Ibn Tofail University

Analysis II — Make-up Exam

Year: 22-23

Exercise 1:

Let $a, b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \to \mathbb{R}$ a non-zero continuous function on $[a, b]$ such that

$$\int_a^b f(x), dx = 0 \quad \text{and} \quad \int_a^b x f(x), dx = 0.$$

1. Using the mean value theorem, show that there exists at least one $c \in [a, b]$ such that $f(c) = 0$.
2. Verify that f necessarily changes sign on $[a, b]$. (Hint: use the fact that f is continuous and not identically zero on $[a, b]$).
3. Suppose that c is the only point in $[a, b]$ such that $f(c) = 0$. In this case, we have

$$\forall x \in [a, c[, f(x) < 0 \quad \text{and} \quad \forall x \in]c, b], f(x) > 0.$$

   (a) Show that $\int_a^b (x - c) f(x), dx > 0$.

   (b) Deduce a contradiction.

4. State a conclusion summarizing the preceding results.

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Exercise 2:

Consider the two integrals I and J defined by:

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}, dx \quad \text{and} \quad J = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}, dx.$$

1. Using an appropriate change of variable, show that $I = J$.
2. Calculate $I + J$. Deduce the common value of I and J.
3. Deduce (using an appropriate change of variable) the integral $\int_0^1 \frac{1}{\sqrt{1-t^2}+t}, dt$.

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Exercise 3:

For all $n \in \mathbb{N}$, we define:

$$I_n = \int_0^1 \frac{x^n}{x+1}, dx.$$

1. Justify the existence of I_n for all $n \in \mathbb{N}$. Then calculate I_0.
2. Verify that for all $n \in \mathbb{N}$, we have the following inequality:

$$\forall x \in [0,1], \frac{x^n}{2} \leq \frac{x^n}{x+1} \leq x^n.$$

3. Deduce that $\lim_{n\to+\infty} I_n = 0$.
4. Calculate for all $n \in \mathbb{N}$, the value of $I_n + I_{n+1}$.
5. Deduce that

$$\lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \right) = \ln 2.$$

Answer Area