Study Notes of Topology

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Contents

Preface

The notes mainly refer to the following materials:

- Topology without tears
- lecture notes from Toronto university
- lecture notes from ustc
- Basic Topology by You Chengye

Chapter 1

Preliminary Knowledge

1.1 Countability

1.2 Inverse Image and Direct Image

Definition 1.1

Let $f : X \to Y$ be a function, and let $U \subset Y$ be a subset. The inverse image(or, preimage) of U is the set $f^{-1}(U) \subset X$ consisting of all elements $a \in X$ such that $f(a) \in U$.

Proposition 1.1

$f : X \to Y$, $A \subseteq X$, $B \subseteq Y$.
(1) $A \subseteq f^{-1}(f(A))$ with equality if f is injective.
(2) $f(f^{-1}(B)) \subseteq B$ with equality if f is surjective.
(3) $B \subseteq f(A)$, then $f^{-1}(B) \subseteq A$ only when f is injective.

Proof. (3) $\forall x \in f^{-1}(B), f(x) \in B$. Since $B \subseteq f(A)$, $\exists y \in f(A)$, $f(x) = y \in f(A)$. Then $x \in f^{-1}(f(A))$. If f is injective, then $A = f^{-1}(f(A))$. Then $f^{-1}(B) \subseteq A$. $\square$

The inverse image commutes with all set operations:

1.3 Reference

- Countability: lecture notes from toronto
- Inverse images and direct images

1.4 Reference

Part I

Topology Space and Continuity

Chapter 2

Topological Space

This chapter opens with the definition of a topology and is then devoted to some simple examples.

Topology, like other branches of pure mathematics such group theory, is an axiomatic subjece. We start with a set of axioms and we use these axioms and we use these axioms to prove propositions and theorems. It is extremely important to develop your skill at writing proofs.

2.1 Topological Space

Definition 2.1

Let X be a non-empty set. A set $\tau \subseteq \mathcal{P}(X)$ is said to be a topology on $\mathcal{X}$ if
(1) $X, \emptyset \in \tau$,
(2) If $U_\alpha \in \tau(\alpha \in I$, I is finite or infinite), then $\cup_{\alpha \in I} U_\alpha \in \tau$,
(3) If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

Example 2.1: Trivial topology

Let X be any non-empty set and $\tau_t = \{X, \emptyset\}$. Then τ_t is called the trival topology on X.

(1) $X, \emptyset \in \tau$; (2) $X \cup \emptyset = X \in \tau$; (3) $X \cap \emptyset = \emptyset \in \tau$.

Example 2.2: Discrete topology

Let X be any non-empty set and $\tau_s = \mathcal{P}(X)$. Then τ_s is called the discrete topology on X.

(1) $X, \emptyset \in \tau$; (2) $\cup U_\alpha \in \tau$; (3) $U_1 \cap U_2 \in \tau$.

Example 2.3: Cofinite topology

Let X be any non-empty set and $\tau_f = \{U \subseteq X : U = \emptyset \; or \; U^c \; is \; finite\}$. Then τ_f is called the cofinite topology on X.

(1) $\emptyset \in \tau$, $X^c = \emptyset$ is finite with cardinality zero, then $X \in \tau$;

(2) If $U_\alpha \in \tau$ and $U_\alpha \neq \emptyset$ ($\emptyset$ has no effect on union). Let $U = \cup U_\alpha$, then $U^c = \cap U_\alpha^c$ is the intersection of finite set and so U^c is finite. Hence, $U \in \tau$;

(3) If $U_1, U_2 \in \tau$, let $U = U_1 \cap U_2$. Then $U^c = U_1^c \cup U_2^c$ is the union of finite set and so U^c is finite. Hence, $U \in \tau$.

Example 2.4: Cocountable topology

Let X be any non-empty set and $\tau_c = \{U \subseteq X : U = \emptyset \textit{ or } U^c \textit{ is countable}\}$. Then τ_c is called the cocountable topology on X.

(1) $\emptyset \in \tau$, $X^c = \emptyset$ is finite with cardinality zero, then $X \in \tau$;

(2) If $U_\alpha \in \tau$ and $U_\alpha \neq \emptyset$ ($\emptyset$ has no effect on union). Let $U = \cup U_\alpha$, then $U^c = \cap U_\alpha^c$ is the intersection of countable set and so U^c is countable. Hence, $U \in \tau$;

(3) If $U_1, U_2 \in \tau$, let $U = U_1 \cap U_2$. Then $U^c = U_1^c \cup U_2^c$ is the union of countable set and so U^c is countable. Hence, $U \in \tau$.

Example 2.5: Euclidean topology

$\tau_e = \{U : U = \cup_{(a,b)\in I}(a,b), a < b \in \mathbb{R}, I$ is a collection of open interval$\}$. I can be infinite, finite or zero. Then τ_e is called the euclidean topology on $\mathbb{R}$. We write $E^1 = (\mathbb{R}, \tau_e)$.

(1) $\emptyset$ = empty union. Then $\emptyset \in \tau$. For every $x \in \mathbb{R}$, there exists (a_x, b_x) s.t. $x \in (a_x, b_x)$, then $\mathbb{R} = \cup_{x\in\mathbb{R}}(a_x, b_x) \in \tau$.

(2) (3) refer to topology without tears page 51.

2.2 Metric Topology

The most important class of topological spaces is the class of metric spaces. Metric spaces provide a rich source of examples in topology. But more than this, most of the applications of topology to analysis are via metirc spaces.

Definition 2.2

Let X be a non-empty set and d a real-valued function defined on $X \times X$ such that for $x, y, z \in X$:
(1) $d(x,y) \geqslant 0$ and $d(x,y) = 0$ iff $x = y$;
(2) $d(x,y) = d(x,y)$;
(3) $d(x,z) \leqslant d(x,y) + d(y,z)$.
Then d is said to be a metric on X, (X,d) is called a metric space and $d(a,b)$ is referred to as the distance between a and b.

Example 2.6

$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) | x_i \in \mathbb{R}, i = 1, 2, ..., n\}$. We defined the metirc in $\mathbb{R}^n$ as

$$d((x_1, ..., x_n), (y_1, ..., y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

$(\mathbb{R}^n, d)$ is called n dimension euclidean space, denoted by E^n.

Definition 2.3

Let (X, d) be a metirc space and ϵ any positive real number. Then the open ball about $x_0 \in X$ of radius ϵ is the set $B(x_0, \epsilon) = \{x \in X : d(x_0, x) \le \epsilon\}$

Example 2.7

In $\mathbb{R}$ with the euclidean metric, $B(x_0, \epsilon)$ is the open interval $(x_0 - \epsilon, x_0 + \epsilon)$.

Lemma 2.1

Let (X, d) be a metric space and $x, y \in X$. Further, let ϵ_1 and ϵ_2 be positive real numbers. If $z \in B(x, \epsilon_1) \cap B(y, \epsilon_2)$, then there exists a $\epsilon > 0$ such that $B(z, \epsilon) \subseteq B(x, \epsilon_1) \cap B(y, \epsilon_2)$.

Proof. Let $\epsilon = \min\{\epsilon_1 - d(x, z), \epsilon_2 - d(y, z)\}$, then for $a \in B(z, \epsilon)$,

$$d(a, x) \leqslant d(a, z) + d(z, x) \leqslant \epsilon + d(x, z) = \epsilon_1,$$
$$d(a, y) \leqslant d(a, z) + d(z, y) \leqslant \epsilon + d(y, z) = \epsilon_2.$$

Hence, $a \in B(x, \epsilon_1) \cap B(y, \epsilon_2)$ and so $B(z, \epsilon) \subseteq B(x, \epsilon_1) \cap B(y, \epsilon_2)$. □

Corollary 2.1

Let (X, d) be a metric space and B_1 and B_2 open balls in (X, d). Then $B_1 \cap B_2$ is a union of open balls in (X, d).

Proof. By lemma2.1, $\forall z \in B_1 \cap B_2$, there exists $\epsilon_z > 0$ such that $B(z, \epsilon_z) \subseteq B_1 \cap B_2$. Then $\cup_{z \in B_1 \cap B_2} B(z, \epsilon_z) \subseteq B_1 \cap B_2 \subseteq \cup_{z \in B_1 \cap B_2} B(z, \epsilon_z)$. Hence, $B_1 \cap B_2 = \cup_{z \in B_1 \cap B_2} B(z, \epsilon_z)$ □

Proposition 2.1

Let (X, d) be a metric space. Then $\tau_d = \{U : U = \cup_\alpha B(x_\alpha, \epsilon_\alpha)\}$ is a topology on X.

Proof. (1) $\varnothing$ = empty union, then $\varnothing \in \tau_d$. $X = \cup_{x \in X} B(x, \epsilon_x) \in \tau_d$.
(2) The union of open ball union is open ball union.
(3) If $U, U' \in \tau_d$, then $U = \cup_\alpha B(x_\alpha, \epsilon_\alpha)$, $U' = \cup_\beta B(x_\beta, \epsilon_\beta)$, then

$$\begin{aligned} U \cap U' &= (\cup_\alpha B(x_\alpha, \epsilon_\alpha)) \cap (\cup_\beta B(x_\beta, \epsilon_\beta)) \\ &= \cup_{\alpha,\beta} (B(x_\alpha, \epsilon_\alpha) \cap B(x_\beta, \epsilon_\beta)). \end{aligned}$$

By corollary2.1, $B(x_\alpha, \epsilon_\alpha) \cap B(x_\beta, \epsilon_\beta)$ is the union of open ball. Then $U \cap U'$ is the union of open ball. □

τ_d is called the topology induced by metirc or simply metric topology.

2.3 Basic Conception in Topological Space

Rather than continually refer to "members of τ", we find it more convenient to give such sets a name. We call them "open sets". We shall also name the complements of open sets. They will be called "closed sets".

2.3.1 Open set

Definition 2.4

Let (X, τ) be any topological space. Then the members of τ are said to be open sets.

Proposition 2.2

Let U be a subset of a topological space (X, τ). Then $U \in \tau$ iff for each $x \in U$ there exists $U_x \in \tau$ such that $x \in U_x \subseteq U$.

Proof. ($\Rightarrow$): Since $U \in \tau$, for each $x \in U$, take $K = U$, then $x \in K \subseteq U$.
($\Leftarrow$): Since $U \subseteq \cup_{x \in U} U_x \subseteq U$, $U = \cup_{x \in U} U_x \in \tau$. □

Remark. This proposition provides a useful test of whether a set is open or not. It says that a set is open iff it contains an open set about each of its points.

2.3.2 Closed set

Definition 2.5

Let (X, τ) be a topological space. A subset A of X is said to be closed set in (X, τ) if its complements in X, denoted by A^c, is open in (X, τ).

Proposition 2.3

If (X, τ) is any topological space, then
(1) $\emptyset$ and X are closed set.
(2) the intersection of any (finite or infinite) number of closed sets is a closed set and
(3) the union of any finite number of closed sets is closed set.

Proof. 1 □

According to our experience in Euclidian space topology, a set is closed if and only if any convergence sequence will converge to a point in that set. A natural question is: In topological spaces, do we still have: a set is closed if and only if it contains all its sequential limits? The answer is No! You can see a counterexample from lecture notes from ustc page 2. From this example, we can know that in a topological space, you can't claim a set A is closed by proving "if $x_n \in A$ and $x_n \to x_0$, then $x_0 \in A$". But the necessity of the proposition is still true in topological sapce. And under the first countability, the sufficiency is true. All proof you can refer to lecture notes from ustc page 4-5.

2.3.3 Neighbourhood, interior point, interior

Definition 2.6

Let (X, τ) be a topological space, A a subset of X and x a point in X. Then If there exists an open set U such that $x \in U \subseteq A$, then x is called a interior point of A and A is called the neighbourhood of x. The collection of all interior point in A is called the interior of A, denoted by $\text{Int}(A)$.

Proposition 2.4

(1) $x \in \text{Int}(A) \Leftrightarrow \exists U \in \tau$ with $x \in U, U \cap A^c = \emptyset$.
(2) If $A \subset B$, then $\text{Int}(A) \subset \text{Int}(B)$;
(3) $\text{Int}(A)$ is the largest open subset of X contained in A;
(4) $\text{Int}(A)$ is the union of all open sets of X contained in A;
(5) $\text{Int}(A) = A$ iff A is open;
(6) $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$;
(7) $\text{Int}(A \cup B) \supset \text{Int}(A) \cup \text{Int}(B)$.

2.3.4 Limit point and closure, exterior, boundary

Definition 2.7

Let A be a subset of a topological space (X, τ). A point $x \in X$ is said to be a limit point (or accumulation point or cluster point) of A if every open set, U, containing x contains a point of $A \setminus \{x\}$, i.e. $\forall U \in \tau$ with $x \in U, U \cap A \setminus \{x\} \neq \emptyset$. The collection of all limit points of A is called derived set, denoted by A'. $\overline{A} := A \cup A'$ is called the closure of A.

Remark. From the definition of $\overline{A}$, we can get $x \in \overline{A} \Leftrightarrow \forall U \in \tau$ with $x \in U, U \cap A \neq \emptyset$.

The conception of limit point derived from Euclidean space. But we should note the current promotion conception has changing in meaning. In Euclidean space , finite sets have no limit points. However, in general topological space, finite set can do.

Example 2.8

Consider the topological space (X, τ) where the set $X = \{a, b, c, d, e\}$, the topology $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$, and $A = \{a, b, c\}$. Then b, d and e are limit points of A but a and c are not limit points of A.

The point x is a limit point of A iff every open set containing x contains another point of the set A. So to show x is a limit point of A, we should writing down all of the open sets containing x and verifying that each contains a point of A other than x. And to show that x is note a limit point of A, it suffices ot find even one open set which contains x but contains no other point of A.

The set $\{a\} \in \tau$ with $a \in \{a\}$, but $\{a\} \cap A \setminus \{a\} = \emptyset$. The set $\{c, d\} \in \tau$ with $c \in \{c, d\}$, but $\{c, d\} \cap A \setminus \{c\} = \emptyset$. Hence, a and c are not limit point of A.

The open sets containing b are X and $\{b, c, d, e\}$. Then $X \cap A \setminus \{b\} = a, c \neq \emptyset$ and

haven't done!

By the following proposition, we can know closure and interior are closely related.

Proposition 2.5

If $A = B^c$, then $\overline{A} = (\text{Int}(B))^c$.

Proposition 2.6

(1) $x \in \overline{A} \Leftrightarrow \forall U \in \tau$ with $x \in U, U \cap A \neq \emptyset$
(2) If $A \subset B$, then $\overline{A} \subset \overline{B}$;
(3) $\overline{A}$ is the smallest closed subset of X containing A;
(4) $\overline{A}$ is the intersection of all closed sets of X containing A;
(5) $\overline{A} = A$ iff A is closed;
(6) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
(7) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Example 2.9: Limit point in discrete topology

Let (X, τ_s) be a discrete space and A a subset of X. Then A has no limit points, since for each $x \in X$, $\{x\}$ is an open set containing no point of A different from x.

