Ibn Tofail University

Analysis II — Normal Exam

Year: 21-22

Exercise 1:

Calculate the following limits:

1. $\lim_{x \to 0} \frac{\sin x - \arcsin x}{\sin^2 x}$.
2. $\lim_{x \to 0} \frac{4^x - 2^x}{5^x - 3^x}$.
3. $\lim_{x \to +\infty} \left(a^{\frac{1}{x}} + b^{\frac{1}{x}} + c^{\frac{1}{x^3}} \right)^x$, where $a, b, c \in \mathbb{R}_+^*$ are fixed.

Answer Area

1.
$$\lim_{x\to 0}\frac{\sin x-\arcsin x}{\sin^2 x}$$

Using Taylor expansions:

$$\sin x = x-\frac{x^3}{6}+o(x^3),\quad \arcsin x = x+\frac{x^3}{6}+o(x^3),\quad \sin^2 x = x^2+o(x^2)$$

Then:
$$\sin x-\arcsin x = -\frac{x^3}{3}+o(x^3),\quad \sin^2 x = x^2+o(x^2)$$

So:
$$\lim_{x\to 0}\frac{\sin x-\arcsin x}{\sin^2 x}=\lim_{x\to 0}\frac{-\frac{x^3}{3}}{x^2}=\lim_{x\to 0}-\frac{x}{3}=0$$

2.
$$\lim_{x\to 0}\frac{4^x-2^x}{5^x-3^x}$$

Using $a^x = 1+x\ln a+o(x)$, we get:

$$4^x-2^x = x\ln 2+o(x),\quad 5^x-3^x = x(\ln 5-\ln 3)+o(x)$$

Therefore:
$$\lim_{x\to 0}\frac{4^x-2^x}{5^x-3^x}=\frac{\ln 2}{\ln 5-\ln 3}$$

3.
$$\lim_{x\to+\infty}\left(a^{1/x}+b^{1/x}+c^{1/x}\right)^x$$

Use $a^{1/x}\approx 1+\frac{\ln a}{x}$, so:

$$a^{1/x}+b^{1/x}+c^{1/x}\approx 3+\frac{\ln(abc)}{x}$$

Then:
$$\ln f(x) = x\ln\left(3+\frac{\ln(abc)}{x}\right)\approx x\cdot\frac{\ln(abc)}{3x}=\frac{\ln(abc)}{3}$$

So:
$$\lim_{x\to+\infty} f(x) = e^{\frac{1}{3}\ln(abc)} = (abc)^{1/3}$$

Exercise 2:

Course questions and applications:

1. State Leibniz's formula.
2. Let f be the function defined by $f(x) = x^3 e^{3x}$. For all $n \in \mathbb{N}$, determine the n-th derivative of f.
3. State Taylor's formula with integral remainder.

   For the remainder of this exercise, assume that $x \geq 0$.
4. Show that $\forall x \geq 0 : \left| e^{-x} - \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} \right| \leq \frac{x^{n+1}}{(n+1)!}$
5. Show that $\forall n \in \mathbb{N}^* : \lim_{n \to +\infty} \frac{x^n}{n!} = 0$.
6. Deduce $\lim_{n \to +\infty} \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!}$.

Answer Area

1. **State Leibniz's formula:**
For two functions u and v that are n-times differentiable, the n-th derivative of their product is:

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

2. **Let $f(x) = x^3 e^{3x}$. Find the n-th derivative of f:**
Apply Leibniz's formula:

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}(x^3) \cdot \frac{d^{n-k}}{dx^{n-k}}(e^{3x})$$

- Derivatives of x^3 vanish for $k \geq 4$ - $\frac{d^{n-k}}{dx^{n-k}}(e^{3x}) = 3^{n-k} e^{3x}$

So:

$$f^{(n)}(x) = \sum_{k=0}^{\min(n,3)} \binom{n}{k} \cdot \frac{d^k}{dx^k}(x^3) \cdot 3^{n-k} e^{3x}$$

Compute derivatives of x^3:

$$\frac{d^0}{dx^0}(x^3) = x^3, \quad \frac{d^1}{dx^1}(x^3) = 3x^2, \quad \frac{d^2}{dx^2}(x^3) = 6x, \quad \frac{d^3}{dx^3}(x^3) = 6$$

Therefore:

$$f^{(n)}(x) = e^{3x} \sum_{k=0}^{\min(n,3)} \binom{n}{k} \cdot \frac{d^k}{dx^k}(x^3) \cdot 3^{n-k}$$

3. **State Taylor's formula with integral remainder:**
Let $f \in C^{n+1}([a,b])$, then for all $x \in [a,b]$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

where the remainder is:

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$$

4. **Show that** $\left| e^{-x} - \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} \right| \leq \frac{x^{n+1}}{(n+1)!}$:
Consider Taylor expansion of e^{-x} around 0:

$$e^{-x} = \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} + R_n(x)$$

Using Lagrange remainder:

$$|R_n(x)| = \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} e^{-c} \right| \leq \frac{x^{n+1}}{(n+1)!}, \quad \text{for some } c \in [0, x]$$

5. **Show that** $\lim_{n\to+\infty} \frac{x^n}{n!} = 0$:
Since factorial grows faster than exponential:

$$\forall x > 0, \quad \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

This follows from ratio test or comparison with geometric series.

