Lecture Notes

of

DSC-11: Multivariate Calculus

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This is a work in progress. Last updated February 22, 2024.

This document contains my class notes for the course *Multivariate Calculus*. This is a work in progress.

CONTENTS

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TIMETABLE (CHANGE REQUIRED)

Monday. **(3–4)** *in 101.*

Tuesday.

Wednessday. **(3)** *in 103.*

Thursday. **(1)** *in 123.*

Friday.

Saturday. **(4)** *in 101.*

SYLLABUS

Unit 1: Calculus of Functions of Several Variables

Basic concepts, Limits and continuity, Partial derivatives, Tangent planes, Total differential, Differentiability, Chain rules, Directional derivatives and the gradient, Extrema of functions of two variables, Method of Lagrange multipliers with one constraint.

Unit 2: Double and Triple Integrals

Double integration over rectangular and nonrectangular regions, Double integrals in polar coordinates, Triple integrals over a parallelopiped and solid regions, Volume by triple integrals, Triple integration in cylindrical and spherical coordinates, Change of variables in double and triple integrals.

Unit 3: Green's, Stokes' and Gauss Divergence Theorem

Vector field, Divergence and curl, Line integrals and applications to mass and work, Fundamental theorem for line integrals, Conservative vector fields, Green's theorem, Area as a line integral, Surface integrals, Stokes' theorem, Gauss divergence theorem.

Essential Readings

1. Strauss, Monty J., Bradley, Gerald L., & Smith, Karl J. (2007). Calculus (3rd ed.). Dorling Kindersley (India) Pvt. Ltd. Pearson Education. Indian Reprint.

Suggestive Readings

1. Marsden, J. E., Tromba, A., & Weinstein, A. (2004). Basic Multivariable Calculus. Springer (SIE). Indian Reprint.

CHAPTER

1
FUNCTIONS OF MULTIPLE VARIABLES

`Lecture 1. 12 Feb, 2024.`

1.1 Introduction

Multivariate Calculus is the study of changes in functions of multiple variables, such as those defined from $\mathbb{R}^2$ to $\mathbb{R}$ by some definition f,

$$(x, y) \to f(x, y).$$

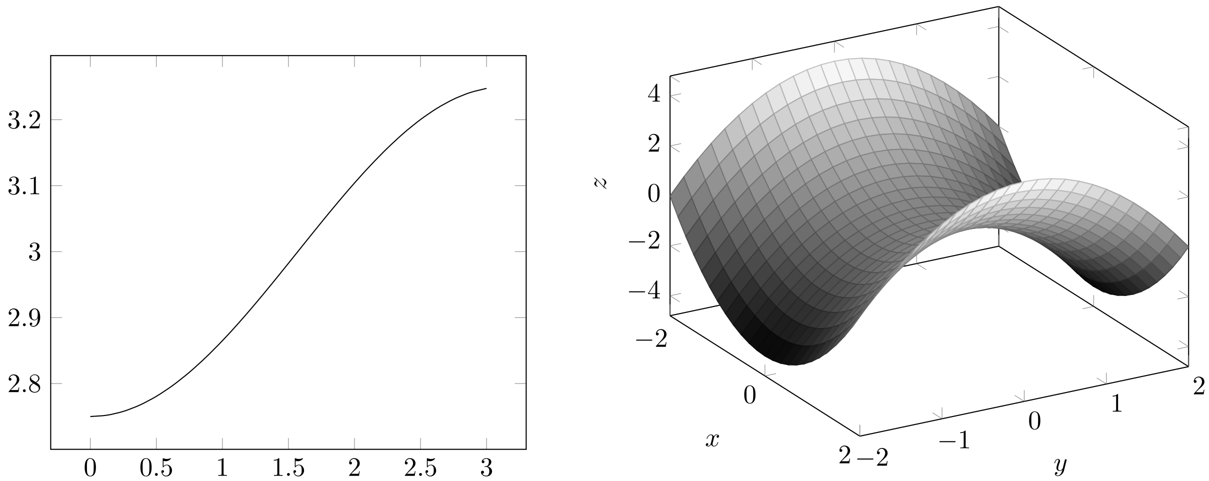
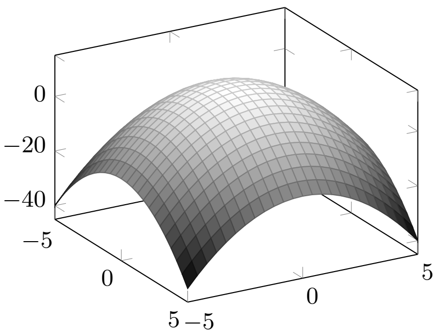


