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Algebra II

Problem Set II

Exercise 1:

Let A be the matrix in $M_3(\mathbb{R})$ defined by $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

1. Calculate A^2 as a function of A and I_3 (the identity matrix of $M_3(\mathbb{R})$), and deduce that A is invertible and calculate its inverse.
2. Calculate A^{-1} using the cofactor method.

Correction

1) Compute A^2:

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Observe that $A^2 = A + 2I_3$, where I_3 is the identity matrix. Rearranging:

$$A^2 - A - 2I_3 = 0 \Rightarrow A(A - I_3) = 2I_3 \Rightarrow A^{-1} = \frac{1}{2}(A - I_3).$$

Compute A^{-1}:

$$A - I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

2) Compute the determinant of A:

$\det(A) = 0 \cdot (0 \cdot 0 - 1 \cdot 1) - 1 \cdot (1 \cdot 0 - 1 \cdot 1) + 1 \cdot (1 \cdot 1 - 0 \cdot 1) = 0 + 1 + 1 = 2.$

Compute the adjugate matrix (transpose of the cofactor matrix):

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Then:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

This matches the result from part 1.

Exercise 2:

Show that:

1. For any matrix $A \in M_{m,n}(\mathbb{R})$, the product $A \cdot^t (A)$ is a symmetric square matrix.
2. If A is a symmetric or antisymmetric matrix, then A^2 is symmetric.
3. Any square symmetric matrix can be written as the sum of a symmetric matrix and an antisymmetric matrix.

Correction

1. **Symmetry of $A \cdot A^\top$:** Let $A \in M_{m,n}(\mathbb{R})$. We prove that $A \cdot A^\top$ is symmetric. The transpose of $A \cdot A^\top$ is:
$$(A \cdot A^\top)^\top = (A^\top)^\top \cdot A^\top = A \cdot A^\top.$$
Thus, $A \cdot A^\top$ equals its own transpose, so it is symmetric.

2. **Symmetry of A^2 for Symmetric/Antisymmetric A:** Let A be symmetric ($A^\top = A$) or antisymmetric ($A^\top = -A$). We show that A^2 is symmetric.
- **Case 1:** If A is symmetric, then:
$$(A^2)^\top = (A^\top)^2 = A^2.$$
Thus, A^2 is symmetric.
- **Case 2:** If A is antisymmetric, then:
$$(A^2)^\top = (A^\top)^2 = (-A)^2 = A^2.$$
Again, A^2 is symmetric.

3. **Decomposition of a Symmetric Matrix:** We show that any square symmetric matrix $S \in M_n(\mathbb{R})$ can be written as the sum of a symmetric matrix and an antisymmetric matrix. For any square matrix M, we can decompose it as:
$$M = \underbrace{\frac{M + M^\top}{2}}_{\text{symmetric}} + \underbrace{\frac{M - M^\top}{2}}_{\text{antisymmetric}}.$$
If S is symmetric ($S^\top = S$), then:
$$S = \frac{S + S^\top}{2} + \frac{S - S^\top}{2} = S + 0.$$
Here, S is symmetric, and 0 is trivially antisymmetric. Thus, the decomposition holds.

Exercise 3:

Consider the following matrices: $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, $B = A - I_3$.

1. Calculate B^n for $n \geq 1$.
2. Deduce the value of A^n for $n \geq 1$.

Correction

1. **Calculate B^n for $n \geq 1$:** Given $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = A - I_3$, we compute B:

$$B = A - I_3 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$B^2 = B \cdot B = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^3 = B^2 \cdot B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. **Deduce A^n for $n \geq 1$:** Since $A = I + B$, we use the binomial theorem for matrices (note I and B commute):

$$A^n = (I + B)^n = \sum_{k=0}^{n} \binom{n}{k} B^k.$$

Because $B^k = 0$ for $k \geq 3$, the expansion terminates:

$$A^n = I + nB + \frac{n(n-1)}{2} B^2.$$

Perform the addition:

$$A^n = \begin{pmatrix} 1 & 2n & 3n(n-1) \\ 0 & 1 & 3n \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4:

