Ibn Tofail University

Algebra II — Make-up Exam

Year: 19-20

Exercise 1:

Let E be a K-vector space, $p \in \mathbb{N}^*$, and $u_1, u_2, ..., u_p, u_{p+1}$ be vectors in E.

1) Assume that $\{u_1, u_2, ..., u_p\}$ is linearly independent. Show the equivalence:
$u_{p+1} \notin \text{span}(\{u_1, u_2, ..., u_p\}) \Leftrightarrow \{u_1, u_2, ..., u_p, u_{p+1}\}$ is linearly independent.

2) Assume that $\{u_1, u_2, ..., u_p, u_{p+1}\}$ is a generating set of E and that
$u_{p+1} \in \text{span}(\{u_1, u_2, ..., u_p\})$. Show that $\{u_1, u_2, ..., u_p\}$ is a generating set of E.

Answer Area

Let E be a K-vector space, $p \in \mathbb{N}^*$, and let $u_1, u_2, \ldots, u_p, u_{p+1}$ be vectors in E.

1. Assume that $\{u_1, u_2, \ldots, u_p\}$ is linearly independent. Show the equivalence:

   $u_{p+1} \notin \text{span}(\{u_1, u_2, \ldots, u_p\}) \iff \{u_1, u_2, \ldots, u_p, u_{p+1}\}$is linearly independent.

   Proof:

   $\Rightarrow$ Suppose $u_{p+1} \notin \text{span}(\{u_1, \ldots, u_p\})$. We want to show that $\{u_1, \ldots, u_p, u_{p+1}\}$ is linearly independent.
   Let
   $$\lambda_1 u_1 + \cdots + \lambda_p u_p + \lambda_{p+1} u_{p+1} = 0.$$
   If $\lambda_{p+1} \neq 0$, then we can solve for u_{p+1}:
   $$u_{p+1} = -\frac{\lambda_1}{\lambda_{p+1}} u_1 - \cdots - \frac{\lambda_p}{\lambda_{p+1}} u_p,$$
   which contradicts the assumption that $u_{p+1} \notin \text{span}(\{u_1, \ldots, u_p\})$. Therefore, $\lambda_{p+1} = 0$.
   Then we are left with:
   $$\lambda_1 u_1 + \cdots + \lambda_p u_p = 0.$$
   Since $\{u_1, \ldots, u_p\}$ is linearly independent, all $\lambda_i = 0$. Hence, $\{u_1, \ldots, u_p, u_{p+1}\}$ is linearly independent.

   $\Leftarrow$ Conversely, suppose $\{u_1, \ldots, u_p, u_{p+1}\}$ is linearly independent. Then u_{p+1} cannot be written as a linear combination of $u_1, \ldots, u_p$, otherwise we would have a nontrivial linear relation among these vectors. Thus:
   $$u_{p+1} \notin \text{span}(\{u_1, \ldots, u_p\}).$$

   This completes the proof of the equivalence.

2. Assume that $\{u_1, u_2, \ldots, u_p, u_{p+1}\}$ is a generating set of E and that
   $$u_{p+1} \in \text{span}(\{u_1, \ldots, u_p\}).$$
   Show that $\{u_1, \ldots, u_p\}$ is a generating set of E.

   Proof:

   Since $\{u_1, \ldots, u_p, u_{p+1}\}$ generates E, every vector $v \in E$ can be written as:
   $$v = \lambda_1 u_1 + \cdots + \lambda_p u_p + \lambda_{p+1} u_{p+1}.$$
   But since $u_{p+1} \in \text{span}(\{u_1, \ldots, u_p\})$, there exist scalars $\mu_1, \ldots, \mu_p \in K$ such that:
   $$u_{p+1} = \mu_1 u_1 + \cdots + \mu_p u_p.$$
   Substituting into the expression for v, we get:
   $$v = \lambda_1 u_1 + \cdots + \lambda_p u_p + \lambda_{p+1}(\mu_1 u_1 + \cdots + \mu_p u_p) = (\lambda_1 + \lambda_{p+1}\mu_1) u_1 + \cdots + (\lambda_p + \lambda_{p+1}\mu_p) u_p.$$
   So $v \in \text{span}(\{u_1, \ldots, u_p\})$, hence $\{u_1, \ldots, u_p\}$ generates E.