Figure 1.1: Functions in one (left) and two (right) variables.

Every point in the xy-plane in (right) gets mapped to a real number in $\mathbb{R}$ (z-axis).

> **Definition 1** $f(x, y)$ Define the graph of $f(x, y)$ to be the collection of all three tuples (x, y, z) such that (x, y) belongs to the domain of f

and $z = f(x,y)$. Hence the graph of $f(x,y)$ is a surface in $\mathbb{R}^3$ whose projection onto the xy-plane is the domain D.



For example, consider $f(x,y) = 10 - x^2 - y^2$ Here $(0,0)$ gets mapped to 10, for example.

1. For a constant c, the plane $z = c$ intersects the surface $z = f(x,y)$ at $f(x,y) = c$. Such an intersection is called the

   $\boxed{\textit{Trace of graph of } f}$

   in the plane $z = c$.
2. Level curve of f at c is the set of all points (x,y) in the xy-plane which satisfy $f(x,y) = c$.

1.2 Neighbourhood in $\mathbb{R}$

Definition 2 ***Open Disk*** An open disk centered at the point $P(a,b)$ in $\mathbb{R}^2$ is the collection of all those points (x,y) in $\mathbb{R}^2$ for which

$$\sqrt{(x-a)^2 + (y-b)^2} < r$$

for $r \in \mathbb{R}$ (radius of the disk).

1. If the boundary is to be included, we then modify the curve as

   $$\sqrt{(x-a)^2 + (y-b)^2} \leq r$$

   and call the disk a "closed disk."
2. A point $P_0(x_0, y_0)$ is said to be an interior point of a set S in $\mathbb{R}^2$ if some open disk centered at P_0 is contained entirely in S.
3. If the set S is the empty set or if every point of S is an interior point, the set S is said to be open.
4. A Point $P_1(x_1, y_1)$ is said to be a boundary point of set S is every open disk centered at P_1 has points (at least one) that belong to S and points that do not.
5. Set S is said to be closed if it includes its boundary.

   Closed if it has infinite boundary points?
6. ϕ and $\mathbb{R}^2$ are both open as well as closed, by convention.

1.3 Open sets in $\mathbb{R}^3$

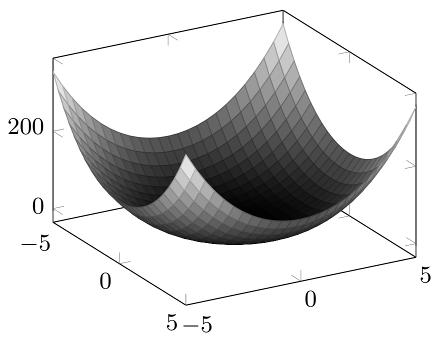
An open sphere centered at $P(a,b,c)$ in $\mathbb{R}^3$ is the set

$$\{(x,y,z) \in \mathbb{R}^3 : 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < r$$

for r being the radius of such sphere.

Limit of a function of two variables We say that

$$\lim_{(x,y)\to(0,0)} f(x,y) = l$$



if and only if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$x we have $|f(x,y) + l| < \epsilon$.

Question 1. Using the definition of limits, prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ where $f(x,y) = (x^2+y^2)^{3/2}$.

