1. Prove.

i) $P(A \cup B) \leq P(A) + P(B)$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

since $P(A \cap B) \geq \phi$.

$P(A \cup B) \leq P(A) + P(B)$.

ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) - P(A \cap B \cap C)$

$\leq P(A) + P(B) + P(C)$.

iii) When $j = 1$.

$P(A_1) = P(A_1)$, true.

Assume:

$P(\cup_1^n A_j) \leq \Sigma_1^n P(A_j)$. is true.

For $j = n+1$.

$P(\cup_1^{n+1} A_j) = P\left[(\cup_1^n A_j) \cup A_{n+1}\right]$

$\leq P(\cup_1^n A_j) + P(A_{n+1})$

$\leq \Sigma_1^n P(A_j) + P(A_{n+1}) = \Sigma_1^{n+1} P(A_j)$

$\therefore P(\cup_1^{n+1} A_j) \leq \Sigma_1^{n+1} P(A_j)$

By the law of mathematical induction,

the assumption is true.

2. (i) Let W be the event of drawing (at least ?) a Q or K.

$$P(W) = 1 - P(W^c) = 1 - \frac{44 \times 43}{52 \times 51} \approx \underline{0.2866}$$

ii) $P(Q \text{ and } K) = P(Q_1 \cap K_2) + P(K_1 \cap Q_2)$

$$= \frac{1}{13} \times \frac{4}{51} \times 2.$$

$$\approx 0.0121$$

3. reward.

$$\frac{2}{6} \times \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{360}$$

4. i) ways of having 2 heads: $\frac{4!}{2!\,2!} = 6$.

hhtt htht htth thth tthh. thht.

$$P(2H) = 6 \times \left(\frac{1}{2}\right)^4 = \frac{6}{16} = \boxed{\frac{3}{8}}$$

ii) $$P(3H) = \frac{4!}{3!} \cdot \left(\frac{1}{2}\right)^4 = \frac{6}{16} = \boxed{\frac{1}{4}}$$

5. 3 dice, S: sum of face values.

i) $P(S=12) = ?$

	3 distinct	2 same	3 same
12 =	1 5 6	5 5 2	4 4 4
	2 4 6	3 3 6	
	3 4 5		

$$P(S=12) = 3! \times 3 \times \left(\frac{1}{6}\right)^3 + \frac{3!}{2!} \times 2 \times \left(\frac{1}{6}\right)^3 + \frac{3!}{3!} \times \left(\frac{1}{6}\right)^3$$

$$= (18 + 6 + 1) \times \frac{1}{216}$$

$$= \boxed{\frac{25}{216}}$$

ii) $m = 11 =$

1 4 6   1 5 5

2 3 6   3 3 5

2 4 5   4 4 3

$3! \times 3 + \frac{3!}{2} \times 3 = 18 + 9 = 27$.

$P(S=11) = \frac{27}{216} > P(S=12)$.

$m = 10 =$

1 3 6   2 2 6

2 3 5   3 3 4

1 4 5   4 4 2

$P(S=10) = \frac{27}{216} > P(S=12)$.

$m = 9 =$ 1 2 6   2 2 5   3 3 3. $P(S=9) = \frac{25}{216} = P(S=12)$

1 3 5   4 4 1

2 3 4

The maximum is when $m = 10, 11$. and it is not unique.

iii) $P(A_1 = 3 \mid S=10) = \dfrac{P(A_1 = 3 \cap S=10)}{P(S=10)}$

$= \dfrac{4/216}{27/216} = \underline{\dfrac{4}{27}}$

6. $P(H) = \frac{1}{2} = p$. $P(T) = \frac{49}{100} = q$ $P(E) = \frac{1}{100} = e$.

Let W_n be winning \$200 starting with n dollars.

$$p(W_n) = P(W_n | H)\,P(H) + P(W_n | T)\,P(T) + P(W_n | E) \cdot P(E)$$

$$p(n) = P(n+1)\,p + p(n-1)\,q + p(n-2)\,e$$

$$\begin{cases} p(0) = 0 \\ p(-1) = 0 \\ p(200) = 1. \end{cases}$$

$$p(n) = a_1 \lambda_1^n + a_2 \lambda_2^n + a_3 \lambda_3^n$$

$$p(n+1)\,p - p(n) + p(n-1)\,q + p(n-2)\,e = 0.$$

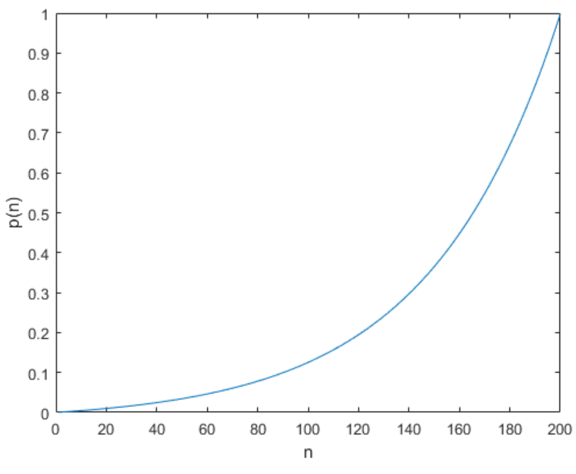
$$p\lambda^3 - \lambda^2 + q\lambda + e = 0.$$