1. Show without expanding the calculations that the following determinants are zero:

$$\Delta_1 = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & 5 & -1 \end{vmatrix}, \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ a+b & b+c & a+c \end{vmatrix}.$$

2. a) Calculate in factored form the following determinant:

$$D_1 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}.$$

b) Calculate the following determinant:

$$D_2 = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 \\ -1 & 2 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{vmatrix}.$$

Correction

1. **Show that the following determinants are zero without expanding:**

   (a) **Determinant Δ_1:** Perform the column operation $C_3 \to C_3 + C_1$. The third column becomes:

   $$\begin{bmatrix} -1+1 \\ 1+(-1) \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

   A matrix with a zero column has determinant zero. Thus, $\Delta_1 = 0$.

   (b) **Determinant Δ_2:** we observe a **linear dependency among rows**:

   $$\text{Row}_3 = (a+b+c) \cdot \text{Row}_1 - \text{Row}_2.$$

   Explicitly:

   $$\text{Row}_3 = \begin{bmatrix} a+b & b+c & a+c \end{bmatrix} = (a+b+c)\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} c & a & b \end{bmatrix}.$$

   Since the rows are linearly dependent, $\Delta_2 = 0$.

2. **Calculate the following determinants:**

   (a) **Determinant D_1:** After subtracting the first row from the second and third rows:

   $$D_1 = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix}.$$

   $$D_1 = 1 \cdot \begin{vmatrix} b-a & b^3-a^3 \\ c-a & c^3-a^3 \end{vmatrix}.$$

   Final factorization:

   $$D_1 = (a+b+c)(b-a)(c-a)(c-b).$$

   (b) **Determinant D_2:** Perform row operations $R_3 \to R_3 + R_1$ and $R_4 \to R_4 + 2R_1$:

   $$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & -1 & -2 & 5 \end{bmatrix}.$$

   Expand along the first column:

   $$D_2 = \begin{vmatrix} 2 & 3 & -1 \\ 2 & 0 & 3 \\ -1 & -2 & 5 \end{vmatrix}.$$

   Compute the 3×3 determinant using cofactor expansion:

   $$D_2 = -23.$$

Exercise 5:

Consider the following matrices: $T = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -10 & 11 \\ -3 & 6 & 5 \\ -6 & 12 & 8 \end{pmatrix}$.

1. Determine the matrix $B = TA$ and calculate the determinant of B.
2. From the previous question, deduce the determinant of A.
3. From the previous question, deduce the determinant of the following matrix:

$$\begin{pmatrix} 3 & 5 & 55 \\ -9 & -3 & 25 \\ -18 & -6 & 40 \end{pmatrix}$$

Correction

1. **Compute** $B = T \cdot A$ **and find** $\det(B)$**:** Perform matrix multiplication $B = T \cdot A$:

$$B = \begin{pmatrix} 1 & -10 & 11 \\ 0 & -24 & 38 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since B is upper triangular, its determinant is the product of diagonal entries:

$$\det(B) = 1 \cdot (-24) \cdot (-2) = 48.$$

2. **Deduce** $\det(A)$**:** Matrix T is lower triangular with 1s on the diagonal, so $\det(T) = 1$. Using the property $\det(T \cdot A) = \det(T) \cdot \det(A)$, we have:

$$\det(B) = \det(T) \cdot \det(A) \Rightarrow 48 = 1 \cdot \det(A) \Rightarrow \det(A) = 48.$$

3. **Deduce the determinant of the matrix:**

$$C = \begin{pmatrix} 3 & 5 & 55 \\ -9 & -3 & 25 \\ -18 & -6 & 40 \end{pmatrix}.$$

Observe that C is obtained from A via column operations: - $C_1 \to 3 \cdot C_1$, - $C_2 \to -\frac{1}{2} \cdot C_2$, - $C_3 \to 5 \cdot C_3$.

