Discrete Revolution

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Digital signal processing has taken over. First used in the 1950s at the service of analog signal processing to simulate analog transforms, digital algorithms have invaded most traditional fortresses, including phones, music recording, cameras, televisions, and all information processing. Analog computations performed with electronic circuits are faster than digital algorithms implemented with microprocessors but are less precise and less flexible. Thus, analog circuits are often replaced by digital chips once the computational performance of microprocessors is sufficient to operate in real time for a given application.

Whether sound recordings or images, most discrete signals are obtained by sampling an analog signal. An analog-to-digital conversion is a linear approximation that introduces an error dependent on the sampling rate. Once more, the Fourier transform is unavoidable because the eigenvectors of discrete time-invariant operators are sinusoidal waves. The Fourier transform is discretized for signals of finite size and implemented with a fast Fourier transform (FFT) algorithm.

3.1 SAMPLING ANALOG SIGNALS

The simplest way to discretize an analog signal f is to record its sample values $\{f(ns)\}_{n \in \mathbb{Z}}$ at interval s. An approximation of $f(t)$ at any $t \in \mathbb{R}$ may be recovered by interpolating these samples. The Shannon-Whittaker sampling theorem gives a sufficient condition on the support of the Fourier transform $\hat{f}$ to recover $f(t)$ exactly. Aliasing and approximation errors are studied when this condition is not satisfied.

Digital-acquisition devices often do not satisfy the restrictive hypothesis of the Shannon-Whittaker sampling theorem. General linear analog-to-discrete conversion is introduced in Section 3.1.3, showing that a stable uniform discretization is a linear approximation. A digital conversion also approximates discrete coefficients, with a given precision, to store them with a limited number of bits. This quantization aspect is studied in Chapter 10.

3.1.1 Shannon-Whittaker Sampling Theorem

Sampling is first studied from the more classic Shannon-Whittaker point of view, which tries to recover $f(t)$ from its samples $\{f(ns)\}_{n \in \mathbb{Z}}$. A discrete signal can

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be represented as a sum of Diracs. We associate to any sample $f(ns)$ a Dirac $f(ns)\delta(t-ns)$ located at $t=ns$. A uniform sampling of f thus corresponds to the weighted Dirac sum

$$f_d(t) = \sum_{n=-\infty}^{+\infty} f(ns)\,\delta(t-ns). \tag{3.1}$$

The Fourier transform of $\delta(t-ns)$ is $e^{-ins\omega}$, so the Fourier transform of f_d is a Fourier series:

$$\hat{f}_d(\omega) = \sum_{n=-\infty}^{+\infty} f(ns)\,e^{-ins\omega}. \tag{3.2}$$

To understand how to compute $f(t)$ from the sample values $f(ns)$ and therefore f from f_d, we relate their Fourier transforms $\hat{f}$ and $\hat{f}_d$.

Theorem 3.1. The Fourier transform of the discrete signal obtained by sampling f at interval s is

$$\hat{f}_d(\omega) = \frac{1}{s} \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{s}\right). \tag{3.3}$$

Proof. Since $\delta(t-ns)$ is zero outside $t=ns$,

$$f(ns)\,\delta(t-ns) = f(t)\,\delta(t-ns),$$

we can rewrite (3.1) as multiplication with a Dirac comb:

$$f_d(t) = f(t) \sum_{n=-\infty}^{+\infty} \delta(t-ns) = f(t)\,c(t). \tag{3.4}$$

Computing the Fourier transform yields

$$\hat{f}_d(\omega) = \frac{1}{2\pi}\hat{f}\star\hat{c}(\omega). \tag{3.5}$$

The Poisson formula (2.4) proves that

$$\hat{c}(\omega) = \frac{2\pi}{s} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{s}\right). \tag{3.6}$$

Since $\hat{f}\star\delta(\omega-\xi) = \hat{f}(\omega-\xi)$, inserting (3.6) into (3.5) proves (3.3). ∎

Theorem 3.1 proves that sampling f at interval s is equivalent to making its Fourier transform $2\pi/s$ periodic by summing all its translations $\hat{f}(\omega - 2k\pi/s)$. The resulting sampling theorem was first proved by Whittaker [482] in 1935 in

a book on interpolation theory. Shannon rediscovered it in 1949 for applications to communication theory [429].

Theorem 3.2: *Shannon, Whittaker.* If the support of $\hat{f}$ is included in $[-\pi/s, \pi/s]$, then

$$f(t) = \sum_{n=-\infty}^{+\infty} f(ns)\,\phi_s(t - ns), \tag{3.7}$$

with

$$\phi_s(t) = \frac{\sin(\pi t/s)}{\pi t/s}. \tag{3.8}$$

Proof. If $n \neq 0$, the support of $\hat{f}(\omega - n\pi/s)$ does not intersect the support of $\hat{f}(\omega)$ because $\hat{f}(\omega) = 0$ for $|\omega| > \pi/s$; so (3.3) implies

$$\hat{f}_d(\omega) = \frac{\hat{f}(\omega)}{s} \text{ if } |\omega| \leq \frac{\pi}{s}. \tag{3.9}$$

The Fourier transform of ϕ_s is $\hat{\phi}_s = s\,\mathbf{1}_{[-\pi/s,\pi/s]}$. Since the support of $\hat{f}$ is in $[-\pi/s, \pi/s]$, it results from (3.9) that $\hat{f}(\omega) = \hat{\phi}_s(\omega)\hat{f}_d(\omega)$. The inverse Fourier transform of this equality gives

$$f(t) = \phi_s \star f_d(t) = \phi_s \star \sum_{n=-\infty}^{+\infty} f(ns)\,\delta(t - ns)$$

$$= \sum_{n=-\infty}^{+\infty} f(ns)\,\phi_s(t - ns). \qquad \blacksquare$$

The sampling theorem supposes that the support of $\hat{f}$ is included in $[-\pi/s, \pi/s]$, which guarantees that f has no brutal variations between consecutive samples; thus, it can be recovered with a smooth interpolation. Section 3.1.3 shows that one can impose other smoothness conditions to recover f from its samples. Figure 3.1 illustrates the different steps for a sampling and reconstruction from samples, in both the time and Fourier domains.

3.1.2 Aliasing

The sampling interval s is often imposed by computation or storage constraints and support of $\hat{f}$ is generally not included in $[-\pi/s, \pi/s]$. In this case the interpolation formula (3.7) does not recover f. We analyze the resulting error and a filtering procedure to reduce it.

Theorem 3.1 proves that

$$\hat{f}_d(\omega) = \frac{1}{s} \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{s}\right). \tag{3.10}$$

Suppose that support of $\hat{f}$ goes beyond $[-\pi/s, \pi/s]$. In general, support of $\hat{f}(\omega - 2k\pi/s)$ intersects $[-\pi/s, \pi/s]$ for several $k \neq 0$, as shown in Figure 3.2. This folding
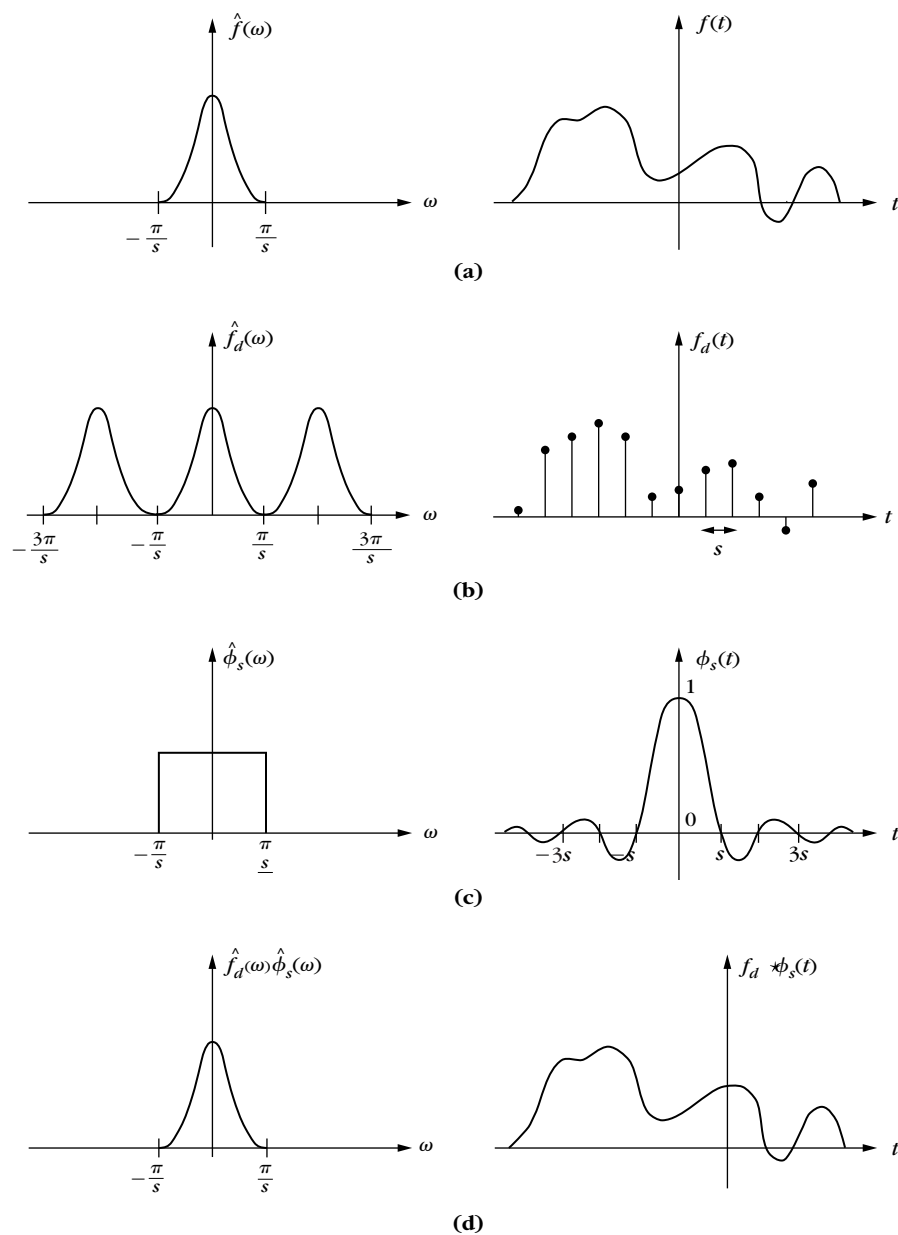
FIGURE 3.1

