HW8.

1. $f_Y(y) = \frac{1}{\lambda} e^{-\frac{y}{\lambda}}$ $\quad 0 \le y < \infty$

$= 0$ $\quad$ otherwise.

$f_{X|Y}(x|y) = \frac{1}{y}$. $\quad 0 \le x \le y$

$= 0$ $\quad$ otherwise.

(i) $f_{XY}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$

$$\boxed{f_{XY}(x,y) = \begin{cases} = \frac{1}{\lambda y} e^{-\frac{y}{\lambda}} & 0 \le x \le y \\ = 0 & \text{otherwise}. \end{cases}}$$

(ii) $f_X = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$ $\qquad t = \frac{y}{\lambda} \quad dt = \frac{1}{\lambda} dy$.

$= \int_{x}^{\infty} \frac{1}{\lambda y} e^{-\frac{y}{\lambda}} dy$

$= \int_{\frac{x}{\lambda}}^{\infty} \frac{1}{\lambda} \cdot \frac{1}{t} e^{-t} dt$.

$\boxed{= \frac{1}{\lambda} \operatorname{expint}\left(\frac{x}{\lambda}\right)}$

(iii) $M_X = \cancel{E}x = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot \frac{1}{\lambda y} e^{-\frac{y}{\lambda}} dx\, dy$.

$= \int_0^{\infty} \int_0^{y} x \cdot \frac{1}{\lambda y} e^{-\frac{y}{\lambda}} dx\, dy$

$= \int_0^{\infty} \frac{y}{2\lambda} e^{-\frac{y}{\lambda}} dy$

$\boxed{= \frac{\lambda}{2}}$

iv). $\sigma_{X}^{2} = EX^2 - \mu_X^2$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\lambda y} e^{-\frac{y}{\lambda}} \, dx \, dy - \frac{\lambda^2}{4}$$

$$= \int_0^{\infty}\int_0^{y} x^2 \cdot \frac{1}{\lambda y} e^{-\frac{y}{\lambda}} \, dx \, dy - \frac{\lambda^2}{4}$$

$$= \int_0^{\infty} \frac{y^2}{3\lambda} e^{-\frac{y}{\lambda}} \, dy - \frac{\lambda^2}{4}$$

$$= \frac{2}{3}\lambda^2 - \frac{\lambda^2}{4}$$

$$\boxed{= \frac{5}{12}\lambda^2}$$

v) $E(X \mid Y=y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x \mid Y=y) \, dx.$

$$= \int_0^{y} x \cdot \frac{1}{y} \, dx$$

$$= \frac{y}{2} \qquad E(X \mid Y) = \frac{Y}{2}.$$

$$EX = E(E(X \mid Y))$$

$$= E\left(\frac{Y}{2}\right) = \int_0^{\infty} \frac{y}{2} \cdot \frac{1}{\lambda} e^{-\frac{y}{\lambda}} \, dy = \boxed{\frac{\lambda}{2}}$$ Adam's Law

vi). $\sigma_{X}^{2} = E(\mathrm{var}(X \mid Y)) + \mathrm{Var}(E(X \mid Y))$

$$= E\left(\frac{Y^2}{12}\right) + \mathrm{Var}\left(\frac{Y}{2}\right)$$

$$= \frac{1}{12}\int_0^{\infty} y^2 \cdot \frac{1}{\lambda} e^{-\frac{y}{\lambda}} \, dy + \int_0^{\infty} \left(\frac{y}{2} - \frac{\lambda}{2}\right)^2 \cdot \frac{1}{\lambda} e^{-\frac{y}{\lambda}} \, dy$$

$$= \frac{\lambda^2}{6} + \frac{\lambda^2}{4}$$

$$= \boxed{\frac{5}{12}\lambda^2}$$ Eve's Law

vii) Since $f_X(x)$ is unsolvable analytically, it will be difficult to integrate it to solve for mean & variance.

