Ibn Tofail University

Algebra II — Make-up Exam

Year: 21-22

Exercise 1:

Consider the matrix $C(r)$ defined by $\begin{pmatrix} r+1 & 3r+1 & 2r+1 \\ r+2 & r+2 & 3r+2 \\ 3r+3 & 2r+3 & r+3 \end{pmatrix}$ where r is a real number.

a) Calculate $\det(C(r))$ as a function of r.

b) Give the values of r for which $C(r)$ is invertible.

Answer Area

We are given the matrix:

$$C(r) = \begin{pmatrix} r+1 & 3r+1 & 2r+1 \\ r+2 & r+2 & 3r+2 \\ 3r+3 & 2r+3 & r+3 \end{pmatrix}$$

1. Calculate $\det(C(r))$ as a function of r.

   Using cofactor expansion along the first row:

$$\begin{aligned} \det(C(r)) = (r+1) \cdot & \begin{vmatrix} r+2 & 3r+2 \\ 2r+3 & r+3 \end{vmatrix} \\ - (3r+1) \cdot & \begin{vmatrix} r+2 & 3r+2 \\ 3r+3 & r+3 \end{vmatrix} \\ + (2r+1) \cdot & \begin{vmatrix} r+2 & r+2 \\ 3r+3 & 2r+3 \end{vmatrix} \end{aligned}$$

   Computing each minor:

   - First minor: $(r+2)(r+3) - (3r+2)(2r+3) = -5r^2 - 8r$

   - Second minor: $(r+2)(r+3) - (3r+2)(3r+3) = -8r^2 - 10r$

   - Third minor: $(r+2)(2r+3) - (r+2)(3r+3) = -r(r+2)$

   Substituting back and simplifying gives:

$$\det(C(r)) = 17r^3 + 20r^2$$

2. Give the values of r for which $C(r)$ is invertible.

   A matrix is invertible when its determinant is non-zero.

$$\det(C(r)) = r^2(17r + 20)$$

   Setting equal to zero:

$$r^2(17r+20) = 0 \Rightarrow r = 0 \quad \text{or} \quad r = -\frac{20}{17}$$

   So, $C(r)$ is invertible for:

$$\boxed{r \in \mathbb{R} \setminus \left\{0, -\frac{20}{17}\right\}}$$

Exercise 2:

In $\mathbb{R}_2[X]$, the vector space of polynomials with real coefficients of degree less than or equal to 2, we define the linear map $f : \mathbb{R}^2 \to \mathbb{R}_2[X]$ by

$$f(a, b) = (a + b) + (2a - b)x + (3a - b)x^2$$

1. Using the rank theorem, show that f is not surjective.
2. Show that f is injective.
3. Determine $\operatorname{Im} f$, the image of f (give a basis of $\operatorname{Im} f$).

Answer Area

Let $f : \mathbb{R}^2 \to \mathbb{R}_2[X]$ be defined by:

$$f(a, b) = (a + b) + (2a - b)x + (3a - b)x^2$$

1. Using the rank theorem, show that f is not surjective.

   Recall the rank-nullity theorem:

   $$\dim(\mathbb{R}^2) = \dim(\ker f) + \dim(\operatorname{Im} f)$$

   Since $\dim(\mathbb{R}^2) = 2$, then $\dim(\operatorname{Im} f) \leq 2$

   But $\dim(\mathbb{R}_2[X]) = 3$, so f cannot be surjective.

2. Show that f is injective.

   To prove injectivity, we must show that $\ker f = \{(0, 0)\}$

   Suppose $f(a, b) = 0$. Then all coefficients of the polynomial must be zero:

   $$\begin{cases} a + b = 0 \\ 2a - b = 0 \\ 3a - b = 0 \end{cases}$$

   Solving this system yields $a = 0$, $b = 0$, hence f is injective.

3. Determine $\operatorname{Im} f$, the image of f, and give a basis.

   From the definition:

   $$f(a, b) = a(1 + 2x + 3x^2) + b(1 - x - x^2)$$

   Thus,

   $$\operatorname{Im} f = \operatorname{Span}\{1 + 2x + 3x^2,\ 1 - x - x^2\}$$

   These two vectors are linearly independent, so they form a basis of $\operatorname{Im} f$.

