# Mathematical Complements

Important mathematical concepts are reviewed without proof. Sections A.1 through A.5 we present results of real and complex analysis, including properties of Hilbert spaces, bases, and linear operators [59]. Random vectors and Dirac distributions are covered in the last two sections.

## A.1 FUNCTIONS AND INTEGRATION

Analog signals are modeled by measurable functions. We first give the main theorems of Lebesgue integration. A function $f$ is said to be *integrable* if $\int_{-\infty}^{+\infty} |f(t)|\, dt < +\infty$. The space of integrable functions is written as $\mathbf{L}^1(\mathbb{R})$. Two functions $f_1$ and $f_2$ are equal in $\mathbf{L}^1(\mathbb{R})$ if

$$\int_{-\infty}^{+\infty} |f_1(t) - f_2(t)|\, dt = 0.$$

This means that $f_1(t)$ and $f_2(t)$ can differ only on a set of points of measure 0. We say that they are *almost everywhere* equal.

The Fatou lemma (A.1) gives an inequality when taking a limit under the Lebesgue integral of positive functions.

**Lemma A.1:** *Fatou.* Let $\{f_n\}_{n\in\mathbb{N}}$ be a family of positive functions $f_n(t) \geqslant 0$. If $\lim_{n\to+\infty} f_n(t) = f(t)$ almost everywhere, then

$$\int_{-\infty}^{+\infty} f(t)\, dt \leqslant \lim_{n\to+\infty} \int_{-\infty}^{+\infty} f_n(t)\, dt.$$

The dominated convergence theorem (A.1) supposes the existence of an integrable upper bound to obtain an equality when taking a limit under a Lebesgue integral.

**Theorem A.1:** *Dominated Convergence.* Let $\{f_n\}_{n\in\mathbb{N}}$ be a family such that almost everywhere $\lim_{n\to+\infty} f_n(t) = f(t)$. If

$$\forall n \in \mathbb{N}\ |f_n(t)| \leqslant g(t) \quad \text{and} \quad \int_{-\infty}^{+\infty} g(t)\, dt < +\infty, \tag{A.1}$$

then $f$ is integrable and

$$\int_{-\infty}^{+\infty} f(t)\, dt = \lim_{n\to+\infty} \int_{-\infty}^{+\infty} f_n(t)\, dt.$$

**753**

The Fubini theorem (A.2) gives a sufficient condition for inverting the order of integrals in multidimensional integrations.

**Theorem A.2:** *Fubini.* If $\int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} |f(x_1, x_2)| dx_1 \right) dx_2 < +\infty$, then

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x_1, x_2)\, dx_1\, dx_2 = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(x_1, x_2)\, dx_1 \right) dx_2$$

$$= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(x_1, x_2)\, dx_2 \right) dx_1.$$

### *Convexity*
A function $f(t)$ is said to be *convex* if for all $p_1, p_2 > 0$ with $p_1 + p_2 = 1$ and all $(t_1, t_2) \in \mathbb{R}^2$,

$$f(p_1 t_1 + p_2 t_2) \leqslant p_1 f(t_1) + p_2 f(t_2).$$

The function $-f$ satisfies the reverse inequality and is said to be *concave*. If $f$ is convex, then the Jensen inequality generalizes this property for any $p_k \geqslant 0$ with $\sum_{k=1}^{K} p_k = 1$ and any $t_k \in \mathbb{R}$:

$$f\left( \sum_{k=1}^{K} p_k t_k \right) \leqslant \sum_{k=1}^{K} p_k f(t_k). \tag{A.2}$$

Theorem A.3 relates the convexity to the sign of the second-order derivative.

**Theorem A.3.** If $f$ is twice differentiable, then $f$ is convex if and only if $f''(t) \geqslant 0$ for all $t \in \mathbb{R}$.

The notion of convexity also applies to sets $\Omega \subset \mathbb{R}^n$. This set is convex if for all $p_1, p_2 > 0$ with $p_1 + p_2 = 1$ and all $(x_1, x_2) \in \Omega^2$, then $p_1 x_1 + p_2 x_2 \in \Omega$. If $\Omega$ is not convex, then its convex hull is defined as the smallest convex set that includes $\Omega$.

