Study Notes of Functional Analysis

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Contents

Preface

The notes mainly refer to:

- A Course in Functional Analysis
- lecture notes from Michigan university

Part I

Hilbert Spaces

Chapter 1

Elementary Properties and Examples

1.1 Elementary Properties and Examples

Remark. $\mathbb{F}$ will mean either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1

If $\mathcal{X}$ is a vector space over $\mathbb{F}$. An inner product on $\mathcal{X}$ is a mapping $u : \mathcal{X} \times \mathcal{X} \to \mathbb{F}$ such that $\forall x, y, z \in X, \alpha, \beta \in F$,
(1) $u(\alpha x + \beta y, z) = \alpha u(x, z) + \beta u(y, z)$
(2) $u(x, y) = \overline{u(y, x)}$
(3) $u(x, x) \geqslant 0, < x, x >= 0$ iff $x = 0$
$\mathcal{X}$ together with such a funtion u is called an inner product space.

Remark. We always denote our innner product by $< x, y > := u(x, y)$.
Remark. By def1.1(1)(2), we can get $u(x, \alpha y + \beta z) = \overline{\alpha} u(x, y) + \overline{\beta} u(x, z)$.

Definition 1.2

$\mathcal{X}$ is an inner product space. The mapping

$$\begin{aligned} ||\cdot|| : \mathcal{X} &\to [0, +\infty] \\ x &\mapsto \sqrt{< x, x >} \end{aligned}$$

defines a norm on $\mathcal{X}$. This norm is called norm induced by the inner space or simply induced norm.

Before checking the induced norm well definited, we introduce the Cauchy-Schwarz inequality.

Theorem 1.1

If $\mathcal{X}$ is an inner product space. Then

$$|<x,y>| \leqslant ||x|| \cdot ||y||.$$

equality occurs iff $x = ky (k \neq 0)$.

Proof. write when available. $\square$

Now let's check the three properties of induced norm.
(a) $||x|| = \sqrt{<x,x>} \geqslant 0$; $||x|| = 0 \Leftrightarrow <x,x>= 0 \Leftrightarrow x = 0$.
(b) $||\alpha x|| = \sqrt{<\alpha x, \alpha x>} = \sqrt{\alpha^2 <x,x>} = |\alpha| ||x||$.
(c)

$$\begin{aligned} ||x+y||^2 &=< x+y, x+y > \\ &=< x,x > + < x,y > + < y,x > + < y,y > \\ &= ||x||^2 + \mathrm{Re} < x,y > + ||y||^2. \end{aligned}$$

Since $\mathrm{Re} < x,y > \leqslant | < x,y > | \leqslant ||x|| ||y||$, $||x+y||^2 \leqslant ||x||^2 + 2||x|| ||y|| + ||y||^2$. Then $||x+y||^2 \leqslant (||x|| + ||y||)^2$. Hence, $||x+y|| \leqslant ||x|| + ||y||$.

Proposition 1.1: Continuity of the Inner Product

Let $\mathcal{X}$ be an inner product space with induced norm $|| \cdot ||$. Then $< \cdot, \cdot >: \mathcal{X} \times \mathcal{X} \to \mathbb{F}$ is continuous.

Proposition 1.2: Parallelogram Law

Let $\mathcal{X}$ be an inner product space. Then $\forall x, y \in \mathcal{X}$

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2).$$

Proposition 1.3: Polarization Identity

Let $\mathcal{X}$ be an inner product space. Then $\forall x, y \in \mathcal{X}$

$$< x,y >= \frac{1}{4}(||x+y||^2 - ||x-y||^2 - \mathrm{i}||y + \mathrm{i}x||^2 + \mathrm{i}||y - \mathrm{i}x||^2).$$

Remark. In an inner product space, the inner product determines the norm. The polarization identity shows that the norm determines the inner product. But not every norm on a vector space X is induced by an inner product, for example $(\ell^\infty, || \cdot ||_\infty)$ is not an inner space.

Theorem 1.2

Suppose $(\mathcal{X}, || \cdot ||)$ is a normed linear space. The norm $|| \cdot ||$ is induced by an inner product iff the parallelogram law holds in $(\mathcal{X}, || \cdot ||)$.

Proof. ($\Rightarrow$):See the proof of prop1.2.
($\Leftarrow$): We need to show that the inner product determined by the norm is well definited. Use the polarization identity to define $< \cdot, \cdot >$. Then immediately

$$< x, x > = \frac{1}{4}(||x + x||^2) = ||x||^2 \Rightarrow < x, x > \geqslant 0, < x, x >= 0 \text{ iff } x = 0,$$
$$< x, y > = \overline{< y, x >}$$

Use parallelogram law, we can get

$$< x, y + z > =< x, y > + < x, z >,$$
$$< \alpha x, y > = \alpha < x, y >$$

The inner product satisfys three properties in def1.1 and so well definited. □

Definition 1.3: Cauchy Sequence, Convergent Sequence

Let $\mathcal{X}$ be a normed space, and let $\{x_n\}$ be a sequence of points in X.
(1) We say that $\{x_n\}$ is a Cauchy sequence if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that

$$i, j \geqslant N \Rightarrow ||x_i - x_j|| < \epsilon.$$

(2) We say that $\{x_n\}$ converges to a point $x \in \mathcal{X}$ if

$$\lim_{n\to\infty} ||x_n - x|| = 0.$$

Remark. In a non empty space $\mathcal{X}$, any constant sequence is a cauchy sequence, so you can always find a cauchy sequence.

Proposition 1.4

Cauchy sequence is always bounded.

