Study Notes of Matrix and Tensor

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Contents

Preface

The notes mainly refer to:

- Introduction to Mathematical Statistics 8th Edition
- lecture note
- Study Guide

Chapter 1

Probability and Distributions

1.1 Introduction

Definition 1.1

If an experiment can be repeated under the same conditions it is a random experiment. The set of every possible outcome of an experiment is the sample space, denoted $\mathcal{C}$.

Remark. For an experiment, the sample space is not unique. For example, When talking about the temperature in an area, we can define the sample space as $\mathcal{C} = (-\infty, \infty)$ or $\mathcal{C} = [a, b]$. For a specific random experiment, we can use different sample spaces to describe it. However, it is worth studying how to describe it with an appropriate sample space.

Note/Definition. Notationally, we denote the elements of the sample space with lower case letters such as a, b, c. Subsets of the sample space are *events* and we denote them with upper case letters such as A, B, C.

Definition 1.2

If an experiment is performed N times and a specific event occurs f times, then f is the frequency of the event and f/N is the relative frequency of the event.

1.2 Sets

1.3 The Probability Set Function

We need to define a set function that assigns a probability to the events (subsets of sample space $\mathcal{C}$). We denote the colletion of events as $\mathcal{B}$. If $\mathcal{C}$ is finite set, then we hope to assign a probability to all events (that is, to define a probability set function on the power set of $\mathcal{C}$). More generally, we require that $\mathcal{B}$ (the colletion of events) to satisfy: (1) the sample space $\mathcal{C}$ itself is an event, (2) the complement of every event is again an event, and (3) every countable union of events is again an event. Symbolically, this means (1) $\mathcal{C} \in \mathcal{B}$, (2) if $A \in \mathcal{B}$ then $A^c \in \mathcal{B}$, and (3) if $A_1, A_2, \ldots \in \mathcal{B}$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{B}$. Combining (2) and (3), we see by DeMorgan's Law (for countable unions) that if $A_1, A_2, \ldots \in \mathcal{B}$ then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$. So the collection of events $\mathcal{B}$ is closed under complements, countable unions, and countable intersections. Such a collection of sets form a *σ-algebra*.

Definition 1.3

A collection of events $\{A_n | n \in I\}$ (where I is some indexing set) such that $A_i \cap A_j = \emptyset$ is a mutually exclusive collection of events.

Definition 1.4

Let $\mathcal{C}$ be a sample space and let $\mathcal{B}$ be the set of all events (thus, $\mathcal{B}$ is a σ-field). Let P be a real-valued function defined on $\mathcal{B}$. Then P is a probability set function if P satisfies the following three conditions:
(1) $P(A) \geqslant 0$ for $A \in \mathcal{B}$.
(2) $P(\mathcal{C}) = 1$.
(3) If $\{A_n\}$ is a mutually exclusive collection of events, then $P(\cup_{n=1}^{+\infty} A_n) = \sum\limits_{n=1}^{+\infty} P(C_n)$.

Theorem 1.1

For each event $A \in \mathcal{B}$, $P(A) = 1 - P(A^c)$.

Theorem 1.2

The probability of the null set is zero; that is, $P(\emptyset) = 0$.

Theorem 1.3

If A and B are events such that $A \subset B$, then $P(A) \leqslant P(B)$.

Theorem 1.4

For each event $A \in \mathcal{B}$ we have $0 \leqslant P(A) \leqslant 1$.

Theorem 1.5

If A and B are events in $\mathcal{C}$, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Theorem 1.6

Let $\{A_n\}$ be a nondecreasing sequence of events (ie. $A_n \subseteq A_{n+1}$). Then

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n) = P(\cup_{n=1}^{\infty} A_n).$$

Let $\{A_n\}$ be a nonincreasing sequence of events (ie. $A_n \supseteq A_{n+1}$). Then

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n) = P(\cap_{n=1}^{\infty} A_n).$$

Theorem 1.7

Let $\{A_n\}$ be an arbitrary sequence of events. Then

$$P(\cup_{n=1}^{\infty} A_n) \leqslant \sum_{n=1}^{\infty} P(A_n).$$

1.4 Homework

Exercise 1.1

Show that the moment generating function of the random variable X having the pdf $f(x) = \frac{1}{3}$, $-1 < x < 2$, zero elsewhere, is

$$M(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & t \neq 0 \\ 1 & t = 0. \end{cases}$$

1.5 Reference

- lecture note
- Probability and Distributions
- Sample space is unique?
- proof of 1.3

Chapter 2

Multivariate Distributions

2.1 Homework

Exercise 2.1

Let the joint pdf of X and Y be given by

$$f(x,y) = \begin{cases} \frac{2}{(1+x+y)^3} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Compute the marginal pdf of X and the conditional pdf of Y, given $X = x$. (b) For a fixed $X = x$, compute $E(1 + x + Y|x)$ and use the result to compute $E(Y|x)$.

Exercise 2.2

Let X_1, X_2, X_3 be iid with common pdf $f(x) = \exp(-x)$, $0 < x < \infty$, zero elsewhere. Evaluste:
(a) $P(X_1 < X_2|X_1 < 2X_2)$.
(b) $P(X_1 < X_2 < X_3|X_3 < 1)$.