Stat 200A Homework 1 Solution

1. Ch. 1, Problem 1

   Solution. There are 26 letters and 10 numbers (0-9), which means there are $26^2(10^5) = 67,600,000$. We need to select 2 numbers in order, of which there are 26(25) outcomes, and 5 letters, of which there are 10(9)(8)(7)(6) outcomes totalling $26(25)(10)(9)(8)(7)(6) = 19656000$ outcomes.

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2. Ch. 1, Problem 5

   Solution. There are $8(2)(9) = 144$ possibilities. If we start with a 4 then there are $2(9) = 18$ possibilities.

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3. Ch. 1, Problem 10

   Solution.

   - $8! = 40320$
   - Let's place A first: 8 possibilities; then B: 2 possibilities except when A is first or last, thus A,B have $8(2) - 2 = 14$ possibilities. Then the rest: 6! which gives us $14(6!) = 10080$.
   - If there first position has a male then it is MFMFMFMF and we can permute the males and females: $(4!)^2 = 576$ and similarly if a female is first, so there are in total $2(576) = 1152$.
   - There are 5 positions for the first male, and with this fixed there are $4!^2$ options, so we have $5(4!)^2 = 2880$ total options.
   - There are 4! positions for the couples and then 2^4 orders for the couples: $2^4(4!) = 384$

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4. Ch. 1, Problem 13

   Solution. What are the ways to choose 2 right hands from 20 people (let's say that everyone shakes with the right hand), then there are $\binom{20}{2} = 190$.

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5. Ch. 1, Problem 27

   Solution. This is the multinomial coefficient: $\binom{12}{3,4,5} = 27720$

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6. Ch. 1, Theoretical Exercise 11

Solution. Think of $\binom{n}{k}$ as the number of subsets of size k taken from the set $\{1, \ldots, n\}$.
For any subset of size k, let's say it has "height" i if the largest number in the subset is equal to i.

Also note that for any subset of size k, its height must be at least k and at most n.

We can break up all the subsets of size k according to their height. Let $N(i)$ denote the number of subsets of size k that have height i.
Then, the number of subsets of size k must be equal to

$$\binom{n}{k} = \sum_{i=k}^{n} N(i).$$

Finally, note that $N(i) = \binom{i-1}{k-1}$, because in order to construct a subset of size k and height i, we must choose $k-1$ numbers from the set $\{1, \ldots, i-1\}$. So, altogether

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

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7. Prove the identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ without using the binomial theorem.

   Solution. It is a basic fact that there are 2^n distinct subsets of any set of n items. This holds because any subset corresponds to making n choices ("in or out") for each item. Since there are n choices with 2 possibilities for each choice, the basic principle of counting implies there are 2^n subsets (including the empty set with no elements.)

   Now, to prove the identity, another way of counting all possible subsets of a set of n items is to ask
   "How many subsets of size 0" are there?" Ans: $\binom{n}{0}$
   "How many subsets of size 1" are there?" Ans: $\binom{n}{1}$
   "How many subsets of size 2" are there?" Ans: $\binom{n}{2}$
   ...
   Hence, the sum $\sum_{k=0}^{n} \binom{n}{k}$ represents a way of counting all possible subsets of a set of size n.

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8. A set of n items contains k defective items, and m are sampled randomly for inspection. How should the value of m be chosen so that the probability that at least one defective item turns up is 0.90? Apply your answer to (a) $n = 1000$, $k = 10$, and (b) $n = 10,000$ and $k = 100$

   Hint: It's ok to derive an approximate answer by assuming k is small compared to n.

   Solution.

$$\# \text{ of defectives} = k, \qquad \# \text{ of non-defectives} = n - k$$

$$P(\text{at least one def}) = 1 - P(\text{non defective}) = 1 - \frac{\binom{n-k}{m}}{\binom{n}{m}} = 0.9 \quad \Longrightarrow \quad \frac{\binom{n-k}{m}}{\binom{n}{m}} = 0.1$$

then using the definition of combination $\binom{x}{y} = \frac{x!}{y!(x-y)!}$, we have

$$\frac{\binom{n-k}{m}}{\binom{n}{m}} = \frac{(n-k)!}{m!(n-k-m)!} \times \frac{m!(n-m)!}{n!} = \frac{(n-m)(n-m-1)...(n-m-k+1)}{n(n-1)...(n-k+1)}$$

here we will be using approximation to the last fraction, i.e.,

$$\frac{(n-m)(n-m-1)...(n-m-k+1)}{n(n-1)...(n-k+1)} = \frac{n-m}{n}\frac{n-m-1}{n-1}...\frac{n-m-k+1}{n-k+1} \approx \left(\frac{n-m}{n}\right)^k$$

Hence we have,

$$\frac{\binom{n-k}{m}}{\binom{n}{m}} \approx \left(\frac{n-m}{n}\right)^k = 0.1$$

(a) when $k = 10$ and $n = 1000$, we have

$$\left(\frac{1000-m}{1000}\right)^{10} = 0.1 \Rightarrow 10 \log\left(\frac{1000-m}{1000}\right) = \log(0.1) \Rightarrow m = 205.67$$

(b) when $k = 100$ and $n = 10000$, we have

$$\left(\frac{10000-m}{10000}\right)^{100} = 0.1 \Rightarrow 100 \log\left(\frac{10000-m}{10000}\right) = \log(0.1) \Rightarrow m = 227.63$$

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9. A group of 60 second graders is to be randomly assigned to two classes of 30 each. (The random assignment is ordered by the school district to ensure against any bias.) Five of the second graders, Marcelle, Sarah, Michelle, Katy, and Camerin, are close friends. What is the probability that they will all be in the same class? What is the probability that exactly four of them will be? What is the probability that Marcelle will be in one class and her friends in the other?

Solution.

$$P(\text{all in same class}) = \frac{2\binom{5}{5}\binom{55}{25}}{\binom{60}{30}} = 0.0522$$

$$P(\text{4 in same class}) = \frac{2\binom{5}{4}\binom{1}{1}\binom{55}{29}}{\binom{60}{30}} = 0.301$$

$$P(\text{Marcelle in one, rest in other}) = \frac{2\binom{4}{4}\binom{1}{1}\binom{55}{29}}{\binom{60}{30}} = 0.0602$$