## A.2  BANACH AND HILBERT SPACES

### *Banach Space*
Signals are often considered as vectors. To define a distance, we work within a vector space **H** that admits a norm. A norm satisfies the following properties:

$$\forall f \in \mathbf{H}, \quad \|f\| \geqslant 0 \quad \text{and} \quad \|f\| = 0 \iff f = 0, \tag{A.3}$$

$$\forall \lambda \in \mathbb{C} \; \|\lambda f\| = |\lambda| \, \|f\|, \tag{A.4}$$

$$\forall f, g \in \mathbf{H}, \quad \|f + g\| \leqslant \|f\| + \|g\|. \tag{A.5}$$

With such a norm, the convergence of $\{f_n\}_{n \in \mathbb{N}}$ to $f$ in **H** means that

$$\lim_{n \to +\infty} f_n = f \iff \lim_{n \to +\infty} \|f_n - f\| = 0.$$

To guarantee that we remain in **H** when taking such limits, we impose a completeness property, using the notion of *Cauchy sequences*. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if for any $\varepsilon > 0$, if $n$ and $p$ are large enough, then $\|f_n - f_p\| < \varepsilon$. The space **H** is said to be *complete* if every Cauchy sequence in **H** converges to an element of **H**.

---

**EXAMPLE A.1**

For any integer $p \geqslant 1$, we define over discrete sequences $f[n]$

$$\|f\|_p = \left( \sum_{n=-\infty}^{+\infty} |f[n]|^p \right)^{1/p}.$$

The space $\ell^p = \{f \,:\, \|f\|_p < +\infty\}$ is a Banach space with the norm $\|f\|_p$.

---

**EXAMPLE A.2**

The space $\mathbf{L^p}(\mathbb{R})$ is composed of the measurable functions $f$ on $\mathbb{R}$ for which

$$\|f\|_p = \left( \int_{-\infty}^{+\infty} |f(t)|^p \, dt \right)^{1/p} < +\infty.$$

This integral defines a norm for $p \geqslant 1$ and $\mathbf{L^p}(\mathbb{R})$ is a Banach space, provided one identifies functions that are equal almost everywhere.

---

### Hilbert Space

Whenever possible, we work in a space that has an inner product to define angles and orthogonality. A *Hilbert space* **H** is a Banach space with an inner product. The inner product of two vectors $\langle f, g \rangle$ is linear with respect to its first argument:

$$\forall \lambda_1, \lambda_2 \in \mathbb{C}, \quad \langle \lambda_1 f_1 + \lambda_2 f_2, g \rangle = \lambda_1 \langle f_1, g \rangle + \lambda_2 \langle f_2, g \rangle. \tag{A.6}$$

It has an Hermitian symmetry:

$$\langle f, g \rangle = \langle g, f \rangle^*.$$

Moreover,

$$\langle f, f \rangle \geqslant 0 \quad \text{and} \quad \langle f, f \rangle = 0 \Leftrightarrow f = 0.$$

One can verify that $\|f\| = \langle f, f \rangle^{1/2}$ is a norm. The positivity (A.3) implies the Cauchy-Schwarz inequality:

$$|\langle f, g \rangle| \leqslant \|f\| \, \|g\|, \tag{A.7}$$

which is an equality if and only if $f$ and $g$ are linearly dependent.

We write $\mathbf{V}^\perp$ the orthogonal complement of a subspace **V** of **H**. All vectors of **V** are orthogonal to all vectors of $\mathbf{V}^\perp$ and $\mathbf{V} \oplus \mathbf{V}^\perp = \mathbf{H}$.

