HW7. 35-38 in "Quadratic Stability"

P128

Exercise 35 Prove the following result: Suppose there exist a positive-definite symmetric matrix P and a positive scalar α which satisfy

$$\begin{pmatrix} PA + A^T P + C^T C + 2\alpha P & PB \\ B^T P & -\gamma^{-2} I \end{pmatrix} \leq 0 . \qquad (11.26)$$

Then system (11.18)-(11.19) is globally exponentially stable about the origin with rate of convergence α.

$\dot{x} = Ax + B\phi(Cx)$ $\quad A : n\times n . \quad B : n\times m \quad C : p\times n$

$z = Cx .$ $\quad \|\phi(z)\| \leq \gamma\|z\| \quad , \exists \gamma \geq 0 .$

$\leq \gamma \|Cx\|$

Let $v = x^T P x$

$\dot{v} = 2 x^T P \dot{x}$

$= 2x^T P (Ax + B\phi(Cx))$

$= 2x^T PAx + 2x^T PB \phi(Cx)$

$= x^T (PA + A^T P) x + 2x^T PB \phi(Cx)$

$\leq x^T (PA + A^T P) x + 2\| B^T Px \| \cdot \|\phi\|$

$\leq x^T (PA + A^T P) x + 2\| B^T Px\| \cdot \gamma \| Cx\|$

$\leq x^T (PA + A^T P) x + \gamma^2 \|B^T Px\| + \|Cx\|^2$

$= x^T (PA + A^T P) x + \gamma^2 x^T PBB^T Px + x^T C^T C x$

$= x^T (PA + A^T P + \gamma^2 PBB^T P + C^T C) x$

$= x^T Q x .$

Thus the system is GES.

using Schur complement result.

$PA + A^T P + C^T C + 2\alpha P - PB \cdot (-\gamma^2) \cdot B^T P \leq 0 \qquad -\gamma^{-2} I \leq 0$

$PA + A^T P + C^T C + 2\alpha P + \gamma^2 PBB^T P \leq 0 .$

$PA + A^T P + C^T C + \gamma^2 PBB^T P \leq -2\alpha P$

$Q \leq -2\alpha P$

Thus the system is GES with rate of α.

P133

Exercise 36 Recall the double inverted pendulum of ~~Example~~ Exercise 34. Using the results of this section, obtain a value of the spring constant k which guarantees that this system is globally exponentially stable about the zero solution.

$$\begin{aligned}\ddot{\theta}_1 + 2\dot{\theta}_1 - \dot{\theta}_2 + 2k\theta_1 - k\theta_2 - \sin\theta_1 &= 0\\ \ddot{\theta}_2 - \dot{\theta}_1 + \dot{\theta}_2 - k\theta_1 + k\theta_2 - \sin\theta_2 &= 0\end{aligned}$$

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix} \qquad x = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\psi_1(\theta_1) = \begin{cases} \dfrac{\sin\theta_1}{\theta_1} & \theta_1 \neq 0 \\ 1 & \theta_1 = 0 \end{cases} \qquad \Delta A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{a_1}{-1} \le \psi_1(\theta) \le \underset{b_1}{1}$$

$$\psi_2(\theta_2) = \begin{cases} \dfrac{-\sin\theta_2}{\theta_2} & \theta_2 \neq 0 \\ 1 & \theta_2 = 0 \end{cases} \qquad \Delta A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\underset{a_2}{-1} \le \psi_2(x) \le \underset{b_2}{1}$$

