Karlsruhe Institute of Technology

High Performance Humanoid Technologies

Robotics I: Introduction to Robotics
Exercise 1: Mathematical Foundations

Tilman Daab, Tamim Asfour

https://www.humanoids.kit.edu

$$T = \begin{pmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \quad T \in SE(3) \quad \text{with } \mathbf{t} \in \mathbb{R}^3 \text{ und } R \in SO(3) \quad T = \begin{pmatrix} n_x & o_x & a_x & u_x \\ n_y & o_y & a_y & u_y \\ n_z & o_z & a_z & u_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 1: Euler Angles, RPY Angles, Quaternions

1. Let R_1 be a general 3×3 rotation matrix, $R_1 = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}$.

   i. Calculate the Euler angles $\mathbf{z\,x'z''}$ corresponding to R_1.

   ii. Calculate the RPY angles ($\mathbf{xyz}$ convention) corresponding to R_1.

2. Let R_2 be a rotation matrix, given by $R_2 = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix}$.

   Calculate the quaternion $\mathbf{q}$ that describes the rotation given by R_2.

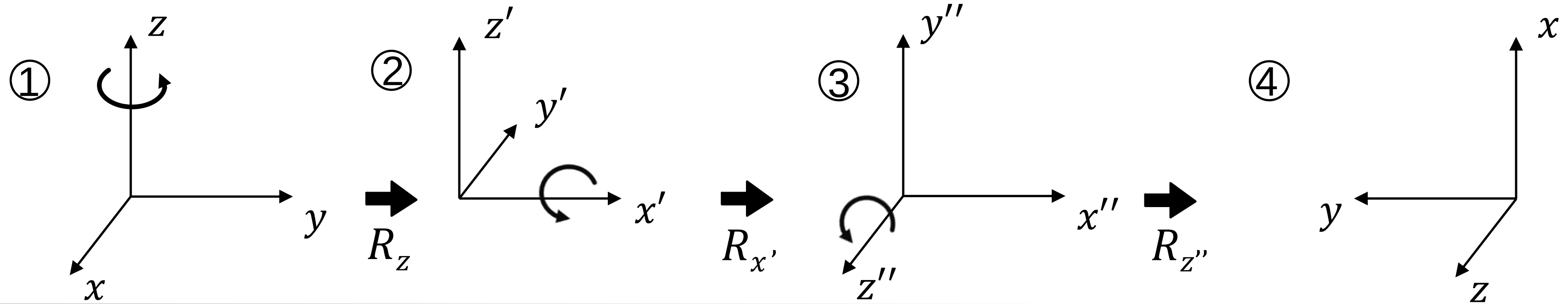
Exercise 1.1 (i): Euler Angle Convention z, x', z''

Euler angles, z, x', z'' convention

- Rotation by α around the z-axis of the BCS $\quad R_z$
- Rotation by β around the new x-axis, x′ $\quad R_{x'}$
- Rotation by γ around the new z-axis, z′′ $\quad R_{z''}$

$$R_s = R_z(\alpha)\, R_{x'}(\beta)\, R_{z''}(\gamma)$$

Important: Rotation around the new/rotated axes!



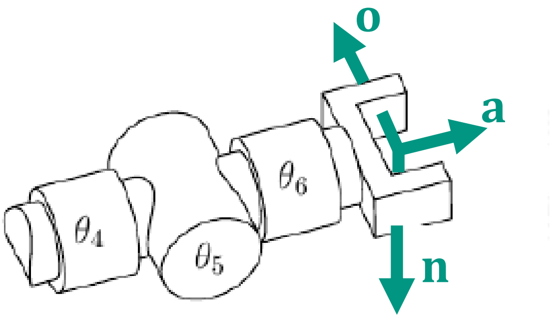
Exercise 1.1 (i): z x′z″ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

$\cos x = cx$

$\sin x = sx$

a: approach
n: normal
o: orientation



$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} =$$

$$R_z = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.1 (i): z x′z″ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

$\cos x = cx$

$\sin x = sx$

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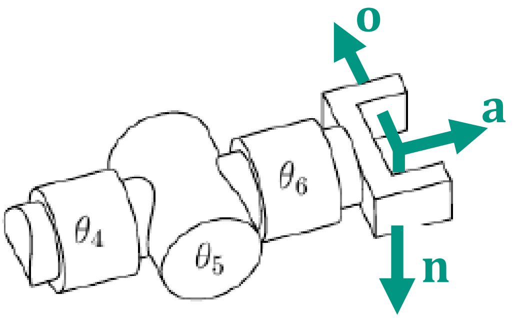


$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\beta & -s\beta \\ 0 & s\beta & c\beta \end{pmatrix} \cdot \begin{pmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.1 (i): $z\, x' z''$ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

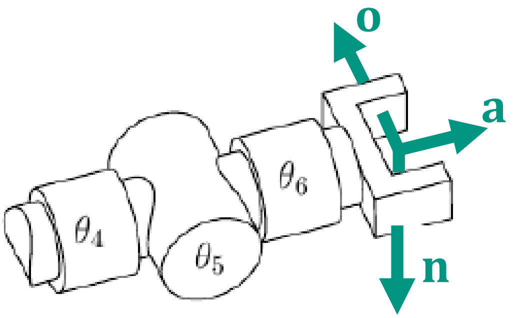
$\cos x = cx$

$\sin x = sx$

a: approach

n: normal

o: orientation



$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\beta & -s\beta \\ 0 & s\beta & c\beta \end{pmatrix} \cdot \begin{pmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c\alpha \cdot c\gamma - s\alpha \cdot c\beta \cdot s\gamma & -c\alpha \cdot s\gamma - s\alpha \cdot c\beta \cdot c\gamma & s\alpha \cdot s\beta \\ s\alpha \cdot c\gamma + c\alpha \cdot c\beta \cdot s\gamma & -s\alpha \cdot s\gamma + c\alpha \cdot c\beta \cdot c\gamma & -c\alpha \cdot s\beta \\ s\beta \cdot s\gamma & s\beta \cdot c\gamma & c\beta \end{pmatrix}$$

Exercise 1.1 (i): z x′z″ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\alpha \cdot c\gamma - s\alpha \cdot c\beta \cdot s\gamma & -c\alpha \cdot s\gamma - s\alpha \cdot c\beta \cdot c\gamma & s\alpha \cdot s\beta \\ s\alpha \cdot c\gamma + c\alpha \cdot c\beta \cdot s\gamma & -s\alpha \cdot s\gamma + c\alpha \cdot c\beta \cdot c\gamma & -c\alpha \cdot s\beta \\ s\beta \cdot s\gamma & s\beta \cdot c\gamma & c\beta \end{pmatrix}$$

Exercise 1.1 (i): $z\ x'z''$ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\alpha \cdot c\gamma - s\alpha \cdot c\beta \cdot s\gamma & -c\alpha \cdot s\gamma - s\alpha \cdot c\beta \cdot c\gamma & s\alpha \cdot s\beta \\ s\alpha \cdot c\gamma + c\alpha \cdot c\beta \cdot s\gamma & -s\alpha \cdot s\gamma + c\alpha \cdot c\beta \cdot c\gamma & -c\alpha \cdot s\beta \\ s\beta \cdot s\gamma & s\beta \cdot c\gamma & c\beta \end{pmatrix}$$

$$a_z = c\beta$$

$$a_x = s\alpha \cdot s\beta \qquad a_y = -c\alpha \cdot s\beta$$

$$n_z = s\beta \cdot s\gamma \qquad o_z = s\beta \cdot c\gamma$$

Exercise 1.1 (i): z x′z″ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\alpha \cdot c\gamma - s\alpha \cdot c\beta \cdot s\gamma & -c\alpha \cdot s\gamma - s\alpha \cdot c\beta \cdot c\gamma & s\alpha \cdot s\beta \\ s\alpha \cdot c\gamma + c\alpha \cdot c\beta \cdot s\gamma & -s\alpha \cdot s\gamma + c\alpha \cdot c\beta \cdot c\gamma & -c\alpha \cdot s\beta \\ s\beta \cdot s\gamma & s\beta \cdot c\gamma & c\beta \end{pmatrix}$$

$$a_z = c\beta = \cos\beta \Rightarrow \beta = \operatorname{acos}(a_z)$$

$$a_x = s\alpha \cdot s\beta \qquad a_y = -c\alpha \cdot s\beta$$

$$\Rightarrow \frac{a_x}{a_y} = \frac{s\alpha \cdot s\beta}{-c\alpha \cdot s\beta} = -\frac{\sin\alpha}{\cos\alpha} = -\tan\alpha \Rightarrow \alpha = \operatorname{atan}\left(-\frac{a_x}{a_y}\right)$$

$$n_z = s\beta \cdot s\gamma \qquad o_z = s\beta \cdot c\gamma$$

$$\Rightarrow \frac{n_z}{o_z} = \frac{s\beta \cdot s\gamma}{s\beta \cdot c\gamma} = \frac{\sin\gamma}{\cos\gamma} = \tan\gamma \Rightarrow \gamma = \operatorname{atan}\left(\frac{n_z}{o_z}\right)$$

Exercise 1.1 (i): z x′z″ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\alpha \cdot c\gamma - s\alpha \cdot c\beta \cdot s\gamma & -c\alpha \cdot s\gamma - s\alpha \cdot c\beta \cdot c\gamma & s\alpha \cdot s\beta \\ s\alpha \cdot c\gamma + c\alpha \cdot c\beta \cdot s\gamma & -s\alpha \cdot s\gamma + c\alpha \cdot c\beta \cdot c\gamma & -c\alpha \cdot s\beta \\ s\beta \cdot s\gamma & s\beta \cdot c\gamma & c\beta \end{pmatrix}$$

$$\alpha = \operatorname{atan}\left(-\frac{a_x}{a_y}\right)$$

$$\beta = \operatorname{acos}(a_z)$$

$$\gamma = \operatorname{atan}\left(\frac{n_z}{o_z}\right)$$

Exercise 1.1 (i): $z\,x'z''$ Euler Angles

$$R_s = R_z(\alpha) \cdot R_{x'}(\beta) \cdot R_{z''}(\gamma)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\alpha \cdot c\gamma - s\alpha \cdot c\beta \cdot s\gamma & -c\alpha \cdot s\gamma - s\alpha \cdot c\beta \cdot c\gamma & s\alpha \cdot s\beta \\ s\alpha \cdot c\gamma + c\alpha \cdot c\beta \cdot s\gamma & -s\alpha \cdot s\gamma + c\alpha \cdot c\beta \cdot c\gamma & -c\alpha \cdot s\beta \\ s\beta \cdot s\gamma & s\beta \cdot c\gamma & c\beta \end{pmatrix}$$

$$\alpha = \operatorname{atan}\left(-\frac{a_x}{a_y}\right)$$

$$\beta = \operatorname{acos}(a_z)$$

$$\gamma = \operatorname{atan}\left(\frac{n_z}{o_z}\right)$$

Assumption: $a_y \neq 0, o_z \neq 0$

Ambiguity: $a_z = \cos\beta$

Example: $a_z = 0 = \cos\beta$

$$\beta = \operatorname{acos}(0) = \frac{\pi}{2} = 90°$$

But also: $\cos\left(\frac{3\pi}{2}\right) = 0$

$$\Rightarrow \beta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \subset [0,2\pi)$$

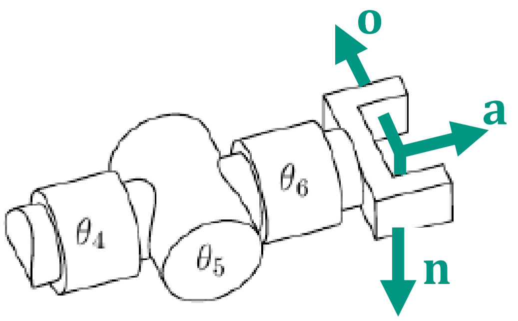
Check other matrix entries for compatibility

