Ex 1. Determine if the following functions are lpd

a). $V(x) = x_1^2 - x_1^4 + x_2^2$

$DV(x) = [2x_1 - 4x_1^3 \quad 2x_2]$. $DV(x^e) = 0$.

$D^2V(x) = \begin{bmatrix} 2-12x_1^2 & 0 \\ 0 & 2 \end{bmatrix}$. $D^2V(0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0$: $\boxed{\text{Lpd}}$.

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b). $V(x) = x_1 + x_2^2$ $\qquad x^e$ at the origin.

$DV(x) = [1 \quad 2x_2] \Rightarrow DV(0) = [1 \quad 0] > 0$

$\boxed{\text{Not lpd}}$.

c) $V(x) = 2x_1^2 - x_1^3 + x_1x_2 + x_2^2$ $\qquad V(0) = 0$.

$DV(x) = [4x_1 - 3x_1^2 + x_2 \quad x_1 + 2x_2]$ $\qquad DV(0) = 0$.

$D^2V(x) = \begin{bmatrix} 4-6x_1 & 1 \\ 1 & 2 \end{bmatrix}$ $\qquad D^2V(0) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} > 0$. $\boxed{\text{Lpd}}$

Ex. 2. By appropriate choice of Lyapunov function, show that the origin is a stable equilibrium state for

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3$$

$DV(x)\, f(x) \leq 0$ for $\| x - x^e \| < R$. $\quad x^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$f(x) = \begin{bmatrix} x_2 \\ -x_1^3 \end{bmatrix}. \qquad \ddot{y} = -y^3 :$$

Let the candidate Lyapunov function be.

$$V(x) = \frac{1}{2} x_2^2 + \frac{1}{4} x_1^4$$

$$DV(x)\, f(x) = \frac{\partial V(x)}{\partial x_1} x_2 + \frac{\partial V(x)}{\partial x_2} \cdot (-x_1^3)$$

$$DV(x)\, f(x) = x_1^3 x_2 + x_2(-x_1^3) = 0$$

Hence the system is stable about origin.

Ex. 3 choose Lyapunov Function.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^3$$

Let the candidate Lyapunov function be

$$\boxed{V(x) = -\frac{1}{4}x_1^4 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}.}$$

$$DV(x)\cdot f(x) = \frac{\partial V(x)}{\partial x_1}\cdot x_2 + \frac{\partial V(x)}{\partial x_2}(-x_1 + x_1^3)$$

$$= (-x_1^3 + x_1)x_2 + (x_2)(-x_1 + x_1^3)$$

$$= 0$$

Hence the origin is stable.

4. Show the system is stable about zero state.

$$\dot{x}_1 = x_2^3$$

$$\dot{x}_2 = -x_2^2 x,$$

Let the lyapunov candidate function be.

$$V(x) = x_1^2 + x_2^2$$

$$DV(x) f(x) = \frac{\partial V(x)}{\partial x_1} f_1(x) + \frac{\partial V(x)}{\partial x_2} f_2(x)$$

$$= 2x_1 x_2^3 + 2x_2 (-x_2^2 x_1)$$

$$= 2x_1 x_2^3 - 2x_1 x_2^3$$

$$= 0.$$

Hence the system is stable about the origin

5, Show the system is GAS about 0

$$\dot{x} = -(2+\cos x)\,x. \qquad -2x - \cos x \cdot x.$$

Let the candidate Lyapunov function be

$$V(x) = x^2$$

$$DV(x)f(x) = 2x(-2-\cos x)\,x.$$

$$= -2(2+\cos x)\,x^2$$

since $2+\cos x > 0$. & $x^2 > 0$.

$DV(x)f(x) < 0$. when $x \neq 0$

Hence the system is GAS about 0

6. GAS about $x = 1$.

$$\dot{x} = -(2 + \cos x)(x-1)$$

Let the candidate Lyapunov function be

$$V(x) = (x-1)^2.$$

$$DV(x) \cdot f(x) = -2(x-1) \cdot (2+\cos x)(x-1)$$

$$= -2(2+\cos x)(x-1)^2$$

$$< 0. \quad \text{when} \quad x \neq 1.$$

Hence the system is GAS about $x = 1$