HW 4 P34 1 & 2.

1. x is uniform over $[0, 10]$. v is uniform over $[0, 4]$.

$y = x + v$, $\mathcal{H}_3 = \text{span}\{1, y, y^2\}$. $g = [1, y, y^2]^T$

Find optimal estimate $\hat{x} = P_{\mathcal{H}_3} x$ & error in estimation $E(x - \hat{x})^2$

Compute $E(x|y)$ then compare it with $\hat{x}$.

Sol:

$y \in [0, 14]$.

$$f_x(x) = \begin{cases} \frac{1}{10} & 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \qquad f_v(y-x) = \begin{cases} \frac{1}{4} & 0 \le y-x \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{xy}(x, y) = f_x(x) f_v(y-x) = \begin{cases} 0.25 \times 0.1 & \begin{matrix} 0 - 10 \\ 0 - 4 \end{matrix} \end{cases}$$

$x = y - v$.

$$f_y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{y-4}^{y} f_x(x) f_v(y-x)\, dx.$$

$$= \int_{y-4}^{y} \frac{1}{10} \cdot \frac{1}{4}\, dx.$$

$$= \frac{1}{40} \cdot x \Big|_{y-4}^{y} = \begin{cases} \frac{y}{40} - 0 = \frac{y}{40} & 0 \le y \le 4 \quad \text{(Shown)} \\ \frac{y}{40} - \frac{y-4}{40} = \frac{1}{10} & 4 \le y \le 10 \\ \frac{10}{40} - \frac{y-4}{40} = \frac{14-y}{40} & 10 \le y \le 14. \end{cases}$$

$$f_{x|y}(x|y) = \frac{f_x(x) \cdot f_v(y-x)}{f_y(y)}$$

$$= \begin{cases} \frac{1}{40} \cdot \frac{40}{y} = \frac{1}{y} & 0 \le y \le 4 \quad \text{(Shown)} \\ \frac{1}{40} \cdot 10 = \frac{1}{4} & 4 \le y \le 10 \\ \frac{1}{40} \cdot \frac{40}{14-y} = \frac{1}{14-y} & 10 \le y \le 14. \end{cases}$$

$$E(X|Y=y) = \int_{y-4}^{y} x\, f_{x|y}(x|y)\, dx$$

$$= f_{x|y}(x|y) \cdot \frac{x^2}{2}\Big|_{y-4}^{y}. = \begin{cases} \frac{1}{y} \cdot \frac{x^2}{2}\Big|_0^y = \frac{y}{2}. & 0 \le y \le 4 \quad \text{(Shown)} \\ \frac{1}{4} \cdot \frac{x^2}{2}\Big|_{y-4}^{y} = \frac{y^2 - y^2 + 8y - 16}{8} = y - 2 & 4 \le y \le 10 \\ \frac{1}{14-y} \cdot \frac{x^2}{2}\Big|_{y-4}^{10} = \frac{100 - y^2 + 8y - 16}{2(14-y)} = \frac{y+6}{2} & 10 \le y \le 14. \end{cases}$$

$$R_{xg} = E x g^* = \iint x g^* f(x) f(v)\, dx\, dv.$$

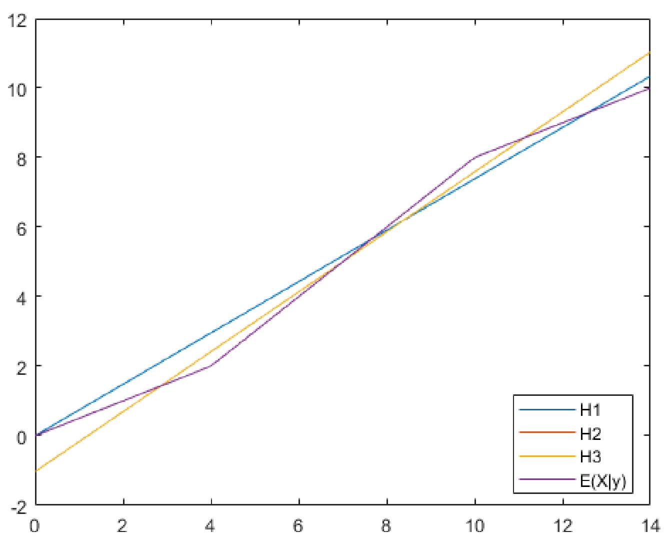
$$R_{xg} = E x g^* = \begin{bmatrix} E x & E xy & E xy^2 \end{bmatrix} = \begin{bmatrix} 5 & \frac{130}{3} & 410 \end{bmatrix}.$$

