Ibn Tofail University

Analysis II — Normal Exam

Year: 23-24

Exercise 1:

Let f be the function defined on $\mathbb{R}$ by:

$$\forall x \in \mathbb{R}, f(x) = \frac{1}{e^x + 1}.$$

1. Find the Taylor expansion of f up to order 3 at zero.
2. Deduce the equation of the tangent line to the curve C_f of f at the point with abscissa 0.
3. Also deduce that the curve C_f crosses the tangent line at 0 (i.e., $(0, f(0))$ is an inflection point).
4. (a) Study the convexity and concavity of the function f on $\mathbb{R}$.
   (b) Deduce that for all $(a, b) \in \mathbb{R}^2$ such that $a \geq 1$ and $b \geq 1$, we have:

$$\frac{2}{1 + \sqrt{ab}} \leq \frac{1}{1+a} + \frac{1}{1+b}.$$

N.B.: Question 4 is independent of the previous questions.

Answer Area

1. Find the Taylor expansion of $f(x) = \frac{1}{e^x+1}$ up to order 3 at zero.

   First, recall that:
   $$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$
   So,
   $$e^x + 1 = 2 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$
   Now compute $\frac{1}{e^x+1}$ using a series expansion. Let:
   $$f(x) = \frac{1}{2 + x + \frac{x^2}{2} + \frac{x^3}{6}} = \frac{1}{2} \cdot \frac{1}{1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12}}$$
   Use the geometric series expansion:
   $$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$$
   where $u = \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12}$. Expand up to order 3:
   $$f(x) = \frac{1}{2}\left(1 - \left(\frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12}\right) + \left(\frac{x}{2} + \frac{x^2}{4}\right)^2 - \left(\frac{x}{2}\right)^3 + \cdots\right)$$
   Compute each term:
   $$\left(\frac{x}{2} + \frac{x^2}{4}\right)^2 = \frac{x^2}{4} + \frac{x^3}{4} + \cdots, \quad \left(\frac{x}{2}\right)^3 = \frac{x^3}{8}$$
   Combine all terms:
   $$f(x) = \frac{1}{2}\left(1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{12} + \frac{x^2}{4} + \frac{x^3}{4} - \frac{x^3}{8} + \cdots\right)$$
   Simplify:
   $$f(x) = \frac{1}{2}\left(1 - \frac{x}{2} + \left(-\frac{x^2}{4} + \frac{x^2}{4}\right) + \left(-\frac{x^3}{12} + \frac{x^3}{4} - \frac{x^3}{8}\right) + \cdots\right)$$
   Final simplification gives:
   $$f(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^3}{48} + o(x^3)$$

2. Deduce the equation of the tangent line to the curve C_f of f at the point with abscissa 0.

   The tangent line at $x = 0$ is given by:
   $$y = f(0) + f'(0)(x - 0)$$
   From the expansion:
   $$f(0) = \frac{1}{2}, \quad f'(0) = -\frac{1}{4}$$
   So the equation is:
   $$y = \frac{1}{2} - \frac{x}{4}$$

3. Show that the curve C_f crosses the tangent line at $x = 0$, i.e., $(0, f(0))$ is an inflection point.

   An inflection point occurs when the concavity changes, or equivalently, when the second derivative changes sign. From the Taylor expansion:

   $$f''(0) = 0, \quad f'''(0) = \frac{1}{8} \neq 0$$

   Since the first non-zero derivative after the second is odd (third), the function changes concavity at $x = 0$, so $(0, \frac{1}{2})$ is an inflection point.

4. Study the convexity and concavity of f on $\mathbb{R}$.

   Recall:

   $$f''(x) = \frac{e^x(e^x - 1)}{(e^x + 1)^3}$$

   Sign of $f''(x)$ depends on $e^x - 1$:

   $$f''(x) > 0 \iff x > 0, \quad f''(x) < 0 \iff x < 0$$

   Therefore: - f is concave on $(-\infty, 0)$ - f is convex on $(0, \infty)$

5. Deduce that for all $a, b \geq 1$, we have:

   $$\frac{2}{1 + \sqrt{ab}} \leq \frac{1}{1 + a} + \frac{1}{1 + b}$$

   Consider the function $g(x) = \frac{1}{1+e^x}$. This is convex on $[0, \infty)$ since $f(x) = \frac{1}{1+e^x}$ is convex there.