4. $X \in \text{Uniform}[0, 6]$. $V \in \text{Uniform}[0, 2]$

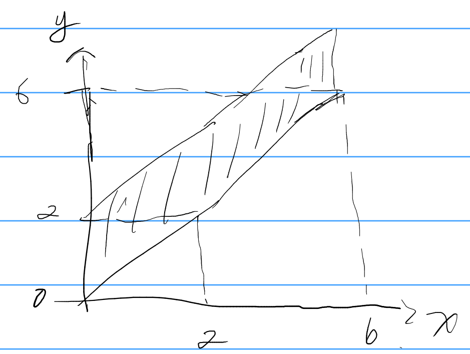
$Y = X + V$ $\qquad$ X & V are independent.

i). $\hat{g}(y) = E(X \mid Y = y) = ?$

$y \in [0, 8]$

$$f_X(x) = \begin{cases} \frac{1}{6} & 0 \le x \le 6 \\ 0 & \text{o.w.} \end{cases} \qquad f_V(y-x) = \begin{cases} \frac{1}{2} & 0 \le y-x \le 2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{XY} = \begin{cases} \frac{1}{12} & 0 \le x \le 6 \\ & 0 \le y-x \le 2 \\ 0 & \text{o.w.} \end{cases}$$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx$$

$$= \begin{cases} \frac{x}{12}\Big|_0^y & 0 \le y \le 2 \\ \frac{x}{12}\Big|_{y-2}^y & 2 \le y \le 6 \\ \frac{x}{12}\Big|_{y-2}^6 & 6 \le y \le 8. \\ 0 & \text{o.w.} \end{cases} = \begin{cases} \frac{y}{12} & 0 \le y \le 2 \\ \frac{1}{6} & 2 \le y \le 6 \\ \frac{8-y}{12}. & 6 \le y \le 8 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X|Y}(x \mid y) = \frac{f_X(x) \cdot f_V(y-x)}{f_Y(y)}$$

$$= \begin{cases} \frac{1}{12} \cdot \frac{12}{y} \\ \frac{1}{12} \cdot 6 \\ \frac{1}{12} \cdot \frac{12}{8-y} \\ 0 \end{cases} = \begin{cases} \frac{1}{y} & 0 \le y \le 12 \\ \frac{1}{2} & 2 \le y \le 6 \\ \frac{1}{8-y} & 6 \le y \le 8 \\ 0 & \text{o.w.} \end{cases}$$

$$\hat{g}(y) = E(X|Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$= \begin{cases} \frac{1}{y}\cdot\frac{x^2}{2}\Big|_0^y \\ \frac{1}{2}\frac{x^2}{2}\Big|_{y-2}^{y} \\ \frac{1}{8-y}\cdot\frac{x^2}{2}\Big|_{y-2}^{6} \\ 0 \end{cases} = \boxed{\begin{cases} \frac{y}{2} & 0 \le y \le 2 \\ y-1 & 2 \le y \le 6 \\ \frac{y}{2}+2 & 6 \le y \le 8 \\ 0 & \text{o.w} \end{cases}}$$

ii) $E(X-\hat{g}(y))^2 = EX^2 - E\hat{g}(y)^2$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \int_{-\infty}^{\infty} \hat{g}(y)^2 \cdot f_Y(y)\, dy$$

$$= \frac{1}{6}\cdot\frac{x^3}{3}\Big|_0^6 - \int_0^2 \left(\frac{y}{2}\right)^2 \frac{y}{12}\,dy - \int_2^6 (y-1)^2\cdot\frac{1}{6}\,dy - \int_6^8 \left(\frac{y}{2}+2\right)^2\cdot\frac{8-y}{12}\,dy$$

$$= 12 - \frac{1}{12} - \frac{62}{9} - \frac{19}{4}$$

$$= \boxed{\frac{5}{18}} \approx 0.2778$$

iii) Let $p = [1, y, y^2, y^3, y^4]$.