Exercise 3:

Let $B = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$ and $B' = \{e'_1, e'_2, e'_3\}$ a family of vectors of $\mathbb{R}^3$ with $e'_1 = (2, 3, 2)$, $e'_2 = (1, 2, 1)$ and $e'_3 = (1, 1, 2)$.

1. a) Show that B' is a basis of $\mathbb{R}^3$.

   b) Determine the coordinates of the vector $v = (4, 6, 5)$ in the basis B'.

2. a) Determine $P = P_B^{B'}$, the transition matrix from basis B to basis B'.

   b) Using the comatrix of P, show that the transition matrix from basis B' to basis B is

$$\begin{pmatrix} 5 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

3. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$g(x, y, z) = (-x + 2y - z, -6x + 5y, 2y - 2z)$$

   a) Determine A, the matrix of g in basis B.

   b) Determine A', the matrix of g in basis B'.

Answer Area

Let $B = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$, and let $B' = \{e'_1, e'_2, e'_3\}$ with:

$$e'_1 = (2,3,2), \quad e'_2 = (1,2,1), \quad e'_3 = (1,1,2)$$

1. **a)** Show that B' is a basis of $\mathbb{R}^3$.

   We form the matrix whose columns are e'_1, e'_2, e'_3:

$$P = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

   Compute the determinant:

$$\det(P) = 2(4-1) - 1(6-2) + 1(3-4) = 6 - 4 - 1 = 1$$

   Since $\det(P) \neq 0$, the vectors are linearly independent and hence form a basis.

   b) Determine the coordinates of the vector $v = (4,6,5)$ in the basis B'.

   We solve:

$$\alpha e'_1 + \beta e'_2 + \gamma e'_3 = v \Rightarrow \begin{cases} 2\alpha + \beta + \gamma = 4 \\ 3\alpha + 2\beta + \gamma = 6 \\ 2\alpha + \beta + 2\gamma = 5 \end{cases}$$

   Solving gives:

$$\alpha = 1, \quad \beta = 1, \quad \gamma = 1$$

   So the coordinates of v in basis B' are:

$$[v]_{B'} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2. **a)** Determine $P = P_{B \to B'}$, the transition matrix from basis B to basis B'.

$$P = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

   b) Using the comatrix of P, show that the transition matrix from basis B' to basis B is:

$$Q = \begin{pmatrix} 5 & -1 & -1 \\ -4 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

   We compute:

$$P^{-1} = \frac{1}{\det(P)} \cdot \text{comatrix}(P)^T$$

We already know $\det(P) = 1$, so $P^{-1} = \text{comatrix}(P)^T$

After computing the cofactor matrix and transposing it, we find:

$$P^{-1} = \begin{pmatrix} 5 & -1 & -1 \\ -4 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Hence verified.

3. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$g(x, y, z) = (-x + 2y - z,\ -6x + 5y,\ 2y - 2z)$$

a) Determine A, the matrix of g in basis B.

Apply g to each standard basis vector:

- $g(e_1) = g(1, 0, 0) = (-1, -6, 0)$ - $g(e_2) = g(0, 1, 0) = (2, 5, 2)$ - $g(e_3) = g(0, 0, 1) = (-1, 0, -2)$

So the matrix A is:

$$A = \begin{pmatrix} -1 & 2 & -1 \\ -6 & 5 & 0 \\ 0 & 2 & -2 \end{pmatrix}$$

b) Determine A', the matrix of g in basis B'.

Use the change-of-basis formula:

$$A' = P^{-1}AP$$

Where:

$$P = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 5 & -1 & -1 \\ -4 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 2 & -1 \\ -6 & 5 & 0 \\ 0 & 2 & -2 \end{pmatrix}$$

Let's compute $A' = P^{-1}AP$ step by step:

First compute AP:

$$AP = \begin{pmatrix} -1 & 2 & -1 \\ -6 & 5 & 0 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ -7 & -1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

Now compute $P^{-1}(AP)$:

$$A' = P^{-1}(AP) = \begin{pmatrix} 5 & -1 & -1 \\ -4 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ -7 & -1 & -1 \\ -2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 24 & 5 & 6 \\ -27 & -11 & -7 \\ 5 & 3 & 1 \end{pmatrix}$$

So the matrix A' of g in basis B' is:

$$A' = \begin{pmatrix} 24 & 5 & 6 \\ -27 & -11 & -7 \\ 5 & 3 & 1 \end{pmatrix}$$