2.3 2)

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \Rightarrow \text{rank} = 3$$

The largest number of independent vectors is $\underset{\sim}{3}$

This number is the dimension of the space spanned by v's

2.3 8) w_1, w_2 & w_3 are independent,

$v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, $v_3 = w_1 + w_2$

show v_1, v_2 & v_3 are independent.

Suppose v_1, v_2 & v_3 are <u>linearly dependent</u>.

Then there are scalars c_1, c_2 & c_3 such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \quad (c_1, c_2, c_3 \neq 0)$$

$$c_1(w_2 + w_3) + c_2(w_1 + w_3) + c_3(w_1 + w_2) = 0$$

$$(c_2 + c_3)w_1 + (c_1 + c_3)w_2 + (c_1 + c_2)w_3 = 0$$

Since w_1, w_2 & w_3 are linearly independent,

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

By proof of contradiction,

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$, only when $c_1, c_2, c_3 = 0$

Thus v_1, v_2, & v_3 are linearly independent.

2.3 14)

a) $x = (0, 0, 0, 0)$

b) $x = (1, 1, 1, 1)$

c) $x = (1, -1, 0, 0)$

d) $x = (1, 0, 0, 0)$

2.3 16

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xRightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

order
$$\left(\begin{array}{l} r_1 \leftrightarrow r_2 \\ r_2 - r_1 \\ r_3 - r_2 \\ r_4 - r_3 \end{array} \right)$$

$$\begin{cases} c_1 + c_4 = 0 \\ c_2 - c_4 = 0 \\ c_3 + c_4 = 0 \end{cases} \qquad \begin{cases} c_4 \in \mathbb{R} \\ c_1 = c_3 = -c_4 \\ c_2 = c_4 \end{cases}$$

Since 0 is not the only solution, v_1, v_2, v_3 & v_4 are not independent.

They do not span $\mathbb{R}^4$, as the last row in the rref $[0\ 0\ 0\ 0 \,\vdots\, 1]$ is invalid.

2.4 4) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A)$ contains $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ Column space

$C(A^T)$ contains $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ row space

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$Ax = 0 \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{cases} x_1 \in \mathbb{R} \\ x_2 = 0 \\ x_3 = 0 \end{cases} \quad \bar{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_1$

$N(A)$ contains $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ null space

$A^T x = 0 \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 \in \mathbb{R} \end{cases} \quad \bar{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3$

$N(A^T)$ contains $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ left-null space

2.4 8)

If the system only has trivial solution,

A has max rank $\underline{n}$.

Columns of A are linearly $\underline{\text{independent}}$

$Ax = 0$

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

If $m = n$. $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & 1 & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$. $X = 0$ is the only solution

A has max rank,

If $m > n$ $\begin{bmatrix} I(n) \\ \cdots \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$. $X = 0$ is the only solution

A has max rank

If $m < n$ $\begin{bmatrix} I(m) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$, $X = 0$ is not the only solution,

A does not have max rank,

Thus this case is not considered.

2.4 10) $Ax = b$ has at least one solution.

Show $A^T y = 0$ only has $y = 0$ as the solution.

If A is $m \times n$, and has at least one solution,

the columns span $\mathbb{R}^m$ $\quad r = m$

The dimension of null space of $A^T = m - r = m - m = 0$

thus it only contains the zero vector.

Therefore, it only have the trivial solution,

no special solution.