---

**EXAMPLE A.3**

An inner product between discrete signals $f[n]$ and $g[n]$ can be defined by

$$\langle f, g \rangle = \sum_{n=-\infty}^{+\infty} f[n] g^*[n].$$

It corresponds to an $\ell^2(\mathbb{Z})$ norm:

$$\|f\|^2 = \langle f, f \rangle = \sum_{n=-\infty}^{+\infty} |f[n]|^2.$$

The space $\ell^2(\mathbb{Z})$ of finite-energy sequences is therefore a Hilbert space. The Cauchy-Schwarz inequality (A.7) proves that

$$\left| \sum_{n=-\infty}^{+\infty} f[n] g^*[n] \right| \leq \left( \sum_{n=-\infty}^{+\infty} |f[n]|^2 \right)^{1/2} \left( \sum_{n=-\infty}^{+\infty} |g[n]|^2 \right)^{1/2}.$$

---

**EXAMPLE A.4**

Over analog signals $f(t)$ and $g(t)$, an inner product can be defined by

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t) g^*(t) \, dt.$$

The resulting norm is

$$\|f\| = \left( \int_{-\infty}^{+\infty} |f(t)|^2 \, dt \right)^{1/2}.$$

The space $\mathbf{L}^2(\mathbb{R})$ of finite-energy functions is thus also a Hilbert space. In $\mathbf{L}^2(\mathbb{R})$, the Cauchy-Schwarz inequality (A.7) is

$$\left| \int_{-\infty}^{+\infty} f(t) g^*(t) \, dt \right| \leq \left( \int_{-\infty}^{+\infty} |f(t)|^2 \, dt \right)^{1/2} \left( \int_{-\infty}^{+\infty} |g(t)|^2 \, dt \right)^{1/2}.$$

Two functions $f_1$ and $f_2$ are equal in $\mathbf{L}^2(\mathbb{R})$ if

$$\|f_1 - f_1\|^2 = \int_{-\infty}^{+\infty} |f_1(t) - f_2(t)|^2 \, dt = 0,$$

which means that $f_1(t) = f_2(t)$ for almost all $t \in \mathbb{R}$.

## A.3 BASES OF HILBERT SPACES

### *Orthonormal Basis*

A family $\{e_n\}_{n\in\mathbb{N}}$ of a Hilbert space $\mathbf{H}$ is orthogonal if for $n \neq p$,

$$\langle e_n, e_p \rangle = 0.$$

If for $f \in \mathbf{H}$ there exists a sequence $a[n]$ such that

$$\lim_{N\to+\infty} \left\| f - \sum_{n=0}^{N} a[n]\, e_n \right\| = 0,$$

then $\{e_n\}_{n\in\mathbb{N}}$ is said to be an *orthogonal basis* of $\mathbf{H}$. The orthogonality implies that necessarily $a[n] = \langle f, e_n \rangle / \|e_n\|^2$, and we write

$$f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n \rangle}{\|e_n\|^2}\, e_n. \tag{A.8}$$

A Hilbert space that admits an orthogonal basis is said to be *separable*.

The basis is *orthonormal* if $\|e_n\| = 1$ for all $n \in \mathbb{N}$. Computing the inner product of $g \in \mathbf{H}$ with each side of (A.8) yields a Parseval equation for orthonormal bases:

$$\langle f, g \rangle = \sum_{n=0}^{+\infty} \langle f, e_n \rangle\, \langle g, e_n \rangle^*. \tag{A.9}$$

When $g = f$, we get an energy conservation called the *Plancherel formula*:

$$\|f\|^2 = \sum_{n=0}^{+\infty} |\langle f, e_n \rangle|^2. \tag{A.10}$$

The Hilbert spaces $\ell^2(\mathbb{Z})$ and $\mathbf{L}^2(\mathbb{R})$ are separable. For example, the family of translated Diracs $\{e_n[k] = \delta[k - n]\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$. Chapters 7 and 8 construct orthonormal bases of $\mathbf{L}^2(\mathbb{R})$ with wavelets, wavelet packets, and local cosine functions.