Exercise 2:

In the $\mathbb{R}$-vector space $\mathbb{R}^4$, consider the following vector subspaces:
$F = \mathrm{span}(\{(1,0,1,0),(0,1,0,1)\})$
$G = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y=0 \text{ and } z+t=0\}$
$H = \{(x,y,z,t) \in \mathbb{R}^4 \mid x-y+z-t=0\}$

1) Determine a basis and the dimension of G.

2) Determine a basis and the dimension of H.

3) Determine a basis of the vector subspace $F \cap G$.

4) Show that $\mathbb{R}^4 = (F \cap G) \oplus H$.

Answer Area

In the $\mathbb{R}$-vector space $\mathbb{R}^4$, consider the following vector subspaces:

$$F = \text{span}\left(\{(1,0,1,0),(0,1,0,1)\}\right)$$

$$G = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y=0 \text{ and } z+t=0\}$$

$$H = \{(x,y,z,t) \in \mathbb{R}^4 \mid x-y+z-t=0\}$$

1. **Determine a basis and the dimension of G.**

   A vector $(x,y,z,t) \in G$ satisfies:

   $$x+y=0 \quad \text{and} \quad z+t=0$$

   Solving these equations:

   $$y=-x, \quad t=-z$$

   So any vector in G can be written as:

   $$(x,-x,z,-z) = x(1,-1,0,0) + z(0,0,1,-1)$$

   Therefore, a basis for G is:

   $$\{(1,-1,0,0),(0,0,1,-1)\}$$

   and

   $$\dim(G) = 2$$

2. **Determine a basis and the dimension of H.**

   A vector $(x,y,z,t) \in H$ satisfies:

   $$x-y+z-t=0$$

   We can express one variable in terms of others. Let's solve for t:

   $$t = x-y+z$$

   Then any vector in H can be written as:

   $$(x,y,z,x-y+z) = x(1,0,0,1) + y(0,1,0,-1) + z(0,0,1,1)$$

   Therefore, a basis for H is:

   $$\{(1,0,0,1),(0,1,0,-1),(0,0,1,1)\}$$

   and

   $$\dim(H) = 3$$

3. **Determine a basis of the vector subspace $F \cap G$.**

Recall:

$$F = \text{span}\left(\{(1,0,1,0),(0,1,0,1)\}\right)$$
$$G = \text{span}\left(\{(1,-1,0,0),(0,0,1,-1)\}\right)$$

Let's find vectors in F that also belong to G.

Any vector in F has the form:

$$a(1,0,1,0) + b(0,1,0,1) = (a,b,a,b)$$

For this vector to be in G, it must satisfy:

$$x + y = a + b = 0 \Rightarrow b = -a$$

$$z + t = a + b = 0 \Rightarrow b = -a$$

So, substituting $b = -a$, we get:

$$(a,-a,a,-a) = a(1,-1,1,-1)$$

Therefore, $F \cap G = \text{span}\left(\{(1,-1,1,-1)\}\right)$, and a basis is:

$$\{(1,-1,1,-1)\}$$

with

$$\dim(F \cap G) = 1$$

4. **Show that $\mathbb{R}^4 = (F \cap G) \oplus H$.**

To prove that $\mathbb{R}^4 = (F \cap G) \oplus H$, we need to show two things:

- $\dim((F \cap G) \oplus H) = 4$
- $(F \cap G) \cap H = \{0\}$

From above:

$$\dim(F \cap G) = 1, \quad \dim(H) = 3 \Rightarrow \dim((F \cap G) + H) \leq 4$$

Since both are subspaces of $\mathbb{R}^4$, their sum is at most 4-dimensional.

