V the second. Furthermore, let

$$X_1 = \begin{cases} 1 & \text{if } U \geq 4 \\ -1 & \text{if otherwise} \end{cases}$$

and

$$X_2 = \begin{cases} 1 & \text{if } V \geq 4 \\ -1 & \text{if otherwise} \end{cases}$$

What is the joint distribution of X_1 and X_2?

answer: Since each marginal distribution is uniform, the joint distribution is also uniform:

$p_{X_1,X_2}(x_1,x_2)$		$x_2 = 1$	$x_2 = -1$	$p_{X_1}(x_1)$
x_1	1	1/4	1/4	1/2
	-1	1/4	1/4	1/2
$p_{X_2}(x_2)$		1/2	1/2	

(3) Suppose that you will roll two fair six-sided dice. Let U be the first result, and V the second. Furthermore, let

$$X_1 = \begin{cases} 1 & \text{if } U \geq 3 \\ -1 & \text{if otherwise} \end{cases}$$

and

$$X_2 = \begin{cases} 1 & \text{if } V \geq 3 \\ -1 & \text{if otherwise} \end{cases}$$

What is the joint distribution of X_1 and X_2?

answer:

$p_{X_1,X_2}(x_1,x_2)$		$x_2 = 1$	$x_2 = -1$	$p_{X_1}(x_1)$
x_1	1	4/9	2/9	2/3
	-1	2/9	1/9	1/3
$p_{X_2}(x_2)$		2/3	1/3	

(4) Consider r.v.'s X and Y with the joint probability distribution given below. X here indicates the occurance of a recession in a given quarter, and Y the number of firms in a sector that declare bankruptcy. Are these independent under this model? If not, find the conditional distributions of Y given $X = 1$, and compare it with the corresponding marginal distribution.

$p_{X,Y}(x,y)$		$y = 0$	$y = 1$	$y = 2$	$y = 3$
x	0	7/16	7/32	15/96	1/16
	1	1/16	1/32	1/96	1/48

answer:

With the marginal distributions:

$p_{X,Y}(x,y)$		$y = 0$	1	2	3	$p_X(x)$
x	0	$7/16$	$7/32$	$15/96$	$1/16$	$7/8$
	1	$1/16$	$1/32$	$1/96$	$1/48$	$1/8$
$p_Y(y)$		$1/2$	$1/4$	$1/6$	$1/12$	

and the joint probabilities with these margins would be different, so no, not independent.

Comparing the marginal distribution of Y with its conditional distribution given $X = 1$:

y	$p_Y(y)$	$p_{Y\|X}(y,1)$
0	$1/2$	$1/2$
1	$1/4$	$1/4$
2	$1/6$	$1/12$
3	$1/12$	$1/6$

So if there is a recession, $Y = 3$ will be more likely, $Y = 2$ will be less likely, $Y = 0$ will be as likely, and $Y = 1$ will be as likely.

(5) Are X and Y, with the joint distribution given below, independent?

$p_{X,Y}(x,y)$		$y = -1$	0	1
x	0	$1/28$	$5/56$	$1/56$
	1	$1/7$	$5/14$	$1/14$
	2	$3/28$	$15/56$	$3/56$

answer:

With the margins given below, under independence, the joint probabilities are the same; so yes, these are independent.

$p_{X,Y}(x,y)$		y			$p_X(x)$
		-1	0	1	
	-1	$1/28$	$5/56$	$1/56$	$1/7$
x	0	$1/7$	$5/14$	$1/14$	$4/7$
	1	$3/28$	$15/56$	$3/56$	$3/7$
$p_Y(y)$		$1/4$	$5/8$	$1/8$	

(6) Find the marginal and conditional distributions of Y given $X = 0$ and given $X = 1$ under the joint distribution below.

$p_{X,Y}(x,y)$		y			$p_X(x)$
		1	2	3	
x	0	$4/9$	$4/27$	$2/27$	$2/3$
	1	$1/27$	$2/27$	$2/9$	$1/3$
$p_Y(y)$		$13/27$	$2/9$	$8/27$	

answer: We have margins

$p_{X,Y}(x,y)$		y			$p_X(x)$
		1	2	3	
x	0	$4/9$	$4/27$	$2/27$	$2/3$
	1	$1/27$	$2/27$	$2/9$	$1/3$
$p_Y(y)$		$13/27$	$2/9$	$8/27$	

and conditional distributions

y	$p_Y(y)$	$p_{Y\|X}(y,0)$	$p_{Y\|X}(y,1)$
1	$13/27$	$2/3$	$1/9$
2	$2/9$	$2/9$	$2/9$
3	$8/27$	$1/9$	$2/3$

(7) What is the missing value in the table of joint probabilities below? Are these r.v.'s independent?

$p_{X,Y}(x,y)$

$p_{X,Y}(x,y)$		$y = -1$	$y = 0$	$y = 1$
x	-1	$7/54$	$5/54$	$7/54$
	0	$5/54$	$7/54$	$5/54$
	1	$7/54$	$5/54$	?

answer: $1/9$; No!