HW 4,

1, $V(x) = x_1^4 - x_1^2 x_2 + x_2^2$ is this PD.

$DV(x) = [4x_1^3 - 2x_1x_2 \quad -x_1^2 + 2x_2] \Rightarrow DV(0) = [0 \quad 0]$

$$D^2V(x) = \begin{bmatrix} 12x_1^2 - 2x_2 & -2x_1 \\ -2x_1 & 2 \end{bmatrix} \not> 0 \quad \text{for all } x$$

Thus the function is not positive definite

2. Show the system is AS about zero.

$$\dot{x} = -(1+\sin x)\,x.$$

Let the candidate Lyapunov function be

$V(x) = x^2$

$DV(x) = 2x \Rightarrow DV(0) = 0.$ $\qquad V(x)$ is PD.

$D^2V(x) = 2 > 0 \quad \Rightarrow \quad V(x) > 0$

$DV(x)f(x) = -2x\cdot(1+\sin x)x = -2(1+\sin x)x^2. < 0$. for $x \neq 0$

Thus the system is GAS about zero

3. Show the system is AS about the origin.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^3 - x_2$$

Let the candidate Lyapunov function be

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4 \quad \Rightarrow V(0) = 0$$

$$DV(x) = [x_1 + x_1^3 \quad x_2] \quad \Rightarrow DV(0) = 0.$$

$$D^2V(x) = \begin{bmatrix} 3x_1^2+1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow D^2V(0) > 0 \qquad V(x) \text{ is } lpd.$$

$$DV(x)f(x) = (x_1 + x_1^3)x_2 \quad + \quad x_2(-x_1 + x_1^3 - x_2).$$

$$= -2x_2^2 < 0. \quad \text{for } x \neq [0,0]^T$$

Thus the system is AS about the origin.

4. System:

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_1 - x_1^3 + u$

Obtain linear controller

$u = -k_1 x_1 - k_2 x_2$

Which results in a closed loop system which is GAS about the origin.

Numerically simulate the open-loop close-loop system.

Let the candidate lyapunov function be

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4 + x_1 x_2 \quad \Rightarrow V(0) = 0.$$

$$DV(x) = [\, x_1 + x_1^3 + x_2 \quad x_2 + x_1 \,] . \quad \Rightarrow DV(0) = 0$$

$$D^2V(x) = \begin{bmatrix} 1+3x_1^2 & 1 \\ 1 & 1 \end{bmatrix} > 0 \quad \forall x \quad V(x) \text{ is pd.}$$

$$\dot{V} = x_1 x_2 + x_2(x_1 - x_1^3 - k_1 x_1 - k_2 x_2) + x_1^3 x_2 + x_2^2 + x_1(x_1 - x_1^3 - k_1 x_1 - k_2 x_2)$$

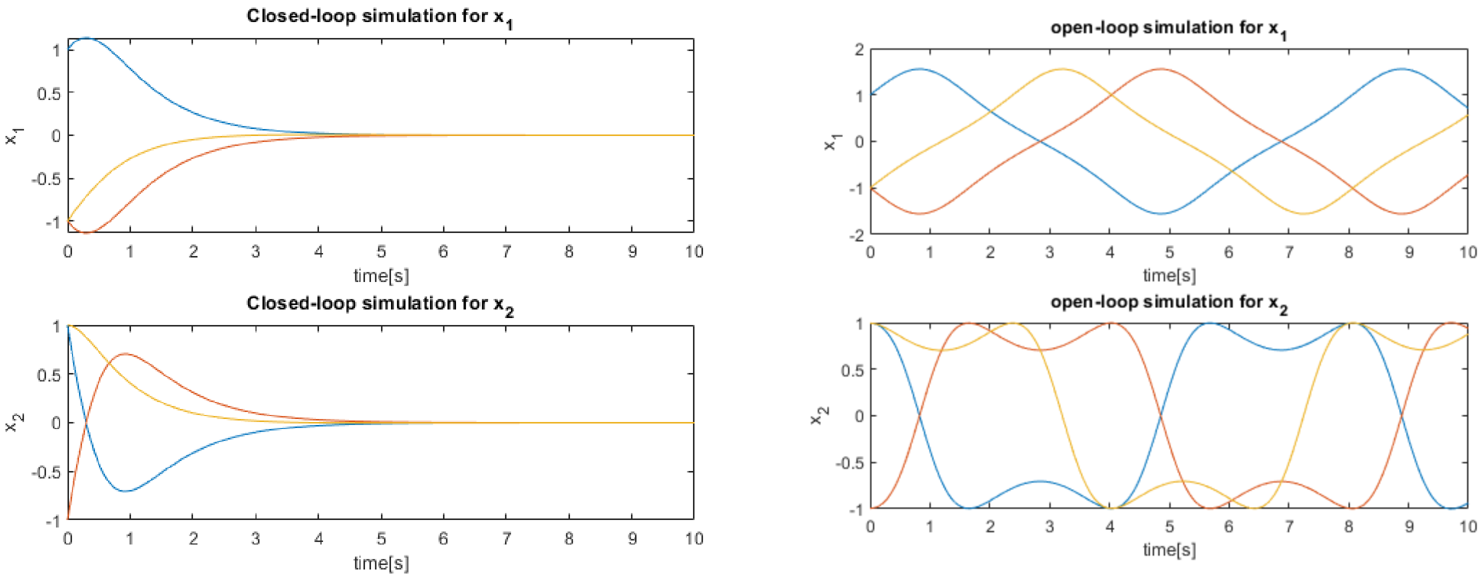
$$= x_1 x_2 + x_2 x_1 - \cancel{x_1^3 x_2} - k_1 x_1 x_2 - k_2 x_2^2 + \cancel{x_1^3 x_2} + x_2^2 + x_1^2 - x_1^4 - k_1 x_1^2 - k_2 x_1 x_2$$

