Lecture Notes

of

DSC-11: Multivariate Calculus

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This document contains my class notes for the course *Multivariate Calculus*. This is a work in progress.

CONTENTS

TIMETABLE (CHANGE REQUIRED)

Monday. **(3–4)** *in 101.*

Tuesday.

Wednessday. **(3)** *in 103.*

Thursday. **(1)** *in 123.*

Friday.

Saturday. **(4)** *in 101.*

SYLLABUS

Unit 1: Calculus of Functions of Several Variables

Basic concepts, Limits and continuity, Partial derivatives, Tangent planes, Total differential, Differentiability, Chain rules, Directional derivatives and the gradient, Extrema of functions of two variables, Method of Lagrange multipliers with one constraint.

Unit 2: Double and Triple Integrals

Double integration over rectangular and nonrectangular regions, Double integrals in polar coordinates, Triple integrals over a parallelopiped and solid regions, Volume by triple integrals, Triple integration in cylindrical and spherical coordinates, Change of variables in double and triple integrals.

Unit 3: Green's, Stokes' and Gauss Divergence Theorem

Vector field, Divergence and curl, Line integrals and applications to mass and work, Fundamental theorem for line integrals, Conservative vector fields, Green's theorem, Area as a line integral, Surface integrals, Stokes' theorem, Gauss divergence theorem.

Essential Readings

1. Strauss, Monty J., Bradley, Gerald L., & Smith, Karl J. (2007). Calculus (3rd ed.). Dorling Kindersley (India) Pvt. Ltd. Pearson Education. Indian Reprint.

Suggestive Readings

1. Marsden, J. E., Tromba, A., & Weinstein, A. (2004). Basic Multivariable Calculus. Springer (SIE). Indian Reprint.

CHAPTER

1

FUNCTIONS OF MULTIPLE VARIABLES

`Lecture 1. 12 Feb, 2024.`

1.1 Introduction

Multivariate Calculus is the study of changes in functions of multiple variables, such as those defined from $\mathbb{R}^2$ to $\mathbb{R}$ by some definition f,

$$(x, y) \to f(x, y).$$

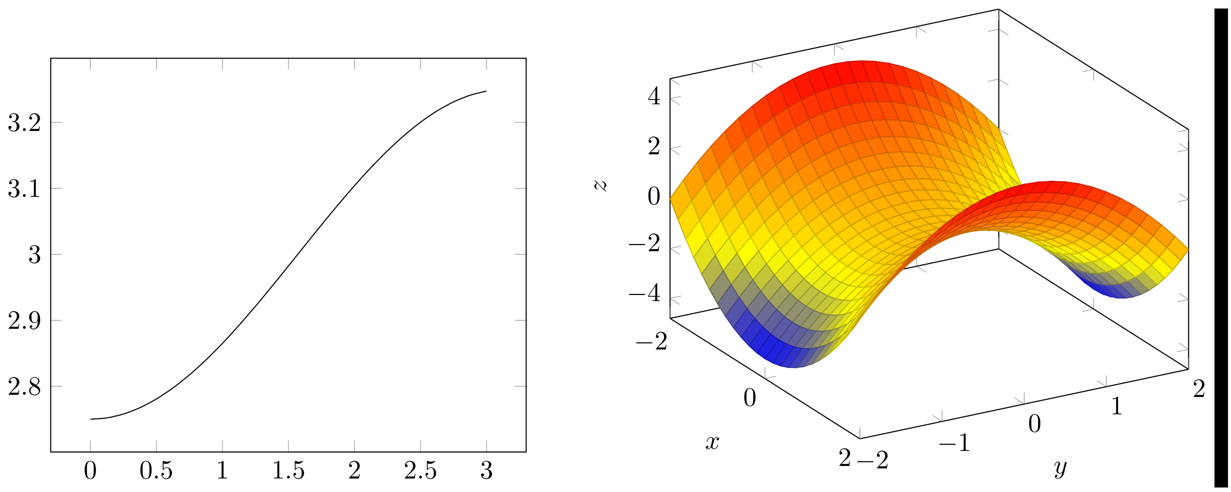
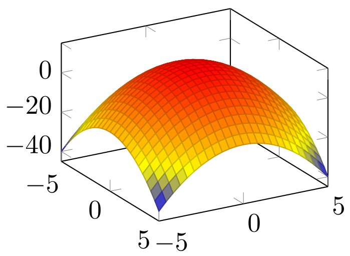


Figure 1.1: Functions in one (left) and two (right) variables.