Solution. Let $\epsilon > 0$ be given. Then, if $0 < \sqrt{x^2+y^2} < \delta$, we have

$$|f(x,y) - 0| = |(x^2+y^2)^{3/2}| < \epsilon$$

that is if and only if

$$(x^2+y^2)^{1/2} < \epsilon^{1/3}.$$

Let $\delta = \epsilon^{1/3}$. Then we have $|f(x,y)| < \epsilon$ hence $\lim_{(0,0)} f(x,y) = 0$.

Question 2. Prove $\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x} = 0$.

Solution. Let $\epsilon > 0$ be given. Then, consider

$$\begin{aligned}|f(x,y) - 0| &= \left|y \sin \frac{1}{x}\right| \\ &= |y| \left|\sin \frac{1}{x}\right| \\ &\leq |y| \\ &= \sqrt{y^2}\} \qquad \leq \sqrt{x^2+y^2} < \epsilon. \qquad \text{(by assumption)}\end{aligned}$$

Then, with $\delta = \epsilon$, we have $|f(x,y)| < \epsilon$, and hence the proof.

For a function f of two variables, when we write $(x,y) \to (x_0,y_0)$, we mean that the point is allowed to approach (x_0,y_0) along any path or axis in $\mathbb{R}^3$.

If limit $f(x,y)$ is not the same for all approaches within the domain of f, then it is said that the limit does not exist (DNE).

Question 3. Show that $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ does not exist.

Solution. Along x and y axes, the limit is 0. However, along $y = x$, it is

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}\bigg|\, y = x = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

Since limit is not the same along all axes, we conclude that it does not exist.

In case of homogenous equations, just set $y = mx$ for $m \in \mathbb{R}$ and proceed as follows:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0} \frac{x^2 m}{x^2 + m^2 x^2} \\ &= \frac{m^2}{1+m^2}\end{aligned}$$

and note that the limit is dependent on m, and hence the path we choose. Therefore, it does not exist.

1.4 Polar Transformations

If $f(x,y)$ involves a term of the type x^2+y^2 then we may use polar transformations

$$x = r\cos\theta, \quad \text{and,} \quad y = r\sin\theta$$

where $x^2 = x^2 + y^2$, to evaluate the limit which becomes

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta).$$

For example, to find $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}$, we may use polar transformations so that the limit becomes $\lim_{r\to 0} \frac{\tan r^2}{r^2}$ which is 1. Similarly, for $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$, we can proceed as

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} &\overset{\text{polar}}{=\!=} \lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^2} \\ &= \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta \\ &= 0\end{aligned}$$

to evaluate the limit.

1.5 Continuity

A function $f(x, y)$ is said to be continuous at (x_0, y_0) if $f(x_0, y_0)$ is defined, if $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, and if $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$. In other words, and mathematically, if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, we have $|f(x, y) - f(x_0, y_0)| < \epsilon$.

Question 4. Test continuity of $f(x, y) = \frac{x-y}{x^2+y^2}$.

Solution. Here, the numerator and denominator — both are continuous functions. Therefore, f is continuous as well for every $(x, y) \neq (0, 0)$. At $(0, 0)$, f is not defined.

Question 5. Test continuity of $f(x, y) = \frac{1}{y-x^2}$.

Solution. Again, discontinuous for all points where the denominator is zero. Note that these points are all (x, y) such that $y = x^2$. Interestingly, this is a parabola. See Figure 1.2 for more detail.