Proof. Suppose $\{x_n\}$ is a Cauchy sequence. Then for $\epsilon = 1$, $\exists N$, s.t., when $n > N$: $||x_n - x_N|| < 1$. Then by triangle inequality, we have

$$||x_n|| = ||x_n - x_N + x_N|| \leqslant ||x_n - x_N|| + ||x_N|| < 1 + ||x_N||.$$

Let $M = \max\{||x_1||, ||x_2||, ..., ||x_N||, 1 + ||x_N||\}$, then $\forall n$, $||x_n|| \leqslant M$. Hence, $\{x_n\}$ is bounded. □

Proposition 1.5: Convergent Sequence are Cauchy

If $\mathcal{X}$ is a normed space, then every convergent sequene in $\mathcal{X}$ is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a sequence that converges to a point $x \in X$, and let $\epsilon > 0$. Since $\lim_{n\to\infty} ||x_n - x|| = 0$, there exists an $N \in \mathbb{N}$ so that

$$n \geqslant N \Rightarrow ||x_n - x|| < \frac{\epsilon}{2}.$$

If $i, j \geqslant N$, it follows that

$$||x_i - x_j|| \leqslant ||x_i - x|| + ||x_j - x|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves that $\{x_n\}$ is a Cauchy sequence. □

Though every convergent sequence is Cauchy, it is not necessarily the case that every Cauchy sequence in a normed space converges. For example, let $\mathbb{Q}$ be the normed space of all rational numbers under the usual norm: $||q_1 - q_2|| = |q_1 - q_2|$. Then there are many Cauchy sequences in $\mathbb{Q}$ that do not converge to any point in $\mathbb{Q}$. For example, the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

is a Cauchy sequence in $\mathbb{Q}$, but it does not converge to any point in $\mathbb{Q}$.

Definition 1.4: Complete Normed Space

A normed space $\mathcal{X}$ is said to be complete if every Cauchy sequence in $\mathcal{X}$ converges to a point in $\mathcal{X}$.

Definition 1.5: Hilbert Space

An inner product space which is complete with respect to the norm induced by the inner product is called a Hilbert space.

Proposition 1.6

If $\mathcal{X}$ is a vector space and $< \cdot, \cdot >_{\mathcal{X}}$ is an inner product on $\mathcal{X}$. The completion of $\mathcal{X}$ is a Hilbert space.

Proof. You can construct the completion of $\mathcal{X}$, denoted by $\overline{X}$, by referring to lecture notes from nw. Then You can construct the inner product in $\overline{X}$ by referring to answer in mathexchage. □

Definition 1.6: L^p function

Let (X, μ) be a measure space, and let $p \in [1, \infty)$. An L^p function on X is a measurable function f on X for which

$$\int_X |f|^p d\mu < \infty.$$

An important special case of L^p function is for the measure space $(\mathbb{N}, \mu)$, where μ is counting measure on $\mathbb{N}$. In this case, a measure function f on $\mathbb{N}$ is just a sequence

$$f(1), f(2), f(3), \ldots$$

and the Lebesgue integral is the same as the sum of the series

$$\int_{\mathbb{N}} f d\mu = \sum_{n\in\mathbb{N}} f(n).$$

The definition of an L^p function on $\mathbb{N}$ takes the following form.

Definition 1.7: ℓ^p sequence

An ℓ^p sequence is a sequence $\{x_n\}$ of real numbers for which

$$\sum_{n\in\mathbb{N}} |x_n|^p < \infty.$$

Proposition 1.7

For any measure space (X, μ), the set

$$L^p = \{f|f\ is\ L^p\ function\}$$

forms a Hilbert space under the inner product

$$< f, g >= \int_X f\overline{g} d\mu.$$

Proposition 1.8

The set

$$\ell^p = \{x = (x_1, x_2, x_3, ...)|x\ is\ \ell^p\ sequence\}$$

forms a Hilbert space under the inner product

$$< x, y >= \sum_i x_i\overline{y_i}.$$

Definition 1.8

Let I be a nonempty index set. For each $\alpha \in I$, let y_α be a nonnegative real number. Define $\sum\limits_{\alpha\in I} y_\alpha = \sup\{\sum\limits_{\alpha\in J} y_\alpha | J \subset I$ and J is finite$\}$.

Proposition 1.9

If $\sum\limits_{\alpha\in I} y_\alpha < \infty$, then $y_\alpha = 0$ for at at most countably many α.

Proof. refer to answer in mathexchage. Let $l \in \mathbb{N}$. We claim that the subset S_l of S

$$S_l = \{\alpha | \ \frac{1}{l} \leqslant |y_\alpha|\}$$

is a finite set. Sine $\sum_{\alpha \in I} y_\alpha < \infty$,there exist $M \in \mathbb{N}$

$$M \geqslant \sum_{\alpha \in S_l} |y_\alpha| \geqslant \sum_{i \in S_l} \frac{1}{l} = \frac{N}{l}$$

where N is the number of elements in S_l. Thus S_l has at most $||x||^2 l$ elements.
Hence $\{\alpha | \ 0 < |y_\alpha|\} = \cup_{l \in \mathbb{N}} S_l$ is countable as the countable union of finite sets. □

1.2 Homework

Exercise 1.1

Let A be any set and let $l^2(A)$ denote the set of all functions $x : A \to \mathbb{F}$ such that $x(i) = 0$ for all but a countable number of i and $\sum_{i \in A} |x(i)|^2 < \infty$. For $x, y \in l^2(A)$ define

$$< x, y >= \sum_i x(i)\overline{y(i)}.$$

Then $l^2(A)$ is a Hilbert space.

Proof. Note that the induced norm in $l^2(A)$ is

$$||x|| = (\sum_i |x(i)|^2)^{\frac{1}{2}}.$$

Let $\{x_n\}$ be Cauchy sequence in $l^2(A)$ and Then,

$$|x_m(i) - x_n(i)| \leqslant (\sum_i |x_m(i) - x_n(i)|^2)^{\frac{1}{2}} = ||x_n - x_m|| \to 0.$$

Thus, $x_n(i)$ is a Cauchy sequence over the real(or complex) numbers. By the completness of $\mathbb{R}$(ro $\mathbb{C}$), it must converge to a limit, denoted by $xl(i)$. Now we show that $x_n \to x(x : i \mapsto xl(i))$ when $n \to \infty$.

Since $\{x_n\}$ is Cauchy, $\forall \epsilon$, $\exists N$, s.t., when $m, n > N, \forall k$:

$$\sum_i^k |x_m(i) - x_n(i)|^2 \leqslant ||x_m - x_n||^2 < \sqrt{\frac{\epsilon}{2}}.$$

Let $m \to \infty$ and $n > N$, we have

$$\sum_{i}^{k} |x(i) - x_n(i)|^2 \leqslant ||x - x_n||^2 \leqslant \sqrt{\frac{\epsilon}{2}}.$$

Let $k \to \infty$ and $n > N$, we have

$$(\sum_{i}^{\infty} |x(i) - x_n(i)|^2)^{\frac{1}{2}} \leqslant ||x - x_n|| \leqslant \frac{\epsilon}{2} < \epsilon.$$

Hence, $x_n \to x$ when $n \to \infty$.Then, we show that $x \in l^2(A)$.

