1. $f_X(x) = 2x \quad 0 \le x \le 1$ $\qquad f_V(v) = 1 \quad 0 \le v \le 1$

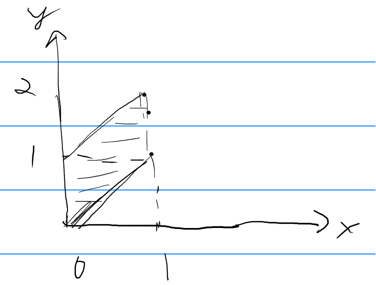
$\quad\;\; = 0 \quad$ o.w. $\qquad\qquad\qquad = 0 \quad$ o.w.

$Y = X + V$ $\qquad$ X & V are independent.

$y \in [0, 2]$

i) $f_{XY}(x,y) = f_X(x) \cdot f_V(y-x)$

$$= \begin{cases} 2x & 0 \le x \le 1,\ 0 \le y - x \le 1 \\ 0 & \text{o.w.} \end{cases}$$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \begin{cases} x^2 \Big|_0^y \\ x^2 \Big|_{y-1}^1 \\ 0 \quad \text{o.w} \end{cases} = \begin{cases} y^2 & 0 \le y \le 1 \\ 2y - y^2 & 1 \le y \le 2 \\ 0 & \text{o.w} \end{cases}$$

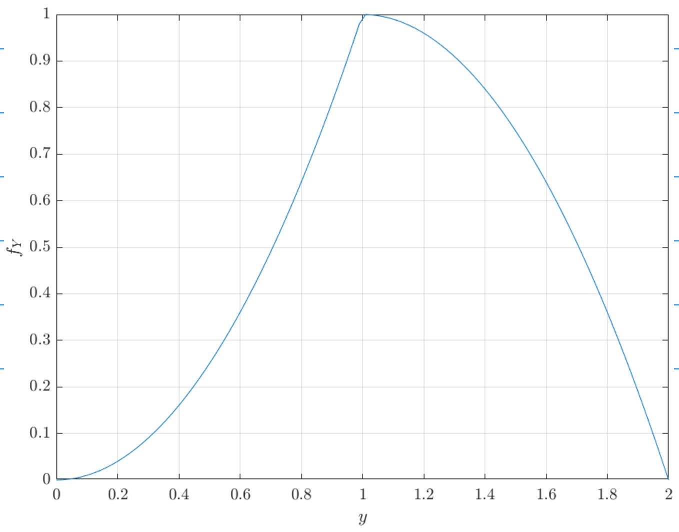
$$\mu_y = E_Y = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_0^1 y^3\, dy + \int_1^2 2y^2 - y^3\, dy$$

$$\boxed{\mu_y = \frac{7}{6}}$$

ii) $E y^2 = \int_0^1 y^4\, dy + \int_1^2 2y^3 - y^4\, dy = \boxed{\frac{3}{2}}$

iii)



iv) $f_{X|Y}(x|Y=y) = \begin{cases} \dfrac{2x}{y^2} & 0 \le y \le 1,\ 0 \le x \le 1 \\ \dfrac{2x}{2y-y^2} & 1 \le y \le 2,\ 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$

$\hat{g}(y) = E(X|Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$

$$= \begin{cases} \dfrac{2x^3}{3y^2}\Big|_0^y \\ \dfrac{2x^3}{6y-3y^2}\Big|_{y-1}^{1} \\ 0 \quad \text{o.w} \end{cases} \boxed{= \begin{cases} \dfrac{2}{3}y & 0 \le y \le 1 \\ \dfrac{2}{3}\left(y + \dfrac{1}{y} - 1\right) & 0 \le y \le 2 \\ 0 & \text{o.w.} \end{cases}}$$

