UNIVERSITY IBN TOFAIL

Algebra II

Problem Set I

Exercise 1:

Study the following propositions. Prove those that are true and provide counterexamples for those that are false.

a) $\mathbb{R}^2$ with the usual addition and the external law: $\lambda \cdot (x, y) = (\lambda x, 0)$ where $\lambda \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ is a vector space over $\mathbb{R}$.

b) $\mathbb{C}^3$ with the usual addition and the external law over $\mathbb{C}$ defined by $\lambda \cdot (x, y, z) = (\lambda x, y, z)$ where $\lambda \in \mathbb{C}$, $(x, y, z) \in \mathbb{C}^3$ is a $\mathbb{C}$-vector space.

c) The set of polynomials with real coefficients divisible by $X^3 + 1$, with the usual addition of polynomials and multiplication of a polynomial by a scalar, is an $\mathbb{R}$-vector space.

Correction

a) The scalar multiplication fails the axiom $1 \cdot \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$. For example, take $\mathbf{v} = (0, 1)$. Then:

$$1 \cdot (0, 1) = (1 \cdot 0, 0) = (0, 0) \neq (0, 1).$$

Since the multiplicative identity axiom is violated, this structure is not a vector space. ■

b) The scalar multiplication fails the distributive property over scalar addition. Let $\lambda, \mu \in \mathbb{C}$ and $\mathbf{v} = (x, y, z) \in \mathbb{C}^3$. Then:

$$(\lambda + \mu) \cdot \mathbf{v} = ((\lambda + \mu)x, y, z),$$

whereas:

$$\lambda \cdot \mathbf{v} + \mu \cdot \mathbf{v} = (\lambda x, y, z) + (\mu x, y, z) = ((\lambda x + \mu x), 2y, 2z).$$

These two results are unequal unless $y = z = 0$. Hence, distributivity fails, and the structure is not a vector space. ■

c) We verify closure under addition and scalar multiplication:

Correction

(a) If $P, Q \in F$, then $P = (X^3+1)P_0$ and $Q = (X^3+1)Q_0$ for some $P_0, Q_0 \in \mathbb{R}[X]$. Thus:

$$P + Q = (X^3+1)(P_0+Q_0) \in F.$$

(b) For $\lambda \in \mathbb{R}$, $\lambda P = (X^3+1)(\lambda P_0) \in F$.

The zero polynomial is in F, and all other vector space axioms hold inherited from $\mathbb{R}[X]$. Therefore, F is an $\mathbb{R}$-vector space. ■

Exercise 2:

Consider the following sets:

$$\begin{aligned} E_1 &= \{(x,y,z) \in \mathbb{R}^3 \mid 3x - 2y + 5z = 0\} \\ E_2 &= \{v \in \mathbb{R}^3 \mid v = (a-b, 2b, a+3b), a, b \in \mathbb{R}\} \\ E_3 &= \{(x,y,z) \in \mathbb{R}^3 \mid x \cdot y = 0\} \end{aligned}$$

1. Among these sets, which ones are vector subspaces of the vector space $\mathbb{R}^3$ over $\mathbb{R}$?
2. Give a basis for each vector subspace.

Correction

1) Subspace Verification:

- **Set** $E_1 = \{(x,y,z) \in \mathbb{R}^3 \mid 3x - 2y + 5z = 0\}$**:**
E_1 is a subspace.

- Contains the zero vector: $3(0) - 2(0) + 5(0) = 0$.

- Closed under addition: If $(x_1,y_1,z_1), (x_2,y_2,z_2) \in E_1$, then:

$$3(x_1+x_2) - 2(y_1+y_2) + 5(z_1+z_2) = 0 + 0 = 0.$$

- Closed under scalar multiplication: If $\lambda \in \mathbb{R}$, then:

$$3(\lambda x) - 2(\lambda y) + 5(\lambda z) = \lambda(3x - 2y + 5z) = 0.$$

- **Set** $E_2 = \{v \in \mathbb{R}^3 \mid v = (a-b, 2b, a+3b), a, b \in \mathbb{R}\}$**:**
E_2 is a subspace.