### *Riesz Bases*

In an infinite-dimensional space, if we loosen up the orthogonality requirement, we must still impose a partial-energy equivalence to guarantee the stability of the basis. A family of vectors $\{e_n\}_{n\in\mathbb{N}}$ is said to be a Riesz basis of $\mathbf{H}$ if it is linearly independent and if there exist $B \geqslant A > 0$ such that

$$\forall f \in \mathbf{H}, \quad A\|f\|^2 \leqslant \sum_{n=0}^{+\infty} |\langle f, e_n \rangle|^2 \leqslant B\,\|f\|^2. \tag{A.11}$$

Section 5.1.2 proves that there exists a unique dual basis $\{\tilde{e}_n\}_{n\in\mathbb{N}}$ characterized by biorthogonality relations

$$\forall (n, p) \in \mathbb{N}^2, \quad \langle e_n, \tilde{e}_p \rangle = \delta[n - p], \tag{A.12}$$

and that satisfies

$$\forall f \in \mathbf{H}, \quad f = \sum_{n=0}^{+\infty} \langle f, \tilde{e}_n \rangle e_n = \sum_{n=0}^{+\infty} \langle f, e_n \rangle \tilde{e}_n.$$

## A.4 LINEAR OPERATORS

Classical signal-processing algorithms are mostly based on linear operators. An operator $U$ from a Hilbert space $\mathbf{H}_1$ to another Hilbert space $\mathbf{H}_2$ is linear if

$$\forall \lambda_1, \lambda_2 \in \mathbb{C}, \ \forall f_1, f_2 \in \mathbf{H}, \ U(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 U(f_1) + \lambda_2 U(f_2).$$

The null space and image spaces of $U$ are defined by

$$\mathbf{Null} U = \{h \in \mathbf{H}_1 \ : \ Uh = 0\} \quad \text{and} \quad \mathbf{Im} U = \{g \in \mathbf{H}_2 \ : \ \exists h \in H_1, g = Uh\}.$$

### Supremum Norm
The supremum operator norm of $U$ is defined by

$$\|U\|_S = \sup_{f \in \mathbf{H}_1} \frac{\|Uf\|}{\|f\|}. \tag{A.13}$$

If this norm is finite, then $U$ is continuous. Indeed, $\|Uf - Ug\|$ becomes arbitrarily small if $\|f - g\|$ is sufficiently small.

### Adjoint
The *adjoint* of $U$ is the operator $U^*$ from $\mathbf{H}_2$ to $\mathbf{H}_1$ such that for any $f \in \mathbf{H}_1$ and $g \in \mathbf{H}_2$,

$$\langle Uf, g \rangle = \langle f, U^*g \rangle.$$

The null and image spaces of adjoint operators are orthogonal complement:

$$\mathbf{Null} U = (\mathbf{Im} U^*)^\perp \quad \text{and} \quad \mathbf{Im} U = (\mathbf{Null} U^*)^\perp.$$

When $U$ is defined from $\mathbf{H}$ into itself, it is *self-adjoint* if $U = U^*$. It is also said to be *symmetric*.

A nonzero vector $f \in \mathbf{H}$ is a called an *eigenvector* if there exists an *eigenvalue* $\lambda \in \mathbb{C}$ such that

$$Uf = \lambda f.$$

In a finite-dimensional Hilbert space (Euclidean space), a self-adjoint operator is always diagonalized by an orthogonal basis $\{e_n\}_{0 \leqslant n < N}$ of eigenvectors

$$Ue_n = \lambda_n e_n.$$

When $U$ is self-adjoint, the eigenvalues $\lambda_n$ are real. For any $f \in \mathbf{H}$,

$$Uf = \sum_{n=0}^{N-1} \langle Uf, e_n \rangle e_n = \sum_{n=0}^{N-1} \lambda_n \langle f, e_n \rangle e_n.$$

For any $U$, the operators $U^*U$ and $UU^*$ are self-adjoint and have the same eigenvalues. These eigenvalues are called *singular values* of $U$.

In an infinite-dimensional Hilbert space, the eigenvalues of symmetric operators are generalized by introducing the spectrum of the operator.

