Ibn Tofail University

Analysis II — Normal Exam

Year: 22-23

Exercise 1:

Consider the function $f : [1, 3] \to \mathbb{R}$ defined by:

$$f(x) = \frac{1}{x}$$

1. Justify that f is integrable (in the Riemann sense) on $[1, 3]$.
2. Calculate the Darboux sums (lower and upper) $D_S^-(f)$ and $D_S^+(f)$ of f with respect to the subdivision S of $[1, 3]$ defined by $S = \{1, 2, 3\}$.
3. State (without proving) the inequalities between $D_S^-(f)$, $D_S^+(f)$ and $\int_1^3 f(x)dx$.
4. Deduce an approximation of $\ln 3$ by rational numbers.

Answer Area

1. The function $f(x) = \frac{1}{x}$ is continuous on $[1,3]$ because it is a rational function and the denominator does not vanish on this interval. Since every continuous function on a closed and bounded interval is Riemann integrable, f is integrable on $[1,3]$.

2. We are given the subdivision $S = \{1,2,3\}$. This divides $[1,3]$ into two subintervals: $[1,2]$ and $[2,3]$. On each subinterval, we compute the infimum and supremum of $f(x) = \frac{1}{x}$:

$$\text{On } [1,2]: \quad m_1 = \min_{x\in[1,2]} f(x) = \frac{1}{2}, \quad M_1 = \max_{x\in[1,2]} f(x) = 1$$
$$\text{On } [2,3]: \quad m_2 = \min_{x\in[2,3]} f(x) = \frac{1}{3}, \quad M_2 = \max_{x\in[2,3]} f(x) = \frac{1}{2}$$

Compute the lower Darboux sum:

$$D_-(S,f) = m_1(2-1) + m_2(3-2) = \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 = \frac{5}{6}$$

Compute the upper Darboux sum:

$$D_+(S,f) = M_1(2-1) + M_2(3-2) = 1\cdot 1 + \frac{1}{2}\cdot 1 = \frac{3}{2}$$

3. The Darboux sums satisfy the following inequality:

$$D_-(S,f) \le \int_1^3 f(x)\,dx \le D_+(S,f)$$

That is:

$$\frac{5}{6} \le \int_1^3 \frac{1}{x}\,dx \le \frac{3}{2}$$

4. Since $\int_1^3 \frac{1}{x}\,dx = \ln 3$, we deduce:

$$\frac{5}{6} \le \ln 3 \le \frac{3}{2}$$

A reasonable approximation can be obtained by taking the average:

$$\ln 3 \approx \frac{\frac{5}{6}+\frac{3}{2}}{2} = \frac{\frac{5}{6}+\frac{9}{6}}{2} = \frac{14}{12}\cdot\frac{1}{2} = \frac{7}{6}$$

So, $\ln 3$ lies between $\frac{5}{6}$ and $\frac{3}{2}$, and one rational approximation is $\frac{7}{6}$.

Exercise 2:

Consider the function $G : \mathbb{R} \to \mathbb{R}$ defined by:

$$G(x) = \int_x^{2x} \frac{dt}{\sqrt{t^2+1}}$$

1. Justify that G is defined on $\mathbb{R}$. Also show that G is an odd function.
2. Verify that G is differentiable on $\mathbb{R}$, and calculate its derivative $G'(x)$. (Hint: use any primitive F of the function $t \mapsto \frac{1}{\sqrt{t^2+1}}$).
3. Deduce that G is strictly increasing on $\mathbb{R}$.
4. Verify that $t^2 \leq t^2 + 1 \leq (t+1)^2$ for all $t > 0$. Deduce the following inequality:

$$\forall x > 0, \ln(2x+1) - \ln(x+1) \leq G(x) \leq \ln 2$$

5. Deduce the limit $\lim_{x \to +\infty} G(x)$.
6. Solve the equation $G(x) = 0$.

Answer Area

1. The function $G(x) = \int_x^{2x} \frac{dt}{\sqrt{t^2+1}}$ is defined for all $x \in \mathbb{R}$ because the integrand $\frac{1}{\sqrt{t^2+1}}$ is continuous on $\mathbb{R}$. Therefore, the integral over any finite interval exists. To show that G is odd, we compute:

$$G(-x) = \int_{-x}^{-2x} \frac{dt}{\sqrt{t^2+1}}$$

Perform the substitution $u = -t$, so $du = -dt$, and the limits become:

$$G(-x) = \int_x^{2x} \frac{-du}{\sqrt{u^2+1}} = -\int_x^{2x} \frac{du}{\sqrt{u^2+1}} = -G(x)$$

Hence, G is an odd function.

2. Since the integrand $f(t) = \frac{1}{\sqrt{t^2+1}}$ is continuous on $\mathbb{R}$, by the Fundamental Theorem of Calculus, $G(x)$ is differentiable on $\mathbb{R}$. Let F be an antiderivative of f, then:

$$G(x) = F(2x) - F(x)$$

Differentiating using the chain rule:

$$G'(x) = 2F'(2x) - F'(x) = 2f(2x) - f(x) = \frac{2}{\sqrt{(2x)^2+1}} - \frac{1}{\sqrt{x^2+1}}$$

3. From the previous part:

$$G'(x) = \frac{2}{\sqrt{4x^2+1}} - \frac{1}{\sqrt{x^2+1}}$$

We analyze the sign of $G'(x)$. For $x > 0$, clearly:

$$\frac{2}{\sqrt{4x^2+1}} > \frac{1}{\sqrt{x^2+1}} \quad \Rightarrow \quad G'(x) > 0$$

For $x < 0$, since G is odd, G' is even (you can verify this), so $G'(x) > 0$ also holds. Thus, G is strictly increasing on $\mathbb{R}$.

