Ibn Tofail University

Analysis II — Make-up Exam

Year: 22-23

Exercise 1:

Let $a, b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \to \mathbb{R}$ a non-zero continuous function on $[a, b]$ such that

$$\int_a^b f(x), dx = 0 \quad \text{and} \quad \int_a^b xf(x), dx = 0.$$

1. Using the mean value theorem, show that there exists at least one $c \in [a, b]$ such that $f(c) = 0$.
2. Verify that f necessarily changes sign on $[a, b]$. (Hint: use the fact that f is continuous and not identically zero on $[a, b]$).
3. Suppose that c is the only point in $[a, b]$ such that $f(c) = 0$. In this case, we have

$$\forall x \in [a, c[, f(x) < 0 \quad \text{and} \quad \forall x \in]c, b], f(x) > 0.$$

   (a) Show that $\int_a^b (x - c)f(x), dx > 0$.

   (b) Deduce a contradiction.

4. State a conclusion summarizing the preceding results.

Answer Area

1. **Using the mean value theorem, show that there exists at least one $c \in [a,b]$ such that $f(c) = 0$.**

   Since f is continuous on $[a,b]$ and $\int_a^b f(x)\,dx = 0$, by the Mean Value Theorem for integrals, there exists a point $c \in [a,b]$ such that:

   $$\int_a^b f(x)\,dx = f(c)(b-a).$$

   Given that $\int_a^b f(x)\,dx = 0$ and $b - a > 0$, it follows that:

   $$f(c)(b-a) = 0 \Rightarrow f(c) = 0.$$

   Therefore, there exists at least one $c \in [a,b]$ such that $f(c) = 0$.

2. **Verify that f necessarily changes sign on $[a,b]$.**

   Suppose f does not change sign on $[a,b]$. Then either $f(x) \geq 0$ or $f(x) \leq 0$ for all $x \in [a,b]$. Since f is continuous and not identically zero, then the integral $\int_a^b f(x)\,dx \neq 0$, which contradicts the given condition $\int_a^b f(x)\,dx = 0$. Hence, f must change sign on $[a,b]$.

3. **Suppose that c is the only point in $[a,b]$ such that $f(c) = 0$. In this case, we have:**

   $$\forall x \in [a,c),\ f(x) < 0 \quad \text{and} \quad \forall x \in (c,b],\ f(x) > 0.$$

   (a) **Show that $\int_a^b (x-c)f(x)\,dx > 0$.**
   Consider the function $g(x) = (x-c)f(x)$. Note that:
   - On $[a,c)$, $x - c < 0$ and $f(x) < 0$, so $g(x) > 0$.
   - At $x = c$, $g(c) = 0$.
   - On $(c,b]$, $x - c > 0$ and $f(x) > 0$, so $g(x) > 0$.
   Thus, $g(x) \geq 0$ on $[a,b]$, and $g(x) > 0$ on a set of positive measure. Since g is continuous and non-negative with positive values on a subset of $[a,b]$, we conclude:

   $$\int_a^b (x-c)f(x)\,dx > 0.$$

   (b) **Deduce a contradiction.**
   Now expand the integral:

   $$\int_a^b (x-c)f(x)\,dx = \int_a^b xf(x)\,dx - c\int_a^b f(x)\,dx.$$

   But from the problem statement:

   $$\int_a^b f(x)\,dx = 0 \quad \text{and} \quad \int_a^b xf(x)\,dx = 0,$$

so the entire expression becomes:

$$\int_a^b (x - c)f(x)\,dx = 0 - c \cdot 0 = 0.$$

This contradicts the earlier result that the integral is strictly positive. Therefore, our assumption that c is the only zero of f must be false.

4. **State a conclusion summarizing the preceding results.**

   From the above, since assuming that f has only one zero leads to a contradiction, we conclude that f must have at least two distinct zeros in $[a, b]$. Additionally, since f changes sign and is continuous, it must vanish at least twice in the interval $[a, b]$.

Exercise 2:

Consider the two integrals I and J defined by:

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}, dx \quad \text{and} \quad J = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}, dx.$$

1. Using an appropriate change of variable, show that $I = J$.
2. Calculate $I + J$. Deduce the common value of I and J.
3. Deduce (using an appropriate change of variable) the integral $\int_0^1 \frac{1}{\sqrt{1-t^2}+t}, dt$.

Answer Area

1. **Using an appropriate change of variable, show that $I = J$.**

   Recall:
   $$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx, \quad J = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}\,dx.$$

   Consider the substitution $u = \frac{\pi}{2} - x$. Then when $x = 0$, $u = \frac{\pi}{2}$; and when $x = \frac{\pi}{2}$, $u = 0$. Also, $dx = -du$.