### Orthogonal Projector

Let $\mathbf{V}$ be a subspace of $\mathbf{H}$. A *projector* $P_\mathbf{V}$ on $\mathbf{V}$ is a linear operator that satisfies

$$\forall f \in \mathbf{H}, \ P_\mathbf{V} f \in \mathbf{V} \quad \text{and} \quad \forall f \in \mathbf{V}, \ P_\mathbf{V} f = f.$$

The projector $P_\mathbf{V}$ is *orthogonal* if

$$\forall f \in \mathbf{H}, \ \forall g \in \mathbf{V}, \ \langle f - P_\mathbf{V} f, g \rangle = 0.$$

The properties in Theorem A.4 are often used.

**Theorem A.4.** If $P_\mathbf{V}$ is a projector on $\mathbf{V}$, then the following statements are equivalent:

1. $P_\mathbf{V}$ is orthogonal.

2. $P_\mathbf{V}$ is self-adjoint.

3. $\|P_\mathbf{V}\|_S = 1$.

4. $\forall f \in \mathbf{H}, \quad \|f - P_\mathbf{V} f\| = \min_{g \in \mathbf{V}} \|f - g\|$.

5. If $\{e_n\}_{n \in \mathbb{N}}$ is an orthogonal basis of $\mathbf{V}$, then

$$P_\mathbf{V} f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n \rangle}{\|e_n\|^2} e_n. \tag{A.14}$$

6. If $\{e_n\}_{n \in \mathbb{N}}$ and $\{\tilde{e}_n\}_{n \in \mathbb{N}}$ are biorthogonal Riesz bases of $\mathbf{V}$, then

$$P_\mathbf{V} f = \sum_{n=0}^{+\infty} \langle f, e_n \rangle \tilde{e}_n = \sum_{n=0}^{+\infty} \langle f, \tilde{e}_n \rangle e_n. \tag{A.15}$$

### Limit and Density Argument

Let $\{U_n\}_{n \in \mathbb{N}}$ be a sequence of linear operators from $\mathbf{H}$ to $\mathbf{H}$. Such a sequence *converges weakly* to a linear operator $U_\infty$ if

$$\forall f \in \mathbf{H}, \quad \lim_{n \to +\infty} \|U_n f - U_\infty f\| = 0.$$

To find the limit of operators it is often preferable to work in a well-chosen subspace $\mathbf{V} \subset \mathbf{H}$ that is dense. A space $\mathbf{V}$ is *dense* in $\mathbf{H}$ if for any $f \in \mathbf{H}$ there exist $\{f_m\}_{m \in \mathbb{N}}$ with $f_m \in \mathbf{V}$ such that

$$\lim_{m \to +\infty} \|f - f_m\| = 0.$$

Theorem A.5 justifies this approach.

**Theorem A.5:** *Density.* Let $\mathbf{V}$ be a dense subspace of $\mathbf{H}$. Suppose that there exists $C$ such that $\|U_n\|_S \leqslant C$ for all $n \in \mathbb{N}$. If

$$\forall f \in \mathbf{V}, \quad \lim_{n \to +\infty} \|U_n f - U_\infty f\| = 0,$$

then

$$\forall f \in \mathbf{H}, \quad \lim_{n \to +\infty} \|U_n f - U_\infty f\| = 0.$$

## A.5 SEPARABLE SPACES AND BASES

### *Tensor Product*

Tensor products are used to extend spaces of one-dimensional signals into spaces of multidimensional signals. A tensor product $f_1 \otimes f_2$ between vectors of two Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ satisfies the following properties.

$$Linearity: \ \forall \lambda \in \mathbb{C}, \ \ \lambda(f_1 \otimes f_2) = (\lambda f_1) \otimes f_2 = f_1 \otimes (\lambda f_2). \tag{A.16}$$

$$\begin{aligned} Distributivity: \ (f_1 + g_1) \otimes (f_2 + g_2) &= (f_1 \otimes f_2) + (f_1 \otimes g_2) \\ &+ (g_1 \otimes f_2) + (g_1 \otimes g_2). \end{aligned} \tag{A.17}$$

