HW6 31 – 34 in "Quadratic Stability"

Exercise 31 Prove the following result: Suppose there exists a positive-definite symmetric matrix P and a positive scalar α which satisfy

$$PA_1 + A_1^T P + 2\alpha P \leq 0 \tag{11.7a}$$
$$PA_2 + A_2^T P + 2\alpha P \leq 0 \tag{11.7b}$$

where $A_1 := A_0 + a\Delta A$ and $A_2 := A_0 + b\Delta A$. Then system (11.2)-(11.4) is globally exponentially stable about the origin with rate of convergence α.

Let $V = x^T P x$.

$\dot{V} = x^T P \dot{x} + \dot{x}^T P x$

$= 2x^T P \dot{x}$

$= 2x^T P (A_0 + \psi(x)\Delta A)x$

This implies that $\dot{V}$ is bounded by either

$\dot{V} \leq 2x^T P A_1 x = x^T(PA_1 + A_1^T P)x$ when $\psi(x) = a$ or b.

or $\dot{V} \leq 2x^T P A_2 x = x^T(PA_2 + A_2^T P)x$.

since $PA_1 + A_1^T P \leq -2\alpha P$

and $PA_2 + A_2^T P \leq -2\alpha P$.

$\dot{V} \leq -2\alpha x^T P x$

$\leq -2\alpha V$ Thus the system is GES with rate of α.

Exercise 32 What is the supremal value of $\gamma > 0$ for which Theorem 16 guarantees that the following system is guaranteed to be stable about the origin?

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + \gamma e^{-x_1^2} x_2 \\ \dot{x}_2 &= -x_1 - 3x_2 - \gamma e^{-x_1^2} x_1\end{aligned}$$

This system can be described by. $\dot{x} = A(x)x$.

$A(x) = A_0 + \psi(x) \Delta A$

$|x_1| = 0$, $\psi(x) = \gamma$.

$|x|_1 \to \infty$, $\psi(x) \to 0$.

$A_0 = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix}$. $\Delta A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $\psi(x) = \gamma e^{-x_1^2} \Rightarrow 0 \le \psi(x) \le \gamma$.

By iterating γ and checking feasibility with LMI toolbox,

It's clear that the supremal value of γ is $+\infty$.

Exercise 33 Consider the pendulum system of Example 119 with $\gamma = 1$. Obtain the largest rate of exponential convergence that can be obtained using the results of Exercise 31 and the LMI toolbox.

33

```
clear;
gamma = 1

A0 = [0 1;-2 -1];
DA = [0 0;1 0];

A1 = A0 - gamma*DA;
A2 = A0 + gamma*DA;

% s1 = ltisys(A1);
% s2 = ltisys(A2);
%
% polsys = psys([s1 s2])
% [drate,P] = decay(polsys)

setlmis([])
p = lmivar(1, [2,1]);

Plmi = newlmi;
lmiterm([-Plmi,1,1,p],1,1)
lmiterm([Plmi,1,1,0],1)

lmi1 = newlmi;
lmiterm([lmi1,1,1,p],1,A1,'s')
lmiterm([-lmi1,1,1,p],1,1)

lmi2 = newlmi;
lmiterm([lmi2,1,1,p],1,A2,'s')
lmiterm([-lmi2,1,1,p],1,1)

lmis = getlmis;

% [tfeas, xfeas] = feasp(lmis)

% P = dec2mat(lmis, xfeas, p)
[alpha, popt] = gevp(lmis,2)
alpha = alpha/(-2)
```

```
gamma = 1
```

```
     12              -0.227179
     13              -0.241276
***                    new lower bound:    -0.318273
     14              -0.243682
***                    new lower bound:    -0.315867
     15              -0.243682
     16              -0.243682
***                    new lower bound:    -0.261728
     17              -0.243964
***                    new lower bound:    -0.252423
     18              -0.243964
     19              -0.243964
***                    new lower bound:    -0.246079

 Result:  feasible solution of required accuracy
          best value of t:    -0.243964
          guaranteed absolute accuracy:  2.11e-03
          f-radius saturation:  0.000% of R =  1.00e+08
```

```
alpha = -0.2440
popt = 3×1
       3.9884
       0.9972
       1.9941

alpha = 0.1220
```