(a) Signal f and its Fourier transform $\hat{f}$. **(b)** A uniform sampling of f makes its Fourier transform periodic. **(c)** Ideal low-pass filter. **(d)** The filtering of (b) with (c) recovers f.
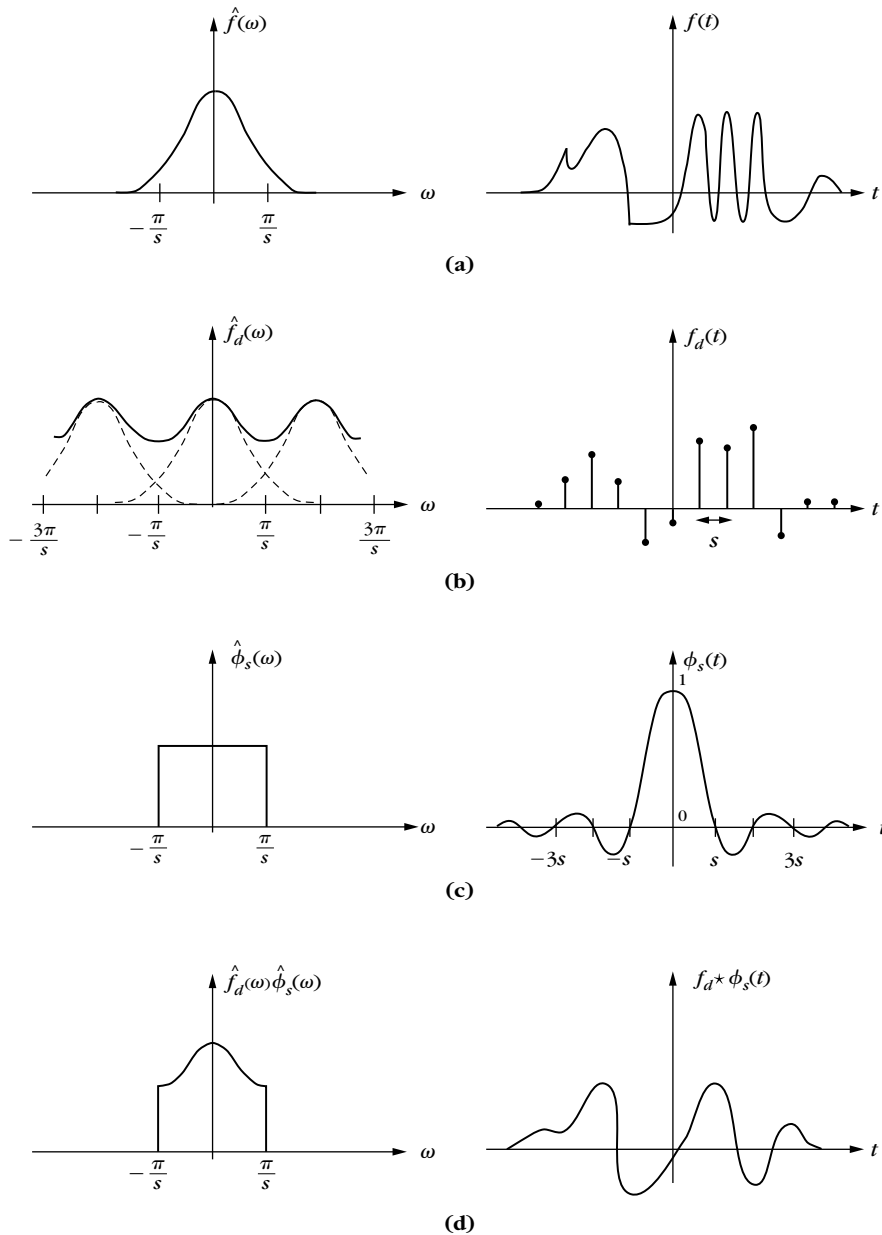
FIGURE 3.2

(a) Signal f and its Fourier transform $\hat{f}$. **(b)** Aliasing produced by an overlapping of $\hat{f}(\omega - 2k\pi/s)$ for different k, shown with dashed lines. **(c)** Ideal low-pass filter. **(d)** The filtering of (b) with (c) creates a low-frequency signal that is different from f.

of high-frequency components over a low-frequency interval is called *aliasing*. In the presence of aliasing, the interpolated signal

$$\phi_s \star f_d(t) = \sum_{n=-\infty}^{+\infty} f(ns) \, \phi_s(t - ns)$$

has a Fourier transform

$$\hat{f}_d(\omega) \, \hat{\phi}_s(\omega) = s\hat{f}_d(\omega) \, \mathbf{1}_{[-\pi/s, \pi/s]}(\omega) = \mathbf{1}_{[-\pi/s, \pi/s]}(\omega) \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{s}\right), \qquad \textbf{(3.11)}$$

which may be completely different from $\hat{f}(\omega)$ over $[-\pi/s, \pi/s]$. The signal $\phi_s \star f_d$ may not even be a good approximation of f, as shown by Figure 3.2.

EXAMPLE 3.1

Let us consider a high-frequency oscillation

$$f(t) = \cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}.$$

Its Fourier transform is

$$\hat{f}(\omega) = \pi\Big(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\Big).$$

If $2\pi/s > \omega_0 > \pi/s$, then (3.11) yields

$$\hat{f}_d(\omega) \, \hat{\phi}_s(\omega)$$

$$= \pi \, \mathbf{1}_{[-\pi/s, \pi/s]}(\omega) \sum_{k=-\infty}^{+\infty} \left(\delta\left(\omega - \omega_0 - \frac{2k\pi}{s}\right) + \delta\left(\omega + \omega_0 - \frac{2k\pi}{s}\right)\right)$$

$$= \pi\left(\delta\left(\omega - \frac{2\pi}{s} + \omega_0\right) + \delta\left(\omega + \frac{2\pi}{s} - \omega_0\right)\right),$$

so

$$f_d \star \phi_s(t) = \cos\left[\left(\frac{2\pi}{s} - \omega_0\right)t\right].$$

The aliasing reduces the high-frequency ω_0 to a lower frequency $2\pi/s - \omega_0 \in [-\pi/s, \pi/s]$. The same frequency folding is observed in a film that samples a fast-moving object without enough images per second. A wheel turning rapidly appears as though turning much more slowly in the film.

Removal of Aliasing

To apply the sampling theorem, f is approximated by the closest signal $\tilde{f}$, the Fourier transform of which has a support in $[-\pi/s, \pi/s]$. The Plancherel formula (2.26) proves that

$$\|f - \tilde{f}\|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega) - \widehat{\tilde{f}}(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{|\omega| > \pi/s} |\hat{f}(\omega)|^2 \, d\omega + \frac{1}{2\pi} \int_{|\omega| \le \pi/s} |\hat{f}(\omega) - \widehat{\tilde{f}}(\omega)|^2 \, d\omega.$$

This distance is minimum when the second integral is zero and therefore

$$\widehat{\tilde{f}}(\omega) = \hat{f}(\omega) \, \mathbf{1}_{[-\pi/s,\pi/s]}(\omega) = \frac{1}{s} \, \hat{\phi}_s(\omega) \hat{f}(\omega). \tag{3.12}$$

It corresponds to $\tilde{f} = \frac{1}{s} f \star \phi_s$.

The filtering of f by ϕ_s avoids aliasing by removing any frequency larger than π/s. Since $\widehat{\tilde{f}}$ has a support in $[-\pi/s, \pi/s]$, the sampling theorem proves that $\tilde{f}(t)$ can be recovered from the samples $\tilde{f}(ns)$. An analog-to-digital converter is therefore composed of a filter that limits the frequency band to $[-\pi/s, \pi/s]$, followed by a uniform sampling at interval s.

3.1.3 General Sampling and Linear Analog Conversions

The Shannon-Whittaker theorem is a particular example of linear discrete-to-analog conversion, which does not apply to all digital acquisition devices. This section describes general analog-to-discrete conversion and reverse discrete-to-analog conversion, with general linear filtering and uniform sampling. Analog signals are approximated by linear projections on approximation spaces.

Sampling Theorems

We want to recover a stable approximation of $f \in \mathbf{L}^2(\mathbb{R})$ from a filtering and uniform sampling, which outputs $\{f \star \bar{\phi}_s(ns)\}_{n \in \mathbb{Z}}$, for some real filter $\bar{\phi}_s(t)$. These samples can be written as inner products in $\mathbf{L}^2(\mathbb{R})$:

$$f \star \phi_s(ns) = \int_{-\infty}^{+\infty} f(t) \, \bar{\phi}_s(ns - t) \, dt = \langle f(t), \phi_s(t - ns) \rangle, \tag{3.13}$$

with $\phi_s(t) = \bar{\phi}_s(-t)$. Let $\mathbf{U}_s$ be the approximation space generated by linear combination of the $\{\phi_s(t - ns)\}_{n \in \mathbb{Z}}$. The approximation $\tilde{f} \in \mathbf{U}_s$, which minimizes the maximum possible error $\|f - \tilde{f}\|$, is the orthognal projection of f on $\mathbf{U}_s$ (Exercice 3.5). The calculation of this orthogonal projection is stable if $\{\phi_s(t - ns)\}_{n \in \mathbb{Z}}$ is a Riesz basis of $\mathbf{U}_s$, as defined in Section 5.1.1.

Following Definition 5.1, a Riesz basis is a family of linearly independent functions that yields an inner product satisfying an energy equivalence. There exists $B \geqslant A > 0$ such that for any $f \in \mathbf{U}_s$

$$A \|f\|^2 \leqslant \sum_{n=-\infty}^{+\infty} |\langle f(t), \phi_s(t - ns) \rangle|^2 \leqslant B \|f\|^2. \tag{3.14}$$

The basis is orthogonal if and only if $A = B$. The following generalized sampling theorem computes the orthogonal projection on the approximation space $\mathbf{U}_s$ [468].

Theorem 3.3: *Linear sampling.* Let $\{\phi_s(t - ns)\}_{n \in \mathbb{Z}}$ be a Riesz basis of $\mathbf{U}_s$ and $\bar{\phi}_s(t) = \phi_s(-t)$. There exists a biorthogonal basis $\{\tilde{\phi}_s(t - ns)\}_{n \in \mathbb{Z}}$ of $\mathbf{U}_s$ such that

$$\forall f \in \mathbf{L}^2(\mathbb{R}), \quad P_{\mathbf{U}_s} f(t) = \sum_{n=-\infty}^{+\infty} f \star \bar{\phi}_s(ns) \, \tilde{\phi}_s(t - ns). \tag{3.15}$$

Proof. For any Riesz basis, Section 5.1.2 proves that a biorthogonal basis $\{\tilde{\phi}_{s,n}(t)\}_{n\in\mathbb{Z}}$ exists that satisfies the biorthogonality relations

$$\forall (n,m) \in \mathbb{Z}^2, \quad \langle \phi_s(t-ns), \tilde{\phi}_{s,m}(t-ms) \rangle = \delta[n-m]. \tag{3.16}$$

Since $\langle \phi_s(t-(n-m)s), \tilde{\phi}_{s,0}(t) \rangle = \langle \phi_s(t-ns), \tilde{\phi}_{s,0}(t-ms) \rangle = 0$ and since the dual basis is unique, necessarily $\tilde{\phi}_{s,m}(t) = \tilde{\phi}_{s,0}(t-ms)$. Section 5.1.2 proves in (5.20) that the orthogonal projection in $\mathbf{U}_s$ can be written

$$P_{\mathbf{U}_s} f(t) = \sum_{n=-\infty}^{+\infty} \langle f(t), \phi_s(t-ns) \rangle \, \tilde{\phi}_s(t-ns)$$

which proves (3.15). ∎

The orthogonal projection (3.15) can be rewritten as an analog filtering of the discrete signal $f_d(t) = \sum_{n=-\infty}^{+\infty} f \star \bar{\phi}_s(ns) \, \delta(t-ns)$:

$$P_{\mathbf{U}_s} f(t) = f_d \star \tilde{\phi}_s(t). \tag{3.17}$$

If $f \in \mathbf{U}_s$, then $P_{\mathbf{U}_s} f = f$ so it is exactly reconstructed by filtering the uniformly sampled discrete signal $\{f \star \bar{\phi}_s(ns)\}_{n\in\mathbb{Z}}$ with the analog filter $\tilde{\phi}(t)$. If $f \notin \mathbf{U}_s$, then (3.17) recovers the best linear approximation of f in $\mathbf{U}_s$. Section 9.1 shows that the linear approximation error $\|f - P_{\mathbf{U}_s} f\|$ essentially depends on the uniform regularity of f. Given some prior information on f, optimizing the analog discretization filter ϕ_s amounts to optimizing the approximation space $\mathbf{U}_s$ to minimize this error. The following theorem characterizes filters ϕ_s that generate a Riesz basis and computes the dual filter.

