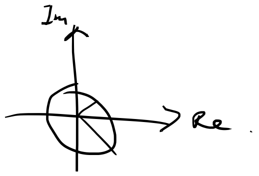
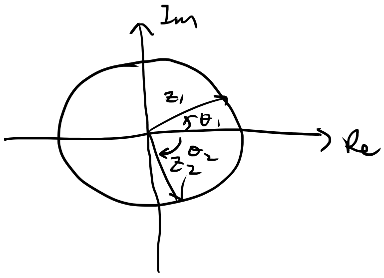
5.5 2

a) It's 2 times the real part.

b) 


The conjugate is the reflection about the real axis, and lies on the unit circle.

c) 


$z_1 = e^{i\theta_1}$

$z_2 = e^{i\theta_2}$

$\underline{z_1 \cdot z_2 = e^{i(\theta_1+\theta_2)}}$

The product still lies on the unit circle, and has an angle of $\theta_1+\theta_2$ from the real axis.

d) $z_1 + z_2 = a+b+i(c+d)$ $\quad \|z_1+z_2\| = \sqrt{(a+b)^2+(c+d)^2} = \sqrt{a^2+b^2+c^2+d^2+2ab+2cd}$

It has a magnitude greater than 0 and smaller than 2.

$0 \leq \|z_1+z_2\| \leq 2$.

5-5 b

$$x = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} \qquad y = \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}$$

$$\|x\| = \sqrt{2^2+4^2+4^2} = \sqrt{36} = \underline{6}$$

$$\|y\| = \sqrt{2^2+4^2+4^2} = \underline{6}$$

$$\bar{x}^T \cdot y = \begin{bmatrix} 2+4i & -4i \end{bmatrix} \cdot \begin{bmatrix} 2+4i \\ 4i \end{bmatrix} = 4+16i-16+16 = \underline{4+16i}$$

5.5 20 $\quad K = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$. find $e^{Kt} = Se^{\Lambda t}S^{-1}$, Verify $e^{Kt}$ is unitary.

$\frac{d}{dt}e^{Kt}$ at $t=0$,

$$K - \lambda I = \begin{bmatrix} i-\lambda & i \\ i & i-\lambda \end{bmatrix}$$

$(i-\lambda)^2 + 1 = 0.$

$-1 - 2i\lambda + \lambda^2 + 1 = 0.$

$\lambda^2 - 2i\lambda = 0.$

$\lambda(\lambda - 2i) = 0$

$\lambda = 0, 2i$

$\lambda_1 = 0,$

$\begin{bmatrix} i & i \\ i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$

$ix_1 = -ix_2.$

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2i$

$\begin{bmatrix} -i & i \\ i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$

$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2i \end{bmatrix}$

$U = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$U^{-1} = \frac{1}{-2} \cdot \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$$\underline{K = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2i \end{bmatrix}\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}}$$

$$e^{Kt} = Ue^{\Lambda t}U^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{2it} \end{bmatrix}\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & e^{i2t} \\ 1 & e^{i2t} \end{bmatrix}\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 + 0.5e^{i2t} & -0.5 + 0.5e^{i2t} \\ -0.5 + 0.5e^{i2t} & 0.5 + 0.5e^{i2t} \end{bmatrix}$$

$$[e^{Kt}]^H \cdot e^{Kt} = \frac{1}{4} \cdot \begin{bmatrix} 1+e^{-i2t} & -1+e^{-i2t} \\ -1+e^{-i2t} & 1+e^{-i2t} \end{bmatrix} \cdot \begin{bmatrix} 1+e^{i2t} & -1+e^{i2t} \\ -1+e^{i2t} & 1+e^{i2t} \end{bmatrix} = \begin{bmatrix} 2+\cancel{e^{-i2t}}+e^{i2t}+2\cancel{-e^{i2t}}\cancel{-e^{2i}} & 0 \\ 0 & 4 \end{bmatrix} \cdot \frac{1}{4}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \underline{\text{unitary}}$$

$$\frac{d}{dt}e^{Kt} = Ke^{Kt} \quad (t=0)$$

$$= K \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= K$$

5.6 2 Describe all matrices that are similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ find two of them.

all similar matrices have the same eigenvalues

$$A-\lambda I = \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}$$

$$(1-\lambda)(-1-\lambda) = 0.$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1.$$

Example similar matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

5.6 14 Show every number is an eigenvalue for $Tf(x) = \frac{df}{dx}$,

but $Tf(x) = \int_0^x f(t)\,dt$ has no eigenvalue.

if $f(x) = e^{ax}$. $\frac{df}{dx} = a e^{ax} = a f(x) \Rightarrow T = a$.