Every point in the $x - y$ plane in (right) gets mapped to a real number in $\mathbb{R}$ (z-axis).

> **Definition 1** $f(x,y)$ Define the graph of $f(x,y)$ to be the collection of all three tuples (x,y,z) such that (x,y) belongs to the domain of f and $z = f(x,y)$. Hence the graph of $f(x,y)$ is a surface in $\mathbb{R}^3$ whose projection onto the $x-y$ plane is the domain D.



For example, consider $f(x,y) = 10 - x^2 - y^2$ Here $(0,0)$ gets mapped to 10, for example.

1. For a constant c, the plane $z = c$ intersects the surface $z = f(x,y)$ at $f(x,y) = c$. Such an intersection is called the

$$\boxed{\textit{Trace of graph of } f}$$

in the plane $z = c$.
2. Level curve of f at c is the set of all points (x,y) in the $x-y$ plane which satisfy $f(x,y) = c$.

1.2 Neighbourhood in $\mathbb{R}$

> **Definition 2** ***Open Disk*** An open disk centered at the point $P(a,b)$ in $\mathbb{R}^2$ is the collection of all those points (x,y) in $\mathbb{R}^2$ for which
>
> $$\sqrt{(x-a)^2 + (y-b)^2} < r$$
>
> for $r \in \mathbb{R}$ (radius of the disk).

1. If the boundary is to be included, we then modify the curve as

$$\sqrt{(x-a)^2 + (y-b)^2} \leq r$$

and call the disk a "closed disk."
2. A point $P_0(x_0, y_0)$ is said to be an interior point of a set S in $\mathbb{R}^2$ if some open disk centered at P_0 is contained entirely in S.
3. If the set S is the empty set or if every point of S is an interior point, the set S is said to be open.
4. A Point $P_1(x_1, y_1)$ is said to be a boundary point of set S is every open disk centered at P_1 has points (at least one) that belong to S and points that do not.
5. Set S is said to be closed if it includes its boundary.

Closed if it has infinite boundary points?

6. ϕ and $\mathbb{R}^2$ are both open as well as closed, by convention.

1.3 Open sets in $\mathbb{R}^3$

An open sphere centered at $P(a,b,c)$ in $\mathbb{R}^3$ is the set

$$\{(x,y,z) \in \mathbb{R}^3 : 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < r$$

for r being the radius of such sphere.

Limit if a function of two variables We say that

$$\lim_{(x,y)\to(0,0)} f(x,y) = l$$

if and only if $\forall\, \epsilon > 0,\ \exists\, \delta > 0$ such that whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$x we have $|f(x,y) + l| < \epsilon$.

Question 1. Using the definition of limits, prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ where $f(x,y) = (x^2+y^2)^{3/2}$.

Solution. Let $\epsilon > 0$ be given. Then, if $0 < \sqrt{x^2+y^2} < \delta$, we have

$$|f(x,y) - 0| = |(x^2+y^2)^{3/2}| < \epsilon$$

that is if and only if

$$(x^2+y^2)^{1/2} < \epsilon^{1/3}.$$

Let $\delta = \epsilon^{1/3}$. Then we have $|f(x,y)| < \epsilon$ hence $\lim_{(0,0)} f(x,y) = 0$.

Question 2. Prove $\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x} = 0$.

Solution. Let $\epsilon > 0$ be given. Then, consider

$$\begin{aligned} |f(x,y) - 0| &= \left| y \sin \frac{1}{x} \right| \\ &= |y| \left| \sin \frac{1}{x} \right| \\ &\leq |y| \\ &= \sqrt{y^2}\} \qquad \leq \sqrt{x^2+y^2} < \epsilon. \qquad \text{(by assumption)} \end{aligned}$$

Then, with $\delta = \epsilon$, we have $|f(x,y)| < \epsilon$, and hence the proof.