Example 2.10: Limit point in trival topology

Let (X, τ_t) be a trival space and A a subset of X with at least two elements. Every point of X is a limit point of A, since for each $x \in X$, $X \cap A \setminus \{x\} \neq \emptyset$. If A is single set $\{x\}$, then every point of X rather than x is a limit point of A.

Example 2.11: Limit point in cofinite topology

Let (X, τ_f) be a cofinite space and A a subset of X.
(1) If X is finite, then $\tau = \mathcal{P}(X)$. Then every point of X is not a limit point of A, since for $x \in X$, $\{x\} \cap A \setminus \{x\} = \emptyset$.
(2) If X is infinte and A is finite, every point of X is not a limit point of A, since $((X \setminus A) \cup$

$\{x\})^c \subset A$ is finite and $((X \setminus A) \cup \{x\}) \cap A \setminus \{x\} = \emptyset$.
(3) If X is infinte and A is infinite, then every point of X is the limit point of A.

Let's check (3). Firstly, we verify that for any $U \in \tau$, $U \cap A$ is infinite. Since $U \in \tau$, U^c is finite. And we have

$$A = A \cap (U \cup U^c) = (A \cap U) \cup (A \cap U^c).$$

Suppose $A \cap U$ is finite. Since $A \cap U^c$ is finite, A is the union of two finite sets. Then, A is finite. This is a contradiction as A is infinite. Hence, $A \cap U$ is infinite. Thus, $(U \cup \{x\}) \cap A \setminus \{x\} \neq \emptyset$. Hence, every point of X is the limit point of A.

Example 2.12: Limit point in cocountable topology

Let (X, τ_c) be a cocountable space and A a subset of X.
(1) If A is uncountable, then every point of X is a limit point of A.
(2) If A is countable or finite, then A contains all its limit points.(that is, A is closed).

(1) For any $U \in \tau$, U^c is countable. Then $(U \cup \{x\})^c = U^c \cap \{x\}^c \subseteq U^c$ is countable. Then $U \cup \{x\} \in \tau$. Suppose $(U \cup \{x\}) \cap A \setminus \{x\} = \emptyset$. Then $A \setminus \{x\} \subseteq U^c$ and so A is countable. So if A is uncountable, it is bound to $(U \cup \{x\}) \cap A \setminus \{x\} \neq \emptyset$. Hence, $x \in X$ is a limit point of A.

(2) For any $U \in \tau$, U^c is countable. Since $(A^c)^c = A$ is countable, $A^c \in \tau$. Then A is closed. Then $A' \subseteq A$. Hence, A contains all its limit points.

Example 2.13: Limit point in euclidean topology

Let $(\mathbb{R}, \tau_e)$ be a euclidean space and $\langle a, b\rangle$ a subset of $\mathbb{R}$. ($\langle a, b\rangle$ is any case in $(a, b), (a, b], [a, b), [a, b]$). The point in $[a, b]$ is the limit point of $\langle a, b\rangle$.

Definition 2.8

Let (X, τ) be a topological space and A a subset of X. Then exterior of A

$$\text{Ext}(A) = \text{Int}(A^c).$$

Proposition 2.7

(1) $x \in \text{Ext}(A) \Leftrightarrow \exists U \in \tau$ with $x \in U, U \cap A \neq \emptyset$.
(2) $\text{Ext}(A) = (\overline{A})^c$.

Definition 2.9

Let (X, τ) be a topological space and A a subset of X. The boundary of A consistis of all the points in $\overline{A}$ but not in $\text{Int}(A)$. Thus, the boundary of A

$$\partial A := \overline{A} \setminus \text{Int}(A).$$

Proposition 2.8

(1) $x \in \partial A \Leftrightarrow \forall U$ with $x \in U, A \cap U \neq \emptyset$ and $A^c \cap U \neq \emptyset$.
(2) $\partial A = \overline{A} \cap \overline{A^c}$
(3) $\partial A = A \setminus (\text{Int}(A) \cup \text{Ext}(A))$

Proposition 2.9

$A = \text{Int}(A) \cup \text{Ext}(A) \cup \partial(A)$.

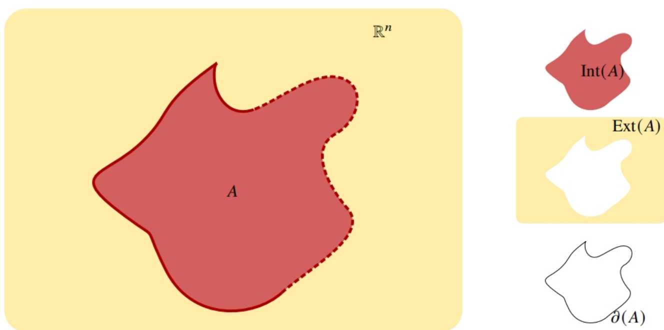

Figure 2.1

Example 2.14

Let $X = \{a, b, c, d, e\}$ and

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Show that $\overline{\{b\}} = \{b, e\}$.

Definition 2.10

Let A be a subset of a topological space (X, τ). Then A is said to be dense in X if $\overline{A} = X$. X is said to be separable if there exists countable dense subset in X.

Proposition 2.10

Let A be a subset of a topological space (X, τ). Then A is dense in X iff every non-empty open subset of X intersects A non-trivially, i.e. if $U \in \tau$ and $U \neq \emptyset$ then $A \cap U \neq \emptyset$.

Example 2.15

(X, τ_s) is separable iff X is countable.

refer to website proof.

Example 2.16

(X, τ_t) is separable.

Let A be a countably infinite subset of X. By example2.10, $\overline{A} = X$. Hence, A is dense in X and so X is separable.

Example 2.17

(X, τ_f) is separable.

Let A be a countably infinite subset of X. By example2.11, $\overline{A} = X$. Hence, A is dense in X and so X is separable.

Example 2.18

(X, τ_c) is inseperable.

Let A be a countably infinite subset of X.By example2.12, $\overline{A} = A$. Hence, X have no countably dense subset and so inseperable.

Example 2.19

$(\mathbb{R}, \tau_e)$ is separable.

prove that $\overline{\mathbb{Q}}$=$\mathbb{R}$.

2.3.5 Sequence convergence

Definition 2.11

Let (X, τ) be a topological space and A a subset of X. Let $(x_n)_{n\in\mathbb{N}}$ be an infinite sequence in A. Then (x_n) converages to the limit $x \in X$(denoted by $x_n \to x$) iff

$$\forall U \in \tau \text{ with } x \in U \Rightarrow \{n \in \mathbb{N} : x_n \notin U\} \text{ is finite.}$$

Or,

$$\forall U \in \tau \text{ with } x \in U \Rightarrow \exists N \in \mathbb{N}, \forall n > N, x_n \in U.$$

In euclidean space, the convergence point of a convergent sequence is unique but this is not true in cocountable space. And in euclidean space, when x is the limit point of the set A, there is a sequence (x_n) in A, which converges to x. But this is not ture in topological space, you can recall the example2.8, in which there is no sequence converges to the limit point.

Proposition 2.11

Let (x_n) be a sequence which elements are different in (R, τ_f), then $\forall x \in X$, $x_n \to x$.

Proof. $\forall U \in \tau$ with $x \in U \Rightarrow \{n \in \mathbb{N} : x_n \notin U\} = \{n \in \mathbb{N} : x_n \in U^c\}$ is finite. □

Proposition 2.12

Let (x_n) be a sequence which elements are different in (R, τ_f), then

$$x_n \to x \Leftrightarrow \exists N \in \mathbb{N}, \forall n > N, x_n = x.$$

Proof. $(\Leftarrow)$ clear by definition2.11
$(\Rightarrow)$ Consider the set $B := \{x_n : x_n \neq x\}$. Since a sequence is countable, B is countable. By example2.12, B is closed. By construction, $x \notin B$, so $U = X \setminus B$ is an open set containing x. But $x_n \to x$, so a tail of this sequence must lie in $X \setminus B$. Since $\{x_n\} \cap (X \setminus B) = \{x\}$, this means that a tail of this sequence is constanst. □

2.4 Subspace

Definition 2.12

Let A be a non-empty subset of a topological space $(X\ \tau)$. The collection

$$\tau_A = \{O \cap A : O \in \tau\}$$

of subsets of A is a topology on A called the subspace topology (or the topology induced on A by τ). The topological space (A, τ_A) is said to be a subspace of (X, τ).

Let's check that τ_A is indeed a topology on A.
(1) $A = X \cap A, \emptyset = \emptyset \cap A$, then $A, \emptyset \in \tau_A$.
(2) If $U_\alpha \in \tau_A$, then $\cup_\alpha U_\alpha = \cup_\alpha (O_\alpha \cap A) = (\cup_\alpha O_\alpha) \cap A \in \tau_A$.
(3) If $U_1, U_2 \in \tau_A$, then $U_1 \cap U_2 = (O_1 \cap A) \cap (O_2 \cap A) = (O_1 \cap O_2) \cap A \in \tau_A$.

In the following content, we follow the convention: a subset of topological Spaces is treated as a subspace.

Proposition 2.13

Let (X, τ) be a topological space and $B \subseteq A \subseteq X$. Then $(\tau_A)_B = \tau_B$.

$$\begin{aligned}
(\tau_A)_B =& \{K \cap B : K \in \tau_A\} \\
=& \{(O \cap A) \cap B : O \in \tau\} \\
=& \{(O \cap B) \cap (A \cap B) : O \in \tau\} \\
\overset{B \subseteq A}{=}& \{(O \cap B) \cap B : O \in \tau\} \\
=& \{(O \cap B) : O \in \tau\} \\
=& \tau_B.
\end{aligned}$$

Hence, there are two way to induce the topology on B: induced by the topology on A or induced by the topology on X.

Example 2.20

Let $X = \{a, b, c, d, e, f\}$,

$$\tau = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\},$$

and $A = \{b, c, e\}$. Then the subspace topology on A is

$$\tau_A = \{A, \varnothing, \{c\}\}.$$

Consider the subset $[1, 2]$ of $(\mathbb{R}, \tau_e)$. Then the topology on $[1, 2]$ is

$$\tau = \{(a, b) \cap [1, 2] : (a, b) \in \tau_e\}.$$

But here we see some surprising things happening; e.g. $[1, \frac{3}{2})$ is certainly not an open set in $\mathbb{R}$, but $[1, \frac{3}{2}) = (1, \frac{3}{2}) \cap [1, 2]$, is an open set in the subspace $[1, 2]$.

Also $(1, 2]$ is not open in $\mathbb{R}$ but is open in $[1, 2]$. Even $[1, 2]$ is not open in $\mathbb{R}$, but is an open set in $[1, 2]$.

So whenever we speak of a set being open we must make perfectly clear in what space or what topology it is an open set.

Proposition 2.14

Let (X, τ) be a topological space and $C \subset A \subset X$, then

$$C \text{ is closed in } A \Leftrightarrow C = A \cap V, \text{ where } V \text{ is closed in } X.$$

Proof.

$$\begin{aligned} C \text{ is closed in } A &\Leftrightarrow A \setminus C \text{ is open in } A \\ &\Leftrightarrow \exists O \in \tau_X, \text{ s.t. } A \setminus C = O \cap A \\ &\Leftrightarrow \exists O \in \tau_X, C = A \setminus (O \cap A) \\ &\qquad = (A \setminus O) \cup (A \setminus A) \\ &\qquad = (A \setminus O) \cup \varnothing = A \setminus O \\ &\qquad = (A \cap X) \setminus O \\ &\qquad = A \cap (X \setminus O) \\ &\qquad = A \cap V, \text{ where } V \text{ is closed in } X. \end{aligned}$$

□

Proposition 2.15

Let (X, τ) be a topological space, $B \subset A \subset X$, then
(1) If B is open(closed) in X, then B is open(closed) in A;
(2) If A is open(closed) in X and B is open(closed) in A, then B is open(closed) in X.

Proof. (1) We know that $B = B \cap A$. If $B = O$, then $B = O \cap A$ and so B is open in A. If B is closed in X, by proposition**??**, $B = X \cap V$, where V is closed in X. Then $B = B \cap A = (X \cap V) \cap A = (X \cap A) \cap V$, by by proposition**??**, B is closed in A.
(2) If $B = O_1 \cap A$ and $A = O_2$, then $B = O_1 \cap O_2 \in \tau$ and so B is open in X. If A is closed in X and B is closed in A, then $A = X \cap V_1, B = A \cap V_2$, where V_1, V_2 is closed in X. Then, $B = X \cap (V_1 \cap V_2)$. As $V_1 \cap V_2$ is closed in X, B is closed in X. □

Corollary 2.2

Let subspace A be closed in X. Then $C \subset A$ is closed in A iff C is closed in X.

Proof. ($\Rightarrow$): By proposition2.14, $\exists V$ which is closed in X such that $C = A \cap V$. Since A and V are closed in X, it follows that $C = A \cap V$ is closed in X.
($\Leftarrow$): By proposition2.15. □

Proposition 2.16

Let (X, τ) be a topological space and $A \subset Y \subset X$. Then $\mathrm{cl}_Y(A) = A \cap \mathrm{cl}_X(A)$.

Proof. $\mathrm{cl}_Y = \cap\{K \subseteq Y : A \subseteq K, k \text{ is closed in} Y\}$

□

2.5 Reference

- lecture notes from uky
- lecture notes from ustc

2.6 Exercise

Chapter 3

Continuous Mappings and Homeomorphisms

3.1 Continuous Mappings

We are already familiar with the notion of a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at $x_0 \in \mathbb{R}$ iff each positive real number ϵ, there exists a positive real number δ such that $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$.

It is not all obvious how to generalize this definition to general topological spaces where we do not have "absolute value" or "subtraction". So we shall seek another(equivalent) definition of continuity which lends itself more to generalization.

It is easily seen that: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ iff for each interval $(f(x_0) - \epsilon, f(x_0) + \epsilon)$, for $\epsilon > 0$, there exists a $\delta > 0$ such that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

This definition is an improvement since it does not involve the concept "absolute value" but it still involves "substraction". The next definition shows how to avoid substraction.

Definition 3.1

Let (X, τ) and (Y, τ') be topological spaces and f a function from X into Y. Then f is continuous at $x_0 \in X$ iff for each $U \in \tau'$ containing $f(x_0)$, there exists $K \in \tau$ containing x_0, such that $f(K) \subseteq U$.

Use neighborhood to describe

Proposition 3.1

Let (X, τ) and (Y, τ') be topological spaces and f a function from X into Y. Then f is continuous at $x_0 \in X$ iff for any neighborhood N of $f(x_0)$ in Y, f^{-1} is the neighborhood of x_0.

