5.1 4

$$\frac{du}{dt} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} u, \quad u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

$\det[P - \lambda I] = 0.$

$$\det \begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{bmatrix} = 0.$$

$(\frac{1}{2}-\lambda)^2 - \frac{1}{4} = 0.$

$\frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4} = 0.$

$\lambda(\lambda - 1) = 0.$

$\lambda = 0, 1.$

$\lambda_1 = 0,$ $\lambda_2 = 1.$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} x_1 = 0. \qquad \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} x_2 = 0.$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$u(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\begin{cases} c_1 + c_2 = 5 \\ -c_1 + c_2 = 3 \end{cases} \quad \begin{cases} c_1 = 1 \\ c_2 = 4 \end{cases}$$

$$\boxed{u(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} e^t}$$

5.1 6

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad r_2 - r_1$$

$(1-\lambda)^2 - 1 = 0.$

$1 - 2\lambda + \lambda^2 - 1 = 0,$

$\lambda(\lambda - 2) = 0$

$\underline{\lambda = 0, 2}$

$(1-\lambda)\cdot\lambda = 0.$

$\underline{\lambda = 0, 1}$

the non-zero eigenvalue is changed.

$Ax = \lambda x.$

$(A - \lambda I)x = 0.$

The zero eigenvalue can't be changed because the original matrix is not full rank.

5.1 10 a) b)

a) $Ax_A = \lambda_A x_A$ ① $\quad ABx_C = \lambda_C x_C$. $\quad \lambda_A \cdot \lambda_B \neq \lambda_C$

$Bx_B = \lambda_B x_B$ ② . $\quad (A+B)x_D = \lambda_D x_D$. $\quad \lambda_A + \lambda_B \neq \lambda_D$

$$\lambda = \frac{trace \pm (trace^2 - 4\,det)^{1/2}}{2}$$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \lambda = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} = 1 \pm \sqrt{2}$. $\quad \lambda_{A_1} \cdot \lambda_{B_1} \neq \lambda_{C_1}$

$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. $\lambda = \frac{3 \pm \sqrt{9-4}}{2} = 1.5 \pm \frac{\sqrt{5}}{2}$. $\quad \lambda_{A_1} + \lambda_{B_1} \neq \lambda_{D_1}$

$AB = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$. $\lambda = \frac{6 \pm \sqrt{36 - 4\cdot(-1)}}{2} = 3 \pm 2\sqrt{10}$ $\quad \lambda_{A_2} \cdot \lambda_{B_2} \neq \lambda_{C2}$

$\lambda_{A_2} + \lambda_{B_2} \neq \lambda_{D_2}$.

$A+B = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$. $\lambda = \frac{5 \pm \sqrt{25 - 4\cdot 0}}{2} = 0, 5$.

b) $tr(A)$ = trace of A.

$\lambda_{A_1} + \lambda_{A_2} + \lambda_{B_1} + \lambda_{B_2} = tr(A) + tr(B) = a_{11} + a_{22} + b_{11} + b_{22}$.

$$\shortparallel$$

$A+B = \begin{bmatrix} a_{11}+b_{11} & \cdots \\ \cdots & a_{22}+b_{22} \end{bmatrix} \Rightarrow tr(A+B) = a_{11} + a_{22} + b_{11} + b_{22}$.

$\lambda_{D_1} + \lambda_{D_2} = tr(A+B) = tr(A) + tr(B)$

Thus. $\lambda_{A_1} + \lambda_{A_2} + \lambda_{B_1} + \lambda_{B_2} = \lambda_{D_1} + \lambda_{D_2}$.

Sum of all eigenvalues of A & B equals sum of all eigenvalues of A+B.

$$\lambda_{A_1}\cdot\lambda_{A_2}\cdot\lambda_{B_1}\cdot\lambda_{B_2}=\frac{\left[\cancel{trA^2}-(\cancel{trA^2}-4\det A)\right]\cdot\left[\cancel{trB^2}-(\cancel{trB^2}-4\det B)\right]}{4\cdot 4}$$

$$=\det A\cdot\det B.$$

$$\lambda_{c_1}\cdot\lambda_{c_2}=\frac{trAB^2-(trAB^2-4\det AB)}{4}=\det AB.=\det A\cdot\det B.$$

Thus. $\underline{\lambda_{A_1}\cdot\lambda_{A_2}\cdot\lambda_{B_1}\cdot\lambda_{B_2}=\lambda_{c_1}\cdot\lambda_{c_2}}$

5.1 12

$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$. $\lambda = \frac{0 \pm \sqrt{4(-25)}}{2} = \underline{\pm 5}$

$\lambda_1 = 5$.

$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \cdot x_1 = 0$.

$-2a+4b=0$. $a=2b$.

$\underline{x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$.

$\lambda_2 = -5$

$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} x_2 = 0$.

$8a+4b=0$.

$2a=-b$.

$\underline{x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$;

$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. $\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2-b^2)}}{2} = \underline{a \pm b}$.

$\lambda_1 = a+b$.

$\begin{bmatrix} -b & b \\ b & -b \end{bmatrix} x_1 = 0$.

$\underline{x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

$\lambda_2 = a-b$.

$\begin{bmatrix} b & b \\ b & b \end{bmatrix} x_2 = 0$.

$\underline{x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$.

5.2 2

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad S = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}.$$

$$A = S\Lambda S^{-1}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}$$

8.2 6

a) $A^2 = I$

$\lambda^2 x = I x$.

eigenvalues of I are 1's

~~t~~hus $\underline{\lambda = \pm 1}$.

b) $\lambda_1 + \lambda_2 = trA$

$trA = 0$.

$\lambda_1 \cdot \lambda_2 = detA$

$detA = -1$

c) $A = \begin{bmatrix} 3 & -1 \\ a & b \end{bmatrix}$.

$3 + b = 0$.

$b = -3$.

$3 \cdot -3 + a = -1$.

$a = 8$

$\underline{A = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}}$

5.4 2 $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. $u(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 0}}{2} = 0, -2.$$

$\lambda_1 = 0$. $\lambda_2 = -2$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot x_1 = 0$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2 = 0$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$u(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$\begin{cases} c_1 + c_2 = 3 \\ c_1 - c_2 = 1 \end{cases}$ $\begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}$

$$\underline{u(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}}$$

as $t \to \infty$.

$\underline{u(\infty) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$ S.S.

5.4 6

$$y''+y=0 \qquad y''=-y.$$

$$\begin{bmatrix} y' \\ y'' \end{bmatrix} = A \cdot \begin{bmatrix} y \\ y' \end{bmatrix}.$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$y(0)=2$

$y'(0)=0.$

$$\lambda = \frac{0 \pm \sqrt{-4\cdot(1)}}{2} = \pm i$$

$\lambda_1 = i$ $\qquad$ $\lambda_2 = -i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \cdot x_1 = 0 \qquad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} x_2 = 0.$$

$ia=b.$ $\qquad$ $ia=-b.$

$$x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} \cdot e^{it} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-it}$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

$$\begin{cases} c_1 + c_2 = 2 \\ (c_1 - c_2) i = 0. \end{cases} \qquad \begin{cases} c_1 = 1 \\ c_2 = 1. \end{cases}$$

$$\boxed{\begin{aligned} y(t) &= e^{it} + e^{-it} = 2\cos(t) \\ y'(t) &= i e^{it} - i e^{-it} = -2\sin(t) \end{aligned}}$$