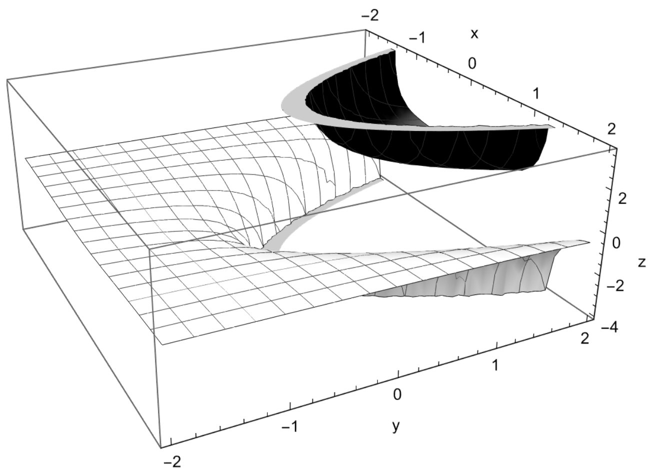
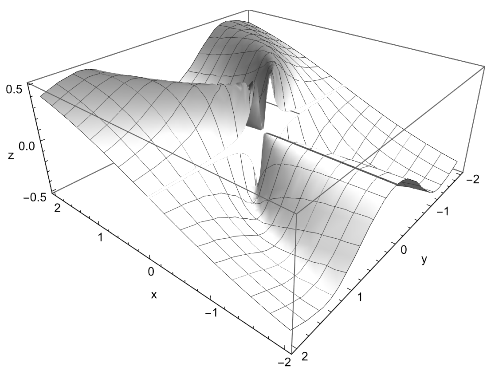


Figure 1.2: Graph of $z = \dfrac{1}{y - x^2}$ continuous everywhere except along the parabola $y = x^2$.



Question 6. Let f be defined as

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Is f continuous at origin? Explain.

Solution. We check if the limit exists at $(0, 0)$. Notice that along $x = 0$, this limit is zero (see also the attached figure in margin notes: along y-axis, the function attains 0 value). However, along some other path, say $y^2 = x$, limit is $\frac{1}{2}$ (confirm graphically).

Question 7. Given that the function

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ A, & (x,y) = (0,0) \end{cases}$$

is continuous at origin, what is the value of A?

Solution. Since f is continuous, hence limit exists. Therefore, the limiting value will be same along all curves that pass through origin. In particular, along $y = 0$, we must have

$$\begin{aligned} A &= \lim_{x\to 0} f(x,0) \\ &= \lim_{x\to 0} \frac{x^3}{x^2} \\ &\boxed{0}. \end{aligned}$$

1.6 Partial Derivatives

If $z = f(x,y)$ is a function, then the partial derivatives are denoted by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Logically, for instance, the partial derivative $\frac{\partial z}{\partial x}$ can be thought of as the change in z when, while x is increasing, but the other all variables (in this case y) are resting constantly.

And at that stage, the two variable function can be thought of as a function in one variable, since only one variable is changing. And hence the followed mathematical definitions.

Mathematically, we define the partials as

$$\begin{aligned} & \frac{\partial z}{\partial x} = f_x(x,y) = \lim_{\Delta_x \to 0} \frac{f(x+\Delta_x, y_0) - f(x,y_0)}{\Delta_x}, \\ \text{and,} \qquad & \frac{\partial z}{\partial y} = f_y(x,y) = \lim_{\Delta_y \to 0} \frac{f(x_0, y+\Delta_y) - f(x_0,y)}{\Delta_y}. \end{aligned}$$

The motive of Figure 1.3 is to show visually that when we set $x = x_0$ or $y = y_0$ and obtain a plane, its intersection with the function z in question is a curve. Partial derivative is then the change in z along this curve.

