Dice	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Notice that odd numbers appear 18 times. This readily implies that $P(A_3) = 18/36 = 1/2$. Moreover, we have

$$P(A_3|A_1) = P(\text{Die 2 is even}) = \frac{1}{2} = P(A_3).$$

Hence A_1 and A_3 are independent. A similar calculation shows that A_2 and A_3. So A_1, A_2 and A_3 are pairwise independent, that is, A_1 and A_2 are independent, A_1 and A_3 are independent, and A_2 and A_3 are independent. Now observe that

$$P(A_1A_2A_3) = P(\phi) = 0 \quad \text{and} \quad P(A_1)P(A_2)P(A_3) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}.$$

Obviously, $0 \neq \frac{1}{8}$. Hence $P(A_1A_2A_3) \neq P(A_1)P(A_2)P(A_3)$. In other words, the events A_1, A_2 and A_3 are not independent.

1.6.1 Exercise

Problem 1. Prove the following

(i) $P(A \cup B) \leq P(A) + P(B)$;

(ii) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$;

(iii) $P(\bigcup_{j=1}^{n} A_j) \leq \sum_{j=1}^{n} P(A_j)$.

Problem 2.

(i) Suppose one draws two cards without replacement from a deck of 52 cards. Find the probability of drawing a queen or a king.

(ii) Suppose one draws two cards without replacement from a deck of 52 cards. Find the probability of drawing a queen and a king.

Problem 3. (Rozanov [46], Problem 12, page 11.) The word "drawer" is spelled out using six scrabble tiles. Then the tiles are randomly rearranged. Find the probability of the rearranged tiles spelling out the word "reward"?

Problem 4. Suppose that one flips a coin four times. Each coin flip is independent of the previous flip. The probability of a head is p while the probability of a tail is q where $p + q = 1$. The following problems are a special case of the binomial distribution.

(i) Find the probability of obtaining exactly two heads. Hint: consider the number of ways exactly two heads can occur in four flips, that is, $HHTT, HTHT, HTTH, \cdots$ and so on.

(ii) Find the probability of obtaining at least three heads.

Problem 5. Consider three die whose sides are labeled $\{1, 2, 3, 4, 5, 6\}$. Suppose that one roles all three die. Let $\mathcal{S}$ be the event corresponding to the sum of all three die. Notice that the values of $\mathcal{S}$ are integers in $[3, 18]$. Let $A_k = j$ be the event that j appears on the k-th die for $k = 1, 2, 3$.

(i) Find $P(\mathcal{S} = 12)$.

(ii) Find a value of m such that $P(\mathcal{S} = m)$ is maximal, that is, $P(\mathcal{S} = m) \geq P(\mathcal{S} = n)$ for all integers n in $[3, 18]$. Is m unique?

(iii) Find $P(A_1 = 3|\mathcal{S} = 10)$.

Problem 6. Consider the gamblers ruin problem where one has a biased coin such that

$$P(H) = \frac{1}{2} \quad \text{and} \quad P(T) = \frac{49}{100} \quad \text{and} \quad P(E) = \frac{1}{100}.$$

Here H, T and E is the event that the coin respectively lands heads up, tail up and on the edge. Starting with n dollars, the gambler plays until he achieves a total of $m = 200$ dollars (a profit of 100 dollars) or goes broke. Assume the gambler bets one dollar at a time. The rules of the game are if the coin is a head, the gambler wins one dollar and loses one dollar if the coin is a tail. However, if the coin lands on its edge, then the gambler looses two dollar. So the gambler could actually end the game with a debt of one dollar. Using Matlab for computations:

(i) Find the difference equation for this problem and its solution. The solution of the difference equation will be of the form

$$p(n) = a_1\lambda_1^n + a_2\lambda_2^n + a_3\lambda_3^n.$$

Here $p(n)$ is the probability that the gambler achieves $m = 200$ dollars starting with n dollars. Plot $p(n)$ for $1 \leq n \leq 200$.

(ii) If the gambler starts with $n = 100$ dollars find the probability he achieves $m = 200$ dollars or a profit of 100 dollars.

(iii) Find the smallest initial amount of dollars n the gambles must start out with to achieve $m = 200$ with a probability of 50%.

Problem 7. Bayes rule can be used to design a spam filter. For a naive example, assume that spam e-mail contains a certain word or phrase more often than normal e-mail. For example, spam may use certain words or phrases such as prize, free stuff, mortgage, loan, zero down, etcetera. Let $\mathcal{S}$ be the event that the mail is spam and W a word or phrase that occurs more often in spam mail. So assume that $P(W|\mathcal{S}) = 0.25$. Moreover, 60% of your e-mail is spam and $P(W|\mathcal{S}^c) = 0.05$.

(i) Find the probability your mail is spam given W appears, that is, find $P(\mathcal{S}|W)$.

(ii) Find the probability your mail is spam given W does not appear, that is, find $P(\mathcal{S}|W^c)$.

Finally, when designing a spam filter, one searches over many different words or phrases, the number of times a special word or phrase is used and domain names to make a more accurate spam filter.

Problem 9. Suppose that there is 1000 lottery tickets sold and there is only one winner. Buying one ticket at a time, what is the minimum number of times one has to play the lottery to guarantee a 50% chance of winning the lottery?

Problem 10. Show that $1 - a < e^{-a}$ for $0 < a \leq 1$.

1.7 PageRank

In this section, we will use conditional probability to derive Google's PageRank for searching for webpages; see PageRank, Wikipedia. We will also introduce Markov matrices. Suppose that we have ν webpages denoted by $\{\xi_j\}_1^\nu$. The probability that you are at site $\xi_j(n)$ at time n is denoted by $P(\xi_j(n))$. Let $\mathcal{D}$ be the event that one uses the link on page $\xi_k(n)$ to move to $\xi_j(n+1)$, and assume that $P(\mathcal{D}) = d$ for all time n. PageRank refers to d as damping. So the probability that the user opens up a new webpage at random and does use the link at time n is $P(\mathcal{D}^c) = 1 - d$. By collecting user data Google determined that the damping $d = 0.85$ in many application. Recall that $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$; see (2.2). This readily implies that

$$P(\xi_j(n+1)) = P(\xi_j(n+1)|\mathcal{D})P(\mathcal{D}) + P(\xi_j(n+1)|\mathcal{D}^c)P(\mathcal{D}^c). \tag{7.1}$$

Assume that if the user does not use a link, then the surfer is equally likely to choose a webpage at random, that is,

$$P(\xi_j(n+1)|\mathcal{D}^c) = \frac{1}{\nu}.$$

Using this in (7.1), we obtain

$$P(\xi_j(n+1)) = P(\xi_j(n+1)|\mathcal{D})\, d \,+\, \frac{1-d}{\nu}. \tag{7.2}$$

Let us compute $P(\xi_j(n+1)|\mathcal{D})$. To this end, let $P(\xi_j(n+1)|\xi_k(n))$ denote the probability that one moves from $\xi_k(n)$ to $\xi_j(n+1)$ using a link in webpage $\xi_k(n)$. Here we assume that