Exercise 1.2 (ii): RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

$$\cos x = cx$$
$$\sin x = sx$$

$\mathbf{a}$: approach
$\mathbf{n}$: normal
$\mathbf{o}$: orientation



$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} =$$

Exercise 1.2 (ii): RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

$\cos x = cx$
$\sin x = sx$

a: approach
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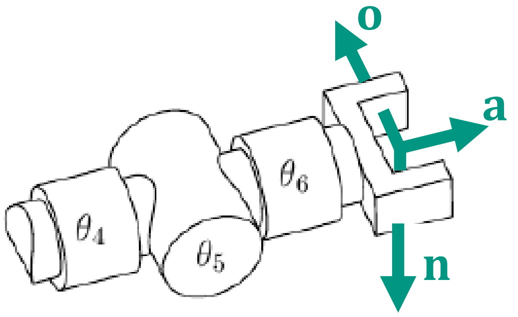

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

$$= \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

Exercise 1.2 (ii): RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

Exercise 1.2 (ii): RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

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$n_z = -s\beta$

$o_z = s\alpha \cdot c\beta$ $\qquad$ $a_z = c\alpha \cdot c\beta$

$n_x = c\beta \cdot c\gamma$ $\qquad$ $n_y = c\beta \cdot s\gamma$

Exercise 1.2 (ii): RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

$$n_z = -s\beta = \sin\beta \Rightarrow \beta = \operatorname{asin}(-n_z)$$

$$o_z = s\alpha \cdot c\beta \qquad a_z = c\alpha \cdot c\beta$$

$$\Rightarrow \frac{o_z}{a_z} = \frac{s\alpha \cdot c\beta}{c\alpha \cdot c\beta} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha \Rightarrow \alpha = \operatorname{atan}\left(\frac{o_z}{a_z}\right)$$

$$n_x = c\beta \cdot c\gamma \qquad n_y = c\beta \cdot s\gamma$$

$$\Rightarrow \frac{n_y}{n_x} = \frac{c\beta \cdot s\gamma}{c\beta \cdot c\gamma} = \frac{\sin\gamma}{\cos\gamma} = \tan\gamma \Rightarrow \gamma = \operatorname{atan}\left(\frac{n_y}{n_x}\right)$$

Exercise 1.2 (ii): RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

$$\alpha = \operatorname{atan}\left(\frac{o_z}{a_z}\right)$$

$$\beta = \operatorname{asin}(-n_z)$$

$$\gamma = \operatorname{atan}\left(\frac{n_y}{n_x}\right)$$

Exercise 1.2 (ii): RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

$$\alpha = \operatorname{atan}\left(\frac{o_z}{a_z}\right)$$

$$\beta = \operatorname{asin}(-n_z)$$

$$\gamma = \operatorname{atan}\left(\frac{n_y}{n_x}\right)$$

Assumption: $a_z \neq 0, n_x \neq 0$

Ambiguity: $n_z = -\sin\beta$

Example: $n_z = 0 = -\sin\beta$

$$\beta = \operatorname{asin} 0 = 0$$

But also: $\sin(\pi) = 0$

$$\Rightarrow \beta \in \{0, \pi\} \subset [0, 2\pi)$$

Check other matrix entries for compatibility

Exercise 1.2: Conversion to Quaternion

Let R_2 be a rotation matrix, given by $R_2 = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix}$.

Calculate the quaternion $\mathbf{q}$ that describes the rotation given by R_2.

To determine the quaternion, the **rotation axis** and **rotation angle** need to be calculated.

Exercise 1.2: Rotation Axis

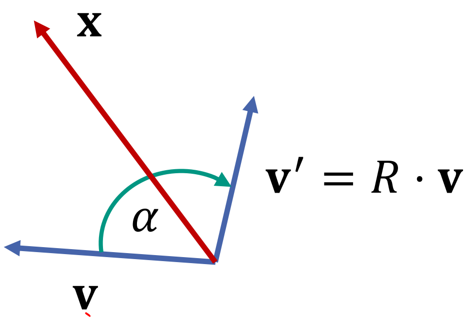
- A rotation in $\mathbb{R}^3$ can be represented by a rotation axis $\mathbf{x} \in \mathbb{R}^3$ and a rotation angle $\alpha \in \mathbb{R}$ around this axis.
- How to determine $\mathbf{x}$ from the rotation matrix $R \in \mathrm{SO}(3)$?

Exercise 1.2: Rotation Axis

- A rotation in $\mathbb{R}^3$ can be represented by a rotation axis $\mathbf{x} \in \mathbb{R}^3$ and a rotation angle $\alpha \in \mathbb{R}$ around this axis.
- How to determine $\mathbf{x}$ from the rotation matrix $R \in SO(3)$?

- Rotating $\mathbf{v} \in \mathbb{R}^3$: $\mathbf{v}' = R \cdot \mathbf{v}$
  ➔ Usually: $\mathbf{v}' \neq \mathbf{v}$

- What does $\mathbf{v}' = R \cdot \mathbf{v} = \mathbf{v}$ imply?
  ➔ $\mathbf{v} \parallel \mathbf{x}$, e.g., $\mathbf{v} = \mathbf{x}$
  Only the rotation axis is invariant under a rotation.



Exercise 1.2: Rotation Axis

- A rotation in $\mathbb{R}^3$ can be represented by a rotation axis $\mathbf{x} \in \mathbb{R}^3$ and a rotation angle $\alpha \in \mathbb{R}$ around this axis.
- How to determine $\mathbf{x}$ from the rotation matrix $R \in SO(3)$?
- Rotating $\mathbf{v} \in \mathbb{R}^3$: $\mathbf{v}' = R \cdot \mathbf{v}$
  ➔ Usually: $\mathbf{v}' \neq \mathbf{v}$
- What does $\mathbf{v}' = R \cdot \mathbf{v} = \mathbf{v}$ imply?
  ➔ $\mathbf{v} \parallel \mathbf{x}$, e.g., $\mathbf{v} = \mathbf{x}$
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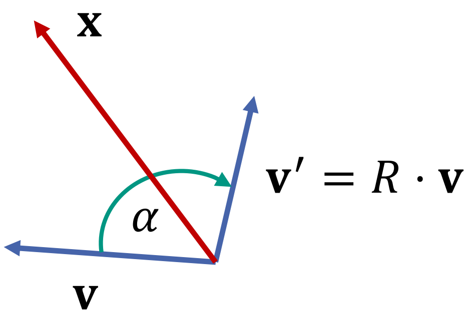


$$\begin{aligned} R \cdot \mathbf{x} &= I \cdot \mathbf{x} \\ R \cdot \mathbf{x} - I \cdot \mathbf{x} &= \mathbf{0} \\ (R - I) \cdot \mathbf{x} &= \mathbf{0} \\ (R - \lambda \cdot I) \cdot \mathbf{x} &= \mathbf{0} \qquad \mid \ \lambda = 1 \end{aligned}$$

Every rotation matrix has the **Eigen value $\lambda = 1$**. The other two Eigen values are complex conjugates of each other.

Exercise 1.2: Rotation Axis

$$(R - I) \cdot \boldsymbol{x} = \mathbf{0}$$

Exercise 1.2: Rotation Axis

$$(\quad R \quad - \quad I \quad) \cdot \boldsymbol{x} = \mathbf{0}$$

$$\left(\begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise 1.2: Rotation Axis

$$(\quad R \quad - \quad I \quad) \cdot \boldsymbol{x} = \boldsymbol{0}$$

$$\left(\begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.64 & 0.48 & -0.8 \\ -0.8 & -0.4 & 0 \\ 0.48 & 0.64 & -0.4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -0.64x_1 + 0.48x_2 - 0.8x_3 &= 0 \\ -0.8x_1 - 0.4x_2 &= 0 \\ 0.48x_1 + 0.64x_2 - 0.4x_3 &= 0 \end{aligned}$$

➔ Solve a system of equations

Exercise 1.2: Rotation Axis

$$
\begin{aligned}
&(1) \quad -0.64x_1 + 0.48x_2 - 0.8x_3 = 0 \\
&(2) \quad -0.8x_1 \; - 0.4x_2 \qquad\qquad\quad = 0 \\
&(3) \qquad 0.48x_1 + 0.64x_2 - 0.4x_3 = 0
\end{aligned}
$$

Exercise 1.2: Rotation Axis

$$
\begin{array}{ll}
(1) & -0.64x_1 + 0.48x_2 - 0.8x_3 = 0 \\
(2) & -0.8x_1 \quad - 0.4x_2 \qquad\qquad\quad = 0 \\
(3) & \quad 0.48x_1 + 0.64x_2 - 0.4x_3 = 0
\end{array}
$$

$$(4) \quad 2 \cdot (3) - (1): \qquad 1.6x_1 + 0.8x_2 = 0$$

$$(5) \quad 2 \cdot (2) + (4): \qquad 0 = 0$$

$\Rightarrow$ linear dependency: $x_1 = c$

$$(2) \quad -0.8c - 0.4x_2 = 0 \qquad \Rightarrow x_2 = \frac{0.8c}{-0.4} = -2c$$

$$(3) \quad 0.48c + 0.64(-2c) - 0.4x_3 = 0$$

$$\Rightarrow \quad -0.8c - 0.4x_3 = 0 \qquad \Rightarrow x_3 = \frac{0.8c}{-0.4} = -2c$$

Exercise 1.2: Rotation Axis

$$
\begin{aligned}
&(1) \quad -0.64x_1 + 0.48x_2 - 0.8x_3 = 0 \\
&(2) \quad -0.8x_1 - 0.4x_2 = 0 \\
&(3) \quad 0.48x_1 + 0.64x_2 - 0.4x_3 = 0
\end{aligned}
$$

$(4) \quad 2 \cdot (3) - (1): \quad 1.6x_1 + 0.8x_2 = 0$

$(5) \quad 2 \cdot (2) + (4): \quad 0 = 0 \qquad \Rightarrow$ linear dependency: $x_1 = c$

$(2) \quad -0.8c - 0.4x_2 = 0 \qquad \Rightarrow x_2 = \frac{0.8c}{-0.4} = -2c$

$(3) \quad 0.48c + 0.64(-2c) - 0.4x_3 = 0$

$\Rightarrow \quad -0.8c - 0.4x_3 = 0 \qquad \Rightarrow x_3 = \frac{0.8c}{-0.4} = -2c$

$$\boxed{\mathbf{x} = \begin{pmatrix} c \\ -2c \\ -2c \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}}$$

Exercise 1.2: Rotation Axis

Solving the equation system led to:

$$\mathbf{x}_c = \begin{pmatrix} c \\ -2c \\ -2c \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Additional constraint:

$$\|\mathbf{x}\| = 1$$

Exercise 1.2: Rotation Axis

Solving the equation system led to:

$$\mathbf{x}_c = \begin{pmatrix} c \\ -2c \\ -2c \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Additional constraint:

$$\|\mathbf{x}\| = 1$$

Rotation axis:

$$\mathbf{x} = \frac{\mathbf{x}_c}{\|\mathbf{x}_c\|} = \frac{1}{\sqrt{c^2 \cdot (1^2 + (-2)^2 + (-2)^2}} \cdot c \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$\mathbf{x} = \frac{1}{|c| \cdot \sqrt{9}} \cdot c \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \pm \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Exercise 1.2: Rotation Angle

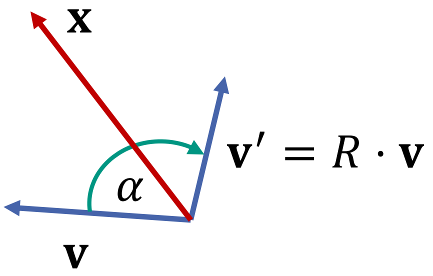
- **Given:** Rotation matrix $R \in \mathrm{SO}(3)$ and rotation axis $\mathbf{x} \in \mathbb{R}^3$
- **Unknown:** Rotation angle $\alpha \in \mathbb{R}$

- **Two approaches:**
  A) Rotation of an orthogonal vector
  B) $\mathrm{Trace}(R)$

Exercise 1.2: Rotation Angle, Approach A

- Vector $\mathbf{v} \in \mathbb{R}^3$, with $\|\mathbf{v}\| \neq \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{x} = 0$ (i.e., $\mathbf{v} \perp \mathbf{x}$),

$$\mathbf{x} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \qquad \mathbf{v} =$$



- Rotation $\mathbf{v}' = \mathrm{R} \cdot \mathbf{v}$

$$\mathbf{v}' = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix} \cdot \mathbf{v} =$$

