Study Notes of Topology

Pei Zhong

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Contents

Preface

The notes mainly refer to the following materials:

- Topology without tears
- lecture notes from Toronto university
- lecture notes from ustc
- Basic Topology by You Chengye

Chapter 1

Preliminary Knowledge

1.1 Countability

1.2 Reference

- Countability: lecture notes from toronto

Part I

Topology Space and Continuity

Chapter 2

Topological Space

This chapter opens with the definition of a topology and is then devoted to some simple examples.

Topology, like other branches of pure mathematics such group theory, is an axiomatic subjece. We start with a set of axioms and we use these axioms and we use these axioms to prove propositions and theorems. It is extremely important to develop your skill at writing proofs.

2.1 Topological Space

Definition 2.1

Let X be a non-empty set. A set $\tau \subseteq \mathcal{P}(X)$ is said to be a topology on $\mathcal{X}$ if
(1) $X, \varnothing \in \tau$,
(2) If $U_\alpha \in \tau(\alpha \in I$, I is finite or infinite), then $\cup_{\alpha \in I} U_\alpha \in \tau$,
(3) If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

Example 2.1: Trivial topology

Let X be any non-empty set and $\tau_t = \{X, \varnothing\}$. Then τ_t is called the trival topology on X.

(1) $X, \varnothing \in \tau$; (2) $X \cup \varnothing = X \in \tau$; (3) $X \cap \varnothing = \varnothing \in \tau$.

Example 2.2: Discrete topology

Let X be any non-empty set and $\tau_s = \mathcal{P}(X)$. Then τ_s is called the discrete topology on X.

(1) $X, \varnothing \in \tau$; (2) $\cup U_\alpha \in \tau$; (3) $U_1 \cap U_2 \in \tau$.

Example 2.3: Cofinite topology

Let X be any non-empty set and $\tau_f = \{U \subseteq X : U = \varnothing \; or \; U^c \; is \; finite\}$. Then τ_f is called the cofinite topology on X.

(1) $\varnothing \in \tau$, $X^c = \varnothing$ is finite with cardinality zero, then $X \in \tau$;

(2) If $U_\alpha \in \tau$ and $U_\alpha \neq \emptyset$ ($\emptyset$ has no effect on union). Let $U = \cup U_\alpha$, then $U^c = \cap U_\alpha^c$ is the intersection of finite set and so U^c is finite. Hence, $U \in \tau$;

(3) If $U_1, U_2 \in \tau$, let $U = U_1 \cap U_2$. Then $U^c = U_1^c \cup U_2^c$ is the union of finite set and so U^c is finite. Hence, $U \in \tau$.

Example 2.4: Cocountable topology

Let X be any non-empty set and $\tau_c = \{U \subseteq X : U = \emptyset \text{ or } U^c \text{ is countable}\}$. Then τ_c is called the cocountable topology on X.

(1) $\emptyset \in \tau$, $X^c = \emptyset$ is finite with cardinality zero, then $X \in \tau$;

(2) If $U_\alpha \in \tau$ and $U_\alpha \neq \emptyset$ ($\emptyset$ has no effect on union). Let $U = \cup U_\alpha$, then $U^c = \cap U_\alpha^c$ is the intersection of countable set and so U^c is countable. Hence, $U \in \tau$;

(3) If $U_1, U_2 \in \tau$, let $U = U_1 \cap U_2$. Then $U^c = U_1^c \cup U_2^c$ is the union of countable set and so U^c is countable. Hence, $U \in \tau$.

Example 2.5: Euclidean topology

$\tau_e = \{U : U = \cup_i (a_i, b_i), a_i < b_i \in \mathbb{R}\}$. The number of (a_i, b_i) can be infinite, finite or zero. Then τ_e is called the euclidean topology on $\mathbb{R}$. We write $E^1 = (\mathbb{R}, \tau_e)$.

(1) $\emptyset$ = empty union. Then $\emptyset \in \tau$. For every $x \in \mathbb{R}$, there exists (a_x, b_x) s.t. $x \in (a_x, b_x)$, then $\mathbb{R} = \cup_{x \in \mathbb{R}} (a_x, b_x) \in \tau$.

(2) (3) refer to topology without tears page 51.

