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***Abstract*—K ’s**

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**d Kane’s method successfully to model**

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**the Kane’s method, especially for the b . A**
**3 DOF**

**involved in modeling using Kane’s method. T**

**L .**

***Index Terms*—D , Kane’s method, N - E , L .**

## I. Introduction

In dynamic modeling, a set of mathematical equations is formulated to represent the dynamic behavior of a system. Mathematical modeling uses mathematical concepts to describe a model. Modeling enables us to extract information about a system under particular conditions without having to practically build the actual system and test it which can be otherwise very expensive

Biomechanical modelling has been of great significance over the past few decades. These models enable the biomechanists to explain a system and to study the effects of different components, and to make predictions about behavior of the system without having to physically construct the model. Newton-Euler and Lagrangian methods for deriving the dynamic equations of motion are very difficult to implement on for 3D as well as 2D systems with many DOFs. On the other is devoid of such limitations and as such is becoming increasing popular among the biomechanists. thod provides a convenient way of deriving the dynamic equations of motion for complex multibody systems having several degrees of freedom [1].

This paper presents a detailed step by step explanation of The previous works on the mathematical model
A dynamic model for a 3DOFs kinematic chain has been developed and verified to illustrate the procedure implemented in this technique. This paper provides simple and quick guidelines for the b

This paper is organized as follows: Section II gives an introduction about the method and also highlights its advantages which distinguish it from the rest. Section III presents the review of previous works which have been modeled using the . Section IV describes the step by step procedure for modeling a dynamic system using s method. Section V presents an example of dynamic

The model development by Lagrange method is also outlined in this section as a comparison. Lastly, the conclusion is drawn in Section VI.

## II. Kane s method

auxiliary quantities called partial angular velocities and partial velocity vectors are created, and then using their dot products with the forces and the torques acting from the external and inertial forces. The dot products form quantities called generalized active forces and generalized inertial forces [2].

s the advantages of both the Newton-Euler and Lagrange methods 3]. Following are the advantages

1. It is vector based approach unlike the previous methods which use differential and integral calculus that proved to be very arduous when applied to complex systems
2. , the forces and torques having no influence on the dynamic equations are eliminated in the beginning of the analysis and hence, greatly simplifying the mathematics. Joint interaction forces are noncontributory.
3. The invention of quantities called partial angular velocities and partial velocity vectors eliminated the need of introducing the imaginary perturbations (in Newton-Euler) which were completely based on a [4].

## III. previous works

Few of the previous works have been highlighted in the Table 1. From Table. 1, it can be observed that both 3D as well as 2D systems with the number of DOFs as high as 10 have been modeled by [5], [7], [9] and [10] have implemented the technique for dynamic

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TABLE I. PREVIOUS WORKS USING METHOD FOR DYNAMIC MODELING

| No. | Authors | Focus of study | Objective | No. of DOFs | 2D or 3D | Input data | Output data | Results/findings |
|---|---|---|---|---|---|---|---|---|
| 1. | F. Ariff and A. Rambely, 2008 [5] | Human upper limb during smash activity in a badminton game | To determine the unknown torques that caused the rotational movement of the upper limb joints using | 3 | 2D | Kinematic data (joint angles and force) from 2 professional badminton players | Torques produced at the upper limb | Elbow joints produced the highest value of torque during contact while performing jumping smash activity |
| 2. | C. Yang, Q. Huang, Z. Ye and J. Han, 2010 [6] | Spatial 6-DOF Parallel Robot | To derive the equations of motion of Spatial 6-DOF method | 6 | 2D | Various parameters of Parallel Robots | Force produced in the 4 legs | Equations derived successfully verified by SimMechanics |
| 3. | F. Ariff, A. Rambely and N. Ghani, 2011 [7] | Human shoulder segment during badminton game | To construct a 3D biomechanics model for movements of a shoulder and to determine the unknown method | 3 | 3D | Kinematic data (joint angles and force) from badminton players | Torques produced at the shoulder | The simulation results showed that during the execution of the smash, the shoulder is in its extension movement in the force producing stage, reaches its maximum before contact. |
| 4. | A. Rambely and Fazrolrozi, 2012 [8] | Six link kinematic chain of human body while jumping, walking and running | To construct the model of the system and to determine torques produced at the joints | 6 | 2D | Kinematic data of 6 segments while walking, jumping and running | Torques produced at the joints | From the results, it was concluded that the highest value of torque was produced at the foot segment. |
| 5. | N. Tumit, et al.,2014 [9] | Upper limb of an oil palm harvester | To construct the model of upper human limbs (six method | 6 | 2D | Not mentioned | Not mentioned | Model successfully created |
| 6. | N. Tumit, et al., 2015 [10] | Full body of an oil palm harvester | To construct the model of full human body (ten segments) | 10 | 2D | Not mentioned | Not mentioned | Model successfully created |

modeling of human body parts, determined the torque during various desired activities like jumping smash activity in

method was used to model six-link chain of human body which included foot, calf, thigh, trunk, upper arm and forearm while jumping, running and walking.

