Problem 1.

Show $P_H f = P_M f + R_{f\varphi} R_{\varphi}^{-1} \varphi$

$$E(f - P_H f)(f - P_H f)^* = E(f - P_M f)(f - P_M f)^* - R_{f\varphi} R^{-1} R_{\varphi f}.$$

using Lemma 2.4.1 in Ch.2.

$$T = \begin{bmatrix} R & M \\ N & Q \end{bmatrix}, \quad \Delta = Q - N R^{-1} M \text{ (Schur complement of } T\text{)}$$

$$T^{-1} = \begin{bmatrix} I & -R^{-1}M \\ 0 & I \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & \Delta^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -NR^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} R^{-1} + R^{-1} M \Delta^{-1} N R^{-1} & -R^{-1} M \Delta^{-1} \\ -\Delta^{-1} N R^{-1} & \Delta^{-1} \end{bmatrix}$$

In this case

$$T = R_h = \begin{bmatrix} R_g & R_{gy} \\ R_{yg} & R_y \end{bmatrix}, \quad \Delta = R_\varphi = R_y - R_{yg} R_g^{-1} R_{gy}$$

$$T^{-1} = R_h^{-1} = \begin{bmatrix} I & -R_g^{-1} R_{gy} \\ 0 & I \end{bmatrix} \begin{bmatrix} R_g^{-1} & 0 \\ 0 & R_\varphi^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -R_{yg} R_g^{-1} & I \end{bmatrix}$$

$$P_H f = R_{fh} \cdot R_h^{-1} \cdot h$$

$$= [R_{fg} \;\; R_{fy}] \begin{bmatrix} I & -R_g^{-1} R_{gy} \\ 0 & I \end{bmatrix} \begin{bmatrix} R_g^{-1} & 0 \\ 0 & R_\varphi^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ R_{yg} R_g^{-1} & I \end{bmatrix} \begin{bmatrix} g \\ y \end{bmatrix}$$

$$= [R_{fg} \;\; R_{fy} - R_{fg} R_g^{-1} R_{gy}] \begin{bmatrix} R_g^{-1} & 0 \\ 0 & R_\varphi^{-1} \end{bmatrix} \begin{bmatrix} g \\ y - R_{yg} R_g^{-1} g \end{bmatrix}$$

$$= [R_{fg} \;\; R_{fy} - R_{fg} R_g^{-1} R_{gy}] \begin{bmatrix} R_g^{-1} g \\ R_\varphi^{-1} y - R_\varphi^{-1} \overbrace{R_{yg} R_g^{-1} g}^{P_M y} \end{bmatrix}$$

$$= R_{fg} R_g^{-1} y + (R_{fy} - R_{fg} R_g^{-1} R_{gy})(R_\varphi^{-1} y - R_\varphi^{-1} P_m y)$$

$$= P_m f + (R_{fy} - R_{fg} R_g^{-1} R_{gy}) R_\varphi^{-1} (y - P_m y)$$

$$\underline{= P_m f + R_{f\varphi} \cdot R_\varphi^{-1} \cdot \varphi} .$$

$$E(f - P_H f)(f - P_H f)^* = R_f - R_{fh} R_h^{-1} R_{hf} .$$

$$R_f = E(f - P_m f)(f - P_m f)^*$$

$$R_{fh} R_h^{-1} R_{hf} = \begin{bmatrix} R_{fg} & R_{fy} \end{bmatrix} \begin{bmatrix} I & -R_g^{-1} R_{gy} \\ 0 & I \end{bmatrix} \begin{bmatrix} R_g^{-1} & 0 \\ 0 & R_\varphi^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -R_{yg} R_g^{-1} & I \end{bmatrix} \begin{bmatrix} R_{gf} \\ R_{yf} \end{bmatrix}$$

$$= \begin{bmatrix} R_{fg} & R_{fy} - R_{fg} R_g^{-1} R_{gy} \end{bmatrix} \begin{bmatrix} R_g^{-1} & 0 \\ 0 & R_\varphi^{-1} \end{bmatrix} \begin{bmatrix} R_{gf} \\ R_{yf} - R_{yg} R_g^{-1} R_{gf} \end{bmatrix}$$

$$= \begin{bmatrix} R_{fg} R_g^{-1} & R_{f\varphi} R_\varphi^{-1} \end{bmatrix} \begin{bmatrix} R_{gf} \\ R_{\varphi f} \end{bmatrix}$$

$$= \cancel{R_{fg} R_g^{-1} R_{gf}} + R_{f\varphi} R_\varphi^{-1} R_{\varphi f}$$

$$= R_{f\varphi} R_\varphi^{-1} R_{\varphi f}$$

$$E(f - P_H f)(f - P_H f)^* = \underline{E(f - P_m f)(f - P_m f)^* - R_{f\varphi} R_\varphi^{-1} R_{\varphi f}}$$

2. Consider SS system:

$$x(n+1) = Ax(n) + Bu(n) \quad \& \quad y(n) = Cx(n) + Dv(n)$$

$$\hat{x}(n) = P_{M_{n-1}} x(n).$$

find state estimate $P_{M_n} x(n)$ in terms of $\hat{x}(n)$ & $y(n)$.