6. **Deduce** $\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!}$:
From previous parts:

$$\sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} = e^{-x} + R_n(x), \quad |R_n(x)| \leq \frac{x^{n+1}}{(n+1)!}$$

As $n \to \infty$, remainder goes to 0, so:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} = e^{-x}$$

Exercise 3:

Let f be the function defined by $f(x) = \frac{e^x - \ln(1+2x)}{1+\sin(x)}$.

1. Why can we state that this function has a Taylor expansion of any order around 0?
2. Determine a third-order Taylor expansion of f around 0. What are the values of $f''(0)$ and $f^{(3)}(0)$?
3. Give the equation of the tangent line to the curve of f at the point with x-coordinate 0.
4. What is the relative position of the curve of f with respect to this tangent line?

Answer Area

1. **Why can we state that** $f(x) = \frac{e^x - \ln(1+2x)}{1+\sin x}$ **has a Taylor expansion of any order around** $x = 0$?
   The function $f(x)$ is a combination of smooth functions:

   - e^x is analytic (infinite differentiable, with Taylor series everywhere).

   - $\ln(1+2x)$ is analytic for $x > -\frac{1}{2}$, so it's analytic at $x = 0$.

   - $\sin x$ is analytic everywhere.

   - Denominator $1 + \sin x \neq 0$ near $x = 0$ since $\sin 0 = 0$, so $1 + \sin x = 1$ at $x = 0$, hence non-zero in a neighborhood of 0.

   Therefore, $f(x)$ is smooth near $x = 0$ and admits a Taylor expansion of any order.

2. **Determine a third-order Taylor expansion of** f **around** $x = 0$. **Find** $f''(0)$ **and** $f^{(3)}(0)$:
   We expand numerator and denominator up to order 3:

   $$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

   $$\ln(1+2x) = 2x - 2x^2 + \frac{8x^3}{3} + o(x^3)$$

   So:

   $$e^x - \ln(1+2x) = (1 + x + \frac{x^2}{2} + \frac{x^3}{6}) - (2x - 2x^2 + \frac{8x^3}{3}) = 1 - x + \frac{5x^2}{2} - \frac{13x^3}{6} + o(x^3)$$

   $$1 + \sin x = 1 + x - \frac{x^3}{6} + o(x^3)$$

   Now divide:

   $$f(x) = \frac{1 - x + \frac{5x^2}{2} - \frac{13x^3}{6}}{1 + x - \frac{x^3}{6}} = (1 - x + \frac{5x^2}{2} - \frac{13x^3}{6})(1 - x + x^2 - x^3 + \cdots)$$

   After simplifying up to order 3:

   $$f(x) = 1 - 2x + 3x^2 - \frac{19x^3}{6} + o(x^3)$$

   Therefore:

   $$f''(0) = 6, \quad f^{(3)}(0) = -\frac{19}{2}$$

3. **Give the equation of the tangent line to the curve of** f **at** $x = 0$:
   From the expansion:

   $$f(x) \approx 1 - 2x \Rightarrow \text{Tangent line: } y = 1 - 2x$$

4. **What is the relative position of the curve of** f **with respect to this tangent line**?
   The next term in the expansion is $+3x^2$, which is positive. So:

   $$f(x) - (1 - 2x) = 3x^2 + o(x^2) > 0 \text{ as } x \to 0$$

   Hence, the curve lies **above** the tangent line near $x = 0$.

Exercise 4:

Let f be the function defined by: $f(x) = e^{\frac{1}{x}}\sqrt{x^2+x+1}$.

1. Study the asymptote of the curve representing f in the neighborhood of $+\infty$.
2. Study the relative position of this asymptote with respect to the curve representing f.

Answer Area

1. **Study the asymptote of the curve representing** $f(x) = e^{\frac{1}{x}}\sqrt{x^2+x+1}$ **as** $x \to +\infty$:
   As $x \to +\infty$:
   $$\sqrt{x^2+x+1} = x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} \approx x\left(1+\frac{1}{2x}+\frac{1}{2x^2}-\frac{1}{8x^2}+\cdots\right)$$
   So:
   $$\sqrt{x^2+x+1} \approx x+\frac{1}{2}+\frac{3}{8x}+o\left(\frac{1}{x}\right)$$
   Also:
   $$e^{1/x} = 1+\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)$$
   Therefore:
   $$f(x) = e^{1/x}\sqrt{x^2+x+1} \approx \left(1+\frac{1}{x}+\frac{1}{2x^2}\right)\left(x+\frac{1}{2}+\frac{3}{8x}\right)$$
   Multiplying and keeping terms up to $\frac{1}{x}$:
   $$f(x) \approx x+\frac{3}{2}+\frac{7}{8x}+o\left(\frac{1}{x}\right)$$
   Hence, the asymptote is:
   $$y = x+\frac{3}{2}$$
2. **Study the relative position of this asymptote with respect to the curve**:
   From above:
   $$f(x)-\left(x+\frac{3}{2}\right) \approx \frac{7}{8x}+o\left(\frac{1}{x}\right) > 0 \text{ as } x \to +\infty$$
   Therefore, the curve lies **above** its asymptote as $x \to +\infty$.