HW3 P25-26. 1-6.

1. $F = -m\omega \times (\omega \times b)$

Show $F = -m\|\omega\|^2 P_{\mathcal{H}} b$. where $P_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H} = \{h \in \mathbb{C}^3 : h \perp \omega\}$.

The cross product matrix A is

$$A_w = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}. \qquad \text{s.t. } w \times b = A \cdot b.$$

proof: $$w \times b = \left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) = \begin{bmatrix} w_2 \cdot b_3 - w_3 b_2 \\ w_3 b_1 - w_1 b_3 \\ w_1 b_2 - w_2 b_1 \end{bmatrix} = \left(\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) = Ab.$$

$A_w^2 = -A_w^* A_w$ (A is skew symmetric)

$$= -\left(\begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} w_2^2 + w_3^2 & -w_1 w_2 & -w_1 w_3 \\ -w_1 w_2 & w_1^2 + w_3^2 & -w_2 w_3 \\ -w_1 w_3 & -w_2 w_3 & w_1^2 + w_2^2 \end{bmatrix}$$

$$= -\begin{bmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{bmatrix} + \begin{bmatrix} w_1^2 + w_2^2 + w_3^2 & 0 & 0 \\ 0 & w_1^2 + w_2^2 + w_3^2 & 0 \\ 0 & 0 & w_1^2 + w_2^2 + w_3^2 \end{bmatrix}$$

$$= \|w\|^2 \cdot I - ww^*$$

$$= I - ww^*, \text{ if } \|w\| = 1$$

$$= P_{\mathcal{H}}$$

$(I - ww^*)(I - ww^*) = I - 2ww^* + (ww^*)(ww^*)$

$P_H \cdot P_H = I - 2\|w\| ww^* + w\|w\| w^*$

$= I - ww^*$

$P_H^2 = P_H$

, thus P_H is a projection matrix.

$F = -m w \times (w \times b)$

$= -m\|w\|^2 \cdot \hat{w} \times (\hat{w} \times b)$ where $\hat{w} = \frac{w}{\|w\|}$, normalized.

$= m\|w\|^2 A_w \cdot (A_w b)$

$= m\|w\|^2 A_w^2 b$

$\boxed{= m\|w\|^2 P_H b}$.

2. $T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $y = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

solve $d = \|y - T\hat{u}\| = \inf\{\|y - Tu\| : u \in \mathbb{C}^2\}$

$$\hat{u} = (T^* T)^{-1} T^* y$$

$$= \left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$\boxed{= \frac{1}{11} \begin{bmatrix} 1 \\ 12 \end{bmatrix}}$$

$$d = \|y - T\hat{u}\|$$

$$= \left\| \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{11} \begin{bmatrix} 1 \\ 12 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{11} \begin{bmatrix} 13 \\ 14 \\ 25 \end{bmatrix} \right\|$$

$$= \left\| \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix} \right\|$$

$$= \sqrt{\frac{81 + 18}{121}}$$

$$= \frac{3}{11}\sqrt{11}$$

$$\boxed{= \frac{3}{\sqrt{11}}}$$

3. $y^* = [1 \quad -2 \quad 3 \quad 5 \quad 10 \quad 8 \quad 4]$.

Find an optimal $p(t) = \sum_0^3 \alpha_j t^j = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$

to solve

$$d^2 = \inf \left\{ \sum_{k=0}^{6} |y_k - p(k)|^2 : p(t) \text{ is a polynomial } \& \deg(p) \le 3 \right\}$$

$$V = \begin{bmatrix} 0^3 & \cdots & 0^0 \\ & \cdots & \\ \vdots & & \vdots \\ 6^3 & \cdots & 6^0 \end{bmatrix}$$

$$d^2 = \sum_{k=0}^{6} | y_k - p(k) |^2 = \| y - V\hat{\alpha} \|^2$$

$$\hat{\alpha} = (V^* V)^{-1} V^* \cdot y.$$

$$\hat{\alpha} = \begin{bmatrix} -0.3889 \\ 3.0952 \\ -4.0635 \\ 0.5952 \end{bmatrix}$$

$$d = \| y - V\hat{\alpha} \| = 2.4785$$

4. find optimal polynomial $p(t) = \sum_{j=0}^{5} \alpha_j t^j$.