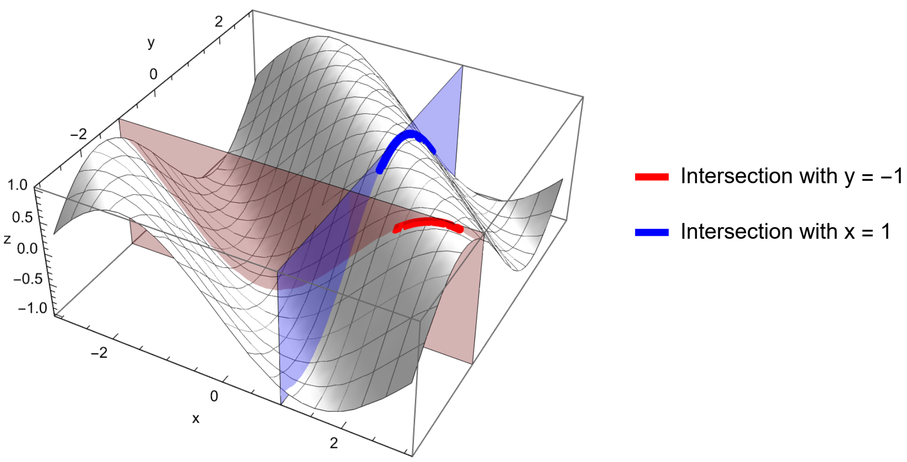


Figure 1.3: Visualising changes in $z = f(x,y)$ when each of the input variables stays constant one at a time.

More elaborately,
1. Consider a surface $z = f(x,y)$ where $(x,y) \in D \subseteq \mathbb{R}^2$ and let $(x_0,y_0) \in D$.
2. Draw the plane $y = y_0$. This plane intersects the surface $z = f(x,y)$ at the curve $c = f(x, y_0)$ which is parallel to the xz-plane, *i.e.*, curve c is the

$\boxed{\textit{trace of the surface in the plane } y = y_0}$

and has equation $z = f(x, y_0)$.

3. We can compute the slope of the tangent line to c at the point $P(x_0, y_0, z_0)$ in the plane $y = y_0$ by differentiating $f(x, y_0)$ w.r.t. x and then evaluating the obtained derivative at $x = x_0$.

Some second order partial derivatives

$$\begin{aligned}
f_{xx}(x,y) &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f_x) \\
&= \lim_{\Delta_x \to 0} \frac{f_x(x+\Delta_x, y_0) - f_x(x, y_0)}{\Delta_x}, \\
f_{yy}(x,y) &= \lim_{\Delta_y \to 0} \frac{f_y(x_0, y+\Delta_y) - f_x(x_0, y)}{\Delta_y}, \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = f_{xy}(x,y) \\
&= \lim_{\Delta_x \to 0} \frac{f_y(x+\Delta_x, y_0) - f_y(x, y_0)}{\Delta_x}, \\
f_{yx}(x,y) &= \lim_{\Delta_y \to 0} \frac{f_x(x_0, y+\Delta_y) - f_x(x_0, y)}{\Delta_y}.
\end{aligned}$$

PD as slope of tangent line The line parallel to the xz-plane and tangent to the surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ has a slope equal to $f_x(x_0, y_0)$.

Question 8. Find slope of the line $\parallel$ xz-plane and tangent to the surface $z = x\sqrt{x+y}$ at the point $P(1, 3, 2)$.

Solution. Simply, we need to find $f_x(1, 3)$, which can be evaluated as:

$$\begin{aligned}
f_x(1,3) &= \lim_{\Delta_x \to 0} \frac{f(1+\Delta_x, 3) - f(1,3)}{\Delta_x} \\
&= \lim_{\Delta_x \to 0} \frac{(1+\Delta_x)\sqrt{1+\Delta_x+3} - \sqrt{1+3}}{\Delta_x} \\
&= \lim_{\Delta_x \to 0} (1+\Delta_x)\frac{1}{2}(4+\Delta_x)^{-\frac{1}{2}} + \sqrt{4+\Delta_x} \\
&= 1\left(\frac{1}{2}\right)(4)^{-\frac{1}{2}} + 2 \\
&= \boxed{2\frac{1}{4}}.
\end{aligned}$$

Question 9. Let

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right), & (x,y) \neq (0,0), \\ 0, (x,y) = (0,0). \end{cases}$$

Show that

1. $f_x(0, y) = -y$
2. $f_y(x, 0) = x$
3. $f_{xy}(0, 0) = -1$
4. $f_{yx}(0, 0) = 1$

