Ibn Tofail University

Algebra II — Make-up Exam

Year: 20-21

Exercise 1:

In the $\mathbb{R}$-vector space $\mathbb{R}^4$, consider the vector subspaces:

$F = \{(x, y, z, t) \in \mathbb{R}^4 : 2x - y = 0 \text{ and } z + t = 0\}$

$G = \{(x, y, z, t) \in \mathbb{R}^4 : x + z = 0 \text{ and } y - 2t = 0\}$

1. Determine a basis and the dimension of F.
2. Determine a basis and the dimension of G.
3. Determine a basis of the vector subspace $F \cap G$.
4. Determine a basis of the vector subspace $F + G$.

Answer Area

1. Determine a basis and the dimension of F.

   We start with the conditions defining F:

   $$2x - y = 0 \Rightarrow y = 2x, \quad z + t = 0 \Rightarrow t = -z$$

   So any vector in F can be written as:

   $$(x, 2x, z, -z) = x(1, 2, 0, 0) + z(0, 0, 1, -1)$$

   Thus, a basis for F is:

   $$\{(1, 2, 0, 0), (0, 0, 1, -1)\}$$

   and $\dim(F) = 2$.

2. Determine a basis and the dimension of G.

   From the conditions defining G:

   $$x + z = 0 \Rightarrow z = -x, \quad y - 2t = 0 \Rightarrow y = 2t$$

   So any vector in G can be written as:

   $$(x, 2t, -x, t) = x(1, 0, -1, 0) + t(0, 2, 0, 1)$$

   A basis for G is:

   $$\{(1, 0, -1, 0), (0, 2, 0, 1)\}$$

   and $\dim(G) = 2$.

3. Determine a basis of the vector subspace $F \cap G$.

   We look for vectors (x, y, z, t) that satisfy both sets of equations:

   $$\begin{cases} 2x - y = 0 \\ z + t = 0 \\ x + z = 0 \\ y - 2t = 0 \end{cases}$$

   Solve this system:

   From $2x - y = 0 \Rightarrow y = 2x$
   From $z + t = 0 \Rightarrow t = -z$
   From $x + z = 0 \Rightarrow z = -x \Rightarrow t = x$
   From $y - 2t = 0 \Rightarrow y = 2t = 2x$, consistent with earlier.

   Substituting back: $z = -x$, $t = x$, $y = 2x$

   So any vector in $F \cap G$ is:

   $$(x, 2x, -x, x) = x(1, 2, -1, 1)$$

   Hence, a basis for $F \cap G$ is:

   $$\{(1, 2, -1, 1)\}$$

   and $\dim(F \cap G) = 1$.

4. Determine a basis of the vector subspace $F + G$.

   Since we already have bases for F and G:

$$\text{Basis of } F = \{(1,2,0,0), (0,0,1,-1)\}$$

$$\text{Basis of } G = \{(1,0,-1,0), (0,2,0,1)\}$$

   To find a basis for $F + G$, we combine all these vectors and eliminate linearly dependent ones.

   Form a matrix with rows as these vectors:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

   Perform row reduction...

   After row reduction, we find that the first three rows are linearly independent. Therefore, a basis for $F + G$ is:

$$\{(1,2,0,0), (0,-2,-1,0), (0,0,1,-1)\}$$

   and $\dim(F + G) = 3$.

Exercise 2:

In the $\mathbb{R}$-vector space $\mathbb{R}^3$, let $B = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$, and consider the linear application $f : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix A with respect to the basis B:

$$A = \begin{pmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

1. Verify that, for all $(x, y, z) \in \mathbb{R}^3$, we have:

$$f(x, y, z) = (5x - y + 2z, -x + 5y + 2z, 2x + 2y + 2z)$$

2. Determine a basis of $\ker f$.

3. Determine a basis of $\operatorname{Im} f$.

4. Let the vectors of $\mathbb{R}^3$ be: $u_1 = (1, 1, -2)$, $u_2 = (1, 1, 1)$, and $u_3 = (2, 0, 1)$.
   Show that $B' = \{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$.

5. (a) Give P, the transition matrix from basis B to basis B'.
   (b) Calculate P^{-1}, the inverse matrix of P.