Proof. □

Definition 3.2

Let (X,τ) and (Y,τ') be topological spaces and f a function from X into Y. Then f is continuous iff for each $x_0 \in X$ and for each $U \in \tau'$ containing $f(x_0)$, there exists $K \in \tau$ containing x_0, such that $f(K) \subseteq U$.

As in analysis, continuity is a local concept.

Proposition 3.2

Let (X,τ) and (Y,τ') be topological spaces and f a function from X into Y , A a subset of X and $x_0 \in A$. We define the restriction of f on A as $f_A = f|A : A \to Y$, then
(1) If f is continuous at x_0, then f_A is continuous at x_0.
(2) When A is open in X, if f_A is continuous at x_0, then f is continuous at x_0.

Proof. (1) We need to prove for each $U \in \tau'$ with $f_A(x_0) \in U$, there exists $O \in \tau_A$ with $x_0 \in O$, $f_A(O) \subseteq U$. f is continuous at x_0 and $x_0 \in A$, then for each $U \in \tau'$ with $f(x_0) = f_A(x_0) \in U$, there exists $K \in \tau$ with $x_0 \in K$, $f(K) \subseteq U$. Since $A \cap K \in \tau_A$ with $x_0 \in A \cap K$ and $f_A(A \cap K) = f(A \cap K) \subseteq f(A) \cap f(K) \subseteq Y \cap U = U$. Hence, f_A is continuous at x_0.

(2) f_A is continuous at x_0, then for each $U \in \tau'$ containing $f_A(x_0) = f(x_0)$, there exists $(K \cap A) \in \tau_A(K \in \tau)$ containing x_0, such that $f_A(K \cap A) \subseteq U$. Since $A \in \tau$, $K \cap A \in \tau$ and $f(A \cap K) = f_A(A \cap K) \subset U$. Hence, f is continuous at x_0. □

Definition 3.3

Let f be a function from a set x into a set Y. If S is any subset of Y, then the set $f^{-1}(S)$ is defined by

$$f^{-1}(S) = \{x : x \in X \text{ and } f(x) \in S\}.$$

Then subset $f^{-1}(S)$ of X is said to be the inverse image of S.

Remark. Note that an inverse function of f exists iff f is bijective. But the inverse image of any subset of Y exists even if f is neither one-to-one nor onto.

Proposition 3.3

Let f be a mapping of a topological space (X,τ) into a topological space (Y,τ'). Then the following conditions are equivalent:
(1) f is continuous;
(2) for each $U \in \tau'$, $f^{-1}(U) \in \tau$;
(3) for each closed set V in Y, $f^{-1}(Y)$ is closed in X.

Proof. □

In $(\mathbb{R},\tau_e)$, we can use sequence convergence to characterize continuity, but in general topological space, we cannot do this.

Proposition 3.4

$f: X \to Y$ is continuous at $x_0 \in X$, then $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.

Proof. f is continous at $x_0 \in X$ and $x_n \to x_0$, then $\forall U \in \tau_Y$ containing $f(x_0)$, there exists $K \in \tau_X$ containing x_0 such that $f(K) \subseteq U$. Since $x_n \to x_0$, $\exists N \in \mathbb{N}$, $\forall n > N$, $x_n \in K$, then $f(x_n) \in f(K) \subseteq U$. So $f(x_n) \to f(x_0)$. □

However, the inverse proposition is not true. Let $f: X \to Y$ be injective, X be a uncountable space with τ_c and Y be a discrete space. Then, by proposition2.12, When $x_n \to x_0$ in X, $\exists N \in N$, $\forall n > N$, $x_n = x$, then $f(x_n) = f(x)$ and so $f(x_n) \to f(x)$. But f is not continuous at x_0, because for $U = \{f(x_0)\} \in \tau_Y$ containing $f(x_0)$, $\forall K \in \tau_X$ containing x_0, $f(K) \supset \{f(x_0)\}$ as f is injective and $K \supset \{x_0\}$.

3.2 The properties of continuous mapping

Firstly, we introduce some simple and common continuous mappings.

Proposition 3.5

Identity mapping $\mathrm{id}: X \to X$ is continuous.

Definition 3.4

Let X be a topological space and $A \subset X$. Then inclusion mapping $i_A : A \to X$ is the mapping defined as:

$$i_A : A \to X : \forall x \in A : i_A(x) = x$$

Proposition 3.6

Let X be a topological space and $A \subset X$. Then inclusion mapping $i_A : A \to X$ is continuous.

Proof. For any open set U in X, $i^{-1}(U) = U \cap A$ is open in A. □

3.3 Homemorphism

Definition 3.5

Let (X, τ) and (Y, τ') be topological spaces, and let $f: X \to Y$ be a bijection. f is said to be a homeomorphism if f is continous and its inverse f^{-1} is continous.
In this case we say that (X, τ) and (Y, τ') are homeomorphism, and write $(X, \tau) \cong (Y, \tau')$, or more often simply $X \cong Y$ if the topologies are understood from context.

Proposition 3.7

Let (X, τ) and (Y, τ') be a topological spaces, and let $f : X \to Y$ be a bijection, Then the following are equivalent.
(1) f is a homeomorphism.
(2) f is continous and open.
(3) f is continuous and closed.
(4) $U \subset X$ is open iff $f(U) \subset Y$ is open.

Definition 3.6: L

t X and Y be topological spaces. Suppose $f : X \to Y$ is an injective continous mapping. If the function $f' : X \to f(X)$ obtained by restricting the range of f is a homeomorphism, then the map $f : X \to Y$ is called a topological embedding.

3.4 Exercise

Exercise 3.1

Let f be a mapping from X to Y, the following statements are equivalent:
(1) f is continous;
(2) $\forall A \subseteq X$, $f(\overline{A}) \subset \overline{f(A)}$;
(3) $\forall B \subseteq Y$, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

Proof. (1)$\Rightarrow$(2): f is continous. Since $\overline{f(A)}$ is closed in Y, then $f^{-1}(\overline{f(A)})$ is closed in X. Since $f(A) \subseteq \overline{f(A)}$, $A \subseteq f^{-1}(\overline{f(A)})$. Since $\overline{A}$ is the smallest closed set containing A, $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Then, $f(\overline{A}) \subseteq f(f^{-1}(\overline{f(A)})) \subseteq \overline{f(A)}$.

□

Exercise 3.2

$f : X \to Y$ is called open(closed) mapping, if $f(X)$ is open(closed). Illustrate that open mapping may not be closed mapping and vice versa.

Proof. □

Exercise 3.3: I

$f : X \to Y$ is bijective, then

$$f \text{ is open mapping } \Leftrightarrow f \text{ is closed mapping } \Leftrightarrow f^{-1} \text{ is continuous.}$$

Proof. $\forall U \in \tau_X$,

$$\begin{aligned} f(U) \in \tau_Y &\Leftrightarrow Y \setminus f(U) \text{ is closed} \\ &\overset{f \text{ is bijective}}{\Leftrightarrow} f(X \setminus U) \text{ is closed} \\ &\Leftrightarrow f \text{ is closed mapping.} \end{aligned}$$

Since f is bijective, f^{-1} exists. As f is open mapping, f^{-1} is continuous. $\square$

Remark. From this exercise, we can know if f is bijective, continuous and open, then f is homeomorphism.

3.5 Reference

- Homeomorphisms
-

Chapter 4

Topological basis and Product Space

4.1 Topological basis

Let's recall the euclidean topology in $\mathbb{R}$,

$$\tau_e = \{U : U = \cup_{(a,b)\in I}(a,b), a < b \in \mathbb{R}, I \text{ is a collection of open interval}\}.$$

It seems like the entire collection of sets in τ_e can be specified by declaring that just the usual open intervals are open. Once these "special sets" are known to be open, we get all the other sets for free by taking unions. These special collections of sets are called bases of topologies.

Definition 4.1

Let (X, τ) be a topological space. A collection $\mathcal{B}$ of subsets of X is said to be a basis for the topology τ if for each $U \in \tau$, $U = \cup_{B\in I}B$, where $I \subseteq \mathcal{B}$.

If we construct a set $\overline{\mathcal{B}} = \{U_{B\in I}B : I \subseteq \mathcal{B}\}$, then we can get a improved definition.

Definition 4.2

Let (X, τ) be a topological space. A collection $\mathcal{B}$ of subsets of X is said to be a basis for the topology τ if $\overline{\mathcal{B}} = \tau$.

Proposition 4.1

Let (X, τ) be a topological space. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for τ if and only if
(1) $\mathcal{B} \subseteq \tau$,
(2) for any $U \in \tau$ and any $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proof. ($\Rightarrow$): Suppose $\overline{\mathcal{B}} = \tau$, then $\mathcal{B} \subseteq \overline{\mathcal{B}} \subseteq \tau$. Since $\tau \subseteq \overline{\mathcal{B}}$, then take $B = U$, $x \in B \subseteq U$.
($\Leftarrow$): (1) implies $\overline{\mathcal{B}} \subseteq \tau$, (2) implies $U \subseteq \cup_{B_x} B_x \subseteq U$ and so $U = \cup_{B_x} B_x \in \overline{\mathcal{B}}$. Hence, $\overline{\mathcal{B}} \subseteq \tau$ and so $\overline{\mathcal{B}} = \tau$. □

Example 4.1

$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for $(\mathbb{R}, \tau_e)$.

Example 4.2

$\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for (X, τ_s).

Observe that $\tau = \mathcal{P}(X)$ is also a basis for the discrete topology on X. Therefore, there can be many different bases for the same topology. indeed if $\mathcal{B}$ is a basis for a topology τ on a set X and $\mathcal{B}_1$ is a collections of subsets of X such that $\mathcal{B} \subseteq \mathcal{B}_1 \subseteq \tau$, then $\mathcal{B}_1$ is also a basis for τ.

The above content show us when $\mathcal{B}$ is basis for a given topology in X. Now we consider when $\mathcal{B}$ is basis for a topology in X.

Proposition 4.2

Let X be a non-empty set and $\overline{\mathcal{B}}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology on X iff $\overline{B}$ is a topology.

Proposition 4.3: Basis for a topology

Let X be a non-empty set and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology on X iff $\mathcal{B}$ has the following properties:
(1) $X = \cup_{B\in\mathcal{B}} B$, and
(2) for any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 = \cup_{B\in I} B$, where $I \subseteq \mathcal{B}$.

Proof. ($\Rightarrow$): $\overline{\mathcal{B}}$ is a topology, then $X \in \overline{\mathcal{B}}$ and so $X = \cup_{B\in I} B$. Since $B \in \mathcal{B}$ is contained in X, $X \subseteq \cup_{B\in\mathcal{B}} \subseteq X$. Hence, $X = \cup_{B\in\mathcal{B}} B$. If $B_1, B_2 \in \mathcal{B}$, then $B_1, B_2 \in \overline{\mathcal{B}}$ and so B_1, B_2 is open. Then $B_1 \cap B_2$ is open and so contained in $\overline{\mathcal{B}}$.

($\Leftarrow$): We should show that $\overline{\mathcal{B}}$ is a topology. By (1), $X \in \overline{\mathcal{B}}$. $\emptyset$ is the empty union of members in $\overline{\mathcal{B}}$. Hence, $\emptyset \in \overline{\mathcal{B}}$. For $U_J = \{U \in J : J \subseteq \overline{\mathcal{B}}\}$, U is the union of members in $\mathcal{B}$. Then $\cup_{U\in U_J} U$ is the union of members in $\mathcal{B}$ and so belong to $\overline{\mathcal{B}}$. If $U_1, U_2 \in \overline{\mathcal{B}}$, then $U_1 = \cup_{B_1\in I_1} B_1, U_2 = \cup_{B_2\in I_2} B_2$. Then $U_1 \cap U_2 = (\cup_{B_1\in I_1} B_1) \cap (\cup_{B_2\in I_2} B_2) = \cup_{B_1\in I_1}(B_1 \cap (\cup_{B_2\in I_2} B_2)) = \cup_{B_1\in I_1, B_2\in I_2}(B_1 \cap B_2)$. By (2), $B_1 \cap B_2$ is the union of members in $\mathcal{B}$, thus $U_1 \cap U_2$ is the union of members in $\mathcal{B}$ and so contained in $\mathcal{B}$. Hence, $\overline{\mathcal{B}}$ is a topology. □

Haven't done! Add the content about basis for a given topology.

4.2 Product space

Part II

Topology Invariant

Chapter 5

THE AXIOMS OF COUNTABILITY

Now we turn to countability features in topology. In topology, an axiom of countability is a topological property that asserts the existence of a countable set with certain properties. There are several different topological properties describing countability.

5.1 First countable spaces

Definition 5.1

Let (X, τ) be a topological space and x be an element of X. A neighborhood base at x is a set $\mathcal{U}$ of neighborhood N of x such that:

$$N \text{ is a neighborhood of } x \Rightarrow \exists U \in \mathcal{U} : U \subseteq N.$$

Definition 5.2

A topological space (X, τ) is called first countable, or an C_1 space, if for every point in X has a countable neighborhood base, i.e. for each $x \in X$, there exists a countable family $\mathcal{U} = \{U_i^x : i \in \mathbb{N}\}$, where U_i^x is neighborhood of x, such that every neighborhood N of x, there exists n s.t. $U_n^x \subseteq N$.

Here are some examples of C_1-spaces and non-C_1-spaces.

Example 5.1

Any metric space is first countable since we can take $U_n^x = B(x, \frac{1}{n})$.

Example 5.2

$(\mathbb{R}, \tau_c)$ is not first countable: for any countable family $\mathcal{U} = \{U_i^x : i \in \mathbb{N}\}$ of open sets containing x, $\cup_{U \in \mathcal{U}} U^c$ is countable. Take $y \notin \cup_{U \in \mathcal{U}} U^c$ and $y \neq x$, then $\forall U \in \mathcal{U}, y \in U$. Then $\mathbb{R} \setminus \{y\}$ is a open set containing x, but it doen't contain any $U \in \mathcal{U}$.

Example 5.3

$(\mathbb{R}, \tau_f)$ is not first countable.

Proposition 5.1

Let (X, τ) be a topological space. If $x \in X$ has a countable neighborhood base $\mathcal{U} = \{U_i^x\}$, then x has a countable neighborhood base $\{V_i^x\}$, which satisfys $V_m^x \subset V_n^x$ when $m > n$.

