Ibn Tofail University

Analysis II — Make-up Exam

Year: 23-24

Exercise 1:

Let $f : [0,1] \to \mathbb{R}$ be an arbitrary continuous function.

1. Show that if $\int_0^1 f(x), dx = 0$, then there exists $c \in [0,1]$ such that $f(c) = 0$.
2. Deduce that if $\int_0^1 f(x), dx = \frac{1}{2}$, then there exists $d \in [0,1]$ such that $f(d) = d$.

Answer Area

1. Since f is continuous on $[0, 1]$, it is integrable. Suppose $\int_0^1 f(x)\,dx = 0$. If $f(x) > 0$ for all $x \in [0, 1]$, then the integral would be positive; similarly, if $f(x) < 0$ everywhere, the integral would be negative. Thus, f must take both non-negative and non-positive values. By the Intermediate Value Theorem, there exists $c \in [0, 1]$ such that $f(c) = 0$.

2. Define $g(x) = f(x) - x$. Then:

$$\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx - \int_0^1 x\,dx = \frac{1}{2} - \frac{1}{2} = 0.$$

   From part (1), since g is continuous and its integral is zero, there exists $d \in [0, 1]$ such that $g(d) = 0$, i.e., $f(d) = d$.

Exercise 2:

Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by:

$$F(x) = \int_x^{2x} \frac{e^{-t}}{t}\, dt.$$

1. Verify that F is defined on $\mathbb{R}^+$, i.e., $D_F = \mathbb{R}^+$.
2. Show that F is differentiable on $\mathbb{R}^+$, and calculate its derivative. Deduce the variations of F on $\mathbb{R}^+$. (Hint: use any primitive F_0 of the function $t \mapsto e^{-t}/t$).
3. Show that $\forall x > 0, (\ln 2) \cdot e^{-2x} \leq F(x) \leq (\ln 2) \cdot e^{-x}$.
4. Deduce $\lim_{x\to 0^+} F(x)$ and $\lim_{x\to +\infty} F(x)$.

Answer Area

1. The function $t \mapsto \frac{e^{-t}}{t}$ is continuous on $(0, \infty)$, hence integrable on any interval $[x, 2x]$ for $x > 0$. Therefore, $F(x) = \int_x^{2x} \frac{e^{-t}}{t}\, dt$ is well-defined for all $x > 0$, so $\mathcal{D}_F = \mathbb{R}_+^*$.

2. Define F_0 as a primitive of $t \mapsto \frac{e^{-t}}{t}$. Then we can write $F(x) = F_0(2x) - F_0(x)$. By the Fundamental Theorem of Calculus, F is differentiable on $\mathbb{R}_+$ and:

$$F'(x) = 2F_0'(2x) - F_0'(x) = 2 \cdot \frac{e^{-2x}}{2x} - \frac{e^{-x}}{x} = \frac{e^{-2x} - e^{-x}}{x}.$$

Since $e^{-2x} < e^{-x}$ for all $x > 0$, we have $F'(x) < 0$, so F is strictly decreasing on $\mathbb{R}_+$.

3. For $t \in [x, 2x]$, we have $e^{-2x} \le e^{-t} \le e^{-x}$, since $x \le t \le 2x$. Dividing by t (which is positive), and integrating over $[x, 2x]$, we get:

$$\int_x^{2x} \frac{e^{-2x}}{t}\, dt \le \int_x^{2x} \frac{e^{-t}}{t}\, dt \le \int_x^{2x} \frac{e^{-x}}{t}\, dt.$$

Evaluating the left and right integrals gives:

$$e^{-2x} \ln 2 \le F(x) \le e^{-x} \ln 2.$$

4. From part (3), as $x \to 0^+$, both bounds tend to $\ln 2$, so by the Squeeze Theorem:

$$\lim_{x \to 0^+} F(x) = \ln 2.$$

As $x \to +\infty$, $e^{-x} \to 0$, so again by squeezing:

$$\lim_{x \to +\infty} F(x) = 0.$$

Exercise 3:

For all $n \in \mathbb{N}$, let:

$$I_n = \int_0^1 x^n \cdot e^{-x}\, dx.$$

1. Justify the existence of I_n for all $n \in \mathbb{N}$. Then calculate I_0.
2. Show that $\forall n \geq 0, 0 \leq I_n \leq \frac{1}{n+1}$.
3. Deduce that the sequence $(I_n)n \geq 0$ is convergent and calculate its limit.
4. Show (using integration by parts) that $\forall n \in \mathbb{N}, In + 1 = (n+1)I_n - e^{-1}$.
5. Deduce that $\forall n \geq 0, 0 \leq I_n - \frac{e^{-1}}{n+1} \leq \frac{1}{(n+1)(n+2)}$.
6. From 5, deduce a simple equivalent of I_n as n approaches infinity (i.e., a non-zero numerical sequence $(J_n)n \geq 0$ such that $\lim n \to +\infty \frac{I_n}{J_n} = 1$).

Answer Area

1. The function $x \mapsto x^n e^{-x}$ is continuous on $[0, 1]$ for all $n \in \mathbb{N}$, hence integrable. Therefore, I_n exists for all $n \in \mathbb{N}$. For $n = 0$, we have:

$$I_0 = \int_0^1 e^{-x} dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}.$$

2. On $[0, 1]$, we have $0 \leq x^n e^{-x} \leq x^n$, since $0 < e^{-x} \leq 1$. Therefore:

$$0 \leq I_n = \int_0^1 x^n e^{-x} dx \leq \int_0^1 x^n dx = \frac{1}{n+1}.$$

3. From part (2), $0 \leq I_n \leq \frac{1}{n+1}$. Since $\frac{1}{n+1} \to 0$ as $n \to \infty$, by the Squeeze Theorem, $I_n \to 0$ as $n \to \infty$.

4. Use integration by parts with $u = x^{n+1}$, $dv = -e^{-x}dx$. Then $du = (n+1)x^n dx$, $v = e^{-x}$. We get:

$$I_{n+1} = \int_0^1 x^{n+1} e^{-x} dx = \left[-x^{n+1} e^{-x}\right]_0^1 + (n+1) \int_0^1 x^n e^{-x} dx = -\frac{1}{e} + (n+1) I_n.$$

Hence,

$$I_{n+1} = (n+1) I_n - \frac{1}{e}.$$

5. From part (4):

$$I_n = \frac{I_{n+1} + \frac{1}{e}}{n+1}.$$

Using $0 \leq I_{n+1} \leq \frac{1}{n+2}$ from part (2), we deduce:

$$0 \leq I_n - \frac{1}{e(n+1)} \leq \frac{1}{(n+1)(n+2)}.$$

6. From part (5), we have:

$$I_n \sim \frac{1}{e(n+1)} \quad \text{as } n \to \infty.$$

So a simple equivalent of I_n is $J_n = \frac{1}{e(n+1)}$, since:

$$\lim_{n\to\infty} \frac{I_n}{J_n} = \lim_{n\to\infty} \frac{I_n}{\frac{1}{e(n+1)}} = 1.$$