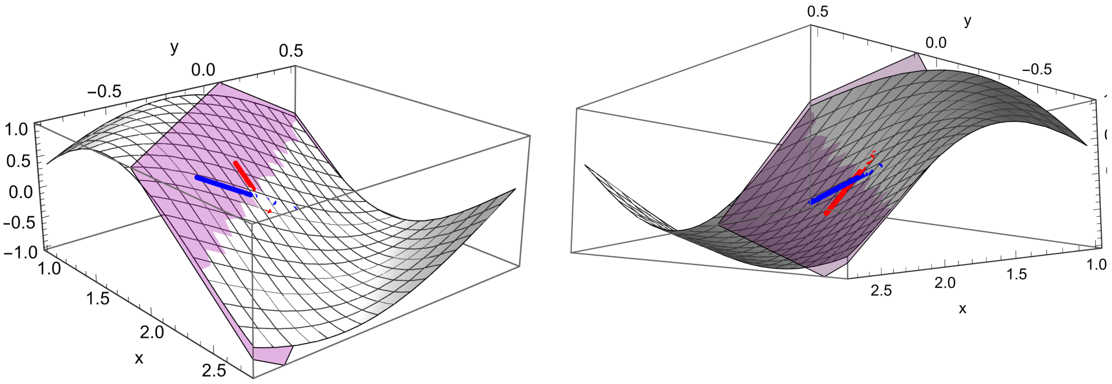
Upon solving, it will be found that $f_{xy} \neq f_{yx}$. This however does not violate the theorem on mixed partials, as f is not continuous at $(0, 0)$.

1.7 Tangent Plane

Suppose S is a surface with the equation $z = f(x, y)$ where f_x and f_y are continuous. Let $P_0(x_0, y_0, z_0)$ be a point on S. Let C_1 be the curve of intersection of S with the plane $x = x_0$ and C_2 be the curve in case of the plane $y = y_0$.

The tangent lines T_1 and T_2 to C_1 and C_2 respectively determine a unique plane. This plane contains the tangent to every smooth curve on S that passes through the point P_0. This plane is called the Tangent Plane to the surface S at the point P_0.

Here, smoothness means existence of partial derivatives.



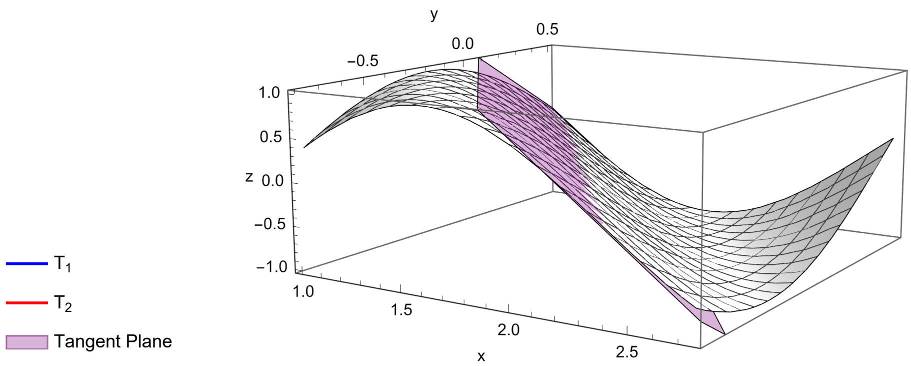


Figure 1.4: Tangent plane for the function $z = f(x, y)$ uniquely determined by tangent lines.