$$= (2 - k_1 - k_2) x_1 x_2 + (1 - k_2) x_2^2 - (1 - k_1) x_1^2$$

$\Rightarrow \dot{V} \le 0$ if $k_1 \ge 1$ & $k_2 \ge 1$. the system is GAS.

pick $u = -2x_1 - 2x_2$ to ensure the closed-loop system is GAS.

$x_0 = [-1, -1], [1, 1], [-1, 1]$



5. It the function radially unbounded.

$$V(x) = x_1 - x_1^3 + x_1^4 - x_2^2 + x_2^4$$

$$D^2V(x) = \begin{bmatrix} -6x_1 + 12x_1^2 & 0 \\ 0 & -2+12x_2^2 \end{bmatrix}$$

If we choose $P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

$$D^2V - P \geq 0.$$

$$\begin{bmatrix} -6x_1 + 12x_1^2 - 1 & 0 \\ 0 & -2+12x_2^2 - 1 \end{bmatrix} \geq 0.$$

① $-6x_1 + 12x_1^2 - a \geq 0.$

② $(-6x_1 + 12x_1^2 - a)(-2 + 12x_2^2 - b) \geq 0.$

$\Rightarrow -2 + 12x_2^2 - b \geq 0.$

solve for x_1, x_2, we can find a scalar $R \geq 0$ such that $D^2V(x) \geq P$

thus V is radially bounded.

6. Show all solutions of the system are bounded.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - c x_2 + 1 \qquad c > 0.$$

Let $V(x) = \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2 + \frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 + \frac{1}{4} x_1^4$.

$$P = \frac{1}{2}\begin{bmatrix} \lambda c^2 & \lambda c \\ \lambda c & 1 \end{bmatrix}, \quad V(x) = x^T P x - \frac{1}{2} x_1^2 + \frac{1}{4} x_1^4$$
$$\geq x^T P x - \frac{1}{4}$$

V is radially unbounded.

$$DV(x) f(x) = (\lambda c^2 x_1 + \lambda c x_2 - x_1 + x_1^3) \cdot x_2$$
$$+ (\lambda c x_1 + x_2) \cdot (x_1 - x_1^3 - c x_2 + 1)$$
$$= \lambda c^2 \cancel{x_1 x_2} + \lambda c x_2^2 - \cancel{x_1 x_2 + x_2 x_1^3}$$
$$+ \lambda c x_1^2 - \lambda c x_1^4 - \cancel{\lambda c^2 x_1 x_2} + \lambda c x_1 + \cancel{x_1 x_2} - \cancel{x_2 x_1^3} - c x_2^2 + x_2$$
$$= \lambda c x_2^2 + \lambda c x_1^2 - \lambda c x_1^4 + \lambda c x_1 - c x_2^2 + x_2$$
$$= \underbrace{\lambda}_{+ve} c (x_1^2 + x_1 - x_1^4) + \underbrace{(\lambda - 1)}_{-ve} x_2^2 + x_2 \qquad 0 < \lambda < 1$$

