Midterm:

1. $d = \|y - A\hat{x}\| = \min\{\|y - Ax\| : x \in \mathbb{C}^2\}$

$$A = \begin{bmatrix} 1 & 1 \\ a & b \\ a^2 & b^2 \\ \vdots & \vdots \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ c \\ c^2 \\ \vdots \end{bmatrix} \quad \left\| \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix} \right\|^2 = \sum_{j=1}^{\infty} |f_j|^2$$

$a = \frac{1}{2}$, $b = \frac{2}{3}$, $c = \frac{3}{4}$.

(i). $\hat{x} = (A^T A)^{-1} A^T y$.

$$= \begin{bmatrix} 1 + a^2 + a^4 + \cdots & 1 + ab + a^2b^2 + \cdots \\ 1 + ab + a^2b^2 + \cdots & 1 + b^2 + b^4 + \cdots \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 + ac + a^2c^2 + \cdots \\ 1 + bc + b^2c^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma a^{2j} & \Sigma (ab)^j \\ \Sigma (ab)^j & \Sigma (b^2)^j \end{bmatrix}^{-1} \cdot \begin{bmatrix} \Sigma (ac)^j \\ \Sigma (bc)^j \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1 - \frac{1}{2}^2} & \frac{1}{1 - \frac{1}{3}} \\ \frac{1}{1 - \frac{1}{3}} & \frac{1}{1 - \frac{4}{9}} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \frac{1}{1 - \frac{3}{8}} \\ \frac{1}{1 - \frac{1}{2}} \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -\frac{4}{5} \\ \frac{16}{9} \end{bmatrix}} \leftarrow \text{(i)}$$

(ii) The optimal solution $\hat{x}$ is unique. $\boxed{\text{True}}$ $\leftarrow$ (ii)

(iii) $d^2 = \|y - A\hat{x}\|^2$

$$= \|y\|^2 - \|A\hat{x}\|^2$$

$$= \Sigma |c^j|^2 - \hat{x}^T (A^T A) \hat{x}$$

$$= \Sigma (c^2)^j - \hat{x}^T (A^T A) \hat{x}$$

$$= \frac{1}{1 - \frac{9}{16}} - \begin{bmatrix} -\frac{4}{5} & \frac{16}{9} \end{bmatrix} \begin{bmatrix} \frac{4}{3} & \frac{3}{2} \\ \frac{3}{2} & \frac{9}{5} \end{bmatrix} \begin{bmatrix} -\frac{4}{5} \\ \frac{16}{9} \end{bmatrix} = \boxed{0.0102} \leftarrow \text{(iii)}$$

2.

$$A = \begin{bmatrix} 2 & 6 \\ -2 & -5 \end{bmatrix}. \qquad C = [2 \ \ 3].$$

$$d^2 = \int_0^\infty |26e^{-3t} - Ce^{At}\hat{x}|^2 \, dt = \min\left\{ \int_0^\infty |26e^{-3t} - Ce^{At}x|^2 dt : x \in \mathbb{C}^2 \right\}.$$

(i)

$$\begin{pmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \quad \text{rank} \Rightarrow 1 < 2 \text{ unobservable.} \quad \boxed{\text{False}} \leftarrow (i).$$

(ii) $\boxed{\text{False}} \leftarrow$ (ii). due to unobservable A & C.

(iii)

$$P = \int_0^\infty e^{A^* t} C^* C e^{At} dt. = \begin{bmatrix} 2 & 3 \\ 3 & 4.5 \end{bmatrix}.$$

$$\hat{X}_{opt} = P^{-1} \cdot \int_0^\infty e^{A^* t} C^* \cdot 26 \cdot e^{-3t} \, dt.$$

$$= \boxed{\begin{bmatrix} -2 \\ 3 \end{bmatrix}.} \leftarrow (iii).$$ This is one of the optimal solution.

$$\mathcal{N}(P) = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$ All optimal solutions are $\hat{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \alpha \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$. $\alpha \in \mathbb{C}$

iv)

$$d^2 = \int_0^\infty (26e^{-3t})^2 dt - \hat{x}_{opt}^* \cdot P \cdot \hat{x}_{opt}$$

$$= \boxed{\frac{169}{6}} \leftarrow (iv)$$

3. $x, u, \& v$ are independent r.v. over $[0,1]$.

$$y = x + u + v \qquad \mathcal{H} = \text{span}\{1, y\}$$

(i) $\hat{x} = P_{\mathcal{H}} x = \alpha + \beta y = ?$

$$g = [1, y].$$

$$R_{xg} = \int_0^1\int_0^1\int_0^1 x \cdot g^* \cdot f_u(u) \cdot f_v(v) \cdot f_x(v) \cdot dx\, dv\, du = \begin{bmatrix} \frac{1}{2} & \frac{5}{6} \end{bmatrix}.$$

$$R_g = \int_0^1\int_0^1\int_0^1 g \cdot g^* \cdot f(x) f(u) f(v)\, dx\, du\, dv = \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}.$$

$$M = R_{xg} \cdot R_g^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \end{bmatrix}.$$

$$\boxed{\hat{x} = 0 + \frac{y}{3}} \longleftarrow (i) \qquad \alpha = 0, \ \beta = \frac{1}{3}$$

ii) $E(x - \hat{x})^2 = R_x - M \cdot R_{xg}' = \boxed{\frac{1}{18}} \longleftarrow$ ii)

4. X & V are independent r.v.

$$f_X(x) = \begin{cases} x e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$y = X + V$$

$$f_{x,y}(x,y) = f_x(x) \cdot f_v(y-x)$$

$$f_V(v) = \begin{cases} e^{-v} & v \geq 0 \\ 0 & v < 0 \end{cases}$$

$$f_y(y) = \int_{-\infty}^{\infty} f_x(x) f_v(y-x)\, dx = \int_0^y f_x(x) f_v(y-x)\, dx \quad y \geq 0$$

i). $\hat{g}(y) = E(X|Y=y) = ?$

$$f_{xy} = \begin{cases} x e^{-y} & x \geq 0,\ y - x \geq 0 \\ 0 \end{cases}$$

$$f_y = \int_0^y x e^{-x} \cdot e^{x-y}\, dx = \frac{y^2 e^{-y}}{2}, \quad y \geq 0.$$

$$f_{x|y} = \frac{x e^{-y}}{\frac{y^2 e^{-y}}{2}} = 2\frac{x}{y^2} \quad \begin{matrix} x \geq 0 \\ y \geq 0 \end{matrix}$$

$$E_{x|y}(X|Y=y) = \int_0^y x f_{x|y}(x,y)\, dx = \int_0^y \frac{2x^2}{y^2}\, dx = \frac{2}{3} \frac{x^3}{y^2}\Big|_0^y = \boxed{\frac{2}{3}y} \leftarrow \text{(i)}$$

ii). $g = [1, y]$.

$$R_{xg} = \int_0^\infty\int_0^\infty x \cdot g^* f_x(x) \cdot f_v(x)\, dx\, dv = [2 \quad 8].$$

$$R_g = \int_0^\infty \int_0^\infty g^* \cdot g \cdot f_x(x) \cdot f_v(v)\, dx\, dv = \begin{bmatrix} 1 & 3 \\ 3 & 12 \end{bmatrix}$$

$$M = R_{xg} \cdot R_g^{-1} = [0 \quad \tfrac{2}{3}].$$

$$P_H X = \alpha + \beta y = \boxed{0 + \frac{2}{3}y} \longleftarrow \text{ii)}.$$