   Apply Jensen's inequality for convex functions:

   $$g\left(\frac{\ln a + \ln b}{2}\right) \leq \frac{g(\ln a) + g(\ln b)}{2}$$

   Note that $g(\ln a) = \frac{1}{1+a}$, etc., and:

   $$g\left(\frac{\ln a + \ln b}{2}\right) = \frac{1}{1 + \sqrt{ab}}$$

   Multiply both sides by 2:

   $$\frac{2}{1 + \sqrt{ab}} \leq \frac{1}{1 + a} + \frac{1}{1 + b}$$

Exercise 2:

Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by:

$$F(x) = \int_x^{2x} e^{-t^2} dt.$$

1. Verify that F is defined on $\mathbb{R}$. Also show that F is odd.
2. Show that F is differentiable on $\mathbb{R}$, and calculate its derivative. Deduce the variations of F on $\mathbb{R}$.
   (Hint: use any antiderivative F_0 of the function $t \mapsto e^{-t^2}$).
3. Show that
   $$\forall x \geq 0, 0 \leq F(x) \leq x \cdot e^{-x^2}.$$
4. Deduce $\lim_{x \to +\infty} F(x)$.
5. Sketch the curve C_F of the function F.

Answer Area

1. Verify that F is defined on $\mathbb{R}$. Also show that F is odd.

   The function $t \mapsto e^{-t^2}$ is continuous on $\mathbb{R}$, so it is integrable over any interval. Hence, for all $x \in \mathbb{R}$, the integral $\int_0^{2x} e^{-t^2} dt$ exists, and therefore $F(x)$ is well-defined on $\mathbb{R}$.

   To show that F is odd:

   $$F(-x) = \int_0^{-2x} e^{-t^2} dt = -\int_{-2x}^{0} e^{-t^2} dt = -\int_0^{2x} e^{-u^2} du = -F(x)$$

   (using substitution $u = -t$). So F is odd.

2. Show that F is differentiable on $\mathbb{R}$, and calculate its derivative. Deduce the variations of F on $\mathbb{R}$.

   Let F_0 be an antiderivative of $t \mapsto e^{-t^2}$. Then:

   $$F(x) = F_0(2x) - F_0(0)$$

   Differentiate using the chain rule:

   $$F'(x) = 2 \cdot e^{-(2x)^2} = 2e^{-4x^2}$$

   Since $e^{-4x^2} > 0$ for all $x \in \mathbb{R}$, we have $F'(x) > 0$ for all x. Therefore, F is strictly increasing on $\mathbb{R}$.

3. Show that $\forall x \geq 0,\ 0 \leq F(x) \leq x \cdot e^{-x^2}$.

   First, since $e^{-t^2} \geq 0$, the integral from 0 to $2x$ is non-negative:

   $$F(x) = \int_0^{2x} e^{-t^2} dt \geq 0$$

   Next, observe that for $t \in [0, 2x]$, we have $t^2 \geq x^2$ when $t \geq x$. But more simply, use the inequality:

   $$e^{-t^2} \leq e^{-x^2}, \quad \text{for } t \in [0, 2x]$$

   because $t \geq x \Rightarrow t^2 \geq x^2$, so $-t^2 \leq -x^2 \Rightarrow e^{-t^2} \leq e^{-x^2}$

   Therefore:

   $$F(x) = \int_0^{2x} e^{-t^2} dt \leq \int_0^{2x} e^{-x^2} dt = 2x \cdot e^{-x^2}$$

   So:

   $$0 \leq F(x) \leq 2x \cdot e^{-x^2}$$

4. Deduce $\lim_{x\to+\infty} F(x)$.

   From the previous inequality:

   $$0 \leq F(x) \leq 2x \cdot e^{-x^2}$$

   As $x \to +\infty$, $2x \cdot e^{-x^2} \to 0$ (because exponential decay dominates polynomial growth), so by the squeeze theorem:

   $$\lim_{x\to+\infty} F(x) = 0$$

5. Sketch the curve C_F of the function F.

   Based on our findings:

   - F is odd symmetric about the origin.

   - $F'(x) = 2e^{-4x^2} > 0$ strictly increasing.

   - $F(x) \to 0$ as $x \to +\infty$, and $F(x) \to 0$ as $x \to -\infty$ due to oddness.

   - At $x = 0$, $F(0) = 0$.

   So the graph starts at the origin, increases smoothly, approaching a horizontal asymptote at $y = 0$ as $x \to \pm\infty$, with symmetry about the origin.

Exercise 3:

For all $(n,m) \in \mathbb{N} \times \mathbb{N}$, let:

$$I_{n,m} = \int_0^1 t^n \cdot (1-t)^m dt.$$

1. Calculate $I_{n,0}$ for all $n \in \mathbb{N}$.
2. Show (using an appropriate change of variable) that $I_{n,m} = I_{m,n}$ for all $(n,m) \in \mathbb{N} \times \mathbb{N}$.
3. Show (using integration by parts) that

$$\forall (n,m) \in \mathbb{N}^* \times \mathbb{N}, I_{n,m} = \frac{n}{m+1} \cdot I_{n-1,m+1}.$$

4. Deduce that

$$\forall (n,m) \in \mathbb{N}^2, I_{n,m} = \frac{n!\,m!}{(n+m+1)!}.$$

5. Show (using the binomial theorem) that

$$\forall (n,m) \in \mathbb{N}^2, I_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{m+k+1}.$$

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