to solve

$$\int_0^{\pi} |\sin(t) - p(t)|^2 \, dt = \inf \left\{ \int_0^{\pi} |\sin(t) - p(t)| \, dt : \alpha_j \in \mathbb{C} \right\}$$

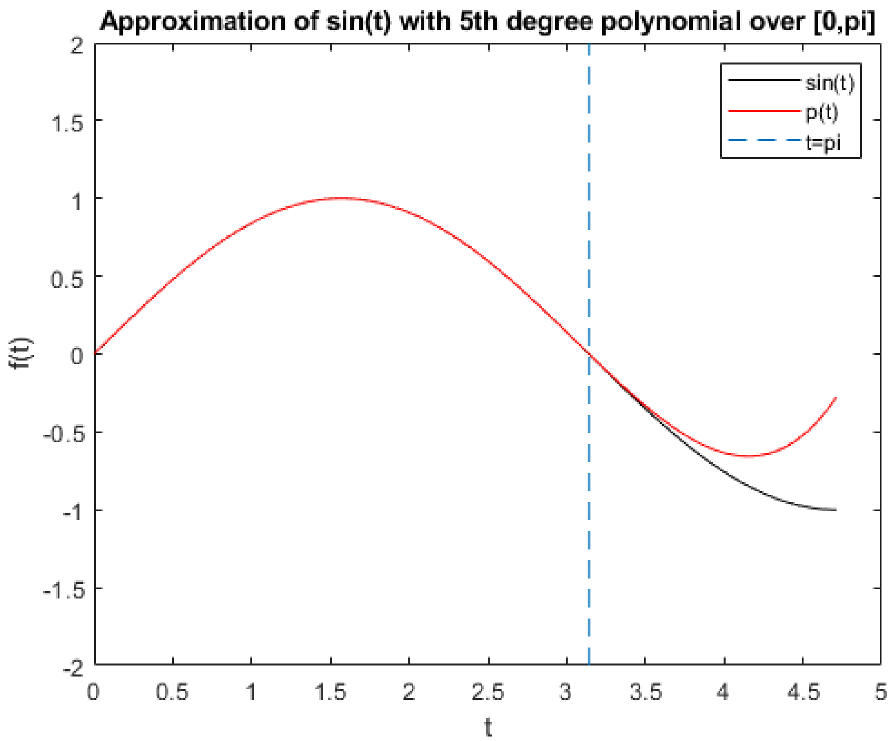
$$T : \alpha = \mathbb{C}^5 \to L^2(0,\pi)$$

$$\hat{\alpha} = (T^* T)^{-1} T^* \sin(t).$$

$$= \begin{bmatrix} 0^5 \cdot 0^5 + \cdots + \pi^5 \pi^5 & \cdots & 0^5 0^0 + \cdots + \pi^5 \pi^0 \\ \vdots & \ddots & \vdots \\ 0^0 0^5 + \cdots + \pi^0 \pi^5 & \cdots & 0^0 0^0 + \cdots + \pi^0 \pi^0 \end{bmatrix}^{-1} T^* \sin(t)$$

$$= \left(\int_0^{\pi} \begin{bmatrix} t^5 \\ \vdots \\ t^0 \end{bmatrix} [t^5 \cdots t^0] \, dt \right)^{-1} \int_0^{\pi} \begin{bmatrix} t^5 \\ \vdots \\ t^0 \end{bmatrix} \sin(t) \, dt.$$

```
alpha = 6×1
   -0.0000
    0.0372
   -0.2338
    0.0545
    0.9826
    0.0013
```



