P22. 1–4. 1.4.2. Due Friday.

1. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\quad C = [1 \;\; 2]$.

Is $\{C, A\}$ observable?. Find optimal $\hat{x}_0$ & d in

$$d^2 = \inf\left\{\int_0^\infty |f(t) - Ce^{At}x_0|^2 dt : x_0 \in \mathbb{C}^2\right\}, \text{ where } f = e^{-t}$$

Is $\hat{x}_0$ unique?.

$CA = [1 \;\; 2]\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = [-4 \;\; -5]$. $\quad \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}$.

$\text{rank}\begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix} = 2 \Rightarrow$ $\{C, A\}$ is observable

Thus there is a unique solution $\hat{x}_0$ to the optimization problem.

$$\hat{x}_0 = P^{-1} \cdot \int_0^\infty e^{A^* t} C^* \cdot f(t)\, dt,$$

where

$$P = \int_0^\infty e^{A^* t} C^* C e^{At} dt.$$

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\boxed{\hat{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}.$$

$$d^2 = \int_0^\infty \|f(t)\|^2 dt - (P\hat{x}_0, \hat{x}_0)$$

$$= \frac{1}{2} - [-1 \;\; 1]\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= 0$$

$$\boxed{d = 0}$$

Problem 2. Repeat P1 with $f = e^{-3t}$

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\hat{x}_0 = P^{-1} \int_0^{\infty} e^{A^* t} C^* f(t)\, dt = \boxed{\begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix}}$$

This a unique solution too because the system is observable.

$$d^2 = \|f(t)\|^2 - (P\hat{x}_0, \hat{x}_0)$$

$$= \frac{1}{6} - \hat{x}_0^* P \hat{x}_0$$

$$= \frac{1}{600}$$

$$\boxed{d = 0.0408}$$

Problem 3. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $c = \begin{bmatrix} 2 & 1 \end{bmatrix}$

Is $\{C, A\}$ observable? Find optimal $\hat{x}_0$ & d in

$d^2 = \inf\left\{ \int_0^\infty |f(t) - Ce^{At} x_0|^2 \, dt : x_0 \in \mathbb{C}^2 \right\}$ $f = e^{-t}$.

$CA = \begin{bmatrix} -2 & -1 \end{bmatrix}$. $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ rank = 1.

The system is <u>unobservable</u>, thus there is no unique solution $\hat{x}_0$.

$P = \int_0^\infty e^{A^* t} C^* \cdot C e^{At} \, dt$

$= \begin{bmatrix} 2 & 1 \\ 1 & 0.5 \end{bmatrix}$. (singular)

$P \hat{x}_0 = \int_0^\infty e^{A^* t} \cdot C^* f(t) \, dt$.

$\boxed{\hat{x}_0 = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}}$

$d^2 = \|f(t)\|^2 - \hat{x}_0^* P \hat{x}_0$

$= \frac{1}{2} - \frac{1}{2}$

$= 0$.

$\boxed{d = 0}$

Problem 4. Repeat P3 with $f(t) = e^{-3t}$

$$P\hat{x}_0 = \int_0^\infty e^{A^* t} C^* f(t)\,dt$$

$$\boxed{\hat{x}_0 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}}$$ Not unique

$$d^2 = \int_0^\infty \|f(t)\|^2 dt - (P\hat{x}_0, \hat{x}_0)$$

$$= \frac{1}{6} - \hat{x}_0^* P \hat{x}_0$$

$$= \frac{1}{24}$$

$$\boxed{d = 0.2041}$$