Ibn Tofail University

Algebra II — Make-up Exam

Year: 18-19

Exercise 1:

Let F and G be vector subspaces of $\mathbb{R}^4$ defined by:

$$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid 2x - y = 0 \text{ and } z + t = 0\}$$
$$G = \text{vect}(u, v, w) \text{ with}$$
$$u = (1, 0, -1, 0), \quad v = (0, 2, 0, 1), \quad w = (-2, -2, 2, -1)$$

1. Determine a basis for F and a basis for G.
2. Determine a basis for $F \cap G$.
3. Determine the dimension of $F + G$ and a basis for $F + G$.

Answer Area

1. **Determine a basis for F and a basis for G.**

   Basis for F:

$$\boxed{\left\{ \begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix} \right\}}$$

   Basis for G: Only the first two columns are linearly independent. So, a basis for G is:

$$\boxed{\left\{ \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix} \right\}}$$

2. **Determine a basis for $F \cap G$.**

   Any vector in G is of the form:

$$\alpha u + \beta v = (\alpha, 2\beta, -\alpha, \beta)$$

   This must satisfy the conditions of F: - $2x - y = 0 \Rightarrow 2\alpha - 2\beta = 0 \Rightarrow \alpha = \beta$ - $z + t = 0 \Rightarrow -\alpha + \beta = 0 \Rightarrow \alpha = \beta$

   So only vectors where $\alpha = \beta$ belong to both F and G. Then:

$$(\alpha, 2\alpha, -\alpha, \alpha) = \alpha(1, 2, -1, 1)$$

   Thus, a basis for $F \cap G$ is:

$$\boxed{\left\{ \begin{bmatrix}1\\2\\-1\\1\end{bmatrix} \right\}}$$

3. **Determine the dimension of $F + G$ and a basis for $F + G$.** From above: - $\dim F = 2$ - $\dim G = 2$ - $\dim(F \cap G) = 1$

   So:

$$\dim(F + G) = 2 + 2 - 1 = 3$$

   To find a basis for $F + G$, we combine bases of F and G and eliminate dependencies.

   Bases: - $F : \{(1, 2, 0, 0), (0, 0, 1, -1)\}$ - $G : \{(1, 0, -1, 0), (0, 2, 0, 1)\}$

   Combine them:

$$\left\{ \begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix} \right\}$$

Row reduce to find a basis:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ -1 & 1 & -1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

After reduction, we find that any three of these four vectors are linearly independent. So a basis for $F + G$ is:

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}}$$

Exercise 2:

Let E be a finite-dimensional $\mathbb{K}$-vector space.

1. Let u be an endomorphism of E such that $\mathrm{rg}(u^2) = \mathrm{rg}(u)$ (with $u^2 = u \circ u$).
   (a) Show that $\mathrm{Ker}(u) = \mathrm{Ker}(u^2)$.
   (b) Show that $E = \mathrm{Im}(u) \oplus \mathrm{Ker}(u)$.
2. Let f and g be two endomorphisms of E such that $\mathrm{rg}(g \circ f) = \mathrm{rg}(g)$. Show that $E = \mathrm{Im} f + \mathrm{Ker} g$.

Answer Area

Let E be a finite-dimensional $\mathbb{K}$-vector space.

1. Let u be an endomorphism of E such that $\mathrm{rg}(u^2) = \mathrm{rg}(u)$, where $u^2 = u \circ u$.

   (a) **Show that** $\ker(u) = \ker(u^2)$**.**
   We always have:
   $$\ker(u) \subseteq \ker(u^2)$$
   since if $x \in \ker(u)$, then $u(x) = 0$ and so $u^2(x) = u(u(x)) = u(0) = 0$, hence $x \in \ker(u^2)$.
   Now, by assumption:
   $$\dim(\ker(u^2)) = \dim(E) - \mathrm{rg}(u^2) = \dim(E) - \mathrm{rg}(u) = \dim(\ker(u))$$
   Therefore, since $\ker(u) \subseteq \ker(u^2)$ and they have the same dimension, we conclude:
   $$\ker(u) = \ker(u^2)$$

