AAE 568 Applied Optimal Control and Estimation
Problem Set 1

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Neatly write/type your homework on a sheet of paper and show the steps to receive full credit.

Scan and upload your solution as a pdf to Gradescope by **11:59 PM ET** on the due date.

Problem 1

Consider the coupled-mass system in Fig. 1 under the influence of applied input force u.

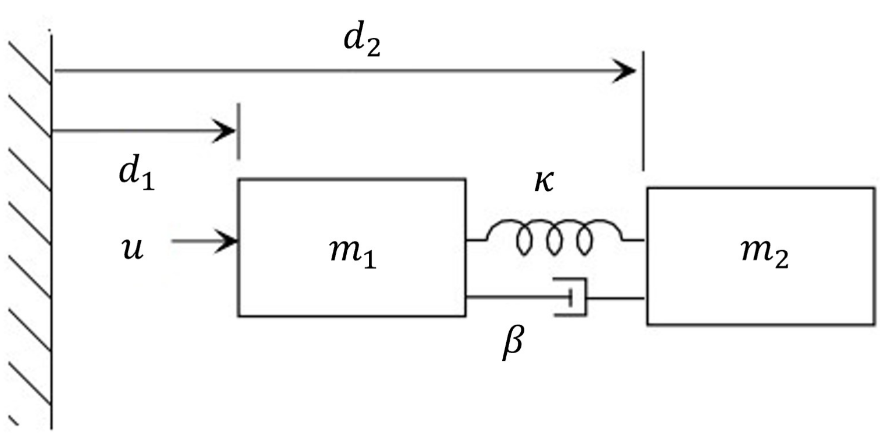


Figure 1: Compliant Mass System

The system has the equations of motion:

$$m_1\ddot{d}_1 + \beta(\dot{d}_1 - \dot{d}_2) + \kappa(d_1 - d_2) = u,$$
$$m_2\ddot{d}_2 + \beta(\dot{d}_2 - \dot{d}_1) + \kappa(d_2 - d_1) = 0.$$

Given that we measure the displacement of the mass m_2 as the output,

1. Obtain a state space representation of the system (A, B, C, D).
2. Check the controllability and observability of the system.

Consider that the following parameters are used: $m_1 = 1, m_2 = 0.1, \kappa = 0.091, \beta = 0.0036$.

3. Design an output feedback controller in Fig. 2 (a) such that the eigenvalues of $A-BK$ are placed at $-1+i$,$-1-i$,-2,-10, the eigenvalues of $(A - LC)$ at $-5, -2, -3, -1$, and the steady-state error of the unit-step response is zero. Note that Figure 2 (a) corresponds to the "Compensator in the feedforward loop" in Lecture Note 04 with $M = -L$, $N = 0$.
4. Design an output feedback controller in Fig. 2(b) such that the requirements in 3 are satisfied. Note that Figure 2 (b) corresponds to the "Compensator in the feedback loop" in Lecture Note 04 with $M = BN$.
5. Plot the unit-step responses obtained in 3 and 4, and compare them and discuss the results.

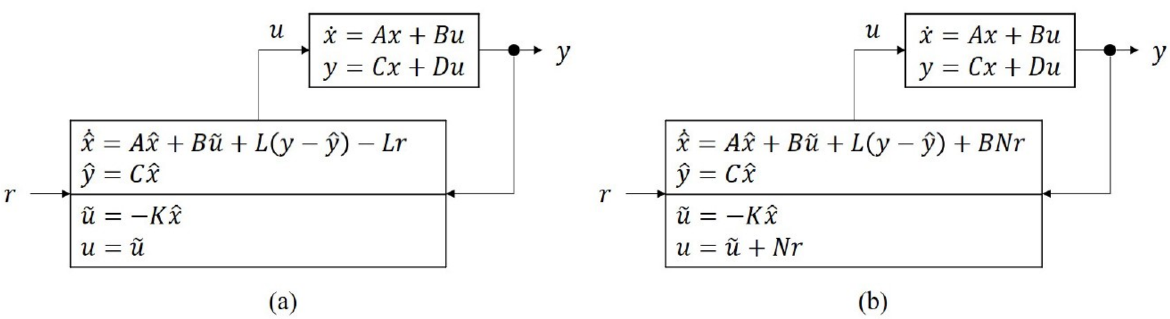


Figure 2: Output Feedback Controller

Problem 2

The vector Gaussian variable $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is completely described by its mean and covariance matrix. In this example, they are

$$E(X) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad Cov(X) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

