Ibn Tofail University

Electricity I — Normal Exam

Year: 20-21

Exercise 1:

A/

An infinite cylinder with axis Oz and radius R carries a uniform positive surface charge density σ.

1. Show that the electrostatic field created at any point M is radial and depends only on the distance $r = HM$ where H is the projection of M on the Oz axis (figure 1).
   We can therefore write: $\vec{E}(r) = E(r)\vec{e_r}$
2. Calculate $E(r)$ at any point M in space.
3. Plot the variation of the magnitude of the electrostatic field E as a function of r.

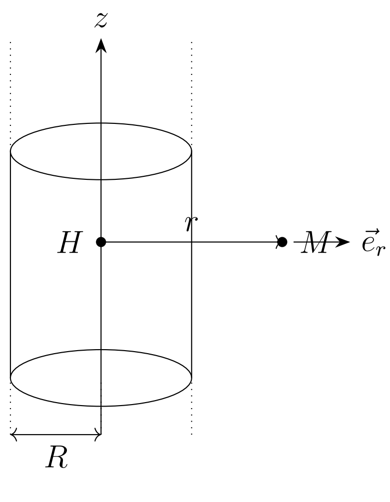

Figure 1: Cylinder figure

B/

Consider a cylindrical capacitor formed by two armatures with the same Oz axis, height h, and respective radii R_1 and R_2 with $R_1 < R_2$.

The internal armature carries a charge $Q > 0$, and the electrostatic potentials of the internal and external armatures are V_1 and V_2 respectively.

1. Determine the capacitance C of the capacitor as a function of ε_0, h, R_1, and R_2.

2. What becomes of the expression for C if the radii of the armatures are very close. Let $R_2 - R_1 = e \ll R_1$. Express C as a function of the surface area s of the internal armature.

Answer Area

A/

1. **Show that the electrostatic field is radial and depends only on $r = HM$:**

   The infinite cylinder has cylindrical symmetry. Due to this symmetry:

   - The electric field must be radial (i.e., directed along $\vec{e}_r$).

   - The magnitude of the field depends only on the radial distance r from the axis (Oz), since there is no dependence on z or angular direction.

   Therefore, we can write:

   $$\vec{E}(r) = E(r)\,\vec{e}_r$$

2. **Calculate $E(r)$ at any point M in space:**

   Use Gauss's Law:

   $$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

   Consider a Gaussian surface as a coaxial cylinder of radius r and height h:

   - For $r < R$: No charge enclosed, so

   $$E(r) = 0$$

   - For $r \geq R$: Enclosed charge is $Q_{\text{enc}} = \sigma \cdot 2\pi R h$, and the flux becomes:

   $$E(r) \cdot 2\pi r h = \frac{\sigma \cdot 2\pi R h}{\varepsilon_0} \Rightarrow E(r) = \frac{\sigma R}{\varepsilon_0 r}$$

   Final result:

   $$E(r) = \begin{cases} 0 & \text{if } r < R \\ \dfrac{\sigma R}{\varepsilon_0 r} & \text{if } r \geq R \end{cases}$$

3. **Plot the variation of the magnitude of the electrostatic field E as a function of r:**

   A qualitative plot shows:

   - $E(r) = 0$ for $r < R$

   - $E(r)$ decreases as $1/r$ for $r \geq R$

   The graph would have a sharp jump at $r = R$, starting from zero to a maximum value of $\frac{\sigma}{\varepsilon_0}$, then decreasing.

B/

1. **Determine the capacitance C of the capacitor:**

   The electric field between two cylinders is given by (from Gauss's law):

   $$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}, \quad \text{where } \lambda = \frac{Q}{h}$$

The potential difference between the two cylinders is:

$$V = \int_{R_1}^{R_2} E(r)\,dr = \frac{Q}{2\pi\varepsilon_0 h}\ln\left(\frac{R_2}{R_1}\right)$$

Then, the capacitance is:

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 h}{\ln\left(\frac{R_2}{R_1}\right)}$$

2. **Approximation when $R_2 - R_1 = e \ll R_1$:**

   When $e \ll R_1$, we can approximate:

   $$\ln\left(\frac{R_2}{R_1}\right) = \ln\left(1 + \frac{e}{R_1}\right) \approx \frac{e}{R_1}$$

   Substituting into the expression for C:

   $$C \approx \frac{2\pi\varepsilon_0 h R_1}{e}$$

   Now, note that the surface area of the internal armature is:

   $$S = 2\pi R_1 h$$

   So:

   $$C \approx \frac{\varepsilon_0 S}{e}$$