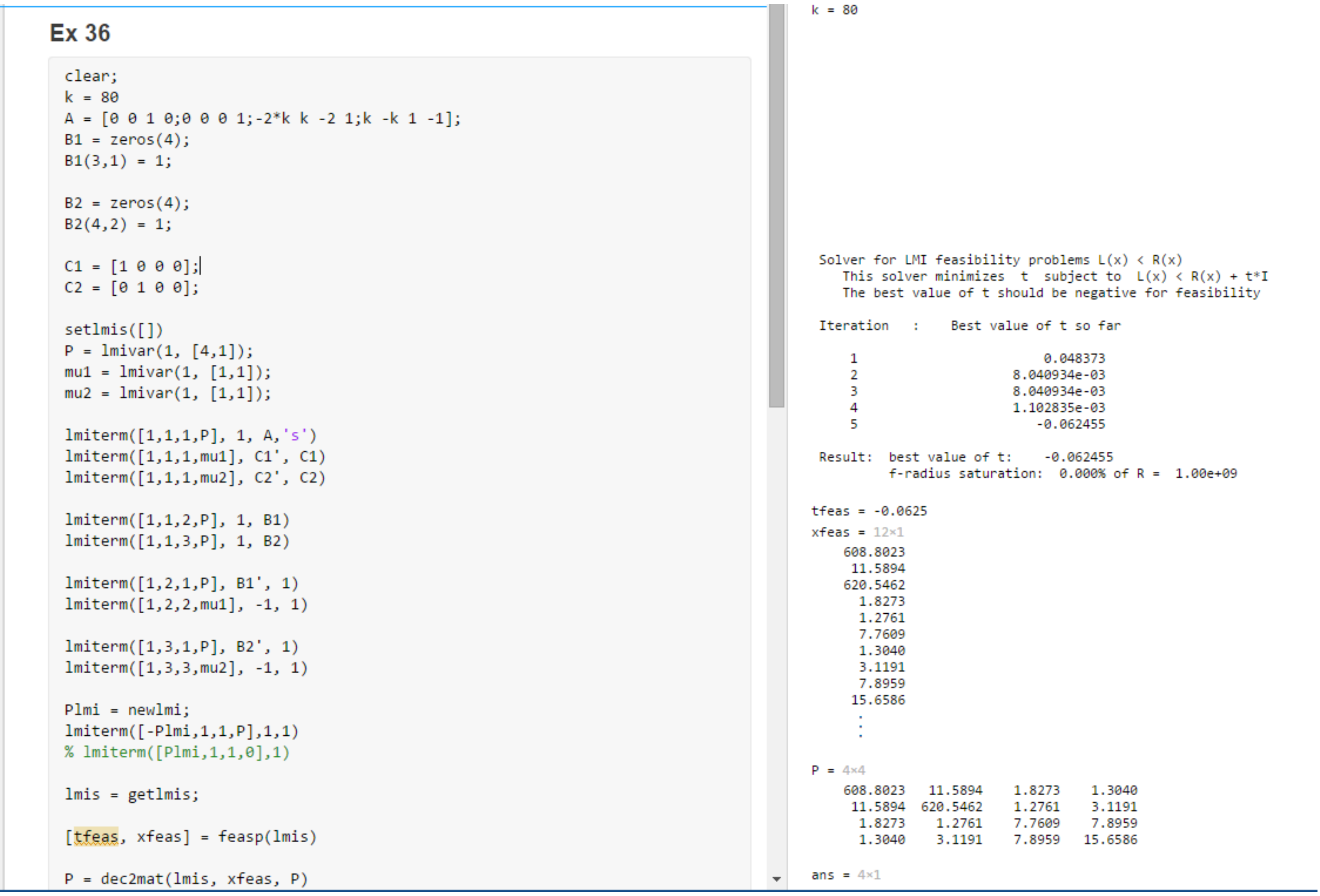
$$\dot{x} = A_0 \cdot x + \Delta A_1 \cdot \psi_1(\theta_1) + \Delta A_2 \psi_2(\theta_2)$$

$$= Ax + B_1 \psi_1(C_1 x) + B_2 \psi_2(C_2 x)$$

$$\Rightarrow \begin{array}{ll} A = A_0 & \\ B_1 = \Delta A_1 & C_1 = [1\,0\,0\,0] \\ B_2 = \Delta A_2 & C_2 = [0\,1\,0\,0] \end{array}$$

$$\begin{bmatrix} PA + A^T P + \sum_{i=1}^{2} \mu_i C_i' C_i & PB_1 & PB_2 \\ B_1^T P & -\mu_1 I & 0 \\ B_2^T P & 0 & -\mu_2 I \end{bmatrix} < 0.$$

After using LMI toolbox to solve the LMI shown above, k = 80 yeilds a feasible solution.



