1. $f_X(x) = a\sin(x)$ if $0 \le x \le \pi$ Find a s.t. $f_X(x)$ is a density function & Find $F_X(x)$.

$= 0$ otherwise.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\pi} a\sin(x)\,dx$$

$$1 = a\left[-\cos(x)\right]_0^{\pi}$$

$$a = \frac{1}{1-(-1)}$$

$$\boxed{a = \frac{1}{2}}$$

$$F_X(x) = \int_0^{x} \frac{1}{2}\sin(x)\,dx = \boxed{\begin{cases} 0 & x \le 0. \\ \dfrac{-\cos(x)+1}{2} & 0 < x \le \pi \\ 1 & x > \pi \end{cases}}$$

2. $f_X(x) = a e^{-|x|}$ for $-\infty < x < \infty$.

$$\int_{-\infty}^{\infty} a e^{-|x|} dx = 1.$$

$$a\left(\int_{-\infty}^{0} e^{-(-x)} dx + \int_{0}^{\infty} e^{-x} dx\right) = 1.$$

$$a\left(e^{x}\Big|_{-\infty}^{0} + -e^{-x}\Big|_{0}^{\infty} \right) = 1.$$

$$a(1 - 0 - 0 + 1) = 1.$$

$$\boxed{a = \frac{1}{2}}$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2} e^{-|x|} dx.$$

$$= \begin{cases} \int_{-\infty}^{x} \frac{1}{2} e^{x} dx & \text{if } x \leq 0 \\ \frac{1}{2} + \int_{0}^{x} \frac{1}{2} e^{-x} dx & \text{if } x \geq 0. \end{cases}$$

$$\boxed{F_X(x) = \begin{cases} \frac{1}{2} e^{x} & \leq 0 \\ 1 - \frac{1}{2} e^{-x} & x \geq 0. \end{cases}}$$

$$P(1 \leq X) = 1 - F_X(1) = 1 - \left(1 - \frac{1}{2e}\right) = \boxed{\frac{1}{2e}}$$

3. $f_X(x) = \frac{c}{\sqrt{x}}$ if $0 < x \le 4$.

$= 0$ otherwise.

$c \int_0^4 \frac{1}{\sqrt{x}} \, dx = 1$

$c \cdot 2x^{\frac{1}{2}} \Big|_0^4 = 1$

$4c = 1$

$\boxed{c = \frac{1}{4}}$

$F_X(x) = \int_0^x \frac{1}{4} \frac{1}{\sqrt{k}} \, dk = \frac{1}{4} \left(2k^{\frac{1}{2}} \right)_0^x$ $\boxed{\begin{array}{ll} = \frac{\sqrt{x}}{2} & \text{if } 0 < x \le 4 \\ = 0 & \text{otherwise} \end{array}}$

$P(|X| < 1) = P(-1 \le X \le 1) = P(-1 \le X < 0) + P(0 \le X \le 1)$

$= 0 + \frac{1}{2}$

$\boxed{= \frac{1}{2}}$

4. $f_X(x) = 1 \quad \text{if } 0 \le x \le 1$ $\quad F_X = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$

$\quad\;\; = 0 \quad \text{otherwise}.$

Let $Y = X^2$

$$F_Y(y) = P(Y < y) = P(X^2 < y) = P(-\sqrt{y} < X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \sqrt{y} - 0$$

$$= \sqrt{y}$$

$$\boxed{= \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y \le 1 \\ 1 & 1 \le y \end{cases}}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \boxed{\begin{cases} \frac{1}{2\sqrt{y}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$$