6.2 2

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$$

a) $2 > 0$.

$\det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 > 0$

$\det[A] = 2 \cdot 3 + 1 \cdot (-2-1) - 1 \cdot (1+2) = 6 - 3 - 3 = 0.$

Since not all submatrices have positive determinant,

A is <u>not</u> positive definite, it is positive semidefinite

b) $\det[B] = 2 \cdot 3 + 1 \cdot (-2+1) - 1 \cdot (-1+2) = 6 - 1 - 1 = 4 > 0.$

$\det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0.$ $2 > 0.$

B is positive definite

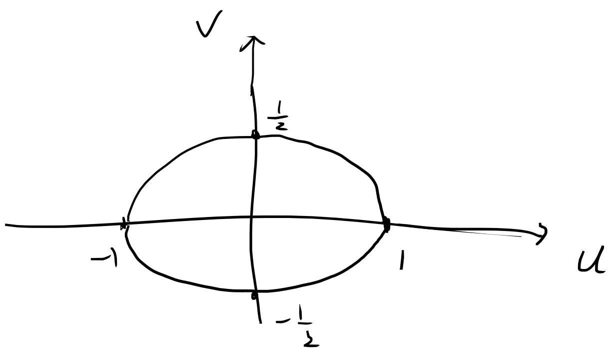
c) $c_{11} = 0$. C is <u>not</u> positive definite.

6.2 10 $\quad u^2+4v^2=1 \Rightarrow A=\begin{bmatrix}1&0\\0&4\end{bmatrix}$. write eigenvalues, eigenvectors. & sketch ellipse

$\det(A-\lambda I)=0$.

$\lambda_1=1,\ \lambda_2=4$.

$v_1=\begin{bmatrix}1\\0\end{bmatrix}$. $v_2=\begin{bmatrix}0\\1\end{bmatrix}$



6.3 2 $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

a) $AA^T = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$

$(17-\lambda)(68-\lambda) - 34^2 = 0.$

$\lambda^2 - 85\lambda + 1156 - \cancel{1156} = 0$

$\sigma_1^2 = \lambda_1 = 85, \ \lambda_2 = 0.$

$\lambda_1 = 85 = \sigma_1^2$

$\begin{bmatrix} -68 & 34 \\ 34 & -17 \end{bmatrix} u_1 = 0.$

$-2x_1 + x_2 = 0$

$2x_1 = x_2$

$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$\lambda_2 = 0$

$\begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix} u_2 = 0.$

$x_1 = -2x_2$

$u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

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$A^TA = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}.$

$(5-\lambda)(80-\lambda) - 400 = 0.$

$\sigma_1^2 = \lambda_1 = 85 \quad \lambda_2 = 0.$

$\lambda_1 = 85$

$\begin{bmatrix} -80 & 20 \\ 20 & -5 \end{bmatrix} v_1 = 0$

$v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix}$

$\lambda_2 = 0$

$\begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix} v_2 = 0.$

$v_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{bmatrix}$

b) choose $\sigma_1 = +\sqrt{85}$

$$[u_1\ u_2]\begin{bmatrix}\sigma_1 & \\ & 0\end{bmatrix}[v_1\ v_2]^T$$

$$= \begin{bmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix} \cdot \begin{bmatrix}\sqrt{85} & 0 \\ 0 & 0\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}}\end{bmatrix}$$

$$= \begin{bmatrix}\sqrt{17} & 0 \\ 2\sqrt{17} & 0\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 4 \\ 2 & 8\end{bmatrix} = A$$

c) orthonormal bases for $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$

$C(A) \Rightarrow u_1$

$N(A) \Rightarrow u_2$

$C(A^T) \Rightarrow v_1$

$N(A^T) \Rightarrow v_2$

6.3 8 Find $U\Sigma V^T$ if $\underset{m\times n}{A}$ has orthogonal columns $w_1 \cdots w_n$, of length $\sigma_1 \cdots \sigma_n$.

$$w_i^T \cdot w_j = \begin{cases} 0 & i \neq j \\ \sigma_i^2 & i = j \end{cases} \quad \text{(orthogonal columns)}$$

$$A^T A = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} w_1^T w_1 & & \\ & \ddots & \\ & & w_n^T w_n \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}$$

eigenvalues of $A^T A$ are $\sigma_1^2 \cdots \sigma_n^2$.

eigenvectors of $A^T A$ are $\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$, $\Rightarrow V = I^{n\times n}$.

With $\boxed{\Sigma = \underset{m\times n}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}}, \quad V^T = I^{n\times n}.}$

$$\left.\begin{aligned} A &= U\Sigma V^T \\ AV &= U\Sigma \\ AI &= U\Sigma \\ A &= U\Sigma \end{aligned}\right\} \Rightarrow \underset{m\times n}{\begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix}} = U \underset{m\times n}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}}$$

$$U^T \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_1^T \cdot w_1 = \sigma_1 \\ \vdots \\ u_n^T \cdot w_n = \sigma_n. \end{cases} \Rightarrow \sigma_n \cdot u_n^T \cdot w_n = \sigma_n^2$$

$$\sigma_n u_n^T = w_n^T$$

$$u_n^T = \frac{w_n^T}{\sigma_n}$$

Thus. $\boxed{U = \begin{bmatrix} \frac{w_1}{\sigma_1} & \cdots & \frac{w_n}{\sigma_n} \end{bmatrix}}$

6.3 10 $A^{2\times2}$ is symmetric, with unit eigenvector $u_1, u_2$, $\lambda_1 = 3$, $\lambda_2 = -2$.

Find $U\Sigma V^T$

Eigenvectors of a square symmetric matrix are orthogonal, since $u_1$ & $u_2$ are unit vectors, they are orthonormal.

Diagonalize $A$:

$$A = S\Lambda S^{-1}$$
$$= S\Lambda S^T$$
$$= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & \\ & -2 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$$
$$\quad\ \ \shortparallel \qquad\qquad \shortparallel \qquad\quad \shortparallel$$
$$= U \cdot \Sigma \cdot V^T$$