UNIVERSITY IBN TOFAIL

Algebra II

Problem Set III

Exercise 1:

Solve the following linear systems using the Gaussian elimination method:

1. (S_1): $\begin{cases} x+y-z=0 \\ x+5y-2z=3 \\ 2x+y-z=1 \end{cases}$

2. (S_2): $\begin{cases} x+y+3z+2t=-2 \\ 2x+3y+4z+t=-1 \\ 3x+7y+z-6t=6 \end{cases}$

3. (S_3): $\begin{cases} x+2y-z=1 \\ 2x+y+2z=2 \\ x-4y+7z=3 \end{cases}$

4. (S_4): $\begin{cases} x-3y-2z=-1 \\ 2x+y-4z=3 \\ x+4y-2z=4 \\ 5x+6y-10z=10 \end{cases}$

Correction

1. **System** (S_1): **Augmented Matrix:**

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 1 & 5 & -2 & | & 3 \\ 2 & 1 & -1 & | & 1 \end{bmatrix}$$

Steps:

- Eliminate x from rows 2 and 3: $R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 4 & -1 & | & 3 \\ 0 & -1 & 1 & | & 1 \end{bmatrix}$$

Correction

- Eliminate y from row 3: $R_3 \leftarrow R_3 + \frac{1}{4}R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & \frac{3}{4} & \frac{7}{4} \end{array}\right]$$

Solution:

$$\boxed{x = 1, \quad y = \frac{4}{3}, \quad z = \frac{7}{3}}$$

System (S_2): **Augmented Matrix:**

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 2 & -2 \\ 2 & 3 & 4 & 1 & -1 \\ 3 & 7 & 1 & -6 & 6 \end{array}\right]$$

Steps: - Eliminate x: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 3R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 2 & -2 \\ 0 & 1 & -2 & -3 & 3 \\ 0 & 4 & -8 & -12 & 12 \end{array}\right]$$

- Eliminate y: $R_3 \leftarrow R_3 - 4R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 2 & -2 \\ 0 & 1 & -2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

General Solution: Let $z = s$, $t = r$ (free parameters):

$$y - 2z - 3t = 3 \Rightarrow y = 3 + 2s + 3r,$$
$$x + y + 3z + 2t = -2 \Rightarrow x = -5 - 5s - 5r.$$

Solution:

$$\boxed{\begin{aligned} x &= -5 - 5s - 5r, \\ y &= 3 + 2s + 3r, \\ z &= s, \\ t &= r. \end{aligned}}$$

System (S_3): **Augmented Matrix:**

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & -4 & 7 & 3 \end{array}\right]$$

Steps: - Eliminate x: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$

Correction

$$\begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & -3 & 4 & | & 0 \\ 0 & -6 & 8 & | & 2 \end{bmatrix}$$

- Eliminate y: $R_3 \leftarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & -3 & 4 & | & 0 \\ 0 & 0 & 0 & | & 2 \end{bmatrix}$$

Conclusion: The last row implies $0 = 2$, which is impossible.

$$\boxed{\text{No solution (inconsistent system)}}$$

System (S_4): **Augmented Matrix:**

$$\begin{bmatrix} 1 & -3 & -2 & | & -1 \\ 2 & 1 & -4 & | & 3 \\ 1 & 4 & -2 & | & 4 \\ 5 & 6 & -10 & | & 10 \end{bmatrix}$$

Steps: - Eliminate x: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$, $R_4 \leftarrow R_4 - 5R_1$

$$\begin{bmatrix} 1 & -3 & -2 & | & -1 \\ 0 & 7 & 0 & | & 5 \\ 0 & 7 & 0 & | & 5 \\ 0 & 21 & 0 & | & 15 \end{bmatrix}$$

- Eliminate y: $R_3 \leftarrow R_3 - R_2$, $R_4 \leftarrow R_4 - 3R_2$

$$\begin{bmatrix} 1 & -3 & -2 & | & -1 \\ 0 & 7 & 0 & | & 5 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

General Solution: Let $z = t$ (free parameter):

$$7y = 5 \Rightarrow y = \frac{5}{7},$$
$$x - 3\left(\frac{5}{7}\right) - 2t = -1 \Rightarrow x = \frac{8}{7} + 2t.$$

Solution:

$$\boxed{x = \frac{8}{7} + 2t, y = \frac{5}{7}, z = t.}$$

Exercise 2:

Solve the following linear systems using Cramer's method:

1. (S_1): $\begin{cases} 2x - 5y + 4z = -3 \\ x - 2y + z = 5 \\ x - 4y + 6z = 10 \end{cases}$

2. (S_2): $\begin{cases} 2x - 5y + 4z + t = -3 \\ x - 2y + z - t = 5 \\ x - 4y + 6z + 2t = 10 \end{cases}$

Correction

1. **System** (S_1): **Coefficient Matrix:**

$$A = \begin{bmatrix} 2 & -5 & 4 \\ 1 & -2 & 1 \\ 1 & -4 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -3 \\ 5 \\ 10 \end{bmatrix}$$

