1. $T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. consider inner product on $\mathbb{C}^3$

$(x, y)_T = (Tx, y) = y' Tx \quad (x \in \mathbb{C}^3 \ \& \ y \in \mathbb{C}^3)$

$\|x\|_T = \sqrt{(Tx, x)}$.

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

solve: $\delta = \min \{ \|e_1 - \alpha e_2 - \beta e_3\|_T^2 : \alpha \in \mathbb{C} \ \& \ \beta \in \mathbb{C} \}$

$\|e_1 - a e_2 - b e_3\|_T^2 = \delta = \min \{ \|e_1 - \alpha e_2 - \beta e_3\|_T^2 : \alpha \in \mathbb{C} \ \& \ \beta \in \mathbb{C} \}$.

$(e_1, e_2)_T = (\alpha e_2, e_2)_T + (\beta e_3, e_2)_T$.

$[1\ 0\ 0]\, T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \alpha [0\ 1\ 0]\, T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta [0\ 1\ 0]\, T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$2 = 3\alpha + 2\beta$ ①

$(e_1, e_3)_T = (\alpha e_2, e_3)_T + (\beta e_3, e_3)_T$

$[1\ 0\ 0]\, T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \alpha [0\ 1\ 0]\, T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \beta [0\ 0\ 1]\, T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$1 = 2\alpha + 3\beta$ ②

solve ① & ②. $\begin{cases} \alpha = \frac{4}{5} \\ \beta = -\frac{1}{5} \end{cases}$

$\|e_1 - \frac{4}{5} e_2 + \frac{1}{5} e_3\|_T^2 = (e_1, -\frac{4}{5} e_2 + \frac{1}{5} e_3)_T$

$= [0\ -\frac{4}{5}\ \frac{1}{5}]\, T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= \boxed{\frac{7}{5}}$

2. $$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-\frac{n}{50}} & 1 \\ 2 & \cos(\frac{n}{50}) \end{bmatrix}}_{A(n)} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} e^{-\frac{n}{50}}\cos(\frac{n}{50}) \\ 1 \end{bmatrix}}_{B(n)} u(n)$$

$$y(n) = \underbrace{\begin{bmatrix} 1+e^{-\frac{n}{50}} & 2+\sin(\frac{n}{50}) \end{bmatrix}}_{C(n)} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \underbrace{\left(1+\frac{1}{2}\sin(\frac{n}{50})\right)}_{D(n)} v(n).$$

u, v are indp. Gaussian white noise processes. The initial conditions $x(0)=0$ & $\hat{x}(0)=0$.

Let $M_n = \text{span}\{y(j)\}_0^n$. Find

i) $\underline{P_{M_{n-1}}} x_1(n)$ for $n = 8, 9, 10$

ii) $\underline{P_{M_n}}\ x_2(n)$ for $n = 8, 9, 10$

The Kalman filter is given as

$$\begin{cases} \hat{x}(n+1) = A\hat{x}(n) + \Delta_n(y(n) - C\hat{x}(n)) \\ \Delta_n = AQ_nC^*(CQ_nC^* + DD^*)^{-1}. \\ Q_{n+1} = AQ_nA^* + BB^* - AQ_nC^*((CQ_nC^* + DD^*)^{-1}CQ_nA^* \end{cases}$$

$$Q_0 = E(x(0)-\mu_0)(x(0)-\mu_0)^* = 0.$$

$$\boxed{P_{M_{n-1}} X_1(8:9) = \hat{x}(8:9) = [-31.5307 \quad -75.9469 \quad -174.4413]}$$

ii) $P_{M_n} x(n) = \hat{x}(n) + Q_nC^*(CQ_nC^* + DD^*)^{-1}(y(n) - C\hat{x}(n))$

$$\boxed{P_{M_n} x_2(8:9) = [-48.0377,\ -111.7605,\ -260.9927]}$$

3. $x \in G(0,1)$. $\{v_n\}_0^\infty$

$y_n = x + v_n$ is a random process. $n \geq 0$.

i) Find best estimate for x given $\{y_j\}_{j=0}^{n-1}$

$$\hat{x}_n = E(x \mid y_0, y_1, \cdots y_{n-1}) = P_{\mathcal{M}_{n-1}} x$$

where $\mathcal{M}_{n-1} = \text{span}\{y_j\}_{j=0}^{n-1}$.

ii) Find error σ_n in the estimate

$$\sigma_n^2 = E(x - \hat{x}_n)^2$$

---

i) $y = \begin{bmatrix} y(0) \\ \vdots \\ y(n-1) \end{bmatrix}$.