$$R_g = E g g^* = \begin{bmatrix} E 1 & E y & E y^2 \\ E y & E y^2 & E y^3 \\ E y^2 & E y^3 & E y^4 \end{bmatrix}$$

$$\hat{x} = R_{xg} R_g^{-1} g = \underline{-\frac{30}{29} + \frac{25}{29} y + 0 \cdot y^2}$$

$$E(x-\hat{x})^2 = R_x - R_{xg} R_g^{-1} R_{gx}$$

$$= \frac{160}{87}$$



P_{H_2} & P_{H_3} gives the same estimation of x, and the error is smaller than P_{H_1}.

2. y is uniform RV over $[0, 1]$. $x = e^y$ over $[1, e]$. $y = \ln(x)$.

$$f(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases} \qquad g = [1, y, y^2]^T$$

$$F_x(x) = P(X \le x) = P(e^Y \le x) = P(Y \le \ln x)$$

$$= \int_0^{\ln x} f(y)\, dy$$

$$= \ln(x). \quad \Rightarrow \quad f(x) = \frac{d}{dx} F_x(x) = \begin{cases} \frac{1}{x} & 1 \le x \le e \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{x} = 1.013 + 0.8511\, y + 0.8392\, y^2.$$

$$R_{xg} = E\, x g = \int_0^1 \int_1^e x \cdot g^* \cdot f(x) \cdot f(y)\, dx\, dy. = [1.7183 \quad 1 \quad 0.7183]$$

$$R_g = E\, g \cdot g^* = \int_0^1 \int_1^e g g^*\, f(x)\, f(y)\, dx dy = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$R_x = E x^2 = \int_1^e x^2 \cdot f(x)\, dx = 3.1945$$

$$M = R_{xg} \cdot R_g^{-1} = [1.0130 \quad 0.8511 \quad 0.8392].$$

$$E(x - \hat{x})^2 = R_x - M \cdot R_{xg}^* = 2.7835 \times 10^{-5}$$

$y = \ln x$. $\quad x = e^{y}$

$Y \le y$ & $Y \le y$

$X \le X$, $X \le X$.

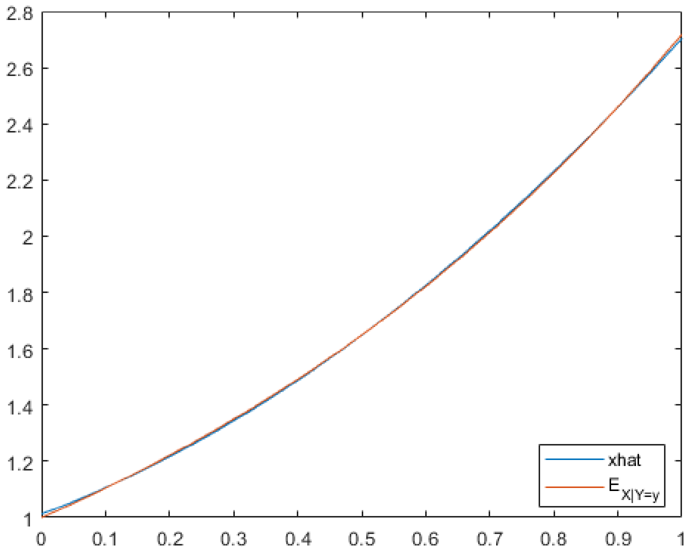
$F_{X,Y} = P(X \le x, Y \le y) = P(X \le x, \log X \le y) = P_X(X \le x, \underbrace{X \le e^{y}}) = P_Y(Y \le \log(x), Y \le y)$.

$\hookrightarrow = \int_{-\infty}^{y} f(y)\,dy. = y.$ $\qquad \underline{= P_X(X \le X)}.$ $=$ $P_Y(Y \le y)$.