Step 1: Compute $\det(A)$**:**

$$\det(A) = 2(-2 \cdot 6 - 1 \cdot (-4)) - (-5)(1 \cdot 6 - 1 \cdot 1) + 4(1 \cdot (-4) - (-2) \cdot 1)$$
$$= 2(-8) + 5(5) + 4(-2) = -16 + 25 - 8 = \boxed{1}$$

Step 2: Compute $\det(A_x), \det(A_y), \det(A_z)$**:**

$$A_x = \begin{bmatrix} -3 & -5 & 4 \\ 5 & -2 & 1 \\ 10 & -4 & 6 \end{bmatrix}, \quad A_y = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 5 & 1 \\ 1 & 10 & 6 \end{bmatrix}, \quad A_z = \begin{bmatrix} 2 & -5 & -3 \\ 1 & -2 & 5 \\ 1 & -4 & 10 \end{bmatrix}$$

$$\det(A_x) = 124, \quad \det(A_y) = 75, \quad \det(A_z) = 31$$

Solution:

$$x = \frac{124}{1} = 124, \quad y = \frac{75}{1} = 75, \quad z = \frac{31}{1} = 31$$
$$\boxed{x = 124, \quad y = 75, \quad z = 31}$$

2. **System** (S_2): **Analysis:** This system has **4 variables** and **3 equations**, making it **underdetermined**. Cramer's Rule is **not applicable** because it requires a square coefficient matrix (equal number of equations and variables).

Exercise 3:

Solve the following linear system using both the Gaussian elimination method and Cramer's method:

$$(S)\colon \begin{cases} x + y + 2z + 2t = -2 \\ 2x + 3y - z + t = 1 \\ x + 2y - 3z + t = 0 \end{cases}$$

Correction

1. **System** (S): **Analysis:** This system has **4 variables** (x, y, z, t) and **3 equations**, making it **underdetermined**. Cramer's Rule is **not applicable** because it requires a **square coefficient matrix** (equal number of equations and variables). We solve it using **Gaussian elimination**.

   Augmented Matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 2 & | & -2 \\ 2 & 3 & -1 & 1 & | & 1 \\ 1 & 2 & -3 & 1 & | & 0 \end{bmatrix}$$

   Steps:

   (a) Eliminate x from rows 2 and 3:

$$R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & | & -2 \\ 0 & 1 & -5 & -3 & | & 5 \\ 0 & 1 & -5 & -1 & | & 2 \end{bmatrix}$$

   (b) Eliminate y from row 3:

$$R_3 \leftarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & | & -2 \\ 0 & 1 & -5 & -3 & | & 5 \\ 0 & 0 & 0 & 2 & | & -3 \end{bmatrix}$$

   (c) Back-substitute to express variables:

$$2t = -3 \Rightarrow t = -\frac{3}{2},$$
$$y - 5z - 3t = 5 \Rightarrow y = \frac{1}{2} + 5z,$$
$$x + y + 2z + 2t = -2 \Rightarrow x = \frac{1}{2} - 7z.$$

   General Solution: Let $z = s$ (free parameter). Then:

$$\boxed{x = \frac{1}{2} - 7s, y = \frac{1}{2} + 5s, z = s, t = -\frac{3}{2}.}$$

Exercise 4:

Solve the following linear system according to the values of the real parameter m:

$$(S)\colon \begin{cases} x + 2y - z = 1 \\ 2x + y + 2z = 2 \\ x - 4y + 7z = m \end{cases}$$

Correction

1. **Gaussian Elimination on Augmented Matrix:** The augmented matrix is:
$$\begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 2 & 1 & 2 & | & 2 \\ 1 & -4 & 7 & | & m \end{bmatrix}$$
Perform row operations:
   (a) Eliminate x from rows 2 and 3: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$
$$\begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & -3 & 4 & | & 0 \\ 0 & -6 & 8 & | & m-1 \end{bmatrix}$$
   (b) Eliminate y from row 3: $R_3 \leftarrow R_3 - 2R_2$
$$\begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & -3 & 4 & | & 0 \\ 0 & 0 & 0 & | & m-1 \end{bmatrix}$$

2. **Analysis Based on m:**
   (a) If $m \neq 1$: The last row implies $0 = m - 1$, which is a contradiction.
$$\boxed{\text{No solution (inconsistent system)}}$$
   (b) If $m = 1$: The last row becomes $0 = 0$, and the system reduces to:
$$\begin{cases} x + 2y - z = 1 \\ -3y + 4z = 0 \end{cases}$$
   Solve for x and y in terms of z:
$$y = \frac{4}{3}z, \quad x = 1 - 2y + z = 1 - \frac{8}{3}z + z = 1 - \frac{5}{3}z$$
   Let $z = t$ (free parameter). The general solution is:
$$\boxed{\begin{aligned} & x = 1 - \frac{5}{3}t, \\ & y = \frac{4}{3}t, \\ & z = t. \end{aligned}}$$