THE SOUTH CHINA NORMAL UNIVERSITY

School of Mathematical Sciences

Numerical Analysis (2023-2024 The Second Term)

Homework 4:

Due Date: April 24, 2024 (Wednesday)

Name: 钟沛 Student No.: 2023021950 Date: April 17, 2024

§3.3 Upper-triangular Linear Systems

Exercise 1. Solve the upper-triangular system and find the value of the determinant of the coefficient matrix.

$$\begin{array}{rcrcrcrcr} 3x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 8 \\ & & 4x_2 & - & x_3 & + & 2x_4 & = & -3 \\ & & & & 2x_3 & + & 3x_4 & = & 11 \\ & & & & & & 5x_4 & = & 15 \end{array}$$

Solve Solving for x_4 in the last equation yields

$$x_4 = \frac{15}{5} = 3.$$

Using $x_4 = 3$ in the third equation, we obtain

$$x_3 = \frac{11 - 3 \cdot 3}{2} = 1.$$

Now $x_3 = 1$ and $x_4 = 3$ are used to find x_2 in the second equation:

$$x_2 = \frac{-3 - 2 \cdot 3 + 1}{4} = -2.$$

Finally, x_1 is obtained using the first equation:

$$x_1 = \frac{8 + 3 - 1 - 4}{3} = 2.$$

And the determinant of the coefficient matrix is

$$\begin{vmatrix} 3 & -2 & 1 & -4 \\ 0 & 4 & -1 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 3 \cdot 4 \cdot 2 \cdot 5 = 120.$$

$\square$

Exercise 2.

(a) Consider the two upper-triangular matrices

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}.$$

Show that their product $\boldsymbol{C} = \boldsymbol{AB}$ is also upper triangular.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $N \times N$ upper-triangular matrices. Show that their product is also upper triangular.

Solve (a) Since

$$\boldsymbol{C} = \boldsymbol{AB} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ 0 & a_{22}b_{22} & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & a_{33}b_{33} \end{pmatrix},$$

the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is also upper triangular.

(b) Consider the product $\boldsymbol{C} = (c_{ij})_{N\times N}$ of two upper-triangular matrices $\boldsymbol{A} = (a_{ij})_{N\times N}$ and $\boldsymbol{B} = (b_{ij})_{N\times N}$, c_{ij} with $i > j$ is given by

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^{n} a_{ik}b_{kj} = 0,$$

since $a_{ik} = 0$ when $k \leqslant i-1$ and $b_{kj} = 0$ when $k \geqslant i > j$. Hence $\boldsymbol{C}$ is an upper triangular. $\square$