$f_{X,y} = 1$

$$f_{X|Y}(X|Y) = \frac{f_Y(y)}{f_Y(y)} = 1.$$

$$E_{X|Y}(X|Y=y) = \int_{-\infty}^{\infty} x \cdot 1\, dx. = \int_{e^{-\infty}}^{e^{y}} dx = e^{y}$$



These two estimates are very close

1.1 Problem 1.

Let $\mathbf{x}$ be a uniform random variable over the interval $[0,4]$. Moreover, $\mathbf{v}$ is a uniform random variable over the interval $[-1,1]$. Assume that $\mathbf{x}$ and $\mathbf{v}$ are independent. Let $\mathbf{y}$ be the random variable given by $\mathbf{y}=\mathbf{x}+\mathbf{v}$.

- Let $\mathcal{H}$ be the space spanned by $\{1,\mathbf{y},\mathbf{y}^2,\mathbf{y}^3\}$. Then compute

$$P_{\mathcal{H}}\mathbf{x}=a+b\mathbf{y}+c\mathbf{y}^2+d\mathbf{y}^3 \quad \text{and} \quad d_4^2=E|\mathbf{x}-P_{\mathcal{H}}\mathbf{x}|^2$$

- Compute the conditional expectation

$$\widehat{g}(y)=E(\mathbf{x}|\mathbf{y}=y)$$

- Plot $\widehat{g}(y)$ and its approximation $a+by+cy^2+dy^3$ on the same graph over the interval $[-1,5]$.

$x = y - v.$

$y \in [-1, 5]$

$f_x = \frac{1}{4} \quad f_v = \frac{1}{2}. \quad f_{x,y} = \frac{1}{8}$

$$f_y = \int_{-\infty}^{\infty} f_x(x)\cdot f_v(y-x)\, dx$$

$$= \int_{y-1}^{y+1} \frac{1}{8}\, dx = \frac{x}{8}\Big|_{y-1}^{y+1} = \begin{cases} \frac{x}{8}\Big|_0^{y+1} = \frac{y+1}{8} & -1 \le y \le 1 \\ \frac{x}{8}\Big|_{y-1}^{y+1} = \frac{1}{4} & 1 \le y \le 3 \\ \frac{x}{8}\Big|_{y-1}^{4} = \frac{5-y}{8} & 3 \le y \le 5 \end{cases}$$

$$f_{x|y}(x|y) = \frac{f_x(x)\cdot f_v(y-x)}{f_y(y)}$$

$$= \begin{cases} \frac{1}{8}\cdot\frac{8}{y+1} = \frac{1}{y+1} & -1 \le y \le 1 \\ \frac{1}{8}\cdot 4 = \frac{1}{2} & 1 \le y \le 3 \\ \frac{1}{8}\cdot\frac{8}{5-y} = \frac{1}{5-y} & 3 \le y \le 5 \end{cases}$$

$$E_{x|y}(x|y=y) = \int_{y-1}^{y+1} x\, f_{x|y}(x|y)\, dx$$

$$= f_{x|y}(x|y)\cdot\frac{x^2}{2}\Big|_{y-1}^{y+1}$$

$$= \begin{cases} \frac{1}{y+1}\cdot\frac{x^2}{2}\Big|_0^{y+1} = \frac{y+1}{2} & -1 \le y \le 1 \\ \frac{1}{2}\cdot\frac{x^2}{2}\Big|_{y-1}^{y+1} = \frac{y^2+2y+1-y^2+2y-1}{4} = y & 1 \le y \le 3 \\ \frac{1}{5-y}\cdot\frac{x^2}{2}\Big|_{y-1}^{4} = \frac{16-y^2+2y-1}{2(5-y)} = \frac{y+3}{2} & 3 \le y \le 5 \end{cases}$$