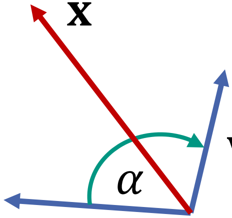
- Rotation angle between $\mathbf{v}$ and $\mathbf{v}'$

$$\cos\alpha = \frac{\mathbf{v} \cdot \mathbf{v}'}{\|\mathbf{v}\| \cdot \|\mathbf{v}'\|}$$

Exercise 1.2: Rotation Angle, Approach A

- Vector $\mathbf{v} \in \mathbb{R}^3$, with $\|\mathbf{v}\| \neq \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{x} = 0$ (i.e., $\mathbf{v} \perp \mathbf{x}$),

$$\mathbf{x} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$



- Rotation $\mathbf{v}' = \mathrm{R} \cdot \mathbf{v}$

$$\mathbf{v}' = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.48 + (-1) \cdot (-0.8) \\ 0.6 \\ 0.64 + (-1) \cdot 0.6 \end{pmatrix} = \begin{pmatrix} 1.28 \\ 0.6 \\ 0.04 \end{pmatrix}$$

- Rotation angle between $\mathbf{v}$ and $\mathbf{v}'$

$$\cos\alpha = \frac{\mathbf{v} \cdot \mathbf{v}'}{\|\mathbf{v}\| \cdot \|\mathbf{v}'\|} = \frac{0.6 - 0.04}{\sqrt{2} \cdot \sqrt{2}} = \frac{0.56}{2} = 0.28$$

$$\Rightarrow \alpha = \operatorname{acos}(0.28) \approx 1.287 \approx 73.7°$$

Exercise 1.2: Rotation Angle, Approach B

- Rotation matrix $R_{\boldsymbol{v}}(\alpha) \in \mathrm{SO}(3)$ dependent on rotation axis $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ as unit vector (i.e., $\|\mathbf{v}\| = 1$) and rotation angle $\alpha \in \mathbb{R}$:

$$R_{\mathbf{v}}(\alpha) = \begin{pmatrix} c\alpha + v_1^2(1 - c\alpha) & v_1 v_2(1 - c\alpha) - v_3 s\alpha & v_1 v_3(1 - c\alpha) + v_2 s\alpha \\ v_2 v_1(1 - c\alpha) + v_3 s\alpha & c\alpha + v_2^2(1 - c\alpha) & v_2 v_3(1 - c\alpha) - v_1 s\alpha \\ v_3 v_1(1 - c\alpha) - v_2 s\alpha & v_3 v_2(1 - c\alpha) + v_1 s\alpha & c\alpha + v_3^2(1 - c\alpha) \end{pmatrix}$$

- Trace of a matrix $A \in \mathbb{R}^{n\times n}$: Sum of the elements on the diagonal

$$\mathrm{Trace}(A) = \sum_{j=1}^{n} a_{jj}$$

Exercise 1.2: Rotation Angle, Approach B

- Rotation matrix $R_v(\alpha) \in \mathrm{SO}(3)$ dependent on rotation axis $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ as unit vector (i.e., $||\mathbf{v}|| = 1$) and rotation angle $\alpha \in \mathbb{R}$:

$$R_{\mathbf{v}}(\alpha) = \begin{pmatrix} c\alpha + v_1^2(1 - c\alpha) & v_1 v_2(1 - c\alpha) - v_3 s\alpha & v_1 v_3(1 - c\alpha) + v_2 s\alpha \\ v_2 v_1(1 - c\alpha) + v_3 s\alpha & c\alpha + v_2^2(1 - c\alpha) & v_2 v_3(1 - c\alpha) - v_1 s\alpha \\ v_3 v_1(1 - c\alpha) - v_2 s\alpha & v_3 v_2(1 - c\alpha) + v_1 s\alpha & c\alpha + v_3^2(1 - c\alpha) \end{pmatrix}$$

- Trace of a matrix $A \in \mathbb{R}^{n \times n}$: Sum of the elements on the diagonal

$$\mathrm{Trace}(A) = \sum_{j=1}^{n} a_{jj}$$

- In this case:

$$\begin{aligned} \mathrm{Trace}(R_{\mathbf{v}}(\alpha)) &= 3 \cdot c\alpha + (1 - c\alpha) \cdot (v_1^2 + v_2^2 + v_3^2) \\ &= 3 \cdot c\alpha + 1 - c\alpha = 1 + 2 \cdot \cos\alpha \end{aligned}$$

Exercise 1.2: Rotation Angle, Approach B

$$R_1 = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix}$$

- Trace of a rotation matrix:

$$\mathrm{Trace}(R_{\mathbf{v}}(\alpha)) = 1 + 2 \cdot \cos\alpha$$

$$\mathrm{Trace}(R_1) =$$

Exercise 1.2: Rotation Angle, Approach B

$$R_1 = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix}$$

- Trace of a rotation matrix:

$$\mathrm{Trace}(R_{\mathbf{v}}(\alpha)) = 1 + 2 \cdot \cos\alpha$$

$$\mathrm{Trace}(R_1) = 0.36 + 0.6 + 0.6 = 1.56 = 1 + 2 \cdot \cos\alpha$$

$$\Rightarrow 0.56 = 2 \cdot \cos\alpha \Rightarrow \cos\alpha = 0.28$$

$$\Rightarrow \alpha = \pm \mathrm{acos}(0.28) \approx \pm 1.287 \approx \pm 73.7^\circ$$

Exercise 1.2: Quaternion

- Quaternion $\mathbf{q}$ from
  - Rotation axis $\mathbf{x} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ and
  - Rotation angle $\alpha = 1.287$

Exercise 1.2: Quaternion

- Quaternion $\mathbf{q}$ from
  - Rotation axis $\mathbf{x} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ and
  - Rotation angle $\alpha = 1.287$

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

Exercise 1.2: Quaternion

- Quaternion $\mathbf{q}$ from
  - Rotation axis $\mathbf{x} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ and
  - Rotation angle $\alpha = 1.287$

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

$$\mathbf{q} = \left(0.8, \quad \frac{1}{3} \cdot 0.6, \quad \frac{-2}{3} \cdot 0.6, \quad \frac{-2}{3} \cdot 0.6\right)$$

$$\mathbf{q} = (0.8, \quad 0.2, \quad -0.4, \quad -0.4)$$

Small Exercises

- Lots of calculations: Were mainly to convert from rotation matrix
- Now, some short quaternion related questions **(interactive "small exercises")**

Small Exercise (1)

Which rotation does the following quaternion describe?

$$\mathbf{q} = (0, 0,1,0)$$

a) 90° around y-axis
b) 90° around z-axis
c) 180° around y-axis
d) 180° around z-axis

Small Exercise (1)

Which rotation does the following quaternion describe?

$$\mathbf{q} = (0, 0,1,0)$$

a) 90° around y-axis

b) 90° around z-axis

c) 180° around y-axis

d) 180° around z-axis

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

Here:

$\mathbf{x} = (0, 1, 0)$, i.e., y-axis

$$\cos\left(\frac{\alpha}{2}\right) = 0 \Rightarrow \frac{\alpha}{2} = 90° \Leftrightarrow \alpha = 180°$$

Small Exercise (2)

Do the quaternions corresponding to the following angle-axis rotations differ?

0° rotation around $(0, 0,1)$ vs. 0° rotation around $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

a) Yes, **in all** coefficients

b) Yes, in all coefficients **except the 2nd** (x component of imaginary part)

c) Yes, but **only in the imaginary** part

d) No, **not at all**

Small Exercise (2)

Do the quaternions corresponding to the following angle-axis rotations differ?

0° rotation around $(0, 0, 1)$ vs. 0° rotation around $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

a) Yes, in all coefficients

b) Yes, in all coefficients except the 2nd (x component of imaginary part)

c) Yes, but only in the imaginary part

d) No, not at all

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

In both cases: $\alpha = 0$, thus $\cos\left(\frac{\alpha}{2}\right) = 1$ and $\sin\left(\frac{\alpha}{2}\right) = 0$. For any $\mathbf{x}$: $\mathbf{x} \cdot 0 = \mathbf{0}$.

Therefore, $\mathbf{q_1} = \mathbf{q_2} = (1, 0, 0, 0)$.

Small Exercise (3)

What is the unit quaternion corresponding to a rotation by 60° around $(0,4,3)^\top$?

a) $\left(\frac{1}{2}, 0, \frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{10}\right)$

b) $\left(\frac{1}{2}, 0, 4, 3\right)$

c) $\left(\frac{\sqrt{3}}{2}, 0, 0.4, 0.3\right)$

d) $\left(\frac{\sqrt{3}}{2}, 0, 0.8, 0.6\right)$

α	$0°$	$30°$	$60°$	$90°$
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

Small Exercise (3)

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What is the unit quaternion corresponding to a rotation by 60° around $(0,4,3)^T$?

a) $\left(\frac{1}{2}, 0, \frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{10}\right)$

b) $\left(\frac{1}{2}, 0, 4, 3\right)$

c) $\left(\frac{\sqrt{3}}{2}, 0, 0.4, 0.3\right)$

d) $\left(\frac{\sqrt{3}}{2}, 0, 0.8, 0.6\right)$

α	0°	30°	60°	90°
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

Here: $\cos\left(\frac{\alpha}{2}\right) = \cos(30°) = \frac{\sqrt{3}}{2}$, $\sin\left(\frac{\alpha}{2}\right) = \sin(30°) = \frac{1}{2}$

$$\mathbf{x} = \frac{1}{\sqrt{0+16+9}}\begin{pmatrix}0\\4\\3\end{pmatrix} = \frac{1}{5}\begin{pmatrix}0\\4\\3\end{pmatrix}$$

$$\Rightarrow \mathbf{q} = \left(\frac{\sqrt{3}}{2}, \underbrace{0, 0.4, 0.3}_{\cdot\frac{1}{2}}\right)$$

$\frac{\alpha}{2} = 30°$

$\sin(30°) = \frac{1}{2}$

$\cos(30°) = \frac{\sqrt{3}}{2}$

H2T

Small Exercise (4)

What is the angle between the rotations represented by following quaternions?

$$\mathbf{q_1} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{2}, \frac{1}{2}\right)$$

$$\mathbf{q_2} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{2}, -\frac{1}{2}\right)$$

a) 0°
b) 90°
c) 180°
d) Something else

Small Exercise (4)

What is the angle between the rotations represented by following quaternions?

$$\mathbf{q_1} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{2}, \frac{1}{2}\right)$$

$$\mathbf{q_2} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{2}, -\frac{1}{2}\right)$$

a) **0°**

b) 90°

c) 180°

d) Something else

Reminder:

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

Equivalent rotation: Rotate by additional 360°, i.e., 2π

$$\underline{\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right)} \triangleq \left(\cos\left(\frac{\alpha + \underline{2\pi}}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha + \underline{2\pi}}{2}\right)\right)$$

$$= \left(\cos\left(\frac{\alpha}{2} + \underline{\pi}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2} + \underline{\pi}\right)\right)$$

$$= \left(\underline{-}\cos\left(\frac{\alpha}{2}\right), \underline{-\mathbf{x}} \cdot \sin\left(\frac{\alpha}{2}\right)\right) = \underline{-\mathbf{q}}$$

$\mathbf{q}$ and $-\mathbf{q}$ **represent the same** rotation (“double coverage”)

Exercise 2: Homogeneous Matrices

Let $T \in SE(3)$ be a homogeneous transformation matrix and Vektor $\mathbf{v} = (1, 2, 3)^\top$ be a vector, with

$$T = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1. Which transformation is described by T?
2. Apply the transformation described by T to $\mathbf{v}$.
3. Determine T^{-1}, being the inverse transformation matrix of T.

