HW8.

1. What are the positive ~~limit~~ sets of the following solutions?

a). $x(t) = \sin(t^2)$ b). $x(t) = e^t \sin(t)$

a) $[-1, 1]$ b) empty.

2. Using La Salle's Theorem, show that all solutions of system,

$$\dot{x}_1 = x_2^2$$
$$\dot{x}_2 = -x_1 x_2.$$

must approach x_1 axis

$\dot{x}_1^e = x_2^{e2} = 0.$ $x_2^e = 0.$

$\dot{x}_2^e = -x_1^e x_2^e = 0.$ $-x_1^e \in \mathbb{R}.$

Equilibrium state of the system is at $x_2 = 0$, thus the set of all points $\underline{M = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}}$ is an invariant set of the system, and it is the largest invariant set in S

Let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \geq 0.$

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= x_1 x_2^2 - x_1 x_2^2$$
$$= 0.$$

By La Salle's theorem: all solution of the system converges to M, which is all points on x_1 axis.

3 .

Exercise 3 Consider the scalar nonlinear mechanical system

$$\ddot{q} + c(\dot{q}) + k(q) = 0$$

If the term $-c(\dot{q})$ is due to damping forces it is reasonable to assume that $c(0) = 0$ and

$$c(\dot{q})\dot{q} > 0 \qquad \text{for all} \quad \dot{q} \neq 0$$

Suppose the term $-k(q)$ is due to conservative forces and define the potential energy by

$$P(q) = \int_0^q k(\eta)\, d\eta$$

Show that if $\lim_{q\to\infty} P(q) = \infty$, then all motions of this system must approach one of its equilibrium positions.

Let $V = P(q) + \frac{1}{2}\dot{q}^2$ $\quad \lim_{q\to\infty} P(q) = \infty$, $\frac{1}{2}\dot{q}^2 \geq 0$. V is radially unbounded.

$\dot{V} = k(q)\cdot\dot{q} + \dot{q}\cdot(\ddot{q})$

$= k(q)\dot{q} + \dot{q}(-c(\dot{q}) - k(q))$

$= -c(\dot{q})\dot{q} < 0 \qquad \forall\, \dot{q} \neq 0.$

$\dot{V} = 0 \qquad$ if $\dot{q} = 0.$

Thus all solution must converges to $[q, 0]$, by La Salle's theorem, The equilibrium of q is q^e, and the invariant set is $[q^e, 0]$.

Exercise 4 Consider a nonlinear mechanical system described by

$$m\ddot{q} + c\dot{q} + k(q) = 0$$

where q is scalar, $m, c > 0$ and k is a continuous function which satisfies

$$\begin{aligned} k(0) &= 0 \\ k(q)q &> 0 \qquad \text{for all } q \neq 0 \\ \lim_{q\to\infty} \int_0^q k(\eta)\, d\eta &= \infty \end{aligned}$$

(a) Obtain a state space description of this system.

(b) Prove that the state space model is GAS about the state corresponding to the equilibrium position $q = 0$.

(i) Use a La Salle type result.

(ii) Do not use a La Salle type result.

a). Let $x_1 = q$, $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ $\qquad \ddot{q} = -\frac{c}{m}\dot{q} - \frac{1}{m}k(q)$

$x_2 = \dot{q}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{m} \end{bmatrix} k(x_1).$$

b)
i) Let $V = \int_0^q k(\eta)\, d\eta + \frac{m}{2}\dot{q}^2$

$$\dot{V} = k(q)\dot{q} + m\dot{q}\left(-\frac{c}{m}\dot{q} - \frac{1}{m}k(q)\right)$$

$$= -c\dot{q}^2$$

$\dot{V} = 0$ iff $\dot{q} = 0$.

The equilibrium points are $[0, 0]$, since $k(0) = 0$

By La Salle's theorem, all solution converges to $[0, 0]$, which is the origin.

c). Using the same $V = \int_0^q k(\eta) d\eta + \frac{m}{2}\dot{q}^2$ is PD, as the first term $\to\infty$ when $q\to\infty$ and the second term $\to\infty$ when $\dot{q}\to\infty$.

