UNIVERSITY IBN TOFAIL

Electricity I

Problem Set I

Exercise 1:

Find the charge Q of a disque with a radius R, charged with a surface density σ, with the element of the surface ds.

Correction

$$
\begin{aligned}
dQ &= \sigma dS \\
Q &= \int_{disque} \sigma dS \\
&= \sigma \int_0^{2\pi} \int_0^R r dr d\theta \\
&= \sigma \int_0^{2\pi} d\theta \int_0^R r dr \\
&= \sigma \pi R^2
\end{aligned}
$$

Exercise 2:

A sphere of radius R is charged with a volumetric charge density ρ such that:

$$
\rho(r) = \begin{cases} \rho_0 \left(1 - \dfrac{r^2}{R^2}\right) & \text{if } r < R \\ 0 & \text{if } r > R \end{cases}
$$

Here, r is the distance from the center O of the sphere, and ρ_0 is the charge density at the center.

Determine the total electric charge of the sphere.

Correction

$$\begin{aligned} dQ &= \rho(r)dV \\ &= \rho(r)r^2 sin(\theta)drd\theta d\phi \\ Q &= \rho_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(1-\frac{r^2}{R^2}\right) r^2 sin(\theta)drd\theta d\phi \\ &= \rho_0 \int_0^{2\pi} d\phi \int_0^{\pi} sin(\theta)d\theta \int_0^{R}\left(1-\frac{r^2}{R^2}\right) r^2 dr \\ &= \frac{8\pi}{15}\rho_0 R^3 \end{aligned}$$

Exercise 3:

Three point charges of the same quantity $q = +10\,\mu\text{C}$ are placed at the vertices A, B, and C of an equilateral triangle with side length $a = 10\,\text{cm}$. Let $\varepsilon_0 = 8.85 \times 10^{-12}$ SI.

1. Give the literal expression of the electric interaction force between two charges. Calculate the value of this force.

2. Determine the resulting force $\vec{F_C}$ acting on the charge at point C due to the charges at A and B.

Correction

1. we have :

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2} \\ &= 0.09N \end{aligned}$$

2. the vector is : $\vec{F_c} = -0.09\hat{j}$.

Exercise 4:

A disk of radius R and center O is uniformly charged with a positive surface charge density σ. A point M, located on its axis, is positioned at a distance $OM = z$.

Determine the electric force $\vec{F}(M)$ exerted by the disk on a positive charge q placed at this point.

Correction

$$
\begin{aligned}
dq &= \sigma \cdot ds = \sigma \cdot 2\pi r dr \\
dE_z &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{z\,dq}{(r^2+z^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{z \cdot \sigma \cdot 2\pi r dr}{(r^2+z^2)^{3/2}} \\
&= \frac{\sigma z}{2\varepsilon_0} \cdot \frac{r dr}{(r^2+z^2)^{3/2}} \\
E_z &= \int_0^R dE_z = \frac{\sigma z}{2\varepsilon_0} \int_0^R \frac{r dr}{(r^2+z^2)^{3/2}} \\
u &= r^2 + z^2 \Rightarrow du = 2r dr \\
r &= 0 \Rightarrow u = z^2, \quad r = R \Rightarrow u = R^2 + z^2 \\
E_z &= \frac{\sigma z}{2\varepsilon_0} \int_{z^2}^{R^2+z^2} \frac{1}{2} u^{-3/2} du = \frac{\sigma z}{4\varepsilon_0} \left[-2u^{-1/2}\right]_{z^2}^{R^2+z^2} \\
&= \frac{\sigma z}{4\varepsilon_0} \left(\frac{2}{z} - \frac{2}{\sqrt{R^2+z^2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) \\
\vec{F}(M) &= q \cdot \vec{E}(z) = q \cdot \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}
\end{aligned}
$$

Exercise 5:

In a Cartesian coordinate system (OXY), two point charges $q > 0$ are placed at points $A(-d, 0)$ and $B(d, 0)$.

Determine the electric field $\vec{E}(M)$ created at point $M(0, y)$.

Correction

$$
\begin{aligned}
r &= \sqrt{d^2 + y^2} \\
E &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{d^2+y^2}
\end{aligned}
$$

$$
\begin{aligned}
\vec{E}_A &= \frac{q}{4\pi\varepsilon_0(d^2+y^2)^{3/2}} \cdot \langle d, y \rangle \\
\vec{E}_B &= \frac{q}{4\pi\varepsilon_0(d^2+y^2)^{3/2}} \cdot \langle -d, y \rangle
\end{aligned}
$$

$$
\begin{aligned}
\vec{E}(M) = \vec{E}_A + \vec{E}_B &= \frac{q}{4\pi\varepsilon_0(d^2+y^2)^{3/2}} \cdot (\langle d-d, y+y \rangle) \\
&= \frac{q}{4\pi\varepsilon_0(d^2+y^2)^{3/2}} \cdot \langle 0, 2y \rangle \\
&= \frac{qy}{2\pi\varepsilon_0(d^2+y^2)^{3/2}}\,\hat{j}
\end{aligned}
$$

Exercise 6:

Consider a circular ring positively charged with a uniform linear charge density λ, of radius R and center O.

1. Determine the electrostatic field $\vec{E}(M)$ created by this distribution at a point M on its axis of revolution, at a distance z from O.
2. Determine the position M for which this electric field is at its maximum value.

Correction

1. Elemental field:

$$E_z = \frac{1}{4\pi\varepsilon_0} \cdot \frac{z\lambda\ell}{(R^2+z^2)^{3/2}}$$

$$\begin{aligned} E_z &= \int_0^{2\pi R} E_z = \frac{z\lambda}{4\pi\varepsilon_0(R^2+z^2)^{3/2}} \cdot 2\pi R \\ &= \frac{\lambda z R}{2\varepsilon_0(R^2+z^2)^{3/2}} \end{aligned}$$

2. Elemental field:

$$E_z = \frac{1}{4\pi\varepsilon_0} \cdot \frac{z\lambda\ell}{(R^2+z^2)^{3/2}}$$

$$\begin{aligned} E_z &= \int_0^{2\pi R} E_z = \frac{z\lambda}{4\pi\varepsilon_0(R^2+z^2)^{3/2}} \cdot 2\pi R \\ &= \frac{\lambda z R}{2\varepsilon_0(R^2+z^2)^{3/2}} \end{aligned}$$

To maximize: $Ez = 0$

$$\begin{aligned} \text{Let } f(z) &= \frac{\lambda R z}{2\varepsilon_0(R^2+z^2)^{3/2}} \\ f'(z) &= \frac{\lambda R(R^2+z^2)^{3/2} - \lambda R z \cdot 3z(R^2+z^2)^{1/2}}{2\varepsilon_0(R^2+z^2)^3} \\ &= \frac{\lambda R(R^2+z^2)^{1/2}\left[(R^2+z^2) - 3z^2\right]}{2\varepsilon_0(R^2+z^2)^3} \end{aligned}$$

$$f'(z) = 0 \Rightarrow (R^2+z^2) - 3z^2 = 0 \Rightarrow R^2 = 2z^2 \Rightarrow z = \frac{R}{\sqrt{2}}$$