Theorem 3.4. A filter ϕ_s generates a Riesz basis $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$ of a space $\mathbf{U}_s$ if and only if there exists $B \geqslant A > 0$ such that

$$\forall \omega \in [0, 2\pi/s], \quad A \leqslant \frac{1}{s} \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - \frac{2k\pi}{s})|^2 \leqslant B. \tag{3.18}$$

The biorthogonal basis $\{\tilde{\phi}_s(t-ns)\}_{n\in\mathbb{Z}}$ is defined by the dual filter $\tilde{\phi}_s$, which satisfies:

$$\widehat{\tilde{\phi}}_s(\omega) = \frac{s \, \hat{\phi}_s^*(\omega)}{\sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - 2k\pi/s)|^2}. \tag{3.19}$$

Proof. Theorem 5.5 proves that $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{U}_s$ with Riesz bounds $B \geqslant A > 0$ if and only if it is linearly independent and

$$\forall a \in \ell^2(\mathbb{Z}), \quad A\|a\|^2 \leqslant \| \sum_{n\in\mathbb{Z}} a[ns] \, \phi_s(t-ns) \|^2 \leqslant B\|a\|^2, \tag{3.20}$$

with $\|a\|^2 = \sum_{n\in\mathbb{Z}} |a[ns]|^2$.

Let us first write these conditions in the Fourier domain. The Fourier transform of $f(t) = \sum_{n=-\infty}^{+\infty} a[ns] \, \phi_s(t-ns)$ is

$$\hat{f}(\omega) = \sum_{n=-\infty}^{+\infty} a[ns] \, \mathrm{e}^{-ins\omega} \, \hat{\phi}_s(\omega) = \hat{a}(\omega) \, \hat{\phi}_s(\omega), \tag{3.21}$$

where $\hat{a}(\omega)$ is the Fourier series $\hat{a}(\omega) = \sum_{n=-\infty}^{+\infty} a[ns]\, e^{-ins\omega}$. Let us relate the norm of f and $\hat{a}$. Since $\hat{a}(\omega)$ is $2\pi/s$ periodic, inserting (3.21) in the Plancherel formula (2.26) gives

$$
\begin{aligned}
\|f\|^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 \, d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi/s} \sum_{k=-\infty}^{+\infty} |\hat{a}(\omega + 2k\pi/s)|^2 \, |\hat{\phi}_s(\omega + 2k\pi/s)|^2 \, d\omega \qquad \textbf{(3.22)} \\
&= \frac{1}{2\pi} \int_0^{2\pi/s} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega + 2k\pi/s)|^2 \, d\omega.
\end{aligned}
$$

Section 3.2.2 on Fourier series proves that

$$
\|a\|^2 = \sum_{n=-\infty}^{+\infty} |a[ns]|^2 = \frac{s}{2\pi} \int_0^{2\pi/s} |\hat{a}(\omega)|^2 \, d\omega. \qquad \textbf{(3.23)}
$$

As a consequence of (3.22) and (3.23), the Riesz bound inequalities (3.20) are equivalent to

$$
\forall \hat{a} \in \mathbf{L}^2[0, 2\pi/s], \quad \frac{1}{2\pi} \int_0^{2\pi/s} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega + 2k\pi/s)|^2 \, d\omega \leqslant \frac{B\,s}{2\pi} \int_0^{2\pi/s} |\hat{a}(\omega)|^2 \, d\omega \qquad \textbf{(3.24)}
$$

and

$$
\forall \hat{a} \in \mathbf{L}^2[0, 2\pi/s], \quad \frac{1}{2\pi} \int_0^{2\pi/s} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega + 2k\pi/s)|^2 \, d\omega \geqslant \frac{A\,s}{2\pi} \int_0^{2\pi/s} |\hat{a}(\omega)|^2 \, d\omega. \qquad \textbf{(3.25)}
$$

If $\hat{\phi}_s$ satisfies (3.18), then clearly (3.24) and (3.25) are valid, which proves (3.22).

Conversely, if $\{\phi_s(ns - t)\}_{n \in \mathbb{Z}}$ is a Riesz basis. Suppose that either the upper or the lower bound of (3.18) is not satisfied for ω in a set of nonzero measures. Let $\hat{a}$ be the indicator function of this set. Then either (3.24) or (3.25) is not valid for this $\hat{a}$. This implies that the Riesz bounds (3.20) are not valid for a and therefore that it is not a Riesz basis, which contradicts our hypothesis. So (3.18) is indeed valid for almost all $\omega \in [0, 2\pi/s]$.

To compute the biorthogonal basis, we are looking for $\tilde{\phi}_s \in \mathbf{U}_s$ such that $\{\tilde{\phi}_s(t - ns)\}_{n \in \mathbb{Z}}$ satisfies the biorthogonal relations (3.16). Since $\tilde{\phi}_s \in \mathbf{U}_s$ we saw in (3.21) that its Fourier transform can be written $\widehat{\tilde{\phi}_s}(\omega) = \hat{a}(\omega)\hat{\phi}_s(\omega)$, where $\hat{a}(\omega)$ is $2\pi/s$ periodic. Let us define $g(t) = \bar{\phi}_s \star \tilde{\phi}_s(t)$. Its Fourier transform is

$$
\hat{g}(\omega) = \hat{\phi}_s^*(\omega)\, \widehat{\tilde{\phi}_s}(\omega) = \hat{a}(\omega)|\hat{\phi}_s(\omega)|^2.
$$

The biorthogonal relations (3.16) are satisfied if and only if $g(ns) = 0$ if $n \neq 0$ and $g(0) = 1$. It results that $g_d(t) = \sum_{n=-\infty}^{+\infty} g(ns)\,\delta(t - ns) = \delta(t)$. Theorem 3.1 derives in (3.3) that

$$
\hat{g}_d(\omega) = \frac{1}{s} \sum_{k=-\infty}^{+\infty} \hat{g}\,(\omega - 2k\pi/s) = \frac{\hat{a}(\omega)}{s} \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - 2k\pi/s)|^2 = 1.
$$

It results that

$$\hat{a}(\omega) = s \left(\sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - 2k\pi/s)|^2 \right)^{-1},$$

which proves (3.19). ∎

This theorem gives a necessary and sufficient condition on the low-pass filter $\bar{\phi}_s(t) = \phi_s(-t)$ to recover a stable signal approximation from uniform sampling at interval s. For various sampling interval s, the low-pass filter can be obtained by dilating a single filter $\phi_s(t) = s^{-1/2}\phi(t/s)$ and thus $\hat{\phi}_s(\omega) = s^{1/2}\hat{\phi}(s\omega)$. The necessary and sufficient Riesz basis condition (3.18) is then satisfied if and only if

$$\forall \omega \in [-\pi, \pi], \quad A \leqslant \sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega - 2k\pi)|^2 \leqslant B. \tag{3.26}$$

It results from (3.19) that the dual filter satisfies $\hat{\tilde{\phi}}_s(\omega) = s^{1/2}\hat{\tilde{\phi}}(s\omega)$ and therefore $\tilde{\phi}_s(t) = s^{-1/2}\tilde{\phi}(t/s)$. When $A = B = 1$, the Riesz basis is an orthonormal basis, which proves Corollary 3.1.

Corollary 3.1. The family $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $\mathbf{U}_s$ it generates, with $\phi_s(t) = s^{-1/2}\phi(t/s)$, if and only if

$$\forall \omega \in [0, 2\pi], \quad \sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega - 2k\pi)|^2 = 1, \tag{3.27}$$

and the dual filter is $\tilde{\phi}_s = \phi_s$.

Shannon-Whittaker Revisited
Shannon-Whittaker, Theorem 3.2, is defined with a sine-cardinal perfect low-pass filter ϕ_s, which we renormalize here to have a unit norm. The following theorem proves that it samples functions on an orthonormal basis.

Theorem 3.5. If $\phi_s(t) = s^{1/2}\sin(\pi s^{-1}t)/(\pi t)$ then $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $\mathbf{U}_s$ of functions whose Fourier transforms have a support included in $[-\pi/s, \pi/s]$. If $f \in \mathbf{U}_s$, then

$$f(nT) = s^{-1/2} f \star \phi_s(ns). \tag{3.28}$$

Proof. The filter satisfies $\phi_s(t) = s^{-1/2}\phi(t/s)$ with $\phi(t) = \sin(\pi t)/(\pi t)$. The Fourier transform $\hat{\phi}(\omega) = \mathbf{1}_{[-\pi,\pi]}(\omega)$ satisfies the condition (3.27) of Corollary 3.1, which proves that $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of a space $\mathbf{U}_s$.

Any $f(t) = \sum_{n=-\infty}^{+\infty} a[ns]\,\phi_s(t - ns) \in \mathbf{U}_s$ has a Fourier transform that can be written

$$\hat{f}(\omega) = \sum_{n=-\infty}^{+\infty} a[ns]\,e^{-ins\omega}\,\hat{\phi}_s(\omega) = \hat{a}(\omega)\,s^{1/2}\,\mathbf{1}_{[-\pi/s,\pi/s]}, \tag{3.29}$$

which implies that $f \in \mathbf{U}_s$ if and only if f has a Fourier transform supported in $[-\pi/s, \pi/s]$.

If $f \in \mathbf{U}_s$, then decomposing it on the orthonormal basis $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$ gives

$$f(t) = P_{\mathbf{U}_s} f(t) = \sum_{n\in\mathbb{Z}} \langle f(u), \phi_s(u-ns)\rangle \, \phi_s(t-ns).$$

Since $\phi_s(ps) = s^{-1/2}\delta[ps]$ and $\phi_s(-t) = \phi_s(t)$, the result is that

$$f(ns) = s^{-1/2}\langle f(u), \phi_s(u-ns)\rangle = s^{-1/2} f \star \phi_s(ns). \qquad \blacksquare$$

This theorem proves that in the particular case of the Shannon-Whittaker sampling theorem, if $f \in \mathbf{U}_s$ then the sampled low-pass filtered values $f \star \phi_s(ns)$ are proportional to the signal samples $f(ns)$. This comes from the fact that the sine-cardinal $\phi(t) = \sin(\pi t/s)/(\pi t/s)$ satisfies the interpolation property $\phi(ns) = \delta[ns]$. A generalization of such multiscale interpolations is studied in Section 7.6.

