UNIVERSITY IBN TOFAIL

Algebra II

Problem Set I

Exercise 1:

Study the following propositions. Prove those that are true and provide counterexamples for those that are false.

a) $\mathbb{R}^2$ with the usual addition and the external law: $\lambda \cdot (x, y) = (\lambda x, 0)$ where $\lambda \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ is a vector space over $\mathbb{R}$.

b) $\mathbb{C}^3$ with the usual addition and the external law over $\mathbb{C}$ defined by $\lambda \cdot (x, y, z) = (\lambda x, y, z)$ where $\lambda \in \mathbb{C}$, $(x, y, z) \in \mathbb{C}^3$ is a $\mathbb{C}$-vector space.

c) The set of polynomials with real coefficients divisible by $X^3 + 1$, with the usual addition of polynomials and multiplication of a polynomial by a scalar, is an $\mathbb{R}$-vector space.

Correction

a)

b)

c)

Exercise 2:

Consider the following sets:

$$E_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - 2y + 5z = 0\}$$
$$E_2 = \{v \in \mathbb{R}^3 \mid v = (a - b, 2b, a + 3b), a, b \in \mathbb{R}\}$$
$$E_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x \cdot y = 0\}$$

1. Among these sets, which ones are vector subspaces of the vector space $\mathbb{R}^3$ over $\mathbb{R}$?
2. Give a basis for each vector subspace.

Correction

1.

2.

Exercise 3:

In the vector space $\mathbb{R}^4$ with its canonical basis, consider the vectors:

$$\begin{aligned} e'_1 &= (1, 2, -1, -2) \\ e'_2 &= (2, 3, 0, -1) \\ e'_3 &= (1, 3, -1, 0) \\ e'_4 &= (1, 2, 1, 4) \end{aligned}$$

a) Show that the family $B' = (e'_1, e'_2, e'_3, e'_4)$ is a basis of $\mathbb{R}^4$.

b) Calculate the coordinates of the vector $v = (7, 14, -1, 2)$ in the basis B'.

Correction

a)

b)

Exercise 4:

Consider the set:

$$F = \{(x, y, z) \in \mathbb{C}^3 \mid x + y + z = 0 \text{ and } 2x + iy - z = 0\}$$

a) Show that F is an $\mathbb{R}$-vector space.

b) Give a basis for F and deduce its dimension.

Correction

a)

b)

Exercise 5:

Let F be the vector subspace of $\mathbb{R}_4[X]$ generated by the following vectors (polynomials):

$$\begin{aligned} P_1 &= X^2 \\ P_2 &= (X-1)^2 \\ P_3 &= (X+1)^2 \end{aligned}$$

a) Show that (P_1, P_2, P_3) is a basis of F.

b) Complete the family (P_1, P_2, P_3) into a basis of $\mathbb{R}_4[X]$ and deduce a supplementary subspace of F in $\mathbb{R}_4[X]$.

Correction

a)

b)

Exercise 6:

In the $\mathbb{R}$-vector space $F(\mathbb{R}, \mathbb{R})$, consider the functions:

$$f_n(x) = \sin(nx), n \geq 1$$

a) Show that for all $n \in \mathbb{N}^*$, the family $(f_1, \dots, f_n)$ is linearly independent.

b) Deduce that $F(\mathbb{R}, \mathbb{R})$ is an $\mathbb{R}$-vector space of infinite dimension.

Correction

a)

b)

Exercise 7:

1. Show that the application $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x - y, x, x + y)$ is linear.
2. Show that f is injective but not surjective.
3. Determine a basis for $\mathrm{Im} f$ (the image of f).

Correction

1.
2.
3.

Exercise 8:

Let $f \in L(\mathbb{R}^3)$ defined by $f(x, y, z) = (2y + z, x - 4y, 3x)$.

1. Determine the matrix A of f with respect to the canonical basis $B = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.
2. Let $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, and $v_3 = (1, 0, 0)$ be vectors in $\mathbb{R}^3$.
    a) Show that the family $B' = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

b) Calculate $f(v_1)$, $f(v_2)$, and $f(v_3)$.

c) Determine A', the matrix of f in the basis B'.

3. a) Determine the matrices P and P^{-1} where P is the change of basis matrix from basis B to basis B'.

   b) Using the change of basis formula, recalculate the matrix A'.

Correction

1.

2. a)

   b)

   c)

3. a)

   b)