UNIVERSITY IBN TOFAIL

Electricity I

Problem Set III

Exercise 1:

A solid conducting sphere S_1 of radius R_1 is brought to a potential V_1. A second hollow conducting sphere S_2, with radius $R_2 > R_1$, is concentric with S_1. The sphere S_2 is brought to a potential V_2.

1. Determine the expressions for:
   - The charge Q_1 on the sphere S_1,
   - The charge Q_2' on the inner surface of S_2,
   - The charge Q_2'' on the outer surface of S_2.
2. Deduce the capacitance and influence coefficients. Verify that $C_{11} > 0$, $C_{22} > 0$, $C_{12} < 0$, and that $C_{11} + C_{12} = 0$.
3. What happens if both spheres are brought to the same potential V_2?

Correction

Charges on the Spheres:
Charge Q_1 on Sphere S1:
The potential V_1 of the solid sphere S1 is due to its own charge Q_1 and the charges on S2. The induced charge on the inner surface of S2 is $Q_2' = -Q_1$. The potential at S1 is:

$$V_1 = \frac{kQ_1}{R_1} + \frac{k(Q_2' + Q_2'')}{R_2} = \frac{kQ_1}{R_1} + \frac{k(-Q_1 + Q_2'')}{R_2}$$

The potential of S2 is:

$$V_2 = \frac{k(Q_1 + Q_2' + Q_2'')}{R_2} = \frac{kQ_2''}{R_2} \implies Q_2'' = \frac{V_2 R_2}{k}$$

Substituting Q_2'' into V_1:

$$V_1 = \frac{kQ_1}{R_1} - \frac{kQ_1}{R_2} + V_2 \implies Q_1 = \frac{4\pi\varepsilon_0 R_1 R_2 (V_1 - V_2)}{R_2 - R_1}$$

Charge Q_2' on Inner Surface of S2:

Correction

From electrostatic induction:

$$Q_2' = -Q_1 = -\frac{4\pi\varepsilon_0 R_1 R_2 (V_1 - V_2)}{R_2 - R_1}$$

Charge Q_2'' on Outer Surface of S2:
From $V_2 = \frac{kQ_2''}{R_2}$:

$$Q_2'' = 4\pi\varepsilon_0 R_2 V_2$$

Capacitance and Influence Coefficients:
Expressing charges in terms of potentials:

$$Q_1 = C_{11}V_1 + C_{12}V_2$$
$$Q_2 = C_{21}V_1 + C_{22}V_2$$

Using previous results:

$$Q_1 = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}(V_1 - V_2)$$
$$Q_2 = -\frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}V_1 + \frac{4\pi\varepsilon_0 R_2^2}{R_2 - R_1}V_2$$

Thus, the capacitance coefficients are:

$$C_{11} = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1} > 0$$
$$C_{12} = -\frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1} < 0$$
$$C_{22} = \frac{4\pi\varepsilon_0 R_2^2}{R_2 - R_1} > 0$$
$$C_{11} + C_{12} = 0$$

Case When Both Spheres Are at Potential V_2:
Setting $V_1 = V_2$:

$$Q_1 = 0, \quad Q_2' = 0, \quad Q_2'' = 4\pi\varepsilon_0 R_2 V_2$$

The inner sphere S1 has no charge, and all charge on S2 resides on its outer surface. This eliminates the electric field between the spheres.

Exercise 2:

A cylindrical capacitor of length L is formed by two coaxial cylinders A_1 and A_2, with radii R_1 and R_2 respectively ($R_1 < R_2$). The capacitor carries a charge Q. The potentials of A_1 and A_2 are V_1 and V_2, respectively. Assuming $L \gg R_2$ to neglect edge effects, determine the capacitance C of this capacitor.

Correction

1. **Capacitance of the Cylindrical Capacitor:**

   Consider two coaxial cylinders A1 and A2 with radii R_1 and R_2 ($R_2 > R_1$) and length L. To determine the capacitance C, we use Gauss's Law and integrate the electric field to find the potential difference between the cylinders.

   (a) **Electric Field Between the Cylinders:**
   For a Gaussian surface of radius r ($R_1 < r < R_2$) and length L, the electric field $E(r)$ is radial and uniform. By Gauss's Law:

   $$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} \implies E(r) \cdot 2\pi r L = \frac{\lambda L}{\varepsilon_0},$$

   where $\lambda = \frac{Q}{L}$ is the linear charge density. Solving for $E(r)$:

   $$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{Q}{2\pi\varepsilon_0 L r}.$$

   (b) **Potential Difference Between the Cylinders:**
   Integrate $E(r)$ from R_1 to R_2 to find the potential difference $\Delta V = V_1 - V_2$:

   $$\Delta V = \int_{R_1}^{R_2} E(r)\, dr = \frac{Q}{2\pi\varepsilon_0 L} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 L} \ln\left(\frac{R_2}{R_1}\right).$$

   (c) **Capacitance:**
   The capacitance is defined as $C = \frac{Q}{\Delta V}$. Substituting ΔV:

   $$C = \frac{Q}{\frac{Q}{2\pi\varepsilon_0 L} \ln\left(\frac{R_2}{R_1}\right)} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{R_2}{R_1}\right)}.$$