For a function f of two variables, when we write $(x,y) \to (x_0,y_0)$, we mean that the point is allowed to approach (x_0,y_0) along any path or axis in $\mathbb{R}^3$.

If limit $f(x,y)$ is not the same for all approaches within the domain of f, then it is said that the limit does not exist (DNE).

Question 3. Show that $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ does not exist.

Solution. Along x and y axes, the limit is 0. However, along $y = x$, it is

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}\bigg| y = x = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

Since limit is not the same along all axes, we conclude that it does not exist.

In case of homogenous equations, just set $y = mx$ for $m \in \mathbb{R}$ and proceed as follows:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0} \frac{x^2 m}{x^2 + m^2 x^2} \\ &= \frac{m^2}{1+m^2} \end{aligned}$$

and note that the limit is dependent on m, and hence the path we choose. Therefore, it does not exist.

1.4 Polar Transformations

If $f(x,y)$ involves a term of the type x^2+y^2 then we may use polar transformations

$$x = r\cos\theta, \quad \text{and,} \quad y = r\sin\theta$$

where $x^2 = x^2 + y^2$, to evaluate the limit which becomes

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta).$$

For example, to find $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}$, we may use polar transformations so that the limit becomes $\lim_{r\to 0} \frac{\tan r^2}{r^2}$ which is 1.

1.5 Differentiability (11.4)

Definition 3 ***Differentiability*** A function f is said to be differentiable at (x_0, y_0) if the increment of f can be expressed as

$$\delta_f = f_x(x_0, y_0)\delta_x + f_y(x_0, y_0)\delta_y + \epsilon_1\delta_x + \epsilon_2\delta_y$$

where $\delta_f = f(x_0 + \delta_x, y_o + \delta_y) - f(x_0, y_0)$ and $\epsilon_1, \epsilon_2 \to 0$ as $\delta_x.\delta_y \to 0$.

In case of one variable for a function,

$$f'(x_0) = \lim_{\delta_x \to 0} \frac{f(x_0 + \delta_x) - f(x_0)}{\delta_x}$$

implies

$$\frac{\delta_f - f'(x_0)\delta_x}{\delta_x} \approx \epsilon$$

and hence $\delta_f = f'(x_0)\delta_x + \epsilon\delta_x$.

Theorem 1 ***Differentiability implies continuity*** If f is differentiable at (x_0, y_0), then it is continuous there.

Proof. Let f be differentiable at (x_0, y_0) then we note that as $(x, y) \to (x_0, y_0)$ as $(\Delta_x, \Delta_y) \to (0, 0)$ where $x - x_0 \stackrel{\text{def}}{=\!=} \Delta_x$ and $y - y_0 \stackrel{\text{def}}{=\!=} \Delta_y$. Since f is differentiable at (x_0, y_0), we have that

$$\Delta_f = f_x(x_0, y)0)\Delta_x + f_y(x_0, y_0)\Delta_y + \epsilon_1\Delta_x + \epsilon_2\Delta_y$$

with $\Delta_f = f(x_0 + \Delta_x, y_o + \Delta_y) - f(x_0, y_0)$ and $\epsilon_1, \epsilon_2 \to 0$ as $\Delta_x, \Delta_y \to 0$. Now, taking limit $(\Delta_x, \Delta_y) \to (0, 0)$, we see that

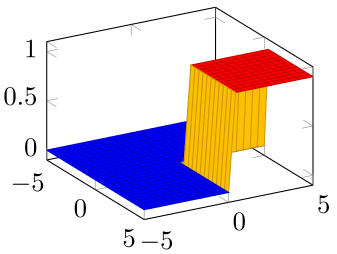
$$\lim_{(\Delta_x, \Delta_y) \to (0,0)} \Delta_f = 0 \implies \lim_{(x,y) \to (0,0)} f(x, y) = f(x_0, y_0).$$

■

Existence of partial derivatives does not guarantee differentiable. For example, consider

$$f(x, y) = \begin{cases} 1, & x > 0, y > 0, \\ 0, & \text{otherwise} \end{cases}$$

with the graph shown below.