We first show that $x(i) = 0$ for all but a countable number of i. Suppose the contrary. Then for any x_n, $||x - x_n||^2$ being a sum of uncountably many nonzero elements, which cannot be bounded. This is a contradiction as $||x - x_n|| \to 0$ ($n \to \infty$).

Finally, we show that $\sum\limits_{i \in A} |x(i)|^2 < \infty$. Since $\{x_n\}$ is Cauchy, $\{x_n\}$ is bounded. Then $\exists M > 0$, $\forall n$, we have $||x_n|| < M$. Then $\forall n, k$, we have

$$\sum_{i}^{k} |x_n(i)|^2 \leqslant ||x_n||^2 \leqslant M^2.$$

Let $n \to \infty$, we have

$$\sum_{i}^{k} |x_n(i)|^2 \leqslant ||x||^2 \leqslant M^2.$$

Let $k \to \infty$, we have

$$\sum_{i} |x_n(i)|^2 \leqslant M^2.$$

Hence, $x \in l^2(A)$. Thus, every Cauchy sequence in $l^2(A)$ is convergent and so $l^2(A)$ is a Hilbert space. □

If $A = \mathbb{N}$, $l^2(A)$ is usually denoted by l^2. Let (X, Ω, μ) be a measure space consisting of a set X, a σ-algebra Ω of subsets of X, and a countably additive measure μ defined on Ω with values in the non-negative extended real numbers. Note that if Ω= the set of all subsets of A and for E in Ω, $\mu(E) := \infty$ if E is infinite and $\mu(E)$=the cardinality of E if E is finite, then $l^2(A)$ and $L^2(A, \Omega, \mu)$ are equal.

1.3 Reference

- lecture notes from brmh
- lecture notes from washington university
- lecture notes from cornell

- the completness of L^p space
  * lectrue notes from princeton
  * lecture notes from cornell
  * lecture notes from ndsu
- homework1

Chapter 2

Orthogonality

Definition 2.1

If $\mathcal{H}$ is a Hilbert space, we say $x, y \in \mathcal{H}$ are orthogonality if $< x, y >= 0$, in which case we write $x \perp y$. We say subsets A, B are orthogonal if $x \perp y$ for all $x \in A$ and $y \in B$, and we write $A \perp B$.

Theorem 2.1: The Pythagorean Theorem

If $x_1, x_2, ..., x_n \in \mathcal{H}$ and $x_j \perp x_k$ for $j \neq k$, then

$$||\sum_{j=1}^{n} x_j||^2 = \sum_{j=1}^{n} ||x_j||^2.$$

Proof. When $n = 2$, expanding the left side. Then use induction. □

Remark. The pythagorean theorem is strongly related to orthogonality.

Definition 2.2

For any subset $A \subset \mathcal{H}$, define $A^{\perp} = \{x \in \mathcal{H} | < y, x >= 0, \forall y \in A\}$.

Definition 2.3: Linear Span, Closed Linear Span

If $A \subset \mathcal{H}$, let $span(A) :=$ the intersetion of all linear subspaces of $\mathcal{H}$ that contain A. $span(A)$ is called the linear span of A. Let $\overline{span(A)} :=$ the intersetion of all closed linear subspaces of $\mathcal{H}$ that contain A. $\overline{span(A)}$ is called the closed linear span of A.

Proposition 2.1

(1) $span(A)$ is the smallest linear subspace of $\mathcal{H}$ that contains A.

(2) $span(A) = \{\sum_{k=1}^{n} \alpha_k x_k | n \geqslant 1, \alpha_k \in \mathbb{F}, x_k \in A\}$.

For convenience, we use the notation $A \leqslant \mathcal{H}$ to signify that A is a closed linear subspace of $\mathcal{H}$.

Proposition 2.2

(1) $\overline{span(A)} \leqslant \mathcal{H}$
(2) $\overline{span(A)}$ is the smallest closed linear subspace of $\mathcal{H}$ that contains A.
(3) $\overline{span(A)}$ = the closure of $\{\sum_{k=1}^{n} \alpha_k x_k | n \geqslant 1, \alpha_k \in \mathbb{F}, x_k \in A\}$.

Proposition 2.3

For any $A \subset \mathcal{H}$,
(1) $A^{\perp} \leqslant \mathcal{H}$
(2) $(span(A))^{\perp} = A^{\perp}$
(3) $\overline{A}^{\perp} = A^{\perp}$
(4) $\overline{span(A)}^{\perp} = A^{\perp}$

Proof. (1) Suppose $\{y_n\} \in A^{\perp}$ with $y_n \to x_0 \in \mathcal{H}$. Since $< x_n$, □

Definition 2.4: Direct Sum

$\mathcal{M}_1, \mathcal{M}_2 \leqslant \mathcal{H}$. We write

$$\mathcal{H} = \mathcal{M}_1 \bigoplus \mathcal{M}_2,$$

if $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$ and for every $x \in \mathcal{H}$ there exist $v_i \in \mathcal{M}_i$ with $v_1 + v_2 = x$.

Proposition 2.4

If $\mathcal{M} \leqslant \mathcal{H}$, then

$$\mathcal{H} = \mathcal{M} \bigoplus \mathcal{M}^{\perp}$$

Definition 2.5: Dense Subset

Let $(\mathcal{X}, ||\cdot||)$ be a normed space and $A \subset \mathcal{X}$. Then A is dense in $\mathcal{X}$ if $\overline{A} = \mathcal{X}$.

In Conway textbook, non-closed subspaces are called ”linear manifolds”, but we will not adopt this convention.

Proposition 2.5

Let $\mathcal{K}$ be a(not necessarily closed) subspace of $\mathcal{H}$. Then $\mathcal{K}$ is dense iff $\mathcal{K}^{\perp} = \{0\}$.

