Ibn Tofail University

Analysis II — Normal Exam

Year: 22-23

Exercise 1:

Consider the function $f : [1, 3] \to \mathbb{R}$ defined by:

$$f(x) = \frac{1}{x}$$

1. Justify that f is integrable (in the Riemann sense) on $[1, 3]$.
2. Calculate the Darboux sums (lower and upper) $D_S^-(f)$ and $D_S^+(f)$ of f with respect to the subdivision S of $[1, 3]$ defined by $S = \{1, 2, 3\}$.
3. State (without proving) the inequalities between $D_S^-(f)$, $D_S^+(f)$ and $\int_1^3 f(x)dx$.
4. Deduce an approximation of $\ln 3$ by rational numbers.

Answer Area

Exercise 2:

Consider the function $G : \mathbb{R} \to \mathbb{R}$ defined by:

$$G(x) = \int_x^{2x} \frac{dt}{\sqrt{t^2+1}}$$

1. Justify that G is defined on $\mathbb{R}$. Also show that G is an odd function.
2. Verify that G is differentiable on $\mathbb{R}$, and calculate its derivative $G'(x)$. (Hint: use any primitive F of the function $t \mapsto \frac{1}{\sqrt{t^2+1}}$).
3. Deduce that G is strictly increasing on $\mathbb{R}$.
4. Verify that $t^2 \leq t^2+1 \leq (t+1)^2$ for all $t > 0$. Deduce the following inequality:

$$\forall x > 0, \ln(2x+1) - \ln(x+1) \leq G(x) \leq \ln 2$$

5. Deduce the limit $\lim_{x \to +\infty} G(x)$.
6. Solve the equation $G(x) = 0$.

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Exercise 3:

For all $n \in \mathbb{N}$, let:

$$I_n = \int_0^1 (1-t^2)^n dt$$

1. Justify the existence of the integral I_n for all $n \in \mathbb{N}$.
2. Show that $\forall n \in \mathbb{N}, I_{n+1} = \frac{2n+2}{2n+3} \cdot I_n$.
3. Deduce that $\forall n \in \mathbb{N}, I_n = \frac{2^n (n!)^2}{(2n+1)!}$.
4. Using Newton's binomial formula, show that $\forall n \in \mathbb{N}, I_n = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{2k+1}$.

Answer Area