P97. 1. $f_{XY}(x,y) = kx^2y$ if $0 \le y \le x \le 1$

i) $= 0$ otherwise

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k x^2 y \, dy \, dx = 1.$$

$$k \int_0^1 \int_0^x x^2 y \, dy \, dx = 1.$$

$$\frac{k}{2} \int_0^1 x^4 \, dx = 1.$$

$$\frac{k}{10} = 1$$

$$\boxed{k = 10}$$

ii)

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, dy = 10 \int_0^x x^2 y \, dy = \boxed{\begin{cases} 5x^4 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, dx = 10 \int_y^1 x^2 y \, dx = \boxed{\begin{cases} \frac{10}{3}(y - y^4) & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}}$$

iii) $f_X(x) \cdot f_Y(y) = 5x^4(y - y^4) \neq f_{XY}(x,y)$.

X & Y are $\boxed{\text{NOT}}$ independent.

7 v). $\frac{1}{2} = \int_{-\infty}^{v_x} f_X(x)\,dx = \int_0^{v_x} 5x^4\,dx = v_x^5$

$$\boxed{v_x = \left(\frac{1}{2}\right)^{\frac{1}{5}}}$$

$\frac{1}{2} = \int_{-\infty}^{v_y} f_Y(y)\,dy = \int_0^{v_y} \frac{10}{3}(y - y^4)\,dy = \frac{10}{3}\left(\frac{v_y^2}{2} - \frac{v_y^5}{5}\right).$

$$\boxed{v_y = 0.5691}$$

P97. 3. $f_{XY}(x,y) = b\sin(x+y)$ if $0<x<\frac{\pi}{2}$ & $0\le y\le \frac{\pi}{2}$.

$= 0$ otherwise.

i). $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$,

$b\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x+y)\,dx\,dy = 1$.

$b \cdot \int_0^{\frac{\pi}{2}} -\cos(x+y)\Big|_0^{\frac{\pi}{2}}\,dx = 1$

$b\int_0^{\frac{\pi}{2}} \sin(x)+\cos(x)\;dx. = 1.$

$b\; -\cos(x)+\sin(x)\Big|_0^{\frac{\pi}{2}} = 1.$

$2b = 1$

$\boxed{b = \frac{1}{2}}$

ii) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(x+y)\,dy$

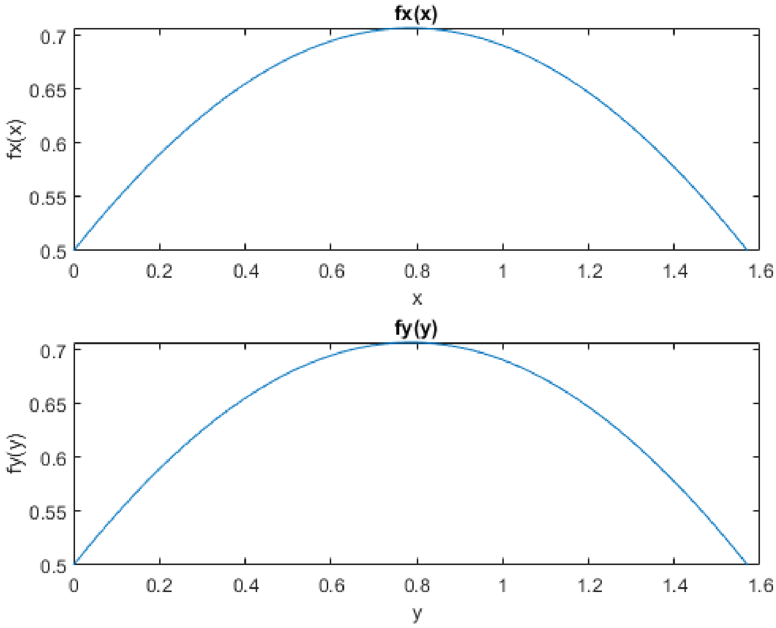
$$\boxed{= \begin{cases} \frac{1}{2}\left[\sin(x)+\cos(x)\right] & 0\le x\le\frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}}$$

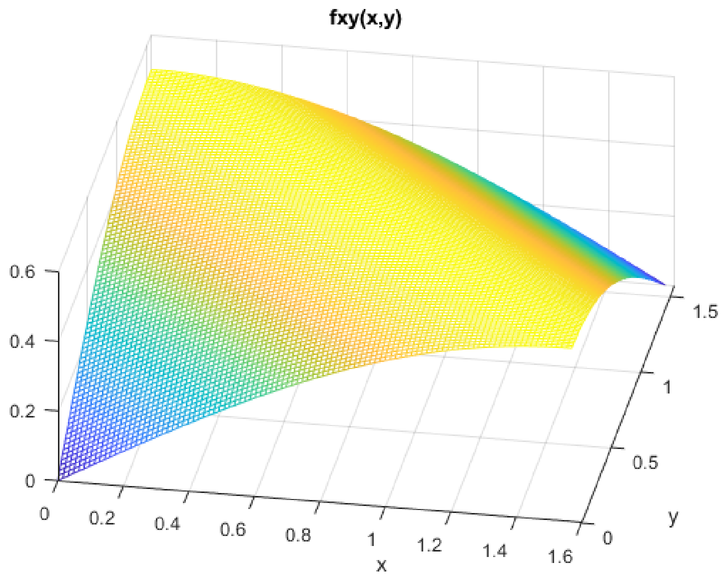
$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(x+y)\,dx$

$$\boxed{= \begin{cases} \frac{1}{2}\left[\sin(y)+\cos(y)\right] & 0\le y\le\frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}}$$

iii) see next page.