For large x_1. For large x_2^2

$-x_1^4$ dominates $\qquad (\lambda - 1) x_2^2$ dominates

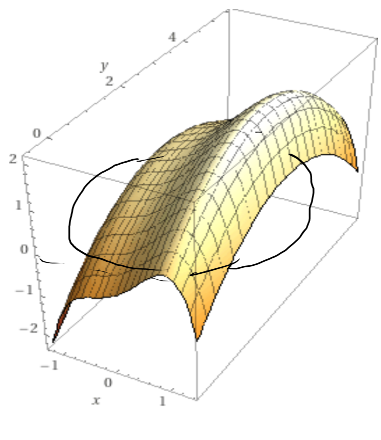
$-x_1^4 + x_1 + x_1^2 < 0$, $\qquad [(\lambda - 1) x_2 + 1] x_2 < 0$.

$x_1 < 0$ or $x_1 > 1.3247$ $\qquad x_2 < 0$ or $x_2 > 2$.

Choose $R = 4$, $\dot{V} < 0$.

So there exists a R such that

$DV(x) f(x) \leq 0$ for all $\|x\| > R$



7. Show all solutions of

$$\dot{x} = \cos x - x^3 + 100$$ are bounded.

Let $V(x) = \frac{1}{2}x^2$

$$\dot{V} = x\dot{x} = x(\cos x - x^3 + 100) = x\cos x - x^4 + 100x.$$

It's easy to see for large $|x|$, $-x^4$ dominates, $\dot{V}(x) < 0$.

one of the R we can choose is 2π.

So for $\|x\| \geq 2\pi$,

$\dot{V} < 0$, thus the solutions are bounded.

8 show all solutions are bounded,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \cos x_1 - x_1^3 + 100. \quad \Rightarrow \quad \ddot{y} = \cos y - y^3 + 100$$

Let $V = -100x_1 + \frac{1}{2}x_2^2 - \sin x_1 + \frac{1}{4}x_1^4$ $\Rightarrow$ clearly radially unbounded,

$$\dot{V} = -100x_2 + \cancel{x_2\cos x_1} - \cancel{x_2 x_1^3} + 100x_2 - \cancel{x_2\cos x_1} + \cancel{x_1^3 x_2}$$

$$= 0$$

Thus all solutions are bounded.

9. Show system is GES about zero.

$$\dot{x} = -(2+\sin x)x$$

Let $V(x) = \frac{1}{2}x^2$

$\frac{1}{4}\|x\|^2 \le V(x) \le \|x\|^2$. $\beta_1 = \frac{1}{4}$, $\beta_2 = 1$.

$\dot{V} = x\dot{x} = -(2+\sin x)x^2$ since $-(2+\sin x) \le 3 \;\forall x$,

$$\begin{aligned}\dot{V} &\le -3x^2 \\ &\le -3 \cdot 2 \cdot \tfrac{1}{2}x^2 \quad \Rightarrow \alpha = 3 \\ &\le -2 \cdot 3\, V(x)\end{aligned}$$

The system is GES about zero.

10. GES about 1.

$\dot{x} = -(2+\sin x)(x-1)$

Let $V(x) = \frac{1}{2}(x-1)^2$

$\frac{1}{4}(x-1)^2 \le V(x) \le (x-1)^2$

$\dot{V} = (x-1)\dot{x} = -(2+\sin x)(x-1)^2$ since $-(2+\sin x) \le 3$

$\dot{V} \le -3(x-1)^2$

$= -6 \cdot V(x)^2$. $\alpha = 3$.

The system is GES about 1.

11, GES about zero.

$$\dot{x}_1 = -x_1 + (I_2 - I_3) x_2 x_3$$

$$\dot{x}_2 = -2x_2 + (I_3 - I_1) x_1 x_3$$

$$\dot{x}_3 = -3x_3 + (I_1 - I_2) x_1 x_2.$$

Let $P = I_3$,

$x^T P f(x)$

$= -x_1^2 + \cancel{(I_2 - I_3) x_1 x_2 x_3} - 2x_2^2 + \cancel{(I_3 - I_1) x_1 x_2 x_3} - 3x_3^2 + \cancel{(I_1 - I_2) x_1 x_2 x_3}$

$= -x_1^2 - 2x_2^2 - 3x_3^2$

$= x^T \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} x$

$\Rightarrow x^T P f(x) = x^T Q x \quad Q = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$ pd.

Thus the origin is GES

$\alpha = \lambda_{min}(P^{-1} Q) = \lambda_{min}(Q) = \underline{1}$