## IV. GENERAL PROCEDURE OF THE METHOD

Before deriving the dynamic equations of motion using

hod. The following notation is employed throughout this paper:
$\boldsymbol{q_i}$ is the generalized coordinate; $\boldsymbol{u_i}$ is the generalized speed, where *i* is the DOF of the system. $\boldsymbol{N}_{\underset{v}{\rightarrow}}\boldsymbol{P}$ is the velocity of point *P* in reference frame *N*. $\boldsymbol{N}_{\underset{p}{\rightarrow}}\boldsymbol{P}$ is the position of point *P* in reference frame *N*. $\boldsymbol{N}_{\underset{v}{\rightarrow}}\boldsymbol{A^*}$ is the velocity of center of mass of body *A* in reference frame *N*. $\boldsymbol{C_i}$ or $\boldsymbol{S_i}$ is cos ($\boldsymbol{q_i}$) or sin ($\boldsymbol{q_i}$). $\boldsymbol{C_{ij}}$ or $\boldsymbol{S_{ij}}$ is cos ($\boldsymbol{q_i}$+ $\boldsymbol{q_j}$) or sin ($\boldsymbol{q_i}$+ $\boldsymbol{q_j}$), where *i* and *j* are any integers.
The following steps should be employed to determine the

1. Develop the *table of direction cosines*. Direction cosines represent the angular relationship between different basis vectors [4].
2. Derive the expressions for the angular velocity ( ) and angula
comprising the system. Determine the velocities of all the points with forces acting through them are known, and all accelerations of mass centers are known.
3. Generate the *partial angular velocities* and *partial velocity vectors* from angular velocity and velocity expressions. The *partial* quantities are picked out of

4. Calculate the *Generalized active forces($F_r$)* which are the scalar quantities that highlight the contributions of active forces and torques (except the inertial quantities) to the dynamic equations of motion [4].
$F_r$ for a system *S* in reference frame *N* having n DOF, *v* points at which forces act and $\mu$ rigid bodies upon which torques [3] act are defined as

$$F_r \equiv \sum_{i=1}^{v} N_{\underset{v_r}{\rightarrow}} P_i \,.\, \overrightarrow{R_i} + \sum_{j=1}^{\mu} N_{\underset{\omega_r}{\rightarrow}} B_j \,.\, \overrightarrow{\tau_j}. \quad (1)$$

for $r$ ,$n$. $\vec{R_i}$ is the sum of all the distance and body forces except the inertial forces acting at each point $P_i$,($i$ $v$). $\vec{\tau_j}$is the torque acting on each body $B_j$ ($j$= $\mu$) *except* the inertial torques acting at each point $P_i$,($i$ $v$). $\vec{\tau_j}$is the torque acting on each body $B_j$ ($j$= $\mu$) *except* the inertial torques.

5. Calculate the *generalized inertia forces($F_r$)* which are also the scalar quantities that highlight the contributions of inertial forces and torques to the dynamic equations of motion.
   $F_r$* for a system having $n$ degrees of freedom in reference frame $N$ [4] are defined as
   $$F_r^* \equiv \sum_{i=1}^{v} N_{\underset{v_r}{\rightarrow}} P_i^* . \overrightarrow{R_i^*} + \sum_{j=1}^{\mu} N_{\underset{\omega_r}{\rightarrow}} B_j \ . \overrightarrow{\tau_j^*}, \qquad (2)$$
   for $r$ $\overrightarrow{\boldsymbol{R_i^*}}$is defined as the inertia force acting at each mass center $\boldsymbol{P_i^*}(\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{v})$.
   $$\overrightarrow{R_i^*} = -m_{P_i^*} N_{\underset{\alpha}{\rightarrow}} P_i^*, \qquad (3)$$
   where $\boldsymbol{m_{P_i^*}}$ is the mass concentrated at point $\boldsymbol{P_i^*}$ and $\boldsymbol{N_{\underset{\alpha}{\rightarrow}} P_i^*}$ is the acceleration of $\boldsymbol{P_i^*}$ in reference frame N. $\overrightarrow{\boldsymbol{\tau_j^*}}$ is defined as the inertia torque for body $\boldsymbol{B_j}$ in N [4],
   $$\overrightarrow{\tau_j^*} = -I_B N_{\underset{\alpha}{\rightarrow}} B_j \ . \qquad (4)$$
6. Add $F_r$ and $F_r$* to obtain the dynamic equations of motion
   $$F_r + F_r^* = 0, \ (5)$$
   for $r$
7. For a system $S$ possessing $n$ DOFs, there exist $n$ dynamic equations of motion which can be written in the matrix form as
   $$M\vec{\ddot{Q}} = \vec{T} + \vec{V} + \vec{G} + \vec{E}, \qquad (6)$$
   where $M$ is the mass matrix, $\vec{\ddot{Q}}$ is the acceleration matrix, $\vec{T}$ is the vector of applied torques, $\vec{V}$ is the vector of moments from centrifugal forces, $\vec{G}$ is the vector of moments from gravitational forces and is the vector of moments from external forces and torques.