Now consider another vector random variable Y that is related to X by the equation

$$Y = AX + b$$

where

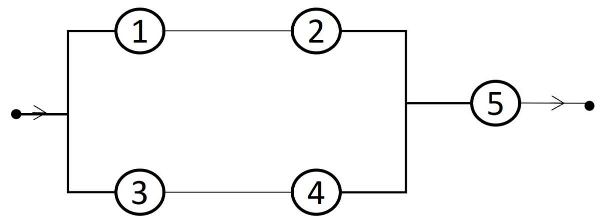
$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find the mean $E(Y)$ and covariance matrix $Cov(Y)$ for Y.

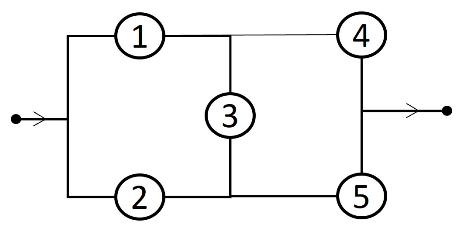
Problem 3

Let c_i be the event of switch i being closed, and E be the event that electricity can flow from left to right for the following circuits. $P(c_i) = 0.7$ for each i.

a. Find $P(E)$, $P(c_1|E)$



b. Find $P(E)$, $P(c_1|E)$ and $P(c_3|E)$



Problem 4

Let X and Y be independent random variables that are each uniformly distributed on the interval $[0, 1]$. Let A be the area of the rectangle formed by X and Y. What are the mean and variance of X, Y, and A?

Problem 5

Let X have the Probability Distribution Function (PDF) $F_X(x)$ that is a mixture of the continuous and discrete types,

$$F_X(x) = \begin{cases} 0, \text{ for } x < 0, \\ \frac{x+1}{4}, \text{ for } 0 \le x < 1, \\ 1, \text{ for } x \ge 1. \end{cases}$$

Find $E(X)$ and $Var(X)$.

Problem 6

Consider a three-dimensional Gaussian random vector, X, one whose probability density is described by

$$f_X(x) = [(2\pi)^{3/2}|P|^{1/2}]^{-1}\exp\{-\frac{1}{2}[x-m]^T P^{-1}[x-m]\}$$

where the mean m and covariance P are

$$m = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad P = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 2.5 & 0.5 \\ 0 & 0.5 & 2.5 \end{bmatrix}$$

The surfaces of constant probability density are called the surfaces of constant likelyhood. They are ellipsoids with the principal axes not generally aligned with the coordinate axes.

1. Determine a transformation of variables $x' = Tx$ so that it is possible to use the principal axes of the ellipsoid as the coordinate axes. When this is done, P' becomes diagonal, i.e.,

$$P' = \begin{bmatrix} \sigma_{11}'^2 & 0 & 0 \\ 0 & \sigma_{22}'^2 & 0 \\ 0 & 0 & \sigma_{33}'^2 \end{bmatrix}$$

   Obtain this form for the given matrix P.

2. Show that now the surface of constant likelyhood is an ellipsoid of the form

$$(x_1'^2/\sigma_{11}'^2) + (x_2'^2/\sigma_{22}'^2) + (x_3'^2/\sigma_{33}'^2) = c^2$$

   Write an expression for the probability that x_1, x_2, and x_3 take values within the ellipsoid.

3. Show that our ellipsoid becomes a sphere by defining new variables

$$x_1'' = x_1'/\sigma_{11}', \quad x_2'' = x_2'/\sigma_{22}', \quad x_3'' = x_3'/\sigma_{33}'.$$

   and that the probability can be written as a volume intergral over the ellipsoid:

$$\text{Prob}\{(x_1, x_2, x_3) \text{ lies within ellipsoid}\} = \iiint \frac{e^{-r^2/2}}{(2\pi)^{3/2}} dx_1'' dx_2'' dx_3''$$

   where $r^2 = x_1''^2 + x_2''^2 + x_3''^2$.

4. Calculate the probability for $c = 1$ and $c = 2$.

Problem 7

For random variables X and Y with joint density function

$$f(x,y) = \begin{cases} 6e^{-2x-3y}, & \text{if } x, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the joint PDF $P(X \leq x, Y \leq y)$.
2. Find the PDF of X, $f_X(x)$.
3. Find the PDF of Y, $f_Y(y)$.
4. Are X and Y independent? Give a reason for your answer.