Shannon-Whittaker sampling approximates signals by restricting their Fourier transform to a low-frequency interval. It is particularly effective for smooth signals with a Fourier transform that has energy concentrated at low frequencies. It is also adapted for sound recordings, which are sufficiently approximated by lower-frequency harmonics.

For discontinuous signals, such as images, a low-frequency restriction produces Gibbs oscillations, as described in Section 2.3.3. The image visual quality is degraded by these oscillations, which have a total variation (2.65) that is infinite. A piecewise constant approximation has the advantage of creating no such spurious oscillations.

Block Sampler

A block sampler approximates signals with piecewise constant functions. The approximation space $\mathbf{U}_s$ is the set of all functions that are constant on intervals $[ns, (n+1)s]$, for any $n \in \mathbb{Z}$. Let $\phi_s(t) = s^{-1/2}\mathbf{1}_{[0,s]}(t)$. The family $\{\phi_s(t-ns]\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{U}_s$ (Exercise 3.1). If $f \notin \mathbf{U}_s$, then its orthogonal projection on $\mathbf{U}_s$ is calculated with a partial decomposition in the block orthonormal basis of $\mathbf{U}_s$

$$P_{\mathbf{U}_s} f(t) = \sum_{n=-\infty}^{+\infty} \langle f(u), \phi_s(u-ns)\rangle \, \phi_s(t-ns), \qquad (3.30)$$

and each coefficient is proportional to the signal average on $[ns, (n+1)s]$:

$$\langle f(u), \phi_s(u-ns)\rangle = f \star \phi_s(ns) = s^{-1/2} \int_{ns}^{(n+1)s} f(u)\, du.$$

This block analog-to-digital conversion is particularly simple to implement in analog electronics, where integration is performed by a capacity.

In domains where f is a regular function, a piecewise constant approximation $P_{\mathbf{U}_s} f$ is not very precise and can be significantly improved. More precise approximations are obtained with approximation spaces $\mathbf{U}_s$ of higher-order polynomial splines. The resulting approximations can introduce small Gibbs oscillations, but these oscillations have a finite total variation.

Spline Sampling

Block samplers are generalized by spline sampling with a space $\mathbf{U}_s$ of spline functions that are $m - 1$ times continuously differentiable and equal to a polynomial of degree m on any interval $[ns, (n + 1)s)$, for $n \in \mathbb{Z}$. When $m = 1$, functions in $\mathbf{U}_s$ are piecewise linear and continuous.

A Riesz basis of polynomial splines is constructed with *box splines*. A box spline ϕ of degree m is computed by convolving the box window $\mathbf{1}_{[0,1]}$ with itself $m + 1$ times and centering it at 0 or 1/2. Its Fourier transform is

$$\hat{\phi}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2} \right)^{m+1} \exp\left(\frac{-i\varepsilon\omega}{2} \right). \tag{3.31}$$

If m is even, then $\varepsilon = 1$ and ϕ have a support centered at $t = 1/2$. If m is odd, then $\varepsilon = 0$ and $\phi(t)$ are symmetric about $t = 0$. One can verify that $\hat{\phi}(\omega)$ satisfies the sampling condition (3.26) using a closed-form expression (7.20) of the resulting series. This means that for any $s > 0$, a box splines family $\{\phi_s(t - ns)\}_{n \in \mathbb{Z}}$ defines a Riesz basis of $\mathbf{U}_s$, and thus is a stable sampling.

3.2 DISCRETE TIME-INVARIANT FILTERS

3.2.1 Impulse Response and Transfer Function

Classic discrete signal-processing algorithms most generally are based on time-invariant linear operators [51, 55]. The time invariance is limited to translations on the sampling grid. To simplify notation, the sampling interval is normalized $s = 1$, and we denote $f[n]$ the sample values. A linear discrete operator L is time-invariant if an input $f[n]$, delayed by $p \in \mathbb{Z}$, $f_p[n] = f[n - p]$, produces an output also delayed by p:

$$Lf_p[n] = Lf[n - p].$$

Impulse Response

We denote by $\delta[n]$ the discrete Dirac

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}. \tag{3.32}$$

Any signal $f[n]$ can be decomposed as a sum of shifted Diracs:

$$f[n] = \sum_{p=-\infty}^{+\infty} f[p]\,\delta[n - p].$$

Let $L\delta[n] = h[n]$ be the discrete *impulse response*. Linearity and time invariance implies that

$$Lf[n] = \sum_{p=-\infty}^{+\infty} f[p]\,h[n - p] = f \star h[n]. \tag{3.33}$$

A discrete linear time-invariant operator is thus computed with a discrete convolution. If $h[n]$ has a finite support, the sum (3.33) is calculated with a finite number of operations. These are called *finite impulse response* (FIR) filters. Convolutions with infinite impulse response filters may also be calculated with a finite number of operations if they can be rewritten with a recursive equation (3.45).

Causality and Stability

A discrete filter L is *causal* if $Lf[p]$ depends only on the values of $f[n]$ for $n \leqslant p$. The convolution formula (3.33) implies that $h[n] = 0$ if $n < 0$.

The filter is *stable* if any bounded input signal $f[n]$ produces a bounded output signal $Lf[n]$. Since

$$|Lf[n]| \leqslant \sup_{n \in \mathbb{Z}} |f[n]| \sum_{k=-\infty}^{+\infty} |h[k]|,$$

it is sufficient that $\sum_{n=-\infty}^{+\infty} |h[n]| < +\infty$, which means that $h \in \ell^1(\mathbb{Z})$. One can verify that this sufficient condition is also necessary. Thus, the filter is stable if and only if $h \in \ell^1(\mathbb{Z})$ (Exercise 3.6).

Transfer Function

The Fourier transform plays a fundamental role in analyzing discrete time-invariant operators because discrete sinusoidal waves $e_\omega[n] = e^{i\omega n}$ are eigenvectors:

$$Le_\omega[n] = \sum_{p=-\infty}^{+\infty} e^{i\omega(n-p)} h[p] = e^{i\omega n} \sum_{p=-\infty}^{+\infty} h[p] e^{-i\omega p}. \tag{3.34}$$

The eigenvalue is a Fourier series

$$\hat{h}(\omega) = \sum_{p=-\infty}^{+\infty} h[p] e^{-i\omega p}. \tag{3.35}$$

It is the filter *transfer function*.

EXAMPLE 3.2

The uniform discrete average

$$Lf[n] = \frac{1}{2N+1} \sum_{p=n-N}^{n+N} f[p]$$

is a time-invariant discrete filter that has an impulse response of $h = (2N+1)^{-1} \mathbf{1}_{[-N,N]}$. Its transfer function is

$$\hat{h}(\omega) = \frac{1}{2N+1} \sum_{n=-N}^{+N} e^{-in\omega} = \frac{1}{2N+1} \frac{\sin(N+1/2)\omega}{\sin \omega/2}. \tag{3.36}$$

3.2.2 Fourier Series

The properties of Fourier series are essentially the same as the properties of the Fourier transform since Fourier series are particular instances of Fourier transforms for Dirac sums. If $f(t) = \sum_{n=-\infty}^{+\infty} f[n]\,\delta(t-n)$, then $\hat{f}(\omega) = \sum_{n=-\infty}^{+\infty} f[n]\,e^{-i\omega n}$.

For any $n \in \mathbb{Z}$, $e^{-i\omega n}$ has period 2π, so Fourier series have period 2π. An important issue to understand is whether all functions with period 2π can be written as Fourier series. Such functions are characterized by their restriction to $[-\pi, \pi]$. We therefore consider functions $\hat{a} \in \mathbf{L}^2[-\pi, \pi]$ that are square integrable over $[-\pi, \pi]$. The space $\mathbf{L}^2[-\pi, \pi]$ is a Hilbert space with the inner product

$$\langle \hat{a}, \hat{b} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{a}(\omega)\,\hat{b}^*(\omega)\,d\omega \tag{3.37}$$

and the resulting norm

$$\|\hat{a}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{a}(\omega)|^2\,d\omega.$$

Theorem 3.6 proves that any function in $\mathbf{L}^2[-\pi, \pi]$ can be written as a Fourier series.

Theorem 3.6. The family of functions $\{e^{-ik\omega}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of $\mathbf{L}^2[-\pi, \pi]$.

Proof. The orthogonality with respect to the inner product (3.37) is established with a direct integration. To prove that $\{\exp(-ik\omega)\}_{k \in \mathbb{Z}}$ is a basis, we must show that linear expansions of these vectors are dense in $\mathbf{L}^2[-\pi, \pi]$.

We first prove that any continuously differentiable function $\hat{\phi}$ with a support included in $[-\pi, \pi]$ satisfies

$$\hat{\phi}(\omega) = \sum_{k=-\infty}^{+\infty} \langle \hat{\phi}(\xi), \exp(-ik\xi) \rangle\,\exp(-ik\omega), \tag{3.38}$$

with a pointwise convergence for any $\omega \in [-\pi, \pi]$. Let us compute the partial sum

$$S_N(\omega) = \sum_{k=-N}^{N} \langle \hat{\phi}(\xi), \exp(-ik\xi) \rangle\,\exp(-ik\omega)$$

$$= \sum_{k=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\phi}(\xi)\,\exp(ik\xi)\,d\xi\,\exp(-ik\omega)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\phi}(\xi) \sum_{k=-N}^{N} \exp[ik(\xi-\omega)]\,d\xi.$$

The Poisson formula (2.37) proves the distribution equality

$$\lim_{N \to +\infty} \sum_{k=-N}^{N} \exp[ik(\xi-\omega)] = 2\pi \sum_{k=-\infty}^{+\infty} \delta(\xi-\omega-2\pi k),$$

and since the support of $\hat{\phi}$ is in $[-\pi, \pi]$, we get

$$\lim_{N \to +\infty} S_N(\omega) = \hat{\phi}(\omega).$$

Since $\hat{\phi}$ is continuously differentiable, following the steps (2.38–2.40) in the proof of the Poisson formula shows that $S_N(\omega)$ converges uniformly to $\hat{\phi}(\omega)$ on $[-\pi, \pi]$.