## V. EXAMPLE

This section presents the development of a dynamic model of a 2-dimensional three DOFs kinematic chain method. The model is also verified by using Lagrange method later. Figure 1 shows the planar three link chain. In Fig. 1, $N$ represents the ground reference frame, $A$ the reference plane of link 1, $B$ the reference plane of link 2 and $C$ the reference frame of link 3. The system is fixed at joint $A_0$ in $N$.

$\vec{\tau}_{N/A}$ is the torque exerted by $N$ on $A$, $\vec{\tau}_{A/B}$ is the torque exerted by $A$ on $B$ and $\vec{\tau}_{B/C}$is the torque exerted by $B$ on $C$.

### A. Method I: Kane's method

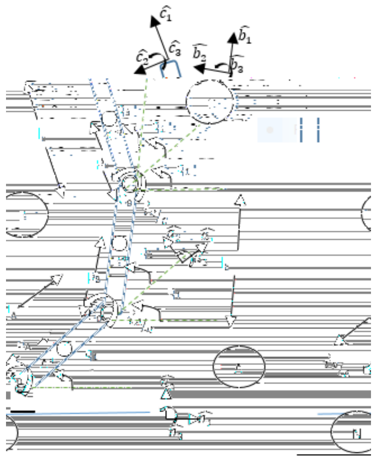

Fig. 1. Planar three-link kinematic chain

The symbols in Fig. 1 represent the following

○ = joints, ● = center of mass

A = segment $A$ (link 1), B = segment $B$ (link 2)

C = segment $C$ (link 3)

where $A^*$, $B^*$ and $C^*$ are the center of masses of segment A, B and C respectively.$q_1$ is the counterclockwise rotation angle from $\widehat{n_1}$ to $\widehat{a_1}$, $q_2$ is the counterclockwise rotation angle from $\widehat{a_1}$ to $\widehat{b_1}$ and $q_3$ is the counterclockwise rotation angle from $\widehat{b_1}$ to $\widehat{c_1}$. $\widehat{n_1}$, $\widehat{n_2}$, $\widehat{n_3}$, $\widehat{a_1}$, $\widehat{a_2}$, $\widehat{a_3}$, $\widehat{b_1}$, $\widehat{b_2}$, $\widehat{b_3}$, $\widehat{c_1}$, $\widehat{c_2}$ and $\widehat{c_3}$are mutually orthogonal unit vectors. An anti-clockwise circular arrow indicates a vector pointing out of the drawing and vice versa for the clockwise circular arrow. Hence, as depicted in Fig.1,$\widehat{n_3}$, $\widehat{a_3}$, $\widehat{b_3}$ and $\widehat{c_3}$ are pointing out of the drawing. $\rho_A$, $\rho_B$ and $\rho_C$ are distances of center of mass from their proximal ends. $l_A$ , $l_B$ and $l_C$ are lengths of the segments $A$ , $B$ and $C$ respectively.

Reference frames $N$, $A$, $B$ and $C$ are related by creating *tables of direction cosines*. Now, following the steps mentioned in Section IV:

Step 1: To construct the direction cosines table e.g., for reference frame $A$ with respect to the ground $N$ ($N_RA$), the following points should be taken into account:

1. A reference frame is affixed each body. During no motion, all basis vectors are aligned with the $N$ basis vectors. When the segments move, the basis vectors move since they are rigidly fixed with the segment.
2. The plane of motion in Fig. 1 is defined by the basis vectors $\hat{n}_1$ and $\hat{n}_2$, with $\hat{n}_2$ pointing vertically upwards and $\hat{n}_1$ pointing in the direction of travel. As segment $A$ moves through an angle of $q_1$, as shown in the Fig. 1, the reference frame $A$ gets aligned in the new orientation with respect to the reference frame $N$.

TABLE II. TABLE OF DIRECTION COSINES BETWEEN REFERENCE FRAMES *N*, *A*, *B* AND *C*

| $N_R A$ | $\widehat{a_1}$ | $\widehat{a_2}$ | $\widehat{a_3}$ | $B_R C$ | $\widehat{c_1}$ | $\widehat{c_2}$ | $\widehat{c_3}$ |
|---|---|---|---|---|---|---|---|
| $\widehat{n_1}$ | $c_1$ | $-s_1$ | 0 | $\widehat{b_1}$ | $c_3$ | $-s_3$ | 0 |
| $\widehat{n_2}$ | $s_1$ | $c_1$ | 0 | $\widehat{b_2}$ | $s_3$ | $c_3$ | 0 |
| $\widehat{n_3}$ | 0 | 0 | 1 | $\widehat{b_3}$ | 0 | 0 | 1 |
| $A_R B$ | $\widehat{b_1}$ | $\widehat{b_2}$ | $\widehat{b_3}$ | $N_R B$ | $\widehat{b_1}$ | $\widehat{b_2}$ | $\widehat{b_3}$ |
| $\widehat{a_1}$ | $c_2$ | $-s_2$ | 0 | $\widehat{n_1}$ | $c_{12}$ | $-s_{12}$ | 0 |
| $\widehat{a_2}$ | $s_2$ | $c_2$ | 0 | $\widehat{n_2}$ | $s_{12}$ | $c_{12}$ | 0 |
| $\widehat{a_3}$ | 0 | 0 | 1 | $\widehat{n_3}$ | 0 | 0 | 1 |
| $N_R C$ | $\widehat{c_1}$ | $\widehat{c_2}$ | $\widehat{c_3}$ | $A_R C$ | $\widehat{c_1}$ | $\widehat{c_2}$ | $\widehat{c_3}$ |
| $\widehat{n_1}$ | $c_{123}$ | $-s_{123}$ | 0 | $\widehat{a_1}$ | $c_{23}$ | $-s_{12}$ | 0 |
| $\widehat{n_2}$ | $s_{123}$ | $c_{12}$ | 0 | $\widehat{a_2}$ | $s_{23}$ | $c_{12}$ | 0 |
| $\widehat{n_3}$ | 0 | 0 | 1 | $\widehat{a_3}$ | 0 | 0 | 1 |