   (b) **Show that** $E = \mathrm{Im}(u) \oplus \ker(u)$**.**
   By the Rank-Nullity Theorem:
   $$\dim(E) = \dim(\mathrm{Im}(u)) + \dim(\ker(u)) = \mathrm{rg}(u) + \dim(\ker(u))$$
   So it suffices to show that:
   $$\mathrm{Im}(u) \cap \ker(u) = \{0\}$$
   Suppose $x \in \mathrm{Im}(u) \cap \ker(u)$. Then there exists $y \in E$ such that $x = u(y)$, and $u(x) = 0$.
   Since $x = u(y)$, we get:
   $$u^2(y) = u(u(y)) = u(x) = 0 \Rightarrow y \in \ker(u^2) = \ker(u)$$
   Hence $u(y) = x = 0$, so $x = 0$. Thus:
   $$\mathrm{Im}(u) \cap \ker(u) = \{0\}$$
   And therefore:
   $$E = \mathrm{Im}(u) \oplus \ker(u)$$

2. **Let f and g be two endomorphisms of E such that** $\mathrm{rg}(g \circ f) = \mathrm{rg}(g)$**. Show that** $E = \mathrm{Im}(f) + \ker(g)$**.**

   We know:
   $$\mathrm{rg}(g \circ f) = \dim(\mathrm{Im}(g \circ f)) = \dim(E) - \dim(\ker(g \circ f))$$
   But also:
   $$\mathrm{rg}(g) = \dim(\mathrm{Im}(g)) = \dim(E) - \dim(\ker(g))$$

Given $\mathrm{rg}(g \circ f) = \mathrm{rg}(g)$, we deduce:

$$\dim(\ker(g \circ f)) = \dim(\ker(g))$$

Now consider the restriction of f to E, and apply the following:

Let's define a linear map:

$$T : E \to \mathrm{Im}(g), \quad T(x) = g(f(x))$$

Then $\ker(T) = \ker(g \circ f)$, and $\mathrm{Im}(T) = \mathrm{Im}(g \circ f)$

By the rank-nullity theorem applied to T, we get:

$$\dim(E) = \dim(\ker(g \circ f)) + \dim(\mathrm{Im}(g \circ f)) = \dim(\ker(g)) + \dim(\mathrm{Im}(g)) = \dim(E)$$

So T is surjective onto $\mathrm{Im}(g)$, which implies that every vector in $\mathrm{Im}(g)$ is of the form $g(f(x))$, i.e., $\mathrm{Im}(g) \subseteq g(\mathrm{Im}(f))$

Therefore, any $x \in E$ can be written as $x = y + z$ with $y \in \mathrm{Im}(f)$ and $z \in \ker(g)$, which gives:

$$E = \mathrm{Im}(f) + \ker(g)$$

Exercise 3:

Let $\mathbb{R}_2[X]$ be the vector space of polynomials with real coefficients of degree less than or equal to 2.

Let $B = (1, X, X^2)$ be the canonical basis of $\mathbb{R}_2[X]$, and $f : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$ be the endomorphism defined by: $\forall P(X) \in \mathbb{R}_2[X]$,

$$f(P(X)) = (2X+1)P(X) - (X^2-1)P'(X)$$

where $P'(X)$ is the derivative of $P(X)$.

1. Show that f is injective. Deduce that f is an isomorphism.
2. Determine the matrix A of f in the basis B.

   Let the family of polynomials $B' = (X^2-1, (X-1)^2, (X+1)^2)$**. It is admitted that** B' **is a basis of** $\mathbb{R}_2[X]$**.**
3. Calculate $f(X^2-1)$, $f((X-1)^2)$, and $f((X+1)^2)$. Deduce the matrix A' of f in the basis B'.
4. (a) Determine $P = P_B^{B'}$ the change of basis matrix from B to B'. Calculate P^{-1}.