$T$ is a $1\times1$ matrix with $\lambda = a \in \mathbb{R}$

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If $T = a$,

$$T f(x) = a f(x) = \int_0^x f(t)\,dt$$

$$a f'(x) = f(x)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{a}$$

$$\int \frac{df}{f(x)} = \int \frac{dx}{a}$$

$$\ln|f(x)| = \frac{x}{a} + C$$

$$f(x) = e^{\frac{x}{a}+C} = e^{\frac{x}{a}} \cdot e^C = A e^{\frac{x}{a}}$$

Integrate $f(x)$ again:

$$\int_0^x A e^{\frac{t}{a}}\,dt = A \cdot \left[ a \cdot e^{\frac{t}{a}} \right]_0^x = A\left(a e^{\frac{x}{a}} - a\right) = a\left(A e^{\frac{x}{a}} - A\right)$$

$$T f(x) = a(f(x) - A)$$

$$a f(x) = a(f(x) - A)$$

If $a \neq 0$, $A = 0 \Rightarrow f(x) = 0$. $\int_0^x f(t)\,dt = 0$.

If $a = 0$, $T = 0 \Rightarrow f(x) = 0$.

So $T$ has no eigenvalue

5.6 22 $A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$ &

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -3 \\ 4 & -2-\lambda \end{bmatrix}$$

$$(5-\lambda)(-2-\lambda) - 4(-3) = 0$$

$$-10 - 3\lambda + \lambda^2 + 12 = 0.$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

$\lambda = 2$

$$\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix} v_1 = 0.$$

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ pick } u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

orthonormal & unitary

$$T = U^{-1} A U$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 9 & -5 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -14 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -7 \\ 0 & 1 \end{bmatrix}$$ triangular

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$$

$$-\lambda \cdot (-\lambda \cdot -\lambda) - 1 \cdot 0 = 0$$

$$\lambda = 0, 0, 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} u_1 = 0$$

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

pick

$$u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ orthonormal & unitary. $u^{-1} = U^T$

$$T = U^{-1} A U$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ Triangular

Appendix B 2 Show special solution $u_2$ satisfies $\frac{du}{dt} = Au$. because

$$\underline{Ax_1 = 8x_1 \, , \quad Ax_2 = 8x_2 + x_1}$$

substitute.

$$u_2 = e^{8t}(tx_1 + x_2)$$

$$\frac{du_2}{dt} = 8e^{t}(tx_1 + x_2) + e^{8t}(x_1)$$

$$Au_2 = Ae^{8t}(tx_1 + x_2) = e^{8t}(tAx_1 + Ax_2)$$

$$= e^{8t}(t8x_1 + 8x_2 + x_1)$$

$$= e^{8t}(t8x_1 + 8x_2) + e^{8t}x_1$$

$$= 8e^{8t}(tx_1 + x_2) + e^{8t}x_1 = \frac{du_2}{dt}$$ (shown).

Appendix B 4 Show $J_i^T = P^{-1} J_i P$. $P = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$ orthonormal,

is similar to $J_i$ $\quad P^{-1} = P^T = P$.

$$J_i = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}. \quad J_i^T = \begin{bmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{bmatrix}. \quad P^{-1} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

$$P^{-1} J_i P = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ & & \lambda \end{bmatrix} \cdot \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & \lambda \\ & \iddots & 1 \\ \lambda & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{bmatrix} = J_i^T$$

6.1 4 a) $F = -1 + 4(e^x - x) - 5x\sin y + 6y^2$ at $x = y = 0$.

b) $F = (x^2 - 2x)\cos y$ at $x = 1$ $y = \pi$.

a). $\frac{\partial F}{\partial x} = 4(e^x - 1) - 5\sin y$

$= 0$ at $(0,0)$

$\frac{\partial F}{\partial y} = -5\cos y + 12y$

$= 0$ at $(0,0)$

$\frac{\partial^2 F}{\partial x^2} = 4e^x$

$= 4$ at $(0,0)$

$\frac{\partial^2 F}{\partial y^2} = 5\sin y + 12$

$= 12$ at $(0,0)$

$\frac{\partial^2 F}{\partial x \partial y} = -5\cos y$

$= -5$ at $(0,0)$.

$4 \cdot 12 = 48 > 25 = (-5)^2$

F has minimum at $(0,0)$.

b) $\frac{\partial F}{\partial x} = (2x - 2)\cos y$

$= 0$ at $(1, \pi)$

$\frac{\partial F}{\partial y} = -(x^2 - 2x)\sin y$

$= 0$ at $(1, \pi)$

$\frac{\partial^2 F}{\partial x^2} = (2)\cos y$

$= -2$

$\frac{\partial^2 F}{\partial y} = (2x - x^2)\cos y$

$= -1$

$\frac{\partial^2 F}{\partial x \partial y} = (2 - 2x)\sin y = 0 < (-2 \cdot -1)$

F has maximum at $(1, \pi)$

6.1 6

$a+c>2b$ Find an example such that $ac<b^2$

the matrix is not positive definite.

$a=0,\ c=7,\ b=3$ $a+c=7>6$.

$ac=0<9$

$A=\begin{bmatrix}0 & 3\\ 3 & 7\end{bmatrix}$ not positive definite.