THE SOUTH CHINA NORMAL UNIVERSITY

School of Mathematical Sciences

Numerical Analysis (2023-2024 The Second Term)

Homework 10:

Due Date: May 29, 2024 (Wednesday)

Name: 钟沛 Student No.: 2023021950 Date: May 23, 2024

§7.1 Exercises for Introduction to Quadrature

Exercise 1. Consider a general interval $[a, b]$. Show that Simpson's rule produces exact results for the functions $f(x) = x^2$, that is,

$$\int_a^b x^2 \mathrm{d}x = \frac{b^3}{3} - \frac{a^3}{3}.$$

Solve For Simpson's rule, $h = \frac{b-a}{2}$, and we get

$$\begin{aligned}
I &= \frac{h}{3}(f_0 + 4f_1 + f_2) \\
&= \frac{b-a}{6}(a^2 + 4(\frac{a+b}{2})^2 + b^2) - \frac{h^5}{90}f^{(4)}(c) \\
&= \frac{b-a}{6}(a^2 + (a^2 + 2ab + b^2) + b^2) \\
&= \frac{a^2b - a^3 + a^2b + 2ab^2 + b^3 - a^3 - 2a^2b - ab^2 + b^3 - ab^2}{6} \\
&= \frac{-2a^3 + 2b^3}{6} = \frac{b^3}{3} - \frac{a^3}{3}.
\end{aligned}$$

Using Newton Leibniz theorem, we get

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

Hence, $I = \int_a^b x^2 dx$ and so the Simpson's rule produces exact results for the functions $f(x) = x^2$. $\square$

§7.2 Exercises for Composite Trapezoidal and Simpson's Rule

Exercise 1.

(a) Verify that the trapezoidal rule ($M = 1, h = 1$) is exact for polynomials of degree $\leqslant 1$ of the form $f(x) = c_1 x + c_0$ over $[0, 1]$.

(b) Use the integrand $f(x) = c_2 x^2$ and verify that the error term for the trapezoidal rule ($M = 1, h = 1$) over the interval $[0, 1]$ is

$$E_T(f, h) = \frac{-(b-a) f^{(2)}(c) h^2}{12}.$$

Solve (a) For the trapezoidal rule, $h = 1$, and we get

$$\begin{aligned} I &= \frac{h}{2}(f_0 + f_1) \\ &= \frac{1}{2}(c_0 + c_1 + c_0) \\ &= \frac{c_1 + 2c_0}{2}. \end{aligned}$$

Using Newton Leibniz theorem, we get

$$\int_0^1 c_1 x + c_0 dx = \frac{c_1}{2} x^2 + c_0 x \mid_{x=0}^{x=1} = \frac{c_1 + 2c_0}{2}.$$

Hence, $I = \int_0^1 f(x) dx$ and so the trapezoidal rule is exact for polynomials of degree $\leqslant 1$ of the form $f(x) = c_1 x + c_0$ over $[0, 1]$.

(b) Integrating the Lagrange polynomial $P_1(x)$ and its remainder over $[0, 1]$ yields

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 P_1(x) dx + \int_0^1 \frac{x(x-1) f^{(2)}(c(x))}{2!} dx \\ &= \int_0^1 f_0 \frac{x-1}{0-1} + f_1 \frac{x-0}{1-0} dx + \int_0^1 \frac{x(x-1) f^{(2)}(c(x))}{2!} dx \\ &= \int_0^1 c_2 x dx + \int_0^1 \frac{x(x-1) f^{(2)}(c(x))}{2!} dx \\ &= \frac{c_2 x^2}{2} \mid_{x=0}^{x=1} + \int_0^1 \frac{x(x-1) f^{(2)}(c(x))}{2!} dx \\ &= \frac{c_2}{2} + \int_0^1 \frac{x(x-1) f^{(2)}(c(x))}{2!} dx. \end{aligned}$$

The term $x(x-1)$ does not change sign on $[x_0, x_1]$, and $f^{(2)}(c(x))$ is continuous. Hence the second Mean Value Theorem for integrals implies that there exits a value c so that

$$\begin{aligned}
\int_0^1 f(x)dx &= \frac{c_2}{2} + f^{(2)}(c) \int_0^1 \frac{x(x-1)}{2!} dx \\
&= \frac{c_2}{2} + f^{(2)}(c) \cdot (\frac{x^3}{6} - \frac{x^2}{4} \mid_{x=0}^{x=1}) \\
&= \frac{c_2}{2} + \frac{-f^{(2)}(c)}{12} \\
&= \frac{h}{2}(f_0 + f_1) + \frac{-(b-a)f^{(2)}(c)h^2}{12} \\
&= T(f,h) + E_T(f,h).
\end{aligned}$$

Hence the error term for the trapezoidal rule $(M = 1, h = 1)$ over the interval $[0,1]$ is $E_T(f,h) = \dfrac{-(b-a)f^{(2)}(c)h^2}{12}$. $\square$