Definition 2.6: Convex Sets

A subset A of a vector space $\mathcal{X}$ is called convex if

$$\forall x, y \in A, t \in [0,1], tx + (1-t)y \in A.$$

Proposition 2.6

(1) Every subspace is convex.
(2) In a normed linear space, the open ball $B(x, \epsilon)$ is convex.
(3) If $A \subset \mathcal{X}$ is convex and $x \in \mathcal{X}$, then $A + x := \{y + x | y \in A\}$ is convex.

Proof. (3) A is convex, then $\forall a, b \in A, t \in [0,1], ta + (1-t)b \in A$. Then $t(a+x) + (1-t)(b+x) = ta + (1-t)b + x \in A + x$. Hence $A + x$ is convex. □

Theorem 2.2

Every none-empty closed convex subset A of $\mathcal{H}$ has a unique element of smallest norm, i.e.

$$\exists! a \in A, ||a|| = \inf_{b \in A} ||b||.$$

Proof. Let $\delta = \inf\{||x|| | x \in A\}$. Since A is convex, $x + y \in A (\forall x, y \in A)$. Then, by the parallelogram law,

$$||x - y||^2 = 2(||x||^2 + ||y||^2) - ||x + y||^2 \leqslant 2(||x||^2 + ||y||^2) - 4\delta^2.$$

Existence follows: by the definition of infimum, we can choose $\{y_n\}_{n=1}^{\infty} \subset A$ for which $||y_n|| \to \delta$. Then when $n, m \to 0$,

$$||y_n - y_m||^2 \leqslant 2(||y_n||^2 + ||y_m||^2) - 4\delta^2 \to 0.$$

Hence, $\{y_n\}$ is Cauchy. By completeness, $\exists y \in \mathcal{H}$ for which $y_n \to y$, and since A is closed, $y \in A$. Also $||y|| = \lim_{n \to \infty} ||y_n|| = \delta$.
Uniqueness follows: if $||x|| = ||y|| = \delta$, then $||x - y||^2 \leqslant 4\delta^2 - 4\delta^2 = 0$, so $x = y$. □

Corollary 2.1

If A is a nonempty closed convex set in $\mathcal{H}$ and $x \in \mathcal{H}$, then there exists a unique closest element of A to x.

Proof. Since A is closed and convex, $A - x$ is closed and convex. By thm2.2, let z_0 be the unique element of smallest norm in $A - x$ and let $y_0 = z_0 + x$. Then $y_0 \in A$ and $||y_0 - x|| = ||z_0|| = \inf_{z \in A - x} ||z|| = \inf_{y \in A} ||y - x||$. Hence, y is the unique closest element of A to x. □

Since the closet point from closed convex set to point can be obtained, we can define the distance about these two object.

Definition 2.7

If A is a nonempty closed convex set in $\mathcal{H}$ and $x \in \mathcal{H}$, we define the distance from A to x:

$$\operatorname{dist}(x, A) := \inf_{y \in A} ||y - x||.$$

If the convex set in cor2.1 is in fact a closed linear subspace of $\mathcal{H}$, more can be said.

Theorem 2.3

If $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$, $x \in \mathcal{H}$, and y_0 is the unique element of $\mathcal{M}$ such that $||y_0 - x|| = \operatorname{dist}(x, \mathcal{M})$, then $x - y_0 \perp \mathcal{M}$. Conversely, if $y_0 \in \mathcal{M}$ such that $x - y_0 \perp \mathcal{M}$, then $||x - y_0|| = \operatorname{dist}(x, \mathcal{M})$.

Proof. You can refer to the lecture notes from brmh. □

Note that Theorem2.3, together with the uniqueness statement in corollary2.1, shows that if $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$ and $x \in \mathcal{H}$, then there is a unique element $y_0 \in A$ such that $x - y_0 \perp \mathcal{M}$. Thus a function $P : \mathcal{H} \to \mathcal{M}$ can be defined by $Px = y_0$.

Definition 2.8

If $\mathcal{M} \leqslant \mathcal{H}$ and $P : \mathcal{H} \to \mathcal{M}$ given by $Px = y_0$, where y_0 is the unique element in A such that $x - y_0 \perp \mathcal{M}$. Then P is called the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. If we wish to show this dependence of P on $\mathcal{M}$, we will denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$ by $P_{\mathcal{M}}$.

Remark. P is onto.

Theorem 2.4

If P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, then
(1) P is a linear transformation on $\mathcal{H}$,
(2) $||Px|| \leqslant ||x||$ for every x in $\mathcal{H}$,
(3) $P^2 = P$,
(4) $N(P) = \mathcal{M}^\perp$ and $R(P) = \mathcal{M}$,
(5) $I - P$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}^\perp$.

Corollary 2.2

If $\mathcal{M} \leqslant \mathcal{H}$, then $(\mathcal{M}^\perp)^\perp = \mathcal{M}$.

2.1 Reference

- lecture notes from washington university
- lecture notes from brmh
- lecture notes from umich

- lecture notes from msu

Chapter 3

The Riesz Representation Theorem

Definition 3.1: Linear Functional

(1) A linear functional on a vector space $\mathcal{X}$ is a linear mapping $f : \mathcal{X} \to \mathbb{F}$:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \forall x, y \in \mathcal{X}, \alpha, \beta \in \mathbb{F}.$$

(2) A linear functional f on a normed space $(\mathcal{X}, || \cdot ||)$ is called a bouned linear functional if there exists $C \geqslant 0$ such that $|f(x)| \leqslant C||x||$ for each $x \in \mathcal{X}$.

Proposition 3.1

Let $\mathcal{H}$ be a Hilbert space and $f : \mathcal{H} \to \mathbb{F}$ a linear functional. The following statements are equivalent.
(1) f is continuous at $x = 0$.
(2) f is continuous.
(3) f is bounded.
(4) $N(f)$ is closed in $\mathcal{H}$.

Note that the bounded linear functionals forms a vector space $\mathcal{H}^*$: $0 \in \mathcal{H}^*$, if $f_i \in \mathcal{H}^*$ and $\alpha_i \in \mathbb{F}$ then $\alpha_1 f_1 + \alpha_2 f_2 \in \mathcal{H}^*$. We will now explain how to define a norm on $\mathcal{H}^*$.