2. **Final Result:**

   The capacitance of the cylindrical capacitor is:

   $$\boxed{C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{R_2}{R_1}\right)}}$$

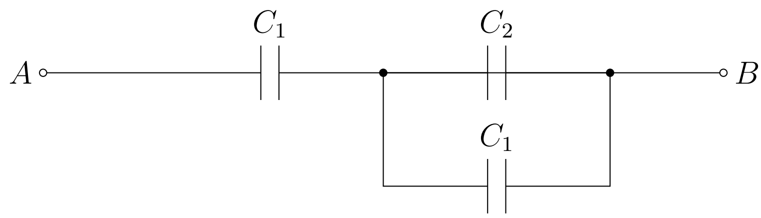
   This result depends only on the geometry (R_1, R_2, L) and the vacuum permittivity ε_0, confirming that capacitance is a purely geometric quantity.

Exercise 3:

Three capacitors are connected as shown in the figure below.

1. What value must C_2 have so that the equivalent capacitance of the system equals C_2, given that $C_1 = 3\,\mu F$?

2. A voltage $U_0 = 400\,\mathrm{V}$ is applied between points A and B. Determine the charge and voltage across each capacitor in the case where C_1 and C_2 have the values found in part (1).



Correction

1. **Finding C_2 for Equivalent Capacitance $C_{\mathbf{eq}} = C_2$:**

   Assume the three capacitors are configured such that one C_1 is in series with a parallel combination of C_2 and another C_1. The equivalent capacitance is:

   $$C_{\text{eq}} = \frac{C_1(C_1 + C_2)}{2C_1 + C_2}.$$

   Setting $C_{\text{eq}} = C_2$, solve:

   $$C_2 = \frac{C_1(C_1 + C_2)}{2C_1 + C_2}.$$

   Multiply both sides by $2C_1 + C_2$:

   $$C_2(2C_1 + C_2) = C_1(C_1 + C_2).$$

   Expand and simplify:

   $$2C_1C_2 + C_2^2 = C_1^2 + C_1C_2 \implies C_1^2 - C_1C_2 - C_2^2 = 0.$$

   Solving the quadratic equation $C_2^2 + C_1C_2 - C_1^2 = 0$:

   $$C_2 = \frac{-C_1 + \sqrt{C_1^2 + 4C_1^2}}{2} = \frac{(\sqrt{5} - 1)}{2}C_1.$$

   For $C_1 = 3\,\mu\mathrm{F}$:

   $$C_2 = \frac{(\sqrt{5} - 1)}{2} \cdot 3\,\mu\mathrm{F} \approx 1.854\,\mu\mathrm{F}.$$

2. **Charge and Voltage Across Each Capacitor:**

   Apply $U_0 = 400\,\mathrm{V}$ between points A and B. The total charge is:

   $$Q_{\text{total}} = C_{\text{eq}} \cdot U_0 = C_2 \cdot 400\,\mathrm{V} \approx 1.854\,\mu\mathrm{F} \cdot 400\,\mathrm{V} = 741.6\,\mu\mathrm{C}.$$

   - **Series C_1:** Charge: $Q_1 = 741.6\,\mu\mathrm{C}$. Voltage: $V_1 = \frac{Q_1}{C_1} = \frac{741.6\,\mu\mathrm{C}}{3\,\mu\mathrm{F}} = 247.2\,\mathrm{V}$.

   - **Parallel C_2:** Voltage: $V_2 = U_0 - V_1 = 400\,\mathrm{V} - 247.2\,\mathrm{V} = 152.8\,\mathrm{V}$. Charge: $Q_2 = C_2 \cdot V_2 = 1.854\,\mu\mathrm{F} \cdot 152.8\,\mathrm{V} \approx 283.3\,\mu\mathrm{C}$.

Correction

- **Parallel C_1:** Voltage: $V_3 = 152.8\,\mathrm{V}$. Charge: $Q_3 = C_1 \cdot V_3 = 3\,\mu\mathrm{F} \cdot 152.8\,\mathrm{V} = 458.4\,\mu\mathrm{C}$.
Verification: Total charge in parallel branch: $Q_2 + Q_3 = 283.3\,\mu\mathrm{C} + 458.4\,\mu\mathrm{C} = 741.7\,\mu\mathrm{C} \approx Q_1$, confirming consistency.