$\dot{V} = -c\dot{q}^2 < 0$ for $\dot{q} \neq 0$.

By Theorem 4 on P 81, the system is GAS about the origin.

Exercise 5 Consider an inverted pendulum $\mathcal{B}$ (or one link manipulator) subject to a control torque u. This system can be described by

$$\ddot{q} - a\sin q = bu$$

where q is the angle between the pendulum and a vertical line, $a = mgl/I$, $b = 1/I$,m is the mass of $\mathcal{B}$, I is the moment of inertia of $\mathcal{B}$ about its axis of rotation through O, l is the distance between O and the mass center of $\mathcal{B}$, and g is the gravitational acceleration constant of YFHB. We wish to stabilize this system about the position corresponding to $q = 0$ by a linear feedback controller of the form

$$u = -k_p q - k_d \dot{q}$$

Using the results of the last problem, obtain the least restrictive conditions on the controller gains k_p, k_d which assure that the closed loop system is GAS about the state corresponding to $q(t) \equiv 0$. Illustrate your results with numerical simulations.

$$\ddot{q} = a\sin q - b(k_p q + k_d \dot{q})$$

$$V = \int_0^q (-a\sin\eta + bk_p \cdot \eta)\, d\eta + \frac{\dot{q}^2}{2} = a\cos q + \frac{b}{2}k_p q^2 + \frac{\dot{q}^2}{2}$$

$$\dot{V} = (-a\sin q + bk_p \cdot q)\cdot \dot{q} + \dot{q}\left[a\sin q - b(k_p q + k_d \dot{q})\right]$$

$$= bk_p q \cdot \dot{q} - \dot{q}\cdot b(k_p q + k_d \dot{q})$$

$$= -b k_d \dot{q}^2.$$

$\dot{V} < 0$ iff $\boxed{k_d > 0}$ as $b > 0$ and $\dot{q}^2 > 0$

Thus the solution approaches the set $[q, 0]$. by La Salle's theorem.

The equilibrium points are $q = q^e$, $\dot{q} = 0$, $\ddot{q} = 0$.

$$0 = a\sin q^e - bk_p q^e$$

$$a\sin q^e = bk_p q^e$$

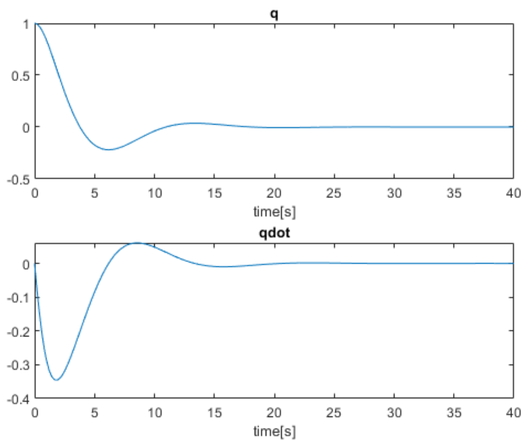
$$\sin q^e = \frac{b}{a} k_p q^e$$

$$\cos q^e = \frac{b}{a} k_p.$$

$$1 = \frac{b}{a} k_p.$$

to converge only to $q^e = 0$,

$$\boxed{k_p > \frac{a}{b}},$$



sim param;

$a = 1$

$b = 1/2$

$k_p = 2.5$

$k_d = 1$

Exercise 6 Consider the system described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 + \theta \sin x_1 + u$$

with control input u where θ is an unknown constant parameter. Obtain an adaptive feedback controller which guarantees that, for any initial conditions, $\lim_{t\to\infty} x(t) = 0$ and $u(\cdot)$ is bounded. (Hint: As a candidate Lyapunov function for the closed loop system, consider something of the form $V(x) + U(\hat{\theta} - \theta)$ where V is a Lyapunov function for the nominal uncontrolled nominal linear system.) Illustrate the effectiveness of your controller with simulations.