Definition 3.2

For a bounded linear functional $f \in \mathcal{H}^*$, its norm is defined as

$$||f||_{\mathcal{H}^*} = \sup_{||x||_{\mathcal{H}} \leqslant 1} |f(x)|.$$

For convenience, the following content will follow the convention: $||f|| = ||f||_{\mathcal{H}^*}, ||x|| = ||x||_{\mathcal{H}}$. Let's check three properties fo the norm:
(1) $||f|| \geqslant 0$; $||f|| = 0 \Leftrightarrow f = 0$
(2) $||\alpha f|| = |\alpha| ||f||$

(3)

$$\begin{aligned}||f_1+f_2|| &= \sup_{||x||\leqslant 1}|(f_1+f_2)(x)| \\ &= \sup_{||x||\leqslant 1}|f_1(x)+f_2(x)| \\ &\leqslant \sup_{||x||\leqslant 1}|f_1(x)|+\sup_{||x||\leqslant 1}|f_2(x)| \\ &= ||f_1||+||f_2||\end{aligned}$$

Proposition 3.2

If f is a bounded linear functional, then
(1) $|f(x)| \leqslant ||f|| \cdot ||x||$ for every $x \in \mathcal{H}$.
(2)

$$\begin{aligned}||f|| &= \sup_{||x||=1}|f(x)| \\ &= \sup_{x\in\mathcal{H}\backslash\{0\}}\frac{|f(x)|}{||x||} \\ &= \inf\{c>0||f(x)|\leqslant c||x||, x\in\mathcal{H}\}.\end{aligned}$$

Proposition 3.3

If $y \in \mathcal{H}$, then

$$f_y(x) =< x, y > \tag{3.1}$$

is a bounded linear funcional on $\mathcal{H}$, with $||f_y|| = ||y||$.

Proof. For $x_1, x_2 \in \mathcal{H}$, $\alpha, \beta \in \mathbb{F}$,

$$f_y(\alpha x_1+\beta x_2) =< \alpha x_1+\beta x_2, y >= \alpha < x_1, y > +\beta < x_2, y >= \alpha f_y(x_1)+\beta f_y(x_2).$$

Hence, f_y is linear. By the Cauchy-Schwarz inequality, for $x \in \mathcal{H}$,

$$|f_y(x)| = | < x, y > | \leqslant ||x||||y||.$$

Hence, f_y is bounded and $||f_y|| = \sup_{x\in\mathcal{H}\backslash\{0\}}\frac{|f_y(x)|}{||x||} \leqslant ||y||$. Then $||f_y|| = \sup_{||x||=1}|f(x)| \leqslant ||y||$. Moreover, $||\frac{y}{||y||}|| = 1$ and $f(\frac{y}{||y||}) =< \frac{y}{||y||}, y >= ||y||$. Hence, $||f_y|| = ||y||$. □

One of the fundamental facts about Hilbert spaces is that all bounded linear functionals are of the form (3.1). In other words, every bounded linear functional on $\mathcal{H}$ can be identified with a unique point in the space itself.

Theorem 3.1

If f is a bounded linear functional on $\mathcal{H}$, then there is a unique vector $y \in \mathcal{H}$ such that

$$f(x) =< x, y >, \forall x \in \mathcal{H}. \tag{3.2}$$

Moreover, $||f|| = ||y||$

Proof. By proposition3.1, $\mathcal{M} = N(f)$ is closed in $\mathcal{H}$. If $\mathcal{M} = \mathcal{H}$, $f(x) = 0$, $y = 0$ is desired requestd. If $\mathcal{M} \neq \mathcal{H}$, then $f \neq 0$ and so there exists some $u_1 \in \mathcal{H}$ so that $f(u_1) \neq 0$, and we take $u_1' = \frac{u_1}{f(u_1)}$ so that $f(u_1') = 1$. We can then define the nonempty set

$$S = \{u \in \mathcal{H} : f(u) = 1\} = f^{-1}(1),$$

which is closed because f is continuous and $\{1\}$ is closed, and the preimage of a closed set by a continuous function is a closed set. We claim that S is convex: indeed, if $u_1, u_2 \in S$ and $t \in [0, 1]$, then

$$f(tu_1 + (1-t)u_2) = tf(u_1) + (1-t)f(u_2) = t + 1 - t = 1,$$

so that $tu_1 + (1-t)u_2 \in S$. So by theorem2.2, there exists $u_0 \in S$ so that $||u_0|| = \inf_{u \in S} ||u||$, and we define $y = \frac{u_0}{||u_0||^2}$ (noting that $u_0 \neq 0$ because $0 \notin S$).
We claim that this is the y that we want; in other words, let's check that $f(x) =< x, y >$. By proposition3.1, $N(f)$ is closed and convex in $\mathcal{H}$. Then we can check that $S = \{u_0 + w : w \in N(f)\}$ ($f(u_0 + w) = 1$,then TRS$\subset S$; $\mathcal{H} = N(f) \bigoplus N(f)^{\perp}$) □

3.1 Reference

- lecture notes from brmh
- Riesz Representation Theorem geometric intuition
- lecture notes from msu
- lecture notes from mit

Chapter 4

Orthonormal Sets of Vectors and Bases

It will be shown in this chapter that, as in Euclidean space, each Hilbert space can be coordinatized. The vehicle for introducing the coordinates is an orthonormal basis. The corresponding vectors in $\mathbb{F}^d$ are the vectors $\{e_1, e_2, ..., e_d\}$, where e_k is the d-tuple having a 1 in the kth place and zeros elsewhere.

Definition 4.1

A subset $A \subset \mathcal{H}$ is said to orthogonal if $< x, y >= 0$, for all $x, y \in A$ such that $x \neq y$. We say it is orthonormal if we further regive $||x|| = 1$ for all $x \in A$.

Remark. Orthogonal set can contain zero vector, but orthonormal set can not.
Subsets $\mathscr{E}, V$ mentioned below can be finite, countably infinite and uncoutably infinite.

