P156. 1. Find $\hat{f} = \sum_0^4 \alpha_j t^j$ and cost d. to solve.

$$\|\sin(t) - \hat{f}\|^2 = d^2 = \inf\left\{ \int_0^{\pi} \left|\sin(t) - \sum_{j=0}^{4} \alpha_j t^j\right|^2 dt \; : \alpha_i \in \mathbb{C} \right\}$$

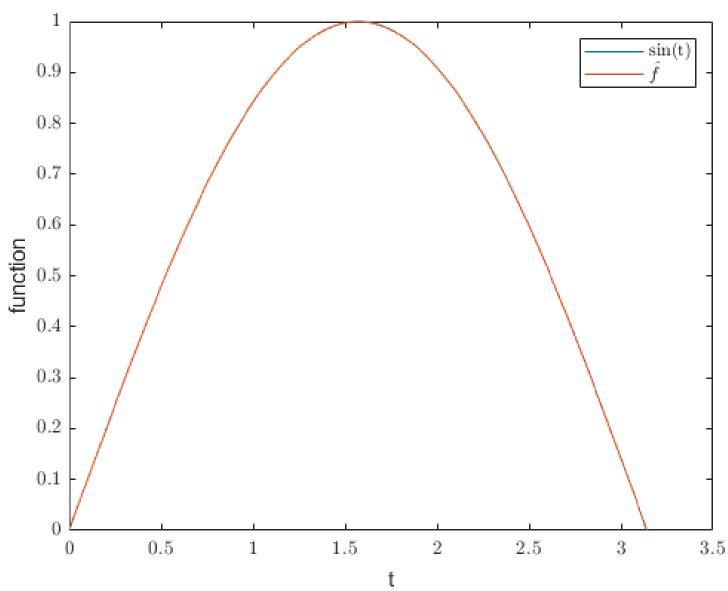
Entries of $\hat{f}$ are linearly independent.

$$G = \begin{bmatrix} (1,1) & \cdots & (1,t^4) \\ (t,1) & & \vdots \\ (t^2,1) & & \vdots \\ (t^3,1) & & \vdots \\ (t^4,1) & \cdots & (t^4,t^4) \end{bmatrix} = \begin{bmatrix} \pi & \cdots & \frac{\pi^5}{5} \\ \frac{\pi^2}{2} & & \vdots \\ \frac{\pi^3}{3} & & \vdots \\ \frac{\pi^4}{4} & & \vdots \\ \frac{\pi^5}{5} & \cdots & \frac{\pi^9}{9} \end{bmatrix}$$

$$y = [(f,1), (f,t), (f,t^2), (f,t^3), (f,t^4)]$$
$$= [2 \quad \pi \quad \pi^2 - 4 \quad \pi^3 - 6\pi \quad \pi^4 - 12\pi^2 + 48]$$

$$\alpha = y G^{-1}$$
$$= \boxed{[0.0013 \quad 0.9826 \quad 0.0545 \quad -0.2338 \quad 0.0372]}$$

$$d^2 = (f,f) - y G^{-1} y^* = 4.2782 \times 10^{-7} \qquad \boxed{d = 6.5408 \times 10^{-4}}$$



Prob. 2. $\hat{f}(t) = \alpha e^{-t} + \beta e^{-2t}$ to solve

$$\int_0^\infty | e^{-3t} - \hat{f}(t)|^2 dt = d^2 = \inf \left\{ \int_0^\infty |e^{-3t} - f(t)|^2 dt : \alpha \in C, \beta \in C \right\}$$

e^{-t} & e^{-2t} are linearly independent.