3. Resolving the new basis vectors $\hat{a}_1$and $\hat{a}_2$w.r.t the previous alignment basis vectors $\hat{n}_1$ and$\hat{n}_2$. All the basis vectors have lengths equal to 1, the horizontal component of $\hat{a}_1$has magnitude $\cos q_1$ and points in the $+\hat{n}_1$ direction. The vertical component has a magnitude of $\sin q_1$ and direction $+\hat{n}_2$. Thus,$\hat{a}_1 = \cos(q_1)\,\hat{n}_1 + \sin(q_1)\,\hat{n}_2$,$\hat{a}_2 = \cos(q_1)\,\hat{n}_2 + \sin(q_1)\,(-\hat{n}_1)$ and $\hat{a}_3$=$\hat{b}_3$. Arranging these in the form of a table gives *the table of direction cosines*. Using the same steps, direction cosines tables of all the references frames relating them to each other are formulated in Table. II.

The relationships between *N, A, B* and *C* basis (set of three noncoplanar vectors) vectors for the linkage in Fig. 1 is presented in Table II

The elements of direction cosines table are the dot products between unit vectors of different coordinate reference frames. This facilitates the calculations done in
of the regular need to compute dot products of vectors in different reference frames [4]. For instance, from Table. II, the dot product of $\widehat{n_2}$ and $\widehat{a_3}$ can be found just by finding the element in the $\widehat{n_2}$ row and element in $\widehat{a_3}$ column, that is, $\widehat{n_2}.\widehat{a_3}$ = 0.

<u>Step 2</u>: Calculating angular velocities of reference frames, *A*, *B* and *C* with respect to the reference frame *N*,

$$N\underset{\omega}{\rightarrow}A = \dot{q}_1\widehat{a_3}. \quad (7)$$

$$N\underset{\omega}{\rightarrow}B = (\dot{q}_1 + \dot{q}_2)\widehat{b_3}. \quad (8)$$

$$N\underset{\omega}{\rightarrow}C = (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)\widehat{c_3}. \quad (9)$$

Angular accelerations of reference frames, *A*, *B* and *C* with respect to the reference frame *N* can be calculated by differentiating Eq. 7- Eq. 9.

$$N\underset{\alpha}{\rightarrow}A = \ddot{q}_1\widehat{a_3}. \quad (10)$$

$$N\underset{\alpha}{\rightarrow}B = (\ddot{q}_1 + \ddot{q}_2)\widehat{b_3}. \quad (11)$$

$$N\underset{\alpha}{\rightarrow}C = (\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)\widehat{b_3}. \quad (12)$$

Next, the velocities of all the points with forces acting through them which are at points $A^*$, $B^*$ and $C^*$ are calculated, using *the two points fixed on a rigid body formula* [4],

$$N\underset{v}{\rightarrow}P = N\underset{v}{\rightarrow}Q + N\underset{\omega}{\rightarrow}B \times \vec{p}^{QP}, \quad (13)$$

where *P* and *Q* are points fixed on the rigid body *B*, which moves with angular velocity $N\underset{\omega}{\rightarrow}B$ and angular acceleration $N\underset{\alpha}{\rightarrow}B$ with respect to reference frame *N* and the velocity of point *Q* is already known. Using Eq.13, the velocities at $A^*$, $B^*$ and $C^*$ can be obtained as

$$N\underset{v}{\rightarrow}A* = \rho_A\dot{q}_1\widehat{a_2}. \quad (14)$$

$$N\underset{v}{\rightarrow}B* = l_A\dot{q}_1\widehat{a_2} + \rho_B(\dot{q}_1 + \dot{q}_2)\widehat{b_2}. \quad (15)$$

$$N\underset{v}{\rightarrow}C* = l_A\dot{q}_1\widehat{a_2} + l_B(\dot{q}_1 + \dot{q}_2)\widehat{b_2} + \rho_c(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)\widehat{c_2}. \quad (16)$$

$N\underset{v}{\rightarrow}A_0$=0 since it is assumed to be fixed.

