UNIVERSITY IBN TOFAIL

Analyses II

Problem Set I

Exercise 1:

let $a, b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \to \mathbb{R}$ a funciton of class C^n in $[a, b]$ and $n+1$ time derivable in $]a, b[$ such that :

$$f(a) = f(b) \text{ and } f'(a) = \cdots = f^n(a) = 0$$

Prove that there is $c \in]a, b[$ such that $f^{(n+1)}(c) = 0$

Correction

usign the Rolls theorem we can prove that there is c_1 such that $f'(c_1) = 0$. usign the theorem we cna prove it for a c_n in a the interval $[a; c_{n-1}]$. then for n we have $f^{(n)}(a) = f^{(n)}(c) = 0$ then usign the Rolls theorem we have $c_{n+1} \in [a, c_n]$ that $f^{n+1}(c_{n+1} = 0$

Exercise 2:

1. using the inequality of Taylor-Lagrange, prove that for all $n \in \mathbb{N}$

$$\forall x \in \mathbb{R} \left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| \leq \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$$

2. we deduce taht the sequence $(U_n)_n >= 0$ defined by :

$$u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

   converges to e, $\lim_\infty \sum_{k=0}^{n} \frac{1}{k!} = e$

3. prove by the same way taht the sequence (v_n) converges to $\ln(2)$.

$$v_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n}$$

Correction

1. we do have from the taylor Taylor-Lagrange formula :

$$\forall x \in \mathbb{R}, e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{x^{n+1}}{(n+1)!} e^c$$

what gives us :

$$\frac{x^{n+1}}{(n+1)!} e^x - \frac{x^{n+1}}{(n+1)!} e^c \geq 0$$
$$\implies x - c \geq 0$$

and this is true because we have $c \in [0, x]$, then :

$$\forall x \in \mathbb{R} \left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| \leq \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$$

2. for $x = 1$ we have :

$$\left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| \leq \frac{e}{(n+1)!}$$

when gettig closer to infinity we get :

$$\left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| \leq 0$$
$$\implies \lim_{\infty} \sum_{k=0}^{n} \frac{1}{k!} = e$$

3. long story short :

$$\left| \ln(2) - \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k} \right| \leq \frac{1}{n+1}$$
$$\implies \lim_{\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k} = \ln(2)$$

Exercise 3:

find the local extremums in teh domaines of those functions:

- $f(x) = x^3 - 3x^2 - 9x + 2$
- $g(x) = e^x + (ln(x) - e - 1)x$

Correction

- $x = 3, x = -1$
- $0.2 < x < 1.6$

Exercise 4:

1. prove that the function $f(x) = ln(e^x + 1)$ is convexe.
2. we deduce :

$$\forall (a, b) \in \mathbb{R} \times \mathbb{R}, 1 + \sqrt{ab} \leq (\sqrt{1+a})(\sqrt{1+b})$$

Correction

1. $f"(x)$ is always postitive then the function is convexe.
2. we know that if f is convexe then : $f\left(\frac{a+b}{2}\right) = \frac{f(a)+f(b)}{2}$
let $a = \ln(a)$ and $b = \ln(b)$

$$f\left(\frac{ln(a) + ln(b)}{2}\right) = \frac{f(ln(a)) + f(ln(b))}{2}$$
$$\implies \ln(\sqrt{ab} + 1) \leq \ln(\sqrt{a+1}) \ln(\sqrt{b+1})$$
$$\implies \sqrt{ab} + 1 \leq \sqrt{a+1}\sqrt{b+1}$$

Exercise 5:

1. prove that the function Ln doesnt admet a DL near 0.
2. prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x^3 lnx & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

admet a $DL_2(0)$, but doesnt admet a $DL_3(0)$.

Correction

1. the ln doesnt admet a DL near 0 because it divergece and also its derivates diverges near 0.
2. the function does have a DL_2 near 0 but not a DL_3, because from the $f(x)$ to the $f"(x)$ we do have xln wich gives us 0 is 0 but the $f^{(3)}(x)$ contains ln and this function doesnt have a DL near 0.

Exercise 6:

calculate the limited developement at 0 of order n of the funcions:

1. $f(x) = e^x + \frac{1}{1-x}, n = 3$
2. $f(x) = cos(x)ln(1+x), n = 4$
3. $f(x) = \frac{sinx}{\sqrt{1+x}}, n = 4$
4. $f(x) = tanx, n = 5$
5. $f(x) = e^{sinx}, n = 4$
6. $f(x) = arctanx, n = 5$

Correction

1. $f(x) = \frac{7}{6}x^3 + \frac{3}{4}x^2 + 2x + 2 + \frac{f^4(c)x^4}{24}$

pretty much all of them the same.

Exercise 7:

calculate teh developement of the function at n :

1. $f(x) = cosx$,in $\frac{\pi}{4}$
2. $f(x) = \frac{\sqrt{x+1}}{x}$,in ∞ with $n = 3$
3. $f(x) = ln(x + \sqrt{x^2+1} - lnx)$,in ∞ with $n = 5$

Correction

1.
2.
3.

Exercise 8:

caldulate the limites:

1. $\lim_0 \frac{sinx-x}{x^3}$
2. $\lim_0 \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$,with $a, b \in \mathbb{R}_+^*$
3. $\lim_0 \frac{ln(1+x)-sinx}{x}$
4. $\lim_0 \frac{e^{x^2}-cosx}{x^2}$

5. $\lim_{-\infty}(\sqrt{x^2+3x+2}+x)$
6. $\lim_0 \frac{ln(1+x)+1-e^x}{1-cosx}$

Correction

1.
2.
3.
4.
5.
6.

Exercise 9:

we considere the funtion $f : \mathbb{R} \to \mathbb{R}$ defined by :

$$f(x) = \sqrt{1+x+x^3}$$

1. calculate $DL_2(0)$ of the function f.
2. we deduce the position of the tangent at teh point $x = 0$.
3. Deletermen the equation of the asymprote in ∞.

Correction

1.
2.
3.