Proposition 4.1

Let V be an orthogonal subset of $\mathcal{H}$. Then

$$\sum_v \{v : v \in V\}$$

converges iff

$$\sum_v ||v||^2 < \infty.$$

Proof. Let s_k be the sequence of partial sums of the given series. By the Pythagorean theorem,

$$||s_i - s_j||^2 = || \sum_{n=i+1}^{j} v_n||^2 = \sum_{n=i+1}^{j} ||v_n||^2.$$

for all $i \leqslant j$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} v_n \ converges \\
&\Leftrightarrow \{s_k\} \ is \ a \ \ Cauchy \ sequence \\
&\Leftrightarrow \sum_{n=i+1}^{j} ||v_n||^2 \to 0, \ i,j \to \infty \\
&\Leftrightarrow \sum_{n=1}^{\infty} ||v_n||^2 < \infty.
\end{aligned}$$

□

Corollary 4.1

Let $\{u_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let an be a sequence of real(complex) numbers. Then the series

$$\sum_{n=1}^{\infty} a_n u_n.$$

converges iff $\{a_n\}$ lies in ℓ^2.

Proof. Let s_k be the sequence of partial sums of the given series. By the Pythagorean theorem,

$$||s_i - s_j||^2 = || \sum_{n=i+1}^{j} ||a_n u_n||^2 = \sum_{n=i+1}^{j} |a_n|^2 ||u_n||^2 = \sum_{n=i+1}^{j} |a_n|^2.$$

for all $i \leqslant j$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n u_n \ converges \\
&\Leftrightarrow \sum_{n=1}^{\infty} |a_n|^2 < \infty.
\end{aligned}$$

□

In general, if $\{a_n\}$ is an ℓ^2 sequence set, then the sum

$$\sum_{n=1}^{\infty} a_n u_n$$

is called a ell^2-linear combintion of the vectors $\{u_n\}$. By corollary4.1, every ℓ^2-linear combination orthonormal vectors in a Hilbert space converges.

Definition 4.2: Fourier Coefficients

If $\{v_n\}$ is an orthogonal set of $\mathcal{H}$ and $x \in \mathcal{H}$. Then $< u_n, x >$ is called the Fourier coefficients of x with respect to the orthonormal set $\{v_n\}$.

When an orthonormal set is finite in $\mathcal{H}$, we have

Theorem 4.1

Let $\{u_1, u_2, ..., u_k\}$ is an orthonormal set in $\mathcal{H}$ and $x = \sum_{i=1}^{k} a_i u_i$, then $a_i =< u_i, x > (1 \leqslant j \leqslant k)$ and $||x||^2 = \sum_{i=1}^{k} |a_i|^2$.

Proof. $< u_i, x >= \sum_{j=1}^{k} < u_i, u_j >= a_i < u_i, u_i >= a_i$. Then use Pythagorean Theorem, we get

$$||x||^2 = ||\sum_{j=1}^{k} a_i u_i||^2 = \sum_{j=1}^{k} ||a_i u_i||^2 = \sum_{j=1}^{k} |a_i|^2.$$

□

Definition 4.3

$\{x_1, ..., x_k\}$ is said to be linearly dependent, if there exist scalars $a_1, a_2, ..., a_k$, not all zero, such that

$$a_1 x_1 + ... + a_k x_k = 0$$

where 0 denotes the zero vector. An infinite set of vectors is linearly independent if every nonempty finite subset is linearly independent.

Corollary 4.2

Every orthonormal set is linearly independent.

Proof. If the set is finite. Suppose $\{u_1, ..., u_k\}$ is an orthonormal set. Assume $b_1 u_1 + ... b_k u_k = 0$, then $b_1 u_1 = -\sum_{i=2}^{k} b_i u_i$. By theorem4.1, $b_i =< u_i, u_1 >= 0 (2 \leqslant i \leqslant k)$. Then $b_1 u_1 = 0$. Since $u_1 \neq 0$, $b_1 = 0$. Hence, $b_i = 0$ for all $1 \leqslant i \leqslant k$ and so $\{u_1, ..., u_k\}$ is linearly independent. If the set is infinite, the above proof can show that every nonempty finite subset is linearly independent and so the set is linearly independent. □

Now we consider the question: Does the orthonormal set always exist? The following theorem tell us we can always construct an orthonormal set.

Theorem 4.2: The Gram-Schmidt Orthogonalization Process

If $\{x_1, ..., x_k\}$ is a linearly independent subset of $\mathcal{H}$, then there is an orthonormal set $\{u_1, ..., u_k\}$ such that $span(\{u_1, ..., u_k\}) = span(\{x_1, ..., x_k\})$.

Proof. Define $\{u_k\}$ inductivey. Start with $u_1 = \frac{x_1}{||x_1||}$. Suppose for $k-1$, $u_1, ..., u_{k-1}$ exist. Let $v_k = x_k - \sum_{j=1}^{k-1} < u_j, x_k > u_j$ and $u_k = \frac{v_k}{||v_k||}$. Then $< u_k, u_j >= \frac{1}{||v_k||}(< x_k, u_j > - < u_j, x_k >< u_j, u_j >) = 0$ and $||u_k|| = 1$. Hence, we can $\{u_1, ..., u_k\}$ is an orthonormal set. from the construct process of $u_j (1 \leqslant j \leqslant k)$, we can know $u_j \in span(\{x_1, ..., x_k\})$ and so $span(\{u_1, ..., u_k\}) \subset span(\{x_1, ..., x_k\})$. Since $\{u_1, ..., u_k\}$ is linearly independent, $dim(span(\{u_1, ..., u_k\})) = n$. Hence, $span(\{u_1, ..., u_k\}) = span(\{x_1, ..., x_k\})$. □

Now we consider the following question: How can we determine Px_0 when x_0 and the subspace $\mathcal{M}$ are given? When $\mathcal{M}$ is finite, we have

Proposition 4.2

Let $\mathcal{M} = \overline{span(\{u_1, ..., u_k\})}$. Then $\forall x \in \mathcal{H}$, the vector

$$y = \sum_{i=1}^{k} < u_i, x > u_i$$

is the projection of x onto $\mathcal{M}$.

Proof. Observe that $< u_i, y >=< u_i, x >$ for each i, and hence $< u_i, x - y >= 0$ for each i. Hence, $x - y \perp \mathcal{M}$. By theorem2.3 and definition2.8, y is the projection of x onto $\mathcal{M}$. □

Now we generalize this proposition when $\mathcal{M}$ is infinite.