2.2 Metric Topology

The most important class of topological spaces is the class of metric spaces. Metric spaces provide a rich source of examples in topology. But more than this, most of the applications of topology to analysis are via metirc spaces.

Definition 2.2

Let X be a non-empty set and d a real-valued function defined on $X \times X$ such that for $x, y, z \in X$:
(1) $d(x, y) \geqslant 0$ and $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) = d(x, y)$;
(3) $d(x, z) \leqslant d(x, y) + d(y, z)$.
Then d is said to be a metric on X, (X, d) is called a metric space and $d(a, b)$ is referred to as the distance between a and b.

Example 2.6

$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) | x_i \in \mathbb{R}, i = 1, 2, ..., n\}$. We defined the metirc in $\mathbb{R}^n$ as

$$d((x_1, ..., x_n), (y_1, ..., y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

$(\mathbb{R}^n, d)$ is called n dimension euclidean space, denoted by E^n.

Definition 2.3

Let (X, d) be a metirc space and ϵ any positive real number. Then the open ball about $x_0 \in X$ of radius ϵ is the set $B(x_0, \epsilon) = \{x \in X : d(x_0, x) \leq \epsilon\}$

Example 2.7

In $\mathbb{R}$ with the euclidean metric, $B(x_0, \epsilon)$ is the open interval $(x_0 - \epsilon, x_0 + \epsilon)$.

Lemma 2.1

Let (X, d) be a metric space and $x, y \in X$. Further, let ϵ_1 and ϵ_2 be positive real numbers. If $z \in B(x, \epsilon_1) \cap B(y, \epsilon_2)$, then there exists a $\epsilon > 0$ such that $B(z, \epsilon) \subseteq B(x, \epsilon_1) \cap B(y, \epsilon_2)$.

Proof. Let $\epsilon = \min\{\epsilon_1 - d(x, z), \epsilon_2 - d(y, z)\}$, then for $a \in B(z, \epsilon)$,

$$\begin{aligned} d(a, x) &\leqslant d(a, z) + d(z, x) \leqslant \epsilon + d(x, z) = \epsilon_1, \\ d(a, y) &\leqslant d(a, z) + d(z, y) \leqslant \epsilon + d(y, z) = \epsilon_2. \end{aligned}$$

Hence, $a \in B(x, \epsilon_1) \cap B(y, \epsilon_2)$ and so $B(z, \epsilon) \subseteq B(x, \epsilon_1) \cap B(y, \epsilon_2)$. □

Corollary 2.1

Let (X, d) be a metric space and B_1 and B_2 open balls in (X, d). Then $B_1 \cap B_2$ is a union of open balls in (X, d).

Proof. By lemma2.1, $\forall z \in B_1 \cap B_2$, there exists $\epsilon_z > 0$ such that $B(z, \epsilon_z) \subseteq B_1 \cap B_2$. Then $\cup_{z \in B_1 \cap B_2} B(z, \epsilon_z) \subseteq B_1 \cap B_2 \subseteq \cup_{z \in B_1 \cap B_2} B(z, \epsilon_z)$. Hence, $B_1 \cap B_2 = \cup_{z \in B_1 \cap B_2} B(z, \epsilon_z)$ □

Proposition 2.1

Let (X, d) be a metric space. Then $\tau_d = \{U : U = \cup_\alpha B(x_\alpha, \epsilon_\alpha)\}$ is a topology on X.

Proof. (1) $\emptyset$ = empty union, then $\emptyset \in \tau_d$. $X = \cup_{x \in X} B(x, \epsilon_x) \in \tau_d$.
(2) The union of open ball union is open ball union.
(3) If $U, U' \in \tau_d$, then $U = \cup_\alpha B(x_\alpha, \epsilon_\alpha)$, $U' = \cup_\beta B(x_\beta, \epsilon_\beta)$, then

$$\begin{aligned} U \cap U' &= (\cup_\alpha B(x_\alpha, \epsilon_\alpha)) \cap (\cup_\beta B(x_\beta, \epsilon_\beta)) \\ &= \cup_{\alpha,\beta} (B(x_\alpha, \epsilon_\alpha) \cap B(x_\beta, \epsilon_\beta)). \end{aligned}$$

By corollary2.1, $B(x_\alpha, \epsilon_\alpha) \cap B(x_\beta, \epsilon_\beta)$ is the union of open ball. Then $U \cap U'$ is the union of open ball. $\square$

τ_d is called the topology induced by metirc or simply metric topology.

