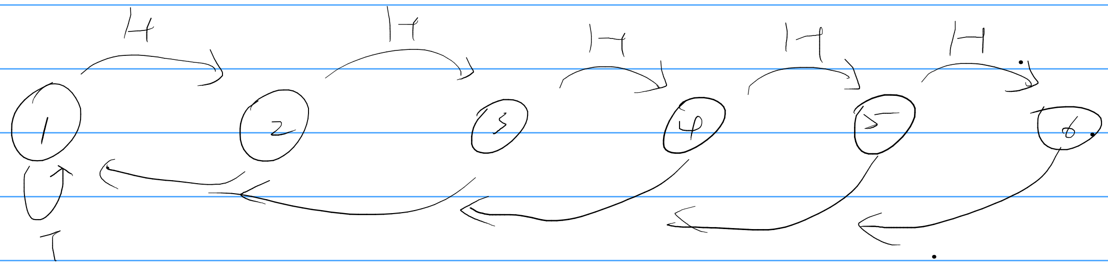
Coin tossing $P(H) = p$ $P(T) = q = 1 - p$

Let $P(X = n)$ is the probability of obtaining the first 5 H in a row with the last head on the n-th toss.

$P(W = n)$ is ...... the first HTHTH with the last H on the n-th toss

Find E_X, σ_X^2, E_W & σ_W^2 for $p = \frac{1}{2}, \frac{2}{3}$.



$$A = \begin{bmatrix} q & q & q & q & q & 0 \\ p & & & & & \\ & \ddots & & & & \vdots \\ & & & & p & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [\,0 \ 0 \ 0 \ 0 \ 0 \ 1\,]$$

$$P(X = n) = CA^nB$$

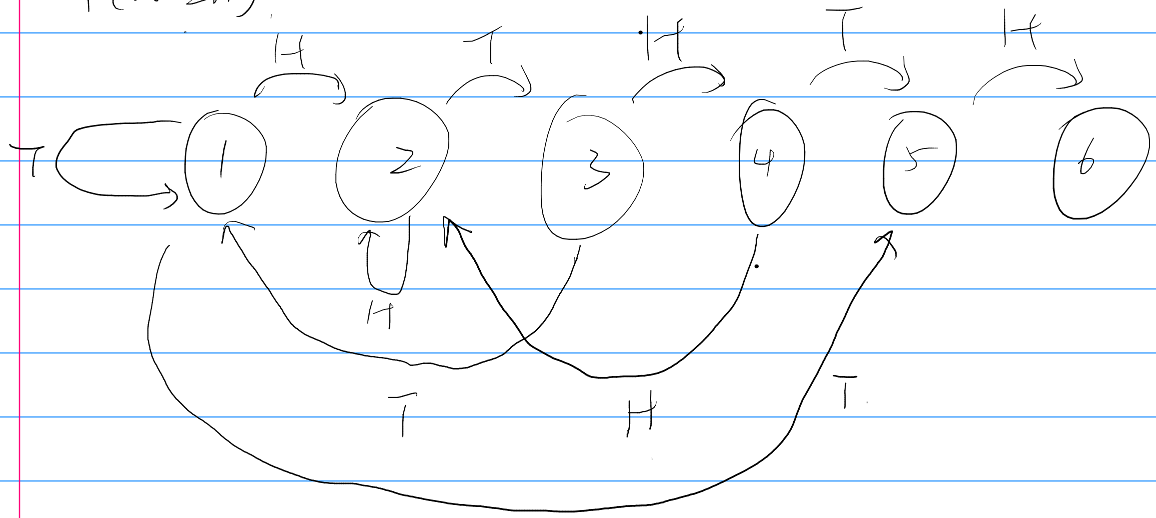
$$\mu_X \ E_X = \sum_0^\infty n P(X = n) = C\left[\sum_0^\infty nA^n\right]B = C(I - A)^{-2}AB$$

$$E_X^2 = C\left[\sum_0^\infty n^2 A^n\right]B = C(I - A)^{-3}(A + A^2)B$$

$$\sigma_X^2 = E_X^2 - \mu_X^2$$

For $\boxed{p = \frac{1}{2}. \quad \mu_X = 62. \quad \sigma_X^2 = 3390 \qquad p = \frac{2}{3} \quad \mu_X = \frac{633}{32} = 19.7813. \quad \sigma_X^2 = 262.3916}$

$P(W=n)$



$$A = \begin{bmatrix} q & 0 & q & 0 & q & 0 \\ p & p & 0 & p & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \end{bmatrix}$$

C & B are the same

μ_W $\quad E W = C(I-A)^{-2} A B$

$E W^2 = C(I-A)^{-3}(A+A^2)B$

$\sigma_W^2 = E W^2 - \mu_W^2$

$$\boxed{\begin{array}{lll} p = \frac{1}{2} & \mu_W = 42 & \sigma_W^2 = 1434 \\ p = \frac{2}{3} & \mu_W = 38.625 & \sigma_W^2 = 1183.3 \end{array}}$$

```
syms p q
q = 1 - p;
A_x = [q q q q q 0; ...
       p 0 0 0 0 0; ...
       0 p 0 0 0 0; ...
       0 0 p 0 0 0; ...
       0 0 0 p 0 0; ...
       0 0 0 0 p 0;]
B = [1 0 0 0 0 0]';
C = [0 0 0 0 0 1];

mu_x = C*(eye(size(A_x)) - A_x)^(-2) * A_x * B;
Ex2 = C*(eye(size(A_x)) - A_x)^(-3) * (A_x + A_x^2) * B;
var_x = Ex2 - mu_x^2;

mu_x1 = subs(mu_x, p, 1/2)
var_x1 = subs(var_x, p, 1/2)
mu_x2 = subs(mu_x, p, 2/3)
eval(mu_x2)
var_x2 = subs(var_x, p, 2/3)
eval(var_x2)
```

A_x =

$$\begin{pmatrix} 1-p & 1-p & 1-p & 1-p & 1-p & 0 \\ p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \end{pmatrix}$$

mu_x1 = 62

var_x1 = 3390

mu_x2 =

$$\frac{633}{32}$$

ans = 19.7813

var_x2 =

$$\frac{268689}{1024}$$

ans = 262.3916

```
A_w = [q 0 q 0 q 0; ...
        p p 0 p 0 0; ...
        0 q 0 0 0 0; ...
        0 0 p 0 0 0; ...
        0 0 0 q 0 0; ...
        0 0 0 0 p 0;]

mu_w = C*(eye(size(A_w)) - A_w)^(-2) * A_w * B;
Ew2 = C*(eye(size(A_w)) - A_w)^(-3) * (A_w + A_w^2) * B;
var_w = Ew2 - mu_w^2;

mu_w1 = subs(mu_w, p, 1/2)
var_w1 = subs(var_w, p, 1/2)
mu_w2 = subs(mu_w, p, 2/3)
eval(mu_w2)
var_w2 = subs(var_w, p, 2/3)
eval(var_w2)
```

A_w =

$$\begin{pmatrix} 1-p & 0 & 1-p & 0 & 1-p & 0 \\ p & p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \end{pmatrix}$$

mu_w1 = 42

var_w1 = 1434

mu_w2 =

$$\frac{309}{8}$$

ans = 38.6250

var_w2 =

$$\frac{75729}{64}$$

ans = 1.1833e+03