Midterm. Zhanpeng (ZP) Yang

Part 1.

1. False.

2. True

3. False

4. True

5. False

6. True

7. True ?

8. True

9. False

10. True

11. False.

12. True

13. True

14. True

15. True

16. False

17. False.

18. True.

19. True

20. False

Part II

1. $L: P_3 \to P_3$, $L(f) = f'$

a). i) if $f = 0$, $L(0) = 0' = 0$. (zero is in it)

ii) if $f = a + bx + cx^2 + dx^3$. $f' = b + 2cx + 3dx^2$

$L(2f) = 2b + 4cx + 6dx^2$

$= 2(b + 2cx + 3dx^2)$

$= 2f'$ (scalar multiple checks out)

iii)

$L(f_1 + f_2) = (b_1 + b_2) + 2(c_1 + c_2)x + 3(d_1 + d_2)x^2$

$= (b_1 + 2c_1x + 3d_1x^2) + (b_2 + 2c_2x + 3d_2x^2)$

$= f_1' + f_2'$

(sumation checks out)

Thus L is a linear transformation.

1.b) $L(f) = 0$. $f' = 0$, polynomials that only has a constant

$a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ basis

1.c)

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2. $\text{span}\left\{ \underset{a_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{a_2}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}, \underset{a_3}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} \right\}, W \in \mathbb{R}^3$ $\quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$w_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$B = a_2 - w_1^T a_2 \cdot w_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} [1 \ 1 \ 0] \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$w_2 = \frac{B}{\|B\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = a_3 - w_1^T a_3 w_1 - w_2^T a_3 w_2$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot [1 \ 1 \ 0] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - [0 \ 0 \ 1] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ -3/2 \\ 0 \end{bmatrix}.$$

$$w_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\left\{ \underset{w_1}{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}} \right\}$$

orthonormal basis of W

$v = w + u$

$\boxed{v = w_2 + [0]}$

$w_2 \in W$

$[0] \in W^\perp$.

$\|w_1 \ 2w_3\|^2$

$$\left\| \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\| = \frac{9}{2} + \frac{1}{2} = 5$$

$$= \|w_1\|^2 + 2\|w_3\|$$

3. $\{u, v, w\}$ orthonormal in $\mathbb{R}^n$, Find $\|u-2v-3w\|^2$

a) $\|u-2v-2w\|^2 = \|u\|^2 + \|2v\|^2 + \|2w\|^2$

$= 1 + 4 + 4$

$= 9$

b) $u, v \in \mathbb{R}^n$, prove $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v^2\|$

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

$\|u+v\|^2 = (u_1+v_1)^2 + \cdots + (u_n+v_n)^2$

$\|u-v\|^2 = (u_1-v_1)^2 + \cdots + (u_n+v_n)^2$

$\|u+v\|^2 + \|u-v\|^2$

$= (u_1+v_1)^2 + \cdots + (u_n+v_n)^2$

$+ (u_1-v_1)^2 + \cdots + (u_n+v_n)^2$

$= u_1^2 + u_1^2 + \cancel{2u_1v_1} + \cdots + u_n^2 + v_n^2 + \cancel{2u_nv_n}$

$+ u_1^2 + v_1^2 - \cancel{2u_1v_1} + \cdots + u_n^2 + v_n^2 - \cancel{2u_nv_n}$

$= 2u_1^2 + 2v_1^2 + \cdots + 2u_n^2 + 2v_n^2$

$= 2(u_1^2 + \cdots + u_n^2) + 2(v_1^2 + \cdots + v_n^2)$

$= 2\|u\|^2 + 2\|v\|^2$

proven

4. $v = \begin{bmatrix} a^2 \\ a \\ 1 \end{bmatrix}$, find a for v in the span $\left\{ \underset{w_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} \right\}$.

$v = c_1 w_1 + c_2 W_2 + c_3 W_3$.

$$\Rightarrow \begin{cases} a^2 = c_1 + c_2 \\ a = 2c_1 + c_2 + c_3 \\ 1 = 3c_1 + c_2 + 2c_3. \end{cases} \qquad 1 - a = c_1 + c_3$$

4 unknowns
3 equations

Let $c_1 = 0$, $c_2 = 1$, $c_3 = 0$.

$$\left.\begin{matrix} a = 1 \\ a^2 = 1 \end{matrix}\right\} \Rightarrow \underline{a = 1}$$

$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = w_2 \in \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

5. Best fit line through $(1,2)$, $(0,3)$, $(-1,5)$

$$y = a + bx.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

$$A^T A \cdot \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}.$$

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 2 \\ -3.5 \end{bmatrix}$$

best fit line is $\underline{y = 2 - 3.5x}$