Correction

- Contains the zero vector: Set $a = 0, b = 0$. - Closed under addition: For $v_1 = (a_1 - b_1, 2b_1, a_1 + 3b_1)$, $v_2 = (a_2 - b_2, 2b_2, a_2 + 3b_2)$, their sum is:

$$(a_1 + a_2 - (b_1 + b_2), 2(b_1 + b_2), (a_1 + a_2) + 3(b_1 + b_2)) \in E_2.$$

- Closed under scalar multiplication: For $\lambda \in \mathbb{R}$, $\lambda v = (\lambda a - \lambda b, 2(\lambda b), \lambda a + 3(\lambda b)) \in E_2$. - **Set $E_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x \cdot y = 0\}$:**
E_3 is **not** a subspace. Counterexample: $v_1 = (1, 0, 0) \in E_3$, $v_2 = (0, 1, 0) \in E_3$, but $v_1 + v_2 = (1, 1, 0) \notin E_3$ since $1 \cdot 1 \neq 0$.

2) Bases for Subspaces:
- **Basis for E_1:**
Solve $3x - 2y + 5z = 0$. Express $x = \frac{2}{3}y - \frac{5}{3}z$, so:

$$E_1 = \text{Span}\left\{\begin{pmatrix} \frac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Clearing denominators, a basis is:

$$\left\{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 3 \end{pmatrix}\right\}.$$

- **Basis for E_2:**
From $v = a(1, 0, 1) + b(-1, 2, 3)$, the vectors:

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}\right\}$$

are linearly independent (determinant test confirms), so they form a basis.

Exercise 3:

In the vector space $\mathbb{R}^4$ with its canonical basis, consider the vectors:

$$\begin{aligned} e_1' &= (1, 2, -1, -2) \\ e_2' &= (2, 3, 0, -1) \\ e_3' &= (1, 3, -1, 0) \\ e_4' &= (1, 2, 1, 4) \end{aligned}$$

a) Show that the family $B' = (e_1', e_2', e_3', e_4')$ is a basis of $\mathbb{R}^4$.

b) Calculate the coordinates of the vector $v = (7, 14, -1, 2)$ in the basis B'.

Correction

a) Proof: To verify that B' is a basis, we check if the vectors are linearly independent. Construct the matrix A with columns $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \mathbf{e}'_4$:

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 2 \\ -1 & 0 & -1 & 1 \\ -2 & -1 & 0 & 4 \end{bmatrix}$$

Compute the determinant of A. Using row operations or cofactor expansion, we find $\det(A) \neq 0$, confirming linear independence. Since $\dim(\mathbb{R}^4) = 4$, the family B' forms a basis. ■

b) Solve $a\mathbf{e}'_1 + b\mathbf{e}'_2 + c\mathbf{e}'_3 + d\mathbf{e}'_4 = \mathbf{v}$. This leads to the system:

$$\begin{cases} a + 2b + c + d = 7 \\ 2a + 3b + 3c + 2d = 14 \\ -a - c + d = -1 \\ -2a - b + 4d = 2 \end{cases}$$

Solving via substitution:

(a) From the third equation: $d = a + c - 1$.

(b) Substitute d into the first and fourth equations:

$$2a + 2b + 2c = 8 \quad \Rightarrow \quad a + b + c = 4,$$
$$2a - b + 4c = 6.$$

(c) Substitute $b = 4 - a - c$ into the remaining equations to find $c = 2$, $a = 0$, $b = 2$, and $d = 1$.

The coordinates of $\mathbf{v}$ in B' are $(0, 2, 2, 1)$. ■

Exercise 4:

Consider the set:

$$F = \{(x, y, z) \in \mathbb{C}^3 \mid x + y + z = 0 \text{ and } 2x + iy - z = 0\}$$

a) Show that F is an $\mathbb{R}$-vector space.

b) Give a basis for F and deduce its dimension.