Ex 36

```
clear;
k = 80
A = [0 0 1 0;0 0 0 1;-2*k k -2 1;k -k 1 -1];
B1 = zeros(4);
B1(3,1) = 1;

B2 = zeros(4);
B2(4,2) = 1;

C1 = [1 0 0 0];
C2 = [0 1 0 0];

setlmis([])
P = lmivar(1, [4,1]);
mu1 = lmivar(1, [1,1]);
mu2 = lmivar(1, [1,1]);

lmiterm([1,1,1,P], 1, A,'s')
lmiterm([1,1,1,mu1], C1', C1)
lmiterm([1,1,1,mu2], C2', C2)

lmiterm([1,1,2,P], 1, B1)
lmiterm([1,1,3,P], 1, B2)

lmiterm([1,2,1,P], B1', 1)
lmiterm([1,2,2,mu1], -1, 1)

lmiterm([1,3,1,P], B2', 1)
lmiterm([1,3,3,mu2], -1, 1)

Plmi = newlmi;
lmiterm([-Plmi,1,1,P],1,1)
% lmiterm([Plmi,1,1,0],1)

lmis = getlmis;

[tfeas, xfeas] = feasp(lmis)

P = dec2mat(lmis, xfeas, P)
```

```
k = 80


 Solver for LMI feasibility problems L(x) < R(x)
    This solver minimizes  t  subject to  L(x) < R(x) + t*I
    The best value of t should be negative for feasibility

 Iteration   :    Best value of t so far

     1                        0.048373
     2                     8.040934e-03
     3                     8.040934e-03
     4                     1.102835e-03
     5                       -0.062455

 Result:  best value of t:    -0.062455
          f-radius saturation:  0.000% of R =  1.00e+09

tfeas = -0.0625
xfeas = 12×1
    608.8023
     11.5894
    620.5462
      1.8273
      1.2761
      7.7609
      1.3040
      3.1191
      7.8959
     15.6586
       ⋮

P = 4×4
    608.8023   11.5894    1.8273    1.3040
     11.5894  620.5462    1.2761    3.1191
      1.8273    1.2761    7.7609    7.8959
      1.3040    3.1191    7.8959   15.6586

ans = 4×1
```

Here we consider systems described by

$$\dot{x} \;=\; Ax - B\phi(Cx) \tag{11.37}$$

where

$$z'\phi(z) \geq 0 \tag{11.38}$$

for all z. Examples of ϕ include $\phi(z) = z, z^3, z^5, \mathrm{sat}(z), \mathrm{sgm}\,(z)$.

Exercise 37 Prove the following result: Suppose there exists a positive-definite symmetric matrix P and a positive scalar α which satisfy which satisfies

$$PA + A'P + 2\alpha P \;\leq\; 0 \tag{11.40a}$$

$$B'P \;=\; C \tag{11.40b}$$

Then system (11.37)-(11.38) is globally exponentially stable about the origin with rate α and with Lyapunov matrix P.

Let $V = x^T P x$,

$\dot{V} = 2x^T P \dot{x}$

$= 2x^T P (Ax - B\phi(Cx))$

$= 2x^T PAx - 2x^T P B\phi(Cx)$

$= 2x^T PAx - 2(Cx)'\phi(Cx)$. since $z'\phi(z) \geq 0$,

$\leq 2x^T PAx$

$= x^T(PA + A^T P)x$.

$\dot{V} \leq 2x^T Q x$, $Q = PA$

Thus the system is GES.