This tensor product yields a new Hilbert space $\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2$ that includes all vectors of the form $f_1 \otimes f_2$ where $f_1 \in \mathbf{H}_1$ and $f_2 \in \mathbf{H}_2$, as well as linear combinations of such vectors. An inner product in $\mathbf{H}$ is derived from inner products in $\mathbf{H}_1$ and $\mathbf{H}_2$ by

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{\mathbf{H}} = \langle f_1, g_1 \rangle_{\mathbf{H}_1} \langle f_2, g_2 \rangle_{\mathbf{H}_2}. \tag{A.18}$$

### *Separable Bases*

Theorem A.6 proves that orthonormal bases of tensor product spaces are obtained with separable products of two orthonormal bases. It provides a simple procedure for transforming bases for one-dimensional signals into separable bases for multidimensional signals.

**Theorem A.6.** Let $\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2$. If $\{e_n^1\}_{n \in \mathbb{N}}$ and $\{e_n^2\}_{n \in \mathbb{N}}$ are two Riesz bases, respectively, of $\mathbf{H}_1$ and $\mathbf{H}_2$, then $\{e_n^1 \otimes e_m^2\}_{(n,m) \in \mathbb{N}^2}$ is a Riesz basis of $\mathbf{H}$. If the two bases are orthonormal, then the tensor product basis is also orthonormal.

### EXAMPLE A.5

A product of functions $f \in \mathbf{L}^2(\mathbb{R})$ and $g \in \mathbf{L}^2(\mathbb{R})$ defines a tensor product:

$$f(x_1) g(x_2) = f \otimes g(x_1, x_2).$$

Let $\mathbf{L}^2(\mathbb{R}^2)$ be the space of $h(x_1, x_2)$ such that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |h(x_1, x_2)|^2 \, dx_1 \, dx_2 < +\infty.$$

One can verify that $\mathbf{L}^2(\mathbb{R}^2) = \mathbf{L}^2(\mathbb{R}) \otimes \mathbf{L}^2(\mathbb{R})$. Theorem A.6 proves that if $\{\psi_n(t)\}_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathbf{L}^2(\mathbb{R})$, then $\{\psi_{n_1}(x_1) \psi_{n_2}(x_2)\}_{(n_1, n_2) \in \mathbb{N}^2}$ is an orthonormal basis of $\mathbf{L}^2(\mathbb{R}^2)$.

### EXAMPLE A.6

A product of discrete signals $f \in \ell^2(\mathbb{Z})$ and $g \in \ell^2(\mathbb{Z})$ also defines a tensor product:

$$f[n_1] g[n_2] = f \otimes g[n_1, n_2].$$

The space $\ell^2(\mathbb{Z}^2)$ of images $h[n_1, n_2]$ such that

$$\sum_{n_1 = -\infty}^{+\infty} \sum_{n_2 = -\infty}^{+\infty} |h[n_1, n_2]|^2 < +\infty$$

is also decomposed as a tensor product $\ell^2(\mathbb{Z}^2) = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$. Thus, orthonormal bases can be constructed with separable products.

## A.6  RANDOM VECTORS AND COVARIANCE OPERATORS

A class of signals can be modeled by a random process (random vector) with realizations that are the signals in the class. Finite discrete signals $f$ are represented by a random vector $Y$ where $Y[n]$ is a random variable for each $0 \leq n < N$. For a review of elementary probability theory for signal processing, the reader may consult [53, 56].

### *Covariance Operator*
If $p(x)$ is the probability density of a random variable $X$, the expected value is

$$E\{X\} = \int x p(x) \, dx,$$

and the variance is $\sigma^2 = E\{|X - E\{X\}|^2\}$. The covariance of two random variables $X_1$ and $X_2$ is

$$\mathrm{Cov}\,(X_1, X_2) = E\left\{ \left(X_1 - E\{X_1\}\right)\left(X_2 - E\{X_2\}\right)^* \right\}. \tag{A.19}$$