The accelerations $N\underset{\alpha}{\rightarrow}A*$, $N\underset{\alpha}{\rightarrow}B*$ and $N\underset{\alpha}{\rightarrow}C*$ of mass centers are calculated using *the two points fixed on a rigid body formula* [4],

$$N\underset{\alpha}{\rightarrow}P = N\underset{\alpha}{\rightarrow}Q + N\underset{\omega}{\rightarrow}B \times \left(N\underset{\omega}{\rightarrow}B \times \vec{p}^{QP}\right) + N\underset{\alpha}{\rightarrow}B \times \vec{p}^{QP}, \quad (17)$$

where *P* and *Q* are points fixed on the rigid body *B*, which moves with angular velocity $N\underset{\omega}{\rightarrow}B$ and angular acceleration $N\underset{\alpha}{\rightarrow}B$ with respect to reference frame *N* and the velocity of point *Q* is already known. Using Eq. 17, the accelerations of mass centers $A^*$, $B^*$ and $C^*$ are calculated as

$$N\underset{\alpha}{\rightarrow}A* = -\rho_A\dot{q}_1^2\widehat{a_1} + \ddot{q}_1\rho_A\widehat{a_2}. \quad (18)$$

$$N\underset{\alpha}{\rightarrow}B* = -l_A\dot{q}_1^2\widehat{a_1} + \ddot{q}_1 l_A\widehat{a_2} - \rho_B(\dot{q}_1 + \dot{q}_2)^2\widehat{b}_1 + \rho_B(\ddot{q}_1 + \ddot{q}_2)\widehat{b_2}. \quad (19)$$

$$N\underset{\alpha}{\rightarrow}C* = -l_A\dot{q}_1^2\widehat{a_1} + \ddot{q}_1 l_A\widehat{a_2} - l_B(\dot{q}_1 + \dot{q}_2)^2\widehat{b}_1 + l_B(\ddot{q}_1 + \ddot{q}_2)\widehat{b_2} - \rho_C(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2\widehat{c}_1 + \rho_C(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)\widehat{c}_2. \quad (20)$$

<u>Step 3</u>: Factorize $N\underset{\omega}{\rightarrow}A = \dot{q}_1\widehat{a_3}$ and assign $u_i = \dot{q}_i$; where $u_i$ is the *generalized speed.* That is,

$$N\underset{\omega}{\rightarrow}A = (\widehat{a_3})u_1 + (\vec{0})u_2 + (\vec{0})u_3, \quad (21)$$

where $u_1$, $u_2$and $u_3$ are known as the *first, second and third generalized speeds* of the system. Hence, the *first partial angular velocity* of body *A* in reference frame *N* is the coefficient of $u_1$ in Eq. 21

$$N\underset{\omega_1}{\rightarrow}A = \widehat{a_3}. \quad (22)$$

Similarly, the *second and third partial angular velocity* of body *A* in reference frame *N* is

$$N\underset{\omega_2}{\rightarrow}A = N\underset{\omega_3}{\rightarrow}A = 0. \quad (23)$$

Likewise, from Eq. 8, the *first, second and third partial angular velocities* of body *B* in reference frame *N* are

$$N\underset{\omega_3}{\rightarrow}B = 0, \quad (24)$$

$$N\underset{\omega_1}{\rightarrow}B = N\underset{\omega_2}{\rightarrow}B = \widehat{b_3}. \quad (25)$$

And from Eq. 9, the *first, second and third partial angular velocities* of body $C$ in reference frame $N$ are

$$N_{\vec{\omega}_1}C = N_{\vec{\omega}_2}C = N_{\vec{\omega}_3}C = \widehat{c_3}. \tag{25}$$

The partial velocities for the center of masses $A^*$, $B^*$ and $C^*$ are calculated in the same manner as shown from Eq. 26 - Eq. 32. It should be noted that since the number of degree of freedom of the system is three, therefore, there are three generalized speeds and, hence there will be three dynamic equations of motion.

$$N_{\vec{v}_1}A^* = \rho_A \widehat{a_2}. \tag{26}$$

$$N_{\vec{v}_2}A^* = 0. \tag{27}$$

$$N_{\vec{v}_1}B^* = l_A \widehat{a_2} + \rho_B \widehat{b_2}. \tag{28}$$

$$N_{\vec{v}_2}B^* = \rho_B \widehat{b_2}. \tag{29}$$

$$N_{\vec{v}_1}C^* = l_A \widehat{a_2} + l_B \widehat{b_2} + \rho_C \widehat{c_2}. \tag{30}$$

$$N_{\vec{v}_2}C^* = l_B \widehat{b_2} + \rho_C \widehat{c_2}. \tag{31}$$

$$N_{\vec{v}_3}C^* = \rho_C \widehat{c_2}. \tag{32}$$

*Step 3*: Using Eq. 1, the *first generalized active force* is calculated as

$$F_1 = \left(N_{\vec{v}_1}A^* . - m_A g \widehat{n_2}\right) + \left(N_{\vec{v}_1}B^* . - m_B g \widehat{n_2}\right) + \left(N_{\vec{v}_1}C^* . - m_C g \widehat{n_2}\right) + \left(N_{\vec{\omega}_1}A . \left(\vec{\tau}_{N/A} - \vec{\tau}_{A/B}\right)\right) + \left(N_{\vec{\omega}_1}B . \left(\vec{\tau}_{A/B} - \vec{\tau}_{B/C}\right)\right) + \left(N_{\vec{\omega}_1}C . \vec{\tau}_{B/C}\right). \tag{33}$$