Exercise 2.1: Described Transformation

Which transformation is described by T?

$$T = \begin{array}{c} \begin{array}{cccc} & R & & t \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Exercise 2.1: Described Transformation

Which transformation is described by T?

$$T = \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{matrix}} & \boxed{\begin{matrix} 5 \\ 0 \\ 0 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

Rotation by 90° around the y-axis:

$$R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

$$R_y(90°) = \begin{pmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}}$$

Translation by 5 along the x-axis:

$$\mathbf{t} = \boxed{\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}}$$

Exercise 2.2: Application to Vector

Apply the transformation described by T to $\mathbf{v}$.

$$\begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$v' = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}$$

Exercise 2.2: Application to Vector

Apply the transformation described by T to $\mathbf{v}$.

$$\begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 5 \cdot 1 \\ 1 \cdot 2 \\ -1 \cdot 1 \\ 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}' = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}$$

Exercise 2.3: Inverse Transformation

Determine T^{-1}, being the inverse transformation matrix of T.

$$T = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 2.3: Inverse Transformation

Determine T^{-1}, being the inverse transformation matrix of T.

$$T = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In general:

$$T \cdot T^{-1} = I_4$$

$$\begin{pmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \cdot \begin{pmatrix} A & \mathbf{b} \\ \mathbf{0}^\top & 1 \end{pmatrix} = \begin{pmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

$$R \cdot A + \mathbf{t} \cdot \mathbf{0}^\top = I_3 \Rightarrow R \cdot A = I_3 \Rightarrow A = R^{-1} = R^\top$$

$$R \cdot \mathbf{b} + \mathbf{t} \cdot 1 = \mathbf{0} \Rightarrow R \cdot \mathbf{b} = -\mathbf{t} \Rightarrow \mathbf{b} = -R^\top \cdot \mathbf{t}$$

Exercise 2.3: Inverse Transformation

Determine T^{-1}, being the inverse transformation matrix of T.

$$T = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} R^\top & -R^\top \cdot \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

Exercise 2.3: Inverse Transformation

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Determine T^{-1}, being the inverse transformation matrix of T.

$$T = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} R^\top & -R^\top \cdot \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

$$R^\top = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$-R^\top \cdot \mathbf{t} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

H2T

Exercise 3: Concatenation of Coordinate Transformations

We consider a service robot with a holonomic platform. The robot's x axis points in the direction of motion, and the z axis points upwards. The y axis is defined so that the coordinate system is right-handed. Let the initial pose of the robot in the basis coordinate system (BCS) be defined as

$$T_{init} = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The following commands are consecutively sent to the service robot and executed:

1. Rotate around the z axis by $90°$.
2. Drive straight for 4 unit lengths.
3. Drive straight for 2 unit lengths, to the right for 3 unit lengths and finally rotate around the z axis by $-45°$.

Calculate the transformation matrices corresponding to the individual commands, and the final pose of the robot in BCS.

Exercise 3: Local Transformations

- Local transformations

$$^{init}T_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotate around the z-axis by 90°

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v \\ 1 \end{pmatrix}$$

$$^{1}T_2 = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Drive straight for 4 length units
(x-axis in direction of motion)

$$= \begin{pmatrix} R \cdot v + t \cdot 1 \\ 1 \end{pmatrix}$$

$$^{2}T_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Drive straight for 2 unit lengths, to the right for 3 unit lengths and finally rotate around the z axis by $-45°$

$$\cos(-45°) = \frac{1}{\sqrt{2}}, \qquad \sin(-45°) = -\frac{1}{\sqrt{2}}$$

Exercise 3: Concatenation of Coordinate Transformations

- Final Pose P of the robot in BCS

$$P = T_{init} \cdot {}^{init}T_1 \cdot {}^1T_2 \cdot {}^2T_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 8 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 4: Distance Between Poses

The current pose T_{TCP} and the target pose T_{Goal} of an endeffector are given as

$$T_{TCP} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } T_{Goal} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate the **translational and rotational distance** between T_{TCP} and T_{Goal}.

Exercise 4: Translational Distance

$$T_{TCP} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{Goal} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Translational distance Δt

Exercise 4: Translational Distance

$$T_{TCP} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{Goal} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Translational distance Δt

$$\boldsymbol{\Delta t} = \mathbf{t}_{Goal} - \mathbf{t}_{TCP} = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$$

$$\Delta \mathrm{t} = \|\mathbf{t}_{Goal} - \mathbf{t}_{TCP}\| = \sqrt{6^2 + 4^2 + 2^2} = \sqrt{36 + 16 + 4} = \sqrt{56} \approx 7.48$$

Exercise 4: Rotational Distance

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$$T_{TCP} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{Goal} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotational distance $\Delta\alpha$

Exercise 4: Rotational Distance

$$T_{TCP} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{Goal} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotational distance $\Delta\alpha$

$$R_{TCP} \cdot {}^{TCP}R_{Goal} = R_{Goal} \qquad \mid R_{TCP}^{-1} \cdot \ldots$$

$$\underline{{}^{TCP}R_{Goal} = R_{TCP}^{-1} \cdot R_{Goal}} = R_{TCP}^{T} \cdot R_{Goal}$$

$$= \underline{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{T}} \underline{\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \underline{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}$$

$$\cancel{R_{TCP}^{-1}} \cdot \cancel{R_{TCP}} \cdot {}^{TCP}R_{Goal} = R_{TCP}^{-1} \cdot R_{Goal}$$

Exercise 4: Rotational Distance

$$T_{TCP} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{Goal} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotational distance $\Delta\alpha$

$$^{TCP}R_{Goal} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Exercise 4: Rotational Distance

$$T_{TCP} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{Goal} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotational distance $\Delta\alpha$

$${}^{TCP}R_{Goal} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Trace}\left({}^{TCP}R_{Goal}\right) = 0 + 0 + 0 = 0 = 1 + 2 \cdot \cos\Delta\alpha$$

$$\Rightarrow \cos\Delta\alpha = -0.5$$

$$\Rightarrow \Delta\alpha = \mathrm{acos}(-0.5) = 120° \approx 2.094$$

Exercise 5: Quaternions

Given are a point $\mathbf{p} = (5, 1, 7)^\top$, a vector $\mathbf{a} = (0, 0, 1)^\top$ and an angle $\Phi = 90°$.

1. Represent $\mathbf{p}$ as a quaternion $\mathbf{v}$.
2. Determine the quaternion $\mathbf{q}$ that describes the rotation by an angle of Φ around the axis $\mathbf{a}$. Also determine $\mathbf{q}^*$, being the conjugated quaternion of $\mathbf{q}$.
3. Transform the point $\mathbf{p}$ by $\mathbf{q}$ and determine the resulting point $\mathbf{p}'$.
4. Let $\mathbf{q_1} = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right)$ and $\mathbf{q_2} = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right)$ be quaternions with $\mathbf{a}_1 = (1, 0, 0)^\top$ and $\boldsymbol{a_2} = (0, 1, 0)^\top$.

   Give the direct formulation of the SLERP interpolation between $\mathbf{q}_1$ and $\mathbf{q}_2$, depending on the parameter $t \in [0, 1]$. Provide the interpolation result for $t = \frac{1}{2}$.

Exercise 5.1

- Represent $\mathbf{p} = (5, 1, 7)^{\top}$ as a quaternion $\mathbf{v}$.

$\mathbf{v} =$

Exercise 5.1

- Represent $\mathbf{p} = (5, 1, 7)^{\mathsf{T}}$ as a quaternion $\mathbf{v}$.

$$\mathbf{v} = (0,\ \mathbf{p})$$

$$= (0,\ 5, 1, 7)$$

Exercise 5.2

- Determine the quaternion $\mathbf{q}$ that describes the rotation by an angle of Φ around the axis $\mathbf{a}$. Also determine $\mathbf{q}^*$, being the conjugated quaternion of $\mathbf{q}$.

$$\mathbf{q} =$$

Exercise 5.2

- Determine the quaternion $\mathbf{q}$ that describes the rotation by an angle of Φ around the axis $\mathbf{a}$. Also determine $\mathbf{q}^*$, being the conjugated quaternion of $\mathbf{q}$.

$$
\begin{aligned}
\mathbf{q} &= \left(\cos\frac{\Phi}{2},\ \mathbf{a}\cdot\sin\frac{\Phi}{2}\right) \\
&= (\cos 45°,\ 0,\ 0,\ \sin 45°) \\
&= \underline{\left(\overset{\cdot 1}{\frac{1}{\sqrt{2}}},\ \overset{\cdot i}{0},\ \overset{\cdot j}{0},\ \overset{\cdot k}{\frac{1}{\sqrt{2}}}\right)} \\
&= \frac{1}{\sqrt{2}} + k\cdot\frac{1}{\sqrt{2}}
\end{aligned}
$$

Exercise 5.2

- Determine the quaternion $\mathbf{q}$ that describes the rotation by an angle of Φ around the axis $\mathbf{a}$. Also determine $\mathbf{q}^*$, being the conjugated quaternion of $\mathbf{q}$.

$$\mathbf{q} = \left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{q}^* =$$

Exercise 5.2

- Determine the quaternion $\mathbf{q}$ that describes the rotation by an angle of Φ around the axis $\mathbf{a}$. Also determine $\mathbf{q}^*$, being the conjugated quaternion of $\mathbf{q}$.

$$\mathbf{q} = \left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$\begin{aligned}\mathbf{q}^* &= (s, -\mathbf{u}) \\ &= \frac{1}{\sqrt{2}} - k \cdot \frac{1}{\sqrt{2}}\end{aligned}$$

Exercise 5.3

- Transform the point $\mathbf{p}$ by $\mathbf{q}$ and determine the resulting point $\mathbf{p}'$.

$$\mathbf{v}' =$$

Exercise 5.3

- Transform the point $\mathbf{p}$ by $\mathbf{q}$ and determine the resulting point $\mathbf{p}'$.

$$\mathbf{v}' = \mathbf{q} \cdot \mathbf{v} \cdot \mathbf{q}^* \qquad \mathbf{v} = (0,\ 5, 1, 7)$$

$$= \left(\frac{1}{\sqrt{2}} + k \cdot \frac{1}{\sqrt{2}}\right) \cdot (5i + 1j + 7k) \cdot \left(\frac{1}{\sqrt{2}} - k \cdot \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}(5i + 1j + 7k + 5ki + 1kj + 7k^2) \cdot \left(\frac{1}{\sqrt{2}} - k \cdot \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}(5i + 1j + 7k + 5j + 1(-i) + 7(-1)) \cdot \left(\frac{1}{\sqrt{2}} - k \cdot \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(-7 + 4i + 6j + 7k) \cdot (1 - k)$$

Exercise 5.3

Karlsruhe Institute of Technology

- Transform the point $\mathbf{p}$ by $\mathbf{q}$ and determine the resulting point $\mathbf{p}'$.

$$\mathbf{v}' = \mathbf{q} \cdot \mathbf{v} \cdot \mathbf{q}^* \qquad \mathbf{v} = (0,\ 5, 1, 7)$$

$i \to j \to k$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} (-7 + 4i + 6j + 7k) \cdot (1 - k)$$

$$= \frac{1}{2} (-7 + 4i + 6j + 7k + 7k - 4ik - 6jk - 7k^2)$$

$$= \frac{1}{2} \big(-7 + 4i + 6j + 7k + 7k - 4(-j) - 6(i) - 7(-1)\big)$$

$$= \frac{1}{2} (0 - 2i + 10j + 14k)$$

H2T

Exercise 5.3

- Transform the point $\mathbf{p}$ by $\mathbf{q}$ and determine the resulting point $\mathbf{p}'$.

$$\mathbf{v}' = \mathbf{q} \cdot \mathbf{v} \cdot \mathbf{q}^* \qquad \mathbf{v} = (0,\ 5, 1, 7)$$

$$= \frac{1}{2}(0 - 2i + 10j + 14k)$$

$$= 0 - i + 5j + 7k = (0,\ \mathbf{-1},\ \mathbf{5},\ \mathbf{7})$$

$$\mathbf{v}' = (0,\ \mathbf{p}')$$

$$\rightarrow \mathbf{p}' = (-1,\ 5,\ 7)$$

Exercise 5.4

Karlsruhe Institute of Technology

- Given:

$$\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right) \text{ with } \mathbf{a}_1 = (1, 0, 0)^\top$$

$$\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right) \text{ with } \mathbf{a}_2 = (0, 1, 0)^\top$$

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, t) = ?$$

Exercise 5.4

- Given:

$$\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right) \text{ with } \mathbf{a}_1 = (1, 0, 0)^\top$$

$$\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right) \text{ with } \mathbf{a}_2 = (0, 1, 0)^\top$$

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, \ t) = ?$$

Angle θ between $\mathbf{q}_1$ and $\mathbf{q}_2$:

Exercise 5.4

- Given:

$$\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right) \text{ with } \mathbf{a}_1 = (1, 0, 0)^\top$$

$$\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right) \text{ with } \mathbf{a}_2 = (0, 1, 0)^\top$$