$$R_{xp} = E_{xp} = \int_0^2\int_0^6 x p^* \cdot f_X(x)\cdot f_V(v)\, dx\, dv$$

Rxp =

$$\begin{pmatrix} 3 & 15 & 82 & \frac{2376}{5} & \frac{14352}{5} \end{pmatrix}$$

$$R_p = E_{pp^*} = \int_0^2\int_0^6 pp^* f_X(x) f_V(v)\, dx\,dv$$

Rp =

$$\begin{pmatrix} 1 & 4 & \frac{58}{3} & 104 & \frac{2992}{5} \\ 4 & \frac{58}{3} & 104 & \frac{2992}{5} & \frac{10816}{3} \\ \frac{58}{3} & 104 & \frac{2992}{5} & \frac{10816}{3} & \frac{157264}{7} \\ 104 & \frac{2992}{5} & \frac{10816}{3} & \frac{157264}{7} & 143680 \\ \frac{2992}{5} & \frac{10816}{3} & \frac{157264}{7} & 143680 & \frac{42219776}{45} \end{pmatrix}$$

$R_x = E x^2 = 12$.

$M = R_{xp} \cdot R_p^{-1} =$

M =

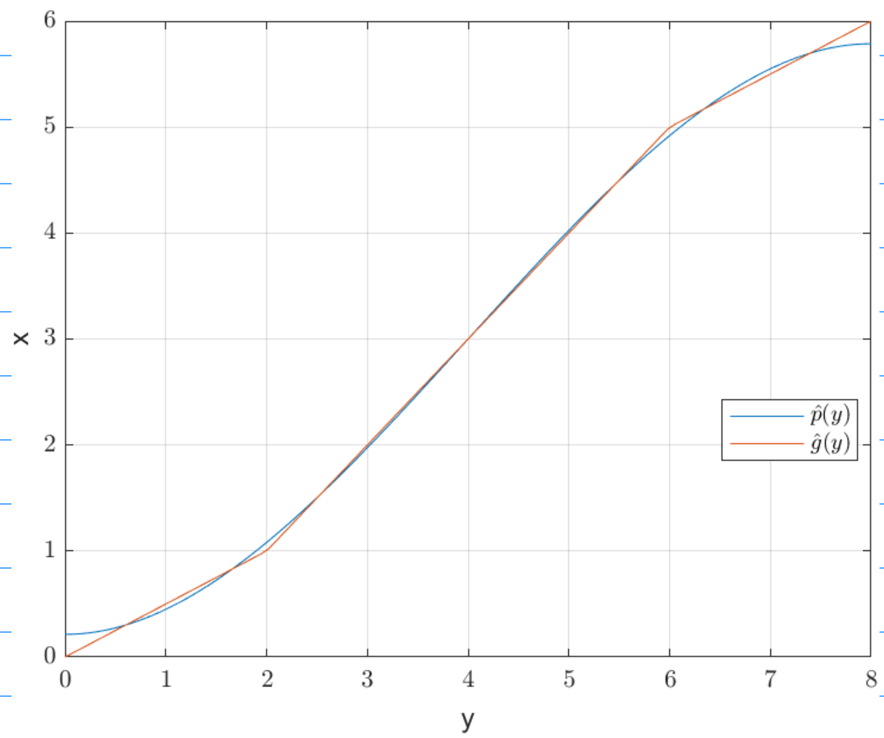
$$\begin{pmatrix} \frac{513}{2393} & -\frac{27}{4786} & \frac{630}{2393} & -\frac{105}{4786} & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\hat{p}(y) = \frac{513}{2393} - \frac{27}{4786} y + \frac{630}{2393} y^2 - \frac{105}{4786} y^3}$$

$$E[(x - \hat{p}(y))]^2 = R_x - R_{xp} R_p^{-1} R_{px}$$

$$\boxed{= \frac{1335}{4786} \approx 0.2789}$$

v)



vi) $\boxed{\hat{g}(4) = 3}$

```
y = 4
v = 2*rand(1, 5000);
x = y - v;
mean_x = mean(x)
```

```
y = 4

mean_x = 3.0085
```

The Monte Carlo sim & $\hat{g}(y)$ are consistent.

