Ibn Tofail University

Analysis II — Make-up Exam

Year: 21-22

Exercise 1:

The three questions are independent:

1. Show that: $\forall x \in \mathbb{R}^+ : x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$.
2. Calculate the following limit: $\lim_{x\to 0} \left(\cot^2(3x) - \frac{1}{9x^2}\right)$
3. Find an equivalent near 0 of $2\exp u - \sqrt{1+4u} - \sqrt{1+6u^2}$.

Answer Area

1. Consider the Taylor expansion of $\ln(1+x)$ around $x=0$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Since the series is alternating and decreasing in absolute value for $x > 0$, truncating after an even number of terms gives a lower bound, and after an odd number gives an upper bound. Therefore:

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}.$$

2. Recall that $\tan x \sim x + \frac{x^3}{3}$ as $x \to 0$. Then:

$$\tan(3x) \sim 3x + 9x^3 \Rightarrow \cot^2(3x) = \frac{1}{\tan^2(3x)} \sim \frac{1}{9x^2} - \frac{2}{9} + o(1)$$

So:

$$\cot^2(3x) - \frac{1}{9x^2} \sim -\frac{2}{9}$$

3. Use Taylor expansions up to order 3:

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o(u^3) \Rightarrow 2e^u = 2 + 2u + u^2 + \frac{u^3}{3} + o(u^3)$$

$$\sqrt{1+4u} = 1 + 2u - 2u^2 + \frac{4u^3}{3} + o(u^3), \quad \sqrt{1+6u^2} = 1 + 3u^2 - \frac{9u^4}{2} + o(u^4)$$

Combine:

$$f(u) = (2 + 2u + u^2 + \frac{u^3}{3}) - (1 + 2u - 2u^2 + \frac{4u^3}{3}) - (1 + 3u^2) + o(u^3)$$

Simplify:

$$f(u) = -u^3 + o(u^3)$$

Exercise 2:

1. Using concavity:

   (a) Show that: $\forall x \in [0; \frac{\pi}{2}] : \sin(x) \leq x$.

   (b) Also show that: $\forall u \in [0; 1] : \sin(\frac{\pi}{2}u) \geq u$. Deduce that $\forall x \in [0; \frac{\pi}{2}] : \sin(x) \geq \frac{2}{\pi}x$ and give the resulting bounds for $\sin x$ on $[0; \frac{\pi}{2}]$.

2. Let $n \in \mathbb{N}^*$ and $a_1, \cdots, a_n \in \mathbb{R}_+^*$. Show that: $\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}} \leq (a_1 \cdots a_n)^{\frac{1}{n}} \leq \frac{a_1+\cdots+a_n}{n}$

Answer Area

1. (a) Consider the function $f(x) = x - \sin x$ on $[0, \frac{\pi}{2}]$. We have:

$$f'(x) = 1 - \cos x \geq 0, \quad \forall x \in [0, \tfrac{\pi}{2}]$$

So f is increasing and since $f(0) = 0$, we get:

$$f(x) \geq 0 \Rightarrow x - \sin x \geq 0 \Rightarrow \sin x \leq x$$

(b) Define $g(u) = u - \sin\left(\frac{\pi}{2}u\right)$ on $[0, 1]$. Compute its derivative:

$$g'(u) = 1 - \frac{\pi}{2}\cos\left(\frac{\pi}{2}u\right)$$

Since cos is decreasing on $[0, \frac{\pi}{2}]$, $g'(u)$ is increasing. Also:

$$g(0) = 0, \quad g(1) = 1 - \sin\left(\frac{\pi}{2}\right) = 0$$

So $g(u) \leq 0$ implies $\sin\left(\frac{\pi}{2}u\right) \geq u$
Let $x = \frac{\pi}{2}u \Rightarrow u = \frac{2}{\pi}x$. Then:

$$\sin x \geq \frac{2}{\pi}x, \quad \forall x \in \left[0, \tfrac{\pi}{2}\right]$$

Combining with the previous inequality:

$$\frac{2}{\pi}x \leq \sin x \leq x, \quad \forall x \in \left[0, \tfrac{\pi}{2}\right]$$

2. This is a standard inequality between the geometric mean and arithmetic mean:

$$(a_1a_2\cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad \text{for } a_i > 0$$

The left inequality follows from the AM–GM inequality applied to reciprocals:

$$\frac{1}{(a_1a_2\cdots a_n)^{1/n}} \leq \frac{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}{n}$$

However, it's also known that:

$$\min(a_1, \ldots, a_n) \leq (a_1\cdots a_n)^{1/n} \leq \max(a_1, \ldots, a_n)$$

Thus:

$$\frac{1}{a_n} \leq (a_1\cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}$$

Exercise 3:

Show that the function below is of class C^1 on $\mathbb{R}$. Using a Taylor expansion, give the equation of the tangent to the curve C_f at the point with abscissa 0, as well as the position of the curve, near 0, with respect to the tangent: $f(x) = \frac{1}{x} \ln \frac{\exp(2x)-1}{2x}$.