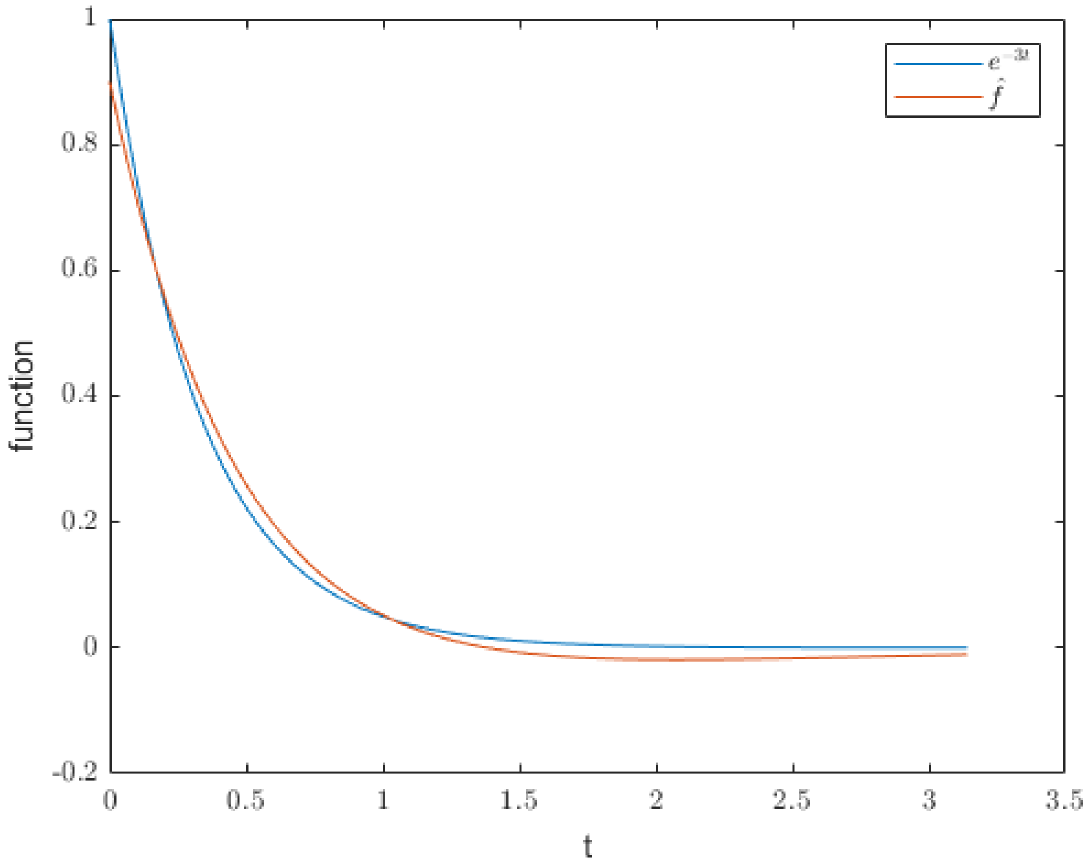
$$G = \begin{bmatrix} (e^{-t}, e^{-t}) & (e^{-2t}, e^{-t}) \\ (e^{-t}, e^{-2t}) & (e^{-2t}, e^{-2t}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$$y = [(e^{-3t}, e^{-t}), (e^{-3t}, e^{-2t})] = [\tfrac{1}{4}, \tfrac{1}{5}].$$

$$[\alpha \ \beta] = yG^{-1} = \boxed{[-0.3 \quad 1.2]}.$$

$$d^2 = \int_0^\infty e^{-6t} dt - y G^{-1} y^* = 0.0017 \quad . \boxed{d = 0.0408}$$

The solution is unique.



197. 3. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. $y = 2x_1 + x_2$.

$$y(t) = Ce^{At} x(0).$$

Find $\hat{x}(0)$ and d to solve.

$$d^2 = \int_0^\infty |e^{-3t} - Ce^{At}\hat{x}(0)|^2 dt = \inf\left\{ \int_0^\infty |e^{-3t} - Ce^{At}\hat{x}(0)|^2 dt : x(0) \in \mathbb{C}^2 \right\}.$$

$CA = [-7 \ -11]$. $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -7 & -11 \end{bmatrix} \rightarrow$ rank $= 2$.

the system is observable thus it has a unique solution

$y(t) = [5e^{-2t} - 3e^{-t} \quad 10e^{-2t} - 9e^{-t}]\ \hat{x}_0$, let $\hat{x}_0 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

which has the same basis at the problem in Q2.

$\therefore$ $-3\alpha - 9\beta = -0.3$
$5\alpha + 10\beta = 1.2$ $\Rightarrow$ $\boxed{\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0.52 \\ -0.14 \end{bmatrix} = \hat{x}_0}$

or.