Correction

a) Since $F \subseteq \mathbb{C}^3$ and $\mathbb{C}^3$ is a vector space over $\mathbb{R}$, it suffices to show F is a subspace. We verify:

(a) **Zero vector:** $(0,0,0) \in F$ because $0+0+0=0$ and $2\cdot 0 + i\cdot 0 - 0 = 0$.

(b) **Closed under addition:** Let $\mathbf{u} = (x_1, y_1, z_1), \mathbf{v} = (x_2, y_2, z_2) \in F$. Then:

$$(x_1+x_2)+(y_1+y_2)+(z_1+z_2) = (x_1+y_1+z_1)+(x_2+y_2+z_2) = 0,$$
$$2(x_1+x_2)+i(y_1+y_2)-(z_1+z_2) = (2x_1+iy_1-z_1)+(2x_2+iy_2-z_2) = 0.$$

Hence, $\mathbf{u}+\mathbf{v} \in F$.

(c) **Closed under scalar multiplication:** For $\lambda \in \mathbb{R}$, $\lambda\mathbf{u} = (\lambda x_1, \lambda y_1, \lambda z_1)$. Then:

$$\lambda x_1 + \lambda y_1 + \lambda z_1 = \lambda(x_1+y_1+z_1) = 0,$$
$$2(\lambda x_1) + i(\lambda y_1) - \lambda z_1 = \lambda(2x_1+iy_1-z_1) = 0.$$

Thus, $\lambda\mathbf{u} \in F$.

Therefore, F is an $\mathbb{R}$-vector space. ■

b) **Basis and Dimension:**

Solution: Solve the system:

$$\begin{cases} x+y+z=0 & (1) \\ 2x+iy-z=0 & (2) \end{cases}$$

From (1): $z = -x-y$. Substitute into (2):

$$2x+iy-(-x-y) = 3x+y(1+i) = 0 \Rightarrow x = -\frac{(1+i)}{3}y.$$

Let $y = t \in \mathbb{C}$. Then:

$$x = -\frac{(1+i)}{3}t, \quad z = \frac{(-2+i)}{3}t.$$

The general solution is $\mathbf{v} = t\cdot\left(-\frac{1+i}{3}, 1, \frac{-2+i}{3}\right)$. Since $t \in \mathbb{C}$, write $t = a+ib$ with $a, b \in \mathbb{R}$. Separating real and imaginary parts:

$$\mathbf{v} = a\cdot\underbrace{\left(-\frac{1}{3}, -\frac{1}{3}, 1, 0, -\frac{2}{3}, \frac{1}{3}\right)}_{\mathbf{v}_1} + b\cdot\underbrace{\left(\frac{1}{3}, -\frac{1}{3}, 0, 1, -\frac{1}{3}, -\frac{2}{3}\right)}_{\mathbf{v}_2}.$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for F over $\mathbb{R}$, and $\dim_{\mathbb{R}}(F) = 2$. ■

Exercise 5:

Let F be the vector subspace of $\mathbb{R}_4[X]$ generated by the following vectors (polynomials):

$$P_1 = X^2$$
$$P_2 = (X-1)^2$$
$$P_3 = (X+1)^2$$

a) Show that (P_1, P_2, P_3) is a basis of F.

b) Complete the family (P_1, P_2, P_3) into a basis of $\mathbb{R}_4[X]$ and deduce a supplementary subspace of F in $\mathbb{R}_4[X]$.

Correction

a) Let $P_1 = X^2$, $P_2 = (X-1)^2 = X^2-2X+1$, and $P_3 = (X+1)^2 = X^2+2X+1$. We verify linear independence by solving:

$$aP_1 + bP_2 + cP_3 = 0.$$

Expanding:

$$aX^2 + b(X^2 - 2X + 1) + c(X^2 + 2X + 1) = 0.$$

Grouping terms:

$$(a+b+c)X^2 + (-2b+2c)X + (b+c) = 0.$$

Equating coefficients:

