HW 2.

Ex 1, Determine the nature (node, focus, etc). of each equilibrium state of the damped duffing system.

$$\ddot{y} + 0.1\dot{y} - y + y^3 = 0$$

linearise: $\delta\ddot{y} = 0.1\,\delta\dot{y} - \delta y + 3\delta y \cdot y^{e^2}$

$\delta\ddot{y} = 0.1\,\delta\dot{y} + (3y^{e^2} - 1)\,\delta y$ let $x = \begin{bmatrix}\delta y \\ \delta\dot{y}\end{bmatrix}$. $\dot{x} = \begin{bmatrix}\delta\dot{y} \\ \delta\ddot{y}\end{bmatrix}$

$y^e = 0, -1, 1.$

For $y^e = 0$,

$\delta\ddot{y} = 0.1\,\delta\dot{y} - \delta y.$

$\dot{x} = \begin{bmatrix}0 & 1 \\ -1 & 0.1\end{bmatrix} x.$

$A = \begin{bmatrix}0 & 1 \\ -1 & 0.1\end{bmatrix}.$

$\lambda_1 = -1.0512,$
$\lambda_2 = 0.9512$

$\boxed{\text{Saddle}}$

For $y^e = -1$,

$\delta\ddot{y} = 0.1\,\delta\dot{y} + 2\,\delta y$

$\dot{x} = \begin{bmatrix}0 & 1 \\ 2 & 0.1\end{bmatrix} x$

For $y^e = 1$

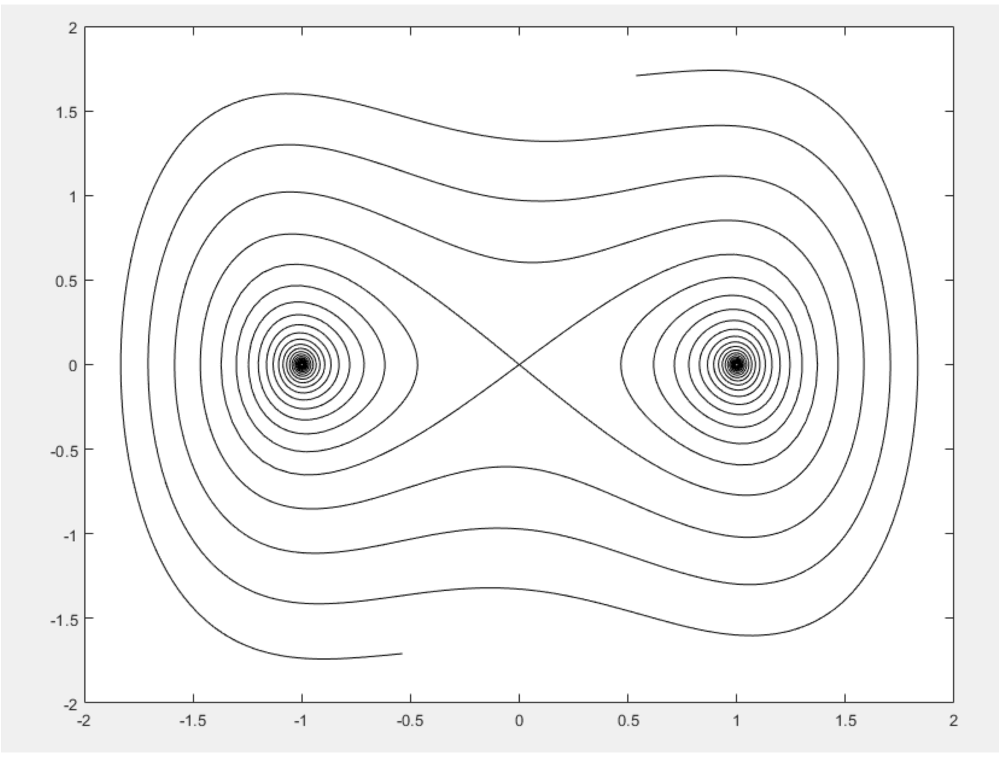
$\delta\ddot{y} = 0.1\,\delta\dot{y} + 2\,\delta y.$

$\dot{x} = \begin{bmatrix}0 & 1 \\ 2 & 0.1\end{bmatrix} x.$

$A = \begin{bmatrix}0 & 1 \\ 2 & 0.1\end{bmatrix}.$

$\lambda = -0.05 \pm 1.4133i$

$\boxed{\text{Stable focus}}$



Ex. 2 Determine nature of equilibrium state.

