200B HW#7 solution

8.8 Fisher Information

4. Suppose that a random variable X has the normal distribution with mean 0 and unknown standard deviation $\sigma > 0$. Find the Fisher information $I(\sigma)$ in X.

solution

$$
\begin{aligned}
f(x|\sigma) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{x^2}{2\sigma^2}\},\\
\frac{\partial \log f(x|\sigma)}{\partial\sigma} &= -\frac{1}{\sigma}+\frac{x^2}{\sigma^3},\\
\frac{\partial^2 \log f(x|\sigma)}{\partial\sigma^2} &= \frac{1}{\sigma^2}-\frac{3x^2}{\sigma^4}.
\end{aligned}
$$

Therefore,

$$I(\sigma) = -E_\sigma\left[\frac{\partial^2 \log f(x|\sigma)}{\partial\sigma^2}\right] = \frac{2}{\sigma^2}.$$

6. Suppose that X is a random variable for which the p.d.f. or the p.f. is $f(x|\theta)$, where the value of the parameter θ is unknown but must lie in an open interval Ω Let $I_0(\theta)$ denote the Fisher information in X . Suppose now that the parameter θ is replaced by a new parameter μ, where $\theta = \Psi(\mu)$, and Ψ is a differentiable function. Let $I_1(\mu)$ denote the Fisher information in X when the parameter is regarded as μ. Show that

$$I_1(\mu) = [\psi'(\mu)]^2 I_0(\psi(\mu)).$$

solution Let $g(x|\mu)$ denote the p.d.f. or the p.f. of X when μ is regarded as the parameter. Then $g(x|\mu) = f(x|\psi(\mu))$. Therefore, by the chain rule of differentiation,

$$\frac{\partial \log g(x|\mu)}{\partial \mu} = \frac{\partial \log f(x|\psi(\mu))}{\partial \mu} = \frac{\partial \log f(x|\psi(\mu))}{\partial \psi}\psi'(\mu).$$

It now follows that

$$I_1(\mu) = E_\mu\{[\frac{\partial \log g(x|\mu)}{\partial \mu}]^2\} = [\psi'(\mu)]^2 E_\mu\{[\frac{\partial \log f(x|\psi(\mu))}{\partial \psi}]^2\} = [\psi'(\mu)]^2 I_0(\psi(\mu)).$$

10. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean 0 and unknown standard deviation $\sigma > 0$. Find the lower bound specified by the information inequality for the variance of any unbiased estimator of $\log \sigma$.

<u>**solution**</u> If $m(\sigma) = \log \sigma$, then $m'(\sigma) = \frac{1}{\sigma}$. Also, it was shown in problem 8.8.4 $I(\sigma) = \frac{2}{\sigma^2}$. Therefore, by Cramér-Rao Bound

$$\text{var}(T) \geq \frac{1}{\sigma^2}\frac{\sigma^2}{2n} = \frac{1}{2n}.$$

11. Suppose that $X_1, \ldots, X_n$ form a random sample from an exponential family for which the p.d.f. or the p.f.$f(x|\theta)$ is as specified in Exercise 23 of Sec. 7.3. Suppose also that the unknown value ofθ must belong to an open interval Ω of the real line. Show that the estimator $T = \sum_{i=1}^n d(X_i)$ is an efficient estimator.

<u>**solution**</u> If $f(x|\theta) = a(\theta)b(x)\exp[c(\theta)d(x)]$, then

$$\log f(x|\theta) = \log a(\theta) + \log b(x) + c(\theta)d(x)$$

and

$$\frac{\partial \log f(x|\theta)}{\partial \theta} = \frac{a'(\theta)}{a(\theta)} + c'(\theta)d(x).$$

Therefore,

$$\frac{\partial \log f_n(\mathbf{X}|\theta)}{\partial \theta} = n\frac{a'(\theta)}{a(\theta)} + c'(\theta)\sum_{i=1}^n d(X_i).$$

If we choose $u(\theta) = \frac{1}{c'(\theta)}$ and $v(\theta) = -\frac{na'(\theta)}{a(\theta)c'(\theta)}$, then $T = \sum_{i=1}^{n} d(X_i)$ satisfies the equation in lecture notes page 41. Hence, this statistic is an efficient estimator of its expectation.

12. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution for which the mean is known and the variance is unknown. Construct an efficient estimator that is not identically equal to a constant, and determine the expectation and the variance of this estimator.

solution Let $\theta = \sigma^2$ denote the unknown variance. Then

$$f(x|\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\{-\frac{1}{2\theta}(x-\mu)^2\}.$$

This p.d.f. $f(x|\theta)$ has the form of an exponential family, with $a(\theta) = \frac{1}{\sqrt{2\pi\theta}}$, $b(x) = 1$, $c(\theta) = -\frac{1}{2\theta}$ and $d(x) = (x-\mu)^2$. Therefore, $T = \sum_{i=1}^{n}(X_i - \mu)^2$ will be an efficient estimator. Since $E[(X_i - \mu)^2] = \sigma^2$ for $i = 1, \ldots, n$, then $E(T) = n\sigma^2$. Also, by $E(X_i - \mu)^4 = 3\sigma^4$ and $\mathrm{var}[(X_i - \mu)^2] = 3\sigma^4 - \sigma^4 = 2\sigma^4$, it follows that $\mathrm{var}(T) = 2n\sigma^4$.

