Part 1

1. F

2. T

3. T

4. F

5. T

6. T $tr(A) = \lambda_1 + \lambda_2$

$tr(B) = \lambda_3 + \lambda_4$

7. T

8. T $P^2 = P$ $det(P^2) = det(P)$

$\lambda^2 = \lambda$

$\lambda = 0, \lambda = 1$

9. T

10. F

11. T

12. F $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} c\theta & s\theta \\ s\theta & c\theta \end{bmatrix}$

13. F $S\Lambda_1 S^{-1} = A$ $S\Lambda_2 S^{-1} = B$. $\Lambda_2 = S^{-1} B S$

$A = P^{-1} B P$?

14. F

15. T

16. T

17. F $(A \cdot B)^T = B^T \cdot A^T = B \cdot A \neq A \cdot B$

18. T?

19. F

20. T $A^2 = S \Lambda^2 S^{-1}$

21. T $\frac{1}{4} \cdot F \cdot \bar{F} = F \cdot F^{-1} = I$

22. T

23. T

24. F

25. T

Part II

1.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \quad b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

$$A^T A \hat{x} = A^T b.$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

A^TA is not invertible ???

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}.$$

2. $\underset{3\times 2}{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$. $\underset{2\times 3}{A^T} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$.

$AA^T = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 5 \end{bmatrix}$. $A^TA = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$. $\underline{\lambda = 3, 6}$

$\underbrace{V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$.

$AA^T - \lambda I = \begin{bmatrix} 2-\lambda & 2 & -1 \\ 2 & 2-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{bmatrix}$

$(2-\lambda)\cdot\left[(2-\lambda)(5-\lambda) - 1\right] - 2\left[2(5-\lambda) - 1\right] - \left[-2 + (2-\lambda)\right] = 0.$

$(2-\lambda)(10 - 7\lambda + \lambda^2 - 1) - 20 + 4\lambda + 2 + \cancel{2} - \cancel{2} + \lambda = 0.$

$(2-\lambda)(\lambda^2 - 7\lambda + 9) - 18 + 5\lambda = 0.$

$2\lambda^2 - 14\lambda \cancel{+18} - \lambda^3 + 7\lambda^2 - 9\lambda \cancel{-18} + 5\lambda = 0.$

$-\lambda^3 + 9\lambda^2 - 18\lambda = 0.$

$\lambda(\lambda^2 - 9\lambda + 18) = 0.$

$\lambda = \underline{0, 3, 6}.$

$\boxed{\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{6} \\ 0 & 0 \end{bmatrix}}$.

$\lambda = 0.$

$\begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 5 \end{bmatrix}$ $u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & -5 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix}$.

$\lambda = 3.$

$\begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}$ $\begin{bmatrix} -1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda = 5$.

$$\begin{bmatrix} -3 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & -5 & -1 \\ 0 & -5 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$u_3 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} = \frac{1}{3\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}.$$

$1+1+25=27$

$$U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{3\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{3\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-5}{3\sqrt{3}} \end{bmatrix}$$

3、 $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. $A^T = A$ $\quad P = A(A^TA)^{-1}A^T$

$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ $(A^TA)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$.

$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. $\quad A-\lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix}$

$$(2-\lambda)(2-\lambda)^2 - 1\cdot(2-\lambda) = 0.$$

$$(2-\lambda)(\lambda^2-4\lambda+3) = 0$$

$$(2-\lambda)(\lambda-3)(\lambda-1) = 0$$

$$\lambda = 1, 2, 3.$$

$\lambda = 1$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad U_1^{-1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U_1^T A U_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & \frac{1}{2} & 2 \end{bmatrix}.$$

$(3-\lambda)(2-\lambda)$ $\quad \lambda = 2, 3.$

$\lambda = 2$

$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$. $\quad u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\quad U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\quad U_2^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$U = U_1 U_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boxed{U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}}$$

5. $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$.

a)

i) $2 > 0$

ii) $2 \times 5 - 4 = 6 > 0$.

iii) $2(40-9) - 2 \cdot 16 = 2 \cdot (31-16) > 0$.

Since all upper submatrices have positive determinent.

A is positive definite.

b). $A - \lambda I$: $\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 3 \\ 0 & 3 & 8-\lambda \end{bmatrix}$

$(2-\lambda)[(5-\lambda)(8-\lambda)-9] - 2 \cdot [2 \cdot (8-\lambda)] = 0$.