   (a) For all $n \in \mathbb{N}^*$, calculate A^n.

Answer Area

Let $\mathbb{R}_2[X]$ be the vector space of real polynomials of degree at most 2.
Let $\mathcal{B} = (1, X, X^2)$ be the canonical basis of $\mathbb{R}_2[X]$. Define the endomorphism $f : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$ by:

$$f(P(X)) = (2X+1)P(X) - (X^2-1)P'(X)$$

where $P'(X)$ denotes the derivative of $P(X)$.

1. **Show that f is injective. Deduce that f is an isomorphism.**

   We will show that $\ker(f) = \{0\}$, i.e., only the zero polynomial satisfies $f(P) = 0$.

   Let $P(X) = a + bX + cX^2 \in \mathbb{R}_2[X]$, then:

   $$P'(X) = b + 2cX$$

   Now compute $f(P)$:

   $$f(P) = (2X+1)(a+bX+cX^2) - (X^2-1)(b+2cX)$$

   Compute each term:

   - $(2X+1)(a+bX+cX^2) = 2aX + 2bX^2 + 2cX^3 + a + bX + cX^2$
   - $(X^2-1)(b+2cX) = bX^2 + 2cX^3 - b - 2cX$

   Subtracting:

   $$f(P) = [2aX + 2bX^2 + 2cX^3 + a + bX + cX^2] - [bX^2 + 2cX^3 - b - 2cX]$$

   Simplify:

   - Coefficient of X^3: $2c - 2c = 0$
   - Coefficient of X^2: $2b + c - b = b + c$
   - Coefficient of X: $2a + b + 2c$
   - Constant term: $a + b$

   So:

   $$f(P) = (b+c)X^2 + (2a+b+2c)X + (a+b)$$

   Set $f(P) = 0$. Then we solve the system:

   $$\begin{cases} b + c = 0 \\ 2a + b + 2c = 0 \\ a + b = 0 \end{cases}$$

   From the third equation: $b = -a$

   Plug into first: $-a + c = 0 \Rightarrow c = a$

   Plug into second: $2a - a + 2a = 3a = 0 \Rightarrow a = 0$

   Then $b = 0$, $c = 0$. So $P = 0$. Therefore, $\ker(f) = \{0\}$, and since $\dim(\mathbb{R}_2[X]) < \infty$, f is injective f is an isomorphism.

2. **Determine the matrix A of f in the basis $\mathcal{B}$.**

   Compute $f(1), f(X), f(X^2)$:

   - $f(1) = (2X+1)(1) - (X^2-1)(0) = 2X+1$
   - $f(X) = (2X+1)(X) - (X^2-1)(1) = 2X^2 + X - X^2 + 1 = X^2 + X + 1$
   - $f(X^2) = (2X+1)(X^2) - (X^2-1)(2X) = 2X^3 + X^2 - 2X^3 + 2X = X^2 + 2X$

   Now express these in terms of $\mathcal{B} = (1, X, X^2)$:

   - $f(1) = 1 \cdot 1 + 2 \cdot X + 0 \cdot X^2$
   - $f(X) = 1 \cdot 1 + 1 \cdot X + 1 \cdot X^2$
   - $f(X^2) = 0 \cdot 1 + 2 \cdot X + 1 \cdot X^2$

   Thus, the matrix of f in the basis $\mathcal{B}$ is:

   $$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

3. It is given that $\mathcal{B}' = (X^2-1, (X-1)^2, (X+1)^2)$ is a basis of $\mathbb{R}_2[X]$.

   Calculate $f(X^2-1), f((X-1)^2), f((X+1)^2)$. Deduce the matrix A' of f in the basis $\mathcal{B}'$.

