THE SOUTH CHINA NORMAL UNIVERSITY

School of Mathematical Sciences

Numerical Analysis (2022-2023 The Second Term)

Homework 9:

Due Date: May 22, 2022 (Wednesday)

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§5.3 Exercises for Interpolation by Spline Functions

Exercise 1 Consider the polynomial $S(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

(a) Show that the conditions $S(1) = 1$, $S'(1) = 0$, $S(2) = 2$, and $S'(2) = 0$ produce the system of equations

$$\begin{array}{ccccccccc} a_0 & + & a_1 & + & a_2 & + & a_3 & = & 1 \\ & & a_1 & + & 2a_2 & + & 3a_3 & = & 0 \\ a_0 & + & 2a_1 & + & 4a_2 & + & 8a_3 & = & 2 \\ & & a_1 & + & 4a_2 & + & 12a_3 & = & 0 \end{array}$$

(b) Solve the system in part (a).

Solve (a) Since $S(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, by the conditions $S(1) = 1$ and $S(2) = 2$, we obtain $a_0 + a_1 + a_2 + a_3 = 1$ and $a_0 + 2a_1 + 4a_2 + 8a_3 = 2$. Since $S'(x) = a_1 + 2a_2x + 3a_3x^2$, by the conditions $S'(1) = 0$ and $S'(2) = 0$, we obtain $a_1 + 2a_2 + 3a_3 = 0$ and $a_1 + 4a_2 + 12a_3 = 0$.
(b) We can use Gaussian Elimination Method and back-substitution algorithm to solve the system

$$\begin{array}{ccccccccc} a_0 & + & a_1 & + & a_2 & + & a_3 & = & 1 \\ a_0 & + & 2a_1 & + & 4a_2 & + & 8a_3 & = & 2 \\ & & a_1 & + & 2a_2 & + & 3a_3 & = & 0 \\ & & a_1 & + & 4a_2 & + & 12a_3 & = & 0 \end{array} \tag{1}$$

The variable a_0 is eliminated from the second equation in (1) by subtracting the first equation from it.

$$\begin{array}{rcrcrcrcr} a_0 & + & a_1 & + & a_2 & + & a_3 & = & 1 \\ & & a_1 & + & 3a_2 & + & 7a_3 & = & 1 \\ & & a_1 & + & 2a_2 & + & 3a_3 & = & 0 \\ & & a_1 & + & 4a_2 & + & 12a_3 & = & 0 \end{array} \tag{2}$$

The variable a_1 is eliminated from the third and forth equations in (2) by subtracting the second equation from them.

$$\begin{array}{rcrcrcrcr} a_0 & + & a_1 & + & a_2 & + & a_3 & = & 1 \\ & & a_1 & + & 3a_2 & + & 7a_3 & = & 1 \\ & & & & -a_2 & + & -4a_3 & = & -1 \\ & & & & a_2 & + & 5a_3 & = & -1 \end{array} \tag{3}$$

The variable a_2 is eliminated from the forth equation in (3) by adding the third equation from it. Finally, we Multiply the third equation by -1 and arrive at the equivalent upper-triangular system:

$$\begin{array}{rcrcrcrcr} a_0 & + & a_1 & + & a_2 & + & a_3 & = & 1 \\ & & a_1 & + & 3a_2 & + & 7a_3 & = & 1 \\ & & & & a_2 & + & 4a_3 & = & 1 \\ & & & & & & a_3 & = & -2 \end{array} \tag{4}$$

Using back-substitution algorithm, we obtain $a_3 = -2, a_2 = 9, a_1 = -12, a_0 = 6$. □

§6.1 Exercises for Approximating The Derivative

Corollary 1 *Assume that* $f \in C^3[a, b]$, $x-h, x, x+h \in [a, b]$, *and that numerical computations are made. If* $|e_{-1}| \leq \epsilon, |e_1| \leq \epsilon$, *and* $M = \max_{a \leq x \leq b}\{|f^{(3)}(x)|\}$, *then*

$$|E(f, h)| \leq \frac{\epsilon}{h} + \frac{Mh^2}{6}, \tag{5}$$

and the value of h *that minimizes the right-hand side of* (5) *is*

$$h = \left(\frac{3\epsilon}{M}\right)^{1/3}. \tag{6}$$

Exercise 1. Show that (6) is the value of h that minimizes the right-hand side of (5).

Solve Let $g(h) = \frac{\epsilon}{h} + \frac{Mh^2}{6}$, which is the right-hand side formula of (5). Since $g'(h) = -\frac{\epsilon}{h^2} + \frac{Mh}{3}$ and for $x, y \in (0, +\infty)$,

$$\begin{aligned}(g'(x) - g'(y))(x - y) &= (-\frac{\epsilon}{x^2} + \frac{Mx}{3} + \frac{\epsilon}{y^2} - \frac{My}{3})(x - y) \\ &= \frac{\epsilon(x^2 - y^2)(x - y)}{x^2y^2} + \frac{M(x - y)^2}{3} \\ &= \frac{\epsilon(x - y)^2(x + y)}{x^2y^2} + \frac{M(x - y)^2}{3} \\ &\geqslant 0,\end{aligned}$$

$g(h)$ is convex. Hence, the local minimizer of $g(h)$ is a global minimizer. Solve $g'(h) = 0$, we obtain $h = \left(\frac{3\epsilon}{M}\right)^{1/3}$. Hence, $h = \left(\frac{3\epsilon}{M}\right)^{1/3}$ is the value that minimizes the right-hand side of (5).

$\square$