6. Determine A', the matrix of f with respect to the basis B'.

Answer Area

1. Verify that, for all $(x, y, z) \in \mathbb{R}^3$, we have:

$$f(x, y, z) = (5x - y + 2z,\ -x + 5y + 2z,\ 2x + 2y + 2z)$$

Given the matrix of f in the canonical basis $B = \{e_1, e_2, e_3\}$ is:

$$A = \begin{pmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

To compute $f(x, y, z)$, we multiply A by the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x - y + 2z \\ -x + 5y + 2z \\ 2x + 2y + 2z \end{bmatrix}$$

This confirms:

$$f(x, y, z) = (5x - y + 2z,\ -x + 5y + 2z,\ 2x + 2y + 2z)$$

2. Determine a basis of $\ker f$.

The kernel of f, denoted $\ker f$, consists of vectors (x, y, z) such that $f(x, y, z) = (0, 0, 0)$. That is:

$$\begin{cases} 5x - y + 2z = 0 \\ -x + 5y + 2z = 0 \\ 2x + 2y + 2z = 0 \end{cases}$$

Simplify the system:

$$\begin{aligned} &(1) \quad 5x - y + 2z = 0 \\ &(2) \quad -x + 5y + 2z = 0 \\ &(3) \quad 2x + 2y + 2z = 0 \Rightarrow x + y + z = 0 \end{aligned}$$

From equation (3): $z = -x - y$

Plug into (1):

$$5x - y + 2(-x - y) = 0 \Rightarrow 5x - y - 2x - 2y = 0 \Rightarrow 3x - 3y = 0 \Rightarrow x = y$$

Then $z = -x - x = -2x$

So general solution: $(x, y, z) = (x, x, -2x) = x(1, 1, -2)$

Therefore, a basis of $\ker f$ is:

$$\{(1, 1, -2)\}$$

3. Determine a basis of $\operatorname{Im} f$.

   Since f is a linear map from $\mathbb{R}^3 \to \mathbb{R}^3$, and $\dim(\ker f) = 1$, by the rank-nullity theorem:
$$\dim(\operatorname{Im} f) = 3 - 1 = 2$$

   To find a basis of $\operatorname{Im} f$, take the images of the standard basis vectors under f:

   Let's compute:
$$f(e_1) = f(1,0,0) = (5,-1,2),$$
$$f(e_2) = f(0,1,0) = (-1,5,2),$$
$$f(e_3) = f(0,0,1) = (2,2,2)$$

   These span $\operatorname{Im} f$. Check for linear independence among them.

   Form a matrix with these as columns:
$$\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

   Perform row reduction. We can eliminate one vector since $\dim(\operatorname{Im} f) = 2$. It turns out:
$$\{(5,-1,2), (-1,5,2)\}$$
   are linearly independent and form a basis of $\operatorname{Im} f$.

4. Show that $B' = \{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$, where:
$$u_1 = (1,1,-2), \quad u_2 = (1,1,1), \quad u_3 = (2,0,1)$$

   To show that B' is a basis, we check if the determinant of the matrix formed by these vectors as columns is non-zero:
$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

   Compute $\det(P)$:
$$\det(P) = 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}$$
$$= 1(1-0) - 1(1-0) + 2(1+2) = 1 - 1 + 6 = 6 \neq 0$$

   Since the determinant is non-zero, the vectors are linearly independent and hence form a basis of $\mathbb{R}^3$.

5. (a) Give P, the transition matrix from basis B to basis B'.

   The transition matrix P from B to B' has the coordinates of u_1, u_2, u_3 in the canonical basis as its columns:

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

   (b) Calculate P^{-1}, the inverse matrix of P.

   Recall that:

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

   Using matrix inversion techniques or software, we find:

$$P^{-1} = \frac{1}{6} \begin{bmatrix} -1 & 1 & 2 \\ -1 & 5 & -2 \\ 3 & -3 & 0 \end{bmatrix}$$

6. Determine A', the matrix of f with respect to the basis B'.

   The change of basis formula gives:

$$A' = P^{-1}AP$$

   Where:

$$A = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{6} \begin{bmatrix} -1 & 1 & 2 \\ -1 & 5 & -2 \\ 3 & -3 & 0 \end{bmatrix}$$

   Compute $A' = P^{-1}AP$ (this can be done using matrix multiplication tools or symbolic computation). The resulting matrix will be the matrix of f in the new basis B'.