$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, \ t) = ?$

Angle θ between $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$\cos\theta = \langle \mathbf{q}_1 | \mathbf{q}_2 \rangle = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Exercise 5.4

Karlsruhe Institute of Technology

- Given:

$$\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right) \text{ mit } \mathbf{a}_1 = (1, 0, 0)^\top$$

$$\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right) \text{ mit } \mathbf{a}_2 = (0, 1, 0)^\top$$

$$\theta = \frac{\pi}{2}$$

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, t) =$$

Exercise 5.4

- Given:

$$\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right) \text{ mit } \mathbf{a}_1 = (1,0,0)^\top$$

$$\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right) \text{ mit } \mathbf{a}_2 = (0,1,0)^\top$$

$$\theta = \frac{\pi}{2}$$

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, t) = \boxed{\frac{\sin(1-t)\theta}{\underbrace{\sin\theta}_{=1}} \cdot \mathbf{q}_1 + \frac{\sin t\theta}{\underbrace{\sin\theta}_{=1}} \cdot \mathbf{q}_2}$$

$$= \sin\left((1-t)\frac{\pi}{2}\right) \cdot \mathbf{q}_1 + \sin\left(t\frac{\pi}{2}\right) \cdot \mathbf{q}_2 \quad // \quad \sin\frac{\pi}{2} = 1$$

Exercise 5.4

- Given:

  $\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right)$ mit $\mathbf{a}_1 = (1, 0, 0)^\mathsf{T}$

  $\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right)$ mit $\mathbf{a}_2 = (0, 1, 0)^\mathsf{T}$

  $t = 0.5$

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2,\ 0.5) = ?$$

Exercise 5.4

- Given:

  $\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \cdot \sin\frac{\pi}{2}\right)$ mit $\mathbf{a}_1 = (1, 0, 0)^\top$

  $\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \cdot \sin\frac{\pi}{2}\right)$ mit $\mathbf{a}_2 = (0, 1, 0)^\top$

  $t = 0.5$

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2,\ 0.5) = \sin\left((1-t)\frac{\pi}{2}\right) \cdot \mathbf{q}_1 + \sin\left(t\frac{\pi}{2}\right) \cdot \mathbf{q}_2$$

$$= \sin\left(\frac{\pi}{4}\right) \cdot \mathbf{q}_1 + \sin\left(\frac{\pi}{4}\right) \cdot \mathbf{q}_2$$

$$= \frac{1}{\sqrt{2}} \cdot \mathbf{q}_1 + \frac{1}{\sqrt{2}} \cdot \mathbf{q}_2$$

$$= \frac{1}{\sqrt{2}} \cdot i + \frac{1}{\sqrt{2}} \cdot j$$

Small Exercise (5)

What happens when interpolating between $\mathbf{q}_1, \mathbf{q}_2$ with an inner product of $\langle \mathbf{q}_1 | \mathbf{q}_2 \rangle = -1$? Why?

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, t) = \frac{\sin((1-t)\theta)}{\sin\theta} \cdot \mathbf{q}_1 + \frac{\sin(t\theta)}{\sin\theta} \cdot \mathbf{q}_2$$

$$\cos(\theta) = -1$$

$$\Rightarrow \theta = \pi$$

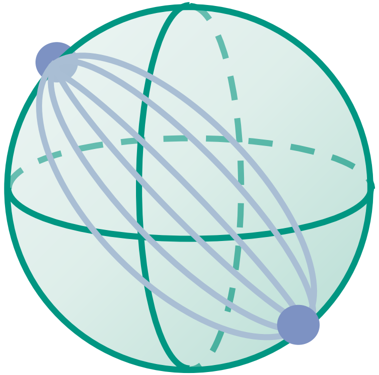
$$\Rightarrow \sin(\theta) = \sin(\pi) = 0$$

Division by zero!

Small Exercise (5)

What happens when interpolating between $\mathbf{q}_1, \mathbf{q}_2$ with an inner product of $\langle \mathbf{q}_1 | \mathbf{q}_2 \rangle = -1$? Why?

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, \ t) = \frac{\sin((1-t)\theta)}{\sin\theta} \cdot \mathbf{q}_1 + \frac{\sin(t\theta)}{\sin\theta} \cdot \mathbf{q}_2$$



No unique shortest path (visualized for S^2)

For $\langle \mathbf{q}_1 | \mathbf{q}_2 \rangle = -1 = \cos\theta$, we have $\theta = 180°$. Thus, $\sin\theta = 0$, which means that division by zero would occur.

When interpolating between two orientations that are opposite points on the sphere, there is no unique shortest path. However, due to the double-coverage, $\mathbf{q}_1$ and $\mathbf{q}_2$ would correspond to the same orientation in such case.

Additional Explanation

- Intuitive Usage of Quaternions
- Rational behind representing a 3D rotation by 4 coordinates
- Why do we need to treat rotations specially?

Intuitive Usage of Quaternions

- As discussed earlier: Quaternion can be related to **rotation angle** α and **rotation axis** $\mathbf{a}$

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{a} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

- Writing all coefficients individually:

Intuitive Usage of Quaternions

- As discussed earlier: Quaternion can be related to **rotation angle** α and **rotation axis** $\mathbf{a}$

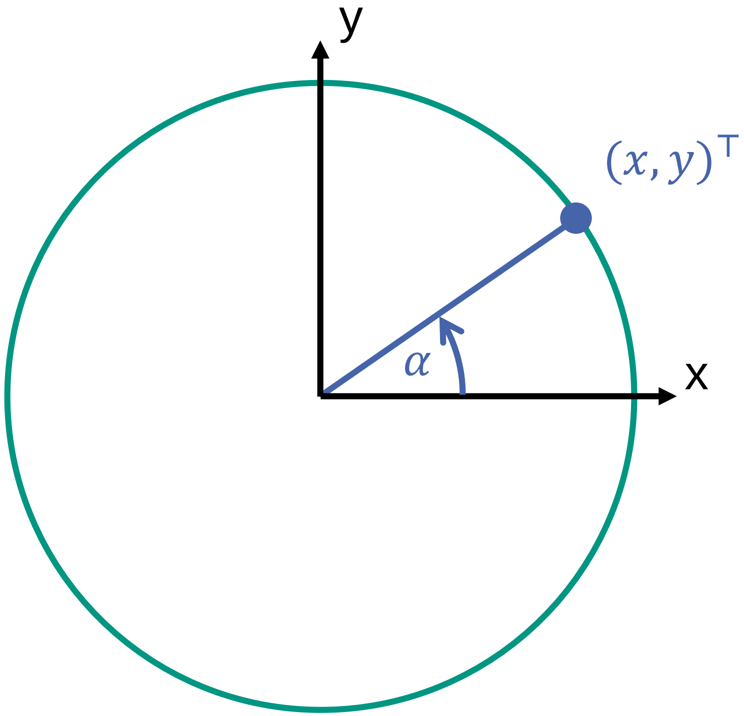
$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{a} \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

- Writing all coefficients individually:

$$\mathbf{q} = (\mathrm{q_w}, \mathrm{qx}, \mathrm{qy}, \mathrm{qz}) = \left(\cos\left(\frac{\alpha}{2}\right), a_x \cdot \sin\left(\frac{\alpha}{2}\right), a_y \cdot \sin\left(\frac{\alpha}{2}\right), a_z \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

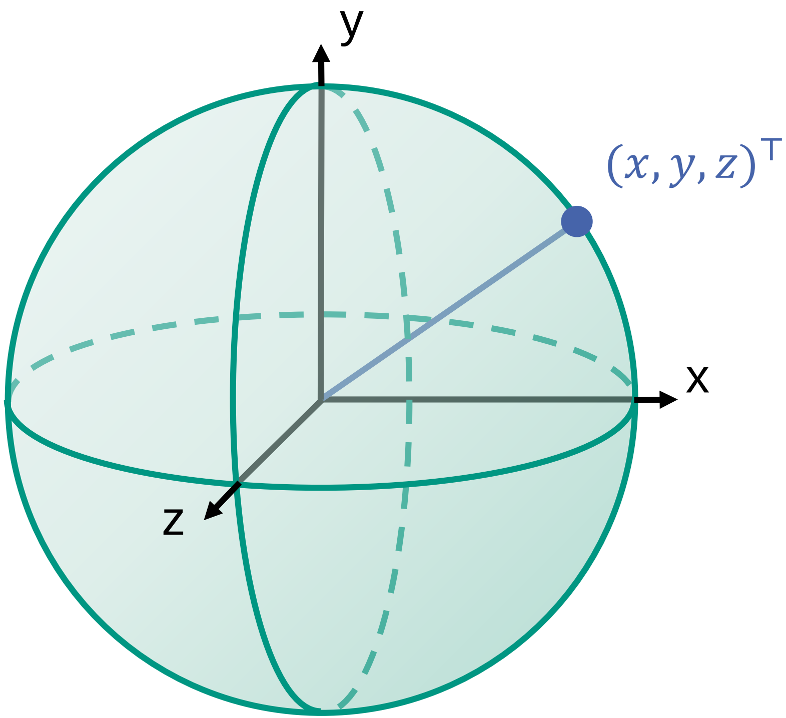
  - Real part $(\mathrm{q_w})$ can **directly be related** to the angle
  - Imaginary part $(\mathrm{q_x}, \mathrm{qy}, \mathrm{qz})$ is **parallel to** axis
    (interdependency between angle and axis only affects the “scaling” of the imaginary part)
    角度和轴之间的相互依存关系只影响虚部的“缩放”

Representing a 3D Rotation by 4 Coordinates



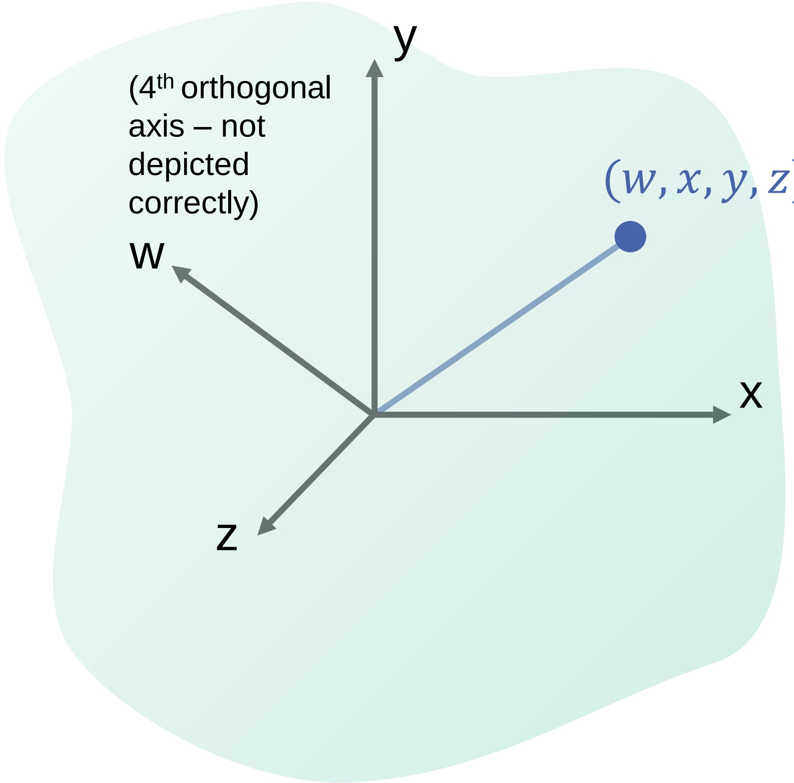
- Example: 1D rotation, embedded in 2D Euclidean plane
- Representation either as
  - $\alpha = 30°$ (compared to x-axis) or
  - $(x, y)^{\top} = (0.87, 0.50)^{\top}$
- Orientation: Described by a point **on the circle** (**"1D sphere"**, as there is 1 DoF, despite being embedded in 2D Euclidean Space)

Representing a 3D Rotation by 4 Coordinates



- Example: 2D rotation, embedded in 3D Euclidean space
- Representation using **1 more coordinate**
  - $(x, y, z)^\top = (0.87, 0.50, 0.00)^\top$
- Orientation: Described by a point on the **surface of a 2D sphere**