I talk about the development of Figure 1.4 here. Specifically, this function is

$$z = \sin(2x + 2y),$$

and the tangent plane is shown for the point $(1.8, -0.2, \approx -0.0583741)$. First of all, I found the curves C_1 and C_2 — intersections of S with the planes $x = 1.8$ and $y = -0.2$ respectively (see figure 1.3). Next, tangent lines at this point both along $x = 1.8$ and $y = -0.2$ were found. Both tangent lines shared the same slope (why?), namely

$$\left.\frac{\partial z}{\partial x}\right|_{y=-0.2} = \left.\frac{\partial z}{\partial y}\right|_{x=1.8} \approx -1.99659.$$

Intercepts for the tangent lines were

$$f(1.8, -0.2) - (-0.2)\left(\left.\frac{\partial z}{\partial x}\right|_{y=-0.2}\right)$$

and

$$f(1.8, -0.2) - (1.8)\left(\left.\frac{\partial z}{\partial y}\right|_{x=1.8}\right)$$

respectively. Finally, the tangent plane was drawn which is, as can be seen, uniquely determined by these tangent lines.

1.8 Differentiability (11.4)

A function f is said to be differentiable at (x_0, y_0) if the increment of f can be expressed as

$$\Delta_f = f_x(x_0, y_0)\Delta_x + f_y(x_0, y_0)\Delta_y + \epsilon_1\Delta_x + \epsilon_2\Delta_y$$

where $\Delta_f = f(x_0 + \Delta_x, y_0 + \Delta_y) - f(x_0, y_0)$ and $\epsilon_1, \epsilon_2 \to 0$ as $\Delta_x, \Delta_y \to 0$.

In the same breath, we can talk about the case of one variable functions. Note that

$$f'(x_0) = \lim_{\Delta_x \to 0} \frac{f(x_0 + \Delta_x) - f(x_0)}{\Delta_x}$$

implies

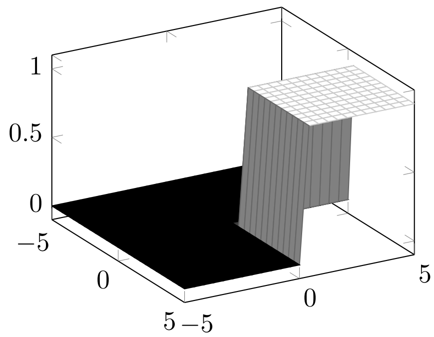
$$\frac{\Delta_f - f'(x_0)\Delta_x}{\Delta_x} \approx \epsilon$$

and hence $\Delta_f = f'(x_0)\Delta_x + \epsilon\Delta_x$.

Important to note that existence of partial derivatives does not guarantee differentiability. For example, consider

$$f(x,y) = \begin{cases} 1, & x > 0, y > 0, \\ 0, & \text{otherwise} \end{cases}$$

with the graph shown below.



Then, clearly

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta_x \to 0} \frac{f(\Delta_x, 0) - f(0,0)}{\Delta_x} \\ &= 0 \end{aligned}$$

and similarly $f_y(0,0) = 0$. However, since the function is not continuous, therefore it is also not differentiable.

Theorem 1 ***Differentiability implies continuity*** If f is differentiable at (x_0, y_0), then it is continuous there.

Proof. Let f be differentiable at (x_0, y_0) then we note that as $(x,y) \to (x_0, y_0)$ as $(\Delta_x, \Delta_y) \to (0,0)$ where $x - x_0 \overset{\text{def}}{=\!=} \Delta_x$ and $y - y_0 \overset{\text{def}}{=\!=} \Delta_y$. Since f is differentiable at (x_0, y_0), we have that

$$\Delta_f = f_x(x_0, y)0)\Delta_x + f_y(x_0, y_0)\Delta_y + \epsilon_1 \Delta_x + \epsilon_2 \Delta_y$$

with $\Delta_f = f(x_0 + \Delta_x, y_o + \Delta_y) - f(x_0, y_0)$ and $\epsilon_1, \epsilon_2 \to 0$ as $\Delta_x, \Delta_y \to 0$. Now, taking limit $(\Delta_x, \Delta_y) \to (0,0)$, we see that

$$\lim_{(\Delta_x, \Delta_y) \to (0,0)} \Delta_f = 0 \implies \lim_{(x,y) \to (0,0)} f(x,y) = f(x_0, y_0).$$

■

Theorem 2 If the function $f(x,y)$ and its partial derivatives f_x and f_y are continuous in a disk centered at (x_0, y_0) then f is differentiable at (x_0, y_0)

Question 10. Show that $f(x,y) = x^2 y + xy^3$ is diff at $(0,0)$.

