Ibn Tofail University

Algebra II — Normal Exam

Year: 21-22

Exercise 1:

Consider the set $E = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y = 0 \text{ and } -2x + y - z = 0\}$.

1. Show that E is a vector subspace of $\mathbb{R}^3$.
2. Determine a basis of E and deduce the dimension of E.
3. Let $F = \text{Vect}\{(1, 0, 0), (0, -1, 1)\}$. Show that $\mathbb{R}^3 = E \oplus F$.

Answer Area

Exercise 2:

Consider the following matrix A:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

1. Calculate A^2 and verify that $A^2 - 3A + 2I_3 = 0$, where I_3 is the identity matrix of $\mathcal{M}_3(\mathbb{R})$.
2. Deduce that matrix A is invertible and determine its inverse A^{-1}.

Answer Area

Exercise 3:

Let f be the endomorphism of $\mathbb{R}^3$ defined by:

$$f(x, y, z) = (2x + y + z, x + 2y + z, x + y + 2z)$$

1. Calculate the matrix A of f in the canonical basis $B = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.
2. Consider the vectors $v_1 = (1, 1, 0)$, $v_2 = (0, 1, 1)$, and $v_3 = (1, 0, 1)$ of $\mathbb{R}^3$.
   (a) Show that the family $B' = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
   (b) Calculate $f(v_1)$, $f(v_2)$, and $f(v_3)$ in the basis B'.
   (c) Determine A', the matrix of f in the basis B'.
   (d) Determine the matrices P and P^{-1} where P is the change of basis matrix from B to B'.
   (e) Using the change of basis formula, calculate again the matrix A'.

Answer Area