3.7.1 Exercise

Problem 1. Consider the state space system

$$x(n+1) = 2x(n) + \frac{1}{\sqrt{5}}u(n) \quad \text{and} \quad y(n) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} x(n) + \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \\ v_4(n) \end{bmatrix}.$$

Moreover, $u(n)$ and $\{v_j(n)\}_1^4$ are all independent Gaussian white noise process with variance one, which are also independent to the initial condition $x(0)$. Find the steady state Kalman filter and the steady state error covariance P. Hint: $A(I + BA)^{-1} = (I + AB)^{-1}A$.

The steady state Kalman filter is

$$\begin{cases} \hat{x}(n+1) = (A - K_p C)\hat{x}(n) + K_p\, y(n) \\ K_p = APC^*(CPC^* + DD^*)^{-1} \\ P = APA^* + BB^* - APC^*(CPC^* + DD^*)^{-1}CPA^* \end{cases}$$

$$P = 4P + \frac{1}{5} - P^2 (4P+1)^{-1} \cdot [2 \ -2 \ 2 \ -2] \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$P = 4P + \frac{1}{5} - \frac{P^2}{4P+1} \cdot 16$$

$P = -\frac{1}{20}, 1$, the positive solution is $\boxed{P = 1}$

$$K_p = 2 \cdot (1+4)^{-1} [1 \ -1 \ 1 \ -1] = \frac{2}{5}[1 \ -1 \ 1 \ -1], \quad K_p C = \frac{8}{5}$$

$$\hat{x}(n+1) = 2\hat{x}(n) - \frac{8}{5}\hat{x}(n) + \frac{8}{5}\hat{x}(n) + \frac{2}{5}[v_1(n) - v_2(n) + v_3(n) - v_4(n)]$$

$$\underline{\hat{x}(n+1) = 2\hat{x}(n) + \frac{2}{5}[v_1(n) - v_2(n) + v_3(n) - v_4(n)]}$$

Exercise. Consider the state space system

$$x(n+1) = Ax(n) + u(n) \qquad \text{and} \qquad y(n) = Cx(n) + v(n) \tag{9.6}$$

where $A = A(n)$ are matrices on a state space $\mathcal{X}$ and $C = C(n)$ are matrices mapping $\mathcal{X}$ into $\mathcal{Y}$. Moreover, $u(n)$ and $v(n)$ are mean zero Gaussian random process which are independent to the initial condition $x(0)$ which is a Gaussian random vector. Furthermore, assume that

$$E\begin{bmatrix} u(n) \\ v(n) \end{bmatrix} \begin{bmatrix} u(m)^* & v(m)^* \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \delta_{n,m}$$

where $R_{ij} = R_{ij}(n)$ can be a function of n. (As expected, $\delta_{n,m}$ is the Kronecker delta, that is, $\delta_{n,m} = 1$ if $n = m$ and zero otherwise.) Let $\mathcal{M}_n = \operatorname{span}\{1, y(k)\}_0^n$ and $\widehat{x}(n) = P_{\mathcal{M}_{n-1}} x(n)$ denote the optimal state estimate. Let $\widetilde{x}(n) = x(n) - \widehat{x}(n)$ and Q_n be the error covariance matrix defined by

$$Q_n = E\widetilde{x}(n)\widetilde{x}(n)^* = E\left(x(n) - \widehat{x}(n)\right)\left(x(n) - \widehat{x}(n)\right)^*.$$

Find the Kalman filter for the state space system in (9.6). To be precise, find a recursive estimate for the optimal state $\widehat{x}(n)$ and a recursive formula for the error covariance Q_n.

(i) Show that the optimal state is given by

$$\begin{aligned} \widehat{x}(n+1) &= A\widehat{x}(n) + L_n\left(y(n) - C\widehat{x}(n)\right) \\ &= \left(A - L_n C\right)\widehat{x}(n) + Ly(n) \\ L_n &= \left(AQ_nC^* + R_{12}\right)\left(CQ_nC^* + R_{22}\right)^{-1}. \end{aligned}$$

The initial conditions is $\widehat{x}(0) = \mu_0$.

i) $\hat{x}(n+1) = P_{M_n} x(n+1)$

$= P_{M_{n-1}} x(n+1) + R_{x(n+1)\psi(n)} R^{-1}_{\psi(n)} \psi(n)$.

$\psi(n) = y(n) - P_{M_{n-1}} y(n) = y(n) - P_{M_{n-1}}(Cx(n) + v(n)) = Cx(n) + v(n) - C\hat{x}(n) - \cancel{P_{M_{n-1}} v(n)}$

$= C\tilde{x}(n) + v(n) = y_n - C\hat{x}(n)$

$P_{M_{n-1}} x(n+1) = P_{M_{n-1}}(Ax(n) + u(n)) = P_{M_{n-1}} A x(n) + \cancel{P_{M_{n-1}} u(n)} = A\hat{x}(n)$

$\Rightarrow \hat{x}(n+1) = A\hat{x}(n) + R_{x(n+1)\psi(n)} R^{-1}_{\psi(n)} \psi(n)$.