Solution. $f_x(x,y) = 2xy + y^3$ and $f_y = x^2 + 3xy^2$. Now, f, f_x, f_y being polynomials in x and y are continuous, hence f is differentiable.

Question 11. Let

$$f(x,y)=\begin{cases} x^2\sin\frac{1}{x}+y^2\sin\frac{1}{y}, & x\neq 0, y\neq 0,\\ x^2\sin\frac{1}{x}, & x\neq 0, y=0,\\ y^2\sin\frac{1}{y}, & x=0, y\neq 0,\\ 0, & x=0, y=0.\end{cases}$$

Prove that f is differentiable at origin and that f_x, f_y both are discontinuous at origin.

Question 12. Show that

$$f(x,y)=\begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & \text{otherwise.}\end{cases}$$

Solution. Since it is not continuous (prove!) at origin, therefore it is not differentiable.

1.9 Chain Rule

Theorem 3 Let $z=f(x,y)$ be a differentiable function of x and y and let $x=x(t), y=y(t)$ be differentiable functions of t. Then, $z=f(x,y)$ is also a differentiable function of t and

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Proof. Given z is a differentiable function of x and y, hence we can write

$$\Delta_z=\frac{\partial z}{\partial x}\Delta_x+\frac{\partial z}{\partial y}\Delta_y+\epsilon_1\Delta_x+\epsilon_2\Delta_y$$

where $\epsilon_1,\epsilon_2\to 0$ as $\Delta_x,\Delta_y\to 0$. For $\Delta_t\neq 0$, we divide by t and obtain

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\Delta_x}{\Delta_t}+\frac{\partial z}{\partial y}\frac{\Delta_y}{\Delta_t}+\epsilon_1\frac{\Delta_x}{\Delta_t}+\epsilon_2\frac{\Delta_y}{\Delta_t}. \tag{1.1}$$

Now increments in x and y are

$$\Delta_x=x(t+\Delta_t)-x(t), \qquad \Delta_y=y(t+\Delta_t)-y(t).$$

Note that

$$\lim_{\Delta_t\to 0}\frac{\Delta_x}{\Delta_t}=\frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \lim_{\Delta_t\to 0}\frac{\Delta_y}{\Delta_t}=\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Hence using (1.1),

$$\lim_{\Delta_t\to 0}\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\Delta_x}{\Delta_t}+\frac{\partial z}{\partial y}\frac{\Delta_y}{\Delta_t}+0\frac{\Delta_x}{\Delta_t}+0\frac{\Delta_y}{\Delta_t}$$

and finally

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\lim_{\Delta_t\to 0}\frac{\Delta_z}{\Delta_t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

■

Question 13. A right circular cylinder is changing in such a way that its radius r is increasing at the rate of 3 inches per minute that height

h is decreasing at the rate of 5 inches per minute. At what rate is the volume of the cylinder changing when radius is 10 inches and height is 8 inches?

Solution. Here,

$$\frac{\partial r}{\partial t} = 3, \qquad \frac{\partial h}{\partial t} = -5$$

so we can find

$$\left.\frac{\partial \pi r^h}{\partial t}\right|_{r=10,h=8} = 2\pi r h(3) + \pi r^2(-5).$$

Substitute and solve.

1.10 Implicit Differentiation

Definition 3 Let $F(x, y) = 0$. Define y implicitly as a differentiable function of x. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

provided $F_y \neq 0$.

To prove the above result, since given $F(x, y) = 0$, differentiate both sides w.r.t. x partially.

Question 14. Let y be a differentiable function on x such that

$$\sin(x + y) + \cos(x - y) = y.$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.