

$$1.. (a) \cdot x^T A x$$

$$= x^T \cdot \begin{bmatrix} A_{11}x \\ \vdots \\ A_{d1}x \end{bmatrix}$$

$A_{i\cdot}$ is i th row of A .

$$= x_1 A_{1\cdot} x + \dots + x_d A_{d\cdot} x$$

$$= x_1 \cdot (A_{11} x_1 + A_{12} x_2 + \dots + A_{1d} x_d) + \dots + x_d (A_{d1} x_1 + \dots + A_{dd} x_d)$$

$$= A_{11} x_1^2 + A_{12} x_1 x_2 + \dots + A_{1d} x_1 x_d$$

$$+ \dots$$

$$+ A_{d1} x_d x_1 + A_{d2} x_d x_2 + \dots + A_{dd} x_d^2$$

let

$$x x^T = \begin{bmatrix} x_1^2 & \dots & x_1 x_d \\ \vdots & \ddots & \vdots \\ x_d x_1 & \dots & x_d^2 \end{bmatrix}$$

$$= B \text{ (symmetric)}$$

$$= \sum_{j=1}^d A_{1j} x_1 x_j + \sum_{j=1}^d A_{2j} x_2 x_j + \dots + \sum_{j=1}^d A_{dj} x_d x_j$$

$$= \sum_{j=1}^d \sum_{i=1}^d A_{ij} x_i x_j$$

$$= A_{1\cdot} B_{\cdot 1} + A_{2\cdot} B_{\cdot 2} + \dots + A_{d\cdot} B_{\cdot d}$$

$$= \sum_{i=1}^d A_{i\cdot} B_{\cdot i}$$

$$= \text{tr}[AB]$$

$$= \text{tr}[A x x^T]$$

1 b).

$$\begin{aligned}
 P(D|\Sigma) &= \prod_{n=1}^N \left\{ \frac{1}{(2\pi)^{d/2} |\Sigma|^{\frac{1}{2}}} \cdot \exp \left\{ -\frac{1}{2} (x_n - \mu)^T \Sigma^{-1} (x_n - \mu) \right\} \right\} \\
 &= \left(\frac{1}{(2\pi)^{d/2} |\Sigma|^{\frac{1}{2}}} \right)^N \cdot \exp \left\{ \sum_{n=1}^N -\frac{1}{2} (x_n - \mu)^T \Sigma^{-1} (x_n - \mu) \right\} \\
 &= \frac{1}{(2\pi)^{Nd/2}} |\Sigma^{-1}|^{N/2} \cdot \exp \left\{ -\frac{1}{2} \sum_{n=1}^N \text{tr} \left[\Sigma^{-1} (x_n - \mu)^T (x_n - \mu) \right] \right\} \\
 &= \frac{1}{(2\pi)^{Nd/2}} |\Sigma^{-1}|^{N/2} \exp \left\{ -\frac{1}{2} \text{tr} \left[\sum_{n=1}^N \Sigma^{-1} (x_n - \mu)^T (x_n - \mu) \right] \right\}
 \end{aligned}$$

1.c) $\tilde{\Sigma} = \frac{1}{N} \sum_{n=1}^N (x_n - \mu)(x_n - \mu)^T$ $A = \Sigma^{-1} \tilde{\Sigma}$ $\text{eig}(A) = \{\lambda_1, \dots, \lambda_d\}$

$$P(D|\Sigma) = \frac{1}{(2\pi)^{Nd/2}} |\Sigma^{-1}|^{N/2} \exp \left\{ -\frac{N}{2} \text{tr} \left[\Sigma^{-1} \tilde{\Sigma} \right] \right\} \quad , \quad \text{let } k = \frac{1}{(2\pi)^{Nd/2}}$$

$$= k \cdot |A \tilde{\Sigma}^{-1}|^{N/2} \exp \left\{ -\frac{N}{2} \text{tr} [A] \right\}$$

$$= k \cdot \frac{1}{|\tilde{\Sigma}|^{N/2}} |A|^{N/2} \exp \left\{ -\frac{N}{2} \sum_{i=1}^d \lambda_i \right\}$$

$$= \frac{1}{(2\pi)^{Nd/2} |\tilde{\Sigma}|^{N/2}} \left(\prod_{i=1}^d \lambda_i \right)^{N/2} \exp \left\{ -\frac{N}{2} \sum_{i=1}^d \lambda_i \right\}$$

1 d). $\hat{\Sigma} = \arg \max_{\Sigma} P(D|\Sigma)$

$$= \arg \min_{\lambda_i} - \ln \left[\frac{1}{(2\pi)^{Nd/2} |\tilde{\Sigma}|^{N/2}} \left(\prod_{i=1}^d \lambda_i \right)^{N/2} \exp \left\{ -\frac{N}{2} \sum_{i=1}^d \lambda_i \right\} \right]$$

$$= \arg \min_{\lambda_i} \frac{Nd}{2} \ln(2\pi) + \frac{N}{2} \ln(|\tilde{\Sigma}|) - \frac{N}{2} \ln \left(\prod_{i=1}^d \lambda_i \right) + \frac{N}{2} \sum_{i=1}^d \lambda_i$$

$$\nabla_{\lambda} -\ln(P(D|\Sigma)) = \nabla_{\lambda} \left[-\frac{N}{2} \sum_{i=1}^d \ln(\lambda_i) + \frac{N}{2} \sum_{i=1}^d \lambda_i \right]$$

$$0 = \nabla_{\lambda} \left[\sum_{i=1}^d \lambda_i - \ln(\lambda_i) \right]$$

$$0 = \left[1 - \frac{1}{\lambda_1} \quad 1 - \frac{1}{\lambda_2} \quad \dots \quad 1 - \frac{1}{\lambda_d} \right] \Rightarrow \lambda = 1 \Rightarrow A = I$$

(e) Prove $\hat{\Sigma}_{ML} = \frac{1}{N} \sum_{n=1}^N (x_n - \mu)(x_n - \mu)^T = \tilde{\Sigma}$

$$A = \tilde{\Sigma} \hat{\Sigma}^{-1} = I.$$

$$\hat{\Sigma} = \tilde{\Sigma} = \frac{1}{N} \sum_{n=1}^N (x_n - \mu)(x_n - \mu)^T.$$

(f) Take derivative w.r.t to Σ , do some tedious matrix calculus, then get $\hat{\Sigma}$.

$$\text{set } \frac{d(P)}{d\Sigma} = 0,$$

(g) $\hat{\Sigma}_{ML}$ is a biased estimator of Σ

$$E(\hat{\Sigma}_{ML}) = \frac{N-1}{N} \Sigma \neq \Sigma$$

by defining $\hat{\mu} = \frac{1}{N} \sum_{n=1}^N x_n$ & $\hat{\Sigma}_{ML} = \frac{1}{N-1} \sum_{n=1}^N (x_n - \hat{\mu})(x_n - \hat{\mu})^T$

we can obtain an unbiased estimator of Σ .

2. See Jupyter Notebook

3. See Jupyter Notebook

4. Currently trying to reimplement MLab's `noise2noise` using the newer version of tensorflow.
and also trying to train the neural net with
different noise distribution other than gaussian & poisson.
Got nothing to report yet for this milestone.