To prove that linear expansions of sinusoidal waves $\{\exp(-ik\omega)\}_{k \in \mathbb{Z}}$ are dense in $\mathbf{L}^2[-\pi, \pi]$, let us verify that the distance between $\hat{a} \in \mathbf{L}^2[-\pi, \pi]$ and such a linear expansion is less than ε, for any $\varepsilon > 0$. Continuously differentiable functions with a support included in $[-\pi, \pi]$ are dense in $\mathbf{L}^2[-\pi, \pi]$; thus, there exists $\hat{\phi}$ such that $\|\hat{a} - \hat{\phi}\| \leqslant \varepsilon/2$. The uniform pointwise convergence proves that there exists N for which

$$\sup_{\omega \in [-\pi, \pi]} |S_N(\omega) - \hat{\phi}(\omega)| \leqslant \frac{\varepsilon}{2},$$

which implies that

$$\|S_N - \hat{\phi}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |S_N(\omega) - \hat{\phi}(\omega)|^2 \, d\omega \leqslant \frac{\varepsilon^2}{4}.$$

It follows that $\hat{a}$ is approximated by the Fourier series S_N with an error

$$\|\hat{a} - S_N\| \leqslant \|\hat{a} - \hat{\phi}\| + \|\hat{\phi} - S_N\| \leqslant \varepsilon. \qquad \blacksquare$$

Theorem 3.6 proves that if $f \in \ell^2(\mathbb{Z})$, the Fourier series

$$\hat{f}(\omega) = \sum_{n=-\infty}^{+\infty} f[n] \, e^{-i\omega n} \qquad (3.39)$$

can be interpreted as the decomposition of $\hat{f}$ in an orthonormal basis of $\mathbf{L}^2[-\pi, \pi]$. The Fourier series coefficients can thus be written as inner products in $\mathbf{L}^2[-\pi, \pi]$:

$$f[n] = \langle \hat{f}(\omega), e^{-i\omega n} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \, e^{i\omega n} \, d\omega. \qquad (3.40)$$

The energy conservation of orthonormal bases (A.10) yields a Plancherel identity:

$$\sum_{n=-\infty}^{+\infty} |f[n]|^2 = \|\hat{f}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 \, d\omega. \qquad (3.41)$$

Pointwise Convergence

The equality (3.39) is meant in the sense of mean-square convergence

$$\lim_{N \to +\infty} \left\| \hat{f}(\omega) - \sum_{k=-N}^{N} f[k] \, e^{-i\omega k} \right\| = 0.$$

It does not imply a pointwise convergence at all $\omega \in \mathbb{R}$.

In 1873, Dubois-Reymond constructed a periodic function $\hat{f}(\omega)$ that is continuous and has a Fourier series that diverges at some point. On the other hand, if $\hat{f}(\omega)$ is continuously differentiable, then the proof of Theorem 3.6 shows that its Fourier series converges uniformly to $\hat{f}(\omega)$ on $[-\pi, \pi]$. It was only in 1966 that Carleson [149] was able to prove that if $\hat{f} \in \mathbf{L}^2[-\pi, \pi]$ then its Fourier series converges almost everywhere. The proof is very technical.

Convolutions

Since $\{e^{-i\omega k}\}_{k \in \mathbb{Z}}$ are eigenvectors of discrete convolution operators, we also have a discrete convolution theorem.

Theorem 3.7. If $f \in \ell^1(\mathbb{Z})$ and $h \in \ell^1(\mathbb{Z})$, then $g = f \star h \in \ell^1(\mathbb{Z})$ and

$$\hat{g}(\omega) = \hat{f}(\omega)\,\hat{h}(\omega). \tag{3.42}$$

The proof is identical to the proof of the convolution, Theorem 2.2, if we replace integrals by discrete sums. The reconstruction formula (3.40) shows that a filtered signal can be written

$$f \star h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}(\omega)\hat{f}(\omega)\,e^{i\omega n}\,d\omega. \tag{3.43}$$

The transfer function $\hat{h}(\omega)$ amplifies or attenuates the frequency components $\hat{f}(\omega)$ of $f[n]$.

EXAMPLE 3.3

An ideal *discrete low-pass filter* has a 2π periodic transfer function that is defined by $\hat{h}(\omega) = \mathbf{1}_{[-\xi,\xi]}(\omega)$, for $\omega \in [-\pi, \pi]$ and $0 < \xi < \pi$. Its impulse response is computed with (3.40):

$$h[n] = \frac{1}{2\pi} \int_{-\xi}^{\xi} e^{i\omega n}\,d\omega = \frac{\sin \xi n}{\pi n}. \tag{3.44}$$

It is a uniform sampling of the ideal analog low-pass filter (2.29).

EXAMPLE 3.4

A *recursive filter* computes $g = Lf$, which is a solution of a recursive equation

$$\sum_{k=0}^{K} a_k f[n-k] = \sum_{k=0}^{M} b_k\, g[n-k], \tag{3.45}$$

with $b_0 \neq 0$. If $g[n] = 0$ and $f[n] = 0$ for $n < 0$, then g has a linear and time-invariant dependency on f and thus can be written $g = f \star h$. The transfer function is obtained by computing the Fourier transform of (3.45). The Fourier transform of $f_k[n] = f[n-k]$ is

$\hat{f}_k(\omega) = \hat{f}(\omega)\,e^{-ik\omega}$, so

$$\hat{h}(\omega) = \frac{\hat{g}(\omega)}{\hat{f}(\omega)} = \frac{\sum_{k=0}^{K} a_k\,e^{-ik\omega}}{\sum_{k=0}^{M} b_k\,e^{-ik\omega}}.$$

It is a rational function of $e^{-i\omega}$. If $b_k \neq 0$ for some $k > 0$, then one can verify that the impulse response h has an infinite support. The stability of such filters is studied in Exercise 3.18. A direct calculation of the convolution sum $g[n] = f \star h[n]$ would require an infinite number of operations, whereas (3.45) computes $g[n]$ with $K + M$ additions and multiplications from its past values.

Window Multiplication

An infinite impulse response filter h, such as the ideal low-pass filter (3.44), may be approximated by a finite response filter $\tilde{h}$ by multiplying h with a window g of finite support:

$$\tilde{h}[n] = g[n]\,h[n].$$

One can verify (Exercise 3.6) that a multiplication in time is equivalent to a convolution in the frequency domain:

$$\widehat{\tilde{h}}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}(\xi)\,\hat{g}(\omega - \xi)\,d\xi = \frac{1}{2\pi}\hat{h} \star \hat{g}(\omega). \tag{3.46}$$

Clearly $\widehat{\tilde{h}} = \hat{h}$ only if $\hat{g} = 2\pi\delta$, which would imply that g has an infinite support and $g[n] = 1$. The approximation $\tilde{h}$ is close to $\hat{h}$ only if $\hat{g}$ approximates a Dirac, which means that all its energy is concentrated at low frequencies. In time, g should therefore have smooth variations.

The rectangular window $g = \mathbf{1}_{[-N,N]}$ has a Fourier transform $\hat{g}$ computed in (3.36). It has important side lobes far away from $\omega = 0$. The resulting $\widehat{\tilde{h}}$ is a poor approximation of $\hat{h}$. The Hanning window

$$g[n] = \cos^2\left(\frac{\pi n}{2N}\right)\,\mathbf{1}_{[-N,N]}[n]$$

is smoother and thus has a Fourier transform better concentrated at low frequencies. The spectral properties of other windows are studied in Section 4.2.2.

3.3 FINITE SIGNALS

Up to now we have considered discrete signals $f[n]$ defined for all $n \in \mathbb{Z}$. In practice, $f[n]$ is known over a finite domain, say $0 \leqslant n < N$. Convolutions therefore must be modified to take into account the border effects at $n = 0$ and $n = N - 1$. The Fourier transform also must be redefined over finite sequences for numerical computations. The fast Fourier transform algorithm is explained as well as its application to fast convolutions.

3.3.1 Circular Convolutions

Let $\tilde{f}$ and $\tilde{h}$ be signals of N samples. To compute the convolution product

$$\tilde{f} \star \tilde{h}[n] = \sum_{p=-\infty}^{+\infty} \tilde{f}[p]\,\tilde{h}[n-p] \quad \text{for } 0 \leqslant n < N,$$

we must know $\tilde{f}[n]$ and $\tilde{h}[n]$ beyond $0 \leqslant n < N$. One approach is to extend $\tilde{f}$ and $\tilde{h}$ with a periodization over N samples, and to define

$$f[n] = \tilde{f}[n \bmod N], \quad h[n] = \tilde{h}[n \bmod N].$$

The *circular convolution* of two such signals, both with period N, is defined as a sum over their period:

$$f \circledast h[n] = \sum_{p=0}^{N-1} f[p]\,h[n-p] = \sum_{p=0}^{N-1} f[n-p]\,h[p].$$

It is also a signal of period N.

The eigenvectors of a circular convolution operator

$$Lf[n] = f \circledast h[n]$$

are the discrete complex exponentials $e_k[n] = \exp(i2\pi kn/N)$. Indeed,

$$Le_k[n] = \exp\left(\frac{i2\pi kn}{N}\right) \sum_{p=0}^{N-1} h[p] \, \exp\left(\frac{-i2\pi kp}{N}\right),$$

and the eigenvalue is the discrete Fourier transform of h:

$$\hat{h}[k] = \sum_{p=0}^{N-1} h[p] \, \exp\left(\frac{-i2\pi kp}{N}\right).$$

3.3.2 Discrete Fourier Transform

The space of signals of period N is an Euclidean space of dimension N and the inner product of two such signals f and g is

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f[n]g^*[n]. \tag{3.47}$$

Theorem 3.8 proves that any signal with period N can be decomposed as a sum of discrete sinusoidal waves.

Theorem 3.8. The family

$$\left\{ e_k[n] = \exp\left(\frac{i2\pi kn}{N}\right) \right\}_{0 \leqslant k < N}$$

is an orthogonal basis of the space of signals of period N.

Since the space is of dimension N, any orthogonal family of N vectors is an orthogonal basis. To prove this theorem, it is sufficient to verify that $\{e_k\}_{0 \leqslant k < N}$ is orthogonal with respect to the inner product (3.47) (Exercise 3.8). Any signal f of period N can be decomposed on this basis:

$$f = \sum_{k=0}^{N-1} \frac{\langle f, e_k \rangle}{\|e_k\|^2} \, e_k. \tag{3.48}$$

By definition, the *discrete Fourier transform* (DFT) of f is

$$\hat{f}[k] = \langle f, e_k \rangle = \sum_{n=0}^{N-1} f[n] \exp\left(\frac{-i2\pi kn}{N} \right). \tag{3.49}$$

Since $\|e_k\|^2 = N$, (3.48) gives an inverse discrete Fourier formula:

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] \exp\left(\frac{i2\pi kn}{N} \right). \tag{3.50}$$

The orthogonality of the basis also implies a Plancherel formula:

$$\|f\|^2 = \sum_{n=0}^{N-1} |f[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\hat{f}[k]|^2. \tag{3.51}$$

The discrete Fourier tranform (DFT) of a signal f of period N is computed from its values for $0 \leqslant n < N$. Then why is it important to consider it a periodic signal with period N rather than a finite signal of N samples? The answer lies in the interpretation of the Fourier coefficients. The discrete Fourier sum (3.50) defines a signal of period N for which the samples $f[0]$ and $f[N-1]$ are side by side. If $f[0]$ and $f[N-1]$ are very different, this produces a brutal transition in the periodic signal, creating relatively high-amplitude Fourier coefficients at high frequencies. For example, Figure 3.3 shows that the "smooth" ramp $f[n] = n$ for $0 \leqslant n < N$ has sharp transitions at $n = 0$ and $n = N$ once made periodic.

Circular Convolutions

Since $\{\exp(i2\pi kn/N)\}_{0 \leqslant k < N}$ are eigenvectors of circular convolutions, we derive a convolution theorem.
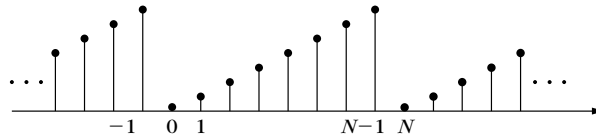

FIGURE 3.3

Signal $f[n] = n$ for $0 \leqslant n < N$ made periodic over N samples.

