Homework 6 (Due 3/13)

Question 1 You are given a sample:

$$\begin{array}{lrrrrrrr} x_1: & 2 & 2 & 2 & 0 & -1 & -2 & -3 \\ x_2: & 2 & 4 & 6 & 0 & -4 & -4 & -4. \end{array}$$

(a) Give the formula for the first and second sample principal components $\hat{y}_1$ and $\hat{y}_2$;
(b) Determine the proportion of total sample variance due to the first sample principal component;
(c) Compare the contributions of the two variates to the determination of the first sample principal component based on loadings;
(d) Compare the contributions of the two variates to the determination of the first sample principal component based on sample correlations;
(e) Redo (a)-(d) on the standardized dataset.

Question 2 If the first principal component of X_1 and X_2 is

$$Y_1 = \frac{\sqrt{2}}{2} X_1 + \frac{\sqrt{2}}{2} X_2,$$

is it possible that $\text{Corr}(X_1, X_2) < 0$? Explain.

Question 3 Problem 8.12 on Page 474.

Question 4 Consider two samples with equal sizes $n_1 = n_2$:

$$\vec{x}_{11}, \ldots, \vec{x}_{1n_1}$$

and

$$\vec{x}_{21}, \ldots, \vec{x}_{2n_2}$$

with summary statistics

$$\bar{\vec{x}}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad \boldsymbol{S}_1 = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix},$$

$$\bar{\vec{x}}_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \quad \boldsymbol{S}_2 = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}.$$

For a new observation $\vec{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$, consider the following classifiers:

1. Classifier 1: Fisher's rule if only x_{01} is observed;
2. Classifier 2: Fisher's rule if only x_{02} is observed;
3. Classifier 3: Fisher's rule based on $\vec{x}_0$.

Does there exist a $\vec{x}_0$, such that Classifier 1 and Classifier 2 give consensus assignment, while Classifier 3 gives a different assignment?

Question 5 Redo Question 4 with the following summary statistics:

$$\bar{\vec{x}}_1 = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \boldsymbol{S}_1 = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix},$$

$$\bar{\vec{x}}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{S}_2 = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}.$$

Question 6 Consider three independent samples from three classes:
π_1: distribution $\mathcal{N}_p(\vec{\mu}_1, \boldsymbol{\Sigma})$, sample size n_1, sample mean $\bar{\vec{x}}_1$, sample covariance $\boldsymbol{S}_1$;
π_2: distribution $\mathcal{N}_p(\vec{\mu}_2, \boldsymbol{\Sigma})$, sample size n_2, sample mean $\bar{\vec{x}}_2$, sample covariance $\boldsymbol{S}_2$;
π_3: distribution $\mathcal{N}_p(\vec{\mu}_3, \boldsymbol{\Sigma})$, sample size n_3, sample mean $\bar{\vec{x}}_3$, sample covariance $\boldsymbol{S}_3$.

Assume that these three populations have equal prior probabilities. For a new observation $\vec{x}_0$, we aim to classify it to one of the three classes with pairwise linear discriminant analyses based on Fisher's rule. Given the three population covariance matrices are assumed to be the same, in all linear discriminant analyses we use

$$\boldsymbol{S}_{pooled} = \frac{1}{n_1 + n_2 + n_3 - 3} \left((n_1 - 1)\boldsymbol{S}_1 + (n_2 - 1)\boldsymbol{S}_2 + (n_3 - 1)\boldsymbol{S}_3 \right).$$

Suppose that in the comparison between π_1 and π_2, $\vec{x}_0$ is allocated to π_2, while in the comparison between π_2 and π_3, $\vec{x}_0$ is allocated to π_3. Show that in the comparison between π_1 and π_3, $\vec{x}_0$ is allocated to π_3.

Question 7 For the dataset on Table 1.6 (Page 42), construct Fisher's rule. Moreover, calculate the apparent error rate as well as the expected actual error rate by Lachenbruch's holdout.