It should be noted that Eq. 1 has been modified to get Eq. 33, where $m_A g \widehat{n_2}$, $m_B g \widehat{n_2}$ and $m_C g \widehat{n_2}$ are the gravitational forces acting on the 3 links $A$, $B$ and $C$ respectively. Substituting $\vec{\tau}_{N/A} = T_1 \hat{a}_3$, $\vec{\tau}_{A/B} = T_2 \hat{b}_3$, $\vec{\tau}_{B/C} = T_3 \hat{c}_3$ and Eq. 22, Eq. 24-Eq. 26, 28 and Eq. 30 in Eq. 33,

$$F_1 = -\rho_A m_A g c_1 - m_B g(l_A c_1 + \rho_B c_{12}) - m_C g(l_A c_1 + l_B c_{12} + \rho_C c_{123}) + T_1 \tag{34}$$

From Eq. 1, the *second generalized active force* can be obtained as

$$F_2 = \left(N_{\vec{v}_2}A^* . - m_A g \widehat{n_2}\right) + \left(N_{\vec{v}_2}B^* . - m_B g \widehat{n_2}\right) + \left(N_{\vec{v}_2}C^* . - m_C g \widehat{n_2}\right) + \left(N_{\vec{\omega}_2}A . \left(\vec{\tau}_{N/A} - \vec{\tau}_{A/B}\right)\right) + \left(N_{\vec{\omega}_2}B . \left(\vec{\tau}_{A/B} - \vec{\tau}_{B/C}\right)\right) + \left(N_{\vec{\omega}_2}C . \vec{\tau}_{B/C}\right). \tag{35}$$

Substituting $\vec{\tau}_{N/A} = T_1 \hat{a}_3$, $\vec{\tau}_{A/B} = T_2 \hat{b}_3$, $\vec{\tau}_{B/C} = T_3 \hat{c}_3$ and Eq. 23-Eq. 25, Eq. 27, Eq. 29 and Eq. 31 in Eq. 35,

$$F_2 = -m_B g \rho_B c_{12} - m_C g l_B c_{12} - m_C g \rho_C c_{123} + T_2 \tag{36}$$

Similarly, the *third generalized active force* can be determined as

$$F_3 = -m_C g \rho_C c_{123} + T_3. \tag{37}$$

Step 4: Using Eq. 2, the *first second and third generalized inertia forces* can be calculated as

$$F_1^* = \left(N_{\vec{v}_1}A^* . \left(-m_A N_{\vec{a}}A^*\right)\right) + \left(N_{\vec{v}_1}B^* . \left(-m_B N_{\vec{a}}B^*\right)\right) + \left(N_{\vec{v}_1}C^* . \left(-m_C N_{\vec{a}}C^*\right)\right) + \left(N_{\vec{\omega}_1}A^* . \left(-I_{A^*} N_{\vec{\alpha}}A\right)\right) + \left(N_{\vec{\omega}_1}B^* . \left(-I_{B^*} N_{\vec{\alpha}}B\right)\right) + \left(N_{\vec{\omega}_1}C^* . \left(-I_{C^*} N_{\vec{\alpha}}C\right)\right). \tag{38}$$

Similarly, substituting *r=2 and 3* in Eq. 2, equations for $F_2^*$ and $F_3^*$ can be obtained.

Step 5: Substituting *r=1,2 and 3* in Eq. 5 and solving for the torques $T_1, T_2$ and $T_3$ respectively, the final equations of motion are

$$\begin{aligned} T_1 = {} & \ddot{q}_1(I_A^* + I_B^* + I_C^* + m_A\rho_A^2 + m_B l_A^2 + m_B\rho_B^2 + m_C l_A^2 + m_C l_B^2 + m_C\rho_C^2 + 2m_B l_A \rho_B C_2 + 2m_C l_A l_B C_2 + 2m_C l_B \rho_C C_3 + 2m_C l_A \rho_C C_{23}) + \\ & \ddot{q}_2(I_B^* + I_C^* + m_B\rho_B^2 + m_C l_B^2 + m_C\rho_C^2 + m_B l_A \rho_B C_2 + m_C l_A l_B C_2 + 2m_C l_B \rho_C C_3 + m_C l_A \rho_C C_{23}) + \\ & \ddot{q}_3(I_C^* + m_C\rho_C^2 + m_C l_B \rho_C C_3 + m_C l_A \rho_C C_{23}) + \\ & \dot{q}_1\dot{q}_2(-2m_B l_A l_B S_2 - 2m_C l_A l_B S_2 - 2m_C l_A \rho_C S_{23}) + \\ & \dot{q}_1\dot{q}_3(-2m_C l_B \rho_C S_3 - 2m_C l_A \rho_C S_{23}) + \\ & \dot{q}_2\dot{q}_3(-2m_C l_B \rho_C S_3 - 2m_C l_A \rho_C S_{23}) + \\ & \dot{q}_2^2(-m_B l_A \rho_B S_2 - m_C l_A l_B S_2 - m_C l_A \rho_C S_{23}) + \\ & \dot{q}_3^2(-m_C l_B \rho_C S_3 - m_C l_A \rho_C S_{23}) + m_A g \rho_A c_1 + \\ & m_B g(l_A c_1 + \rho_B c_{12}) + m_C g(l_A c_1 + l_B c_{12} + \rho_C c_{123}), \end{aligned} \tag{39}$$