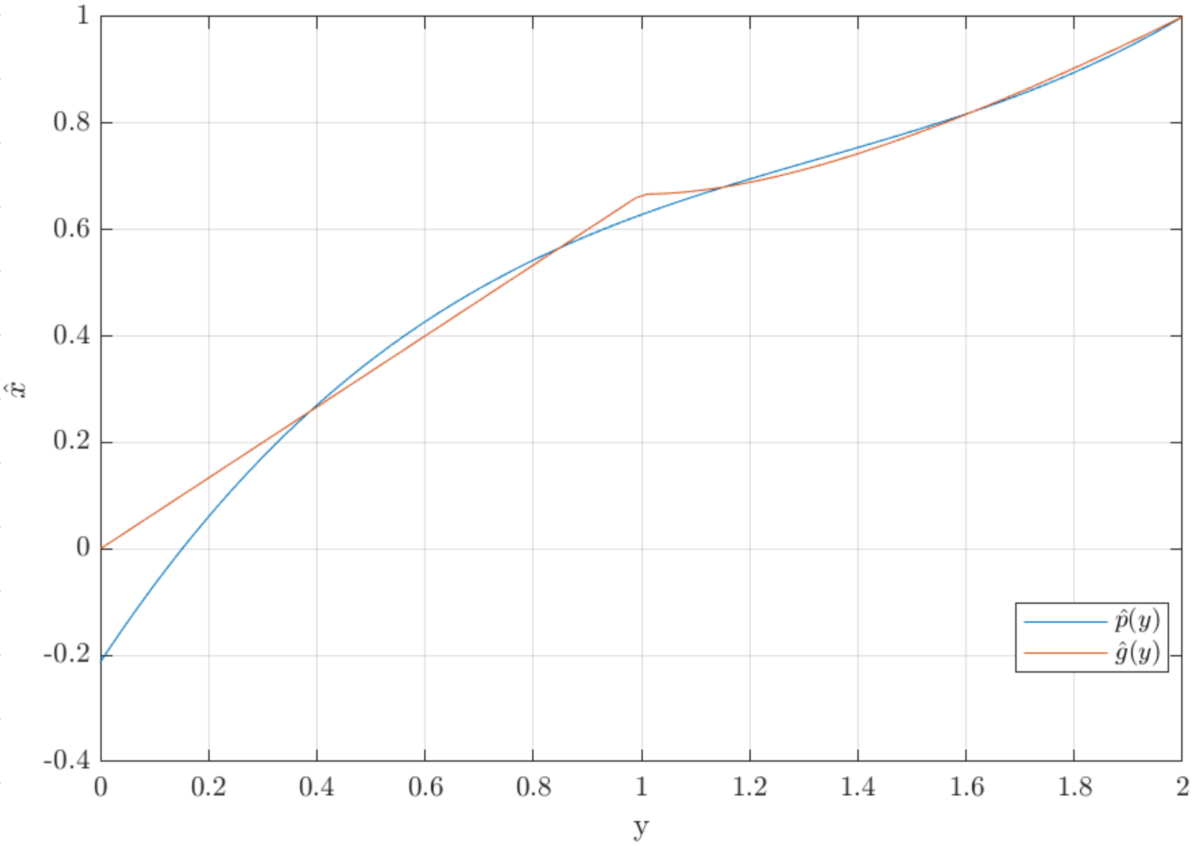
v) Let $p = [1, y, y^2, y^3]$.

$R_{xp} = Exp^* = \left[\dfrac{2}{3} \quad \dfrac{5}{6} \quad \dfrac{101}{90} \quad \dfrac{8}{5}\right]$.

$$R_p = Epp^* = \begin{bmatrix} 1 & & & \\ \frac{7}{6} & & & \\ \frac{3}{2} & & & \\ \frac{31}{15} & 3 & \frac{127}{28} & \frac{85}{12} \end{bmatrix}$$

$P_H x = \hat{p}(y) = R_{xp} R_p^{-1} p$

$$\boxed{= 0.2345\, y^3 - 0.9380\, y^2 + 1.5444\, y - 0.2127}$$



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
$\hat{x}$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
y
$\hat{p}(y)$
$\hat{g}(y)$

win lose

2. $p = \frac{1}{100}$ $q = 1-p$ independent trials.

