Taylor Formulas, Limited Development and Applications

Key Options

-

- Taylor's Theorem (Taylor-Young and Taylor-Lagrange forms)
- Maclaurin Series Expansion
- Limited Developments (DL) and their applications
- Rolle's Theorem and Mean Value Theorem
- Calculation of limits using DL
- Convexity and Extremum conditions

Content

Introduction to Taylor Formulas

The chapter primarily focuses on Taylor's theorem and its varied applications. It includes:

- **Taylor-Young Formula**: For a function f that is n-times differentiable at x_0,

$$f(x) = f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\cdots+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+(x-x_0)^n\varepsilon(x),$$

where $\lim_{x\to x_0}\varepsilon(x) = 0$.

- **Taylor-Lagrange Formula**: For a function f of class C^n on $[a,b]$ and $(n+1)$-times differentiable on $]a,b[$,

$$f(b) = f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\cdots+\frac{(b-a)^n}{n!}f^{(n)}(a)+\frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(c),$$

for some $c \in]a,b[$.

Maclaurin Series Expansion

A special case when $x_0 = 0$, known as the Maclaurin series:

-
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + o(x^n).$$

Limited Developments (DL)

The chapter discusses limited developments extensively:

- Definition: A function f has a limited development of order n around 0 if it can be written as:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n).$$

- Operations on DLs: Sum, scalar multiplication, product, and quotient of functions with DLs.

- Applications: Calculating indeterminate limits, approximating functions, and solving problems involving convexity and extremum conditions.

Applications

- **Calculation of Limits**: Using DLs to resolve indeterminate forms.

- **Convexity and Extremum Conditions**:

- A function f is convex if it lies above all its tangents.

- If $f'(a) = 0$ and $f''(a) > 0$, f has a local minimum at a. Conversely, if $f''(a) < 0$, f has a local maximum.

Notes

- The chapter heavily emphasizes practical applications of Taylor expansions, particularly in evaluating limits and understanding function behavior near critical points.

- Several examples are provided, including expansions for e^x, $\sin x$, $\cos x$, and $\ln(1+x)$.

- The uniqueness of DLs is highlighted, ensuring that for any given order, there is only one valid expansion.

- The text also covers conditions under which functions do not admit DLs, such as $\frac{1}{x}$, $\sin(\frac{1}{x})$, and $\cos(\frac{1}{x})$ at $x = 0$.