$$\begin{aligned} T_2 = {} & \ddot{q}_1(I_B^* + I_C^* + m_B\rho_B^2 + m_C l_B^2 + m_C\rho_C^2 + m_B l_A \rho_B C_2 + m_C l_A l_B C_2 + 2m_C l_B \rho_C C_3 + m_C l_A \rho_C C_{23}) + \ddot{q}_2(I_B^* + I_C^* + m_B\rho_B^2 + m_C l_B^2 + m_C\rho_C^2 + 2m_C l_B \rho_C C_3) + \ddot{q}_3(I_C^* + m_C\rho_C^2 + m_C l_B \rho_C C_3) + \dot{q}_1\dot{q}_3(-2m_C l_B \rho_C S_3) + \\ & \dot{q}_2\dot{q}_3(-2m_C l_B \rho_C S_3) + \dot{q}_1^2(m_B l_A \rho_B s_2 + m_C l_A l_B s_2 + m_C l_A \rho_C s_{23}) + \dot{q}_3^2(-m_C l_B \rho_C S_3) + m_B g \rho_B c_{12} + \\ & m_C g(l_B c_{12} + \rho_C c_{123}), \end{aligned} \tag{40}$$

and

$$\begin{aligned} T_3 = {} & \ddot{q}_1(I_C^* + m_C\rho_C^2 + m_C l_B \rho_C C_3 + m_C l_A \rho_C C_{23}) + \\ & \ddot{q}_2(I_C^* + m_C\rho_C^2 + m_C l_B \rho_C C_3) + \ddot{q}_3(I_C^* + m_C\rho_C^2) + \\ & \dot{q}_1\dot{q}_2(2m_C l_B \rho_C S_3) + \dot{q}_1\dot{q}_2(2m_C l_B \rho_C S_3) + \\ & \dot{q}_1^2(m_C \rho_C l_B s_3 + m_C l_A \rho_C s_{23}) + \dot{q}_2^2(m_C l_B \rho_C S_3) + \\ & m_C g \rho_C c_{123}. \end{aligned} \tag{41}$$

These equations of motion can then be arranged in the matrix form of Eq. 6, and solving for $T$.

### *B. Method II: Lagrangian method*

Lagrangian mechanics is based on the differentiation of the energy terms (kinetic and potential energies) with respect to the system variables and time. A Lagrangian can be defined as

$$L = K - P, \tag{42}$$

where $L$ is the Lagrangian, $K$ is the kinetic energy of the system and $P$ is the potential energy of the system. Thus, for $i$ ,

$$T_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}, \tag{43}$$

Referring to the Fig. 1, the kinetic energy $K$ of the linkage can be derived as

$$K = \dot{q}_1^2\left(\tfrac{1}{2}I_A^* + \tfrac{1}{2}I_B^* + \tfrac{1}{2}I_C^* + \tfrac{1}{2}m_A\rho_A^2 + \tfrac{1}{2}m_B l_A^2 + \tfrac{1}{2}m_B\rho_B^2 + \tfrac{1}{2}m_C l_A^2 + \tfrac{1}{2}m_C l_B^2 + \tfrac{1}{2}m_C\rho_C^2 + m_B l_A\rho_B C_2 + m_C l_A l_B C_2 + m_C l_B\rho_C C_3 + m_C l_A\rho_C C_{23}\right) + \dot{q}_2^2\left(\tfrac{1}{2}I_B^* + \tfrac{1}{2}I_C^* + \tfrac{1}{2}m_B\rho_B^2 + \tfrac{1}{2}m_C l_B^2 + \tfrac{1}{2}m_C\rho_C^2 + m_C l_B\rho_C C_3\right) + \dot{q}_3^2\left(\tfrac{1}{2}I_C^* + \tfrac{1}{2}m_C\rho_C^2\right) + \dot{q}_1\dot{q}_2(I_B^* + I_C^* + m_B\rho_B^2 + m_C l_B^2 + m_C\rho_C^2 + m_B l_A\rho_B C_2 + m_C l_A l_B C_2 + 2m_C l_B\rho_C C_3 + m_C l_A\rho_C C_{23}) + \dot{q}_2\dot{q}_3(I_C^* + m_C\rho_C^2 + m_C l_B\rho_C C_3) + \dot{q}_1\dot{q}_3(I_C^* + m_C\rho_C^2 + m_C l_B\rho_C C_3 + m_C l_A\rho_C C_{23}), \quad (44)$$

and the potential energy $P$ as

$$P = m_A g\rho_A s_1 + m_B g(l_A s_1 + \rho_B s_{12}) + m_C g(l_A s_1 + l_B s_{12} + \rho_C s_{123}). \quad (45)$$