Proof. If one has a countable neighborhood base $\{U_i^x : i \in \mathbb{N}\}$ at x, then one can take Then $\{V_i^x : i \in \mathbb{N}\}$ is countable. $V_n^x = \cap_n U_n^x$. Since $V_n^x \subset U_n^x$, $\{V_i^x : i \in \mathbb{N}\}$ is a neighborhood base at x and $V_m^x \subset V_n^x$ when $m > n$. □

Proposition 5.2

Let (X, τ) be first countable and A a subset of X, then

$$x \in \overline{A} \Leftrightarrow \exists \{x_n\} \subset A, \text{ s.t. } x_n \to x.$$

Proof. ($\Rightarrow$): By propsition5.1, we can take a countable neighborhood base $\{V_i^x\}$ which satisfys $V_m \subseteq V_n$ when $m > n$. Since $x \in A$, by the property of closure, $V_n \cap A \neq \emptyset$. We take $x_n \in V_n \cap A$, then we get a sequence $\{x_n\} \subset A$. Since $\{V_n^x\}$ is a neighborhood base, for any neighborhood U^x of x, there exsits n such that $x \in V_n^x \subseteq U^x$. Then $\forall m \geqslant n$, $V_m^x \subseteq U^x$. Then $\forall m \geqslant n$, $x_m \in U^x$. By definition2.11, $x_n \to x$. □

Corollary 5.1

Let (X, τ) be first countable and A a subset of X, then

$$A \text{ is closed } \Leftrightarrow \text{for any sequence } \{x_n\} \subset A \text{ with } x_n \to x, \text{one has } x \in A.$$
$$i.e. A \text{ contains all its sequential limits.}$$

Proof. ($\Rightarrow$): If A is closed, then A contains all its limits including sequential limits.
($\Leftarrow$): $\forall x \in \overline{A}$, by proposition5.2, $\exists \{x_n\} \subset A$, s.t. $x_n \to x$. Then by the condition, $x \in A$. Then, $\overline{A} \subseteq A$. Since $A \subseteq \overline{A}$, $A = \overline{A}$ and so A is closed. □

Corollary 5.2

Suppose (X, τ) is first countable and f is a map from X to Y.

$$f : X \to Y \text{ is continuous at } x_0 \in X \text{ iff } x_n \to x_0 \text{ implies } f(x_n) \to f(x_0).$$

Proof. ($\Rightarrow$): proof of proposition3.4.
($\Leftarrow$): Suppose f is not continous at x_0. Then there exists a open set U containing $f(x_0)$, for all open set K containing x_0, $f(K) \not\subseteq U$, which means $K \cap (f^{-1}(U))^c \neq \emptyset$. Then by the property of closure, $x_0 \in \overline{(f^{-1}(U))^c}$. Then by proposition 5.2, $\exists \{x_n\} \in \overline{(f^{-1}(U))^c}$, $x_n \to x$. Then by the condition, $f(x_n) \to f(x_0)$. Then by the definition of convergent sequence, all most $f(x_n) \in U$.

But $f(x_n) \in f(\overline{(f^{-1}(U))^c}) \subseteq \overline{f((f^{-1}(U))^c)} = \overline{f(f^{-1}(U^c))} \subseteq \overline{U^c}$, a contradiction. Hence, f is continuous at x_0. $\square$

5.2 Second countable spaces

Definition 5.3

A topological space (X, τ) is called second countable, or an C_2 space, if it has a countable topological base, i.e. there exists a countable family $\{U_i\}$ of open sets such that, for each $x \in X$, every open set U containing x, there exists n s.t. $x \in U_n \subseteq U$.

Remark. In C_1, the open set family is related to x. But, In C_2, the open set family doesn't need. Obviously any second countable space is a first countable space. But the converse is not true, for example, $(\mathbb{R}, \tau_s)$ is first countable as it is a metric space, but it is not second countable. Since every base for a discrete topology must include all singleton sets (since for each $x \in X$, the set $\{x\}$ is an open neighborhood of x, and so if $\mathcal{B}$ is any base for X, then there is a $B \in \mathcal{B}$ with $x \in B \subsetneq \{x\}$, which implies $B = x \in \mathcal{B}$). Since $\mathbb{R}$ is not countable, there is no a conntable family of open sets.

5.3 Separable spaces

Proposition 5.3

Any second countable topological space is separable.

Proof. It is equivalent to prove the space has a coutable dense subset. Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis of (X, τ). For each n, we choose a point $x_n \in U_n$ and let $A = \{x_n : n \in \mathbb{N}\}$. Then A is a countable subset in X. We claim that $\overline{A} = X$. In fact, for any $x \in X$ and any open neighborhood U of x, there exists n s.t. $x \in U_n \subseteq U$. In particular, $U \cap A \neq \emptyset$. So we get $\overline{A} = X$. $\square$

Proposition 5.4

A metric sapce is second countable iff it is separable.

Proof. ($\Rightarrow$): proof of proposition5.3.
($\Leftarrow$): Suppose that (X, d) is a separable metric space. Then X has a countable dense subset A. So

$$\mathcal{B} = \{B(a, \frac{1}{n}) : a \in A, n \in \mathbb{N}\}$$

is a countable family of open balls in X. Show that $\mathcal{B}$ is a base for the metric topology on X. In other words, show that if U is a non-empty open set in X, and $x \in U$, then $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$. Since there is some $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$; thus, you need only show that there is some $B(a, \frac{1}{n}) \in \mathcal{B}$ such that $x \in B(a, \frac{1}{n}) \subseteq B(x, \epsilon)$. We take $n > \frac{2}{\epsilon}$ and $a \in A$ such that $d(x, a) < \frac{1}{n}$, which means $x \in B(a, \frac{1}{n})$. For $y \in B(a, \frac{1}{n})$, $d(a, y) < \frac{1}{n}$. Then $d(x, y) < d(x, a) + d(a, y) < \frac{2}{n} < \epsilon$. So $y \in B(x, \epsilon)$ and $B(a, \frac{1}{n}) \subseteq B(x, \epsilon)$. Hence, (X, d) has a countable topological base and so second countable. $\square$

5.4 Multiplicability and heritability

5.5 Reference

- THE AXIOMS OF COUNTABILITY

Chapter 6

SEPARATION AXIOMS

6.1 Four separation axioms

By "separation axioms" we mean properties of topological spaces concerning separating certain disjoint sets via (disjoint) open sets. [Caution: It is very different from the conception separable that we learned last time!] There are many different separations axioms 1 , four of them are used more often than the others, and we have seen two of them which are most important:

Definition 6.1

(T_1)

$$\forall x_1 \neq x_2 \in X, \exists \text{ open sets } U, K \text{ s.t.}$$
$$x_1 \in U \text{ but } x_2 \notin U \text{ and } x_2 \in K \text{ but } x_1 \notin K.$$

(T_2)

$$\forall x_1 \neq x_2 \in X, \exists \text{ open sets } U, K \text{ s.t.}$$
$$x_1 \in U, x_2 \in K \text{ and } U \cap K = \emptyset.$$

(T_3)

$$\forall \text{ closed sets } V \text{ and } x \notin V, \exists \text{ open sets } U, K \text{ s.t.}$$
$$V \subset U, x \in K \text{ and } U \cap K = \emptyset.$$

(T_4)

$$\forall \text{ closed sets } V_1 \text{ and } V_2 \text{ with } V_1 \cap V_2 = \emptyset, \exists \text{ open sets}$$
$$U, K \text{ s.t. } V_1 \subset U, V_2 \subset K \text{ and } U \cap K = \emptyset.$$

Remark. In some books, "both (T1) and (T3)" in our sense means "regular" "both (T1) and (T4)" means "normal"; in some other books, "(T3)" means "regular" and "(T4)" means "normal". In order to reduce ambiguity, we only talk about (T1-T4) in our sense but not "regular" and "normal".

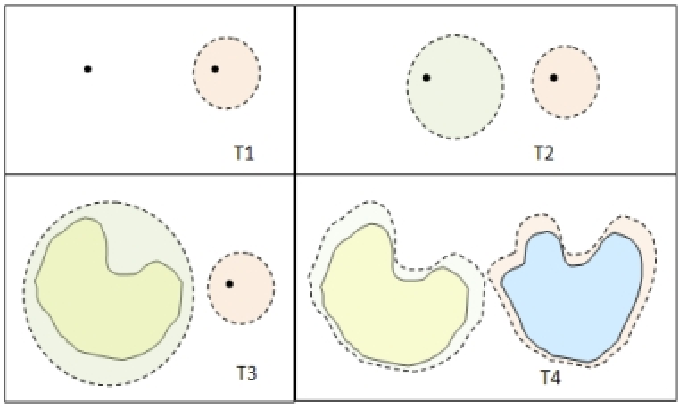

Figure 6.1

6.2 Equivalent characterizations

First we give equivalent characterizations of these axioms.

Proposition 6.1

Let (X, τ) be a topological space.
(1) (X, τ) is (T1) iff any single point set is closed.
(2) (X, τ) is (T2) iff the diagonal $\triangle = \{(x, x) : x \in X\}$ is closed in $X \times X$.
(3) (X, τ) is (T3) iff $\forall$ open set U with $x \in U$, $\exists$ open set K such that $x \in K \subset \overline{K} \subset U$
(4) (X, τ) is (T4) iff $\forall$ closed set $V \subset U$(U is open), $\exists$ open set K such that $V \subset K \subset \overline{K} \subset U$.

Proof. (1) and (2) follow from definitions, while (3) and (4) follow from open-closed duality:
(1) ($\Rightarrow$): For $\forall y \neq x$, $\exists U_y \in \tau$ such that $y \in U_y$ but $x \notin U$, then $U_y \subset \{x\}^c$ and $\{x\}^c = \cup_{y \neq x} U_y$. is open, i.e. $\{x\}$ is closed.
($\Leftarrow$): For $\forall x \neq y$, take $U = y^c$ and $K = x^c$. Then U, K is open and $x \in U, y \in K$ but $x \notin K, y \notin U$.
(2) The proof is trivial if $|X| = 1$, assume that $|X| > 1$.
($\Rightarrow$): Suppose X is Hausdorff. Let $(a, b) \in X \times X - \triangle$. (Note that such an element exists, since $|X| > 1$). Then, $a \neq b$. Since X is Hausdorff, we can pick open sets U_a and U_b such that $a \in U_a, b \in U_b$ and $U_a \cap U_b = \emptyset$. Now, note that $(U_a \times U_b) \cap \triangle = \emptyset$ (If not, $\exists (p, q) \in (U_a \times U_b) \cap \triangle$. Then $(p, q) \in \triangle$ and so $p = q$. But then $p \in U_a$ and $p \in U_b$, contradicting the fact that $U_a \cap U_b = \emptyset$.) Since $U_a \times U_b \in \tau_{X \times X}$ and $(a, b) \in U_a \times U_b \subseteq X \times X - \triangle$, by proposition2.2, $X \times X - \triangle$ is open in $X \times X$. Hence, $\triangle$ is closed in $X \times X$.
($\Leftarrow$): For $x \neq y \in X$, i.e. $(x, y) \in \triangle^c$, there exists U, K such that $x \in U, y \in K$ and $(x, y) \in U \times K \subseteq \triangle^c$. It follows $U \cap K = \emptyset$, if not, there exists $z \in U \cap K$, then $(z, z) \in U \times K \cap \triangle = \emptyset$. This is a contradiction. Hence, X is Hausdorff.
(3) ($\Rightarrow$): $\forall$ open set U with $x \in U$, then $x \notin U^c$ (closed set). Then, there exists $K_1, K_2 \in \tau$ such that $x \in K_1, U^c \subset K_2$ and $K_1 \cap K_2 = \emptyset$. Then by proposition2.5, $x \in K_1 \subset \overline{K_1} \subset K_2^c \subset U$.
($\Leftarrow$): Suppose $x \notin V$ closed, i.e. $x \in V^c$ open, then there exists $K \in \tau$ such that $x \in K \subset \overline{K} \subset V^c$. It follows $K \cap \overline{K}^c = \emptyset$, $x \in K$ and $V \subset \overline{K}^c$.
(4) ($\Rightarrow$): Suppose closed $V \subset U$ open, then $V \cap U^c = \emptyset$. So there exists $K_1, K_2 \in \tau$ such that

$K_1 \cap K_2 = \emptyset, V \subset K_1$ and $U^c \subset K_2$. So $V \subset K_1 \subset \overline{K_1} \subset K_2^c \subset U$.
($\Leftarrow$): Suppose V_1, V_2 are closed and $V_1 \cap V_2 = \emptyset$. Then $V_1 \subset V_2^c$ open. So there exists $K \in \tau$ such that $V \subset K \subset \overline{K} \subset V_2^c$. It follows that $K \cap \overline{K}^c = \emptyset$, $V \subset K$ and $V_2 \subset \overline{K}^c$. □

6.3 Relations between different separation axioms

We can also study the relations between these axioms. Obviously we have

Proposition 6.2

(1) $T_2 \Rightarrow T_1$.
(2) $T_1 + T_3 \Rightarrow T_2, T_1 + T_4 \Rightarrow T_2, T_1 + T_4 \Rightarrow T_3$.

Proof. (1)

$$\begin{aligned} \forall x_1 \neq x_2 \in X &\overset{T2}{\Rightarrow} \exists U, K \in \tau : x_1 \in U, x_2 \in K \text{ and } U \cap K = \emptyset \\ &\Rightarrow x_1 \in U, x \notin K \text{ and } x_2 \in K, x_2 \notin U \\ &\Rightarrow x_1 \in U \setminus K \text{ and } x_2 \in K \setminus U. \end{aligned}$$

(2)

$$\begin{aligned} \forall x_1 \neq x_2 \in X &\overset{T1}{\Rightarrow} \{x_1\} \text{ is closed and } x_2 \notin \{x_1\} \\ &\overset{T3}{\Rightarrow} \exists U, K \in \tau \text{ s.t. } \{x_1\} \subseteq U, x_2 \in K \text{ and } U \cap K = \emptyset. \\ &\Rightarrow x_1 \in U, x_2 \in K \text{ and } U \cap K = \emptyset. \end{aligned}$$

1

□

Proposition 6.3

A metric sapce satisfys T_1, T_2, T_3 and T_4.

6.4 Productive and hereditary

Proposition 6.4

T_1 is hereditary.

Proof. Let (X, τ) be a T_1 space and $A \subset X$. That is:

$$\begin{gathered} \forall x \neq y \in A \subset X, \exists U, K \in \tau : \\ x \in U \text{ but } y \notin U \text{ and } y \in K \text{ but } x \notin K. \end{gathered}$$

Then one can find

$$U_A := U \cap A, K_A := K \cap A.$$

It follows that $U_A, K_A \in \tau_A$ s.t. $x \in U_A$ but $y \notin U_A$ and $y \in K_A$ but $x \notin K_A$ $\square$

Proposition 6.5

T_2 is hereditary.

Proof. Let (X, τ) be a T_2 space and $A \subset X$. That is:

$$\forall x \neq y \in A \subset X, \exists U, K \in \tau :$$
$$x \in U, y \in K \text{ and } U \cap K = \emptyset.$$

Then one can find

$$U_A := U \cap A, K_A := K \cap A.$$

It follows that $U_A, K_A \in \tau_A$ s.t. $x \in U_A, y \in K_A$ and $U_A \cap K_A = (U \cap A) \cap (K \cap A) = (U \cap K) \cap A = \emptyset$. $\square$

Proposition 6.6

T_3 is hereditary.