Answer Area

First, define:

$$f(x) = \frac{1}{x}\ln(e^{2x} - 1)$$

To study regularity and behavior at $x = 0$, note that f is undefined at 0. We define $f(0)$ by continuity if possible.
As $x \to 0$:

$$e^{2x} - 1 = 2x + 2x^2 + \frac{4x^3}{3} + o(x^3) \Rightarrow \ln(e^{2x} - 1) = \ln(2x + 2x^2 + \tfrac{4x^3}{3} + o(x^3))$$

Factor out $2x$:

$$\ln(2x) + \ln\left(1 + x + \tfrac{2x^2}{3} + o(x^2)\right) = \ln(2x) + x + \tfrac{x^2}{6} + o(x^2)$$

So:

$$f(x) = \frac{1}{x}\left(\ln(2x) + x + \tfrac{x^2}{6} + o(x^2)\right) = \frac{\ln(2x)}{x} + 1 + \frac{x}{6} + o(x)$$

But:

$$\frac{\ln(2x)}{x} = \frac{\ln 2 + \ln x}{x} \to 0 \quad \text{as } x \to 0^+$$

Similarly from the left:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x}\ln(e^{2x} - 1)$$

Since $e^{2x} - 1 \to 0^-$, $\ln(e^{2x} - 1)$ is not defined in real numbers $\to$ f is only defined on $\mathbb{R}^*$
Extend f continuously at 0 by defining:

$$f(0) = \lim_{x\to 0} f(x) = 1 + \ln 2$$

This extension is smooth around 0, so $f \in C^1(\mathbb{R})$
The Taylor expansion gives:

$$f(x) = (1 + \ln 2) + \frac{x}{6} + o(x)$$

So the tangent line at $x = 0$ is:

$$y = (1 + \ln 2) + \frac{x}{6}$$

The curve lies **above** the tangent since the first non-zero correction term is positive.

Exercise 4:

Let f be the function defined by: $f(x) = (x-2)\exp\left(\frac{x-1}{x+1}\right)$.

1. Does the curve C_f, representing f, have a vertical asymptote? Justify.
2. Study the equation of the asymptote to the representative curve of f in the neighborhood of $+\infty$ and $-\infty$.
3. Study the relative position of this asymptote with respect to the curve C_f.

Answer Area

1. Does the curve $\mathcal{C}_f$, representing $f(x) = (x-2)e^{\frac{x-1}{x+1}}$, have a vertical asymptote? Justify.

   Solution: The function involves an exponential, which is always defined except where the exponent may blow up. The exponent is:

$$\frac{x-1}{x+1}$$

   This expression becomes undefined when $x + 1 = 0 \Rightarrow x = -1$. Let's check the behavior near $x = -1$:

   As $x \to -1^-$, the denominator goes to 0 from the negative side, so:

$$\frac{x-1}{x+1} \to \frac{-2}{0^-} \to +\infty \Rightarrow f(x) \to (x-2)e^{+\infty} \to \pm\infty$$

   As $x \to -1^+$, the denominator goes to 0 from the positive side:

$$\frac{x-1}{x+1} \to \frac{-2}{0^+} \to -\infty \Rightarrow f(x) \to (x-2)e^{-\infty} \to 0$$

   So there is a vertical asymptote at $x = -1$.

2. Study the equation of the asymptote to the representative curve of $f(x)$ in the neighborhood of $+\infty$ and $-\infty$.

   Solution: First, analyze the exponent:

$$\frac{x-1}{x+1} = 1 - \frac{2}{x+1} \Rightarrow e^{\frac{x-1}{x+1}} = e^{1-\frac{2}{x+1}} = e \cdot e^{-\frac{2}{x+1}} \sim e\left(1 - \frac{2}{x+1}\right)$$

   Then:

$$f(x) = (x-2)e^{\frac{x-1}{x+1}} \sim (x-2) \cdot e\left(1 - \frac{2}{x+1}\right) = e(x-2) - \frac{2e(x-2)}{x+1}$$

   As $x \to \pm\infty$, the second term tends to 0, so:

$$f(x) \sim e(x-2)$$

   Therefore, the oblique asymptote is:

$$y = e(x-2)$$

3. Study the relative position of this asymptote with respect to the curve $\mathcal{C}_f$.

   Solution: From the previous expansion:

$$f(x) - e(x-2) \sim -\frac{2e(x-2)}{x+1}$$

As $x \to +\infty$, this difference behaves like:

$$-\frac{2e(x)}{x} = -2e < 0 \Rightarrow f(x) < e(x-2)$$

So the curve lies **below** the asymptote as $x \to +\infty$.

As $x \to -\infty$, we also get:

$$f(x) - e(x-2) \sim -\frac{2e(x)}{x} = -2e < 0 \Rightarrow f(x) < e(x-2)$$

So again, the curve lies **below** the asymptote as $x \to -\infty$.