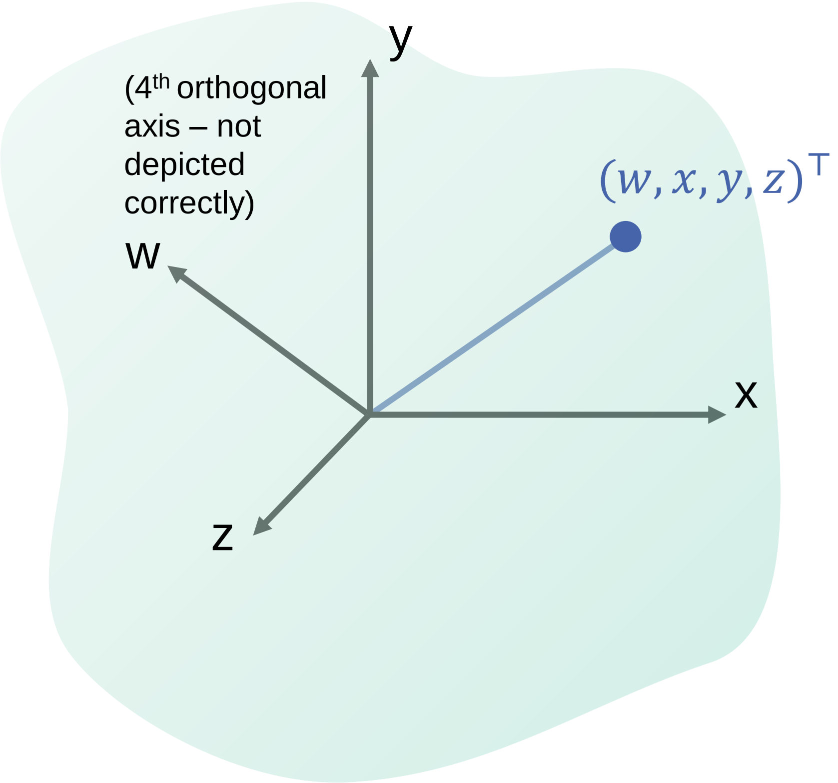
Representing a 3D Rotation by 4 Coordinates



- Example: 3D rotation, embedded in 4D Euclidean space
- Representation using **(again) 1 more coordinate**
  - $(w, x, y, z)^\top = (0.966, 0.00, 0.00, 0.259)^\top$
- Orientation: Described by a point on the **surface of a 3D hypersphere**

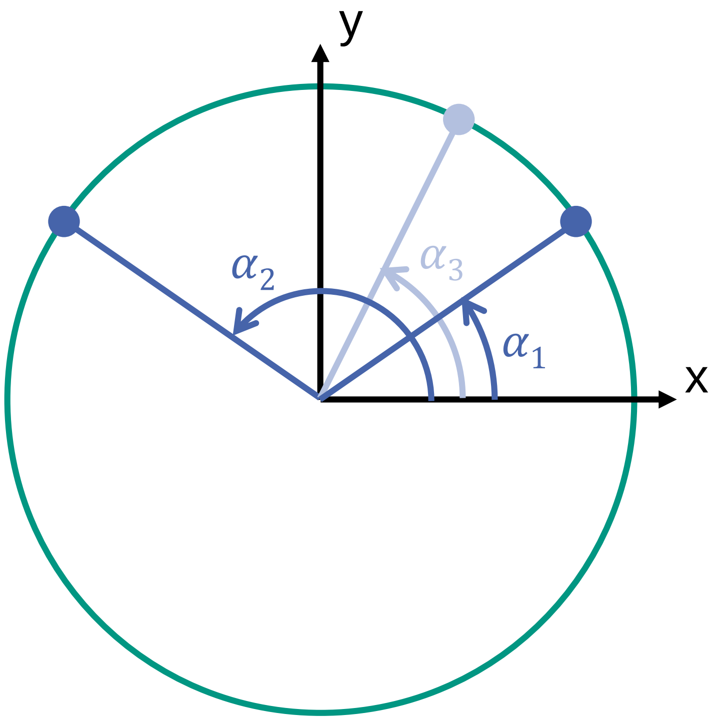
(Difficult to depict, but it's just one more coordinate, following the same principle)

Representing a 3D Rotation by 4 Coordinates



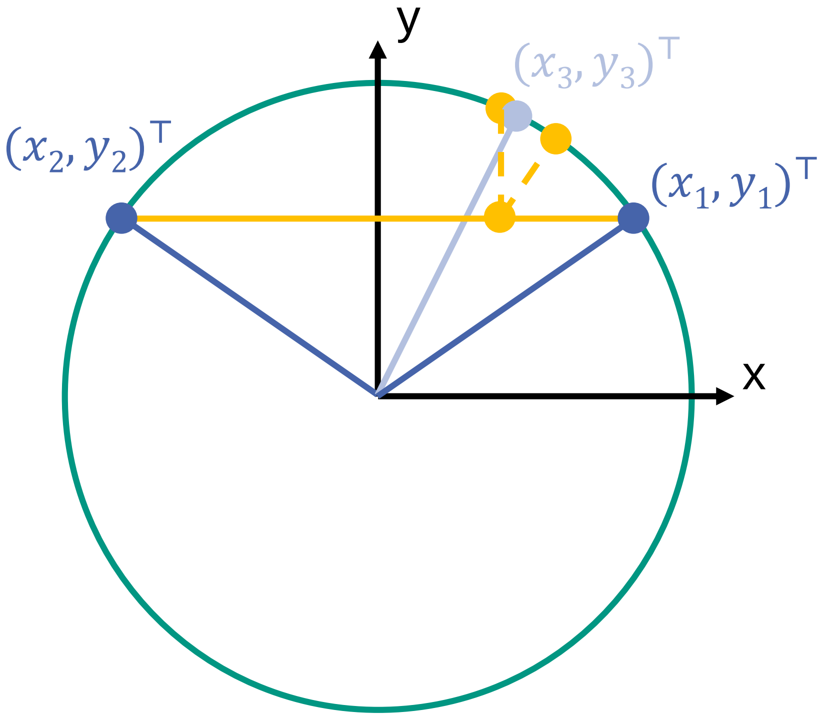
- To summarize:
  - Principle remains the same, across dimensions
  - 3D rotations would require to visualize hypersphere in 4D → **use lower dimensional examples for understanding**

Why do we need to treat rotations specially?



- Low dimensional example: 1D rotation
- Interpolate to 25% between rotation 1 and rotation 2
- Correct result (geometric approach):
  - $\alpha_1 = 30°, \alpha_2 = 150° \Rightarrow \alpha_3 = 30° + \frac{1}{4} \cdot 120° = 60°$

Why do we need to treat rotations specially?

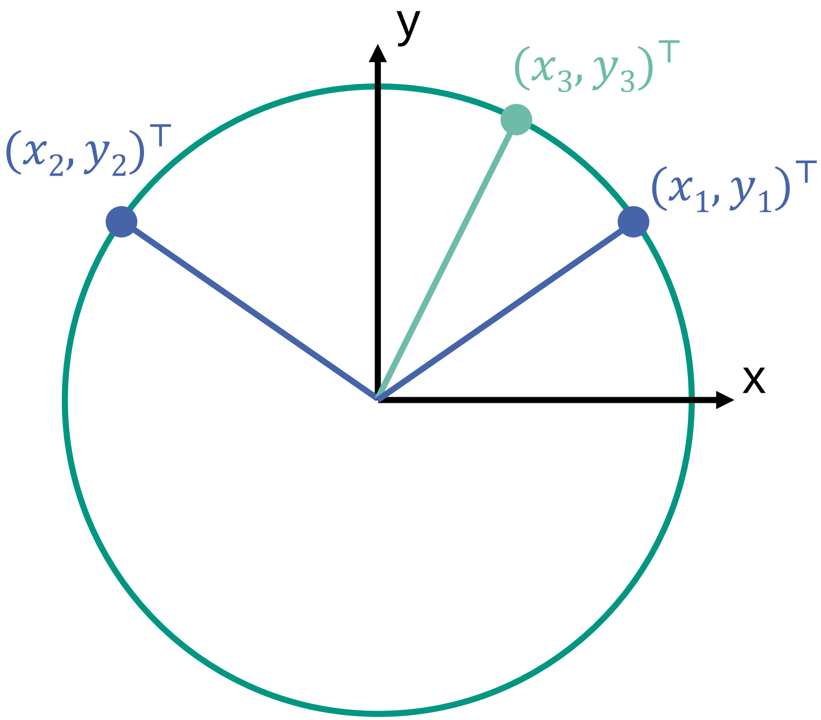


- Low dimensional example: 1D rotation
- Interpolate to 25% between rotation 1 and rotation 2
- **Euclidean approach: incorrect**
  - $p_1 = (0.87, 0.50)^\top, p_2 = (-0.87, 0.50)^\top$
    $\Rightarrow\ p_3 = \left(0.87 - \frac{1}{4} \cdot 1.73, 0.50\right)^\top = (0.43, 0.50)^\top$
  - Initially not on circle ↯
  - Normalizing (radially): $\alpha_3 = 49.1°$ ↯
  - Normalizing (perpendicular to connection): $\alpha_3 = 64.3°$ ↯

⇒ Normalizing does not make the result correct.

⇒ Similarly in higher dimension: Although quaternions embed rotations in a Euclidean space, applying Euclidean methods to their coordinates is not appropriate.

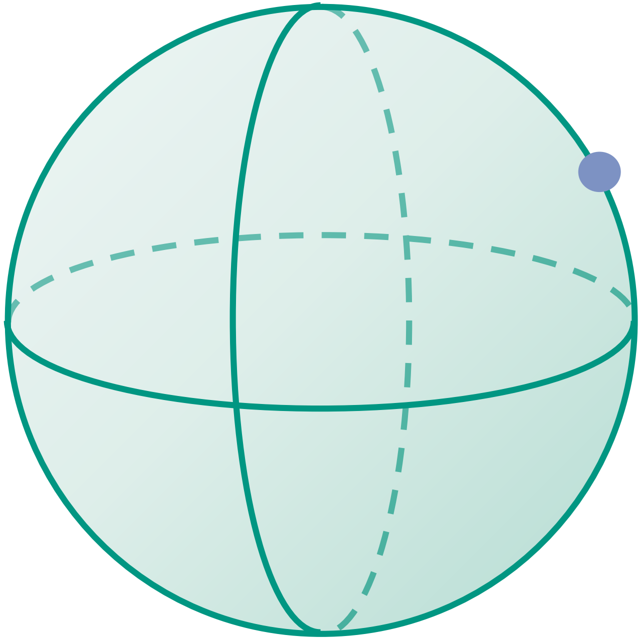
Why do we need to treat rotations specially?



- Low dimensional example: 1D rotation
- Interpolate to 25% between rotation 1 and rotation 2
- SLERP: correct
  - $\boldsymbol{q_1} = (0.966, 0, 0, 0.259)^\top, \boldsymbol{q_2} = (0.259, 0, 0, 0.966)^\top$
    $\Rightarrow \boldsymbol{q_3} = (0.866, 0, 0, 0.5)^\top$
  - Is correctly normalized
  - Corresponds to 60°

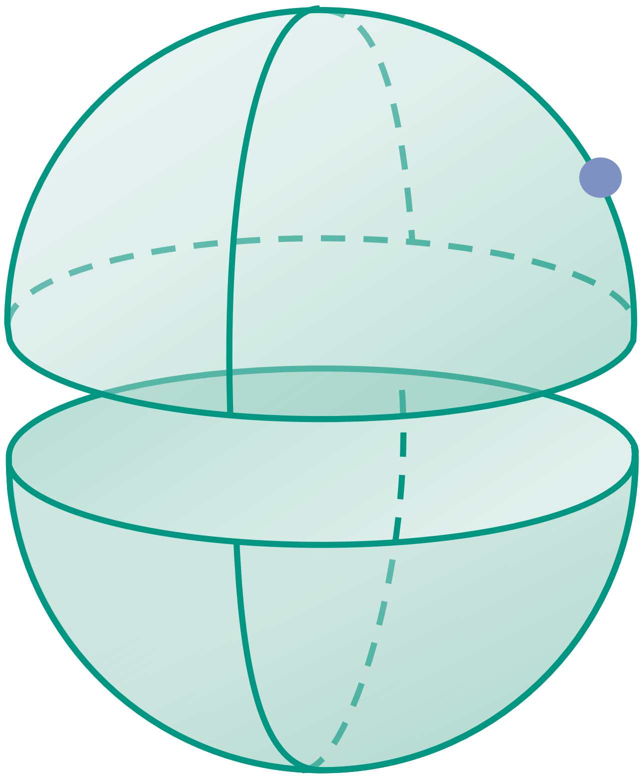
$\Rightarrow$ When dealing with orientations, use appropriate tools.

