THE SOUTH CHINA NORMAL UNIVERSITY

School of Mathematical Sciences

Numerical Analysis (2023-2024 The Second Term)

Homework 6:

Due Date: April 27, 2024 (Wednesday)

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§4.1 Exercises for Taylor Series and Calculation of Functions

Exercise 1.

(a) Find a Taylor polynomial of degree $N = 5$ for $f(x) = x^{1/2}$ expanded about $x_0 = 4$.

(b) Find a Taylor polynomial of degree $N = 5$ for $f(x) = x^{1/2}$ expanded about $x_0 = 9$.

(c) Determine which of the polynomials in parts (a) and (b) best approximates $(6.5)^{1/2}$.

Solve A Taylor polynomial expanded at x_0 has the form:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}.$$

And the first five derivatives of $f(x)$ are

$$\begin{aligned}
f^{(0)} &= f(x) = x^{\frac{1}{2}} \\
f' &= \frac{1}{2}x^{-\frac{1}{2}} \\
f'' &= -\frac{1}{4}x^{-\frac{3}{2}} \\
f''' &= \frac{3}{8}x^{-\frac{5}{2}} \\
f^{(4)} &= -\frac{15}{16}x^{-\frac{7}{2}} \\
f^{(5)} &= \frac{105}{32}x^{-\frac{9}{2}}.
\end{aligned}$$

(a) Plugging in $x_0 = 4$, we get

$$\begin{aligned}
T_5^1(x) &= 4^{\frac{1}{2}} + \frac{1}{2}4^{-\frac{1}{2}}(x-4) - \frac{1}{2!}\frac{1}{4}4^{-\frac{3}{2}}(x-4)^2 + \frac{1}{3!}\frac{3}{8}4^{-\frac{5}{2}}(x-4)^3 - \frac{1}{4!}\frac{15}{16}4^{-\frac{7}{2}}(x-4)^4 + \frac{1}{5!}\frac{105}{32}4^{-\frac{9}{2}}(x-4)^5 \\
&= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16384}(x-4)^4 + \frac{7}{131072}(x-4)^5.
\end{aligned}$$

(b) Plugging in $x_0 = 9$, we get

$$\begin{aligned}
T_5^2(x) &= 9^{\frac{1}{2}} + \frac{1}{2}9^{-\frac{1}{2}}(x-9) - \frac{1}{2!}\frac{1}{4}9^{-\frac{3}{2}}(x-9)^2 + \frac{1}{3!}\frac{3}{8}9^{-\frac{5}{2}}(x-9)^3 - \frac{1}{4!}\frac{15}{16}9^{-\frac{7}{2}}(x-9)^4 + \frac{1}{5!}\frac{105}{32}9^{-\frac{9}{2}}(x-9)^5 \\
&= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3 - \frac{5}{279936}(x-9)^4 + \frac{7}{5038848}(x-9)^5.
\end{aligned}$$

(c) The error term $E_N(x)$ has the form

$$E_N(x) = \frac{f^{N+1}(c)}{(N+1)!}(x-x_0)^{N+1}$$

for some value $c = c(x)$ that lies between x and x_0. Hence, the bound for the error term of $T_5^1(x)$ is

$$|E_5^1(6.5)| = |\frac{f^{(6)}(c)}{6!}(6.5-4)^6| \leqslant |\frac{\frac{105}{32}\cdot\frac{9}{2}\cdot 4^{-\frac{11}{2}}}{6!}2.5^6| = 0.002444721758365631$$

and the bound for the error term of $T_5^2(x)$ is

$$|E_5^2(6.5)| = |\frac{f^{(6)}(c)}{6!}(6.5-9)^6| \leqslant |\frac{\frac{105}{32}\cdot\frac{9}{2}\cdot 6.5^{-\frac{11}{2}}}{6!}3.5^6| = 0.001274395245867137.$$

Since $|E_5^2(6.5)| < |E_5^1(6.5)|$, polynomials in parts (b) best approximates $(6.5)^{1/2}$.

$\square$