3.5.2

$$F P = \bar{F}$$

$$P = F^{-1} \bar{F}$$

$$P = \frac{1}{n} \bar{F} \cdot \bar{F}$$

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix}$$

$$\bar{F} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^{-1} & w^{-2} & \cdots & w^{1-n} \\ 1 & w^{-2} & w^{-4} & & w^{2(1-n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{1-n} & w^{2(1-n)} & \cdots & w^{(1-n)^2} \end{bmatrix}$$

$F(0,:) \cdot P = \bar{F}(0,:)$, $[1\ 1\ 1 \cdots 1] \cdot P = [1\ 1\ 1 \cdots 1]$,

$F(1,:) \cdot P = \bar{F}(1,:)$

$$[1 \quad w \quad w^2 \cdots w^{n-1}]\ P = [1 \quad w^{-1} \quad w^{-2} \cdots w^{1-n}]$$

Since w is a root of unity, $w^n = 1$.

$$w^{-1} = (1) \cdot w^{-1} = w^{n-1}$$

$$w^{-2} = w^{n-2}$$

$$\cdots\cdots$$

$$w^{1-n} = w^1$$

$$[1 \quad w \quad w^2 \cdots w^{n-1}]\ P = [1 \quad w^{n-1} \quad w^{n-2} \cdots w]$$

It is clear that the 0th column stayed same, 1st column swapped with $(n-1)$th column, 2nd $\leftrightarrow$ $(n-2)$th, so on and so ~~forth~~.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots 1\ 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \vdots & \vdots & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

3.5 6

$$\sqrt{w_{128}} = e^{i \cdot \frac{2\pi}{128} \cdot \frac{1}{2}} = e^{i\frac{2\pi}{256}} = w_{256}$$

$$w_{128}^2 = e^{i\frac{2\pi}{64}} = w_{64}$$

4.2 2. $A^{3\times 3}$ has $\det(A) = -1$.

Find $\det(\frac{1}{2}A)$, $\det(-A)$, $\det(A^2)$, $\det(A^{-1})$

$\det(\frac{1}{2}A) = (\frac{1}{2})^3 \cdot (-1) = \boxed{-\frac{1}{8}}$

$\det(-A) = \det(-I \cdot A) = \det(-I) \cdot \det(A) = -1 \cdot -1 = \boxed{1}$

$\det(A^2) = \det(A) \cdot \det(A) = \boxed{1}$

$\det(A^{-1}) = \frac{1}{\det(A)} = \boxed{-1}$

4.2 4.

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$ r3 & r4 exchanged.

$$\det(A) = \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{vmatrix} \begin{matrix} \\ r_2 - 2r_1 \\ r_3 + r_1 \\ r_4 + 2r_2 \end{matrix}$$

$$= \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix} \begin{matrix} \\ \\ \\ r_4 + 2.5r_3 \end{matrix}$$

$$\boxed{= 20}$$

$$\det(B) = -\begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \\ 2 & -1 & 0 & 0 \end{vmatrix} \begin{matrix} r_1 \leftrightarrow r_2 & -1 \\ r_2 \leftrightarrow r_3 & \times -1 \\ r_3 \leftrightarrow r_4 & \times -1 \\ & -1 \end{matrix}$$

$$= -\begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 3 & -2 & 0 \end{vmatrix} \begin{matrix} \\ \\ \\ r_4 + r_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & -2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4 & -3 \end{vmatrix} \begin{matrix} r_1 \leftrightarrow -r_1 \\ \\ \\ r_4 + r_2 \end{matrix}$$

$$= \begin{vmatrix} 1 & -2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -11 \end{vmatrix} \begin{matrix} \\ \\ \\ r_4 + 4r_3 \end{matrix}$$

$$= \boxed{-11}$$

$$\det(C) = -\det(B)$$

$$= \boxed{11}$$

4.2 6

number of changes

Exchange nth row with 1st row,

$(n-1)$th with 2nd.

....

If n is even. # exchange $= \frac{n}{2}$

n is odd, # exchange $= \frac{n-1}{2}$

$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor$ ~ floor function.