iv). $\frac{1}{2}\left[\sin(x)+\cos(x)\right]\frac{1}{2}\left[\sin(y)+\cos(y)\right] = \frac{1}{4}\sin(x+y). \neq \frac{1}{2}\sin(x+y).$


fx(x)
fx(x)
0.7
0.65
0.6
0.55
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
x
fy(y)
fy(y)
0.7
0.65
0.6
0.55
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
y


fxy(x,y)
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
x
1.5
1
0.5
0
y

P102 P1.

$$f_{XY}(x,y) = k(x^2+y^2) \quad \text{if } \sqrt{x^2+y^2} \le 1.$$

$$= 0 \quad \text{otherwise}.$$

i)

$$\int_0^\infty\int_0^\infty f_{XY}(x,y)\,dx\,dy = K\int_0^1\int_0^{2\pi} r^3\,d\theta\,dr$$

$$= K\,\frac{\pi}{2} = 1.$$

$$\boxed{K = \frac{2}{\pi}}$$

ii) $$f_X(x) = \int_{-\infty}^{\infty} \frac{2}{\pi}\cdot(x^2+y^2)\,dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\pi}(x^2+y^2)\,dy.$$

$$= \frac{2}{\pi}\cdot\left(x^2 y + \frac{y^3}{3}\right)\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$= \frac{2}{\pi}\left[x^2\sqrt{1-x^2} + \frac{(1-x^2)\sqrt{1-x^2}}{3} + x^2\sqrt{1-x^2} + \frac{(1-x^2)\sqrt{1-x^2}}{3}\right]$$

$$= \frac{2}{3\pi}\left[(2x^2+1)\sqrt{1-x^2}\right]\cdot 2.$$

$$= \boxed{\begin{cases} \frac{4}{3\pi}(2x^2+1)\sqrt{1-x^2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{2}{\pi}(x^2+y^2)\,dx = \int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}(x^2+y^2)\,dy = \boxed{\begin{cases} \frac{4}{3\pi}(2y^2+1)\sqrt{1-y^2} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

iii) $$f_X(x)\cdot f_Y(y) \ne f_{XY}(x,y)$$

$\boxed{\text{They are NOT independent.}}$

iv) $F_r(r) = K \int_0^r \int_0^{2\pi} r^3 \, d\theta \, dr$

$= \frac{2}{\pi} \frac{\pi}{2} r^4 = r^4.$

$\Rightarrow \boxed{F_r(r) = \begin{cases} 0 & r < 0 \\ r^4 & 0 \le r \le 1 \\ 1 & r > 1 \end{cases}}$

$\boxed{f_r(r) = \frac{d}{dr} F_r(r) = \begin{cases} 4r^3 & 0 < r < 1. \\ 0 & \text{otherwise} \end{cases}}$

P108. 2. $Y = \max\{X, Z_1\}$

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad f_{Z_1}(z) = \begin{cases} \frac{1}{2} & 0 \leq z \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$F_x(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad F_z(z) = \begin{cases} 0 & \\ \frac{z}{2} & 0 \leq z \leq 2 \\ 1 & 2 < z. \end{cases}$$

$$1 - e^{-y} > \frac{y}{2}.$$

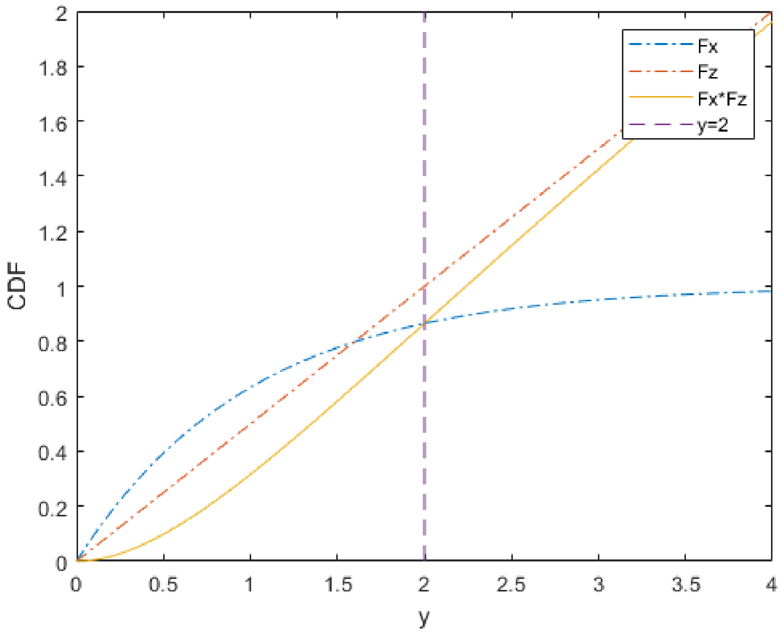
$$F_Y(y) = P(Y < y) = P(\max\{X, z\} < y)$$