$$\begin{cases} a+b+c = 0 \\ -2b+2c = 0 \\ b+c = 0 \end{cases}$$

From the third equation: $b = -c$. Substituting into the second equation:

$$-2(-c) + 2c = 4c = 0 \Rightarrow c = 0 \Rightarrow b = 0.$$

Substituting into the first equation: $a = 0$. Thus, (P_1, P_2, P_3) is linearly independent. Since F is the span of these vectors, they form a basis. ■

b) The space $\mathbb{R}_4[X]$ has dimension 5, with standard basis $\{1, X, X^2, X^3, X^4\}$. Since F has dimension 3, we extend (P_1, P_2, P_3) with X^3 and X^4 to form a basis:

$$\mathcal{B} = \{P_1, P_2, P_3, X^3, X^4\}.$$

Correction

The supplementary subspace S is the span of X^3 and X^4, satisfying:

$$\mathbb{R}_4[X] = F \oplus S.$$

Verification: - $F \cap S = \{0\}$: No non-zero polynomial in S has degree 2, while F contains only polynomials of degree 2. - $\dim(F) + \dim(S) = 3 + 2 = 5 = \dim(\mathbb{R}_4[X])$. Hence, $S = \text{Span}\{X^3, X^4\}$ is a valid supplementary subspace. ■

Exercise 6:

In the $\mathbb{R}$-vector space $F(\mathbb{R}, \mathbb{R})$, consider the functions:

$$f_n(x) = \sin(nx), n \geq 1$$

a) Show that for all $n \in \mathbb{N}^*$, the family $(f_1, \dots, f_n)$ is linearly independent.

b) Deduce that $F(\mathbb{R}, \mathbb{R})$ is an $\mathbb{R}$-vector space of infinite dimension.

Correction

a) Suppose there exist scalars $a_1, \dots, a_n \in \mathbb{R}$ such that:

$$\sum_{k=1}^{n} a_k \sin(kx) = 0 \quad \forall x \in \mathbb{R}.$$

To prove linear independence, we show $a_k = 0$ for all k. Multiply both sides by $\sin(mx)$ (for fixed $m \in \{1, \dots, n\}$) and integrate over $[0, 2\pi]$:

$$\sum_{k=1}^{n} a_k \int_0^{2\pi} \sin(kx) \sin(mx) \, dx = 0.$$

Using orthogonality:

$$\int_0^{2\pi} \sin(kx) \sin(mx) \, dx = \begin{cases} \pi & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$

Thus, the equation reduces to $a_m \pi = 0 \Rightarrow a_m = 0$. Since m was arbitrary, $a_k = 0$ for all k. Hence, the family is linearly independent. ■

b) For every $n \in \mathbb{N}^*$, the family $(f_1, \dots, f_n)$ is linearly independent by part (a). In a finite-dimensional vector space of dimension d, no set of size greater than d can be linearly independent. Since $\mathcal{F}(\mathbb{R}, \mathbb{R})$ contains arbitrarily large linearly independent sets, it must be infinite-dimensional. ■

Exercise 7:

1. Show that the application $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x - y, x, x + y)$ is linear.
2. Show that f is injective but not surjective.
3. Determine a basis for $\mathrm{Im} f$ (the image of f).

Correction

1) Let $\mathbf{u} = (x_1, y_1), \mathbf{v} = (x_2, y_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

- **Additivity:**

$$\begin{aligned} f(\mathbf{u} + \mathbf{v}) &= f(x_1 + x_2, y_1 + y_2) \\ &= f(\mathbf{u}) + f(\mathbf{v}). \end{aligned}$$

- **Homogeneity:**

$$\begin{aligned} f(\lambda \mathbf{u}) &= f(\lambda x_1, \lambda y_1) = (\lambda x_1 - \lambda y_1, \lambda x_1, \lambda x_1 + \lambda y_1) \\ &= \lambda(x_1 - y_1, x_1, x_1 + y_1) = \lambda f(\mathbf{u}). \end{aligned}$$

Since both properties hold, f is linear.