Since $PA + A'P \leq -2\alpha P$

The system has a convergence rate of α.

Exercise 38 Consider the transfer function

$$\hat{g}(s) = \frac{\beta s + 1}{s^2 + s + 2}$$

Using Lemma 12, determine the range of β for which this transfer function is SPR. Verify your results with the KYPSPR lemma.

$\hat{g}(s)$ has poles at $\frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm j\sqrt{7}}{2}$.

$Re(s_p) < 0$ thus $\hat{g}(s)$ is stable, Condition (a) holds.

$$\hat{g}(jw) = \frac{j\beta w + 1}{-w^2 + jw + 2}.$$

$$\hat{g}'(jw) = \frac{-j\beta w + 1}{-w^2 - jw + 2}$$

$$\hat{g}(jw) + \hat{g}'(jw) = \frac{j\beta w + 1}{-w^2 + jw + 2} + \frac{-j\beta w + 1}{-w^2 - jw + 2}$$

$$= \frac{2\beta w^2 - 2w^2 + 4}{(w^2)^2 + (2 - jw)^2}$$

$$= \frac{2[w^2(\beta - 1) + 2]}{(w^2)^2 + (2 - jw)^2}$$

It's clear that the denominator is ≥ 0, and the numerator is only > 0, $\forall w \in \mathbb{R}$, iff $\beta \geq 1$

Thus condition b) holds.

The state space st this transfer function are

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad C = [\beta \ 1], \quad D = 0$$

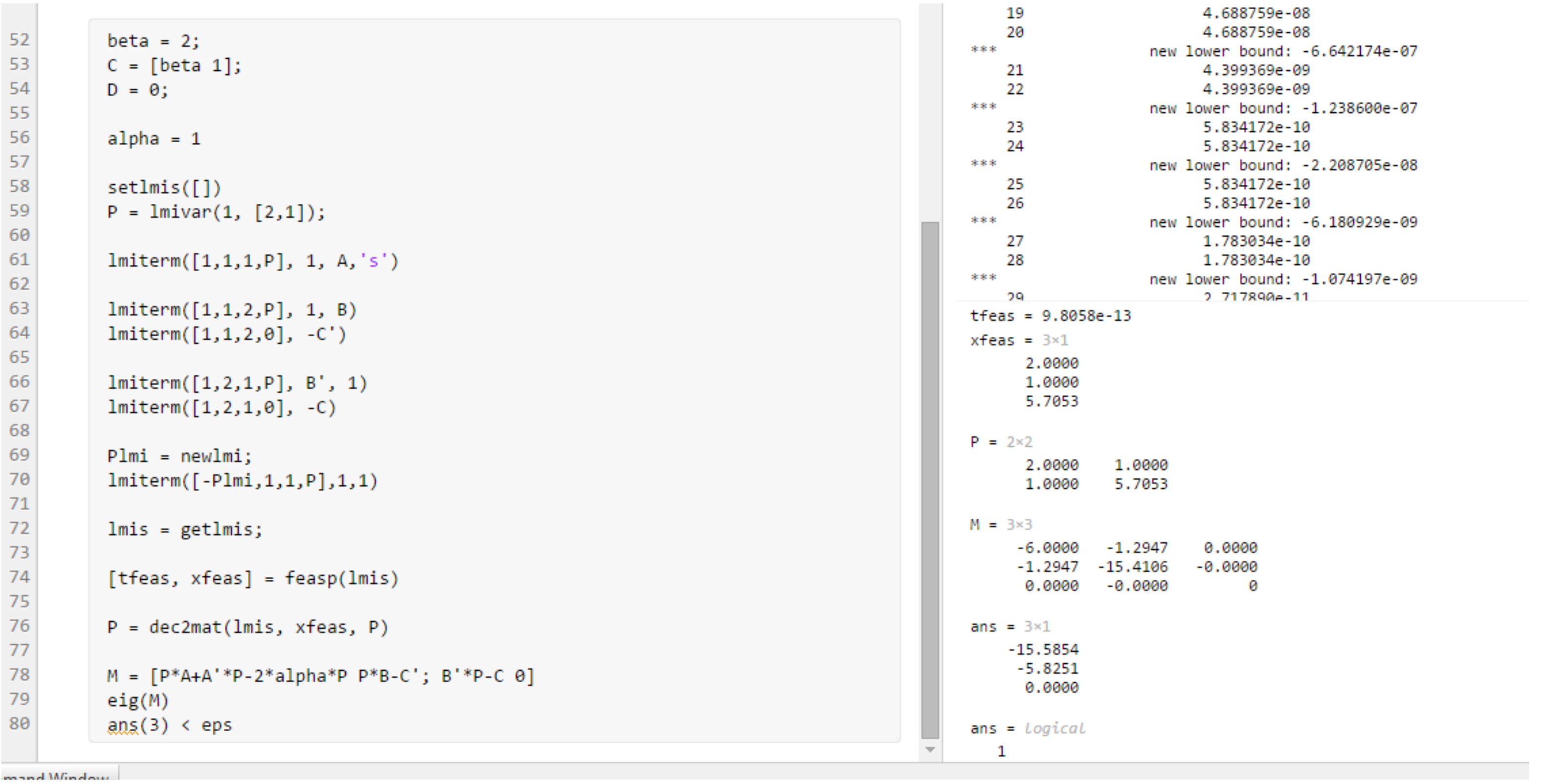
$$\lim_{|w| \to \infty} w^2 [\hat{G}(jw) + \hat{C}(jw)'] = 2(\beta - 1) \quad \neq 0 \quad \text{iff } \beta > 1.$$

