UNIVERSITY IBN TOFAIL

Analyses II

Problem Set II

Exercise 1:

Let n be a fixed positive integer. Consider the function $f : [0,1] \to \mathbb{R}$ defined for all integers i such that $0 \leq i \leq n-1$ by:

$$f(x) = \begin{cases} \frac{i^2}{n^2} & \text{if } x \in \left[\frac{i}{n}, \frac{i+1}{n}\right) \\ 1 & \text{if } x = 1. \end{cases}$$

1. Show that f is a step function.
2. Calculate, as a function of n, the integral $\int_0^1 f(x)\,dx$ of f over $[0,1]$.

Correction

1. f is a constant fuction in all the intervals $\left[\frac{i}{n}, \frac{i+1}{n}\right) anditisconstantfor 1 andweknowthatthefunctinis [0,\ 1] \to \mathbb{R}$.
2. we have :

$$\begin{aligned} \int_0^1 f(x)dx &= \sum_0^1 \int_{\frac{i}{n}}^{\frac{i+1}{n}} f(x)dx \\ &= \sum_0^1 \frac{i^2}{n^2}\frac{1}{n^3} \\ &= \frac{n(n-1)(2n-1)}{6}\frac{1}{n^3} \\ &= \frac{(n-1)(2n-1)}{6n^2} \end{aligned}$$

Exercise 2:

Let $a, b \in \mathbb{R}$ such that $a < b$. Consider the function $f : [a,b] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

1. Show that if ϕ and ψ are two step functions such that $\phi \leq f \leq \psi$, then $\phi \leq 0$ and $1 \leq \psi$.

2. Deduce that f is not integrable (in the Riemann sense) on $[a, b]$.

Correction

the solutions for this exercise does exists with the course.

Exercise 3:

Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by:

$$\forall x \in \mathbb{R},\ f(x) = x^2 \text{ and } g(x) = e^x.$$

1. Show that the function f is integrable on any closed bounded interval of $\mathbb{R}$.
2. Using the definition, calculate the integral $\int_0^1 f(x)\,dx$.
3. Repeat questions 1. and 2. for the function g.
4. Study the integrability of the functions $h, k : [0, 2] \to \mathbb{R}$ defined by:

$$h(x) = x - [x] \text{ and } k(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 2] \\ 0 & \text{if } x = 0. \end{cases}$$

Correction

1. f is continuous function cuz it is a polynomial, then by the Riemann sense it is integrable function.
2. we have this :

$$\begin{aligned} S_n &= \sum_1^n f(\frac{i}{n})\Delta x \\ &= \sum_1^n \left(\frac{i}{n}\right)^2 \frac{1}{n} \\ &= \frac{(n-1)(2n-1)}{6n} \\ &= \frac{2n^2+3n+1}{6n} \\ &= \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6n} \\ \lim_{\infty} S_n &= \frac{1}{3} \end{aligned}$$

   then we have $\int_0^1 f(x)dx = \frac{1}{3}$.

3. g is continuous function cuz it is and expenential functions, then by the Riemann sense it is integrable function.

$$\begin{aligned} S_n &= \sum_1^n g(\frac{i}{n})\Delta x \\ &= \sum_1^n e^{\left(\frac{i}{n}\right)} \frac{1}{n} \\ \int_0^1 e^x dx &= e-1 \end{aligned}$$

4.

Exercise 4:

Let $a, b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \to \mathbb{R}$ a bounded function.

1. Show that if f is zero except at a finite number of points in $[a, b]$, then f is integrable on $[a, b]$, and $\int_a^b f(x)\,dx = 0$.
2. Deduce that if f is integrable and if we change the values of f at a finite number of points in $[a, b]$, then f is still integrable and the value of $\int_a^b f(x)\,dx$ does not change.
3. Show that if f is integrable on $[a, b]$, then its restriction to any interval $[c, d] \subset [a, b]$ is still integrable on $[c, d]$.

Correction

1.
2.
3.

Exercise 5:

Consider the function $f : [0, 1] \to \mathbb{R}$ defined by:

$$f(x) = \frac{1}{1 + x^2}.$$

1. Calculate the Darboux sums (lower and upper) of f with respect to the following subdivision $S_0 = \{0, 1/2, 1\}$ of $[0, 1]$.
2. Same question for the following subdivision $S_1 = \{0, 1/4, 1/2, 3/4, 1\}$ of $[0, 1]$.
3. Assuming that $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$, give bounds for π using rational numbers.
4. For a regular (uniform) subdivision with step $1/n$, what value of n ensures an approximate value of π by excess to within 10^{-3}?

Correction

1.
2.
3.
4.

Exercise 6:

Let $a, b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \to \mathbb{R}$ a continuous function.

1. Show that if $\int_a^b f(x)\,dx = 0$, then f vanishes at least once on $[a, b]$.
2. Deduce that if $\int_a^b f(x)\,dx = \frac{b^2-a^2}{2}$, then f has at least one fixed point on $[a, b]$.
3. Show that if f is non-negative, then $\int_a^b f(x)\,dx = 0 \Leftrightarrow \forall x \in [a, b], f(x) = 0$.
4. Deduce that if P is a real polynomial, then $\int_a^b P^2(x)\,dx = 0 \Rightarrow P = 0$.

Correction

1.
2.
3.
4.

Exercise 7:

Using Riemann sums, calculate the limit of the following sequences:

1. $R_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$
2. $S_n = \frac{\pi}{2n} \sum_{k=1}^{n} \sin\left(\frac{k\pi}{2n}\right)$
3. $T_n = \frac{1}{n^3} \sum_{k=1}^{n} k^2 \sin\left(\frac{k\pi}{n}\right)$
4. $U_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k}$
5. $V_n = \frac{1}{n\sqrt{n}} \sum_{k=1}^{n} E(\sqrt{k})$
6. $W_n = \sum_{k=1}^{2n} \frac{1}{n+k}$
7. $X_n = \sum_{k=n}^{2n-1} \frac{1}{2k}$
8. $Y_n = \left(\prod_{k=1}^{n} (n+k)^{1/n}\right)$
9. $Z_n = \left(\frac{(2n)!}{n! n^n}\right)^{1/n}$

Correction

1.
2.
3.
4.
5.
6.
7.
8.
9.