Lemma 4.1

Let $\{u_n\}$ be an orthonormal set in $\mathcal{H}$ and $x \in \mathcal{H}$. Then

$$\sum_{n=1}^{\infty} < u_n, x >^2 \leqslant ||x||^2,$$

which implies $\{< u_n, x >\}$ is a ℓ^2 sequence.

Proof. Let $N \in \mathbb{N}$, then by proposition4.2,

$$y_N = \sum_{n=1}^{N} < u_n, x > u_n$$

be the projection fo x onto $span(\{u_1, ..., u_N\})$. Then $x - y_N \perp y_N$, so by the Pythagorean theorem

$$||x||^2 = ||y_N + x - y_N||^2 = ||y_N||^2 + ||x - y_N||^2 \geqslant ||y_N||^2 = \sum_{n=1}^{N} < u_n, x >^2 .$$

This holds for all $N \in \mathbb{N}$, so the desired inequality follows. □

Theorem 4.3

Let $\{u_n\}$ be an orthonormal set in $\mathcal{H}$ and $x \in \mathcal{H}$. $\mathcal{M}$ is $\overline{span(\{u_n\})}$. Then

$$y = \sum_{n=1}^{\infty} < u_n, x > u_n$$

is the projection of x onto the $\mathcal{M}$.

Proof. Since $\{u_n, x\}$ is a ell^2 sequence, and thus the sum for y converges. Then,

$$< u_i,$$

□

In particular, if $\overline{span(\{u_n\})} = \mathcal{H}$, then every $x \in \mathcal{H}$ may be expanded in terms of elements of $\{u_n\}$. The following theorem gives equivalent conditions for this property of $\{u_n\}$.

Theorem 4.4

If $\{u_n\}$ is an orthonormal subset of $\mathcal{H}$, then the following conditions are equivalent:
(1) $\{u_n\}$ is a maximal orthonormal set.
(2) $\overline{span(\{u_n\})} = \mathcal{H}$;
(3) $||x|| = \sum_{n=1}^{\infty} | < u_n, x > |^2$ for all $x \in \mathcal{H}$ (Parsevars Identity);
(4) $x = \sum_{n=1}^{\infty} < u_n, x > u_n$ for all $x \in \mathcal{H}$;
(5) $< u_n, x >= 0$ for all n implies $x = 0$;

Proof. (1)⇒ (2): Since $\{u_n\}$ is maximal, $span(\{u_n\})^{\perp} = \{0\}$. Let $\mathcal{M} = \overline{span(\{u_n\})}$, then $\mathcal{M}^{\perp} = \{0\}$. Since $\mathcal{M}^{\perp} \bigoplus \mathcal{M} = \mathcal{H}$, $\mathcal{M} = \overline{span(\{u_n\})} = \mathcal{H}$.
(2)⇒ (3): □

Definition 4.4

An orthonormal set $\{u_n\}$ in $\mathcal{H}$ satisfying any of the equivalent conditions (1)-(5) in theorem4.4 is called a complete orthonormal set(or a complete orthonormal system) or an orthonormal basis in $\mathcal{H}$.

Remark. If $\mathcal{H}$ is infinite dimensional, an orthonormal basis is not a basis in the usual definition of a basis for a vector space (i.e., each $x \in \mathcal{H}$ has a unique representation as a finite linear combination of basis elements). Such a basis in this context is called a Hamel basis.

Theorem 4.5

Every Hilbert space $\mathcal{H}$ has an orthonormal basis. If $\{u_n\}$ is an orthonormal set, then $\mathcal{H}$ has an orthonormal basis containing $\{u_n\}$.

Proof. referring to lecture notes from ucdavis □

4.1 Reference

- lecture notes from brmh
- lecture notes from mit
- lecture notes from ucdavis
- lecture notes from washington
- lecture notes from cornell
- lecture notes from cuhk

Chapter 5

Isomorphic Hilbert Spaces and the Fourier Transform for the Circle

Definition 5.1

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, an isomorphism between $\mathcal{H}$ and $\mathcal{K}$ is linear surjection $U : \mathcal{H} \to \mathcal{K}$ such that

$$< Uh, Ug >=< h, g >$$

for all $h, g \in \mathcal{H}$. In this case $\mathcal{H}$ and $\mathcal{K}$ are said to be isomorphic.

Remark. Many call what we call an isomorphism a unitary operator

Isomorphism defined above is an equivalent relation:

(1) $< idh, idg >=< h, g >$

(2) $< h, g >=< U^{-1}Uh, U^{-1}Ug >=< Uh, Ug >$

(3) $< h, g >=< Uh, Ug >=< VUh, VUg >$

from the previous definition, we know that isomorphism preserves inner product. Now we claim that isomorphism preserves distance and completeness.

Proposition 5.1

If $V : \mathcal{H} \to \mathcal{K}$ is a linear map between Hilbert space, then V is an isometry iff $< Vh, Vg >=< h, g >$ for all $h, g \in \mathcal{H}$.

Part II

Operators on Hilbert Spaces

Chapter 6

Elementary Properties and Examples

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert space over $\mathbb{F}$. Recall that a map $A : \mathcal{H} \to \mathcal{K}$ is a linear transformation if for all $x_1, x_2 \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{F}$,

$$A(\alpha x_1 + \beta x_2) = \alpha A(x_1) + \beta A(x_2).$$

Then $N(A) = \{x \in \mathcal{H} : Ax = 0\}$ and $R(A)$ are subspaces of $\mathcal{H}$ and $\mathcal{K}$ respectively.

The collection of all linear operators from $\mathcal{H}$ to $\mathcal{K}$ forms a vector space $\mathcal{L}(\mathcal{H}, \mathcal{K})$ under pointwise addition and scalar multiplication of functions.

Let's recall the definition of bounded linear transformation and the norm of bounded linear transformation. The proof of the next proposition is similar to the proofs of the corresponding results for linear funtionals in proposition3.1.