2.3 Basic Conception in Topological Space

Rather than continually refer to "members of τ", we find it more convenient to give such sets a name. We call them "open sets". We shall also name the complements of open sets. They will be called "closed sets".

2.3.1 Open set

Definition 2.4

Let (X, τ) be any topological space. Then the members of τ are said to be open sets.

Proposition 2.2

Let U be a subset of a topological space (X, τ). Then $U \tau$ iff for each $x \in U$ there exists $U_x \in \tau$ such that $x \in U_x \subseteq U$.

Proof. ($\Rightarrow$): Since $U \in \tau$, for each $x \in U$, take $K = U$, then $x \in K \subseteq U$.
($\Leftarrow$): Since $U \subseteq \cup_{x \in U} U_x \subseteq U$, $U = \cup_{x \in U} U_x \in \tau$. $\square$

Remark. This proposition provides a useful test of whether a set is open or not. It says that a set is open iff it contains an open set about each of its points.

2.3.2 Closed set

Definition 2.5

Let (X, τ) be a topological space. A subset A of X is said to be closed set in (X, τ) if its complements in X, denoted by A^c, is open in (X, τ).

Proposition 2.3

If (X, τ) is any topological space, then
(1) $\emptyset$ and X are closed set.
(2) the intersection of any (finite or infinite) number of closed sets is a closed set and
(3) the union of any finite number of closed sets is closed set.

Proof. 1 $\square$

2.3.3 Neighbourhood, interior point, interior

Definition 2.6

Let (X, τ) be a topological space, A a subset of X and x a point in X. Then If there exists an open set U such that $x \in U \subseteq A$, then x is called a interior point of A and A is called the neighbourhood of x. The collection of all interior point in A is called the interior of A, denoted by $\text{Int}(A)$.

Proposition 2.4

(1) $x \in \text{Int}(A) \Leftrightarrow \exists U \in \tau$ with $x \in U, U \cap A^c = \emptyset$.
(2) If $A \subset B$, then $\text{Int}(A) \subset \text{Int}(B)$;
(3) $\text{Int}(A)$ is the largest open subset of X contained in A;
(4) $\text{Int}(A)$ is the union of all open sets of X contained in A;
(5) $\text{Int}(A) = A$ iff A is open;
(6) $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$;
(7) $\text{Int}(A \cup B) \supset \text{Int}(A) \cup \text{Int}(B)$.

2.3.4 Limit point and closure, exterior, boundary

Definition 2.7

Let A be a subset of a topological space (X, τ). A point $x \in X$ is said to be a limit point (or accumulation point or cluster point) of A if every open set, U, containing x contains a point of $A \setminus \{x\}$, i.e. $\forall U \in \tau$ with $x \in U$, $U \cap A \setminus \{x\} \neq \emptyset$. The collection of all limit points of A is called derived set, denoted by A'. $\overline{A} := A \cup A'$ is called the closure of A.

Remark. From the definition of $\overline{A}$, we can get $x \in \overline{A} \Leftrightarrow \forall U \in \tau$ with $x \in U, U \cap A \neq \emptyset$.

The conception of limit point derived from Euclidean space. But we should note the current promotion conception has changing in meaning. In Euclidean space , finite sets have no limit points. However, in general topological space, finite set can do.

Example 2.8

Consider the topological space (X, τ) where the set $X = \{a, b, c, d, e\}$, the topology $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$, and $A = \{a, b, c\}$. Then b, d and e are limit points of A but a and c are not limit points of A.

The point x is a limit point of A iff every open set containing x contains another point of the set A. So to show x is a limit point of A, we should writing down all of the open sets containing x and verifying that each contains a point of A other than x. And to show that x is note a limit point of A, it suffices ot find even one open set which contains x but contains no other point of A.

The set $\{a\} \in \tau$ with $a \in \{a\}$, but $\{a\} \cap A \setminus \{a\} = \emptyset$. The set $\{c, d\} \in \tau$ with $c \in \{c, d\}$, but $\{c, d\} \cap A \setminus \{c\} = \emptyset$. Hence, a and c are not limit point of A.