$(2-\lambda) \cdot (40 - 13\lambda + \lambda^2 - 9) - 32 + 4\lambda = 0$.

$80 - 26\lambda + 2\lambda^2 - 18 - 40\lambda + 13\lambda^2 - \lambda^3 + 9\lambda - 32 + 4\lambda = 0$.

$40 - 53\lambda + 15\lambda^2 - \lambda^3 = 0$.

$\lambda =$ - - - - no time $S = \begin{bmatrix} v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$.

$A^{\frac{1}{2}} = S \Lambda^{\frac{1}{2}} S^{-1}$

6. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

$$(-1-\lambda)(-1-\lambda) - 1 = 0$$

$$1 + 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda(\lambda+2) = 0$$

$$\lambda = 0, -2.$$

$\lambda = 0$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \quad S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$$e^{At} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -e^{-2t} \\ 1 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \boxed{\frac{1}{2}\begin{bmatrix} 1+e^{-2t} & 1-e^{-2t} \\ 1-e^{-2t} & 1+e^{-2t} \end{bmatrix}}$$

b) $$x(t) = e^{At}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 + \cancel{e^{-2t} - e^{-2t}} \\ 2 + \cancel{e^{-2t} - e^{-2t}} \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

7.

$$P = A(A^TA)^{-1}A^T$$

$$W = N(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}) \qquad \left\{ \underset{a_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{a_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

$x_2 = 1,\ x_3 = 0$ | $x_2 = 0,\ x_3 = 1$

$a_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ | $a_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (A^TA)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$P = \frac{1}{3}\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & -1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$P = QQ^T$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = a_1 \qquad \underbrace{q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$$

$$v_2 = a_2 - \frac{(v_1 \cdot a_2)}{(v_1 \cdot v_1)} \cdot v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$q_2 = \underbrace{\frac{1}{\sqrt{3/2}}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}} \qquad \boxed{Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}}$$

8. $u_1 = 1$

$$u_2 = x^2 - \frac{(u_2 \cdot u_1)}{(u_1 \cdot u_1)} \cdot u_1$$

$$= x^2 - \int_0^1 x^2\,dx \quad 1$$

$$= x^2 - \frac{1}{3}$$

$$(u_2 \cdot u_2) = \int_0^1 (x^2 - \tfrac{1}{3})(x^2 - \tfrac{1}{3})\,dx \qquad \|u_2\| = \sqrt{\frac{4}{45}} = \frac{2}{3\sqrt{5}}$$

$$= \int_0^1 x^4 - \frac{2}{3}x^2 + \frac{1}{9}\,dx \qquad q_2 = u_2 / \|u_2\|$$

$$= \left[\frac{x^5}{5} - \frac{2}{9}x^3 + \frac{1}{9}x \right]_0^1 \qquad = \frac{3\sqrt{5}}{2}(x^2 - \tfrac{1}{3})$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$= \frac{1}{5} - \frac{1}{9}$$

$$= \frac{4}{45}$$

$$(x^3, 1) = \int_0^1 x^3\,dx = \frac{1}{4}x\Big|_0^1 = \frac{1}{4}$$

$$(x^3, \tfrac{3\sqrt{5}}{2}(x^2 - \tfrac{1}{3})) = \frac{3\sqrt{5}}{2}\int_0^1 x^3 (x^2 - \tfrac{1}{3})\,dx$$

$$= \frac{3\sqrt{5}}{2}\int_0^1 x^5 - \frac{x^3}{3}\,dx$$

$$= \frac{3\sqrt{5}}{2} \cdot \left[\frac{1}{6}x^5 - \frac{1}{12}x^4 \right]_0^1$$

$$= \frac{3\sqrt{5}}{2} \cdot \frac{1}{12} = \frac{\sqrt{5}}{8}$$

The closest function is

$$\frac{1}{4}\cdot 1+\frac{\sqrt{5}}{8}\cdot\frac{3\sqrt{5}}{2}\left(x^2-\frac{1}{3}\right)$$

$$=\frac{1}{4}+\frac{15}{16}x^2-\frac{5}{16}$$

$$=\frac{15}{16}x^2-\frac{1}{16}$$

$$\boxed{c=-\frac{1}{16}\quad d=\frac{15}{16}}$$