Proof. Let (X, τ) be a T_3 space and $A \subset X$. Then $\forall$ closed set V in A and $x \in A \setminus V$, by proposition2.14, $\exists C$ which is closed in X such that $V = A \cap C$. Also $x \notin C$. That is $\exists U, K \in \tau$ s.t. $x \in U, C \subset K$ and $U \cap K = \emptyset$. Then one can find $U_A := U \cap A, K_A := K \cap A \in \tau_A$ s.t. $x \in U_A, V \subset K_A$ and $U_A \cap K_A = \emptyset$. $\square$

Proposition 6.7

T_4 preserved in closed subspace.

Proof. Let (X, τ) be a T_4 space and $A \subset X$ is closed. $\forall$ closed set V_1, V_2 in A with $V_1 \cap V_2 = \emptyset$, by corollary2.2, we can know that V_1, V_2 is closed in X and $V_1 \cap V_2 = \emptyset$. Then $\exists U, K \in \tau$ s.t. $V_1 \subset U, V_2 \subset K$ and $U \cap K = \emptyset$. Then one can find $U_A := U \cap A, K_A := K \cap A \in \tau_A$ such that $V_1 \subset U_A, V_2 \subset K_A$ and $U_K \cap K_A = \emptyset$. $\square$

Remark. If A is not closed. If we use the method in proposition6.6: Let (X, τ) be a T_4 space and $A \subset X$ is closed. Then $\forall$ closed set V_1, V_2 in A with $V_1 \cap V_2 = \emptyset$, by proposition2.14, $\exists C_1, C_2$ which are closed in X such that $V_1 = A \cap C_1, V_2 = A \cap C_2$. It follows that $\emptyset = V_1 \cap V_2 = A \cap (C_1 \cap C_2)$, but we can not get $C_1 \cap C_2 = \emptyset$. Then the proof can not go on.

6.5 exercise

Exercise 6.1: P43 T7

The Hausdorff property is hereditary, that is, if (X, τ) is a Hausdorff topological space and $A \subseteq X$ then (A, τ_A) is a Hausdorff topological space where $\tau_A = \{A \cap U : U \in \tau\}$ is the subspace topology on A.

Proof. For $x \neq y \in A \subseteq X$, there exists $U, K \in \tau$ such that $x \in U, y \in K$ and $U \cap K = \emptyset$. Then $x \in A \cap U, y \in A \cap K$ and $(A \cap U) \cap (A \cap K) = A \cap (U \cap K) = \emptyset$. Since $A \cap U, A \cap K \in \tau_A$, it follows that (A, τ_A) is Hausdorff. □

Exercise 6.2

The Hausdorff property is productive, that is, product of two Hausdorff spaces is Hausdorff.

Proof. Suppose $(X, \tau_X), (Y, \tau_Y)$ are Hausdorff spaces. For $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$, then $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, let $x_1 \neq x_2$. Since X is Hausdorff, it follows that there exists $U, K \in \tau_X$ such that $x_1 \in U, x_2 \in K$ and $U \cap K = \emptyset$. Then $U \times Y, K \times Y$ are open in $X \times Y$, $(x_1, y_1) \in U \times Y, (x_2, y_2) \in K \times Y$ and $(U \times Y) \cap (K \times Y) = \emptyset$ (If not, $\exists (p, q) \in (U \times Y) \cap (K \times Y)$, then $p \in U \cap K$, contradicting the fact $U \cap K = \emptyset$). Hence, $X \times Y$ is Hausdorff. □

Exercise 6.3: P43 T9

Let (X, τ) be a T_3 space , F be a closed subset of X and $x \notin F$. Then there exists open neighborhood U of F and open neighborhood V of x such that $\overline{U} \cap \overline{V} = \emptyset$.

Proof. Since X is T_3 space, it follows that there exists open neighborhood U of F and open neighborhood K of x such that $U \cap K = \emptyset$. And then there exists open neighborhood V of x such that $x \in V \subset \overline{V} \subset K$. Since $U \subset K^c$ and K is open, it follows that $\overline{U} \subset (\text{Int}(K))^c = K^c$. Hence, $\overline{U} \cap \overline{V} = \emptyset$ as required.

□

6.6 Reference

- Separation Axioms and Urysohn's lemma
- T2 equivalent proof

Chapter 7

Compactness

7.1 Definitions of various compactness

Definition 7.1

Let (X, τ) be a topological space, and $A \subset X$ be a subset.
(1) A family of subsets $\mathscr{U} = \{U_\alpha\}$ is called a covering of A if $A \subset \cup_\alpha U_\alpha$.
(2) A covering $\mathscr{U}$ is called a finite covering if it is a finite collection.
(3) A covering $\mathscr{U}$ is called an open covering if each U_α is open.
(4) A covering $\mathscr{V}$ is a sub-covering of $\mathscr{U}$ if $\mathscr{V} \subset \mathscr{U}$ and $A \subset \cup_{v \in \mathscr{V}} V$

Definition 7.2

Let (X, τ) be a topological space.
(1) We say X is compact in X if any open covering $\mathscr{U} = \{U_\alpha\}$ of X admits a finite sub-covering, i.e. there exists $\{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_k}\} \subset \mathcal{U}$ s.t. $X = \cup_{i=1}^k U_{\alpha_i}$.
(2) We say X is sequentially compact if any sequence $x_1, x_2, ... \in X$ admits a convergent subsequence $x_{n_1}, x_{n_2}, ... \to x_0 \in X$.

Remark. Suppose $A \subset X$ be a subset, then we say A is compact/sequentially compact if, when endowed with the subspace topology, (A, τ_A) is compact/sequentially compact.

Proposition 7.1

Let (X, τ) be a topological space, then

$$\begin{aligned} &A \subset X \text{ is compact } \Leftrightarrow \\ &\text{for any family of open sets } \mathcal{U} = \{U_\alpha\} \text{ in } X \text{ satisfying } A \subset \cup_\alpha U_\alpha, \\ &\text{one can find } U_{\alpha_1}, ..., U_{\alpha_k} \in \mathcal{U} \text{ s.t. } A \subset \cup_{j=1}^k U_{\alpha_j}. \end{aligned}$$

In a word, $A \subset X$ is compact $\Leftrightarrow$ any open covering of A in X admits finite sub-covering.

Proof. For any family of open sets $\mathcal{U} = \{U_\alpha\}$ in X satisfying $A \subset \cup_\alpha U_\alpha$, then $A = A \cap A \subset A \cap (\cup_\alpha U_\alpha) = \cup_\alpha (A \cap U_\alpha)$. Then $\mathscr{U}_A = \{U_\alpha \cap A : U_\alpha \in \mathcal{U}\}$ is a open covering of (A, τ_A).

($\Rightarrow$): Since (A, τ_A) is compact, one can find $U_{\alpha_1}, ..., U_{\alpha_k}$ such that $A = \cup_{i=1}^k (U_{\alpha_i} \cap A)$ and so $A \subset \cup_{i=1}^k U_{\alpha_i}$.
($\Leftarrow$): Since $A \subset \cup_{j=1}^k U_{\alpha_j}$, it follows that $A = A \cap A \subset A \cap \cup_{j=1}^k U_{\alpha_j} = \cup_{j=1}^k (A \cap U_{\alpha_j})$. Also $\cup_{j=1}^k (A \cap U_{\alpha_j}) \subset A$ and so $A = \cup_{j=1}^k (A \cap U_{\alpha_j})$. Then any open covering of (A, τ_A) adimits finite open sub-covering and so A is compact. □

7.2 Examples of compactness

Example 7.1

Any finite topological space, including the empty set, is compact. More generally, any space with a finite topology (only finitely many open sets) is compact; this includes in particular the trivial topology.

Example 7.2

Any space carrying the cofinite topology is compact and sequentially compact.

Proof. 1 □

Example 7.3

In the cocountable topology on an uncountable set, no infinite set is compact.

Example 7.4

No discrete space with an infinite number of points is compact. The collection of all singletons of the space is an open cover which admits no finite subcover. Finite discrete spaces are compact.

Example 7.5

The closed unit interval $[0, 1]$ is compact. This follows from the Heine-Borel theorem. The open interval $(0, 1)$ is not compact: the open cover $(\frac{1}{n}, 1 - \frac{1}{n})$ for $n = 3, 4, ...$ does not have a finite subcover.

Example 7.6

The set $\mathbb{R}$ of all real numbers is not compact as there is a cover of open intervals that does not have a finite subcover. For example, intervals $(n-1, n+1)$, where n takes all integer values in $\mathbb{Z}$, cover $\mathbb{R}$ but there is no finite subcover.

Remark. (1) We will see later: for topological spaces,

- compact $\not\Rightarrow$ sequentially compact;
- sequentially compact $\not\Rightarrow$ compact.

(2) We will prove: for metric spaces, compact $\Leftrightarrow$ sequentially compact

7.3 Characterization of compactness via closed sets or basis

7.4 Compactness in metric space

Now, we prove that for metric spaces, compact $\Leftrightarrow$ sequentially compact

Proposition 7.2

Compact C_1 space is sequentially compact.

Proof. Suppose X is a compact C_1 space. We need to show that for any sequence $x_n \in X$, there exists convergent subsequence $x_{n_k} \underset{k\to\infty}{\longrightarrow} x_0 \in X$. Firstly, we claim that there exists $x_0 \in X$ such that any neighborhood of x has infinite elements of $\{x_n\}$. Suppose not, then $\forall x \in X$, there exists a open neighborhood U_x of x such that U_x has finite elements of $\{x_n\}$. Then, $\mathscr{U} = \{U_x : x \in X\}$ is open covering of X, but $\{x_n\}$ can not be covered by any finite covering of $\mathscr{U}$, contradicting the fact that X is compact. Secondly, we will construct subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \underset{k\to\infty}{\longrightarrow} x_0$. By X is C_1 space, we can take a countable neighborhood basis $\{U_n\}$ of x such that $U_m \subset U_n$ when $m > n$. Then for any neighborhood U of x_0 and $x_i \in \{x_n\}$, $\exists U_n \in \{U_n\}$ such that $x_i \in U_n \subset U$. Let $x_{n_i} \in U_i$, then we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, and U has infinite elements of $\{x_{n_k}\}$. Then, $x_{n_k} \underset{k\to\infty}{\longrightarrow} x_0$. ☐

Remark. Metric space is C_1 space so in metric space, compact $\Rightarrow$ sequentially compact. The proof for the converse is a bit difficult. We need to use a few lemmas.

Lemma 7.1

Suppose K is a subset of a metric space X and

7.5 Proposition of compactness

7.5.1 Compactness v.s. continuous map

compactness and sequentially compactness are preserved under continuous maps:

Proposition 7.3

Let $f : X \to Y$ be continous.
(1) If $A \subset X$ is compact, then $f(A)$ is compact in Y.
(2) If $A \subset X$ is sequentially compact, then $f(A)$ is sequentially compact in Y.

Proof. (1) Suppose A is compact. Given any open covering $\mathscr{V} = \{V_\alpha\}$ of $f(A)$ in Y, then $\mathscr{U} = \{f^{-1}(V_\alpha)\}$ is an open convering of A in X (Since $A \subset f^{-1}(f(A)) = f^{-1}(\cup_\alpha V_\alpha) = \cup_\alpha f^{-1}(V_\alpha)$). By compactness of A, there exists $\alpha_1, \ldots, \alpha_k$ such that $A \subset U^k_{i=1} f^{-1}(V_{\alpha_i})$. It follows that $f(A) \subset f(U^k_{i=1} f^{-1}(V_{\alpha_i})) = \cup^k_{i=1} f(f^{-1}(V_{\alpha_i})) \subset U^k_{i=1} V_{\alpha_i}$, i.e. $f(A)$ is compact. ☐

Proposition 7.4

Let A be a closed subset of $\mathbb{R}$. Then the following statements are equivalent.
(1) A is compact.
(2) A is closed and bounded.

Proof. referring to Compacts sets in $\mathbb{R}$ Th1.7 □

Corollary 7.1

Let $f : X \to \mathbb{R}$ be any continuous map. If X is compact or sequentially compact, then $f(X)$ is bounded in $\mathbb{R}$. Moreover, there exists $a, b \in X$ s.t. $f(a) = \inf_{x\in X} f(x)$ and $f(b) = \sup_{x\in X} f(x)$.

Proof. By proposition7.3, $f(X)$ is compact. By proposition7.4, $f(X)$ is closed and bounded. Then $f(X)$ has a least upper bound and a greatest lower bound. By definition of bound, $\inf_{x\in X} f(x)$ and $\sup_{x\in X} f(x)$ are limit points of sequences in $f(X)$. Since $f(X)$ is closed, it follows that $\inf_{x\in X} f(x)$, $sup_{x\in X} f(x) \in f(X)$ and so $a, b \in X$ exist. □

7.5.2 Subspace of a compact space

As usual, we would like to construct new compact spaces from old compact spaces, or even non-compact spaces. The first candidates one can look at is: subspaces of a compact space. Unfortunately, it is easy to see that a compact space could have non-compact subspace, e.g. $(0, 1)$ is a subspace of $[0, 1]$.

We can take a closer look at the problem: which subsets of $[0, 1]$ remain to be compcat? We know that a set in $\mathbb{R}$ is compact iff it is bounded and closed.

If A is a subset of $[0, 1]$, it is automatically bounded. So far a subset $A \subset [0, 1]$ to be compact, it is enough to require A to be closed.

It turns out that for more general topological spaces, it is also enough to require closedness for a subset to be compact.

Proposition 7.5

Let $A \subset X$ be a closed subset.
(1) If X is compact, then A compact.
(2) If X is sequentially compact, then A is a sequentially compact.

Proof. (1) For any open covering $\mathscr{U}_A$ of A in X (i.e. $A \subset \cup_{U\in\mathscr{U}} U$), $\mathscr{U} \cup \{A^c\}$ is an open covering of X, which adimits a finite sub-covering $U_1, ..., U_m, A^c$ as X is compact. It follows $A \subset \cup_{i=1}^m U_i$, then by proposition7.1, A is compact. □

Proposition 7.6

Suppose $K \subset A \subset X$. Then K is compact relative to A if and only if it is compact relative to X.

Proof. referring to proof of transitive compactness □

7.5.3 Compact v.s. Hausdorff

Although it seems that compactness and Hausdorff property are very different, it turns out that they are "the dual" to each other in the following sense:

Proposition 7.7

(1) If (X, τ) is compact, then
(a) Every closed subset in X is compact.
(b) If $\tau' \subset \tau$, then (X, τ') is compact.
(2) If (X, τ) is Hausdorff, then
(a) Every compact subset in X is closed.
(b) If $\tau' \supset \tau$, then (X, τ') is Hausdorff.

