THE SOUTH CHINA NORMAL UNIVERSITY

School of Mathematical Sciences

Numerical Analysis (2023-2024 The Second Term)

Homework 5:

Due Date: April 24, 2024 (Wednesday)

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§3.4 Gaussian Elimination and Pivoting

Exercise 1. Find the parabola $y = A + Bx + Cx^2$ that passes through $(1, 6), (2, 5)$, and $(3, 2)$.

Solve For each point we obtain an equation relating the value of x to the value of y. The result is the linear system

$$\begin{array}{ccccccl} A & + & B & + & C & = & 6 \qquad \text{at } (1,6) \\ A & + & 2B & + & 4C & = & 5 \qquad \text{at } (2,5) \\ A & + & 3B & + & 9C & = & 2 \qquad \text{at } (3,2). \end{array} \tag{1}$$

The variable A is eliminated from the second and third equations by subtracting the first equation from them, and the resulting equivalent linear system is

$$\begin{array}{ccccccc} A & + & B & + & C & = & 6 \\ & & B & + & 3C & = & -1 \\ & & 2B & + & 8C & = & -4. \end{array} \tag{2}$$

The variable B is eliminated from the third equation in (2) by subtracting from it two times the second equation. We arrive at the equivalent upper-triangular system:

$$\begin{array}{ccccccc} A & + & B & + & C & = & 6 \\ & & B & + & 3C & = & -1 \\ & & & & 2C & = & -2. \end{array} \tag{3}$$

The back-substitution algorithm is now used to find the coefficients $C = -1, B = 2, A = 5$, and the equation of the parabola is $y = 5 + 2x - x^2$.

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§3.5 Triangular Factorization

Exercise 1. Find the triangular factorization $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ for the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix}.$$

Solve We've seen how to use elimination to convert a matrix $\boldsymbol{A}$ into an upper triangular matrix $\boldsymbol{U}$. This leads to the factorization $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$. we can describe the elimination of the entries of matrix $\boldsymbol{A}$ in terms of multiplication by a succession of elimination matrices L_i, so that $\boldsymbol{A} \longrightarrow \boldsymbol{L}_1\boldsymbol{A} \longrightarrow \boldsymbol{L}_2\boldsymbol{L}_1\boldsymbol{A} \longrightarrow \ldots \boldsymbol{U}$. The first step of Gaussian elimination looks like this: twice the first row is subtracted from the second one, five times the first row is subtracted from the third one, and three times first row is added to forth one. This can be written as:

$$\boldsymbol{L}_1\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & -3 & 1 & -18 \\ 0 & 3 & 2 & 18 \end{bmatrix}$$

Next we subtract the second row from the third and add it to the fourth row:

$$\boldsymbol{L}_2\boldsymbol{L}_1\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & -3 & 1 & -18 \\ 0 & 3 & 2 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 7 & 10 \end{bmatrix}$$

Finally we eliminate the fourth row:

$$\boldsymbol{L}_3\boldsymbol{L}_2\boldsymbol{L}_1\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{7}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & -\frac{15}{2} \end{bmatrix}$$

To exhibit the full factorization $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ we need to compute the product $\boldsymbol{L} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1}\boldsymbol{L}_3^{-1}$. And the inverse of L_j, $j = 1, 2, 3$ is just L_j itself, but with each entry below the diagonal negated:

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}, L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{7}{4} & 1 \end{bmatrix}$$

The product $L_1^{-1}L_2^{-1}L_3^{-1}$ is just the unit lower-triangular matrix with the nonzero sub-diagonal entries of L_1^{-1}, L_2^{-1} and L_3^{-1} inserted in the appropriate places:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -1 & -\frac{7}{4} & 1 \end{bmatrix}$$

Together we have

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -1 & -\frac{7}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & -\frac{15}{2} \end{bmatrix} = LU.$$

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