$$R_{xy} = E\, x\, y^*$$

$$= E\, \underline{x}\, [\underline{x+v(0)\ \cdots\ x+v(n-1)}]$$

$$= E[x^2 + xv(0)\ \cdots\ x^2 + xv(n-1)]$$

$$= E[x^2\ \cdots\ x^2]$$

$$= [1\ \cdots\ 1].$$

$$R_y = E\, y\, y^* = \begin{bmatrix} E(x+v_0)\cdot(x+v_0)^* & \cdots & E(x+v_0)(x+v_{n-1})^* \\ \vdots & \ddots & \vdots \\ E(x+v_{n-1})\cdot(x+v_0)^* & \cdots & E(x+v_{n-1})(x+v_{n-1})^* \end{bmatrix}$$

$$= \begin{bmatrix} E(x^2 + \cancel{2xv_0} + v_0 v_0^*) & \cdots & E(x^2 + \cancel{v_{n-1} v_0^*}) \\ & \ddots & \\ E(x^2 + \cancel{v_0 v_{n-1}^*}) & & E(x^2 + v_{n-1} v_{n-1}^*) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \cdots & 1 \\ & \ddots & \\ 1 & & 2 \end{bmatrix}.$$

$$R_g = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & \cdots & 1 \end{bmatrix} = I + \mathbb{1}$$

let $A = I_n \quad B = \mathbb{1}_n$

$$(A+B)^{-1} = A^{-1} - \frac{1}{1+\operatorname{trace}(BA^{-1})} A^{-1} B A^{-1}$$

$$R_g^{-1} = I - \frac{1}{n+1} \mathbb{1}$$

$$P_{M_{n-1}} x = R_{xg} R_g^{-1} g$$

$$= \underbrace{[1 \cdots 1]}_{n-1} \cdot \left[I_n - \frac{1}{n+1} \mathbb{1}_n \right] \cdot \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$= [1 \cdots 1] \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} - \frac{1}{n+1} [1 \cdots 1] \cdot \mathbb{1}_n \cdot \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$= \sum_i^{n-1} y_i - \frac{1}{n+1} \cdot [n \cdots n] \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$= \sum_i^{n-1} y_i - \frac{n}{n+1} [1 \cdots 1] \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$\boxed{= \left(1 - \frac{n}{n+1}\right) \cdot \sum_i^{n-1} y_i}$$

ii) $\sigma_n^2 = E(x - \hat{x}_n)^2$

$$= R_x - R_{xg} R_g^{-1} R_{xg}^*$$

$$= 1 - [1 \cdots 1] \left[I_n - \frac{1}{n+1} \mathbb{1}_n \right] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= 1 - \left(n - \frac{1}{n+1} [1 \cdots 1] \mathbb{1}_n \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)$$

$$= 1 - \left(n - \frac{n^2}{n+1}\right)$$

$$= \frac{n^2 + (1-n)(n+1)}{n+1} = \boxed{\frac{1}{n+1}}$$

$$[1 \quad 1] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[n \quad n] \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

4.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.5 & 30 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\{u\}^2, \{v\}^2 \in G(0,1)$. w is input

All initial conditions are zeros

Design a feedback controller $w = -K\hat{x}$ based on the steady state Kalman filter such that $|x(t)| \le 1$ & $|x_3(t)| \le 0.35$

$K = [k_1 \ k_2 \ k_3 \ k_4]$, $|k_j| \le 25$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.5 & 30 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix}$$

Dynamics:

$$\dot{x} = Ax + Bu + B_1 w = Ax + [B \ B_1] \begin{bmatrix} u \\ w \end{bmatrix}$$

$$y = Cx + Dv$$

Estimator:

$$0 = AP + PA^{*} + BB^{*} - PC^{*}(DD^{*})^{-1}CP$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + B_1 w = (A - LC)\hat{x} + [L \ B_1] \begin{bmatrix} y \\ w \end{bmatrix}$$

$$L = PC^{*}(DD^{*})^{-1}$$

$4\times4 \quad 4\times5$

$4\times4 \quad 4\times5$

Answer Sheet NAME: Zhanpeng Yang.