$\ddot{y} + \sin y = 0$. $\sin y^e = 0$ $y^e = 0, \pi, 2\pi \cdots$

$\delta\ddot{y} + \delta y \cdot \cos y^e = 0$.

$\delta\ddot{y} = -\cos y^e \cdot \delta y$.

let $x = \begin{bmatrix} \delta y \\ \delta\dot{y} \end{bmatrix}$. $\dot{x} = \begin{bmatrix} \delta\dot{y} \\ \delta\ddot{y} \end{bmatrix}$.

$y^e = 0, 2\pi, 4\pi, \ldots$

$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$

$\lambda^2 + 1 = 0$.

$\lambda = \pm i$

$\boxed{\text{center}}$

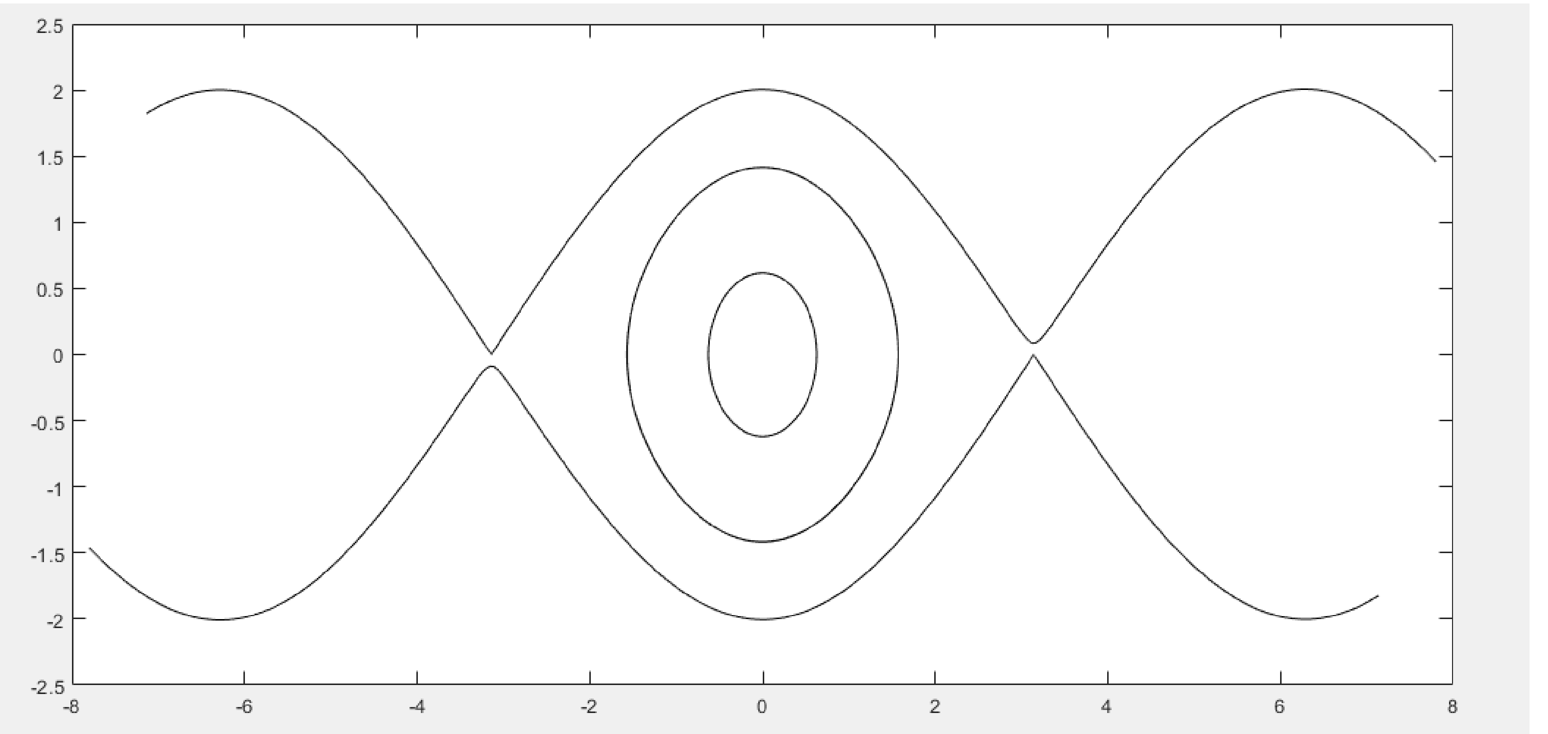
$y^e = \pi, 3\pi, \cdots$

$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$.