If we know θ, choose $u = -\theta \sin x_1$ to cancel out the non-linear term,

the system becomes $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$, which is linear and GAS,

If θ were unknow, consider $\hat{\theta}$ as an estimate of θ. and $\delta\theta = \hat{\theta} - \theta$

We can introduce a new state $\delta\theta$ for the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + \theta \sin x_1 - \hat{\theta} \sin x \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 - \delta\theta \sin x_1 \end{cases}$$

Consider a Lyapunov function

$L(x, \delta\theta) = V(x) + U(\delta\theta)$. $V(x) = \frac{3}{2} x_1^2 + x_2^2 + x_1 x_2$

$$\dot{V}(x) = 3x_1x_2 + 2x_2\dot{x}_2 + x_2^2 + x_1\dot{x}_2$$
$$= 3x_1x_2 + x_2^2 + (x_1 + 2x_2)(-x_1 - x_2) + (x_1 + 2x_2)\delta\theta \sin x_1$$
$$= \cancel{3x_1x_2} + \cancel{x_2^2} - x_1^2 - \cancel{2}x_2^2 - \cancel{3x_1x_2} + (x_1 + 2x_2)\delta\theta \sin x_1$$
$$= -x_1^2 - x_2^2 + (x_1 + 2x_2)\delta\theta \sin x_1$$

let $U(\delta\theta) = \frac{k}{2}\delta\theta^2$

$\dot{U}(\delta\theta) = k\delta\theta \cdot \dot{\delta\theta}$

$k > 0$.

$$\dot{L}(x, \delta\theta) = -x_1^2 - x_2^2 + (x_1 + 2x_2)\delta\theta \sin x_1 + k\delta\theta \cdot \dot{\delta\theta}$$

Let $\dot{\delta\theta} = -\frac{1}{k}(x_1 + 2x_2)\sin x_1 \Rightarrow \dot{L}(x, \delta\theta) = -x_1^2 - x_2^2 < 0$ iff $x \neq 0$

the solution approaches the set $[0, 0, \delta\theta^e]$

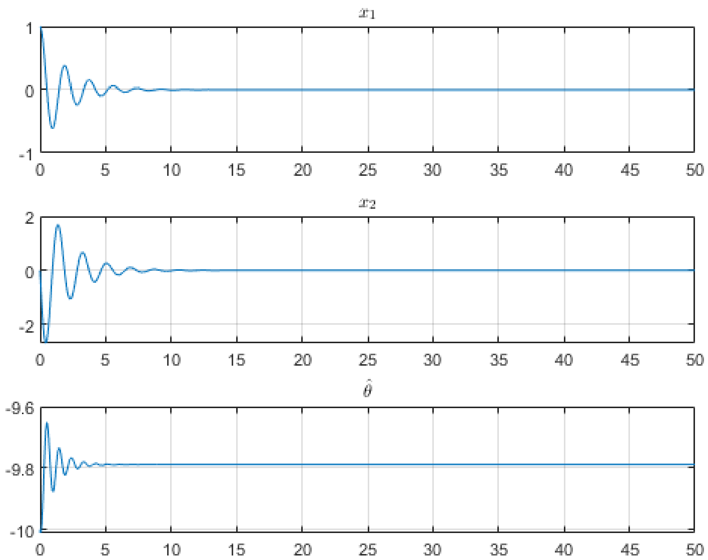
Thus the solution goes to 0 as $t \to \infty$. , and $\delta\theta$ is bounded to $\delta\theta^e$.

since θ is constant. $\delta\dot{\theta} = \dot{\hat{\theta}} - \dot{\theta} = \dot{\hat{\theta}}$

$$\dot{\hat{\theta}} = -\frac{1}{k}(x_1 + 2x_2)\sin x_1$$

The new system is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 - \theta \sin x_1 + \hat{\theta} \sin x_1 \\ \dot{\hat{\theta}} = -\frac{1}{k}(x_1 + 2x_2)\sin x_1 \end{cases}$$



sim param:

$x_0 = [1, 0]$ $k = 2$.

$\theta = 1.2$.

$\hat{\theta}_0 = -10$.

As you can see as $t \to \infty$, $x \to 0$. and $\hat{\theta}$ is bounded