The expected number of wins out of 10000 games should be $\frac{1}{100} \cdot 10000 \doteq 100$ games.

Winning 200 games is unlikely and the player might $\boxed{\text{be cheating}}$

OR. by CLT.

$\mu_x = 100$, $\sigma_x = \sqrt{np(1-p)} = 9.9499$

using 95% confidence interval. $\alpha = 2.5\%$

$P(n=200) = 1 - \Phi\left(\frac{200-100}{9.9499}\right) = 0 < \alpha$.

Thus the player is likely to be cheating.

3. $f_{XY}(x,y) = \frac{4}{\pi}$ for $x^2+y^2 \le 1$ & $x \ge 0$ & $y \ge 0$.

$= 0$ o.w.

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\sqrt{1-x^2}} \frac{4}{\pi}\,dy = \frac{4}{\pi}\sqrt{1-x^2}$

i) $\mu_X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 \frac{4}{\pi}\sqrt{1-x^2}\,x\,dx = \boxed{0.4244}$

ii) $\sigma_X = Ex^2 - \mu_X^2 = \boxed{0.0699}$

iii) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{\sqrt{1-y^2}} \frac{4}{\pi}\,dx = \boxed{\begin{array}{ll} \frac{4}{\pi}\sqrt{1-y^2} & 0 \le y \le 1 \\ = 0 & \text{o.w} \end{array}}$

$\int_0^1 f_Y(y)\,dx = 1.$

iv) $\hat{g}(y) = E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{1-y^2}}\,dx$

$= \int_0^{\sqrt{1-y^2}} \frac{x}{\sqrt{1-y^2}}\,dx$

$\boxed{\begin{array}{ll} = \frac{1}{2}\cdot\sqrt{1-y^2} & 0 < y \le 1 \\ = 0 & \text{o.w.} \end{array}}$

v) $E|X-\hat{g}(Y)|^2 = Ex^2 - E(\hat{g}(y))^2$

$= \int_{-\infty}^{\infty} x^2 \frac{4}{\pi}\sqrt{1-x^2}\,dx - \int_{-\infty}^{\infty} \hat{g}(y)^2 \cdot f_Y(y)\,dy$

$= \frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx - \frac{1}{\pi}\int_0^1 \sqrt{1-y^2}\cdot(1-y^2)\,dy$

$= \frac{1}{4} - \frac{3}{16}$

$\boxed{= \frac{1}{16}} = 0.0625$

vi) $E X = E(E(X|Y)) = \int_{-\infty}^{\infty} \hat{g}(y) f_Y(y)\, dy$

$= \int_0^1 \frac{2}{\pi}(1-y^2)\, dy$

$= \boxed{0.4244}$ Adam's Law checks out.

vii). ~~let~~ $p = [1, y, y^2, y^3]^*$

$R_{xp} = E x p^* = \int_0^1 \int_0^{\sqrt{1-y^2}} x \cdot p^* \cdot \frac{4}{\pi}\, dx\, dy$

$= [0.4244 \quad 0.1592 \quad 0.0849 \quad 0.0531].$

$R_p = E p p^* = \int_0^1 \int_0^{\sqrt{1-y^2}} p p^* \frac{4}{\pi}\, dx\, dy$

$$= \begin{bmatrix} 1 & & & \\ 0.4244 & & & \\ 0.25 & & & \\ 0.1698 & 0.1250 & 0.0970 & 0.0781 \end{bmatrix}$$

$$\boxed{P_H X = \hat{p}(y) = R_{xp} \cdot R^{-1} \cdot p = -0.47300 y^3 + 0.1953 y^2 - 0.1197 y + 0.5067}$$

viii) $E|X-\hat{X}|^2 = E x^2 - E\hat{x}^2 = E x^2 - R_{xp} R^{-1} R_{px} = \boxed{0.062522}$

ix) $\boxed{\text{Yes}}$

x)