The open sets containing b are X and $\{b, c, d, e\}$. Then $X \cap A \setminus \{b\} = a, c \neq \emptyset$ and

haven't done!

Example 2.9: Limit point in discrete topology

Let (X, τ_d) be a discrete space and A a subset of X. Then A has no limit points, since for each $x \in X$, $\{x\}$ is an open set containing no point of A different from x.

Example 2.10: Limit point in trival topology

Let (X, τ_t) be a trival space and A a subset of X with at least two elements. Every point of X is a limit point of A, since for each $x \in X$, $X \cap A \setminus \{x\} \neq \emptyset$. If A is single set $\{x\}$, then every point of X rather than x is a limit point of A.

Example 2.11: Limit point in cofinite topology

Let (X, τ_f) be a cofinite space and A a subset of X.
(1) If X is finite, then $\tau = \mathcal{P}(X)$. Then every point of X is not a limit point of A, since for $x \in X$, $\{x\} \cap A \setminus \{x\} = \emptyset$.
(2) If X is infinte and A is finite, every point of X is not a limit point of A, since $((X \setminus A) \cup \{x\})^c \subset A$ is finite and $((X \setminus A) \cup \{x\}) \cap A \setminus \{x\} = \emptyset$.
(3) If X is infinte and A is infinite, then every point of X is the limit point of A.

Let's check (3). Firstly, we verify that for any $U \in \tau$, $U \cap A$ is infinite. Since $U \in \tau$, U^c is finite. And we have

$$A = A \cap (U \cup U^c) = (A \cap U) \cup (A \cap U^c).$$

Suppose $A \cap U$ is finite. Since $A \cap U^c$ is finite, A is the union of two finite sets. Then, A is finite. This is a contradiction as A is infinite. Hence, $A \cap U$ is infinite. Thus, $(U \cup \{x\}) \cap A \setminus \{x\} \neq \emptyset$. Hence, every point of X is the limit point of A.

Example 2.12: Limit point in cocountable topology

Let (X, τ_c) be a cocountable space and A a subset of X.
(1) If A is uncountable, then every point of X is a limit point of A.
(2) If A is countable or finite, then A contains all its limit points.(that is, A is closed).

(1) For any $U \in \tau$, U^c is countable. Then $(U \cup \{x\})^c = U^c \cap \{x\}^c \subseteq U^c$ is countable. Then $U \cup \{x\} \in \tau$. Suppose $(U \cup \{x\}) \cap A \setminus \{x\} = \emptyset$. Then $A \setminus \{x\} \subseteq U^c$ and so A is countable. So if A is uncountable, it is bound to $(U \cup \{x\}) \cap A \setminus \{x\} \neq \emptyset$. Hence, $x \in X$ is a limit point of A.

(2) For any $U \in \tau$, U^c is countable. Since $(A^c)^c = A$ is countable, $A^c \in \tau$. Then A is closed. Hence, A contains all its limit points.

Example 2.13: Limit point in euclidean topology

Let $(\mathbb{R}, \tau_e)$ be a euclidean space and $\langle a, b\rangle$ a subset of $\mathbb{R}$. ($\langle a, b\rangle$ is any case in $(a, b), (a, b], [a, b), [a, b]$). The point in $[a, b]$ is the limit point of $\langle a, b\rangle$.

Definition 2.8

Let (X, τ) be a topological space and A a subset of X. Then exterior of A

$$\text{Ext}(A) = \text{Int}(A^c).$$

Proposition 2.5

(1) $x \in \text{Ext}(A) \Leftrightarrow \exists U \in \tau$ with $x \in U, U \cap A \neq \emptyset$.
(2) $\text{Ext}(A) = (\overline{A})^c$.

Definition 2.9

Let (X, τ) be a topological space and A a subset of X. The boundary of A consistis of all the points in $\overline{A}$ but not in $\text{Int}(A)$. Thus, the boundary of A

$$\partial A := \overline{A} \setminus \text{Int}(A).$$

Proposition 2.6

(1) $x \in \partial A \Leftrightarrow \forall U$ with $x \in U, A \cap U \neq \emptyset$ and $A^c \cap U \neq \emptyset$.
(2) $\partial A = \overline{A} \cap \overline{A^c}$
(3) $\partial A = A \setminus (\text{Int}(A) \cup \text{Ext}(A))$

Proposition 2.7

$A = \text{Int}(A) \cup \text{Ext}(A) \cup \partial(A)$.