Proposition 6.1

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $A : \mathcal{H} \to \mathcal{K}$ be a linear transformation. Then the following are equivalent
(1) A is

As in definition3.2, if A is a bounded linear transformation, the norm of A defined as

$$||A|| = \sup_{x \in \mathcal{H}, ||x|| \leqslant 1} ||Ax||.$$

And then

$$\begin{aligned} ||A|| &= \sup_{||x||=1} ||Ax|| \\ &= \sup_{x \neq 0} \frac{||Ax||}{||x||} \\ &= \inf\{c > 0 : ||Ax|| \leqslant C||x||, x \in \mathcal{H}\}. \end{aligned}$$

Also,

$$||Ax|| \leqslant ||x||.$$

We denote the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$. It is a subspace of $\mathcal{L}(\mathcal{H},\mathcal{K})$. They coincide when $\mathcal{H}$ and $\mathcal{K}$ are of finite dimension, of course. For $\mathcal{K}=\mathcal{H}$, $\mathcal{B}(\mathcal{H}):=\mathcal{B}(\mathcal{H},\mathcal{H})$. Note that $\mathcal{B}(\mathcal{H},\mathbb{F})$= all the bounded linear functionals on $\mathcal{H}$.

Now we introduce some example of bounded linear transformation.

Chapter 7

The Adjoint of an Operator

Definition 7.1

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a function $u : \mathcal{H} \times \mathcal{K} \to \mathbb{F}$ is a sesquilinear form if for $h, g \in \mathcal{H}$, $k, l \in \mathcal{K}$,and $\alpha, \beta \in \mathbb{F}$,
(1) $u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k)$;
(2) $u(h, \alpha k + \beta l) = \overline{\alpha} u(h, k) + \overline{\beta} u(h, l)$.

The prefix "sesqui" is used because the function is linear in one variable but (for $\mathbb{F} = \mathbb{C}$) only conjugate linear in the other. ("Sesqui" means "one-and -a-half.")

A sesquilinear form is bounded if there is a constant M such that $|u(h, k)| \leqslant M||h|| \cdot ||k|||$ for all h in $\mathcal{H}$ and k in $\mathcal{K}$. The constant M is called a bound for u.

Proposition 7.1: I

$A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $u(h, k) :=< Ah, k >$ is a bounded sesquilinear form.

Proof.

$$
\begin{aligned}
u(\alpha h + \beta g, k) &=< A(\alpha h + \beta g), k > \\
&=< \alpha Ah + \beta Ag, k > \\
&= \alpha < Ah, k > + \beta < Ag, k > \\
&= \alpha u(h, k) + \beta u(g, k),
\end{aligned}
$$

$$
\begin{aligned}
u(h, \alpha k + \beta l) &=< Ah, \alpha k + \beta l > \\
&= \overline{\alpha} < Ah, k > + \overline{\beta} < Ag, l > \\
&= \overline{\alpha} u(h, k) + \overline{\beta} u(g, l),
\end{aligned}
$$

$$
|u(h, k)| = | < Ah, k > | \leqslant ||Ah|| \cdot ||k|| \leqslant ||h|| \cdot ||k||.
$$

□

Also, if $B \in \mathcal{B}(\mathcal{H}, \mathcal{H})$, $u(h, k) =< h, Bk >$ is a bounded sesquilinear form. Are there any more? Are these two forms related?

Theorem 7.1

If $u : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ is a bounded sesquilinear form with bound M, then there are unique operators A in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and B in $\mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$u(h, k) =< Ah, k >=< h, Bk > \tag{7.1}$$

for all h in $\mathcal{H}$ and k in $\mathcal{K}$ and $||A||, ||B|| \leqslant M$.

Definition 7.2

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the unique operator B in $\mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying (7.1) is call the adjoint of A and is denote by $B = A^*$.

Proposition 7.2

If $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then U is an isomorphism iff U is invertible and $U^{-1} = U^*$.

Proof. U is isomorphism, then for $h, g \in \mathcal{H}$,

$$< h, g >=< Uh, Ug >=< h, U^*Ug > .$$

So, $U^*U = I$. Since, U is surjection, U is invertible and $U^{-1} = U^*$.

Conversely, let U be invertible with $U^{-1} = U^*$. Then, u is a surjection, and

$$< Ux, Uy >=< x, U^*Uy >=< x, Iy >=< x, y > .$$

□

From now on we will examine and prove results for the adjoint of operators in $\mathcal{B}(\mathcal{H})$. Often, as in the next proposition, there are analogous results for the adjoint of operators in $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

Proposition 7.3

If $A, B \in \mathcal{H}$ and $\alpha \in \mathbb{F}$, then:
(1) $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$.
(2) $(AB)^* = B^* A^*$.
(3) $A^{**} = (A^*)^* = A$.
(4) If A is invertible in $\mathcal{B}(\mathcal{H})$ and A^{-1} is its inverse, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proof. (1) $< (\alpha A + B)h, g >= \alpha < Ah, g > + < Bh, g >= \alpha < h, A^*g > + < h, B^*g >=< h, (\overline{\alpha} A^* + B^*)g >$.
(2)
(3)
(4) □

Proposition 7.4

If $A \in \mathcal{B}(\mathcal{H})$, $||A|| = ||A^*|| = \sqrt{||A^*A||}$.

Definition 7.3: I

$A \in \mathcal{B}(\mathcal{H})$, then
(1) A is hermitian or self-adjoint if $A^* = A$.
(2) A is normal if $AA^* = A^*A$.

In the analogy between the adjoint and the complex conjugate, hermitian operators become the analogues of real numbers and, unitaries are the analogues of complex numbers of modulus 1. Normal operators, as we shall see, are the true analogues of complex numbers. Notice that hermitian and unitary operators are normal.

Proposition 7.5

If $\mathcal{H}$ is a $\mathbb{C}$-Hilbert space and $A \in \mathcal{B}(\mathcal{H})$, then A is self-adjoint iff $< Ah, h > \in \mathbb{R}$ for all h in $\mathcal{H}$.

Proposition 7.6

If $A = A^*$, then

$$||A|| = \sup_{||h||=1} | < Ah, h > |.$$

Corollary 7.1

If $A = A^*$ and $< Ah, h >= 0$ for all h, then $A = 0$.

Proposition 7.7

If $\mathcal{H}$ is $\mathbb{C}$-Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ such that $< Ah, h >= 0$ for all h in $\mathcal{H}$, then $A = 0$.

Part III

Banach Space