$\alpha_{max} = 0.1220$

Exercise 34 Consider the double inverted pendulum described by

$$
\begin{aligned}
\ddot{\theta}_1 + 2\dot{\theta}_1 - \dot{\theta}_2 + 2k\theta_1 - k\theta_2 - \sin\theta_1 &= 0 \\
\ddot{\theta}_2 - \dot{\theta}_1 + \dot{\theta}_2 - k\theta_1 + k\theta_2 - \sin\theta_2 &= 0
\end{aligned}
$$

Using the results of Theorem 17, obtain a value of the spring constant k which guarantees that this system is globally exponentially stable about the zero solution.

$$\ddot{\theta}_1 = -2\dot{\theta}_1 + \dot{\theta}_2 - 2k\theta_1 + k\theta_2 + \sin\theta_1 \qquad x = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\ddot{\theta}_2 = \dot{\theta}_1 - \dot{\theta}_2 + k\theta_1 - k\theta_2 + \sin\theta_2$$

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix}$$

$$\psi_1(x) = \begin{cases} \dfrac{\sin\theta_1}{\theta_1} & \theta_1 \neq 0 \\ 1 & \theta_1 = 0 \end{cases} \qquad \Delta A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{a_1}{-1} \le \psi_1(x) \le \underset{b_1}{1}$$

$$\Delta A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\psi_2(x) = \begin{cases} \dfrac{\sin\theta_2}{\theta_2} & \theta_2 \neq 0 \\ 1 & \theta_2 = 0 \end{cases}$$

$$\underset{a_2}{-1} \le \psi_2(x) \le \underset{b_2}{1}$$

using LMI toolbox, the LMI is feasible at

$$\underline{k = 20}$$

MATLAB code on the next page.

34

```
clear;
k = 20
A0 = [0 0 1 0;0 0 0 1;-2*k k -2 1;k -k 1 -1];
DA1 = zeros(4);
DA1(3,1) = 1;
DA2 = zeros(4);
DA2(4,2) = 1;

A1 = A0 - DA1 - DA2;
A2 = A0 - DA1 + DA2;
A3 = A0 + DA1 - DA2;
A4 = A0 + DA1 + DA2;

setlmis([])
p = lmivar(1, [4,1]);

lmi1 = newlmi;
lmiterm([lmi1,1,1,p],1,A1,'s')

lmi2 = newlmi;
lmiterm([lmi2,1,1,p],1,A2,'s')

lmi3 = newlmi;
lmiterm([lmi3,1,1,p],1,A3,'s')

lmi4 = newlmi;
lmiterm([lmi4,1,1,p],1,A4,'s')

Plmi = newlmi;
lmiterm([-Plmi,1,1,p],1,1)
lmiterm([Plmi,1,1,0],1)

lmis = getlmis;

[tfeas, xfeas] = feasp(lmis)

P = dec2mat(lmis, xfeas, p)
```

```
 Solver for LMI feasibility problems L(x) < R(x)
    This solver minimizes  t  subject to  L(x) < R(x) + t*I
    The best value of t should be negative for feasibility

 Iteration   :    Best value of t so far

     1                        1.008533
     2                        0.981506
     3                        0.747386
     4                       -0.060982

 Result:  best value of t:    -0.060982
          f-radius saturation:  0.000% of R =  1.00e+09

tfeas = -0.0610
xfeas = 10×1
     78.2698
    -21.2570
     57.2606
      0.4033
      0.2455
      2.8630
      0.3265
      0.6876
      1.7993
      4.6671

P = 4×4
     78.2698  -21.2570    0.4033    0.3265
    -21.2570   57.2606    0.2455    0.6876
      0.4033    0.2455    2.8630    1.7993
      0.3265    0.6876    1.7993    4.6671
```