Condition c) holds.

Thus for $\hat{g}(s)$ to be SPR,

$$\boxed{\beta > 1}$$

$$M = \boxed{\begin{pmatrix} PA + A'P + 2\alpha P & PB - C' \\ B'P - C & -(D + D') \end{pmatrix} \leq 0}$$



```
beta = 2;
C = [beta 1];
D = 0;

alpha = 1

setlmis([])
P = lmivar(1, [2,1]);

lmiterm([1,1,1,P], 1, A,'s')

lmiterm([1,1,2,P], 1, B)
lmiterm([1,1,2,0], -C')

lmiterm([1,2,1,P], B', 1)
lmiterm([1,2,1,0], -C)

Plmi = newlmi;
lmiterm([-Plmi,1,1,P],1,1)

lmis = getlmis;

[tfeas, xfeas] = feasp(lmis)

P = dec2mat(lmis, xfeas, P)

M = [P*A+A'*P-2*alpha*P P*B-C'; B'*P-C 0]
eig(M)
ans(3) < eps
```

```
    19                      4.688759e-08
    20                      4.688759e-08
***                 new lower bound: -6.642174e-07
    21                      4.399369e-09
    22                      4.399369e-09
***                 new lower bound: -1.238600e-07
    23                      5.834172e-10
    24                      5.834172e-10
***                 new lower bound: -2.208705e-08
    25                      5.834172e-10
    26                      5.834172e-10
***                 new lower bound: -6.180929e-09
    27                      1.783034e-10
    28                      1.783034e-10
***                 new lower bound: -1.074197e-09
    29                      2.717890e-11
tfeas = 9.8058e-13
xfeas = 3×1
      2.0000
      1.0000
      5.7053

P = 2×2
      2.0000    1.0000
      1.0000    5.7053

M = 3×3
     -6.0000   -1.2947    0.0000
     -1.2947  -15.4106   -0.0000
      0.0000   -0.0000         0

ans = 3×1
    -15.5854
     -5.8251
      0.0000

ans = Logical
   1
```

Though the LMI solver shows marginal feasibility,

the M matrix has two negative eigenvalues,

one zero eigenvalue. thus $M \leq 0$.

The KYPSPR holds.