Distorted Visualization in Lower Dimension



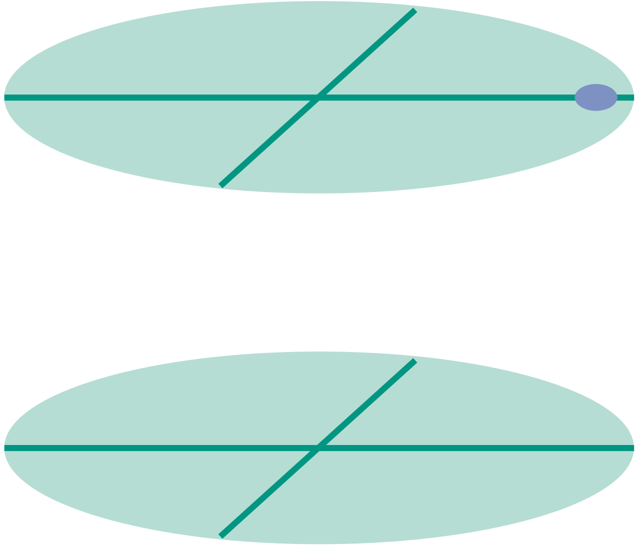
- Using “Dirac’s belt trick”
  - All 2D rotations: on surface of sphere embedded in 3D Euclidean space

Distorted Visualization in Lower Dimension



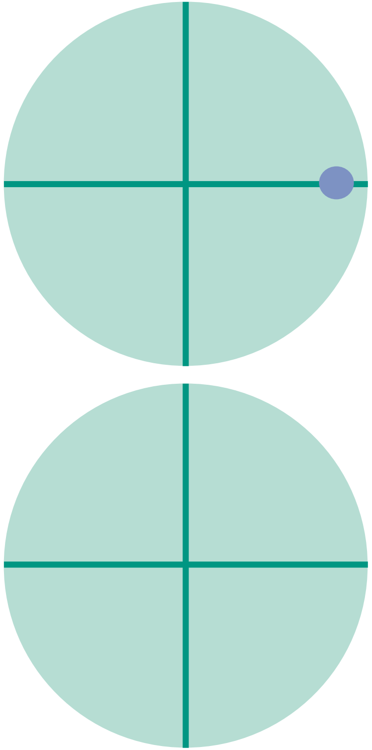
- Using “Dirac’s belt trick”
  - All 2D rotations: on surface of sphere embedded in 3D Euclidean space
  - Imagine cutting the surface on the equator, and flattening each half

Distorted Visualization in Lower Dimension



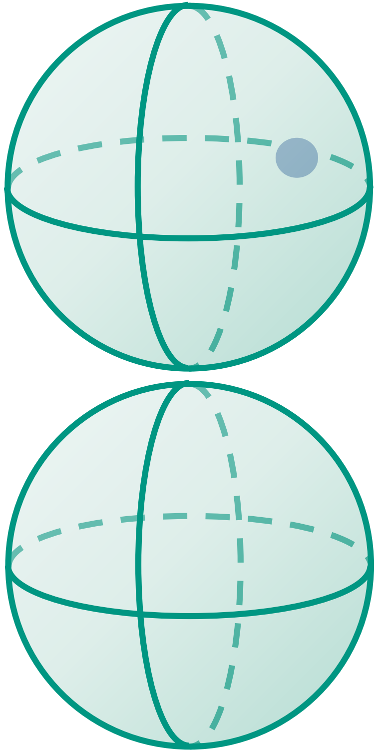
- Using “Dirac’s belt trick”
  - All 2D rotations: on surface of sphere embedded in 3D Euclidean space
  - Imagine cutting the surface on the equator, and flattening each half

Distorted Visualization in Lower Dimension



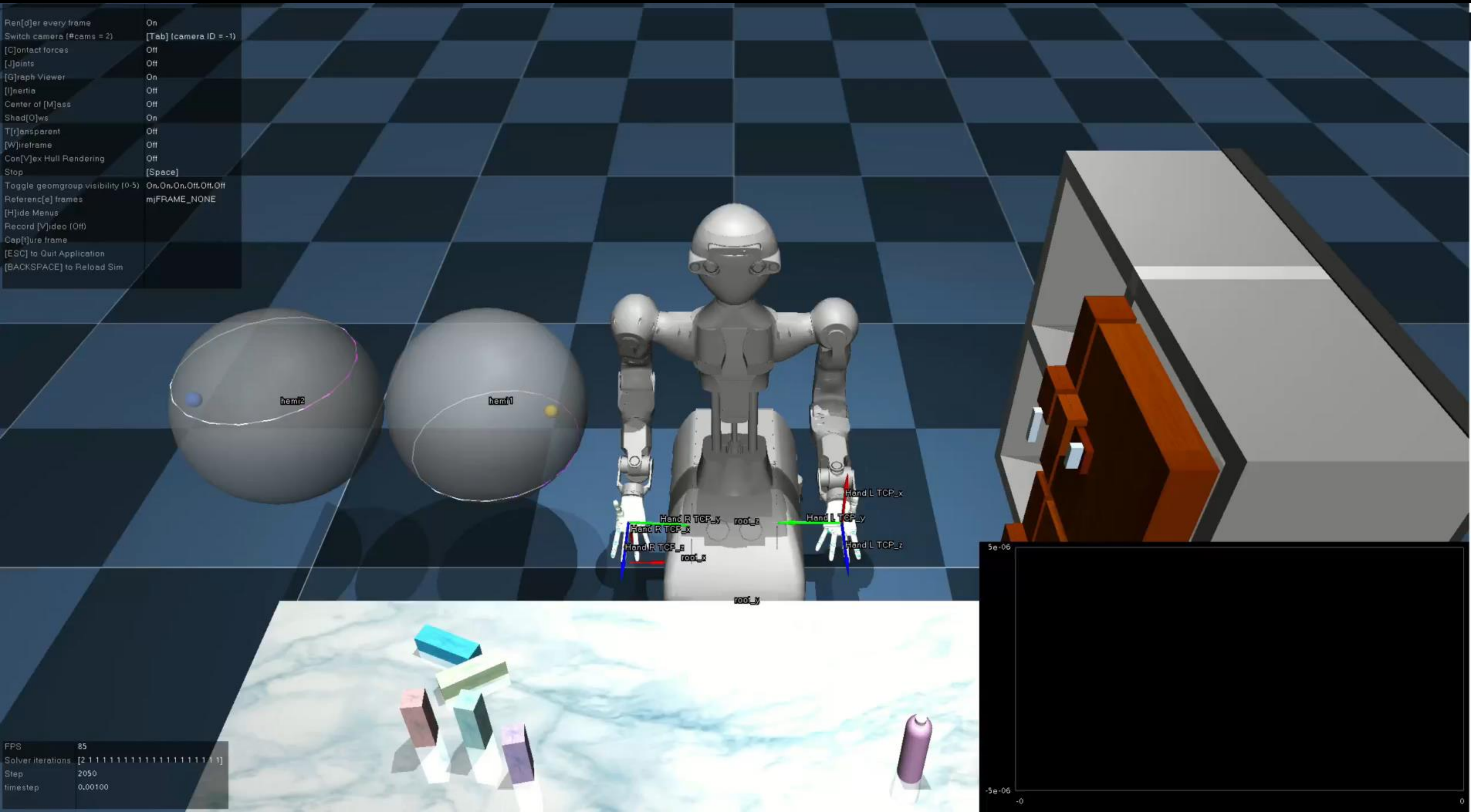
- Using “Dirac’s belt trick”
  - All 2D rotations: on surface of sphere embedded in 3D Euclidean space
  - Imagine cutting the surface on the equator, and flattening each half
  - Allows to represent all 2D rotations
    (*surface* of sphere in 3D Euclidean space)
    in 2D Euclidean space
    (2 *solid* circles in 2D Euclidean space)
  - Be aware of distortion (pole vs. equator)!

Distorted Visualization in Lower Dimension



- Using "Dirac's belt trick"
  - Similarly: Allows to represent all 3D rotations
    (*surface* of sphere in 4D Euclidean space)
    in 3D Euclidean space
    (2 *solid* spheres in 3D Euclidean space)
  - Be aware of distortion!

You can have a look at https://youtu.be/ACZC_XEyg9U for further explanations and illustrations.
Also, you can visualize it interactively using a jupyter notebook provided in our Gitlab.


Ren[d]er every frame On
Switch camera (#cams = 2) [Tab] (camera ID = -1)
[C]ontact forces Off
[J]oints Off
[G]raph Viewer On
[I]nertia Off
Center of [M]ass Off
Shad[O]ws On
T[r]ansparent Off
[W]ireframe Off
Con[V]ex Hull Rendering Off
Stop [Space]
Toggle geomgroup visibility (0-5) On,On,On,Off,Off,Off
Referenc[e] frames mjFRAME_NONE
[H]ide Menus
Record [V]ideo (Off)
Cap[t]ure frame
[ESC] to Quit Application
[BACKSPACE] to Reload Sim
hemi2
hemi1
Hand L TCP_x
Hand R TCP_y
root_z
Hand L TCP_y
Hand R TCP_x
Hand R TCP_z
Hand L TCP_z
root_x
root_y
5e-06
-5e-06
-0
0
FPS 85
Solver iterations [2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
Step 2050
timestep 0.00100

Exercise 6: Quaternions

Show that the space of unit quaternions S^3 is a subgroup of the quaternions $\mathbb{H}$.

Remark: G is a group $(G, \ \cdot)$ if and only if:

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$
2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. Identity element: $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$
4. Inverse element: $\forall a \in G : \exists a^{-1} : a \cdot a^{-1} = e$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 =$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = (\boldsymbol{a} \cdot \boldsymbol{b}) \cdot (\boldsymbol{a} \cdot \boldsymbol{b})^*$$

$$= (\boldsymbol{a} \cdot \boldsymbol{b}) \cdot (\boldsymbol{b}^* \cdot \boldsymbol{a}^*) \qquad \text{// Involutive antiautomorphism}$$

$$= \boldsymbol{a} \cdot (\boldsymbol{b} \cdot \boldsymbol{b}^*) \cdot \boldsymbol{a}^* \qquad \text{// Associativity}$$

$$= \boldsymbol{a} \cdot \|\boldsymbol{b}\|^2 \cdot \boldsymbol{a}^* = \boldsymbol{a} \cdot 1 \cdot \boldsymbol{a}^* = \|\boldsymbol{a}\|^2 = 1$$

Exercise 6.1: Closure (Tricky)

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Exercise 6.1: Closure (Tricky)

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$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2$$

$$\begin{aligned} &= a_3^2 b_3^2 + a_2^2 b_3^2 + a_1^2 b_3^2 + a_0^2 b_3^2 \\ &+ a_3^2 b_2^2 + a_2^2 b_2^2 + a_1^2 b_2^2 + a_0^2 b_2^2 \\ &+ a_3^2 b_1^2 + a_2^2 b_1^2 + a_1^2 b_1^2 + a_0^2 b_1^2 \\ &+ a_3^2 b_0^2 + a_2^2 b_0^2 + a_1^2 b_0^2 + a_0^2 b_0^2 \end{aligned}$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