5. Find $p(t)$ of degree at most 5 to solve

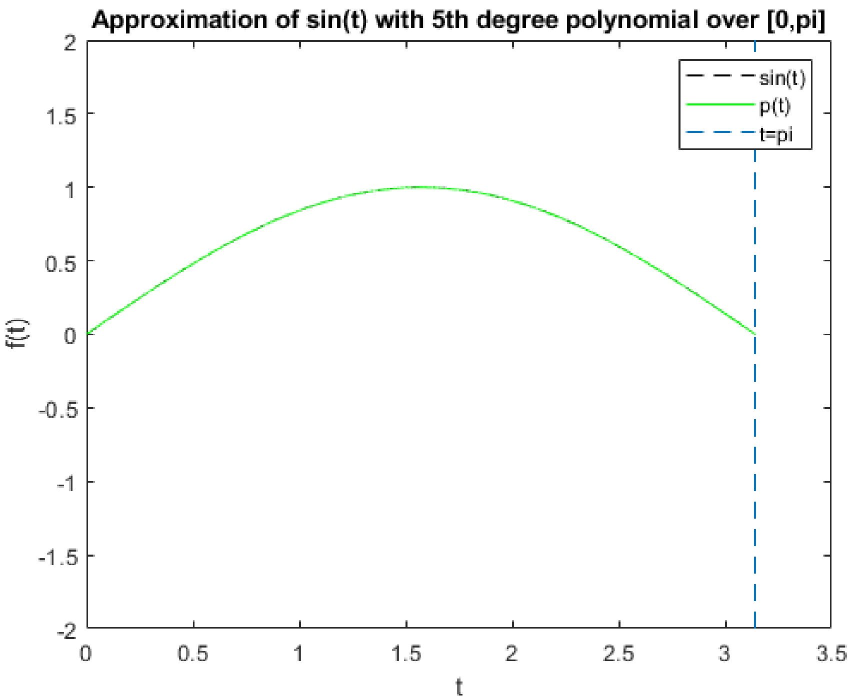
$$d^2 = \inf \left\{ \sum_{k=1}^{m} |y(k) - p(t(k))|^2 : \deg(p(t)) \leq 5 \right\}$$

$$P(t) = T\alpha = \begin{bmatrix} t(1)^5 & \cdots & t(1)^0 \\ & & \\ t(m)^5 & \cdots & t(m)^0 \end{bmatrix} \alpha = y$$

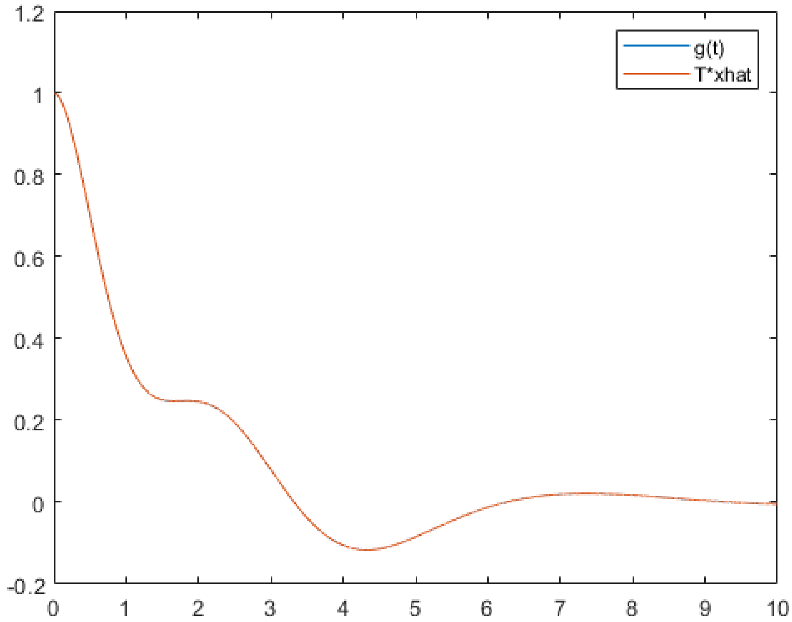
$$\hat{\alpha} = (T^* T)^{-1} T^* y$$

```
alpha = 6×1
    0.0000
    0.0372
   -0.2338
    0.0545
    0.9826
    0.0013
```

The optimal $\hat{\alpha}$ obtained is the same for both questions.



6,



The approximation is right on top of $g(t)$

The integration term $\int_0^{10} |g(t) - Ce^{At}\hat{x}_0|^2 dt$

is basically summation of $|g(t(k)) - Ce^{A t(k)}\hat{x}_0|^2$ to infinity.

Here we are approximating infinite summation with 12000 summation, i.e, least square regression of infinite points with 12000 points.