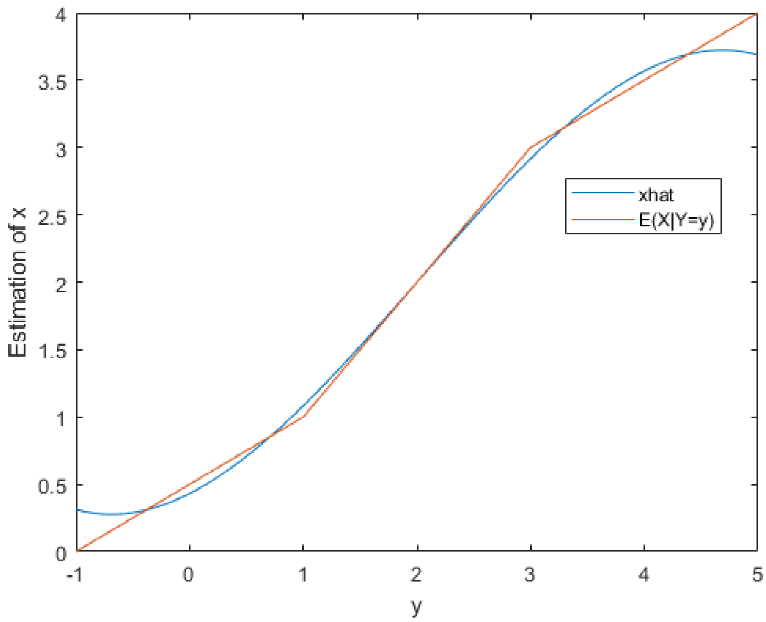
$$R_{xg} = E\,xg = \int_{-1}^{1}\int_{0}^{4} x \cdot g^* \cdot f(x) \cdot f(v)\, dx\, dv = \begin{bmatrix} 2 & \frac{16}{3} & \frac{50}{3} & \frac{848}{15} \end{bmatrix}$$

$$R_g = E\, g \cdot g^* = \int_{-1}^{1}\int_{0}^{4} g g^* f(x) f(v)\, dx\, dv = \begin{pmatrix} 1 & 2 & \frac{17}{3} & 18 \\ 2 & \frac{17}{3} & 18 & \frac{931}{15} \\ \frac{17}{3} & 18 & \frac{931}{15} & 226 \\ 18 & \frac{931}{15} & 226 & \frac{6001}{7} \end{pmatrix}$$

$$R_x = E x^2 = \int_0^4 x^2 \cdot f(x)\, dx = \frac{16}{3}$$

$$M = R_{xg} \cdot R_g^{-1} = \begin{bmatrix} 0.4320 & 0.4286 & 0.2660 & -0.0644 \end{bmatrix}$$

$$E(x-\hat{x})^2 = R_x - M \cdot R_{xg}^* = 0.2525$$



1.2 Problem 2.

Let $\mathbf{x}$ and $\mathbf{y}$ be two independent uniform random variables both over the interval $[0,1]$. Let $\mathbf{a}$ be the random variable defined by the area $\mathbf{a} = \mathbf{xy}$. Clearly, the area $0 \le \mathbf{a} \le 1$. Our problem is given the area $\mathbf{a}$ find the best estimate $\widehat{\mathbf{x}}$ of $\mathbf{x}$.

$f_x = 1 \quad f_y = 1.$

- Let $\mathcal{H}$ be the space spanned by $\{1, \mathbf{a}, \mathbf{a}^2, \mathbf{a}^3\}$. Then compute

$$P_{\mathcal{H}}\mathbf{x} = \alpha + \beta\mathbf{a} + \gamma\mathbf{a}^2 + \delta\mathbf{a}^3 \quad \text{and} \quad d_4^2 = E|\mathbf{x} - P_{\mathcal{H}}\mathbf{x}|^2$$

$g = [1, a, a^2, a^3]$

$x = \frac{a}{y}. \in [\frac{a}{1}, \frac{a}{0}].$

- Compute the conditional expectation

$$\widehat{g}(a) = E(\mathbf{x}|\mathbf{a} = a) \quad \text{and} \quad d_\infty^2 = E|\mathbf{x} - \widehat{g}(\mathbf{a})|^2$$

- Plot $\widehat{g}(a)$ and its approximation $\alpha + \beta a + \gamma a^2 + \delta a^3$ on the same graph over the interval $[0,1]$. Is $d_\infty < d_4$? Explain why or why not.

