Ibn Tofail University

Analysis II — Normal Exam

Year: 23-24

Exercise 1:

Let f be the function defined on $\mathbb{R}$ by:

$$\forall x \in \mathbb{R}, f(x) = \frac{1}{e^x + 1}.$$

1. Find the Taylor expansion of f up to order 3 at zero.
2. Deduce the equation of the tangent line to the curve C_f of f at the point with abscissa 0.
3. Also deduce that the curve C_f crosses the tangent line at 0 (i.e., $(0, f(0))$ is an inflection point).
4. (a) Study the convexity and concavity of the function f on $\mathbb{R}$.

   (b) Deduce that for all $(a, b) \in \mathbb{R}^2$ such that $a \geq 1$ and $b \geq 1$, we have:

$$\frac{2}{1 + \sqrt{ab}} \leq \frac{1}{1+a} + \frac{1}{1+b}.$$

N.B.: Question 4 is independent of the previous questions.

Answer Area

Exercise 2:

Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by:

$$F(x) = \int_x^{2x} e^{-t^2} dt.$$

1. Verify that F is defined on $\mathbb{R}$. Also show that F is odd.
2. Show that F is differentiable on $\mathbb{R}$, and calculate its derivative. Deduce the variations of F on $\mathbb{R}$.
   (Hint: use any antiderivative F_0 of the function $t \mapsto e^{-t^2}$).
3. Show that
   $$\forall x \geq 0, 0 \leq F(x) \leq x \cdot e^{-x^2}.$$
4. Deduce $\lim_{x \to +\infty} F(x)$.
5. Sketch the curve C_F of the function F.

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Exercise 3:

For all $(n, m) \in \mathbb{N} \times \mathbb{N}$, let:

$$I_{n,m} = \int_0^1 t^n \cdot (1-t)^m dt.$$

1. Calculate $I_{n,0}$ for all $n \in \mathbb{N}$.

2. Show (using an appropriate change of variable) that $I_{n,m} = I_{m,n}$ for all $(n, m) \in \mathbb{N} \times \mathbb{N}$.

3. Show (using integration by parts) that

$$\forall (n, m) \in \mathbb{N}^* \times \mathbb{N}, I_{n,m} = \frac{n}{m+1} \cdot I_{n-1,m+1}.$$

4. Deduce that

$$\forall (n, m) \in \mathbb{N}^2, I_{n,m} = \frac{n!\, m!}{(n+m+1)!}.$$

5. Show (using the binomial theorem) that

$$\forall (n, m) \in \mathbb{N}^2, I_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{m+k+1}.$$

Answer Area