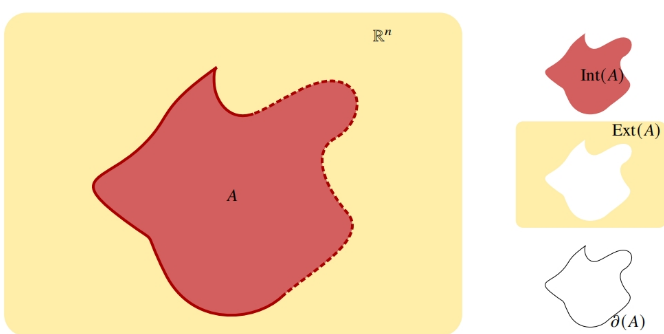

Figure 2.1

By the following proposition, we can know closure and interior are closely related.

Proposition 2.8

If $A = B^c$, then $\overline{A} = (\text{Int}(B))^c$.

Proposition 2.9

(1) If $A \subset B$, then $\overline{A} \subset \overline{B}$;
(2) $\overline{A}$ is the smallest closed subset of X containing A;
(3) $\overline{A}$ is the intersection of all closed sets of X containing A;
(4) $\overline{A} = A$ iff A is closed;
(5) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
(6) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Example 2.14

Let $X = \{a, b, c, d, e\}$ and

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Show that $\overline{\{b\}} = \{b, e\}$.

Definition 2.10

Let A be a subset of a topological space (X, τ). Then A is said to be dense in X if $\overline{A} = X$. X is said to be separable if there exists countable dense subset in X.

Proposition 2.10

Let A be a subset of a topological space (X, τ). Then A is dense in X iff every non-empty open subset of X intersects A non-trivially, i.e. if $U \in \tau$ and $U \neq \emptyset$ then $A \cap U \neq \emptyset$.

Example 2.15

(X, τ_d) is separable iff X is countable.

refer to website proof.

Example 2.16

(X, τ_t) is separable.

Let A be a countably infinite subset of X. By example2.10, $\overline{A} = X$. Hence, A is dense in X and so X is separable.

Example 2.17

(X, τ_f) is separable.

Let A be a countably infinite subset of X. By example2.11, $\overline{A} = X$. Hence, A is dense in X and so X is separable.

Example 2.18

(X, τ_c) is inseperable.

Let A be a countably infinite subset of X.By example2.12, $\overline{A} = A$. Hence, X have no countably dense subset and so inseperable.

Example 2.19

$(\mathbb{R}, \tau_e)$ is separable.

prove that $\overline{\mathbb{Q}}$=$\mathbb{R}$.

2.3.5 Sequence convergence

Definition 2.11

Let (X, τ) be a topological space and A a subset of X. Let $(x_n)_{n\in\mathbb{N}}$ be an infinite sequence in A. Then (x_n) converages to the limit $x \in X$(denoted by $x_n \to x$) iff

$$\forall U \in \tau \text{ with } x \in U \Rightarrow \{n \in \mathbb{N} : x_n \notin U\} \text{ is finite.}$$

Or,

$$\forall U \in \tau \text{ with } x \in U \Rightarrow \exists N \in \mathbb{N}, \forall n > N, x_n \in U.$$

In euclidean space, the convergence point of a convergent sequence is unique and when x is the limit point of the set A, there is a sequence (x_n) in A, which converges to x. However, In general topological space, some difference appear.

Proposition 2.11

Let (x_n) be a sequence which elements are different in (R, τ_f), then $\forall x \in X$, $x_n \to x$.