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$.

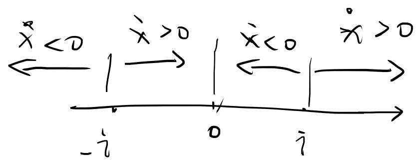
$\boxed{\text{saddle}}$



Ex. 3. Determine stability, and region of attraction for AS states.

$-2i - 8i^3 = -2i + 8i = 6i$

a). $\dot{x} = -x - x^3$

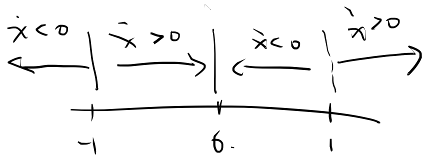


$0 = -x^e - x^{e^3}$

$x^e = 0, \pm i$

AS, at $x^e = 0$, with region of attraction $(-i, i)$

b). $\dot{x} = -x + x^3$



$0 = -x^e(1 - x^{e^2})$

$x^e = 0, \pm 1.$

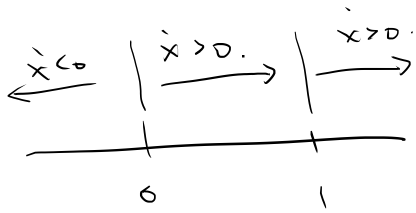
Unstable at $x^e = \pm 1$.

AS at $x^e = 0$. with region of attraction $(-1, 1)$

c). $\dot{x} = x - 2x^2 + x^3$

$0 = x^e - 2x^{e^2} + x^{e^3}$

$x^e = 0, 1$



unstable.

Ex. 4. Show all non-zero solution of the following system "blow up" in a finite time. Compute blow-up time as a function of x_0.

$$\dot{x} = x^3$$

$$\frac{1}{x^3}\,dx = dt$$

$$\int_{x_0}^{x} \frac{1}{x^3}\,dx = \int_{t_0}^{t} dt.$$

$$-\frac{1}{2x^2}\Big|_{x_0}^{x} = t - t_0.$$

$$-\frac{1}{2x^2} + \frac{1}{2x_0^2} = t - \cancel{t_0}^{\,0}$$ assume $t_0 = 0$. & $x_0 \neq 0$.

$$\lim_{x\to\infty}\left\{-\frac{1}{2x^2} + \frac{1}{2x_0^2}\right\} = \frac{1}{2x_0^2}. \Rightarrow \boxed{t = \frac{1}{2x_0^2}}$$

Ex. 5 . show no solution can "blow up" in a finite time.

$$\dot{x} = \frac{x}{1+x^2} + \sin(x).$$

$f(x) = \frac{x}{1+x^2} + \sin(x).$ $\quad \|f(x)\| \leq \alpha \|x\| + \beta$

$$\|f(x)\| = \left\| \frac{x}{1+x^2} + \sin(x) \right\|$$

$$\leq \underline{\left\| \frac{x}{1+x^2} \right\| + \|\sin(x)\|}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

$$\left\| \frac{x}{1+x^2} \right\| = \left\| \frac{1}{1+x^2} \right\| \cdot \|x\|$$

where $\left\| \frac{1}{1+x^2} \right\|$ approaches $\begin{cases} 0 \text{ as } x \to \pm\infty. \\ 1 \text{ as } x \to 0. \end{cases}$

$\|\sin(x)\|$ is bounded by 1 between $(-\infty, \infty)$ because it is sinusoidal.

Thus. we can find an upper bound for $\|f(x)\|$ as

$$\boxed{\|f(x)\| \leq 1 \cdot \|x\| + 1}$$

which implies that the system has no finite escape time.

Ex. 6. What initial state x_0 can you guarantee the following equation has a unique solution.

$$\dot{x} = -\sqrt{(1-x)^2} = -|1-x|$$

Since $\dot{x}$ is not differentiable at $x=1$, there's no guaranteed unique solution. Thus, the solution is only unique if $x_0 \neq 1$.