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Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2$$

$$\begin{aligned} &= b_3^2 \cdot (a_3^2 + a_2^2 + a_1^2 + a_0^2) \\ &+ b_2^2 \cdot (a_3^2 + a_2^2 + a_1^2 + a_0^2) \\ &+ b_1^2 \cdot (a_3^2 + a_2^2 + a_1^2 + a_0^2) \\ &+ b_0^2 \cdot (a_3^2 + a_2^2 + a_1^2 + a_0^2) \end{aligned}$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2$$

$$= b_3^2 \cdot \|\boldsymbol{a}\|^2 + b_2^2 \cdot \|\boldsymbol{a}\|^2 + b_1^2 \cdot \|\boldsymbol{a}\|^2 + b_0^2 \cdot \|\boldsymbol{a}\|^2$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2$$

$$= b_3^2 \cdot \|\boldsymbol{a}\|^2 + b_2^2 \cdot \|\boldsymbol{a}\|^2 + b_1^2 \cdot \|\boldsymbol{a}\|^2 + b_0^2 \cdot \|\boldsymbol{a}\|^2$$

$$= (b_3^2 + b_2^2 + b_1^2 + b_0^2) \cdot \|\boldsymbol{a}\|^2$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\begin{aligned}
\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 &= \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2 \\
&= b_3^2 \cdot \|\boldsymbol{a}\|^2 + b_2^2 \cdot \|\boldsymbol{a}\|^2 + b_1^2 \cdot \|\boldsymbol{a}\|^2 + b_0^2 \cdot \|\boldsymbol{a}\|^2 \\
&= (b_3^2 + b_2^2 + b_1^2 + b_0^2) \cdot \|\boldsymbol{a}\|^2 \\
&= \|\boldsymbol{b}\|^2 \cdot \|\boldsymbol{a}\|^2
\end{aligned}$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\begin{aligned}
\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 &= \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2 \\
&= b_3^2 \cdot \|\boldsymbol{a}\|^2 + b_2^2 \cdot \|\boldsymbol{a}\|^2 + b_1^2 \cdot \|\boldsymbol{a}\|^2 + b_0^2 \cdot \|\boldsymbol{a}\|^2 \\
&= (b_3^2 + b_2^2 + b_1^2 + b_0^2) \cdot \|\boldsymbol{a}\|^2 \\
&= \|\boldsymbol{b}\|^2 \cdot \|\boldsymbol{a}\|^2 = 1 \cdot 1 = 1 \qquad \text{q.e.d.}
\end{aligned}$$

Exercise 6.2 & 6.3: Associativity and Identity Element

2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$

3. Identity element: $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$

Exercise 6.2 & 6.3: Associativity and Identity Element

2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Unit quaternions are a subset of quaternions. Multiplications of quaternions are associative.

3. Identity element: $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$

The identity element is $\boldsymbol{e} = (1, 0, 0, 0)$.

Exercise 6.4: Inverse Element

4. Inverse element: $\forall a \in G : \exists a^{-1} : a \cdot a^{-1} = e$

$$\boldsymbol{q} \in S^3 \Rightarrow \boldsymbol{q}^{-1} \in S^3$$

Exercise 6.4: Inverse Element

4. Inverse element: $\forall a \in G : \exists a^{-1} : a \cdot a^{-1} = e$

$$\boldsymbol{q} \in S^3 \Rightarrow \boldsymbol{q}^{-1} \in S^3$$

$$\|\boldsymbol{q}^{-1}\|^2 = \left\|\frac{\boldsymbol{q}^*}{\|\boldsymbol{q}\|^2}\right\|^2$$

$$= \left\|\frac{\boldsymbol{q}^*}{1}\right\|^2$$

$$= 1$$

Exercise 6: Quaternions

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$ ☑
2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$ ☑
3. Identity element: $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$ ☑
4. Inverse element: $\forall a \in G : \exists a^{-1} : a \cdot a^{-1} = e$ ☑

Exercise 7: Rotations and Machine Learning

Rotations as input and output of learned models

1. Compare the **representations of rotations** as
   - Euler angles,
   - Quaternions, and
   - Rotation matrices

   with respect to how suitable they are as the **output** of a machine learning approach (e.g., neural networks)

2. A neural network, which has been trained to output rotation matrices, yields the matrix A:

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

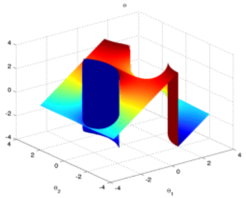
   Determine a rotation matrix R that is as “close” to A as possible.

Exercise 7.1: Euler Angles and ML

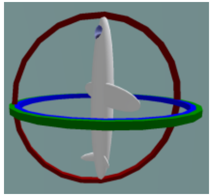
Euler angles: $\alpha, \beta, \gamma \in [0, 2\pi]$
Rotation around three axes (several conventions)

+ Minimal representation for 3 Degree of Freedom
+ All values are valid, even beyond the intervall $[0, 2\pi]$

– Not continuous



– Multi-coverage
A rotation can be describes by multiple tupels α, β, γ
(e.g., Gimbal lock)



Exercise 7.1: Quaternions and ML

$$S^3 = \{ \mathbf{q} \in \mathbb{H} \mid \|\mathbf{q}\|^2 = 1 \}$$

$$\mathbf{q} = \left(\cos\frac{\Phi}{2}, \quad \mathbf{a} \cdot \sin\frac{\Phi}{2}\right)$$

- Easy to normalize $\quad \mathbf{p} \in \mathbb{H},\ \mathbf{p} \notin S^3: \quad \mathbf{q} = \frac{\mathbf{p}}{\|\mathbf{p}\|} \in S^3$
- Local interpolation is linear:

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, t) = \frac{\sin(1-t)\,\theta}{\sin\theta} \cdot \mathbf{q}_1 + \frac{\sin t\theta}{\sin\theta} \cdot \mathbf{q}_2$$

- Representation is not minimal: 1 redundant value ($\|\mathbf{q}\|^2 = 1$)
- Double coverage: Each rotation can be described by two different unit quaternions

Exercise 7.1: Rotation Matrices and ML

$$R_{z,\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- (+) Single coverage:
  A rotation corresponds to exactly one rotation matrix
- (+) Local interpolation is linear: $\sin\alpha \approx \alpha,\ \cos\alpha \approx 1,$ for very small α

- (○) Normalization is possible, but complex
  ➔ Gram-Schmidt, QR decomposition, SVD

- (−) Highly redundant representation: 6 redundant values

Exercise 7.2: Rotation Matrices and ML

A neural network, which has been trained to output rotation matrices, yields the matrix A:

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

Determine a rotation matrix R that is as “close” to A as possible.

Exercise 7.2: Rotation Matrices and ML

A neural network, which has been trained to output rotation matrices, yields the matrix A:

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

Determine a rotation matrix R that is as “close” to A as possible.

A is not a rotation matrix:

$$A \cdot A^T = \begin{pmatrix} 0.38 & 0.44 & 0.13 \\ 0.44 & 1.31 & 0.4 \\ 0.13 & 0.4 & 0.5 \end{pmatrix} \neq I$$

$$\det A = 0.339 \neq 1$$

Exercise 7.2: Orthonormalization

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

A rotation matrix R is to be determined from A:

$$R \cdot R^\top = I, \qquad \det R = 1$$

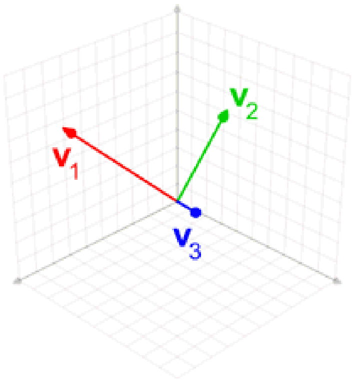
Different orthogonalization algorithms:

- Gram-Schmidt
- QR decomposition
- SVD (singular value decomposition)

Exercise 7.2: Gram-Schmidt Orthogonalization

- Orthogonalization
  - Given: Linearly independent vectors $w_1, \dots, w_3$
  - Unknown: Pairwise orthogonal vectors $v_1, \dots, v_3$ that span the same subspace

- Gram-Schmidt
  - $v_1 = w_1$
  - $v_2 = w_2 - \frac{v_1 \cdot w_2}{v_1 \cdot v_1} \cdot v_1$
  - $v_3 = w_3 - \frac{v_1 \cdot w_3}{v_1 \cdot v_1} \cdot v_1 - \frac{v_2 \cdot w_3}{v_2 \cdot v_2} \cdot v_2$



- Projection on previous vectors
- Subtract the projected part

Exercise 7.2: Orthogonalization of A

$$A = (w_1, w_2, w_3) \qquad w_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}, w_2 = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.0 \end{pmatrix}, w_3 = \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix}$$

$$v_1 = w_1 =$$

$$v_2 = w_2 - \frac{v_1 \cdot w_2}{v_1 \cdot v_1} \cdot v_1$$

$$=$$

Exercise 7.2: Orthogonalization of A

$A = (w_1, w_2, w_3) \qquad w_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}, w_2 = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.0 \end{pmatrix}, w_3 = \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix}$

$v_1 = w_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}$

$$v_2 = w_2 - \frac{v_1 \cdot w_2}{v_1 \cdot v_1} \cdot v_1$$

$$= \begin{pmatrix} 0.1 \\ 0.9 \\ 0.0 \end{pmatrix} - \frac{1}{0.62} \left(\begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix} \cdot \begin{pmatrix} 0.1 \\ 0.9 \\ 0.0 \end{pmatrix} \right) \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1 \\ 0.9 \\ 0.0 \end{pmatrix} - \frac{0.51}{0.62} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix} \approx \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix}$$

Exercise 7.2: Orthogonalization of A

$$A = (w_1, w_2, w_3) \qquad w_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}, w_2 = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.0 \end{pmatrix}, w_3 = \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}, v_2 = \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix}$$

$$v_3 = w_3 - \frac{v_1 \cdot w_3}{v_1 \cdot v_1} \cdot v_1 - \frac{v_2 \cdot w_3}{v_2 \cdot v_2} \cdot v_2$$

$$\approx$$

Exercise 7.2: Orthogonalization of A

$$A = (w_1, w_2, w_3) \qquad w_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}, w_2 = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.0 \end{pmatrix}, w_3 = \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}, v_2 \approx \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix}$$

$$v_3 = w_3 - \frac{v_1 \cdot w_3}{v_1 \cdot v_1} \cdot v_1 - \frac{v_2 \cdot w_3}{v_2 \cdot v_2} \cdot v_2$$

$$\approx \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix} - \frac{1}{0.62} \left(\begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix} \cdot \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix} \right) \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix} - \frac{1}{0.401} \left(\begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix} \cdot \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix} \right) \cdot \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.1 \\ 0.5 \\ 0.7 \end{pmatrix} - \frac{0.38}{0.62} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix} - \frac{0.148}{0.401} \cdot \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix} \approx \begin{pmatrix} -0.122 \\ 0.013 \\ 0.669 \end{pmatrix}$$

Exercise 7.2: Normalization

$$v_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}, v_2 = \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix}, v_3 = \begin{pmatrix} -0.122 \\ 0.013 \\ 0.669 \end{pmatrix}$$

The vectors v_1, v_2, v_3 are pairwise orthogonal: $v_1 \perp v_2, v_1 \perp v_3, v_2 \perp v_3$

However, they are not yet normalized: $\|v_i\| \neq 1$

$$e_1 = \frac{v_1}{\|v_1\|} \approx \frac{1}{\sqrt{0.62}} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix} \approx \begin{pmatrix} 0.762 \\ 0.635 \\ 0.127 \end{pmatrix}$$

$$e_2 = \frac{v_2}{\|v_2\|} \approx \frac{1}{\sqrt{0.401}} \cdot \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix} \approx \begin{pmatrix} -0.622 \\ 0.772 \\ -0.129 \end{pmatrix}$$

$$e_3 = \frac{v_3}{\|v_3\|} \approx \frac{1}{\sqrt{0.463}} \cdot \begin{pmatrix} -0.122 \\ 0.013 \\ 0.669 \end{pmatrix} \approx \begin{pmatrix} -0.179 \\ 0.019 \\ 0.984 \end{pmatrix}$$

Exercise 7.2: Result

- Input:

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

- Orthonormal basis vectors:

$$e_1 = \begin{pmatrix} 0.762 \\ 0.635 \\ 0.127 \end{pmatrix}, e_2 = \begin{pmatrix} -0.622 \\ 0.772 \\ -0.129 \end{pmatrix}, e_3 = \begin{pmatrix} -0.179 \\ 0.019 \\ 0.984 \end{pmatrix}$$

- Rotation matrix:

$$\mathrm{R} = \begin{pmatrix} 0.762 & -0.622 & -0.179 \\ 0.635 & 0.772 & 0.019 \\ 0.127 & -0.129 & 0.984 \end{pmatrix}$$