$\det P = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}$

or if we exchange rows sequentially. $n \leftrightarrow n-1$, $n \leftrightarrow n-2$ .... $n \leftrightarrow 1$

exchange $= (n-1)+(n-2)+(n-3)+\cdots+1$

$= \frac{n\cdot(n-1)}{2}$

$= \frac{n^2-n}{2}$

$\det P = (-1)^{\frac{n^2-n}{2}}$

4.3 2

$$\det A = -1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \cdot \left(1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \right)$$

$$= -1 \cdot (-1)$$

$$= \boxed{1}$$

$$\det B = 1 \cdot \begin{vmatrix} 0 & 3 & 5 \\ 6 & 7 & 9 \\ 0 & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 3 & 4 \\ 6 & 7 & 8 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 6 & 9 \\ 0 & 1 \end{vmatrix} + 5 \cancel{\begin{vmatrix} 6 & 7 \\ 0 & 0 \end{vmatrix}} - 2 \left[-3 \cancel{\begin{vmatrix} 6 & 8 \\ 0 & 0 \end{vmatrix}} + 4 \cancel{\begin{vmatrix} 6 & 7 \\ 0 & 0 \end{vmatrix}} \right]$$

$$= -3 \cdot 6$$

$$\boxed{= -18}$$

4.3 4

a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} \\ r_2 - r_1 \\ r_3 - r_1 \\ r_4 - r_1 \end{matrix}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \\ r_3 - r_2 \\ r_4 - r_2 \end{matrix}$

$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ r_4 - r_3 \end{matrix}$ $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$; pivots, $\{1, 1, 1, 1\}$

$\det(A) = \det(L) \cdot \det(U)$
$= 1 \cdot 1$
$= 1$

b) $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 6 & 6 & 6 \\ 2 & 6 & 8 & 8 \\ 2 & 6 & 8 & 10 \end{bmatrix} = LU$, $U = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$\det A = \det U \cdot \det L$
$= \boxed{32}$

If $n_1 < n_2 < n_3 < n_4$. $\det L = 1$.

$\det A = \det U = n_1 \cdot (n_2 - n_1) \cdot (n_3 - n_2) \cdot (n_4 - n_3)$

4.3 8

$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\det A = -1$

$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. $\det A = -1 \cdot -1 + 1 \cdot 1 = 2$

$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, $\det A = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

$= -[-1-(-1)+1] + [-1] - [1]$

$= -1 - 1 - 1$

$= -3$

$\det A_n = (n-1) \cdot (-1)^{n-1}$

44 4

a) $M = \begin{bmatrix} 1 & & x_1 & & \\ & 1 & \vdots & & \\ 0 & 0 \cdots & x_j & \cdots 0 & 0 \\ & & \vdots & 1 & \\ & & x_n & & 1 \end{bmatrix}$,

$r_n \to r_n - \frac{x_n}{x_j} r_j = [0 \ 0 \cdots 0 \cdots 1]$.

$r_{n-1} \to r_{n-1} - \frac{x_{n-1}}{x_j} \cdot r_j \ [0 \ 0 \cdots 0 \cdots 1 \ 0]$.

$M \Rightarrow \begin{bmatrix} 1 & & x_1 & & \\ & 1 & \vdots & & \\ & & x_j & & \\ & & \vdots & 1 & \\ & & 0 & & 1 \end{bmatrix}$ $\boxed{\det M = x_j}$

b) $Ax = b$

$AM = \begin{bmatrix} a_{11} \cdots & a_{1j} & \cdots a_{1n} \\ & \vdots & \\ a_{j1} \cdots & a_{jj} & \cdots a_{jn} \\ a_{n1} \cdots & a_{nj} & \cdots a_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & & x_1 & & \\ & 1 & \vdots & & \\ & & x_j & & \\ & & \vdots & & \\ & & x_n & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} a_{11} \cdots & m_1 & \cdots a_{1n} \\ a_{j1} \cdots & m_j & \cdots a_{jn} \\ a_{n1} \cdots & m_n & \cdots a_{nn} \end{bmatrix}$ $m_1 = a_{11} \cdot x_1 + \cdots + a_{1j} \cdot x_j + \cdots + a_{1n} \cdot x_n = b_1$

$\underbrace{m_i = b_i}$

$= \begin{bmatrix} a_{11} \cdots & b_1 & \cdots a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} \cdots & b_n & \cdots a_{nn} \end{bmatrix}$

$= B_j$

c) $\det(A \cdot M) = \det(B_j)$

$\det(A) \cdot \det(M) = \det(B_j)$

$\det(A) \cdot x_j = \det(B_j)$

$x_j = \dfrac{\det(B_j)}{\det(A)}$

4.4 6

In n-dimensional space, if each vector is scaled by factor of 3, the total volume is scaled by 3^n.

Since the determinant represents the total volume constructed by these row vectors in A, the determinant would also be scaled by 3^n.