Proof. (2)(a): Let $A \subset X$ be compact, $x_0 \in X \setminus A$. Since X is Hausdorff, it follows that for any $y \in A$ we can find U_y and V_y such that $x_0 \in U, y \in V_y$ and $U_y \cap V_y = \emptyset$. Since $A \subset \cup_{y \in A} V_y$, one can find $y_1, ..., y_m$ s.t. $A \subset \cup_{i=1}^m V_{y_i}$. Then $\cup_i^m U_{y_i} \subset (\cup_i^m V_{y_i})^c \subset A^c$. Hence, A^c is open and so A is closed. □

Theorem 7.1

Let X be compact and Y be Hausdorff space. If $f : X \to Y$ is continous and bijective, then f is homeomorphism.

Proof. □

7.5.4 Compactness "enhances" the separation axioms T_2 and T_3

Theorem 7.2

Any compact Hausdorff space is T_4.

This a consequence of the following proposition, which show how compactness "enhance" the separation axioms (via a simple "local-to-global" argument):

Proposition 7.8

For topological spaces, we have
(1) Compact + $T_2 \Rightarrow T_3$.
(2) Compact + $T_3 \Rightarrow T_4$.

Proof. (1) Let $x \in X, A \subset X$ be closed(and thus compact as X is Hausdorff), and $x \notin A$. Then for any $y \in A$, there exists open sets $U_{x,y}, V_y$ such that $x \in U_{x,y}, y \in V_y$ and $U_{x,y} \cap V_y = \emptyset$. Since $A \subset \cup_{y \in A} V_y$, by compactness of A, one can find $V_{y_1}, ..., V_{y_n}$ covering A. It follows that

$$U := U_{x,y_1} \cap ... \cap U_{x,y_n} \text{ and } V := V_{y_1} \cup ... \cup V_{y_n}$$

are open neighborhood of x and A, and $U \cap V = \emptyset$($U \cap V = \cap_{i=1}^n U_{x,y_i} \cap \cup_{i=1}^n V_{y_i} = \cap_{i=1}^n (U_{x,y_i} \cap \cup_{i=1}^n V_{y_i}) = \cap_{i=1}^n \cup_{i=1}^n (U_{x,y_i} \cap V_{y_i})$).

(2) Let $A, B \subset X$ be closed and $A \cap B = \emptyset$, then for any $y \in A$, there exists open sets U_y and V_y such that $y \in V_y, B \subset U_y$ and $V_y \cap U_y = \emptyset$. Since $A \subset \cup_{y\in A} V_y$, by compactness of A, one can find $V_{y_1}, ..., V_{y_n}$ covering A. It follows that

$$U := U_{y_1} \cap ... \cap U_{y_n} \text{ and } V := V_{y_1} \cup ... \cup V_{y_n}$$

are open neighborhood of B and A, and $U \cap V = \emptyset$. □

7.6 Compactness of product space

7.7 Exercise

Exercise 7.1: P59 T3

Let (X, τ) be a topological space. Then finite union of compact subset of X is compact.

Proof. Suppose $A_1, ..., A_n$ are compact subset of X. Then for any family of open sets $\mathcal{U}$ in X satisfying $\cup_{i=1}^n A_i \subset \cup_{U\in\mathcal{U}} U$, then $A_i \subset \cup_{U\in\mathcal{U}} U$. Since A_i is compact, one can find $U_i^{\alpha_1}, ..., U_i^{\alpha_{n_i}} \in \mathcal{U}$ such that $A_i \subset \cup_{j=1}^{n_i} U_i^{\alpha_j}$. Then $\cup_{i=1}^n A_i \subset \cup_{i=1}^n (\cup_{j=1}^{n_i} U_i^{\alpha_j})$. Hence, $\cup_{i=1}^n A_i$ is compact. □

Exercise 7.2: P59 T5

If A is an infinite subset of a compact space X, then A has a limit point in X.

Proof. It suffices to show that there exists $x_0 \in X$ such that any open neighborhood U of x_0 has infinite points of A. Suppose not, then $\forall x \in X$, there exists an open neighborhood U_x of x such that $U_x \cap A$ is finite. Since $X \subset \cup_{x\in X} U_x$ and X is compact, one can find $U_{x_1}, ..., U_{x_n}$ such that $X = \cup_{i=1}^n U_{x_i}$. Then $A = A \cap X = \cup_{i=1}^n (A \cap U_{x_i})$ and so A is finite. This is a contradiction as A is finite. □

Exercise 7.3: P59 T12

Let X be a Hausdorff space and $(A_\alpha)_{\alpha\in J}$ is a family of compact subsets of X. Then $\cap_{\alpha\in J} A_\alpha$ is compact.

Proof. Since X is Hausdorff, it follows that $A_\alpha (\forall \alpha \in J)$ is closed in X. Then $K = \cap_{\alpha\in J} A_\alpha$ is closed in X. For all $\alpha \in J$, since $K \subset A_\alpha \subset X$, it follows that K is closed in A_α. Since A_α is compact, one can get K is compact relative to A_α. Then K is compact relative to X. □

Exercise 7.4: P59 T13

Let X be a T_3 space , A be a compact subset of X and U be a neighborhood of A. Then there exists a neighborhood V of A such that $\overline{V} \subset U$.

Proof. Since U is a neighborhood of A, it follows that $\exists$ open set K such that $A \subset K \subset U$. Then $A \cap K^c = \emptyset$. Since X is T_3, $\forall a \in A$, one can find open sets U_a, K_a such that $a \in U_a, K^c \subset K_a$ and $U_a \cap K_a = \emptyset$. Then $A \subset \cup_{a\in A} U_a$. Since A is compact in X, one can find $U_{a_1}, ..., U_{a_n}$ such that $A \subset \cup_{i=1}^n U_{a_i}$. Let $V = \cup_{i=1}^n U_{a_i}$. Since $U_a \subset K_a^c$, it follows that $V \subset \cup_{i=1}^n K_{a_i}^c \subset (\cap_{i=1}^n K_{a_i})^c$.

Let $O = \cap_{i=1}^{n} K_{a_i}$. Then $V \subset O^c$ and O is open. So $V \subset \overline{V} \subset O^c$. Since $K^c \subset O$, it follows that $A \subset V \subset \overline{V} \subset O^c \subset K \subset U$. □

7.8 Reference

- Compactness: various definitions and examples
- Sequentially compact metric spaces
- The Lebesgue Number of a Covering
- COMPACTNESS OF PRODUCT SPACE

Chapter 8

Connectedness

Connectedness is one of the simplest/most useful topological properties. It is intuitive and is relatively easy to understand, and, it is a powerful tool in proving many well-known results, e.g. the intermediate value theorem.

For topological spaces which have simple pictures, it is easy to tell whether the space is connected or not. But for more complicated spaces, it may be more complicated to tell whether the space is connected or not.

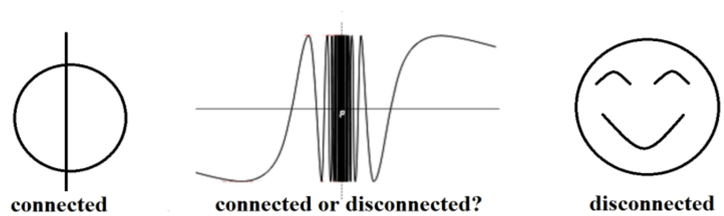

Figure 8.1

So we need a rigorous definition of connectedness (via the collection of open sets). Before we give such a rigorous definition, let's first look at a couple sets in $\mathbb{R}$

$$(a)\ (0,3)\ (b)\ (0,1)\cup[2,3)\ (c)\ (0,1)\cup(1,3]\ (d)\ (0,1]\cup(1,3)$$

Of course (a) is connected, (b) and (c) are disconnected, while (d) is connected! Although (d) looks like a union of two intervals, they are really one interval $(0,3)$ written as a disjoint union of two subsets. The two subsets (0, 1] and (1, 3) are "attached" together at the point 1, which is an element of (0, 1], but sits inside the closure of (1, 3]. For the case (c), although the two "components" (0, 1) and (1, 3] sit "next to each other", it is still disconnected because (0, 1) contains no element in the closure of (1, 3], and (1, 3] contains no element in the closure of (0, 1). This example motivates us to define connectedness. Unlike most other conceptions that you learned, connectedness is defined by its opposite.

8.1 Connectedness: The definition

Definition 8.1

Let (X, τ) be topological space.
(1) We say X is disconnected, if there exists non-empty sets $A, B \subset X$ such that

$$X = A \cup B \text{ and } A \cap \overline{B} = \overline{A} \cap B = \emptyset.$$

(2) We say X is connected if it is not disconnected.
(3) We say a subset X is connected/disconnected if it is connected/disconnected with respect to the subspace topology.

Remark. Note that by definition, the empty set is connected!

Remark. Suppose $A \subset X$ be a subset, then we say A is connected if, when endowed with the subspace topology, (A, τ_A) is connected.

Proposition 8.1

Let X be a topological space. Let $A \subset B \subset X$. Then A is connected in B iff A is connected in X.

Proof. Let τ_A be the subspace topology on A induced by τ. Let τ'_Abe the subspace topology on A induced by τ_B. Then A is connected in X iff (A, τ_A) is connected. Similarly, A is connected in B iff (A, τ'_A) is connected. By proposition2.13, $\tau_A = \tau'_A$. Hence, A is connected in X iff A is connected in B. □

8.2 Connectedness: Equivalent characterizations.

The definition above is intuitive but is also a little bit complicated. Fortunately we have several other equivalent ways to describe connectedness.

Proposition 8.2

For a topological space X, the following are equivalent:
(1) X is disconnected;
(2) there exists non-empty disjoint open sets $A, B \subset X$ s.t. $X = A \cup B$;
(3) there exists non-empty disjoint closed sets $A, B \subset X$ s.t. $X = A \cup B$;
(4) there exists $A \neq \emptyset$, $A \neq X$ such that A is both open and closed in X.

Proof. $(2) \Leftrightarrow (3) \Leftrightarrow (4)$ because

$$A \cap B = \emptyset \Leftrightarrow A^c = B, B^c = A. \tag{8.1}$$

□

Proposition 8.3

For a topological space X, the following are equivalent:
(1) X is connected;
(2) there is no non-empty disjoint open sets $A, B \subset X$ s.t. $X = A \cup B$;
(3) X and $\varnothing$ are the only sets which are both open and closed in X.

8.3 Examples of connected and disconnected spaces

Example 8.1

(X, τ_t) is connected, while (X, τ_s) is disconnected for $|X| \geqslant 2$.

Example 8.2

Infinite set with cofinite topology is connected.

Proof. Let (X, τ_f) be an infinite set with cofinite topology. Suppose (X, τ_f) is disconnected, then $\exists$ non-empty open sets $A, B \subset X$ s.t. $A \cap B = \varnothing$ and $X = A \cup B$. Then $X = \varnothing^c = (A \cap B)^c = A^c \cup B^c$. Since A, B is open in (X, τ_f), it follows that A^c, B^c is finite and so X is finite, contradicting with the fact X is infinite. $\square$

Example 8.3: L

t (X, τ_c) be a co-countable topological space. Show that X is connected iff it is uncountable.

Proof. ($\Leftarrow$): Suppose (X, τ_c) is disconnected when X is uncountable, then $\exists$ non-empty open sets $A, B \subset X$ s.t. $A \cap B = \varnothing$ and $X = A \cup B$. Then $X = \varnothing^c = (A \cap B)^c = A^c \cup B^c$. Since A, B is open in (X, τ_c), it follows that A^c, B^c is countable and so X is countable, contradicting with the fact X is uncountable. $\square$

Example 8.4

$\mathbb{Q} \subset \mathbb{R}$ is disconnected.

$$\mathbb{Q} = ((-\infty, -\sqrt{2}) \cap \mathbb{Q}) \cup ((-\sqrt{2}, +\infty) \cap \mathbb{Q}).$$

Example 8.5

S^1(the unit circle in $\mathbb{R}^2$) is connected.

Example 8.6

More generally, if $A \subset \mathbb{R}^2$ is countable, then $R^2 \setminus A$ is connected. In particular, $R^2 \setminus Q^2$ is connected. (Careful, this is not the set of all points with both coordinates irrational; it is the set of points such that at least one coordinate is irrational.)

8.4 Connectedness in $\mathbb{R}$

Definition 8.2

A subset S of $\mathbb{R}$ is said to be an interval if it has the following property: if $x, z \in S$ and $y \in \mathbb{R}$ are such that $x < y < z$, then $y \in S$.

Remark. Each singleton set $\{x\}$ is an interval.

Remark. Every interval has one of the following forms: $\{a\}$, $[a, b]$, (a, b), $[a, b)$, $(a, b]$, $(-\infty, a)$, $(-\infty, a]$, (a, ∞), $[a, \infty]$, $(-\infty, \infty)$.

Proposition 8.4

$\mathbb{R}$ is connected with respect to the euclidean topology.

Proof. Suppose $\mathbb{R}$ is disconnected. Then there exists an open set □

Remark. By the same proof, one can show that all intervals (a, b), $[a, b]$, $(a, b]$, $[a, b)$,$(a, +\infty)$, $[a, +\infty)$,$(-\infty, b]$, $(-\infty, b)$,$(-\infty, +\infty)$. are connected.

8.5 Properties of connected spaces

8.5.1 Generalized intermediate value theorem

Proposition 8.5

Suppose $f : X \to Y$ is continuous. Then for any connected subset $A \subset X$, the image $f(A) \subset Y$ is connected.

Proposition 8.6

A subspace of R is connected if and only if it is an interval.

Corollary 8.1: I

$f : X \to Y$ is a homeomorphisom, then X is connected iff Y is connected.

Corollary 8.2

If X is connected, $f : X \to \mathbb{R}$ is continous, and if there exist $x_1, x_2 \in X$ s.t. $f(x_1) = a < b = f(x_2)$, then for any $a < c < b$, there exists $x \in X$ s.t. $f(x) = c$.

8.5.2 The closure

Lemma 8.1

Let $X_0 \subset X$ be both open and closed and $A \subset X$ be connected. Then either $A \cap X_0 = \emptyset$ or $A \subset X_0$.

Proof. If X_0 is both open and closed in X, then $A \cap X_0$ is both open and closed in A. Since A is connected, it follows that the only sets which is both open and closed in A is A and $\emptyset$. Then either $A \cap X_0 = \emptyset$ or $A \cap X_0 = A$(implies $A \subset X_0$). □

Proposition 8.7

If X has a connected dense subset, then X is connected.