The covariance matrix of a random vector $Y$ is composed of the $N^2$ covariance values

$$R_Y[n, m] = \text{Cov}\Big(Y[n], Y[m]\Big).$$

It defines the covariance operator $K_Y$, which transforms any $h[n]$ into

$$K_Y h[n] = \sum_{m=0}^{N-1} R_Y[n, m]\, h[m].$$

For any $h$ and $g$,

$$\langle Y, h \rangle = \sum_{n=0}^{N-1} Y[n]\, h^*[n] \quad \text{and} \quad \langle Y, g \rangle = \sum_{n=0}^{N-1} Y[n]\, g^*[n]$$

are random variables, and

$$\text{Cov}\Big(\langle Y, h \rangle, \langle Y, g \rangle\Big) = \langle K_Y g, h \rangle. \tag{A.20}$$

Thus, the covariance operator specifies the covariance of linear combinations of the process values. If $E\{Y[n]\} = 0$ for all $0 \leqslant n < N$, then $E\{\langle Y, h \rangle\} = 0$ for all $h$.

### Karhunen-Loève Basis

The covariance operator $K_Y$ is self-adjoint because $R_Y[n, m] = R_Y^*[m, n]$ and positive because

$$\langle K_Y h, h \rangle = E\{|\langle Y, h \rangle - E\{\langle Y, h \rangle\}|^2\} \geqslant 0. \tag{A.21}$$

This guarantees the existence of an orthogonal basis $\{e_k\}_{0 \leqslant k < N}$ that diagonalizes $K_Y$:

$$K_Y e_k = \sigma_k^2\, e_k.$$

This basis is called a *Karhunen-Loève basis* of $Y$, and the vectors $e_k$ are the *principal directions*. The eigenvalues are the variances

$$\sigma_k^2 = \langle K_Y e_k, e_k \rangle = E\{|\langle Y, e_k \rangle - E\{\langle Y, e_k \rangle\}|^2\}. \tag{A.22}$$

### Wide-Sense Stationarity

We say that $Y$ is *wide-sense stationary* if

$$\text{Cov}(Y[n], Y[m]) = R_Y[n, m] = R_Y[n - m]. \tag{A.23}$$

The covariance between two points depends only on the distance between these points. The operator $K_Y$ is then a convolution with a kernel $R_Y[k]$ that is defined for $-N < k < N$. A wide-sense stationary process is *circular stationary* if $R_Y[n]$ is $N$ periodic:

$$R_Y[n] = R_Y[N + n] \quad \text{for} \quad -N \leqslant n \leqslant 0. \tag{A.24}$$

This condition implies that a periodic extension of $Y[n]$ on $\mathbb{Z}$ remains wide-sense stationary on $\mathbb{Z}$. The covariance operator $K_Y$ of a circular stationary process is a discrete circular convolution. Section 3.3.1 proves that the eigenvectors of circular convolutions are the discrete Fourier vectors

$$\left\{ e_k[n] = \frac{1}{\sqrt{N}} \exp\left(\frac{i2\pi kn}{N}\right) \right\}_{0 \leqslant k < N}.$$

The discrete Fourier basis is therefore the Karhunen-Loève basis of circular stationary processes. The eigenvalues (A.22) of $K_Y$ are the discrete Fourier transform of $R_Y$ and are called the *power spectrum*,

$$\sigma_k^2 = \hat{R}_Y[k] = \sum_{n=0}^{N-1} R_Y[n] \exp\left(\frac{-i2k\pi n}{N}\right). \tag{A.25}$$

Theorem A.7 computes the power spectrum after a circular convolution.

**Theorem A.7.** Let $Z$ be a wide-sense circular stationary random vector. The random vector $Y[n] = Z \circledast h[n]$ is also wide-sense circular stationary and its power spectrum is

$$\hat{R}_Y[k] = \hat{R}_Z[k] \, |\hat{h}[k]|^2. \tag{A.26}$$

---

## A.7 DIRACS

Diracs are useful in making the transition from functions of a real variable to discrete sequences. Symbolic calculations with Diracs simplify computations, without worrying about convergence issues. This is justified by the theory of distributions [61, 64]. A Dirac $\delta$ has a support reduced to $t = 0$ and associates to any continuous function $\phi$ its value at $t = 0$,