Proof. $\forall U \in \tau$ with $x \in U \Rightarrow \{n \in \mathbb{N} : x_n \notin U\} = \{n \in \mathbb{N} : x_n \in U^c\}$ is finite. □

Proposition 2.12

Let (x_n) be a sequence which elements are different in (R, τ_f), then

$$x_n \to x \Leftrightarrow \exists N \in \mathbb{N}, \forall n > N, x_n = x.$$

Proof. ($\Leftarrow$) clear by definition2.11
($\Rightarrow$) Consider the set $B := \{x_n : x_n \neq x\}$. Since a sequence is countable, B is countable. By example2.12, B is closed. By construction, $x \notin B$, so $U = X \setminus B$ is an open set containing x. But $x_n \to x$, so a tail of this sequence must lie in $X \setminus B$. Since $\{x_n\} \cap (X \setminus B) = \{x\}$, this means that a tail of this sequence is constanst. □

2.4 Subspace

Definition 2.12

Let A be a non-empty subset of a topological space $(X\ \tau)$. The collection

$$\tau_A = \{O \cap A : O \in \tau\}$$

of subsets of A is a topology on A called the subspace topology (or the topology induced on A by τ). The topological space (A, τ_A) is said to be a subspace of (X, τ).

Let's check that τ_A is indeed a topology on A.
(1) $A = X \cap A, \emptyset = \emptyset \cap A$, then $A, \emptyset \in \tau_A$.
(2) If $U_\alpha \in \tau_A$, then $\cup_\alpha U_\alpha = \cup_\alpha (O_\alpha \cap A) = (\cup_\alpha O_\alpha) \cap A \in \tau_A$.
(3) If $U_1, U_2 \in \tau_A$, then $U_1 \cap U_2 = (O_1 \cap A) \cap (O_2 \cap A) = (O_1 \cap O_2) \cap A \in \tau_A$.

In the following content, we follow the convention: a subset of topological Spaces is treated as a subspace.

Let (X, τ) be a topological space and $B \subseteq A \subseteq X$. Then

$$\begin{aligned}
(\tau_A)_B =& \{K \cap B : K \in \tau_A\} \\
=& \{(O \cap A) \cap B : O \in \tau\} \\
=& \{(O \cap B) \cap (A \cap B) : O \in \tau\} \\
\overset{B \subseteq A}{=}& \{(O \cap B) \cap B : O \in \tau\} \\
=& \{(O \cap B) : O \in \tau\} \\
=& \tau_B.
\end{aligned}$$

Hence, there are two way to induce the topology on B: induced by the topology on A or induced by the topology on X.

Example 2.20

Let $X = \{a, b, c, d, e, f\}$,

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\},$$

and $A = \{b, c, e\}$. Then the subspace topology on A is

$$\tau_A = \{A, \emptyset, \{c\}\}.$$

Consider the subset $[1, 2]$ of $(\mathbb{R}, \tau_e)$. Then the topology on $[1, 2]$ is

$$\tau = \{(a, b) \cap [1, 2] : (a, b) \in \tau_e\}.$$

But here we see some surprising things happening; e.g. $[1, \frac{3}{2})$ is certainly not an open set in $\mathbb{R}$, but $[1, \frac{3}{2}) = (1, \frac{3}{2}) \cap [1, 2]$, is an open set in the subspace $[1, 2]$.

Also $(1, 2]$ is not open in $\mathbb{R}$ but is open in $[1, 2]$. Even $[1, 2]$ is not open in $\mathbb{R}$, but is an open set in $[1,2]$.

So whenever we speak of a set being open we must make perfectly clear in what space or what topology it is an open set.

Proposition 2.13

Let (X,τ) be a topological space and $C \subset A \subset X$, then

$$C \text{ is closed in } A \Leftrightarrow C = A \cap V, \text{ where } V \text{ is closed in } X.$$

Proof.

$$\begin{aligned} C \text{ is closed in } A &\Leftrightarrow A \setminus C \text{ is open in } A \\ &\Leftrightarrow \exists O \in \tau_X, \text{ s.t. } A \setminus C = O \cap A \\ &\Leftrightarrow \exists O \in \tau_X, C = A \setminus (O \cap A) \\ &\qquad = (A \setminus O) \cup (A \setminus A) \\ &\qquad = (A \setminus O) \cup \emptyset = A \setminus O \\ &\qquad = (A \cap X) \setminus O \\ &\qquad = A \cap (X \setminus O) \\ &\qquad = A \cap V, \text{ where } V \text{ is closed in } X. \end{aligned}$$

□

Proposition 2.14

Let (X,τ) be a topological space, $B \subset A \subset X$, then
(1) If B is open(closed) in X, then B is open(closed) in A;
(2) If A is open(closed) in X and B is open(closed) in A, then B is open(closed) in X.