Proof. Suppose A is a connected dense subset of X and X_0 is a subset which is both open and closed. If $X_0 \neq \emptyset$, then $X_0 \cap A \neq \emptyset$, by lemma8.1, $A \subset X_0$. Then $X = \overline{A} \subset \overline{X_0} = X_0$. Then the only sets which is both open and closed in X are X and $\emptyset$. Hence, X is connected. □

Corollary 8.3

Let A be a connected subset of X. If $A \subset Y \subset \overline{A}$, then Y is connected.

Proof. Since $\text{cl}_Y(A) = Y \cap \text{cl}_X(A) = Y$, it follows that A is a connected dense subset of Y and so Y is connected. □

Corollary 8.4

If A is connected, so is $\overline{A}$.

Proof. $A \subset \overline{A} \subset \overline{A}$. □

Corollary 8.5: Topologist's sine curve

The set

$$S = \{(x, y) : x \in (0, 1), y = sin\frac{1}{x}\} \cup \{(0, y) : y \in [-1, 1]\} \subset \mathbb{R}^2$$

is connected.

8.5.3 The union

Proposition 8.8

Let $A_\alpha \subset X$ be a collection of non-empty connected subsets in X, and assume $\cap_\alpha A_\alpha \neq \emptyset$. Then $\cup_\alpha A_\alpha$ is connected.

Proof. Denote $Y = \cup_\alpha A_\alpha$. Suppose X_0 is both open and closed in Y, we should show that $X_0 = \emptyset$ or Y. □

8.6 The product

Proposition 8.9

If X, Y are connected, so is $X \times Y$.

8.7 Locally connected

Definition 8.3

A topological space X is locally connected at a point $x \in X$ if every neighbourhood U of x contains a connected neighbourhood K of x. The space X is locally connected if it is locally connected at every point $x \in X$.

8.8 Exercise

Exercise 8.1

Open subset of locally connected space is locally connected.

Proof. Suppose (X, τ) is locally connected and $A \subset X$ is open. For $x \in A$ and any neighbourhood N of x in A, $\exists U \in \tau_A$ such that $x \in U \subset N \subset A$. Then $U = O \cap A$ where $O \in \tau$. Then $U \in \tau$. Then N is neighbourhood of x in X. Since X is locally connected, it follows that there exists connected neighbourhood K of x in X and $V \in \tau$ such that $x \in V \subset K \subset N \subset A \subset X$. Then K is connected in A and $V = V \cap A \in \tau_A$. Hence, K is a connected neighbourhood of x in A. and so A is locally connected. □

Exercise 8.2: P66 T7

X is disconnected $\Leftrightarrow$ there exists a continuous function $f : X \to E^1$ such that $f(X)$ only has two points.

Proof. ($\Rightarrow$): If X is disconnected, then there exists non-empty open sets U, V s.t. $X = U \cup V$ and $U \cap V = \emptyset$. Define $f : X \to E^1$ such that $f(U) = 0$ and $f(V) = 1$. We claim that f is continuous. For any open set W in E^1, if $0, 1$ are not in W, then $f^{-1}(W) = \emptyset$, which is open. If $0 \in W$, $f^{-1}(W) = U$ which is open. If $1 \in W$, then $f^{-1}(W) = V$ which is open. Hence, f is continous.
($\Leftarrow$): Suppose X is connected. Since f is continous, it follows that $f(X)$ is connected. But $f(X) = \{a, b\}$ $(a, b \in E^1)$ is disconnected. □

Exercise 8.3: P66 T8

Let X be a subset of E^2 and $X = \{(x, y) : \text{ not all } x, y \text{ are irrational}\}$. Then X is connected.

Proof. $\forall r \in \mathbb{Q}$, let $A_r = \{(x, y) : \text{either } x \text{ or } y \text{ is rational}\}$. Let $A = E^1 \times \{r\}$ and $B = \{r\} \times E^1$, then A, B are connected and $A_r = A \cup B$, $A \cap \{(r, r)\} \neq \emptyset$ and $B \cap \{(r, r)\} \neq \emptyset$. Hence, A_r is connected. Since $X = \cup_{r \in \mathbb{Q}} A_r$ and $A_r \cap A_0 \neq \emptyset$, it follows that X is connected. □

8.9 Reference

- Lecture 18: Connectedness
- 18. Connectedness
- CONNECTEDNESS

Chapter 9

Path Connectedness

Chapter 10

Topological Properties and Homeomorphism

Proposition 10.1

If a, b, c, d are any real numbers with $a < b$ and $c < d$, then

$$(0,1) \cong (a,b) \cong (c,d) \cong \mathbb{R}.$$

Proposition 10.2

If a, b, c, d are any real numbers with $a < b$ and $c < d$, then

$$[a,b] \cong [c,d].$$

Proposition 10.3

If a, b are any real numbers, then

$$(-\infty, a] \cong (-\infty, b] \cong [a, \infty) \cong [b, \infty).$$

Proposition 10.4

If c, d, e and f are any real numbers with $c < d$ and $e < f$, then

$$[c,d) \cong [e,f) \cong (c,d] \cong (e,f].$$

Proposition 10.5

If a, b are any real numbers, then

$$[0,1) \cong (-\infty, a] \cong [a, \infty) \cong [a,b) \cong (a,b].$$

Proposition 10.6

Let $f : (X, \tau) \to (Y, \tau')$ be a homeomorphism. Let $a \in X$, so that $X \setminus \{a\}$ is a subspace of X and has induced topology τ_1. Also $Y \setminus \{f(a)\}$ is subspace of Y and has induced topology τ_1'. Then $(X \setminus \{a\}, \tau_1)$ is homeomorphic to $(Y \setminus \{f(a)\}, \tau_1')$.

Proof. Suppose $f : X \to Y$ is a homeomorphism. Define $g : X \setminus \{a\} \to Y \setminus \{f(a)\}$. Since f is bijection, it follows that g is bijection. For open set U in $(X \setminus \{a\}, \tau_1)$, then $\exists O \in \tau$ such that $U = O \cap (X \setminus \{a\})$. Then $g(U) = f(O \cap (X \setminus \{a\})) \overset{f \text{ is injective}}{=} f(O) \cap (Y \setminus \{f(a)\}) \in \tau_1'$. For open set K in $(Y \setminus \{f(a)\}, \tau_1')$, then $\exists V \in \tau'$ such that $K = V \cap (Y \setminus \{f(a)\})$ Then $g^{-1}(K) = f^{-1}(V \cap (Y \setminus \{f(a)\})) = f^{-1}(V) \cap (X \setminus \{a\}) \in \tau_1$. □

Proposition 10.7

Any topological space homeomorphic to a connected space is connected.

This proposition gives us one way to show two topological spaces are not homeomorphic, by finding a property "preserved by homeomorphisms" which one space has and the other does not. We have met many properties "preserved by homeomorphisms":

- T_1
- T_2
- T_3
- T_4
- separable
- C_1
- C_2
- connected
- locally connected
- path-connected
- locally path-connected
- compact
- sequentially compact

Corollary 10.1

If a, b, c and d are real numbers with $a < b$ and $c < d$, then
(1) $(a, b) \ncong [c, d)$,
(2) $(a, b) \ncong [c, d]$,
(3) $[a, b) \ncong [c, d]$.

Proof. (1) If $(a, b) \cong [c, d)$, then $(a, b) \setminus \{x\} \cong (c, d)$, but $(a, b) \setminus \{x\}$ is disconnected and (c, d) is connected. □

Theorem 10.1

Let X be compact and Y be Hausdorff space. If $f : X \to Y$ is continous and bijective, then f is homeomorphism.

Definition 10.1

$E^1 := \mathbb{R}$, $E^2 := \mathbb{R}^2$,$E^n := \mathbb{R}^n$
$S^1 := \{(x, y) \in E^2 : x^2 + y^2 = 1\}$, $S^2 := \{(x, y, z) \in E^3 : x^2 + y^2 + z^2 = 1\}$
$S^{n-1} = \{(x_1, ..., x_n) \in E^n : \sum_{i=1}^{n} x_i^2 = 1\}$
$D^n = \{x \in E^n : ||x|| \leqslant 1\}$ where $||x||$ is the distance from x to origin.

Proposition 10.8

Int(D^n)$\cong E^n$.

Proposition 10.9

$S^n \setminus \{N\} \cong E^n$ (N is the north point of S^n).

Proposition 10.10

$E^n \setminus \{O\} \cong E^n \setminus D^n$ (O is origin).

Proof. $f : E^n \setminus \{O\} \to E^n \setminus D^n$ given by $f(x) = x + \frac{x}{||x||}$ □

Proposition 10.11

$E^n \setminus \{O\} \cong S^{n-1} \times E^1$.

$\mathbb{R}^+ \cong \mathbb{R} \Rightarrow S^{n-1} \times \mathbb{R}^+ \cong S^{n-1} \times \mathbb{R}$
$f : \mathbb{R}^n \setminus \{O\} \to S^{n-1} \times \mathbb{R}^+$ given by $f(x) = (\frac{x}{||x||}, ||x||)$

Proposition 10.12

$E^1 \ncong E^n$ ($n > 1$).

$E^1 \setminus \{O\}$ is not connected.
$E^n \setminus \{O\} \cong S^{n-1} \times E^1$ is connected.

Proposition 10.13

$I \ncong S^1$.

For $x \in$ Int(I), $I \setminus \{x\}$ is not connected. $S^1 \setminus \{f(x)\} \cong E^2$ is connected.

Proposition 10.14

$S^2 \ncong S^1$.

$S^2 \setminus \{N_2\} \cong E^2$, $S_1 \setminus \{N_1\} \cong E^1$, but $E^1 \ncong E^2$.

Proposition 10.15: I

$f : S^1 \to E^1$ is continuous, then f is not injective and surjective.

Proposition 10.16

10.1 Reference

- topology without tears chp4

Part III

Chapter 11

Quotient Space and Quotient Mapping

We have seen how to create new topological spaces from given topological spaces using the operation of forming a subspace and the operation of forming a product of a set of topological spaces. In this chapter we introduce a third operation, namely that of forming a quotient space(of a topological space). As examples we shall see the Klein bottle and Möbius strip.

11.1 Quotient spaces

Definition 11.1

Let (X,τ) be a topological space and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective (onto) map that carries each point of X to the element of X^* containing it. p is called the projection map from X to X^* . In the topology on X^* induced by p ($\tau^* = \{U \subset X^* : p^{-1}(U) \in \tau\}$), the space X^* under this topology is the quotient space of X.

Note. Recall that we have a partition of a set if and only if we have an equivalence relation on the set. So the approach in terms of partitions can be replaced with an approach based on equivalence relations. The idea of the quotient space is that points of the subsets in the partition (or the equivalent points under the equivalence relation) are "identified" with each other. For this reason, quotient spaces are sometimes called "identifying spaces" or "decomposition spaces." You will notice the parallel between quotient spaces and quotient groups in which all elements of a coset are identified.

Then we can get the definition of quotient space by the equivalence relation.

Definition 11.2

Let (X,τ) be a topological space and and $\sim$ any equivalence relation on X. Let $X/\sim$ be the set of all equivalence classes of $\sim$ and $p : X \to X/\sim$ given by $x \mapsto [x]$. In the topology on $X/\sim$ induced by p ($\tilde{\tau} = \{U \subset X/\sim : p^{-1}(U) \in \tau\}$), the space $X/\sim$ under this topology is the quotient space of X.

Definition 11.3

Let (X, τ) and (Y, τ_1) be topological spaces. Then (Y, τ_1) is said to be a quotient space of (X, τ) if there exists a mapping $f : (X, \tau) \to (Y, \tau_1)$ satisfying the following properties:
(1) f is surjective.
(2) For each subset U of Y, $U \in \tau_1 \Leftrightarrow f^{-1}(U) \in \tau$. And f is said to be a quotient mapping.

Remark. From the definition, it is clear that every quotient mapping is continous.

Remark. Property (2) is equivalent to: For each subset A of Y, A is closed in $(Y, \tau_1) \Leftrightarrow f^{-1}(A)$ is closed in (X, τ).

Proposition 11.1

If $f : (X, \tau) \to (Y, \tau_1)$ is a surjective, continous and open mapping, then f is a quotient mapping. If f is a surjective, continous and closed mapping, then it is a quotient mapping.

Proof. If f is continous and surjective, then (1) and necessary condition of (2) hold. If f is open, then if $f^{-1}(U) \in \tau$,then $U = f(f^{-1}(U))\tau$. □

Proposition 11.2

$f : (X, \tau) \to (Y, \tau_1)$ is injective. Then f is a homeomorphism iff it is a quotient mapping.

Proof. ($\Rightarrow$): f is homeomorphism, then f is continous and surjective. And for $U \in Y$, if $f^{-1}(U) \in \tau$, then $U = f(f^{-1}(U)) \in \tau_1$.
($\Leftarrow$): f is quotient mapping then f is continous and surjective. Then f has an inverse f^{-1}. For $K \in \tau$, $K = f^{-1}(f(K)) \in \tau$ as f is bijective. Then, $f(K) \in \tau_1$. Hence, f^{-1} is continous. □

Proposition 11.3

Let X be compact and Y be Hausdorff space. If $f : X \to Y$ is continous and surjective, then f is a quotient mapping.

Proof. Let A be a closed set in X. Since X is compact, it follows that A is compact. Then $f(A)$ is compact as f is continous. Since Y is Hausdorff, it follows that $f(A)$ is closed. Then f is continous, surjective and closed mapping. Hence, f is a quotient mapping. □

Proposition 11.4: Universality

Let $f : X \to X'$ be a quotient mapping and $g : X' \to Y$ be a map. Then g is continous iff $g \circ f$ is continous.

Proof. ($\Rightarrow$): f is a quotient mapping, then f is continous. Then $g \circ f$ is continous as g is continous.
($\Leftarrow$): For open set U in Y, then $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. Then $g^{-1}(U)$ is open in X' as f is a quotient mapping. Hence, g is continous. □

Let $f : X \to Y$ be a mapping, we define a equivalence relation $\overset{f}{\sim}$: $\forall x, x' \in X, x \sim x' \Leftrightarrow f(x) = f(x')$.

Proposition 11.5

If $f : X \to Y$ is a quotient mapping, then $X/ \overset{f}{\sim} \cong Y$.

11.2 Exercise

Exercise 11.1: P86 T1

Let $f : X \to Y$ and $g : Y \to Z$ be continuous mapping such that $g \circ f$ is a quotient mapping. Then g is quotient mapping.

Proof. Since $g \circ f$ is a quotient mapping, it follows that $g \circ f$ is surjective and for $U \subset Z$, U is open in $Z \Leftrightarrow f^{-1}(g^{-1}(U))$ is open in X. Then for any $z \in Z$, $\exists x \in X$ such that $z = g(f(x))$. Let $y = f(x)$, then $y \in Y$ and $z = g(y)$. Hence, g is surjective. If U is open in Z, then $g^{-1}(U)$ is open in Y as g is continous. If $g^{-1}(U)$ is open in Y, then $f^{-1}(g^{-1}(U))$ is open in X as f is continous, then U is open in Z. Hence, g is a quotient mapping. $\square$

Exercise 11.2: P86 T2

Let $f : X \to Y$ be a quotient mapping , B be open(or closed) set of Y and $A = f^{-1}(B)$. Then $f_A : A \to B$ is a quotient mapping.