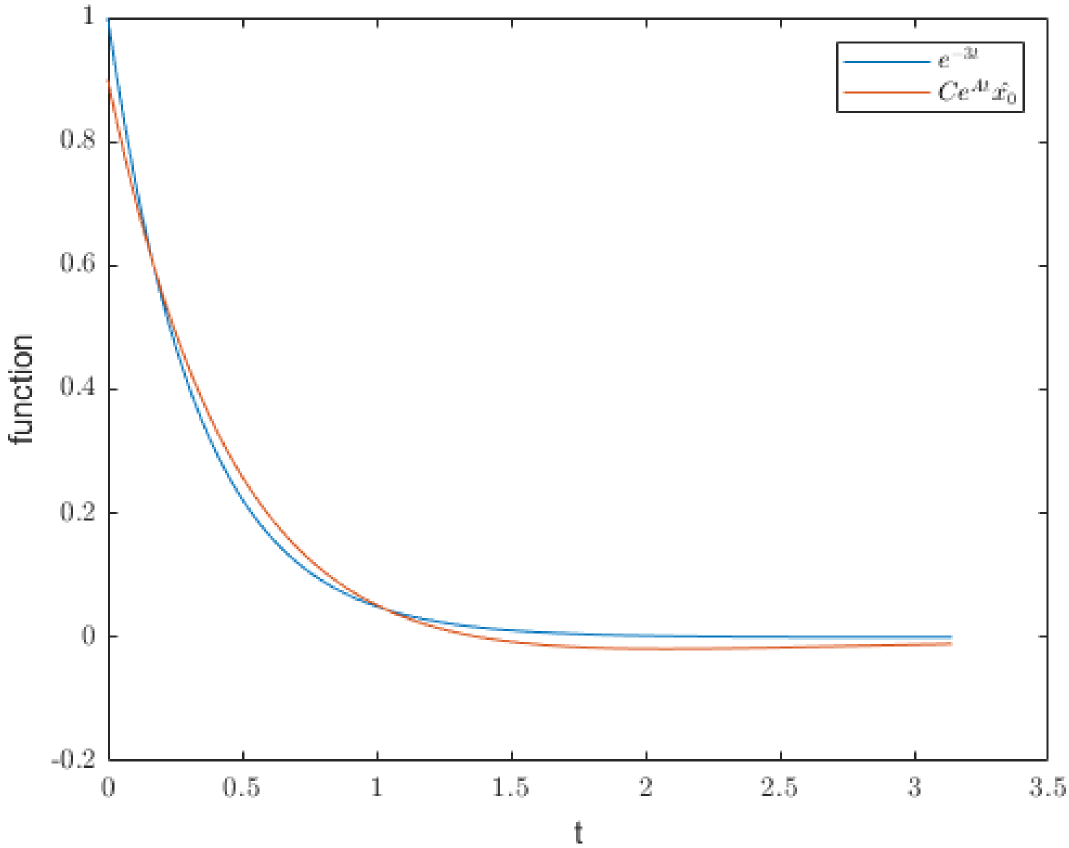
$$P = \int_0^\infty e^{A^* t} C^* C e^{At} dt = \begin{bmatrix} \frac{3}{4} & 1 \\ 1 & \frac{11}{2} \end{bmatrix}$$

$$P\hat{x}_0 = \int_0^\infty e^{A^* t} C^* \cdot e^{-3t} dt.$$

$$\hat{x}_0 = \begin{bmatrix} \frac{13}{25} \\ -\frac{7}{50} \end{bmatrix}$$

$d = \sqrt{\|e^{-3t}\|^2 - \hat{x}_0^* P \hat{x}_0} = \boxed{0.0408}$ The cost is the same because in both problems the subspaces has the same basis and the function to approximate is also the same.

$\mathrm{Col}\{Ce^{At}\} = \mathrm{span}\{e^{-t}, e^{-2t}\}$



e^{-3t}
$Ce^{At}\hat{x}_0$
function
t
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
3.5

P201. 1. $T = \begin{bmatrix} 3 & 2 \\ 4 & -2 \\ 3 & 2 \\ -1 & 3 \end{bmatrix}$ $y = \begin{bmatrix} -2 \\ 4 \\ 3 \\ 8 \end{bmatrix}$

T has full rank, thus it is one-to-one

There is a unique solution.

$$\hat{u} = (T^*T)^{-1} T^* y = \boxed{\begin{bmatrix} 0.6226 \\ 1.2084 \end{bmatrix}}$$

$$d = \sqrt{\|y\|^2 - (T^*T\hat{u}, \hat{u})} = \boxed{6.8746}.$$

P201. 2. $T = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 8 \end{bmatrix}$.

T has rank 1, thus the solution is not unique.

$$T^*T\hat{u} = T^*y$$

$$\begin{bmatrix} 15 & 30 \\ 30 & 60 \end{bmatrix} \hat{u} = \begin{bmatrix} 36 \\ 72 \end{bmatrix}.$$

$$[1 \quad 2]\hat{u} = 2.4$$

$$u_1 + 2u_2 = 2.4.$$

$$\hat{u} = \begin{bmatrix} 2.4 - 2u_2 \\ u_2 \end{bmatrix} = \boxed{\begin{bmatrix} 2.4 \\ 0 \end{bmatrix} + k\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$