Proof. (1) We know that $B = B \cap A$. If $B = O$, then $B = O \cap A$ and so B is open in A. If B is closed in X, by proposition2.13, $B = X \cap V$, where V is closed in X. Then $B = B \cap A = (X \cap V) \cap A = (X \cap A) \cap V$, by by proposition2.13, B is closed in A.
(2) If $B = O_1 \cap A$ and $A = O_2$, then $B = O_1 \cap O_2 \in \tau$ and so B is open in X. If A is closed in X and B is closed in A, then $A = X \cap V_1, B = A \cap V_2$, where V_1, V_2 is closed in X. Then, $B = X \cap (V_1 \cap V_2)$. As $V_1 \cap V_2$ is closed in X, B is closed in X. □

2.5 Reference

- lecture notes from uky

2.6 Exercise

Chapter 3

Continuous Mappings and Homeomorphisms

3.1 Continuous Mappings

We are already familiar with the notion of a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at $x_0 \in \mathbb{R}$ iff each positive real number ϵ, there exists a positive real number δ such that $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$.

It is not all obvious how to generalize this definition to general topological spaces where we do not have "absolute value" or "subtraction". So we shall seek another(equivalent) definition of continuity which lends itself more to generalization.

It is easily seen that: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ iff for each interval $(f(x_0) - \epsilon, f(x_0) + \epsilon)$, for $\epsilon > 0$, there exists a $\delta > 0$ such that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

This definition is an improvement since it does not involve the concept "absolute value" but it still involves "substraction". The next definition shows how to avoid substraction.

Definition 3.1

Let (X, τ) and (Y, τ') be topological spaces and f a function from X into Y. Then f is continuous at $x_0 \in X$ iff for each $U \in \tau'$ containing $f(x_0)$, there exists $K \in \tau$ containing x_0, such that $f(K) \subseteq U$.

Definition 3.2

Let (X, τ) and (Y, τ') be topological spaces and f a function from X into Y. Then f is continuous iff for each $x_0 \in X$ and for each $U \in \tau'$ containing $f(x_0)$, there exists $K \in \tau$ containing x_0, such that $f(K) \subseteq U$.

As in analysis, continuity is a local concept.

Proposition 3.1

Let (X, τ) and (Y, τ') be topological spaces and f a function from X into Y , A a subset of X and $x_0 \in A$. We define the restriction of f on A as $f_A = f|A : A \to Y$, then
(1) If f is continuous at x_0, then f_A is continuous at x_0.
(2) When A is open in X, if f_A is continuous at x_0, then f is continuous at x_0.

Proof. (1) We need to prove for each $U \in \tau'$ with $f_A(x_0) \in U$, there exists $O \in \tau_A$ with $x_0 \in O$, $f_A(O) \subseteq U$. f is continuous at x_0 and $x_0 \in A$, then for each $U \in \tau'$ with $f(x_0) = f_A(x_0) \in U$, there exists $K \in \tau$ with $x_0 \in K$, $f(K) \subseteq U$. Since $A \cap K \in \tau_A$ with $x_0 \in A \cap K$ and $f_A(A \cap K) = f(A \cap K) \subseteq f(A) \cap f(K) \subseteq Y \cap U = U$. Hence, f_A is continuous at x_0.

(2) f_A is continuous at x_0, then for each $U \in \tau'$ containing $f_A(x_0) = f(x_0)$, there exists $(K \cap A) \in \tau_A (K \in \tau)$ containing x_0, such that $f_A(K \cap A) \subseteq U$. Since $A \in \tau$, $K \cap A \in \tau$ and $f(A \cap K) = f_A(A \cap K) \subset U$. Hence, f is continuous at x_0. □

Definition 3.3

Let f be a function from a set x into a set Y. If S is any subset of Y, then the set $f^{-1}(S)$ is defined by

$$f^{-1}(S) = \{x : x \in X \text{ and } f(x) \in S\}.$$

Then subset $f^{-1}(S)$ of X is said to be the inverse image of S.

Remark. Note that an inverse function of f exists iff f is bijective. But the inverse image of any subset of Y exists even if f is neither one-to-one nor onto.

Proposition 3.2

Let f be a mapping of a topological space (X, τ) into a topological space (Y, τ'). Then the following conditions are equivalent:
(1) f is continuous;
(2) for each $U \in \tau'$, $f^{-1}(U) \in \tau$;
(3) for each closed set V in Y, $f^{-1}(Y)$ is closed in X.

