THE SOUTH CHINA NORMAL UNIVERSITY

School of Mathematical Sciences

Numerical Analysis (2022-2023 The Second Term)

Homework 2:

Due Date: March 19, 2024 (Tuesday)

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§2.1 Exercises for Iteration for solving $x = g(x)$

Exercise 1. Let $g(x) = x\cos(x)$. Solve $x = g(x)$ and find all the fixed points of g (there are infinitely many). Can fixed-point iteration be used to find the solution(s) to the equation $x = g(x)$? Why?

Solve For $x = g(x)$, if $x \neq 0$, then $x = x\cos(x)$ could be simplification to $\cos(x) = 1$ and so $x = 2k\pi$ with $k = \pm1, \pm2, \ldots$ are the solutions for the equation. If $x = 0$, then the equation $x = x\cos(x)$ can be established and so $x = 0$ is the solution for the equation. Therefore, $x = 0$ and $x = 2k\pi$ with $k = \pm1, \pm2, \ldots$ are the fixed points of g.

The fixed-point iteration can not be used to find the solutions to the equation $x = g(x)$. Since for any interval I containing $x = 0$, $\max_{x\in I} |g'(x)| \geqslant |\cos(0) - 0\sin(0)| = 1$ and for any interval I_k containing $x = 2k\pi$, $\max_{x\in I_k} |g'(x)| \geqslant |\cos(2k\pi) - 2k\pi\sin(2k\pi) = 2k\pi > 1$, then the iterative sequence obtained by $x_{n+1} = g(x_n)$ diverges. □

§2.2 Exercises for Bracketing Methods

Exercise 1. What will happen if the bisection method is used with the function $f(x) = \tan(x)$ and

(a) the interval is [3, 4]? (b) the interval is [1, 3]?

Solve The roots of $f(x) = \tan(x)$ in $\mathbb{R}$ are $k\pi$ with $k = 0, \pm1, \pm2, \ldots$.

(a) For the interval $[3, 4]$, there is only one solution within this range for the function $f(x) = \tan(x)$. Therefore, using the bisection method in $[1, 3]$ can be effective in finding the solution. A sample iteration process is performed in table 1.

Table 1: Bisection Method Solution of $f(x) = \tan(x)$ in interval is $[3, 4]$

k	Left point a_k	Right point b_k	Center point c_k	Function value $f(c_k)$
0	3	4	3.5	0.374585640158595
1	3	3.5	3.25	0.108834025513330
2	3	3.25	3.125	-0.016594176499358
3	3.125	3.25	3.1875	0.045939623292659
4	3.125	3.1875	3.15625	0.014658396151121
5	3.125	3.15625	3.140625	$-9.676538918152837 \times 10^{-4}$
6	3.140625	3.15625	3.1484375	0.006844953310281
7	3.140625	3.1484375	3.14453125	0.002938604868838

(b) For the interval $[1, 3]$, the function $f(x) = \tan(x)$ has no solution within this range due to the periodic nature of the tangent function. This mean that using the bisection method may not be effective as it may lead to no possible solution. A sample iteration process is performed in table 2.

Table 2: Bisection Method Solution of $f(x) = \tan(x)$ in interval is $[1, 3]$

k	Left point a_k	Right point b_k	Center point c_k	Function value $f(c_k)$
0	1	3	2	-2.185039863261519
1	1	2	1.5	14.101419947171719
2	1.5	2	1.75	-5.520379922509330
3	1.5	1.75	1.625	-18.430862762369620
4	1.5	1.625	1.5625	$1.205325057225426 \times 10^{2}$
5	1.5625	1.625	1.59375	-43.558360406739730
6	1.5625	1.59375	1.578125	$-1.364479038428448 \times 10^{2}$
7	1.5625	1.578125	1.5703125	$2.066855189746604 \times 10^{3}$

$\square$

Exercise 2. Suppose that the bisection method is used to find a zero of $f(x)$ in the interval $[2, 7]$. How many times must this interval be bisected to guarantee that the approximation c_N has an accuracy of 5×10^{-9}?

Solve Assume the root of $f(x)$ in $[a,b]=[2,7]$ is x^*. To achive accuracy of $|c_N - x^*| \leqslant \epsilon = 5\times 10^{-9}$, it suffices to take

$$N \geqslant \frac{\log(b-a)-\log\epsilon}{\log 2} = \frac{\log 5 - (\log 5 - 9\log 10)}{\log 2} \approx 29.89735285398626.$$

Hence, 30 times must this interval be divided to guarantee that the approximation c_N has an accuracy of 5×10^{-9}. $\square$