5. $f_{XY}(x,y) = cxy$. if $0 \le y \le x^2 \le 1$ & $0 \le x \le 1$

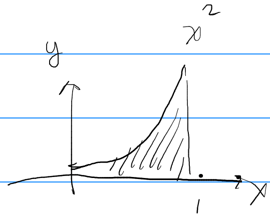
$= 0$ o.w.

i). $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy\, dx = 1.$

$\int_0^1 \int_0^{x^2} cxy\, dy\, dx = 1.$

$\frac{c}{12} = 1 \Rightarrow \boxed{c = 12}$

ii)



$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

$= \int_{\sqrt{y}}^{1} 12xy\, dx$

$$\begin{cases} = 6y - 6y^2 & 0 \le y \le 1. \\ = 0 & \text{o.w.} \end{cases}$$

$f_{X|Y}(x|y) = \frac{f_{XY}}{f_Y} = \frac{12xy}{6y-6y^2} = \frac{2x}{1-y}$ $\quad 0 < y \le x^2 < 1.\ 0 \le x \le 1$

$= 0$ o.w

$\hat{g}(y) = E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}\, dx$

$= \int_{\sqrt{y}}^{1} x \cdot \frac{2x}{1-y}\, dx$

$\hat{g}(y) = \boxed{\frac{2y^{3/2} - 2}{3y - 3}}$

$f_X(x) = 6x^5 \quad 0 \le x \le 1$

$0 \quad$ o.w.

$Ex^2 = \frac{3}{4}.$

iii) $E(X - \hat{g}(y))^2 = Ex^2 - Eg(y)^2$

$= \frac{3}{4} - \int_{-\infty}^{\infty} g(y)^2 \cdot f_Y(y)\, dy$

$= \frac{3}{4} - \int_0^1 g(y)^2 \cdot f_Y(y)$

$\approx \boxed{0.0102}$

iv) Let $P = [1, y, y^2, y^3]$.

$$R_{xp} = E_{xp^*} = \int_0^1 \int_0^{x^2} x \cdot p^* \cdot f_{XY}(x,y)\, dx\, dy$$

Rxp =

$$\begin{pmatrix} \frac{6}{7} & \frac{4}{9} & \frac{3}{11} & \frac{12}{65} \end{pmatrix}$$

$$R_p = E_{p^* p} = \int_0^1 \int_0^{x^2} p \cdot p^* \, f_{XY}(x,y)\, dx\, dy$$

Rp =

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{1}{2} & \frac{3}{10} & \frac{1}{5} & \frac{1}{7} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{7} & \frac{3}{28} \\ \frac{1}{5} & \frac{1}{7} & \frac{3}{28} & \frac{1}{12} \end{pmatrix}$$

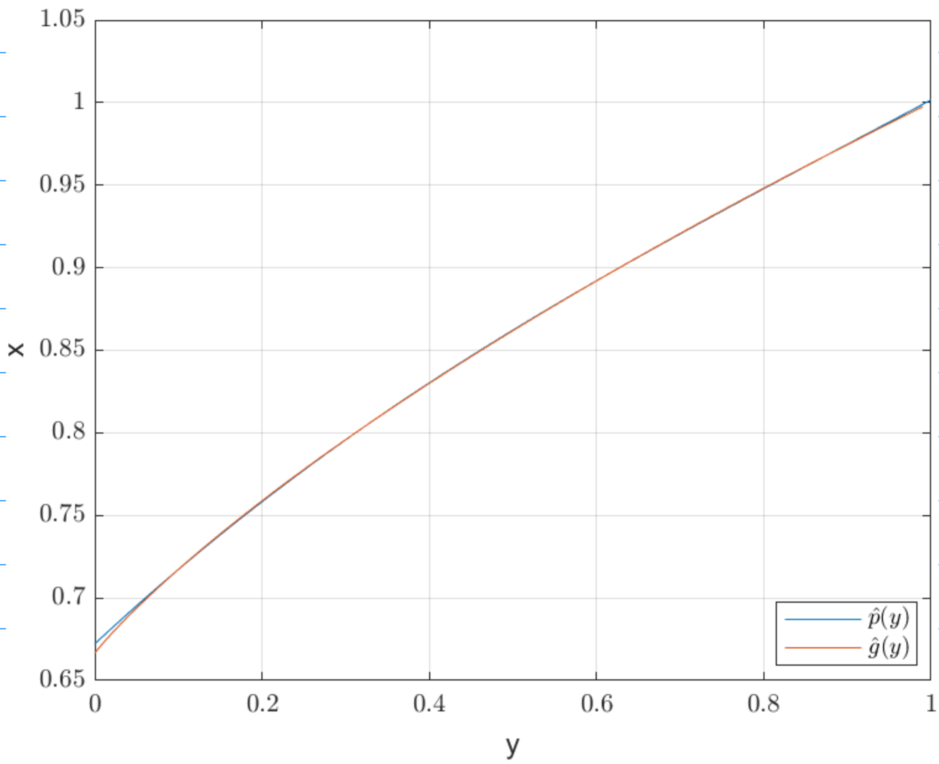
$$\hat{p}(x) = R_{xp} \cdot R_p^{-1} \cdot P$$