Problem 1:

i) $a = \frac{4}{5}$

ii) $b = -\frac{1}{5}$

iii) $\delta = \frac{7}{5}$

Problem 2:

i) $P_{M_{n-1}} x_1(n)$ for $n = 8, 9, 10$:

```
ans = 1×3
   -31.5387  -75.9469 -174.4413
```

ii) $P_{M_n} x_2(n)$ for $n = 8, 9, 10$:

```
ans = 1×3
   -48.0377 -111.7605 -260.9927
```

Problem 3

i). $\hat{x}_n = \boxed{\left(1 - \frac{n}{n+1}\right) \cdot \sum_i^{n-1} y_i}$

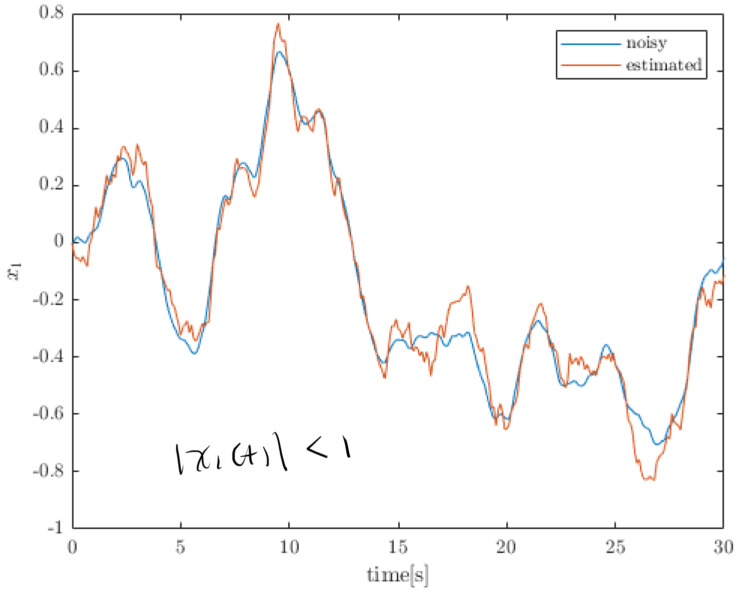
ii) $\sigma_n^2 = E(x - \hat{x}_n)^2 = \boxed{\frac{1}{n+1}}$

Problem 4 :

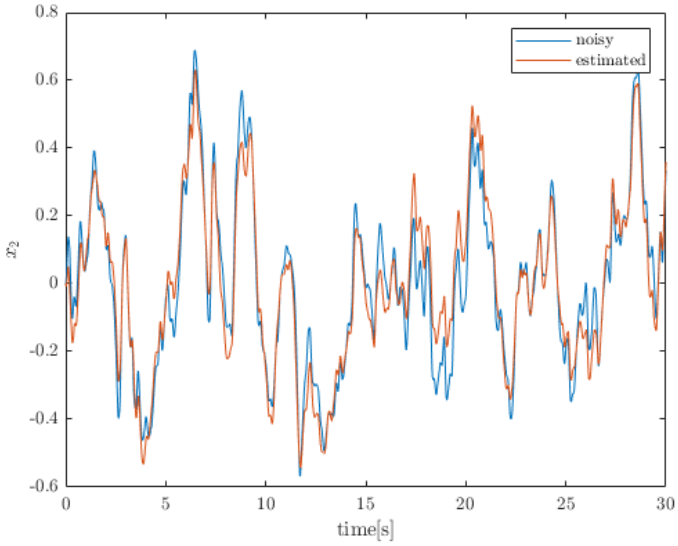
$K =$ K = 1×4

-1.4142 -4.4316 18.8017 5.0905

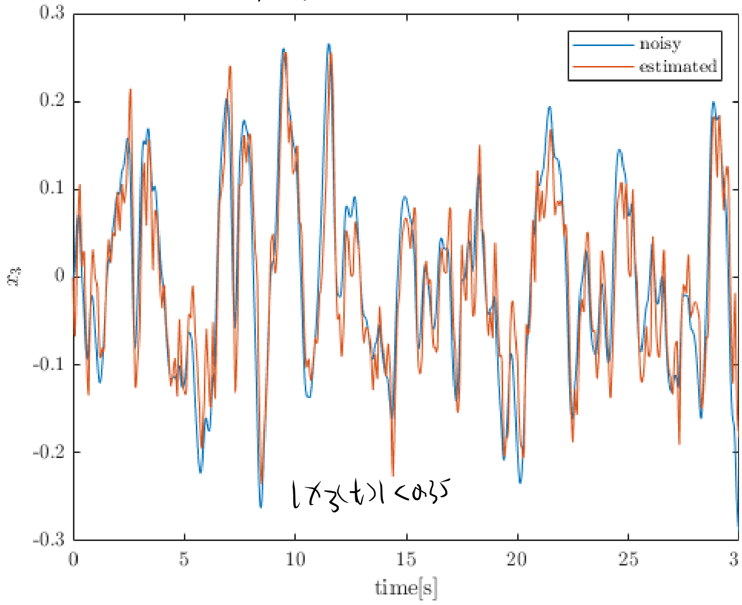
$x_1 , \hat{x}_1$



$x_2 , \hat{x}_2$



$x_3 , \hat{x}_3$



$x_4 , \hat{x}_4$