Proof. Since f is a quotient mapping, it follows that f is surjective and for $U \subset Y$, U is open in Y $\Leftrightarrow f^{-1}(U)$ is open in X. Then $A = f^{-1}(B)$ is open in X as B is open in Y and f is continous. Since f is surjective, it follows that $f_A(A) = f(A) = f(f^{-1})(B) = B$. Then f_A is surjective. For $K \subset B$, if K is open in B, then $K = O_Y \cap B$ (O_Y is open in Y) is open in Y, then $f_A^{-1}(K) = f^{-1}(K) \subset A$ is open in X , then $f_A^{-1}(K) = f_A^{-1}(K) \cap A$ and so open in A. If $f_A^{-1}(K)$ is open in A, then $f^{-1}(K) = f_A^{-1}(K) = O_X \cap A$ (O_X is open in X) is open in X, then $K \subset B$ is open in Y, then $K = K \cap B$ and so open in B. Hence, f_A is a quotient mapping. $\square$

11.3 Reference

- THE QUOTIENT TOPOLOGY
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Chapter 12

Homotopies

Recall that a path is a continuous function $a : [0,1] \to X$. The idea here is that we have two paths a_0 and a_1 in R^2 with $a_0(0) = a_1(0)$ and $a_0(1) = a_1(1)$; i.e. the paths have the same endpoints.

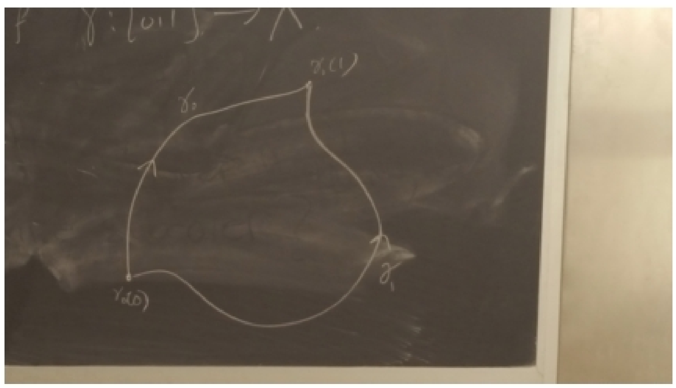

Figure 12.1

Our paths are different and may even have different images, but we want to say that one can be continuously deformed into the other.

Figure 12.2

Intuitively, we want to create a family of paths $\{a_t\}_{t\in[0,1]}$ "from a_0 to a_1." You can also say we want to interpolate continuously between a_0 and a_1. Think of $\{a_t : t \in [0,1]\}$ as one function $a : [0,1] \times [0,1] \to X$, where $a(x,t) := a_t(x)$.

The notion $C(X,Y)$ is the set of all the continuous maps from X to Y.

Two continuous functions from one topological space to another are called homotopic if one can be "continuously deformed" into the other, such a deformation being called a homotopy between the two functions. More precisely, we have the following definition.

Definition 12.1

Let X, Y be topological spaces, and $f, g : X \to Y$ continuous maps. A homotopy from f to g (denoted by $H : f \simeq g$) is a continuous function $H : X \times [0,1] \to Y$ satisfying $H(x,0) = f(x)$ and $H(x,1) = g(x)$, for all $x \in X$. If such a homotopy exists, we say that f is homotopic to g, and denote this by $f \simeq g : X \to Y$ or $f \simeq g$.

Definition 12.2

If f is homotopic to a constant map, i.e., if $f \simeq \mathrm{const}_y$, for some $y \in Y$, then we say that f is nullhomotopic.

Proposition 12.1

Let $f, g : E^n \to E^n$ any two continuous, real functions. Then $f \simeq g$.

To see why this is the case, define a function $F : E^n \times [0,1] \to E^n$ by $H(x,t) = (1-t)\cdot f(x) + t \cdot g(x)$. H is continuous and $H(x,0) = f(x), H(x,1) = g(x)$. Thus, H is a homotopy between f and g.

Proposition 12.2

Let A be a convex subset of E^n, endowed with the subspace topology, and let X be any topological space. Then any two continuous maps $f, g : X \to A$ are homotopic. The homotopy from f to g called the straight line homotopy.

Proposition 12.3

If Y is convex, $p \in Y$, $f : X \to Y$ is continuous, and $g : X \to Y$ is $g(x) = p$ for all $x \in X$, then $f \simeq g$ via the straight line homotopy.

Proposition 12.4

Homotopy is an equivalence relation on $C(X,Y)$.

Proposition 12.5

If $f_0 \simeq f_1 : X \to Y$, $g_0 \simeq g_1 : Y \to Z$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 : X \to Z$.

Proposition 12.6

Let $y_1, y_2 \in Y$ and $f_{y_i} : X \to Y$ given by $f(X) = y_i$. Then $f_{y_1} \simeq f_{y_2} \Leftrightarrow y_1$ and y_2 is in the same path component.

Proposition 12.7

If Y is path-connected, then the set $[I, Y]$ (the homotopy classes of maps from I to Y) has a single element.

For our paths, we want to fix the start and end, so $a_t(0) = a_t(1) = a_0(1)$ for all $t \in [0, 1]$.

Definition 12.3

Let $A \subseteq X$ and $f, g \in C(X, Y)$. We say f and g are homotopic relative to A iff there exists a homotopy H between f and g, and:
(1) $\forall a \in A$: $f(a) = g(a)$
(2) $\forall a \in A, t \in [0, 1]$: $H(a, t) = f(a)$
and denote this by $f \simeq g$ rel$A$ and $H : f \simeq g$ rel$A$.

Proposition 12.8

If $A \subseteq X$, homotopy relative to A is an equivalence relation on $C(X, Y)$.

Proposition 12.9

If $f_0 \simeq f_1 : X \to Y$ rel$A$, $g_0 \simeq g_1 : Y \to Z$ rel$B$ and $f_0(A) \subset B$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$ rel$A$.

Definition 12.4

Let a and b be paths in X. a is path-homotopic to b if $a \simeq b$ rel$\{0, 1\}$ and denoted by $a \underset{\cdot}{\simeq} b$.
i.e. there exists a homotopy H between a and b such that:
(1) $a(0) = b(0), a(1) = b(1)$
(2) $\forall t \in [0, 1]$: $H(0, t) = a(0), H(1, t) = a(1)$.

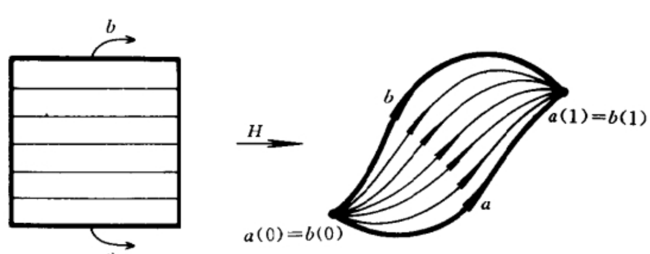

Figure 12.3

Since homotopy relative to a subset is an equivalence relation, it follows that path-homotopy is a equivalence relation on the set $P(X)$ of all paths in X. Then path-homotopy be a partition of $P(X)$ into disjoint subsets whose union is $P(X)$. These disjoint subsets are called path classes in X. The collection of all path classes is denoted by $[X]$. Given path a, the path classes a belongs to denoted by $< a >$. The startpoint and endpoint of a are the startpoint and endpoint of $< a >$. The path class is call closed path class if its startpoint and endpoint coincide, then its startpoint(endpoint) is called base point.

12.1 Reference

- Path Homotopy
- Path-homotopy
- HOMOTOPY AND PATH HOMOTOPY
- Homotopy

Chapter 13

Fundamental Group

13.1 Reference

- The Fundamental Group
- The fundamental group

Chapter 14

The Fundamental Group of S^n

Proposition 14.1

$\pi_1(S_1) \cong \mathbb{Z}$.

Proposition 14.2

Let X, Y be path-connected. Then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.

Proposition 14.3

$\pi_1(S^n) = \{0\}$ for $n > 2$.

Proposition 14.4

$T^2 \ncong S^2$.

Proof. $\pi_(S^1) \cong \mathbb{Z}$, $T^2 = S^1 \times S^1$, then $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$, but $\pi_1(S^2) = \{0\}$. $\square$

14.1 Reference

- $\pi_1(S^1) \cong Z$

Chapter 15

Exam Exercise1

Exercise 15.1

$X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}\}$, then $\overline{\{b\}} =?$

Proof. To find the closure of a particular set, we shall find all the closed set containing that set and then select the smallest. The closed sets in X are $\emptyset, X, \{b, c, d\}$. Then, $\overline{\{b\}} = \{b, c, d\}$. □

Exercise 15.2

$X = \{a, b, c, d\}$, $\tau = X, \emptyset, \{a\}, \{b, c, d\}$, the number of proper subsets of X which are both open and closed is ?

Proof. The closed sets of X are $X, \emptyset, \{b, c, d\}, \{a\}$. Then the proper subsets of X which are both open and closed are $\{b, c, d\}, \{a\}$. □

Exercise 15.3

In $\mathbb{R}$, $\mathbb{Q}^{\circ}(\text{Int}(\mathbb{Q})) =?$

Proof. $\mathbb{Q}^{\circ} = \emptyset$. The interior of set in topological space is the largest open set contained in the set. In the euclidean topology, there is no non-empty open interval contained entirely in $\mathbb{Q}$. since between any two rational numbers, there is an irrational number. □

Exercise 15.4

In $\mathbb{R}$, $\partial(\mathbb{Q}) =?$

Proof. $\partial(\mathbb{Q}) = \mathbb{R}$. Since $\partial(A) = \overline{A} \setminus \text{Int}(A)$ and $\overline{Q} = \mathbb{R}$, it follows that $\partial(\mathbb{Q}) = \mathbb{R} \setminus \emptyset = \mathbb{R}$. □

Exercise 15.5

In $\mathbb{R}$, $\mathbb{Z}^{\circ}(\text{Int}(\mathbb{Z})) =?$

Proof. $\mathbb{Z}^{\circ} = \emptyset$. The interior of set in topological space is the largest open set contained in the set. In the euclidean topology, there is no non-empty open interval contained entirely in $\mathbb{Z}$. since between any two rational numbers, there is an irrational number. □

Exercise 15.6

In $\mathbb{R}$, $\partial(\mathbb{Z}) =?$

Proof. $\overline{\mathbb{Z}} = \mathbb{Z}$. $\mathbb{Z}$ is closed since $\mathbb{Z} = \mathbb{R} \setminus (\cup_{n\in\mathbb{R}}(n, n+1))$. □

Exercise 15.7

(1) $(A \cup B)' = A' \cup B'$
(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Exercise 15.8

Let X be a discret space and $A \subset X$, then $A' =?$

Proof. $A' = \emptyset$. For every $x \in X$, $x \in \{x\}$ which is open,$\{x\} \cap A \setminus \{x\} = \emptyset$. □

Exercise 15.9

Let X be a trival space and $A \subset X$. Then
(1) $A = \emptyset$,then $A' = \emptyset$
(2) If $A = \{x_0\}$, then $A' = X \setminus A$
(3) If $A = \{x_1, x_2\}$, then $A' = X$.

Proof. (1) If $A = \emptyset$, $\emptyset \subset X$ but $X \cap A \setminus \emptyset = \emptyset$.
(2) For $x \in X \setminus A$, $x \in X$ which is open, $X \cap A \setminus \{x\} = \{x_0\}$. If $x \in A$, then $X \cap A \setminus \{x\} = \emptyset$.
(3) For $x \in X \setminus A$, $x \in X$ which is open, $X \cap A \setminus \{x\} = \{x_1, x_2\}$. If $x \in A$, then $X \cap A \setminus \{x\} = \{x_1\}$ or $\{x_2\}$. □

Exercise 15.10

$X = \{a, b, c, d\}$, $B = \{\{a, b, c\}, \{c\}, \{d\}\}$, then the topology induced by $\mathcal{B}$ is $\{X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, b, c\}\}$.

Proof. Let (X, τ) be a topological space. A collection $\mathcal{B}$ of subsets of X is said to be a basis for the topology τ if $\overline{\mathcal{B}} = \tau$. □

Exercise 15.11

Any subset of discret space is both open and closed.

Proof. □

Exercise 15.12

Any subset of trival space is neither open nor closed.

Exercise 15.13

Any singleton of $\mathbb{R}$ is closed.

Proof. $\mathbb{R} \setminus \{x\} = \cup_{a,b\in\mathbb{R}\setminus\{x\}}(a, b) \cup (x, x+1) \cup (x-1, x)$ is open. □

Exercise 15.14

In $\mathbb{R}$, $A = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$, then $\overline{A}$ =?

Proof. $\overline{A} = A \cup \{0\}$. A is not closed, since 0 is a limit point of A but $0 \notin A$. $(A \cup \{0\})^c = \cup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n}) \cup (-\infty, 0) \cup (1, +\infty)$ which is open. Then $A \cup \{0\}$ is closed and $A \subset A \cup \{0\}$. $\square$

Exercise 15.15

$X = X_1 \times X_2$ and $P_1 : X \to X_1$, then P_1 is surjective, continuous and open.

Proof. P_1 is a homeomorphism. $\square$

Exercise 15.16

(1) $\overline{A \times B} = \overline{A} \times \overline{B}$.
(2) Int($A \times B$) = Int(A) $\times$ Int(B).

Exercise 15.17

$\mathbb{Q}$ is disconnected in $\mathbb{R}$.

Proof. $\mathbb{Q} = ((-\infty, -\sqrt{2}) \cap \mathbb{Q}) \cup ((-\sqrt{2}, +\infty) \cap \mathbb{Q})$. $\square$

Exercise 15.18

$X = \{1, 2, 3\}$, $\tau = \{X, \varnothing, \{1\}\}$, then (X, τ) is T_1 or T_2?

Proof. neither T_1 nor T_2. Since $T_2 \Rightarrow T_1$, we just need to show whether it is T_2. For $x = 1, y = 2$, $x \in \{1\}, y \in X$, but $\{1\} \cap X \neq \varnothing$. $\square$

Exercise 15.19

$X = \{1, 2, 3\}$,$\tau = \{X, \varnothing, \{1\}, 2, 3\}$, (X, τ) is T_1, T_2 or T_3.

Proof. X is T_3. $T_1 + T_3 \Rightarrow T_2$, then we just need to show T_1 and T_3. $\{1\}, \{2, 3\}, X, \varnothing$ is closed. And $1 \in \{1\}$ $\square$