$$= P(X < y \cap z < y).$$

$$= (1 - e^{-y})\frac{y}{2} \qquad 0 \leq y \leq 2.$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0. \\ (1 - e^{-y})\frac{y}{2} & 0 \leq y \leq 2. \\ (1 - e^{-y}) & y > 2. \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 0 & y < 0 \\ \frac{1 - e^{-y} + e^{y}y}{2} & 0 \leq y \leq 2. \\ e^{-y} & y > 2. \end{cases}$$



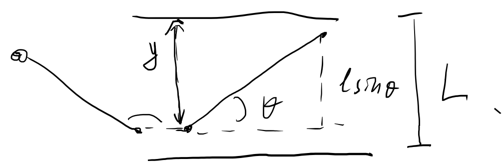
P1|1 P1 (ii)

$$f_Y(y) = \begin{cases} \frac{1}{L} & 0 \le y \le L \\ 0 & \text{otherwise} \end{cases} \qquad f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & 0 \le \theta \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

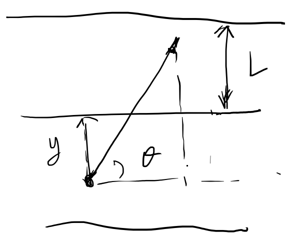
$$f_{\Theta Y}(\theta, y) = \begin{cases} \frac{1}{\pi L} & 0 \le \theta \le \pi, \quad 0 \le y \le L. \\ 0 & \end{cases}$$

$L < l < 2L$.

Let H_0 : hit no line,
H_1 : hit 1 line,
H_2 : hit 2 line.

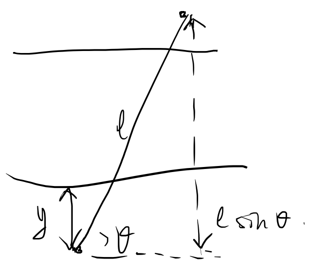


$P(H_0) = P(y > l\sin\theta) \cap P(l\sin\theta < L)$



$P(H_2) = P(y \le l\sin\theta - L) \cap P(l\sin\theta > L)$.

$P(H_1) = 1 - P(H_0) - P(H_2)$.



$$P(H_0) = \int_0^{\sin^{-1}\frac{L}{l}} \int_{l\sin\theta}^{L} \frac{1}{\pi L}\, dy\, d\theta + \int_{\pi - \sin^{-1}\frac{L}{l}}^{\pi} \int_{l\sin\theta}^{L} \frac{1}{\pi L}\, dy\, d\theta$$

$$\simeq 0.1107 + 0.1107$$

$$\boxed{= 0.2214}$$

$$P(H_2) = 2 \cdot \int_{\sin^{-1}\frac{L}{l}}^{\frac{\pi}{2}} \int_0^{l\sin\theta - L} \frac{1}{\pi L}\, dy\, d\theta \simeq \boxed{0.1763}$$

$$P(H_1) = 1 - P(H_0) - P(H_2) \simeq \boxed{0.6023}$$

P116 2: $f_{XYZ}(x,y,z) = \begin{cases} a(x^2+y^2+z^2) & \text{if } x^2+y^2+z^2 \le 1. \\ 0 & \text{otherwise}. \end{cases}$

i). $1 = \iiint_{-\infty}^{\infty} f_{XYZ}(x,y,z)\,dx\,dy\,dz$

$= a\int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2 \cdot r^2 \sin\varphi \cdot dr\,d\theta\,d\varphi.$

$= \frac{4}{5}\pi a \quad \Rightarrow \boxed{a = \frac{5}{4\pi}}$

ii) $P(\frac{1}{9} \le x^2+y^2+z^2 \le \frac{1}{4}) = P(\frac{1}{9} \le r^2 \le \frac{1}{4}).$

$= \int_0^{\pi}\int_0^{2\pi}\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{5}{4\pi} r^4 \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi$

$= \frac{211}{7776} \approx \boxed{0.02713}$

iii) $P(x^2+y^2 \le 1) = 1.$

iv). $P(\{0 \le x\} \cap \{0 \le y\} \cap \{0 \le z\}) = \frac{1}{8}.$ (one of 8 quadrants in a 3D coordinate system).

v). $F_r(r) = a\int_0^{\pi}\int_0^{2\pi}\int_0^{r} r^2 \cdot r^2 \sin\varphi \cdot dr\,d\theta\,d\varphi.$

$= r^5 \Rightarrow \boxed{F_r(r) = \begin{cases} 0 & r < 0 \\ r^5 & 0 \le r \le 1 \\ 1 & r > 1. \end{cases}}$

vi) $f_r(r) = \boxed{\begin{cases} 5r^4 & 0 \le r \le 1 \\ 0 & \text{otherwise} \end{cases}}.$