$$f_{xa}(x,a) = f_{xy}(x,y) \cdot \left| \begin{matrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial a} \end{matrix} \right| \rightarrow \left| \begin{matrix} 1 & \frac{1}{y} \\ 0 & \frac{1}{x} \end{matrix} \right|.$$

$$= \frac{1}{x} f_{xy}(x,y) = \frac{1}{x} f_x(x) f_y(\tfrac{a}{x})$$

$$f_a(a) = \int_{-\infty}^{\infty} \frac{1}{x} f_x(x) f_y(\tfrac{a}{x})\, dx.$$

$$= \int_a^\infty \frac{1}{x}\, dx = \begin{cases} \ln x \Big|_a^1 = -\ln a. & 0 < a \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

$$f_{x|a} = \frac{f_{xa}}{f_a} = \begin{cases} -\frac{1}{x} \cdot \frac{1}{\ln a} & 0 < a \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(x|a=a) = \int_{-\infty}^{\infty} x \cdot f_{x|a}(x|a=a)\, dx = \int_a^\infty -x \cdot \frac{1}{x \ln a}\, dx. = \frac{1}{\ln a} \cdot -x\Big|_a^1 = \frac{(a-1)}{\ln a} = \hat{g}(a)$$

$$E|x - \hat{g}(a)|^2 = E x^2 - E\, \hat{g}(z)^2 = \int_0^1 x^2 dx + \int_0^1 \frac{(a-1)}{\ln(a)}\, da = \frac{1}{3} + \ln(\tfrac{3}{4}) = 0.0457$$

$$d_\infty = \sqrt{0.0457} = \underline{0.2138}$$

$$R_{xg} = E\,xg = \int_0^1\int_0^1 x \cdot g^* \cdot f(x) \cdot f(y)\, dx\, dy = \left[\tfrac{1}{2} \quad \tfrac{1}{6} \quad \tfrac{1}{12} \quad \tfrac{1}{20}\right]$$

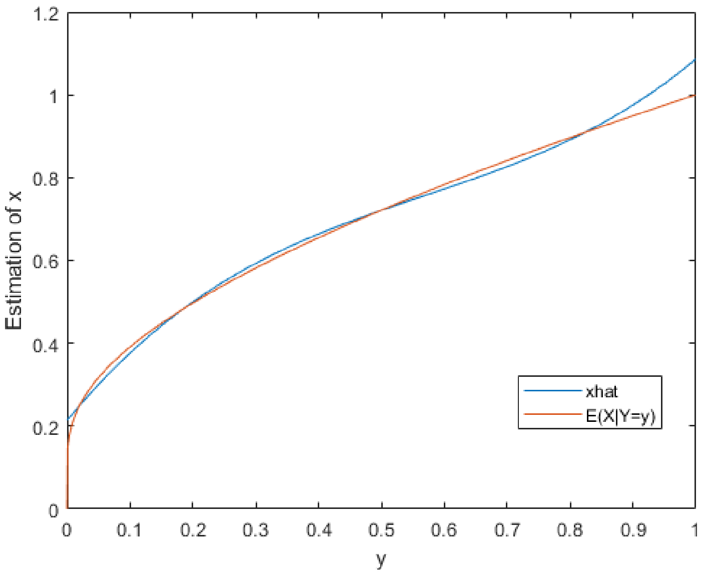
$$R_g = E\,g \cdot g^* = \int_0^1\int_0^1 g g^* f(x) f(y)\, dx\, dy = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{9} & \frac{1}{16} & \frac{1}{25} \\ \frac{1}{9} & \frac{1}{16} & \frac{1}{25} & \frac{1}{36} \\ \frac{1}{16} & \frac{1}{25} & \frac{1}{36} & \frac{1}{49} \end{pmatrix}$$

$$R_x = E x^2 = \int_0^1 x^2 \cdot f(x)\, dx = \frac{1}{3}$$

$$M = R_{xg} \cdot R_g^{-1} = [0.2140 \quad 1.8416 \quad -2.3412 \quad 1.3717]$$

$$E(x-\hat{x})^2 = R_x - M \cdot R_{xg}^* = 0.0459$$

$$d_4 = \sqrt{0.0459} = 0.2143 > d_\infty$$



$d_\infty < d_4$,

Because the expectation optimizes over infinite subspaces but the projection only optimizes over 4 subspaces.