Ibn Tofail University

Analysis II — Make-up Exam

Year: 23-24

Exercise 1:

Let $f : [0, 1] \to \mathbb{R}$ be an arbitrary continuous function.

1. Show that if $\int_0^1 f(x), dx = 0$, then there exists $c \in [0, 1]$ such that $f(c) = 0$.
2. Deduce that if $\int_0^1 f(x), dx = \frac{1}{2}$, then there exists $d \in [0, 1]$ such that $f(d) = d$.

Answer Area

Exercise 2:

Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by:

$$F(x) = \int_x^{2x} \frac{e^{-t}}{t}\, dt.$$

1. Verify that F is defined on $\mathbb{R}^+$, i.e., $D_F = \mathbb{R}^+$.
2. Show that F is differentiable on $\mathbb{R}^+$, and calculate its derivative. Deduce the variations of F on $\mathbb{R}^+$. (Hint: use any primitive F_0 of the function $t \mapsto e^{-t}/t$).
3. Show that $\forall x > 0, (\ln 2) \cdot e^{-2x} \leq F(x) \leq (\ln 2) \cdot e^{-x}$.
4. Deduce $\lim_{x \to 0^+} F(x)$ and $\lim_{x \to +\infty} F(x)$.

Answer Area

Exercise 3:

For all $n \in \mathbb{N}$, let:

$$I_n = \int_0^1 x^n \cdot e^{-x}\,dx.$$

1. Justify the existence of I_n for all $n \in \mathbb{N}$. Then calculate I_0.
2. Show that $\forall n \geq 0, 0 \leq I_n \leq \frac{1}{n+1}$.
3. Deduce that the sequence $(I_n)n \geq 0$ is convergent and calculate its limit.
4. Show (using integration by parts) that $\forall n \in \mathbb{N}, In + 1 = (n+1)I_n - e^{-1}$.
5. Deduce that $\forall n \geq 0, 0 \leq I_n - \frac{e^{-1}}{n+1} \leq \frac{1}{(n+1)(n+2)}$.
6. From 5, deduce a simple equivalent of I_n as n approaches infinity (i.e., a non-zero numerical sequence $(J_n)n \geq 0$ such that $\lim n \to +\infty \frac{I_n}{J_n} = 1$).

Answer Area