HW5.

1. $q^{(4)} - \sin(q) = 0$.

$\delta q^{(4)} - \delta q \cos(q^e) = 0$.

$\delta q^{(4)} - \delta q = 0$. $s^4 - 1 = 0$. $s = \pm 1, \pm i$

unstable

2. $\ddot{q} + \dot{q} - q^3 = 0$.

$\delta\ddot{q} + \delta\dot{q} - \cancel{3q^{e^2}}\,\delta q = 0$

$\delta\ddot{q} + \delta\dot{q} = 0$. $s^2 + s = 0$. undetermined,

$s(s+1) = 0$

$s = 0, -1$

3. (i) $\dot{x}_1 = (1 + x_1^2) x_2$

$\dot{x}_2 = -x_1^3$

$\delta\dot{x}_1 = (1 + \cancel{x_1^{e^2}})\delta x_2 + \cancel{2x_1^e}\,\delta x_1 \cdot \cancel{x_2^e} = \delta x_2$ $\Rightarrow \begin{cases} \delta\dot{x}_1 = \delta x_2 \\ \delta\dot{x}_2 = 0. \end{cases}$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$\delta\dot{x}_2 = -3\cancel{x_1^{e^2}} \cdot \delta x_1 = 0$

undetermined $\alpha = 0$.

(ii) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = (\cos x_1) x_3 \\ \dot{x}_3 = e^{x_1} x_2 \end{cases}$

$\delta\dot{x}_1 = \delta x_2 \cos x_2^e = \delta x_2$

$\delta\dot{x}_2 = \cos x_1^e\, \delta x_3 + 0 = \delta x_3$ $\Rightarrow \begin{cases} \delta\dot{x}_1 = \delta x_2 \\ \delta\dot{x}_2 = \delta x_3 \\ \delta\dot{x}_3 = \delta x_2 \end{cases}$ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\alpha = 0, \pm 1$

$\delta\dot{x}_3 = \delta x_1 e^{x_1^e} x_2^e + \delta x_2 e^{x_1^e} = \delta x_2$

unstable.

4. $x_1(k+1) = x_1(k)^2 + \sin(x_2(k))$

$x_2(k+1) = 0.4\cos(x_2(k))\,x_1(k)$

$\delta x_1(k+1) = 2x_1^e\cancel{(k)}\,\delta x_1(k) + \cos(x_2^e(k))\,\delta x_2(k) = \delta x_2(k)$

$\delta x_2(k+1) = 0.4\cos(x_2^e(k))\,\delta x_1(k) - 0.4\sin\cancel{(x_2^e(k))}\,\delta x_2\,\delta x_1 = 0.4\,\delta x_1(k)$

$A = \begin{bmatrix} 0 & 1 \\ 0.4 & 0 \end{bmatrix}$. $\lambda = \pm 0.6325 < 1$ stable.

5. $x_1(k+1) = (1 + x_1(k)^3)\,x_2(k)$

$x_2(k+1) = x_1(k)^3 + x_2(k)^5$

$\delta x_1(k+1) = \delta x_2(k)$

$\delta x_2(k+1) = 0$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $\lambda = 0$ exponentially stable

6. $x_1(k+1) = x_2(k)$

$x_2(k+1) = \sin(x_1(k)) + x_2(k)^5$

$\delta x_1(k+1) = \delta x_2(k)$

$\delta x_2(k+1) = \delta x_1(k)$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\lambda = \pm 1$, $|\lambda| = 1$, undetermined.

7. $\dot{x}_1 = \sigma(x_2 - x_1)$ $\sigma, r, b > 0$.

$\dot{x}_2 = r x_1 - x_2 - x_1 x_3$ prove boundness of all solutions.

$\dot{x}_3 = -b x_3 + x_1 x_2$

Let $V(x) = r x_1^2 + \sigma x_2^2 + \sigma (x_3 - 2r)^2$

$DV(x) = [2 r x_1 \quad 2\sigma x_2 \quad 2\sigma(x_3 - 2r)]$

$D^2 V(x) = \begin{bmatrix} 2r & 0 & 0 \\ 0 & 2\sigma & 0 \\ 0 & 0 & 2\sigma \end{bmatrix} = P$, where P is symmetric PD

Thus $V(X)$ is radially unbounded.

$\dot{V}(x) = 2 r x_1 \dot{x}_1 + 2\sigma x_2 \dot{x}_2 + 2\sigma(x_3 - 2r) \cdot \dot{x}_3$

$= 2 r x_1 \cdot \sigma(x_2 - x_1) + 2\sigma x_2 (r x_1 - x_2 - x_1 x_3) + 2\sigma(x_3 - 2r)(-b x_3 + x_1 x_2)$

$= \cancel{2 r \sigma x_1 x_2} - 2 r \sigma x_1^2 + \cancel{2\sigma r x_1 x_2} - 2\sigma x_2^2 - \cancel{2\sigma x_1 x_2 x_3}$

$+ 2\sigma(-b x_3^2 + \cancel{x_1 x_2 x_3} + 2 r b x_3 - \cancel{2 r x_1 x_2})$

$= -2 r \sigma x_1^2 - 2\sigma x_2^2 - 2\sigma b x_3^2 + 4 r b \sigma x_3$

Since the x^2 terms dominates and are negative, there exists an R such that $\dot{V}(x) \le 0$ for $\|X\| > R$.

Thus all solutions of the system are bounded.