$$= \frac{12\, y^3}{143} - \frac{292\, y^2}{1287} + \frac{608\, y}{1287} + \frac{6056}{9009}$$

v) $E(x - \hat{p}(y))^2$

$= E_{x^2} - R_{xp} \cdot R_p^{-1} \cdot R_{px}$ $\boxed{\approx 0.0102}$, consistent with $E(x-\hat{g}(y))^2$

vi)



7. $f_{XY}(x,y) = \gamma e^{-\left(\frac{x}{2y+1}\right)} \quad 0 \le x < \infty \;\&\; 0 \le y \le 1$

$= 0 \quad$ o.w.

i). $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma e^{-\left(\frac{x}{2y+1}\right)} \, dx\, dy = 1.$

$= \gamma \int_0^1 \int_0^\infty e^{-\left(\frac{x}{2y+1}\right)} dx\, dy = 1.$

$= \gamma \int_0^1 -(2y+1) \cdot e^{-\left(\frac{x}{2y+1}\right)} \Big|_0^\infty dy = 1$

$= \gamma \left(y^2 + y \Big|_0^1 \right) = 2\gamma = 1$

$\boxed{\gamma = \frac{1}{2}}$

ii) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

$= \frac{1}{2}\int_0^\infty e^{-\frac{x}{2y+1}} dx$

$= \frac{1}{2}(2y+1) = y + \frac{1}{2} \quad 0 \le y \le 1$

$= 0 \quad$ o.w

$f_{X|Y}(x|y) = \frac{e^{-\frac{x}{2y+1}}}{2y+1} \quad 0 \le x < \infty,\; 0 \le y \le 1.$

$= 0 \quad$ o.w.

$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$

$= \int_0^\infty x \cdot \frac{e^{-\frac{x}{2y+1}}}{2y+1}$

$\boxed{\hat{g}(y) = 2y+1.}$

$E(X - \hat{g}(y)) = EX^2 - E(g(y)^2) = 10 - 5 = 5.$

iii) Let $p = [1, y]$.

$$R_{xp} = E\, x p^* = \int_{0}^{1}\int_{0}^{\infty} x p^* f_{XY}(x,y)\,dx\,dy = \left[\tfrac{13}{6} \quad \tfrac{17}{2}\right].$$

$$R_p = E\, p p^* = \int_0^1\int_0^{\infty} p p^* f_{XY}(x,y)\,dx\,dy = \begin{bmatrix} 1 & \frac{7}{12} \\ \frac{7}{12} & \frac{5}{12} \end{bmatrix}$$

$$\hat{p}(y) = R_{xp} \cdot R_p^{-1} \cdot p$$

$$\boxed{= 2y + 1}$$

$$E(x - \hat{p}(y)) = Ex^2 - R_{xp} \cdot R_p^{-1} R_{px}$$

$$= 10 - 5$$

$$\boxed{= 5.}$$

Both solution yield the same result, and equally bad because the error is big.