$R_{x(n+1)\psi(n)} = E[Ax(n) + u(n)][C\tilde{x}(n) + v(n)]^*$

$= AE\, x(n)\tilde{x}(n) C^* + \cancel{AE\, x(n) v(n)} + \cancel{E u(n)\tilde{x}(n)} C^* + E u(n) v(n)$

$= AE[\hat{x}(n) + \tilde{x}(n)]\tilde{x}(n) C^* + R_{12}$

$= AE\,\tilde{x}(n)\tilde{x}(n) C^* + R_{12} = AQ_nC^* + R_{12}$

$R_{\psi(n)} = E\,\psi(n)\psi(n)^* = E[C\tilde{x}(n) + v(n)][C\tilde{x}(n) + v(n)]^* = CE\,\tilde{x}(n)\tilde{x}(n)^* C^* + E v(n) v(n)^*$

$= CQ_nC^* + R_{22}$

$\hat{x}(n+1) = A\hat{x}(n) + (AQ_nC^* + R_{12})(CQ_nC^* + R_{22})^{-1}(y_n - C\hat{x}(n))$

$= A\hat{x}(n) + L_n(y_n - C\hat{x}(n)) = (A - L_nC)\hat{x}(n) + L_n y(n)$

where $L_n = (AQ_nC^* + R_{12})(CQ_nC^* + R_{22})^{-1}$

(ii) Show that the error covariance Q_n is given by the solution to the Riccati difference equation

$$Q_{n+1} = AQ_nA^* + R_{11} - (AQ_nC^* + R_{12})(CQ_nC^* + R_{22})^{-1}(AQ_nC^* + R_{12})^*. \tag{9.7}$$

The initial condition $Q_0 = E\big(x(0) - \mu_0\big)\big(x(0) - \mu_0\big)^*$. Another form for the Riccati difference equation is given by

$$Q_{n+1} = (A - L_nC)Q_n(A - L_nC)^* + \begin{bmatrix} I & -L_n \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I \\ -L_n^* \end{bmatrix}.$$

ii) $Q_{n+1} = E\,\tilde{x}(n+1)\,\tilde{x}(n+1)$

$= E\,[x(n+1) - \hat{x}(n+1)][x(n+1) - \hat{x}(n+1)]^*$

where $x(n+1) - \hat{x}(n+1) = Ax(n) + u(n) - L_n(C\tilde{x}(n) + v(n))$

$= (A - L_nC)\tilde{x}(n) + u(n) - L_nv(n)$

$Q_{n+1} = E[(A - L_nC)\tilde{x}(n) + u(n) - L_nv(n)][(A - L_nC)\tilde{x}(n) + u(n) - L_nv(n)]^*$

$= (A - L_nC)E\,\tilde{x}(n)\tilde{x}(n)(A - L_nC)^* + (A - L_nC)\,\cancel{E\,\tilde{x}(n)u(n)} - (A - L_nC)\,\cancel{E\,\tilde{x}(n)v(n)}\,L_n^*$

$+ E\,u(n)u(n)^* + E\,u(n)v(n)\,L_n^* - L_nE\,v(n)u(n) + L_nE\,v(n)v(n)^*L_n^*$

$= (A - L_nC)Q_n(A - L_nC)^* + R_{11} + R_{12}L_n^* - L_nR_{21} + L_nR_{22}L_n^*$

$= (A - L_nC)Q_n(A - L_nC)^* + [I \;\; -L_n]\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} I \\ -L_n^* \end{bmatrix}$

(iii) Show that $\widehat{x}(n|n) = P_{\mathcal{M}_n} x(n)$ is determined by

$$\widehat{x}(n|n) = \widehat{x}(n) + Q_n C^* \left(C Q_n C^* + R_{22}\right)^{-1} \left(y(n) - C\widehat{x}(n)\right).$$

$$\hat{x}(n|n) = P_{M_n} x(n) = P_{M_{n-1}} x(n) + R_{x(n)\varphi(n)} R^{-1}_{\varphi(n)} \varphi(n)$$

$$= \hat{x}(n) + R_{x(n)\varphi(n)} R^{-1}_{\varphi(n)} \varphi(n)$$

$$R_{x(n)\varphi(n)} = E\, x(n) \psi(n)$$

$$= E\, x(n) (C\tilde{x}(n) + Dv(n))^*$$

$$= E\, x(n) \tilde{x}(n)^* C^* + E\, \cancel{x(n) v(n)^*} D^*$$

$$= E\, [\hat{x}(n) + \tilde{x}(n)]\, \tilde{x}(n)^* C^*$$

$$= E\, \tilde{x}(n) \tilde{x}(n)^* C^*$$

$$= Q_n C^*$$

$$R_{\varphi(n)} = C Q_n C^* + R_{22}$$

$$\Rightarrow \hat{x}(n|n) = \hat{x}(n) + Q_n C^* (C Q_n C^* + R_{22})^{-1} (y(n) - C\hat{x}(n))$$