Study Notes of Numerical Optimization

Pei Zhong

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Contents

Update progress

- writing ch1... 2023.10.20

Preface

Notes mainly refer to the following resources:

- lecture notes from class Matrix and tensor factorization techniques for machine learning

all the codes in the notes are completed by python. You can get the codes in the following website:

- -

Part I

Fundamental Concepts of Optimization

Chapter 1

Convex Optimization

"The great watershed in Optimization is not between linearity and nonlinearity, but convexity and nonconvexity. "

1.1 General Convex Optimization Problems

Definition 1.1 (Convex Set)

a set $\Omega \in \mathbb{R}^n$ is convex if

$$\forall x, y \in \Omega, t \in [0, 1] : x + t(y - x) \in \Omega. \tag{1.1}$$

Remark. This definition is equivalent to saying that all connecting lines lie inside set.

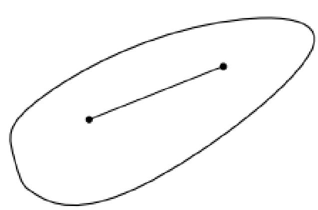

Figure 1.1: an example of a convex set

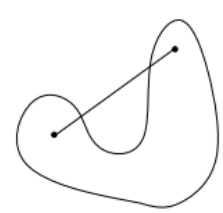

Figure 1.2: an example of a non convex set

Definition 1.2 (Convex Function)

a function $f : \Omega \to \mathbb{R}$ is convex, if Ω is convex and if

$$\forall x, y \in \Omega, t \in [0, 1] : f(x + t(y - x)) \leq f(x) + t(f(y) - f(x)). \tag{1.2}$$

Remark. This definition is equivalent to saying that all secants are above graph.

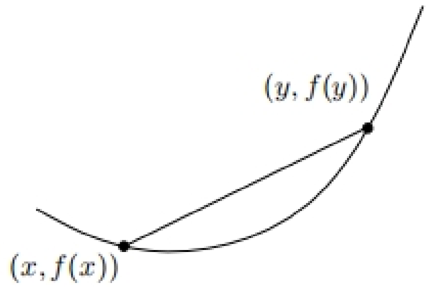

Figure 1.3: For a convex function, the line segment between any two points on the graph lies above the graph

Definition 1.3 (Convex Optimizaiton Problem)

an optimization problem with

- a convex feasible set Ω and
- a convex objective function $f : \Omega \to \mathbb{R}$

is called a "convex Optimization problem"

Theorem 1.1 (Local Implies Global Optimality for Convex Problems)

for a convex Optimization problem, every local minimum is also a global one.

Proof. Consider a local minimum x^* of the convex optimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$
$$s.t. x \in \Omega$$

We will show that for each y $\in \Omega$ it holds $f(y) \geq f(x^*)$.

Suppose that x^* is not the global minmium, that is $\exists\, \widetilde{x} \in \Omega \; s.t. \; f(\widetilde{x}) < f(x^*)$.

Consider the line segement $x(t) = tx^* + (1-t)\widetilde{x}, t \in [0,1]$, noting that $x(t) \in \Omega$ by the convexity of Ω. By the convexity of f,

$$f(x(t)) \leq tf(x^*) + (1-t)f(\widetilde{x}) < tf(x^*) + (1-t)f(x^*) = f(x^*), \forall t \in [0,1]$$

As x^* is a local minmium, we know that $\exists N$(N is a neighbourhood of x^*), $\forall x \in N$, $f(x) \leq f(x^*)$. We can pick t sufficiently to 1 such that $x(t) \in N$. Then $f(x(t)) < f(x^*)$. This is a contradiction as $f(x(t)) < f(x^*)$ by the above inequality.

Hence, x^* is the global minimum. □

1.2 How to Check Convexity of Functions?

Theorem 1.2 (Local Implies Global Optimality for Convex Problems)

for a convex Optimization problem, every local minimum is also a global one.

1.3 convex optimization problems

Chapter 2

Linear Programming

2.1 Introduction to LP

2.2 LP Relaxation

2.3 Algorithms Solving LP