4. First, observe that for $t > 0$,

$$t^2 \leq t^2 + 1 \leq (t+1)^2$$

Taking square roots:

$$t \leq \sqrt{t^2+1} \leq t+1$$

Inverting (and reversing inequalities):

$$\frac{1}{t+1} \leq \frac{1}{\sqrt{t^2+1}} \leq \frac{1}{t}$$

Now integrate from x to $2x$:

$$\int_x^{2x} \frac{dt}{t+1} \leq G(x) \leq \int_x^{2x} \frac{dt}{t}$$

Compute both sides:

$$\ln(2x+1) - \ln(x+1) \leq G(x) \leq \ln(2x) - \ln(x) = \ln 2$$

5. From the inequality:

$$\ln(2x+1) - \ln(x+1) \leq G(x) \leq \ln 2$$

As $x \to +\infty$, the left-hand side tends to:

$$\ln\left(\frac{2x+1}{x+1}\right) \to \ln 2$$

So by the Squeeze Theorem:

$$\lim_{x\to+\infty} G(x) = \ln 2$$

6. We solve $G(x) = 0$, i.e.,

$$\int_x^{2x} \frac{dt}{\sqrt{t^2+1}} = 0$$

This implies $x = 0$, since the integrand is positive for all t, and the only way the integral is zero is if the lower and upper limits are equal. Therefore:

$$x = 0$$

Exercise 3:

For all $n \in \mathbb{N}$, let:

$$I_n = \int_0^1 (1 - t^2)^n dt$$

1. Justify the existence of the integral I_n for all $n \in \mathbb{N}$.
2. Show that $\forall n \in \mathbb{N}, I_{n+1} = \frac{2n+2}{2n+3} \cdot I_n$.
3. Deduce that $\forall n \in \mathbb{N}, I_n = \frac{2^n (n!)^2}{(2n+1)!}$.
4. Using Newton's binomial formula, show that $\forall n \in \mathbb{N}, I_n = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{2k+1}$.

Answer Area

1. The function $(1-t^2)^n$ is continuous on the closed interval $[0,1]$ for all $n \in \mathbb{N}$, because it is a composition and power of continuous functions. Since any continuous function on a closed and bounded interval is Riemann integrable, the integral

$$I_n = \int_0^1 (1-t^2)^n \, dt$$

exists for all $n \in \mathbb{N}$.

2. We use integration by parts to prove the recurrence relation. Let:

$$I_{n+1} = \int_0^1 (1-t^2)^{n+1} dt$$

Write:

$$(1-t^2)^{n+1} = (1-t^2)(1-t^2)^n$$

Now integrate by parts. Set:

$$u = (1-t^2)^{n+1}, \quad dv = dt \quad \Rightarrow \quad du = -2(n+1)t(1-t^2)^n dt, \quad v = t$$

Then:

$$I_{n+1} = t(1-t^2)^{n+1}\Big|_0^1 + 2(n+1)\int_0^1 t^2(1-t^2)^n dt$$

The boundary term vanishes at both ends. So:

$$I_{n+1} = 2(n+1)\int_0^1 t^2(1-t^2)^n dt$$

Now observe that:

$$t^2 = 1-(1-t^2)$$

Hence:

$$I_{n+1} = 2(n+1)\left[\int_0^1 (1-t^2)^n dt - \int_0^1 (1-t^2)^{n+1} dt\right] = 2(n+1)(I_n - I_{n+1})$$

Solving for I_{n+1}:

$$I_{n+1}(1+2(n+1)) = 2(n+1)I_n \quad \Rightarrow \quad I_{n+1} = \frac{2(n+1)}{2n+3} I_n$$

3. We now deduce the general formula:

$$I_n = \frac{2^n (n!)^2}{(2n+1)!}$$

This can be proved by induction using the recurrence:

$$I_{n+1} = \frac{2(n+1)}{2n+3} I_n$$

The base case $n = 0$:

$$I_0 = \int_0^1 (1-t^2)^0 dt = \int_0^1 1 dt = 1$$

Also:

$$\frac{2^0 (0!)^2}{1!} = \frac{1}{1} = 1$$

Assume the formula holds for n. Then:

$$I_{n+1} = \frac{2(n+1)}{2n+3} \cdot \frac{2^n (n!)^2}{(2n+1)!} = \frac{2^{n+1}(n+1)(n!)^2}{(2n+3)(2n+1)!} = \frac{2^{n+1}((n+1)!)^2}{(2n+3)!}$$

Thus, the formula holds for all $n \in \mathbb{N}$.

4. Using Newton's binomial formula:

$$(1-t^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^{2k}$$

Integrate term by term from 0 to 1:

$$\begin{aligned} I_n &= \int_0^1 (1-t^2)^n dt = \int_0^1 \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^{2k} dt \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 t^{2k} dt \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1} \end{aligned}$$