Theorem 3.9. If f and h have period N, then the discrete Fourier transform of $g = f \circledast h$ is

$$\hat{g}[k] = \hat{f}[k]\,\hat{h}[k]. \tag{3.52}$$

The proof is similar to the proof of the two previous convolution theorems—2.2 and 3.7. This theorem shows that a circular convolution can be interpreted as a discrete frequency filtering. It also opens the door to fast computations of convolutions using the fast Fourier transform.

3.3.3 Fast Fourier Transform

For a signal f of N points, a direct calculation of the N discrete Fourier sums

$$\hat{f}[k] = \sum_{n=0}^{N-1} f[n]\,\exp\left(\frac{-i2\pi kn}{N}\right), \quad \text{for } 0 \leqslant k < N, \tag{3.53}$$

requires N^2 complex multiplications and additions. The FFT algorithm reduces the numerical complexity to $O(N \log_2 N)$ by reorganizing the calculations.

When the frequency index is even, we group the terms n and $n + N/2$:

$$\hat{f}[2k] = \sum_{n=0}^{N/2-1} \left(f[n] + f[n+N/2]\right)\exp\left(\frac{-i2\pi kn}{N/2}\right). \tag{3.54}$$

When the frequency index is odd, the same grouping becomes

$$\hat{f}[2k+1] = \sum_{n=0}^{N/2-1} \exp\left(\frac{-i2\pi n}{N}\right)\left(f[n] - f[n+N/2]\right)\exp\left(\frac{-i2\pi kn}{N/2}\right). \tag{3.55}$$

Equation (3.54) proves that even frequencies are obtained by calculating the DFT of the $N/2$ periodic signal

$$f_e[n] = f[n] + f[n+N/2].$$

Odd frequencies are derived from (3.55) by computing the Fourier transform of the $N/2$ periodic signal:

$$f_o[n] = \exp\left(\frac{-i2\pi n}{N}\right)\left(f[n] - f[n+N/2]\right).$$

A DFT of size N may thus be calculated with two discrete Fourier transforms of size $N/2$ plus $O(N)$ operations.

The inverse FFT of $\hat{f}$ is derived from the forward fast Fourier transform of its complex conjugate $\hat{f}^*$ by observing that

$$f^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}^*[k]\,\exp\left(\frac{-i2\pi kn}{N}\right). \tag{3.56}$$

Complexity

Let $C(N)$ be the number of elementary operations needed to compute a DFT with the FFT. Since f is complex, the calculation of f_e and f_o requires N complex additions and $N/2$ complex multiplications. Let KN be the corresponding number of elementary operations. We have

$$C(N) = 2\,C(N/2) + K\,N, \tag{3.57}$$

since the Fourier transform of a single point is equal to itself, $C(1) = 0$. With the change of variable $l = \log_2 N$ and the change of function $T(l) = \frac{C(N)}{N}$, from (3.57) we derive

$$T(l) = T(l-1) + K.$$

Since $T(0) = 0$, we get $T(l) = K\,l$ and therefore

$$C(N) = K\,N\,\log_2(N).$$

Several variations of this fast algorithm exist [49, 237]. The goal is to minimize the constant K. The most efficient fast DFT to this date is the split-radix FFT algorithm, which is slightly more complicated than the procedure just described; however; it requires only $N \log_2 N$ real multiplications and $3N \log_2 N$ additions. When the input signal f is real, there are half as many parameters to compute, since $\hat{f}[-k] = \hat{f}^*[k]$. The number of multiplications and additions is thus reduced by 2.

3.3.4 Fast Convolutions

The low computational complexity of a FFT makes it efficient to compute finite discrete convolutions by using the circular convolution, Theorem 3.9. Let f and h be two signals with samples that are nonzero only for $0 \leqslant n < M$. The causal signal

$$g[n] = f \star h[n] = \sum_{k=-\infty}^{+\infty} f[k]\,h[n-k] \tag{3.58}$$

is nonzero only for $0 \leqslant n < 2M$. If h and f have M nonzero samples, calculating this convolution product with the sum (3.58) requires $M(M + 1)$ multiplications and additions. When $M \geqslant 32$, the number of computations is reduced by using the FFT [11, 49].

To use the fast Fourier transform with the circular convolution, Theorem 3.9, the noncircular convolution (3.58) is written as a circular convolution. We define two signals of period $2M$:

$$a[n] = \begin{cases} f[n] & \text{if } 0 \leqslant n < M \\ 0 & \text{if } M \leqslant n < 2M \end{cases} \tag{3.59}$$

$$b[n] = \begin{cases} h[n] & \text{if } 0 \leqslant n < M \\ 0 & \text{if } M \leqslant n < 2M \end{cases} \tag{3.60}$$

By letting $c = a \circledast b$, one can verify that $c[n] = g[n]$ for $0 \leqslant n < 2M$. The $2M$ nonzero coefficients of g are thus obtained by computing $\hat{a}$ and $\hat{b}$ from a and b and then calculating the inverse DFT of $\hat{c} = \hat{a}\,\hat{b}$.

With the fast Fourier transform algorithm, this requires a total of $O(M \log_2 M)$ additions and multiplications instead of $M(M + 1)$. A single FFT or inverse FFT of a real signal of size N is calculated with $2^{-1}N \log_2 N$ multiplications, using a split-radix algorithm. The FFT convolution is thus performed with a total of $3M \log_2 M + 11M$ real multiplications. For $M \geqslant 32$, the FFT algorithm is faster than the direct convolution approach. For $M \leqslant 16$, it is faster to use a direct convolution sum.

Fast Overlap–Add Convolutions

The convolution of a signal f of L nonzero samples with a smaller causal signal h of M samples is calculated with an overlap–add procedure that is faster than the previous algorithm. The signal f is decomposed with a sum of L/M blocks f_r having M nonzero samples:

$$f[n] = \sum_{r=0}^{L/M-1} f_r[n - rM], \quad \text{with} \quad f_r[n] = f[n + rM]\,\mathbf{1}_{[0,M-1]}[n]. \tag{3.61}$$

For each $0 \leqslant r < L/M$, the $2M$ nonzero samples of $g_r = f_r \star h$ are computed with the FFT-based convolution algorithm, which requires $O(M \log_2 M)$ operations. These L/M convolutions are thus obtained with $O(L \log_2 M)$ operations. The block decomposition (3.61) implies that

$$f \star h[n] = \sum_{r=0}^{L/M-1} g_r[n - rM]. \tag{3.62}$$

The addition of these L/M translated signals of size $2M$ is done with $2L$ additions. The overall convolution is thus performed with $O(L \log_2 M)$ operations.

3.4 DISCRETE IMAGE PROCESSING

Two-dimensional signal processing poses many specific geometric and topological problems that do not exist in one dimension [21, 33]. For example, a simple concept, such as causality, is not well defined in two dimensions. We can avoid the complexity introduced by the second dimension by extending one-dimensional algorithms with a separable approach. This not only simplifies the mathematics but also leads to fast numerical algorithms along the rows and columns of images. Section A.5 in the Appendix reviews the properties of tensor products for separable calculations.

3.4.1 Two-Dimensional Sampling Theorems

The light intensity measured by a camera is generally sampled over a rectangular array of picture elements, called *pixels*. One-dimensional sampling theorems are

extended to this two-dimensional sampling array. Other two-dimensional sampling grids such as hexagonal, are also possible, but nonrectangular sampling arrays are not used often.

Let s_1 and s_2 be the sampling intervals along the x_1 and x_2 axes of an infinite rectangular sampling grid. The following renormalizes the axes so that $s_1 = s_2 = s$. A discrete image obtained by sampling $f(x)$ with $x = (x_1, x_2)$ can be represented as a sum of Diracs located at the grid points:

$$f_d(x) = \sum_{n \in \mathbb{Z}^2} f(sn)\,\delta(x - ns).$$

The two-dimensional Fourier transform of $\delta(x - sn)$ is $e^{-isn \cdot \omega}$ with $\omega = (\omega_1, \omega_2)$ and $n \cdot \omega = n_1 \omega_1 + n_2 \omega_2$. Thus, the Fourier transform of f_d is a two-dimensional Fourier series:

$$\hat{f}_d(\omega) = \sum_{n \in \mathbb{Z}^2} f(sn)\,e^{-isn \cdot \omega}. \tag{3.63}$$

It is $2\pi/s$ periodic along ω_1 and along ω_2. An extension of Theorem 3.1 relates $\hat{f}_d$ to the two-dimensional Fourier transform $\hat{f}$ of f.

Theorem 3.10. The Fourier transform of the discrete image $f_d(x)$ is

$$\hat{f}_d(\omega) = \frac{1}{s^2} \sum_{k \in \mathbb{Z}^2} \hat{f}(\omega - 2k\pi/s), \quad \text{with} \quad k = (k_1, k_2). \tag{3.64}$$

We derive the following two-dimensional sampling theorem, which is analogous to Theorem 3.2.

Theorem 3.11. If $\hat{f}$ has a support included in $[-\pi/s, \pi/s]^2$, then

$$f(x) = s \sum_{n \in \mathbb{Z}^2} f(ns)\,\phi_s(x - ns), \tag{3.65}$$

where

$$\phi_s(x_1, x_2) = \frac{1}{s}\,\frac{\sin(\pi x_1/s)}{\pi x_1/s}\,\frac{\sin(\pi x_2/s)}{\pi x_2/s}. \tag{3.66}$$

If the support of $\hat{f}$ is not included in the low-frequency rectangle $[-\pi/s, \pi/s]^2$, the interpolation formula (3.65) introduces aliasing errors. Such aliasing is eliminated by prefiltering f with the ideal low-pass separable filter $\phi_s(x)$ having a Fourier transform equal to 1 on $[-\pi/s, \pi/s]^2$.

General Sampling Theorems
As explained in Section 3.1.3, the Shannon-Whittaker sampling theorem is a particular case of more general linear sampling theorems with low-pass filters. The following theorem is a two-dimensional extension of Theorems 3.3 and 3.4; it characterizes these filters to obtain a stable reconstruction.

Theorem 3.12. If there exists $B \geqslant A > 0$ such that the Fourier transform of $\phi_s \in \mathbf{L}^2(\mathbb{R}^2)$ satisfies

$$\forall \omega \in [0, 2\pi/s]^2 \quad A \leqslant \hat{h}(\omega) = \sum_{k \in \mathbb{Z}^2} |\hat{\phi}_s(\omega - 2k\pi/s)|^2 \leqslant B,$$

then $\{\phi_s(x - ns)\}_{n \in \mathbb{Z}^2}$ is a Riesz basis of a space $\mathbf{U}_s$. The Fourier transform of the dual filter $\tilde{\phi}_s$ is $\widehat{\tilde{\phi}}_s(\omega) = \hat{\phi}_s^*(\omega)/h(\omega)$, and the orthogonal projection of $f \in \mathbf{L}^2(\mathbb{R}^2)$ in $\mathbf{U}_s$ is

$$P_{\mathbf{U}_s} f(x) = \sum_{n \in \mathbb{Z}^2} f \star \bar{\phi}_s(ns) \, \tilde{\phi}_s(x - ns), \quad \text{with} \quad \bar{\phi}_s(x) = \phi_s(-x). \tag{3.67}$$

This theorem gives a necessary and sufficient condition to obtain a stable linear reconstruction from samples computed with a linear filter. The proof is a direct extension of the proofs of Theorems 3.3 and 3.4. It recovers a signal approximation as an orthogonal projection by filtering the discrete signal $f_d(x) = \sum_{n \in \mathbb{Z}^2} f \star \bar{\phi}_s(ns) \, \delta(x - ns)$:

$$P_{\mathbf{U}_s} f(x) = f_d \star \tilde{\phi}_s(x).$$

The same as in one dimension, the filter ϕ_s can be obtained by scaling a single filter $\phi_s(x) = s^{-1}\phi(s^{-1}x)$. The two-dimensional Shannon-Whittaker theorem is a particular example, where $\hat{\phi}_s = s\mathbf{1}_{[-\pi/s, \pi/s]^2}$, which defines an orthonormal basis of the space $\mathbf{U}_s$ of functions having a Fourier transform supported in $[-\pi/s, \pi/s]^2$.