$$\int_{-\infty}^{+\infty} \delta(t) \, \phi(t) \, dt = \phi(0). \tag{A.27}$$

### Weak Convergence
A Dirac can be obtained by squeezing an integrable function $g$ such that $\int_{-\infty}^{+\infty} g(t) \, dt = 1$. Let $g_s(t) = s^{-1} g(s^{-1}t)$. For any continuous function $\phi$,

$$\lim_{s \to 0} \int_{-\infty}^{+\infty} g_s(t) \, \phi(t) \, dt = \phi(0) = \int_{-\infty}^{+\infty} \delta(t) \, \phi(t) \, dt. \tag{A.28}$$

Thus, a Dirac can be formally defined as the limit $\delta = \lim_{s \to 0} g_s$, which must be understood in the sense of (A.28). This is called *weak convergence*. A Dirac is not a function since it is zero at $t \neq 0$, although its "integral" is equal to 1. The integral at the right of (A.28) is only a symbolic notation, which means that a Dirac applied to a continuous function $\phi$ associates its value at $t = 0$.

General distributions are defined over the space $\mathbf{C}_0^\infty$ of *test functions* that are infinitely continuously differentiable with a compact support. A distribution $d$ is a

linear form that associates to any $\phi \in \mathbf{C}_0^\infty$ a value that is written as $\int_{-\infty}^{+\infty} d(t)\phi(t)dt$. It must also satisfy some weak continuity properties [61, 64] that we do not discuss here, and that are satisfied by a Dirac. Two distributions $d_1$ and $d_2$ are equal if

$$\forall \phi \in \mathbf{C}_0^\infty, \quad \int_{-\infty}^{+\infty} d_1(t)\,\phi(t)\,dt = \int_{-\infty}^{+\infty} d_2(t)\,\phi(t)\,dt. \tag{A.29}$$

### Symbolic Calculations

The symbolic integral over a Dirac is a useful notation because it has the same properties as a usual integral, including change of variables and integration by parts. A translated Dirac $\delta_\tau(t) = \delta(t - \tau)$ has a mass concentrated at $\tau$ and

$$\int_{-\infty}^{+\infty} \phi(t)\,\delta(t-u)\,dt = \int_{-\infty}^{+\infty} \phi(t)\,\delta(u-t)\,dt = \phi(u).$$

This means that $\phi \star \delta(u) = \phi(u)$. Similarly, $\phi \star \delta_\tau(u) = \phi(u - \tau)$.

A Dirac can also be multiplied by a continuous function $\phi$ and since $\delta(t - \tau)$ is zero outside $t = \tau$, it follows that

$$\phi(t)\,\delta(t - \tau) = \phi(\tau)\,\delta(t - \tau).$$

The derivative of a Dirac is defined with an integration by parts. If $\phi$ is continuously differentiable, then

$$\int_{-\infty}^{+\infty} \phi(t)\,\delta'(t)\,dt = -\int_{-\infty}^{+\infty} \phi'(t)\,\delta(t)\,dt = -\phi'(0).$$

The $k$th derivative of $\delta$ is similarly obtained with $k$ integrations by parts. It is a distribution that associates to $\phi \in \mathbf{C}^k$,

$$\int_{-\infty}^{+\infty} \phi(t)\,\delta^{(k)}(t)\,dt = (-1)^k\,\phi^{(k)}(0).$$

The Fourier transform of $\delta$ associates to any $\mathrm{e}^{-i\omega t}$ its value at $t = 0$:

$$\hat{\delta}(\omega) = \int_{-\infty}^{+\infty} \delta(t)\,\mathrm{e}^{-i\omega t}\,dt = 1,$$

and after translation, $\hat{\delta}_\tau(\omega) = \mathrm{e}^{-i\tau\omega}$. The Fourier transform of the Dirac comb $c(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$ is therefore $\hat{c}(\omega) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-inT\omega}$. The Poisson formula (2.4) proves that

$$\hat{c}(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

This distribution equality must be understood in the sense of (A.29).