2) - **Injectivity:** Suppose $f(x, y) = (0, 0, 0)$. Then:

$$\begin{cases} x - y = 0 \\ x = 0 \\ x + y = 0 \end{cases} \Rightarrow x = 0,\ y = 0.$$

Thus, $\ker(f) = \{(0, 0)\}$, so f is injective.

- **Non-surjectivity:** The image $\mathrm{Im}(f) \subseteq \mathbb{R}^3$ has dimension at most 2 (since $\dim(\mathbb{R}^2) = 2$), while $\dim(\mathbb{R}^3) = 3$. Hence, $\mathrm{Im}(f) \neq \mathbb{R}^3$, so f is not surjective.

3) **Basis for** $\mathrm{Im}(f)$**:**
Compute $f(1, 0) = (1, 1, 1)$ and $f(0, 1) = (-1, 0, 1)$. These vectors span $\mathrm{Im}(f)$. To verify linear independence, solve:

$$a(1, 1, 1) + b(-1, 0, 1) = (0, 0, 0).$$

This gives the system:

$$\begin{cases} a - b = 0 \\ a = 0 \\ a + b = 0 \end{cases} \Rightarrow a = 0,\ b = 0.$$

Hence, $\{(1, 1, 1), (-1, 0, 1)\}$ is a basis for $\mathrm{Im}(f)$.

Exercise 8:

Let $f \in L(\mathbb{R}^3)$ defined by $f(x, y, z) = (2y + z, x - 4y, 3x)$.

1. Determine the matrix A of f with respect to the canonical basis $B = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.
2. Let $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, and $v_3 = (1, 0, 0)$ be vectors in $\mathbb{R}^3$.
   a) Show that the family $B' = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
   b) Calculate $f(v_1)$, $f(v_2)$, and $f(v_3)$.
   c) Determine A', the matrix of f in the basis B'.
3. a) Determine the matrices P and P^{-1} where P is the change of basis matrix from basis B to basis B'.
   b) Using the change of basis formula, recalculate the matrix A'.

Correction

1) Matrix A of f in the canonical basis:

The linear map $f(x, y, z) = (2y + z, x - 4y, 3x)$ is represented by the matrix:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

This is obtained by evaluating f on the canonical basis vectors e_1, e_2, e_3 and writing the results as columns.

2.a) Claim: The family $B' = \{v_1, v_2, v_3\}$ with $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 0)$ is a basis of $\mathbb{R}^3$.

Proof: The determinant of the matrix P formed by v_1, v_2, v_3 as columns is:

$$\det(P) = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1 \neq 0.$$

Since $\det(P) \neq 0$, the vectors are linearly independent and span $\mathbb{R}^3$, hence B' is a basis.

2.b) Images of v_1, v_2, v_3 under f:

Compute $f(v_1), f(v_2), f(v_3)$:

$$\begin{aligned} f(v_1) &= f(1, 1, 1) = (2(1) + 1, 1 - 4(1), 3(1)) = (3, -3, 3), \\ f(v_2) &= f(1, 1, 0) = (2(1) + 0, 1 - 4(1), 3(1)) = (2, -3, 3), \\ f(v_3) &= f(1, 0, 0) = (2(0) + 0, 1 - 4(0), 3(1)) = (0, 1, 3). \end{aligned}$$

Correction

2.c) Matrix A' of f in the basis B':

Express $f(v_1), f(v_2), f(v_3)$ in terms of B':

$$\begin{aligned} f(v_1) &= 3v_1 - 6v_2 + 6v_3, \\ f(v_2) &= 3v_1 - 6v_2 + 5v_3, \\ f(v_3) &= 3v_1 - 2v_2 - v_3. \end{aligned}$$

Thus, the matrix A' is:

$$A' = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}.$$

3.a) Change of basis matrices P and P^{-1}:

The change of basis matrix P from B to B' is:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

3.b) Verification of $A' = P^{-1}AP$:

Compute $A' = P^{-1}AP$:

$$A' = P^{-1} \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix} P = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}.$$

This matches the earlier result, confirming correctness.