Now check if the intersection is trivial: suppose $v \in (F \cap G) \cap H$

Then $v = a(1,-1,1,-1)$, and also $v \in H \Rightarrow x - y + z - t = 0$

Compute:

$$x - y + z - t = a - (-a) + a - (-a) = a + a + a + a = 4a$$

Set equal to zero:

$$4a = 0 \Rightarrow a = 0 \Rightarrow v = 0$$

Therefore, $(F \cap G) \cap H = \{0\}$, and since the dimensions add up to 4, we conclude:

$$\mathbb{R}^4 = (F \cap G) \oplus H$$

Exercise 3:

Let $E = \mathbb{R}_2[X]$ be the vector space of polynomials with real coefficients of degree less than or equal to 2. And let $f : E \to E$ be the linear map defined by:
for all $P(X) \in E$, $f(P(X)) = 2P(X) - (X-1)P'(X)$
(where $P'(X)$ denotes the derivative of the polynomial $P(X)$).

1) Determine $\mathrm{Ker} f$.

2) Determine a complement of $\mathrm{Ker} f$ in E.

Answer Area

Let $E = \mathbb{R}_2[X]$ be the vector space of real polynomials of degree less than or equal to 2. Define the linear map $f : E \to E$ by:

$$f(P(X)) = 2P(X) - (X-1)P'(X)$$

where $P'(X)$ denotes the derivative of $P(X)$.
We are asked to:

1. **Determine** $\ker(f)$**.**

   Recall that:
   $$\ker(f) = \{P(X) \in E \mid f(P(X)) = 0\}$$

   Let $P(X) = a + bX + cX^2$, where $a, b, c \in \mathbb{R}$. Then:

   $$P'(X) = b + 2cX$$

   So,

   $$\begin{aligned} f(P(X)) &= 2P(X) - (X-1)P'(X) \\ &= 2(a + bX + cX^2) - (X-1)(b + 2cX) \end{aligned}$$

   Compute each part:
   $$2P(X) = 2a + 2bX + 2cX^2$$
   $$(X-1)(b+2cX) = X(b+2cX) - 1(b+2cX) = bX + 2cX^2 - b - 2cX$$
   Combine:

   $$f(P(X)) = 2a + 2bX + 2cX^2 - (bX + 2cX^2 - b - 2cX)$$

   Simplify:

   $$f(P(X)) = 2a + 2bX + 2cX^2 - bX - 2cX^2 + b + 2cX$$

   Group like terms:

   $$f(P(X)) = (2a+b) + (2b - b + 2c)X + (2c - 2c)X^2 = (2a+b) + (b+2c)X$$

   Set this equal to the zero polynomial:

   $$(2a+b) + (b+2c)X = 0$$

   This gives the system:
   $$\begin{cases} 2a + b = 0 \\ b + 2c = 0 \end{cases}$$

   Solve: From the second equation: $b = -2c$

   Plug into the first: $2a - 2c = 0 \Rightarrow a = c$

Therefore, the general form of $P(X) \in \ker(f)$ is:

$$P(X) = c + (-2c)X + cX^2 = c(1 - 2X + X^2)$$

Thus:

$$\ker(f) = \operatorname{span}\left(\{1 - 2X + X^2\}\right)$$

and

$$\dim(\ker(f)) = 1$$

2. **Determine a complement of** $\ker(f)$ **in** E**.**

Since $E = \mathbb{R}_2[X]$ has dimension 3, and $\dim(\ker(f)) = 1$, we need to find a subspace $W \subset E$ such that:

$$E = \ker(f) \oplus W$$

i.e., W has dimension 2 and intersects $\ker(f)$ trivially.

A standard basis for E is $\{1, X, X^2\}$. We already know that:

$$\ker(f) = \operatorname{span}\left(\{1 - 2X + X^2\}\right)$$

To find a complement, pick two vectors from the standard basis that are not in $\ker(f)$, and whose span does not intersect $\ker(f)$ except at 0.

Consider the subspace:

$$W = \operatorname{span}(\{1, X\})$$

Check if $W \cap \ker(f) = \{0\}$:

Suppose $a + bX = c(1 - 2X + X^2)$

Then:

$$a = c, \quad b = -2c, \quad 0 = cX^2 \Rightarrow c = 0 \Rightarrow a = b = 0$$

So $W \cap \ker(f) = \{0\}$

Also, $\dim(W) = 2$, $\dim(\ker(f)) = 1$, so:

$$E = \ker(f) \oplus W$$

Therefore, $W = \operatorname{span}(\{1, X\})$ is a complement of $\ker(f)$ in E.