Then, clearly

$$f_x(0, 0) = \lim_{\Delta_x \to 0} \frac{f(0 + \Delta_x, 0) - f(0, 0)}{\Delta_x} = 0$$

and similarly $f_y(0, 0) = 0$. However, since the function is not continuous, therefore it is also not differentiable.

Theorem 2 If the function $f(x, y)$ and its partial derivatives f_x and f_y are continuous in a disk centered at (x_0, y_0) then f is differentiable at (x_0, y_0)

Converse of the above theorem is not true. That is, differentiability of f at a point may not guarantee continuity of first order partial derivatives at that point.

Question 4. Show that $f(x, y) = x^2y + xy^3$ is diff at $(0, 0)$.

Solution. $f_x(x, y) = 2xy + y^3$ and $f_y = x^2 + 3xy^2$. Now, f, f_x, f_y being polynomials in x and y are continuous, hence f is differentiable.

Question 5. Let

$$f(x, y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{x}, & x \neq 0 \text{ and } y \neq 0, \\ x^2 \sin\frac{1}{x}, & x \neq 0, y = 0, \\ y^2 \sin\frac{1}{y}, & y \neq 0, x = 0, \\ 0, & \text{if } x = 0, y = 0. \end{cases}$$

Prove that f is differentiable at origin and that f_x, f_y both are discontinuous at origin.

Question 6. Show that

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & \text{otherwise.} \end{cases}$$

Solution. Since it is not continuous (prove!) at origin, therefore it is not differentiable.

1.6 Chain Rule

Theorem 3 Let $z = f(x,y)$ be a differentiable function of x and y and let $x = x(t), y = y(t)$ be differentiable functions of t. Then, $z = f(x,y)$ is also a differentiable function of t and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Proof. Given z is a differentiable function of x and y, hence we can write

$$\Delta_z = \frac{\partial z}{\partial x}\Delta_x + \frac{\partial z}{\partial y}\Delta_y + \epsilon_1\Delta_x + \epsilon_2\Delta_y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta_x, \Delta_y \to 0$. For $\Delta_t \neq 0$, we divide by t and obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\Delta_x}{\Delta_t} + \frac{\partial z}{\partial y}\frac{\Delta_y}{\Delta_t} + \epsilon_1\frac{\Delta_x}{\Delta_t} + \epsilon_2\frac{\Delta_y}{\Delta_t}. \tag{1.1}$$

Now increments in x and y are

$$\Delta_x = x(t+\Delta_t) - x(t), \qquad \Delta_y = y(t+\Delta_t) - y(t).$$

Note that

$$\lim_{\Delta_t\to 0}\frac{\Delta_x}{\Delta_t} = \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \lim_{\Delta_t\to 0}\frac{\Delta_y}{\Delta_t} = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Hence using (1.1),

$$\lim_{\Delta_t\to 0}\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\Delta_x}{\Delta_t} + \frac{\partial z}{\partial y}\frac{\Delta_y}{\Delta_t} + 0\frac{\Delta_x}{\Delta_t} + 0\frac{\Delta_y}{\Delta_t}$$

and finally

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \lim_{\Delta_t\to 0}\frac{\Delta_z}{\Delta_t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

∎

Question 7. A right circular cylinder is changing in such a way that its radius r is increasing at the rate of 3 inches per minute that height h is decreasing at the rate of 5 inches per minute. At what rate is the volume of the cylinder changing when radius is 10 inches and height is 8 inches?

Solution. Here,

$$\frac{\partial r}{\partial t} = 3, \qquad \frac{\partial h}{\partial t} = -5$$

so we can find

$$\left.\frac{\partial \pi r^h}{\partial t}\right|_{r=10,h=8} = 2\pi r h(3) + \pi r^2(-5).$$

Substitute and solve.

1.7 Implicit Differentiation

Definition 4 Let $F(x,y)=0$. Define y implicitly as a differentiable function of x. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

provided $F_y \neq 0$.

To prove the above result, since given $F(x,y)=0$, differentiate both sides w.r.t. x partially.

Question 8. Let y be a differentiable function on x such that

$$\sin(x+y) + \cos(x-y) = y.$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.