13.Determine what is wrong with the following argument:
Suppose that the random variable X has the uniform distribution on the interval $[0, \theta]$, where the value of θ is unknown $(\theta > 0)$ .Then $f(x|\theta) = \frac{1}{\theta}$, $\lambda(x|\theta) = -\log\theta$ and $\lambda'(x|\theta) = -(1/\theta)$. Therefore,

$$I(\theta) = E_\theta\{[\lambda'(x|\theta)]^2\} = \frac{1}{\theta^2}$$

Since $2X$ is an unbiased estimator of θ, the information inequality states that,

$$Var(2X) \geq 1/I(\theta) = \theta^2.$$

But

$$Var(2X) = 4Var(X) = 4.\frac{\theta^2}{12} = \frac{\theta^2}{3} < \theta^2$$

Hence, the information inequality is not correct.

solution The incorrect part of the argument is that the information inequality cannot be applied to the uniform distribution. The assumption (A2) in page 38 of lecture notes is not satisfied.

18. Let X have the negative binomial distribution with parameters r and p. Assume that r is known. Show that the Fisher information in X is $I(p) = r/[p^2(1-p)]$.

solution The derivative of the log-likelihood with respect to p is

$$\frac{\partial}{\partial p}[\log\binom{r+x-1}{x} + r\log(p) + x\log(1-p)] = \frac{r}{p} - \frac{x}{1-p} = \frac{r - rp - xp}{p(1-p)}.$$

The Fisher information is

$$I(p) = \mathrm{var}(\frac{\partial \log f(X|p)}{\partial p}) = \mathrm{var}(\frac{-Xp}{p(1-p)}) = \frac{\mathrm{var}(X)}{(1-p)^2} = \frac{r}{p^2(1-p)}.$$

19. Let X have the gamma distribution with parameters n and θ with θ unknown. Show that the Fisher information in X is $I(\theta) = n/\theta^2$.

solution The derivative of the log-likelihood with respect to θ is

$$\frac{\partial}{\partial \theta}[n\log\theta - \log(\Gamma(n)) + (n-1)\log(x) - \theta x] = \frac{n}{\theta} - x.$$

The Fisher information is

$$I(\theta) = \mathrm{var}(\frac{\partial \log f(X|\theta)}{\partial \theta}) = \mathrm{var}(X) = \frac{n}{\theta^2}.$$

8.9 Supplementary Exercises

14. Suppose that $X_1, \ldots, X_n$ form a random sample from the Poisson distribution with unknown mean θ, and let $Y = \sum_{i=1}^n X_i$.

(a) Determine the value of a constant c such that the estimator e^{-cY} is an unbiased estimator of $e^{-\theta}$.

(b) Use the information inequality to obtain a lower bound for the variance of the unbiased estimator found in part (a).

solution (a) Since Y has a Poisson distribution with mean $n\theta$, it follows that

$$\begin{aligned} E(exp(-cY)) &= \sum_{y=0}^{\infty} \frac{\exp(-cy)\exp(-n\theta)(n\theta)^y}{y!} = \exp(-n\theta)\sum_{y=0}^{\infty}\frac{(n\theta\exp(-c))^y}{y!} \\ &= \exp(-n\theta)\exp[n\theta\exp(-c)] = \exp(n\theta[\exp(-c)-1]). \end{aligned}$$

Set this expectation to $\exp(-\theta)$, it follows that $c = \log(\frac{n}{n-1})$.

(b)

$$\begin{aligned} f(x|\theta) &= \frac{\exp(-\theta)\theta^x}{x!}, \\ \frac{\partial \log f(x|\theta)}{\partial\theta} &= \frac{x}{\theta} - 1. \end{aligned}$$

The Fisher information is

$$I(\theta) = \mathrm{var}(\frac{\partial \log f(X|\theta)}{\partial\theta}) = \frac{\mathrm{var}(X)}{\theta^2} = \frac{1}{\theta}.$$

Since $m(\theta) = \exp(-\theta)$, $m'(\theta) = -\exp(-\theta)$, by Craémer-Rao Bound

$$\mathrm{var}(\exp(-cY)) \geq \frac{\theta\exp(-2\theta)}{n}.$$

18. Suppose that$X_1, \ldots, X_n$ form a random sample from the exponential distribution with unknown parameter β. Construct an efficient estimator that is not identically equal to a constant, and determine the expectation and the variance of this estimator.

solution $f(x|\beta) = \beta\exp(-\beta x)$. This p.d.f. has the form of an exponential familty with $a(\beta) = \beta$, $b(x) = 1$, $c(\beta) = -\beta$ and $d(x) = x$. Therefore, $T = \sum_{i=1}^n X_i$ is an efficient estimator. $E(T) = \frac{n}{\beta}$ and $\mathrm{var}(T) = \frac{n}{\beta^2}$.