Lemma 3.3.1: $P_{M_n} f = P_{M_{n-1}} f + R_{f\varphi(n)} R^{-1}_{\varphi(n)} \varphi(n)$

$$\varphi(n) = y(n) - P_{M_{n-1}} y(n)$$

$$= y(n) - P_{M_{n-1}} [Cx(n) + Dv(n)]$$

$$= y(n) - P_{M_{n-1}} Cx(n) - \cancel{P_{M_{n-1}} D v(n)}$$

$$= y(n) - C\hat{x}(n)$$

$$= C\tilde{x}(n) + Dv(n) \qquad \tilde{x}(n) = x(n) - \hat{x}(n) \text{ (estimation error)}$$

$$\hat{x}(n) = P_{M_n} x(n) = P_{M_{n-1}} x(n) + R_{x(n)\varphi(n-1)} R^{-1}_{\varphi(n-1)} \varphi(n-1)$$

$$= P_{M_{n-1}} (Ax(n-1) + Bu(n-1)) + R_{x(n)\varphi(n-1)} R^{-1}_{\varphi(n-1)} \varphi(n-1)$$

$$= A\hat{x}(n-1) + R_{x(n)\varphi(n-1)} R^{-1}_{\varphi(n-1)} \varphi(n-1)$$

$$R_{x(n)\varphi(n-1)} = E\, x(n)\varphi(n-1)^* \qquad R_{\varphi(n-1)} = E\,\varphi(n-1)\varphi(n-1)^* = CQ_{n-1}C^* + DD^*$$

$$= E(Ax(n-1) + Bu(n-1))\varphi(n-1)^*$$

$$= AE\, x(n-1)\varphi(n-1)^*$$

$$= AE\, x(n-1)(C\tilde{x}(n-1) + Dv(n-1))^*$$

$$= AE\, x(n-1)\tilde{x}(n-1)^* C^*$$

$$= AE(\hat{x}(n-1) + \tilde{x}(n-1))\tilde{x}(n-1)^* C^*$$

$$= AE\,\tilde{x}(n-1)\tilde{x}(n-1)^* C^* = AQ_{n-1}C^*$$

$$\underline{\hat{x}(n) = A\hat{x}(n-1) + AQ_{n-1}C^*(CQ^*_{n-1}C^* + DD^*)^{-1}\varphi(n-1)}$$

3. Consider $x(n+1) = a x(n) + u(n)$,

$$y(n) = x(n) + v(n).$$

a is a scalar, $\{u(0), v(0), v(1), x(0)\}$ are IID independent $G(0,1)$ r.v.

Let $M_0 = \text{span}\{y(0)\}$, $M_1 = \text{span}\{y(0), y(1)\}$

Find: (i) $\hat{x}(0) = P_{M_1} x(0)$

(ii) $E|x(0) - \hat{x}(0)|^2$

(iii) α & β such that $\varphi(1) = y(1) - P_{M_0} y(1) = \alpha y(1) + \beta y(0)$.

i)

$\hat{y}_0 = x_0 + v_0$ $\qquad$ $y_1 = x_1 + v_1$

$x_1 = a x_0 + u_0$ $\qquad$ $= a x_0 + u_0 + v_1$

$$\hat{x}_0 = R_{x_0 y} R_y^{-1} y \quad , \text{where } y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 + v_0 \\ a x_0 + u_0 + v_1 \end{bmatrix}$$

$$R_y = E y y^* = \begin{bmatrix} E y_0^2 & E y_0 y_1 \\ E y_1 y_0 & E y_1^2 \end{bmatrix}$$

$$E y_0^2 = E(x_0 + v_0)^2 = E x_0^2 + \cancel{2 E x_0 v_0} + E v_0^2 = 1 + 1 = 2.$$

$$E y_0 y_1 = E(x_0 + v_0)(a x_0 + u_0 + v_1) = E a x_0^2 + E x_0 u_0 + E x_0 v_1 + E v_0 a x_0 + E v_0 u_0 + E v_0 v_1$$
$$= a$$

$$E y_1^2 = E(a x_0 + u_0 + v_1)^2 = E a^2 x_0^2 + \cdots + E u_0^2 + \cdots + E v_1^2 = a^2 + 2$$

$$\Rightarrow R_y = \begin{bmatrix} 2 & a \\ a & a^2+2 \end{bmatrix} \qquad R_y^{-1} = \frac{1}{a^2+4} \begin{bmatrix} a^2+2 & -a \\ -a & 2 \end{bmatrix}$$

$$R_{x_0 y} = E x_0 y^* = E[x_0^2 + x_0 v_0 \quad a x_0^2 + x_0 u_0 + x_0 v_1] = [1 \quad a]$$

$$\hat{x}_0 = \frac{1}{a^2+4} [1 \quad a] \begin{bmatrix} a^2+2 & -a \\ -a & 2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$= \frac{1}{a^2+4} [2 \quad a] \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$= \underline{\frac{2 y_0 + a y_1}{a^2+4}}$$

ii) $E|x_0 - \hat{x}_0|^2 = E\, x_0^2 - R_{x_0 g} R_g^{-1} R_{x_0 g}^*$

$R_{x_0 g} R_g^{-1} R_{x_0 g}^* = \frac{1}{a^2+4} [1 \quad a] \begin{bmatrix} a^2+2 & -a \\ -a & 2 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = \frac{2+a^2}{a^2+4}$

$= 1 - \frac{2+a^2}{a^2+4}$

$= \frac{2}{a^2+4}$

iii) $g = y_0$

$P_{m_0} y_1 = R_{y_1} R_g^{-1} g = \frac{a\, y_0}{2}$

$\begin{cases} R_{y_1 g} = E\, y_1 g^* = a \\ R_g = E\, g g^* = 2 \end{cases}$

$\varphi(a) = y_1 - \frac{a y_0}{2}$

$\alpha = 1 \qquad \beta = \frac{a}{2}$