The determinant scales by the product of these factors:

$$\det(C) = \det(A) \cdot 3 \cdot \left(-\frac{1}{2}\right) \cdot 5 = 48 \cdot \left(-\frac{15}{2}\right) = -360.$$

Exercise 6:

let $m \in \mathbb{R}$ and $A_m = \begin{vmatrix} m-1 & 2 & 2 \\ 2 & m+1 & 1 \\ -2 & -3 & m-3 \end{vmatrix}$

a) calculate the determinanat of A_m.

b) for what value of m, A_m are invertible.

c) calculate teh range of A_m, by the values of m.

Correction

a) **Calculate the determinant of A_m:**

$$\det(A_m) = m(m-1)(m-2) + (-4m+8) + (4m-8) = m(m-1)(m-2).$$

Thus, the determinant is:

$$\det(A_m) = m(m-1)(m-2).$$

b) **For what values of m is A_m invertible?** A matrix is invertible if and only if its determinant is non-zero. From part (a), $\det(A_m) = m(m-1)(m-2)$. Therefore, A_m is invertible for all $m \in \mathbb{R}$ except $m = 0$, $m = 1$, and $m = 2$.

c) **Determine the rank of A_m for all values of m:** The rank of a matrix is the maximum number of linearly independent rows or columns. Since A_m is a 3×3 matrix:

$$\text{rank}(A_m) = \begin{cases} 3 & \text{if } m \neq 0, 1, 2, \\ 2 & \text{if } m = 0, 1, \text{ or } 2. \end{cases}$$

- For $m \neq 0, 1, 2$, $\det(A_m) \neq 0$, so $\text{rank}(A_m) = 3$. - For $m = 0, 1, 2$, $\det(A_m) = 0$, but there exists at least one non-zero 2×2 minor (e.g., for $m = 0$, the minor $\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5 \neq 0$), so $\text{rank}(A_m) = 2$.

Exercise 7: Vandermonde Determinant

Let $n \in \mathbb{N}^*$ and $a_1, a_2, \ldots, a_n \in \mathbb{C}$. The Vandermonde determinant is defined by:

$$V_n(a_1, \ldots, a_n) = \begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{vmatrix}$$

Show that for all $n \geq 2$, we have:

$$V_n = \prod_{1 \leq i < j \leq n} (a_j - a_i)$$

Correction

1. **Base Case ($n = 2$):** For $n = 2$, the Vandermonde matrix is:

$$V_2 = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 - a_1.$$

The product formula gives $\prod_{1 \leq i < j \leq 2}(a_j - a_i) = a_2 - a_1$, which matches.

2. **Inductive Step:** Assume the formula holds for $n = k$, i.e.,

$$V_k = \prod_{1 \leq i < j \leq k} (a_j - a_i).$$

We prove it for $n = k + 1$.

(a) **Polynomial in a_{k+1}:** The determinant V_{k+1} is a polynomial in a_{k+1} of degree k, since the last row contains powers $a_{k+1}^0, a_{k+1}^1, \ldots, a_{k+1}^k$.

(b) **Roots of the Polynomial:** If $a_{k+1} = a_i$ for any $i \leq k$, two rows of the matrix become identical, making $V_{k+1} = 0$. Thus, $(a_{k+1} - a_i)$ divides V_{k+1} for all $i \leq k$.

(c) **Factorization:** Therefore, V_{k+1} must be divisible by:

$$\prod_{i=1}^{k}(a_{k+1} - a_i).$$

(d) **Inductive Hypothesis:** The remaining factor is the determinant of the $k \times k$ Vandermonde matrix for $a_1, \ldots, a_k$, which by the inductive hypothesis is:

$$\prod_{1 \leq i < j \leq k} (a_j - a_i).$$

Combining these results:

$$V_{k+1} = \left(\prod_{1 \leq i < j \leq k+1} (a_j - a_i) \right).$$

This completes the inductive step.