THE SOUTH CHINA NORMAL UNIVERSITY

School of Mathematical Sciences

Numerical Analysis (2022-2023 The Second Term)

Homework 7:

Due Date: **May 8, 2024 (Wednesday)**

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§4.3 Exercise for Lagrange Approximation

Exercise 1. Find Lagrange polynomials that approximate $f(x) = x^3$.

(a) Find the linear interpolation polynomial $P_1(x)$ using the nodes $x_0 = -1$ and $x_1 = 0$.

(b) Find the quadratic interpolation polynomial $P_2(x)$ using the nodes $x_0 = -1, x_1 = 0$, and $x_2 = 1$.

(c) Find the cubic interpolation polynomial $P_3(x)$ using the nodes $x_0 = -1, x_1 = 0, x_2 = 1$, and $x_3 = 2$.

Solve A Lagrange polynomial $P_N(x)$ of degree at most N that passes through the $N+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$ has the form

$$P_N(x) = \sum_{k=0}^{N} y_k \cdot L_{N,k}(x), \tag{1}$$

where $L_{N,k}$ is the Lagrange coefficient polynomial based on these nodes:

$$L_{N,k}(x) = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_N)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1}\ldots(x_k-x_N))}. \tag{2}$$

(a) Using (2) with the abscisas $x_0 = -1$ and $x_1 = 0$ and the ordinates $y_0 = f(x_0) = -1$ and $y_1 = f(x_1) = 0$, we can produce

$$\begin{aligned} P_1(x) &= y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0} \\ &= -1 \frac{x-0}{-1} + 0 \\ &= x. \end{aligned}$$

(b) Using (2) with the abscisas $x_0 = -1, x_1 = 0$, and $x_2 = 1$ and the ordinates $y_0 = f(x_0) = -1$, $y_1 = f(x_1) = 0$ and $y_2 = f(x_2) = 1$, we can produce

$$\begin{aligned}
P_2(x) &= y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= -1\frac{x(x-1)}{-1(-1-1)} + 0 + 1\frac{(x+1)x}{(1+1)1} \\
&= -\frac{x(x-1)}{2} + \frac{x(x+1)}{2} \\
&= \frac{2x}{2} = x.
\end{aligned}$$

(c) Using (2) with the abscisas $x_0 = -1, x_1 = 0, x_2 = 1$, and $x_3 = 2$ and the ordinates $y_0 = f(x_0) = -1$, $y_1 = f(x_1) = 0$, $y_2 = f(x_2) = 1$ and $y_3 = f(x_3) = 8$, we can produce

$$\begin{aligned}
P_2(x) &= y_0\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\
&\quad + y_2\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\
&= -1\frac{x(x-1)(x-2)}{-1(-1-1)(-1-2)} + 0 + 1\frac{(x+1)x(x-2)}{(1+1)1(1-2)} + 8\frac{(x+1)x(x-1)}{(2+1)2(2-1)} \\
&= \frac{x(x-1)(x-2)}{6} - \frac{x(x+1)(x-2)}{2} + \frac{4x(x+1)(x-1)}{3} \\
&= \frac{x(x^2-3x+2) - 3x(x^2-x-2) + 8x(x^2-1)}{6} \\
&= \frac{6x^3}{6} = x^3.
\end{aligned}$$

□

§4.4 Exercises for Newton Polynomials

Exercise 1. Use the centers x_0, x_1, x_2, and x_3 and the coefficients a_0, a_1, a_2, a_3, and a_4 to find the Newton polynomials $P_1(x), P_2(x)$ and $P_3(x)$, and evaluate them at the value $x = c$.

$$\begin{array}{llll}
a_0 = 4 & a_1 = -1 & a_2 = 0.4 & a_3 = 0.01 \\
x_0 = 1 & x_1 = 3 & x_2 = 4 & c = 2.5
\end{array}$$

Solve A Newton polynomial $P_N(x)$ of degree at most N has the form

$$\begin{aligned}
P_N(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots \\
+ a_N(x-x_0)(x-x_1)\dots(x-x_{N-1}).
\end{aligned} \tag{3}$$

Then we have

$$\begin{aligned}
P_1(x) &= 4 - (x-1) \\
P_2(x) &= 4 - (x-1) + 0.4(x-1)(x-3) \\
P_3(x) &= 4 - (x-1) + 0.4(x-1)(x-3) + 0.01(x-1)(x-3)(x-4)
\end{aligned}$$

Evatuating the polynomials at $c = 2.5$ results in

$$\begin{aligned}
P_1(x) &= 4 - (2.5-1) = 4 - 1.5 = 2.5 \\
P_2(x) &= 4 - (2.5-1) + 0.4(2.5-1)(2.5-3) = 4 - 1.5 - 0.3 = 2.2 \\
P_3(x) &= 4 - (2.5-1) + 0.4(2.5-1)(2.5-3) + 0.01(2.5-1)(2.5-3)(2.5-4) \\
&= 4 - 1.5 - 0.3 + 0.01125 = 2.21125.
\end{aligned}$$

$\square$