3.4.2 Discrete Image Filtering

The properties of two-dimensional space-invariant operators are essentially the same as in one dimension. The sampling interval s is normalized to 1. A pixel value located at $n = (n_1, n_2)$ is written $f[n]$. A linear operator L is space-invariant if $Lf_p[n] = Lf[n - p]$ for any $f_p[n] = f[n - p]$, with $p = (p_1, p_2) \in \mathbb{Z}^2$. A discrete Dirac is defined by $\delta[n] = 1$ if $n = (0, 0)$ and $\delta[n] = 0$ if $n \neq (0, 0)$.

Impulse Response

Since $f[n] = \sum_{p \in \mathbb{Z}^2} f[p] \delta[n - p]$, linearity and time invariance implies

$$Lf[n] = \sum_{p \in \mathbb{Z}^2} f[p] \, h[n - p] = f \star h[n], \tag{3.68}$$

where $h[n]$ is the response of the impulse $h[n] = L\delta[n]$. If the impulse response is separable

$$h[n_1, n_2] = h_1[n_1] \, h_2[n_2], \tag{3.69}$$

then the two-dimensional convolution (3.68) is computed as one-dimensional convolutions along the columns of the image followed by one-dimensional convolutions along the rows (or vice versa):

$$f \star h[n_1, n_2] = \sum_{p_1 = -\infty}^{+\infty} h_1[n_1 - p_1] \sum_{p_2 = -\infty}^{+\infty} h_2[n_2 - p_2] f[p_1, p_2]. \tag{3.70}$$

This factorization reduces the number of operations. If h_1 and h_2 are finite impulse response filters of size M_1 and M_2, respectively, then the separable calculation (3.70) requires $M_1 + M_2$ additions and multiplications per point (n_1, n_2) as opposed to $M_1 M_2$ in a nonseparable computation (3.68).

Transfer Function

The Fourier transform of a discrete image f is defined by the Fourier series:

$$\hat{f}(\omega) = \sum_{n \in \mathbb{Z}^2} f[n] \, e^{-i\omega \cdot n}, \quad \text{with} \quad \omega \cdot n = n_1 \omega_1 + n_2 \omega_2. \tag{3.71}$$

The two-dimensional extension of the convolution, Theorem 3.7, proves that if $g[n] = Lf[n] = f \star h[n]$ then its Fourier transform is $\hat{g}(\omega) = \hat{f}(\omega) \, \hat{h}(\omega)$, and $\hat{h}(\omega)$ is the transfer function of the filter. When a filter is separable $h[n_1, n_2] = h_1[n_1] h_2[n_2]$, its transfer function is also separable:

$$\hat{h}(\omega_1, \omega_2) = \hat{h}_1(\omega_1) \, \hat{h}_2(\omega_2). \tag{3.72}$$

3.4.3 Circular Convolutions and Fourier Basis

The discrete convolution of a finite image $\tilde{f}$ raises border problems. As in one dimension, these border issues are solved by extending the image, making it periodic along its rows and columns:

$$f[n_1, n_2] = \tilde{f}[n_1 \bmod N_1, n_2 \bmod N_2],$$

where $N = N_1 N_2$ is the image size. The resulting periodic image $f[n_1, n_2]$ is defined for all $(n_1, n_2) \in \mathbb{Z}^2$, and each of its rows and columns are periodic one-dimension signals.

A discrete convolution is replaced by a circular convolution over the image period. If f and h have a periodicity N_1 and N_2 along (n_1, n_2), then

$$f \circledast h[n_1, n_2] = \sum_{p_1=0}^{N_1-1} \sum_{p_2=0}^{N_2-1} f[p_1, p_2] \, h[n_1 - p_1, n_2 - p_2]. \tag{3.73}$$

Discrete Fourier Transform

The eigenvectors of circular convolutions are two-dimensional discrete sinusoidal waves:

$$e_k[n] = e^{\omega_k \cdot n}, \quad \text{with} \quad \omega_k = (2\pi k_1 / N_1, 2\pi k_2 / N_2) \quad \text{for} \quad 0 \leq k_1 < N_1, 0 \leq k_2 < N_2.$$

This family of $N = N_1 N_2$ discrete vectors is the separable product of two one-dimensional discrete Fourier bases $\{e^{i2\pi k_1 n/N_1}\}_{0 \leq k_1 < N_1}$ and $\{e^{i2\pi k_1 n/N_2}\}_{0 \leq k_2 < N_2}$. Thus, Theorem A.3 proves that the family $\{e_k[n]\}_{0 \leq k_1 < N_1, 0 \leq k_2 < N_2}$ is an orthogonal basis of $\mathbb{C}^N = \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ (Exercise 3.23). Any image $f \in \mathbb{C}^N$ can be decomposed in this orthogonal basis:

$$f[n] = \frac{1}{N} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \hat{f}[k] \, e^{i\omega_k \cdot n}, \tag{3.74}$$

where $\hat{f}$ is the two-dimensional DFT of f

$$\hat{f}[k] = \langle f, e_k \rangle = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} f[n] \, e^{-i\omega_k \cdot n}. \tag{3.75}$$

Fast Convolutions

Since $e^{i\omega_k \cdot n}$ are eigenvectors of two-dimensional circular convolutions, the DFT of $g = f \circledast h$ is

$$\hat{g}[k] = \hat{f}[k] \, \hat{h}[k]. \tag{3.76}$$

A direct computation of $f \circledast h$ with the summation (3.73) requires $O(N^2)$ multiplications. With the two-dimensional FFT described next, $\hat{f}[k]$ and $\hat{h}[k]$ as well as the inverse DFT of their product (3.76) are calculated with $O(N \log N)$ operations. Noncircular convolutions are computed with a fast algorithm by reducing them to circular convolutions, with the same approach as in Section 3.3.4.

Separable Basis Decomposition

Let $\mathcal{B}_1 = \{e_{k_1}^1\}_{0 \le k_1 < N_1}$ and $\mathcal{B}_2 = \{e_{k_2}^2\}_{0 \le k_2 < N_2}$ be two orthogonal bases of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$. Suppose the calculation of decomposition coefficients of $f_1 \in \mathbb{C}^{N_1}$ in the basis $\mathcal{B}_1$ requires $C_1(N_1)$ operations and of $f_2 \in \mathbb{C}^{N_1}$ in the basis $\mathcal{B}_2$ requires $C_2(N_2)$ operations. One can verify (Exercise 3.23) that the family $\mathcal{B} = \{e_k[n] = e_{k_1}^1[n_1] \, e_{k_2}^2[n_2]\}_{0 \le k_1 < N_1, 0 \le k_2 < N_2}$ is an orthogonal basis of the space $\mathbb{C}^N = \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ of images $f[n_1, n_2]$ of $N = N_1 N_2$ pixels. We describe a fast separable algorithm that computes the decomposition coefficients of an image f in $\mathcal{B}$ with $N_2 C_1(N_1) + N_1 C_2(N_2)$ operations as opposed to N^2. A fast two-dimensional FFT is derived.

Two-dimensional inner products are calculated with

$$\langle f, e_{k_1}^1 e_{k_2}^2 \rangle = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} f[n_1, n_2] \, e_{k_1}^{1*}[n_1] \, e_{k_2}^{2*}[n_2]$$
$$= \sum_{n_1=0}^{N_1-1} e_{k_1}^{1*}[n_1] \sum_{n_2=0}^{N_2-1} f[n_1, n_2] \, e_{k_2}^{2*}[n_2]. \tag{3.77}$$

For $0 \le n_1 < N_1$, we must compute

$$Uf[n_1, k_2] = \sum_{n_2=0}^{N_2-1} f[n_1, n_2] \, e_{k_2}^{2*}[n_2],$$

which are the decomposition coefficients of the N_1 image rows of size N_2 in the basis $\mathcal{B}_2$. The coefficients $\{\langle f, e_{k_1}^1 e_{k_2}^2 \rangle\}_{0 \le k_1 < N_1, 0 \le k_2 < N_2}$ are calculated in (3.77) as the inner products of the columns of the transformed image $Uf[n_1, k_2]$ in the basis $\mathcal{B}_1$. The overall algorithm thus requires performing N_1 one-dimensional transforms

in the basis $\mathcal{B}_2$ plus N_2 one-dimensional transforms in the basis $\mathcal{B}_1$; it therefore requires $N_2\,C_1(N_1) + N_1\,C_2(N_2)$ operations.

The fast Fourier transform algorithm of Section 3.3.3 decomposes signals of size N_1 and N_2 in the discrete Fourier bases $\mathcal{B}_1 = \{e_{k_1}^1[n_1] = e^{i2\pi k_1 n_1/N_1}\}_{0 \leqslant k_1 < N_1}$ and $\mathcal{B}_2 = \{e_{k_2}^2[n_2] = e^{i2\pi k_2 n_2/N_2}\}_{0 \leqslant k_2 < N_2}$, with $C_1(N_1) = KN_1\log_2 N_1$ and $C_2(N_2) = KN_2\log_2 N_2$ operations. A separable implementation of a two-dimensional FFT thus requires $N_2\,C_1(N_1) + N_1\,C_2(N_2) = KN\log_2 N$ operations, with $N = N_1 N_2$. A split-radix FFT corresponds to $K = 3$.

3.5 EXERCISES

3.1 [1] Show that if $\phi_s(t) = s^{-1/2}\,\mathbf{1}_{[0,s)}(t)$, then $\{\phi_s(t - ns)\}_{n \in \mathbb{Z}}$ is an orthonormal basis of the space of piecewise constant function on intervals $[ns,\,(n+1)s)$ for any $n \in \mathbb{Z}$.

3.2 [2] Prove that if f has a Fourier transform included in $[-\pi/s,\,\pi/s]$, then

$$\forall u \in \mathbb{R}, \quad f(u) = \frac{1}{s}\,\langle f(t),\, \phi_s(t-u)\rangle \quad \text{with} \quad \phi_s(t) = \frac{\sin(\pi t/s)}{\pi t/s}.$$

3.3 [2] An interpolation function $f(t)$ satisfies $f(n) = \delta[n]$ for any $n \in \mathbb{Z}$.
   (a) Prove that $\sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi) = 1$ if and only if f is an interpolation function.
   (b) Suppose that $f(t) = \sum_{n=-\infty}^{+\infty} h[n]\,\theta(t-n)$ with $\theta \in \mathbf{L}^2(\mathbb{R})$. Find $\hat{h}(\omega)$ as a function of $\hat{\theta}(\omega)$ so that $f(t)$ is an interpolation function. Relate $\hat{f}(\omega)$ to $\hat{\theta}(\omega)$, and give a sufficient condition on $\hat{\theta}$ to guarantee that $f \in \mathbf{L}^2(\mathbb{R})$.

