Homework 2 (Due on 1/31)

Question 1 Let

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^\top \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{P}\boldsymbol{\Gamma}\boldsymbol{P}^\top$$

where

$$\boldsymbol{P} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$$

is a $k \times k$ orthogonal matrix, and

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Gamma} = \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_k \end{bmatrix}$$

are two diagonal matrices. Find all pairs of eigenvalues and eigenvectors of $\boldsymbol{AB}$.

Question 2 Consider a p-variate sample $\vec{x}_1, \dots, \vec{x}_n$ with sample covariance $\boldsymbol{S}_X$. For some $\boldsymbol{C} \in \mathbb{R}^{q\times p}$ and $\vec{a} \in \mathbb{R}^q$, consider the linear transformation

$$\vec{y}_i = \boldsymbol{C}\vec{x}_i, \quad i = 1, \dots, n.$$

For any $j = 1, \dots, q$ and $k = 1, \dots, p$, denote by s_{Y_j,X_k} the sample covariance between $\{y_{ij}\}_{i=1}^n$ and $\{x_{ik}\}_{i=1}^n$. Define the matrix

$$\boldsymbol{S}_{Y,X} = \begin{bmatrix} s_{Y_1,X_1} & s_{Y_1,X_2} & \cdots & s_{Y_1,X_p} \\ s_{Y_2,X_1} & s_{Y_2,X_2} & \cdots & s_{Y_2,X_p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{Y_q,X_1} & s_{Y_q,X_2} & \cdots & s_{Y_q,X_p} \end{bmatrix}.$$

Express $\boldsymbol{S}_{Y,X}$ by $\boldsymbol{S}_X$ and $\boldsymbol{C}$.

Question 3 Suppose we are given the data for two variates:

$X:$	2	3	4	0	-1	-2	-3
$Y:$	1	1	1	0	-1	-1	-1

(a) Fit the simple linear regression model $Y = \beta_0 + \beta_1 X + \epsilon$. Find $\hat{\beta}_0$ and $\hat{\beta}_1$.

(b) Find the sample mean and sample covariance of $\begin{bmatrix} X \\ Y \end{bmatrix}$.

(c) Find α such that $Y - \alpha X$ and X have zero sample correlation.

Question 4 Suppose we are given the data

$X:$	x_1	x_2	x_3	$\dots x_n$
$Y:$	y_1	y_2	y_3	$\dots y_n$

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the solutions that fit the simple linear regression model $Y = \beta_0 + \beta_1 X + \epsilon$. Suppose α satisfies that $Y - \alpha X$ and X have zero sample correlation. Show that $\hat{\beta}_1 = \alpha$.

Question 5 A p-variate sample $\vec{x}_1, \ldots, \vec{x}_n$ is transformed into $\vec{y}_1, \ldots, \vec{y}_n$ by

$$y_{ij} = c_j x_{ij} + d_j,\ j = 1, \ldots, p,\ i = 1, \ldots, n.$$

Here $c_j > 0$ for $j = 1, \ldots, p$. In other words, the p variates $X_1, \ldots, X_p$ are transformed into $Y_1, \ldots, Y_p$ in that $Y_j = c_j X_j + d_j$. Denote by r^x_{jk} the sample correlation between X_j and X_k, and denote by r^y_{jk} the sample correlation between Y_j and Y_k. Prove that $r^x_{jk} = r^y_{jk}$.

Question 6 For a sample of $\overrightarrow{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$ with sample covariance matrix

$$\boldsymbol{S} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find the sample covariance matrix of $\begin{bmatrix} \frac{1}{2}(X_1 + X_2) \\ \frac{1}{2}(X_2 + X_3) \\ \frac{1}{2}(X_3 + X_4) \end{bmatrix}$.

Question 7 Suppose we are given the data for two variates

$$\begin{array}{lrrrrrrr} X_1: & 2 & 3 & 4 & 0 & -1 & -2 & -3 \\ X_2: & 1 & 1 & 1 & 0 & -1 & -1 & -1 \end{array}$$

The sample mean and sample covariance matrix of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ are denoted by $\bar{\boldsymbol{x}}$ and $\boldsymbol{S}$, respectively. (a) Find the spectral decompositions of $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$, respectively.
(b) Sketch the mean-centered ellipse

$$(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \boldsymbol{S}^{-1} (\boldsymbol{x} - \bar{\boldsymbol{x}}) \leq 4.$$

(c) Determine the sample correlation matrix $\boldsymbol{R}$. Find the spectral decompositions of $\boldsymbol{R}$ and $\boldsymbol{R}^{-1}$, respectively.
(d) Sketch the mean-centered ellipse

$$\boldsymbol{x}^\top \boldsymbol{R}^{-1} \boldsymbol{x} \leq 4.$$