Proof. □

In $(\mathbb{R}, \tau_e)$, we can use sequence convergence to characterize continuity, but in general topological space, we cannot do this.

Proposition 3.3

$f : X \to Y$ is continuous at $x_0 \in X$, then $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.

Proof. f is continous at $x_0 \in X$ and $x_n \to x_0$, then $\forall U \in \tau_Y$ containing $f(x_0)$, there exists $K \in \tau_X$ containing x_0 such that $f(K) \subseteq U$. Since $x_n \to x_0$, $\exists N \in \mathbb{N}$, $\forall n > N$, $x_n \in K$, then $f(x_n) \in f(K) \subseteq U$. So $f(x_n) \to f(x_0)$. □

However, the inverse proposition is not true. Let $f : X \to Y$ be injective, X be a uncountable space with τ_c and Y be a discrete space. Then, by proposition2.12, When $x_n \to x_0$ in X, $\exists N \in N$, $\forall n > N$, $x_n = x$, then $f(x_n) = f(x)$ and so $f(x_n) \to f(x)$. But f is not continuous at x_0, because for $U = \{f(x_0)\} \in \tau_Y$ containing $f(x_0)$, $\forall K \in \tau_X$ containing x_0, $f(K) \supset \{f(x_0)\}$ as f is injective and $K \supset \{x_0\}$.

3.2 The properties of continuous mapping

Firstly, we introduce some simple and common continuous mappings.

Proposition 3.4

Identity mapping $\mathrm{id} : X \to X$ is continuous.

3.3 Exercise

Exercise 3.1

$f : X \to Y$ is called open(closed) mapping, if $f(X)$ is open(closed). Illustrate that open mapping may not be closed mapping and vice versa.

Proof. □

Exercise 3.2: I

$f : X \to Y$ is bijective, then

$$f \text{ is open mapping } \Leftrightarrow f \text{ is closed mapping } \Leftrightarrow f^{-1} \text{ is continuous.}$$

Proof. $\forall U \in \tau_X$,

$$\begin{aligned} f(U) \in \tau_Y &\Leftrightarrow Y \setminus f(U) \text{ is closed} \\ &\overset{f \text{ is bijective}}{\Leftrightarrow} f(X \setminus U) \text{ is closed} \\ &\Leftrightarrow f \text{ is closed mapping.} \end{aligned}$$

Since f is bijective, f^{-1} exists. As f is open mapping, f^{-1} is continuous. □

Remark. From this exercise, we can know if f is bijective, continuous and open, then f is homeomorphism.

Chapter 4

Topological basis and Product Space

Let's recall the euclidean topology in $\mathbb{R}$,

$$\tau_e = \{U : U = U_i(a_i, b_i), a_i < b_i \in \mathbb{R}\}.$$

It seems like the entire collection of sets in τ_e can be specified by declaring that just the usual open intervals are open. Once these "special sets" are known to be open, we get all the other sets for free by taking unions. These special collections of sets are called bases of topologies.

Definition 4.1

Let (X, τ) be a topological space. A collection $\mathcal{B}$ of open subsets of X is said to be a basis for the topology τ if for each $U \in \tau$, $U = \cup_{i \in I} B_i$, where I is a index set and $B_i \in \mathcal{B}$.

Remark. If $\mathcal{B}$ is a basis for a topology τ on a set X then a subset U of X is in τ iff it is a union of members of $\mathcal{B}$. So $\mathcal{B}$ "generates" the topology τ in the following sense: if we told what sets are members of $\mathcal{B}$ then we can determine the members of τ – they are just all the sets which are unions of members of $\mathcal{B}$.

Example 4.1

$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for $(\mathbb{R}, \tau_e)$.

Example 4.2

$\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for (X, τ_d).

Part II

Topology Invariant

Chapter 5

The Axioms of Countablity and Separation

Chapter 6

Compactness

Theorem 6.1

In metric space, compactness and sequential compactness are equivalent.

Remark. In general topology space, compactness and sequential compactness are not equivalent. You can refer to answer in mathexchange.

6.1 Reference

- lecture note from Florida
- The Lebesgue Number of a Covering