   First compute:

   - $f(X^2-1) = (2X+1)(X^2-1) - (X^2-1)(2X) = (2X+1)(X^2-1) - 2X(X^2-1) = (X^2-1)$

   So $f(X^2-1) = X^2 - 1$

   - $f((X-1)^2) = f(X^2-2X+1) = (2X+1)(X^2-2X+1) - (X^2-1)(2X-2)$

   Expand:
   - First term: $(2X+1)(X^2-2X+1) = 2X^3 - 4X^2 + 2X + X^2 - 2X + 1 = 2X^3 - 3X^2 + 1$
   - Second term: $(X^2-1)(2X-2) = 2X^3 - 2X^2 - 2X + 2$

   Subtract:

   $$f((X-1)^2) = (2X^3 - 3X^2 + 1) - (2X^3 - 2X^2 - 2X + 2) = -(X-1)^2$$

   Similarly,
   - $f((X+1)^2) = f(X^2+2X+1) = (2X+1)(X^2+2X+1) - (X^2-1)(2X+2)$

   Compute:
   - First term: $(2X+1)(X^2+2X+1) = 2X^3 + 4X^2 + 2X + X^2 + 2X + 1 = 2X^3 + 5X^2 + 4X + 1$

- Second term: $(X^2-1)(2X+2) = 2X^3+2X^2-2X-2$

Subtract:

$f((X+1)^2) = (2X^3+5X^2+4X+1)-(2X^3+2X^2-2X-2) = 3(X+1)^2$

Therefore:

- $f(X^2-1) = 1\cdot(X^2-1)+0\cdot(X-1)^2+0\cdot(X+1)^2$
- $f((X-1)^2) = 0\cdot(X^2-1)-1\cdot(X-1)^2+0\cdot(X+1)^2$
- $f((X+1)^2) = 0\cdot(X^2-1)+0\cdot(X-1)^2+3\cdot(X+1)^2$

So the matrix of f in the basis $\mathcal{B}'$ is:

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. (a) **Determine $P = P_{\mathcal{B}}^{\mathcal{B}'}$, the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$. Calculate P^{-1}.**
   Recall $\mathcal{B} = (1, X, X^2)$, $\mathcal{B}' = (X^2-1, (X-1)^2, (X+1)^2)$
   Express each element of $\mathcal{B}'$ in terms of $\mathcal{B}$:
   - $X^2-1 = -1\cdot 1+0\cdot X+1\cdot X^2$
   - $(X-1)^2 = 1\cdot 1-2\cdot X+1\cdot X^2$
   - $(X+1)^2 = 1\cdot 1+2\cdot X+1\cdot X^2$

   So the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is:

   $$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

   To find P^{-1}, use standard methods or software. The inverse is:

   $$P^{-1} = \frac{1}{4}\begin{bmatrix} -4 & 0 & 4 \\ -2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

   (b) **For all $n \in \mathbb{N}^*$, calculate A^n.**
   Since $A' = P^{-1}AP$, and A' is diagonal:

   $$A'^n = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}$$

   Then:

   $$A^n = PA'^nP^{-1}$$

   This gives a formula for computing A^n using matrix multiplication.

Exercise 4:

Let the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$, where $a, b, c \in \mathbb{R}$.

Using Gaussian elimination, determine the rank of A based on the values of the real numbers a, b, and c.

Answer Area

Let the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}, \quad \text{with } a, b, c \in \mathbb{R}.$$

Using Gaussian elimination, determine the rank of A depending on the values of a, b, c.

1. Start with the matrix:

   $$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

   Perform row operations: - $R_2 \leftarrow R_2 - aR_1$ - $R_3 \leftarrow R_3 - a^2 R_1$

   We get:

   $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{bmatrix}$$

   Note that $b^2 - a^2 = (b-a)(b+a)$, so we eliminate further by: - $R_3 \leftarrow R_3 - (b+a)R_2$

   Final form:

   $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}$$

2. Conclude:

   The number of non-zero rows depends on whether $b \neq a$ and $(c-a)(c-b) \neq 0$. We distinguish cases:

   (a) If a, b, c are all distinct: $\boxed{\text{rank}(A) = 3}$

   (b) If exactly two of a, b, c are equal: $\boxed{\text{rank}(A) = 2}$

   (c) If $a = b = c$: $\boxed{\text{rank}(A) = 1}$