   Apply this substitution to I:
   $$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx \\ &= \int_{\frac{\pi}{2}}^0 \frac{\cos\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)}(-du) \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin u}{\sin u + \cos u}\,du \\ &= J. \end{aligned}$$

   Therefore, $I = J$.

2. **Calculate $I + J$. Deduce the common value of I and J.**

   We compute:
   $$\begin{aligned} I + J &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}\,dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x + \sin x}\,dx \\ &= \int_0^{\frac{\pi}{2}} 1\,dx \\ &= \frac{\pi}{2} \end{aligned}$$

   Since $I = J$, we have:
   $$2I = \frac{\pi}{2} \Rightarrow I = J = \frac{\pi}{4}.$$

3. **Deduce (using an appropriate change of variable) the integral $\int_0^1 \frac{1}{\sqrt{1-t^2}+t}\,dt$.**

   Consider the substitution $t = \sin x$. Then $dt = \cos x\,dx$, and when $t = 0$,

$x = 0$; when $t = 1$, $x = \frac{\pi}{2}$. The integral becomes:

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1-t^2}+t}\,dt &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1-\sin^2 x}+\sin x}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx \\
&= I \\
&= \frac{\pi}{4}
\end{aligned}$$

.

Exercise 3:

For all $n \in \mathbb{N}$, we define:

$$I_n = \int_0^1 \frac{x^n}{x+1}, dx.$$

1. Justify the existence of I_n for all $n \in \mathbb{N}$. Then calculate I_0.
2. Verify that for all $n \in \mathbb{N}$, we have the following inequality:

$$\forall x \in [0,1], \frac{x^n}{2} \leq \frac{x^n}{x+1} \leq x^n.$$

3. Deduce that $\lim_{n\to+\infty} I_n = 0$.
4. Calculate for all $n \in \mathbb{N}$, the value of $I_n + I_{n+1}$.
5. Deduce that

$$\lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \right) = \ln 2.$$

Answer Area

1. **Justify the existence of I_n for all $n \in \mathbb{N}$. Then calculate I_0.**

   The function $f_n(x) = \frac{x^n}{x+1}$ is continuous on $[0,1]$ for all $n \in \mathbb{N}$, since both numerator and denominator are continuous and the denominator does not vanish on $[0,1]$. Therefore, the integral

   $$I_n = \int_0^1 \frac{x^n}{x+1}\,dx$$

   exists for all $n \in \mathbb{N}$.

   For $n = 0$, we have:

   $$I_0 = \int_0^1 \frac{1}{x+1}\,dx = \Big[\ln|x+1|\Big]_0^1 = \ln(2) - \ln(1) = \ln(2).$$

2. **Verify that for all $n \in \mathbb{N}$, we have the inequality: $\forall x \in [0,1],\ \frac{x^n}{x+1} \leq x^n$.**

   On $[0,1]$, we know that $x+1 \geq 1$, so $\frac{1}{x+1} \leq 1$. Multiplying both sides by $x^n \geq 0$, we get:

   $$\frac{x^n}{x+1} \leq x^n, \quad \forall x \in [0,1].$$

3. **Deduce that $\lim_{n\to+\infty} I_n = 0$.**

   Since $\frac{x^n}{x+1} \leq x^n$, integrating both sides over $[0,1]$ gives:

   $$0 \leq I_n = \int_0^1 \frac{x^n}{x+1}\,dx \leq \int_0^1 x^n\,dx = \frac{1}{n+1}.$$

   As $n \to \infty$, $\frac{1}{n+1} \to 0$, so by the squeeze theorem:

   $$\lim_{n\to+\infty} I_n = 0.$$

4. **Calculate for all $n \in \mathbb{N}$, the value of $I_n + I_{n+1}$.**

   We compute:

   $$\begin{aligned} I_n + I_{n+1} &= \int_0^1 \frac{x^n}{x+1}\,dx + \int_0^1 \frac{x^{n+1}}{x+1}\,dx \\ &= \int_0^1 \frac{x^n(1+x)}{x+1}\,dx \\ &= \int_0^1 x^n\,dx = \frac{1}{n+1} \end{aligned}$$

   .

5. **Deduce that $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln 2$.**

   From the recurrence:

   $$I_k + I_{k+1} = \frac{1}{k+1},$$

summing from $k = 0$ to $n - 1$, we get:

$$\sum_{k=0}^{n-1}(I_k + I_{k+1}) = \sum_{k=0}^{n-1} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k}.$$

But the left-hand side telescopes:

$$\sum_{k=0}^{n-1}(I_k + I_{k+1}) = I_0 + 2(I_1 + I_2 + \cdots + I_{n-1}) + I_n.$$

Alternatively, consider the partial sums of the alternating harmonic series:

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}.$$

It is known that this converges to $\ln 2$ as $n \to \infty$. Therefore:

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln 2.$$