Exercise 4:

Determine the electrostatic energy of a sphere of radius R charged with a uniform volumetric charge density ρ using two different methods:

1. By using the expression for energy in terms of the potential.
2. By using the expression for local energy density.

Correction

[label=(0)]**Electrostatic Energy via Potential:** For a sphere of radius R with uniform volumetric charge density ρ, the total charge is:

$$Q = \rho \cdot \frac{4}{3}\pi R^3.$$

The electrostatic energy U is calculated by integrating the potential energy over the charge distribution:

$$U = \frac{1}{2}\int V(\mathbf{r})\,dq = \frac{1}{2}\int V(\mathbf{r})\rho\,dV.$$

The potential inside the sphere at radius r is:

$$V(r) = \frac{\rho}{6\varepsilon_0}(3R^2 - r^2).$$

Substituting into the integral in spherical coordinates:

$$U = \frac{1}{2}\rho\int_0^R \left[\frac{\rho}{6\varepsilon_0}(3R^2 - r^2)\right] 4\pi r^2\,dr.$$

Simplifying:

$$U = \frac{2\pi\rho^2}{3\varepsilon_0}\int_0^R (3R^2 - r^2)r^2\,dr = \frac{4\pi\rho^2 R^5}{15\varepsilon_0}.$$

Substituting $\rho = \frac{3Q}{4\pi R^3}$:

$$U = \frac{3Q^2}{20\pi\varepsilon_0 R}.$$

Electrostatic Energy via Energy Density: The energy density is $u = \frac{\varepsilon_0}{2}E^2$. The electric field inside ($r \le R$) and outside($r > R$) the sphere is:

$$E_{\text{in}}(r) = \frac{\rho r}{3\varepsilon_0}, \quad E_{\text{out}}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}.$$

Correction

Total energy:

$$U = \int_0^R \frac{\varepsilon_0}{2} E_{\text{in}}^2 \, dV + \int_R^\infty \frac{\varepsilon_0}{2} E_{\text{out}}^2 \, dV.$$

Computing the integrals:

$$U_{\text{in}} = \frac{2\pi\rho^2}{9\varepsilon_0} \int_0^R r^4 \, dr = \frac{4\pi\rho^2 R^5}{45\varepsilon_0},$$

$$U_{\text{out}} = \frac{Q^2}{8\pi\varepsilon_0} \int_R^\infty \frac{1}{r^2} \, dr = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

Substituting $\rho = \frac{3Q}{4\pi R^3}$:

$$U = \frac{4\pi\rho^2 R^5}{45\varepsilon_0} + \frac{Q^2}{8\pi\varepsilon_0 R} = \frac{3Q^2}{20\pi\varepsilon_0 R}.$$

Final Result: Both methods yield the same electrostatic energy:

$$\boxed{U = \frac{3Q^2}{20\pi\varepsilon_0 R}}.$$

This confirms that the energy of a uniformly charged sphere depends quadratically on the total charge Q, inversely on the radius R, and is proportional to the square of the charge density ρ.

Exercise 5:

A capacitor is formed by two horizontal circular plates of surface area S, parallel to each other, with radius R and separated by a distance e. The capacitor is charged using a voltage generator V. Express all results in terms of R.

1. Determine the charge Q acquired by the capacitor (its capacitance is $C = \frac{\varepsilon_0 S}{e}$).
2. Determine the energy W_c stored in the capacitor.
3. What is the energy density W? Deduce the intensity E of the electric field.
4. Determine the energy W_G supplied by the generator. Compare it with W_c and interpret the result.

Correction

[label=(0)]**Charge Acquired by the Capacitor:** The capacitance of a parallel-plate capacitor is:
$$C = \frac{\varepsilon_0 S}{e}.$$
For circular plates of radius R, the surface area is $S = \pi R^2$. Substituting:
$$C = \frac{\varepsilon_0 \pi R^2}{e}.$$
The charge Q on the capacitor is:
$$Q = CV = \frac{\varepsilon_0 \pi R^2}{e} V.$$
Energy Stored in the Capacitor: The energy stored W_c is:
$$W_c = \frac{1}{2}CV^2 = \frac{1}{2} \cdot \frac{\varepsilon_0 \pi R^2}{e} \cdot V^2 = \frac{\varepsilon_0 \pi R^2 V^2}{2e}.$$
Energy Density and Electric Field: The energy density w is energy per unit volume. The volume between the plates is $Se = \pi R^2 e$:
$$w = \frac{W_c}{\pi R^2 e} = \frac{\varepsilon_0 \pi R^2 V^2}{2e} \cdot \frac{1}{\pi R^2 e} = \frac{\varepsilon_0 V^2}{2e^2}.$$
The electric field E between the plates is:
$$E = \frac{V}{e} \implies w = \frac{1}{2}\varepsilon_0 E^2.$$
Substituting $E = \frac{V}{e}$ confirms consistency. **Energy Supplied by the Generator:** The generator supplies energy W_G equal to the work done to move charge Q across voltage V:
$$W_G = QV = \frac{\varepsilon_0 \pi R^2}{e} V \cdot V = \frac{\varepsilon_0 \pi R^2 V^2}{e}.$$
Comparing W_G and W_c:
$$W_G = 2W_c.$$
Interpretation: The generator provides twice the energy stored in the capacitor. The remaining energy is dissipated as heat in the circuit (if resistance is present) or radiated as electromagnetic energy during charging. **Final Results:**
$$(1) \quad Q = \boxed{\frac{\varepsilon_0 \pi R^2 V}{e}},$$
$$(2) \quad W_c = \boxed{\frac{\varepsilon_0 \pi R^2 V^2}{2e}},$$
$$(3) \quad E = \boxed{\frac{V}{e}},$$
$$(4) \quad W_G = \boxed{\frac{\varepsilon_0 \pi R^2 V^2}{e}} = 2W_c.$$