Substituting Eq. 44 and Eq. 45 into Eq. 42, $T_i$, for $i$=1, 2 and 3 can be solved as

$$T_1 = \ddot{q}_1(I_A^* + I_B^* + I_C^* + m_A\rho_A^2 + m_B l_A^2 + m_B\rho_B^2 + m_C l_A^2 + m_C l_B^2 + m_C\rho_C^2 + 2m_B l_A\rho_B C_2 + 2m_C l_A l_B C_2 + 2m_C l_B\rho_C C_3 + 2m_C l_A\rho_C C_{23}) + \ddot{q}_2(I_B^* + I_C^* + m_B\rho_B^2 + m_C l_B^2 + m_C\rho_C^2 + m_B l_A\rho_B C_2 + m_C l_A l_B C_2 + 2m_C l_B\rho_C C_3 + m_C l_A\rho_C C_{23}) + \ddot{q}_3(I_C^* + m_C\rho_C^2 + m_C l_B\rho_C C_3 + m_C l_A\rho_C C_{23}) + \dot{q}_1\dot{q}_2(-2m_B l_A l_B S_2 - 2m_C l_A l_B S_2 - 2m_C l_A\rho_C S_{23}) + \dot{q}_1\dot{q}_3(-2m_C l_B\rho_C S_3 - 2m_C l_A\rho_C S_{23}) + \dot{q}_2\dot{q}_3(-2m_C l_B\rho_C S_3 - 2m_C l_A\rho_C S_{23}) + \dot{q}_2^2(-m_B l_A\rho_B S_2 - m_C l_A l_B S_2 - m_C l_A\rho_C S_{23}) + \dot{q}_3^2(-m_C l_B\rho_C S_3 - m_C l_A\rho_C S_{23}) + m_A g\rho_A c_1 + m_B g(l_A c_1 + \rho_B c_{12}) + m_C g(l_A c_1 + l_B c_{12} + \rho_C c_{123}), \quad (46)$$

for the second link,

$$T_2 = \ddot{q}_1(I_B^* + I_C^* + m_B\rho_B^2 + m_C l_B^2 + m_C\rho_C^2 + m_B l_A\rho_B C_2 + m_C l_A l_B C_2 + 2m_C l_B\rho_C C_3 + m_C l_A\rho_C C_{23}) + \ddot{q}_2(I_B^* + I_C^* + m_B\rho_B^2 + m_C l_B^2 + m_C\rho_C^2 + 2m_C l_B\rho_C C_3) + \ddot{q}_3(I_C^* + m_C\rho_C^2 + m_C l_B\rho_C C_3) + \dot{q}_1\dot{q}_3(-2m_C l_B\rho_C S_3) + \dot{q}_2\dot{q}_3(-2m_C l_B\rho_C S_3) + \dot{q}_1^2(m_B l_A\rho_B s_2 + m_C l_A l_B s_2 + m_C l_A\rho_C s_{23}) + \dot{q}_3^2(-m_C l_B\rho_C S_3) + m_B g\rho_B c_{12} + m_C g(l_B c_{12} + \rho_C c_{123}), \quad (47)$$

and for the third link,

$$T_3 = \ddot{q}_1(I_C^* + m_C\rho_C^2 + m_C l_B\rho_C C_3 + m_C l_A\rho_C C_{23}) + \ddot{q}_2(I_C^* + m_C\rho_C^2 + m_C l_B\rho_C C_3) + \ddot{q}_3(I_C^* + m_C\rho_C^2) + \dot{q}_1\dot{q}_2(2m_C l_B\rho_C S_3) + \dot{q}_1\dot{q}_2(2m_C l_B\rho_C S_3) + \dot{q}_1^2(m_C\rho_C l_B s_3 + m_C l_A\rho_C s_{23}) + \dot{q}_2^2(m_C l_B\rho_C S_3) + m_C g\rho_C c_{123}\,. \quad (48)$$

Comparing the resulting Eq. 39- Eq.41

method and the final dynamic equations Eq. 46-Eq. 48 using Lagrange method, it can be seen that both sets of dynamic equations of motion are the same, hence, verifying the

As evident

from both the methods, a lot of calculation is required to

simpler than Lagrangian. With the increase in the DOFs of the system involved, the complexity of the system increases and hence, the derivation of the dynamic equations of motion becomes complicated too. As such to facilitate the derivation and simulation of complex dynamic systems, software packages like AUTOLEV, MATLAB, etc. are necessary to simplify the calculation [11].

## VI. Conclusion

In this paper, the step by step procedure in deriving a

presented through a 3 DOF linkage kinematic chain example.

thod has proved to be a versatile method and efficient in modeling complex multi-body systems. The model has been verified by using Lagrange method. This paper

for the beginners who would like to study this technique. The

is in

modeling a multi-dimensional system (more than 3 DOFs) and gradually making it more complex by adding more joints and DOFs. The researchers with the knowledge of modeling software like AUTOLEV and MATLAB can facilitate and reduce the calculation effort.

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