$$\lambda = 1,\ 1.0196,\ -0.0196$$

$$p(n) = a_1 + a_2\, 1.0196^n - a_3 \cdot 0.0196^n$$

$$\begin{cases} 0 = a_1 + a_2 - a_3 \\ 0 = a_1 + \frac{a_2}{1.0196} - \frac{a_3}{0.0196} \\ 1 = a_1 + a_2\, 1.0196^{200} \end{cases} \qquad \begin{cases} a_1 \approx -0.021 \\ a_2 \approx 0.021 \\ a_3 \approx 0 \end{cases}$$

$$p(n) = -0.021 + 0.021 \cdot 1.0196^n.$$



ii) $p(100) = \underline{0.1253}$.

iii) $\underline{n = 166}$.

7. $P(W|S) = 0.25$, $P(W|S^c) = 0.05$. $P(S) = 0.6$.

i). $$P(S|W) = \frac{P(W|S)P(S)}{P(W|S)P(S) + P(W|S^c)P(S^c)}$$

$$= \frac{0.25 \times 0.6}{0.25 \times 0.6 + 0.05 \times 0.4}$$

$$= \underline{0.5824}$$

ii) $$P(S|W^c) = \frac{P(W^c|S)P(S)}{P(W^c|S)P(S) + P(W^c|S^c)P(S^c)}$$

$$= \frac{0.75 \times 0.6}{0.75 \times 0.6 + 0.95 \times 0.4}$$

$$= \underline{0.5422}$$

9. $P(W) = 50\% \rightarrow P(L) = 50\%$.

Let L_n be the event of losing on the nth try.

$$P(L_n) = P(L_n \mid L_{n-1})P(L_{n-1}) + \cancel{P(L_n \mid L_{n-1}^c)P(L_{n-1}^c)}$$

0. can't loose if already won.

$$= \frac{999}{1000} \cdot P(L_{n-1})$$

$$y_n = \frac{999}{1000} y_{n-1}.$$

$$y_1 = \frac{999}{1000}$$

$$P(L_n) = y_n = \left(\frac{999}{1000}\right)^n.$$

$$P(W_n) = 1 - P(L_n)$$

$$= 1 - \left(\frac{999}{1000}\right)^n \geq \frac{1}{2}.$$

$$n \geq 692.8$$

minimum number to play is <u>693</u>.

10. Show $1-a < e^{-a}$ for $0 < a \leq 1$.

Taylor series expansion:

$$e^{-a} = 1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots$$

$$= 1 - a + \sum_{2}^{\infty} \left(\frac{a^n}{n!} - \frac{a^{n+1}}{(n+1)!} \right)$$

$$\frac{a^n}{n!} - \frac{a^{n+1}}{(n+1)!} = \frac{(n+1)a^n - a^{n+1}}{(n+1)!} = \frac{(n+1-a)\,a^n}{(n+1)!}$$

Since $n \geq 2$. $a \leq 1$. $n+1-a > 0$ $\therefore$ $a^n > 0$.

$$\sum_{2}^{\infty} \left(\frac{a^n}{n!} - \frac{a^{n+1}}{(n+1)!} \right) = K > 0.$$

$e^{-a} = 1 - a + K$

$e^{-a} - (1-a) > 0$.

thus. $e^{-a} > (1-a)$ for $0 < a \leq 1$

Page 42. 1.7.3 Problem 1.

when driving, equally likely to choose any road.

When flying, equally likely to land in any city.

i) Top two cities the traveler will probably land in when time goes to infinity. Find the probabilities.

ii) Least likely to end up in. & the probability.

Let probability of ending up at city $\xi_j(n)$ at time n be $P(\xi_i(n))$, and D be the event that one uses a road to move from city $\xi_k(n)$ to city $\xi_j(n+1)$.

$P(D) = d = 0.85$ (choose going on road).

$$P(\xi_j(n+1)) = P(\xi_j(n+1)|D)P(D) + P(\xi_j(n+1)|D^c)P(D^c)$$

$$= P(\xi_j(n+1)|D) \cdot 0.85 + \frac{0.15}{10}.$$

Let $x_n = \begin{bmatrix} P(\xi_1(n)) \\ \vdots \\ P(\xi_{10}(n)) \end{bmatrix}$, $r = \frac{1-d}{N}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

$$x_{n+1} = dT x_n + r.$$

$$x_\infty = (I - dT)^{-1} r$$ since T is stable.

x_inf = 10×1

ii) → $\boxed{0.0510}$

$\boxed{0.1749}$ ← i)

0.1036

0.1019

$\boxed{0.1308}$ ← i)

0.1059

0.0815

0.0807

0.1050

0.0647

i) The traveller is most likely to end up in city 2 and 5, with probability of 17.49% & 13.08%.

ii) The traveller is least likely to end up in city 1, with probability of 5.10%.