3.4 [2] Prove that if $f \in \mathbf{L}^2(\mathbb{R})$ and $\sum_{n=-\infty}^{+\infty} f(t-n) \in \mathbf{L}^2[0,\,1]$, then

$$\sum_{n=-\infty}^{+\infty} f(t-n) = \sum_{k=-\infty}^{+\infty} \hat{f}(2k\pi)\,e^{i2\pi kt}.$$

3.5 [1] We want to approximate f by a signal $\tilde{f}$ in an approximation space $\mathbf{U}_s$. Prove that the approximation $\tilde{f}$ that minimizes $\|\tilde{f} - f\|$, is the orthogonal projection of f in $\mathbf{U}_s$.

3.6 [2] Prove that the discrete filter $Lf[n] = f \star h[n]$ is stable if and only if $h \in \ell^1(\mathbb{Z})$.

3.7 [2] If $\hat{h}(\omega)$ and $\hat{g}(\omega)$ are the Fourier transforms of $h[n]$ and $g[n]$, we write

$$\hat{h} \star \hat{g}(\omega) = \int_{-\pi}^{+\pi} \hat{h}(\xi)\,\hat{g}(\omega - \xi)\,d\xi.$$

Prove that if $f[n] = g[n]\,h[n]$, then $\hat{f}(\omega) = (2\pi)^{-1}\,\hat{h} \star \hat{g}(\omega)$.

3.8 [1] Prove that $\{e^{i2\pi kn/N}\}_{0\leqslant k<N}$ is an orthogonal family and thus an orthogonal basis of $\mathbb{C}^N$. What renormalization factor is needed to obtain an orthonormal basis?

3.9 [2] Suppose that $\hat{f}$ has a support in $[-(n+1)\pi/s,\, -n\pi/s]\cup[n\pi/s,\, (n+1)\pi/s]$ and that $f(t)$ is real. Find an interpolation formula that recovers $f(t)$ from $\{f(ns)\}_{n\in\mathbb{Z}}$.

3.10 [3] Suppose that $\hat{f}$ has a support in $[-\pi/s,\, \pi/s]$.
  (a) Give the filter $\phi_s(t)$ such that for any f,

$$\forall n\in\mathbb{Z}, \quad \tilde{f}(ns) = \int_{(n-1/2)s}^{(n+1/2)s} f(t)\,dt = f\star\phi_s(ns).$$

  (b) Show that $\tilde{f}(t)=f\star\phi_s(t)$ can be recovered from $\{\tilde{f}(ns)\}_{n\in\mathbb{Z}}$ with an interpolation formula.
  (c) Reconstruct f from $\tilde{f}$ by inverting ϕ_s.
  (d) Prove that the reconstruction of $f(t)$ from $\{\tilde{f}(ns)\}_{n\in\mathbb{Z}}$ is stable.

3.11 [2] The linear box spline $\phi(t)$ is defined in (3.31) for $m=1$.
  (a) Give an analytical formula for $\phi(t)$ and specify its support.
  (b) Prove with (7.20) that $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of the space of finite-energy functions that are continuous and linear on intervals $[ns,\, (n+1)]$ for $n\in\mathbb{Z}$.
  (c) Does the dual filter $\tilde{\phi}(t)$ have a compact support? Compute its graph numerically.

3.12 [1] If $f[n]$ is defined for $0\leqslant n<N$, prove that $|\hat{f}[k]|\leqslant\sum_{n=0}^{N-1}|f[n]|$ for any $0\leqslant<N$.

3.13 [2] The discrete and periodic total variation is

$$\|f\|_V = \sum_{n=0}^{N-1}|f[n]-f[n-1]| + |f[N-1]-f[0]|.$$

  (a) Prove that $\|f\|_V = \sum_{n=0}^{N-1}|f\circledast h[n]|$ where $h[n]$ is a filter and specify $\hat{h}[k]$.
  (b) Derive an upper bound of $|\hat{f}[k]|$ as a function of k^{-1}.

3.14 [1] Let $g[n]=(-1)^n\,h[n]$. Relate $\hat{g}(\omega)$ to $\hat{h}(\omega)$. If h is a low-pass filter, what kind of filter is g?

3.15 [2] Prove the convolution Theorem 3.7.

3.16 [2] Let h^{-1} be the inverse of h defined by $h\star h^{-1}[n]=\delta[n]$.
  (a) Compute $\hat{h}^{-1}(\omega)$ as a function of $\hat{h}(\omega)$.
  (b) Prove that if h has a finite support, then h^{-1} has a finite support if and only if $h[n]=\delta[n-p]$ for some $p\in\mathbb{Z}$.

3.17 [1] *All pass filters*:
  (a) Verify that

$$\hat{h}(\omega) = \prod_{k=1}^{K} \frac{a_k^* - e^{-i\omega}}{1 + a_k\, e^{i\omega}}$$

  is an all-pass filter; that is, $|\hat{h}(\omega)| = 1$.
  (b) Prove that $\{h[n-m]\}_{m\in\mathbb{Z}}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$.

3.18 [2] *Recursive filters*:
  (a) Compute the Fourier transform of $h[n] = a^n\, \mathbf{1}_{[0,+\infty)}[n]$ for $|a| < 1$. Compute the inverse Fourier transform of $\hat{h}(\omega) = (1 - a\, e^{-i\omega})^{-p}$.
  (b) Suppose that $g = f \star h$ is calculated by a recursive equation with real coefficients

$$\sum_{k=0}^{K} a_k f[n-k] = \sum_{k=0}^{M} b_k\, g[n-k].$$

  Write $\hat{h}(\omega)$ as a function of the parameters a_k and b_k.
  (c) Show that h is a stable filter if and only if the equation $\sum_{k=0}^{M} b_k\, z^{-k} = 0$ has roots with a modulus strictly smaller than 1.

3.19 [1] *Discrete interpolation*. Let $\hat{f}[k]$ be the DFT of a signal $f[n]$ of size N. We define a signal $\tilde{f}[n]$ of size $2N$ by $\widehat{\tilde{f}}[N/2] = \widehat{\tilde{f}}[3N/2] = \widehat{f}[N/2]$ and

$$\widehat{\tilde{f}}[k] = \begin{cases} 2\hat{f}[k] & \text{if } 0 \leqslant k < N/2 \\ 0 & \text{if } N/2 < k < 3N/2 \\ 2\hat{f}[k-N] & \text{if } 3N/2 < k < 2N. \end{cases}$$

  Prove that $\tilde{f}$ is an interpolation of f that satisfies $\tilde{f}[2n] = f[n]$.

3.20 [2] *Decimation*. Let $x[n] = y[Mn]$ with $M > 1$.
  (a) Show that $\hat{x}(\omega) = M^{-1} \sum_{k=0}^{M-1} \hat{y}\left(M^{-1}(\omega - 2k\pi)\right)$.
  (b) Give a sufficient condition on $\hat{y}(\omega)$ to recover y from x and give the interpolation formula that recovers $y[n]$ from x.

3.21 [3] We want to compute numerically the Fourier transform of $f(t)$. Let $f_d[n] = f(ns)$ and $f_p[n] = \sum_{p=-\infty}^{+\infty} f_d[n-pN]$.
  (a) Prove that the DFT of $f_p[n]$ is related to the Fourier series of $f_d[n]$ and to the Fourier transform of $f(t)$ by

$$\hat{f}_p[k] = \hat{f}_d\left(\frac{2\pi k}{N}\right) = \frac{1}{s} \sum_{l=-\infty}^{+\infty} \hat{f}\left(\frac{2k\pi}{Ns} - \frac{2l\pi}{s}\right).$$

  (b) Suppose that $|f(t)|$ and $|\hat{f}(\omega)|$ are negligible when $t \notin [-t_0, t_0]$ and $\omega \notin [-\omega_0, \omega_0]$. Relate N and s to t_0 and ω_0 so that one can compute

an approximate value of $\hat{f}(\omega)$ for all $\omega \in \mathbb{R}$ by interpolating the samples $\hat{f}_p[k]$. Is it possible to compute exactly $\hat{f}(\omega)$ with such an interpolation formula?

(c) Let $f(t) = \left(\sin(\pi t)/(\pi t) \right)^4$. What is the support of $\hat{f}$? Sample f appropriately and compute $\hat{f}$ numerically with an FFT algorithm.

3.22 [2] The analytic part $f_a[n]$ of a real discrete signal $f[n]$ of size N is defined by

$$\hat{f}_a[k] = \begin{cases} \hat{f}[k] & \text{if } k = 0,\ N/2 \\ 2\hat{f}[k] & \text{if } 0 < k < N/2 \\ 0 & \text{if } N/2 < k < N. \end{cases}$$

(a) Compute $f_a[n]$ for $f[n] = \cos(2\pi kn/N)$ with $0 < k < N/2$.

(b) Prove that the real part $g[n] = \mathrm{Re}(f[n])$ is what satisfies

$$\hat{g}[k] = (\hat{f}[k] + \hat{f}^*[-k])/2.$$

(c) Prove that $\mathrm{Re}(f_a) = f$.

3.23 [1] Prove that if $\{e_{k_1}[n_1]\}_{0 \leqslant k_1 < N_1}$ is an orthonormal basis of $\mathbf{C}^{N_1}$ and $\{e_{k_2}[n_2]\}_{0 \leqslant k_1 < N_1}$ is an orthonormal basis of $\mathbf{C}^{N_2}$, then $\{e_{k_1}[n_1]$ $e_{k_2}[n_2]\}_{0 \leqslant k_1 < N_1, 0 \leqslant k_2 < N_2}$ is an orthogonal basis of the space $\mathbb{C}^N = \mathbb{C}^{N_1 N_2}$ of images $f[n_1, n_2]$ of $N = N_1 N_2$ pixels.

3.24 [2] Let $h[n_1, n_2]$ be a nonseparable filter that is nonzero for $0 \leqslant n_1, n_2 < M$. Let $f[n_1, n_2]$ be a square image defined for $0 \leqslant n_1, n_2 \leqslant LM$ of $N = (LM)^2$ pixels. Describe an overlap–add algorithm to compute $g[n_1, n_2] = f \star h[n_1, n_2]$. By using an FFT that requires $KP \log P$ operators to compute the Fourier transform of an image of P pixels, how many operations does your algorithm require? If $K = 6$, for what range of M is it better to compute the convolution with a direct summation?

3.25 [2] Let $f[n_1, n_2, n_3]$ be a three-dimensional signal of size $N = N_1 N_2 N_3$. The discrete Fourier transform is defined as a decomposition in a separable discrete Fourier basis. Give a separable algorithm that decomposes f in this basis with $KN \log N$ operations, by using a one-dimensional FFT algorithm that requires $KP \log P$ operations for a one-dimensional signal of size P.