MGroups

Finite Group Theory in Mathematica

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***MGroups** is a Mathematica Package that implements a part of Finite Group Theory. It facilitates studying group operations, order and inverses of the group elements, Cayley Tables, (ordinary and normal) subgroup structures of a group, and group morphisms easily and quickly. **MGroups** is licensed under the MIT Open-Source License.*

1. Preface

In **MGroups,** groups are defined in the form of “Associations” that are the Mathematica version of mappings. Quite literally, a group in this package is nothing but its Cayley Table. For example, to define the Z_2 group, all that is needed is to define

1. How 0 operates with 0,
2. How 0 operates with 1,
3. How 1 operates with 0, and
4. How 1 operates with 1.

In Mathematica, Associations are of the form

```
In[1]:= <|a → b, c → d, e → f|>;
```

which means that *a* maps to *b, c* maps to *d,* and *e* maps to *f*. Now, it is only appropriate to see how Z_2 will be defined in such a way:—

```
In[2]:= Z2 = <|0 → <|0 → 0, 1 → 1|>, 1 → <|0 → 1, 1 → 0|>|>;
```

... which is nothing but nested Associations. Now, if we need to see what is 0*0 in Z_2, we can type

```
In[3]:= Z2[0][0]
Out[3]= 0
```

to get the required output. Similarly, other complex groups are also defined.

2. Installation

To install the package, you can head over to my GitHub page (`github.com/zplus11/MGroups`) and download the `MGroups.m` file from there. In any Mathematica Notebook, type

```
In[4]:= $UserBaseDirectory <> "\\Applications" (* run this to get your directory *)
Out[4]= C:\Users\Naman Taggar\AppData\Roaming\Mathematica\Applications
```

which should give you a folder path. Place **MGroups.m** file into this path on your device. Then, whenever you need to use **MGroups** in a Mathematica notebook, type

```
In[5]:= << MGroups`
```

which will call the package. Now type

```
In[6]:= AdditiveGroup[2]

Out[6]= <| 0 → <| 0 → 0, 1 → 1 |>, 1 → <| 0 → 1, 1 → 0 |> |>
```

If you get the same output as above, then congratulations — you have successfully installed **MGroups!**

3. Introduction

This documentation covers each aspect of the package, from basic to advanced. We define a group first:—

Definition *(Group)***.** A group is a non-empty set in Mathematics, elements of which follow 4 properties namely Closure, Associativity, Existence of Identity, and Existence of Inverses under a certain binary operation.

Available Groups

This package provides the groups as tabulated below.

Group	Description
AdditiveGroup	Z_n : The group $\{0, 1, ..., n-1\}$ formed under addition modulo *n.*
MultiplicativeGroup	U_n : The group $\{0 \le x < n : \gcd(x, n) == 1\}$ formed under multiplication modulo *n.*
DihedralGroupp	D*n* : The group of symmetries of a regular polygon formed under their composition.
K_4 & Q_8	Klein ' s 4 Group and the Quaternion Group.
ExternalDirectProduct	The External Direct Product of given groups.

They can be called by their respective function names.

Defining a Group

Groups can be freely formed with the following command:—

```
In[7]:= FormGroup[{0}, #⟦1⟧ + #⟦1⟧ &]

Out[7]= <| 0 → <| 0 → 0 |> |>
```

without having to type out the Associations yourself.

Basic Operations

Import the package by running

```
In[8]:= << MGroups`
```

Define a group using one of the functions as shown below.

```
In[9]:= D4 = DihedralGroupp[4]; (* Caution: DihedralGroup is an inbuilt function,
        so this package uses DihedralGroupp *)
```

and check the domain of this group:—

```
In[10]:= FindDomain[D4]

Out[10]= {r0, r1, r2, r3, s0, s1, s2, s3}
```

Group operations can be applied just by indexing those elements through the associations.

```
In[11]:= D4["r2"]["s3"]

Out[11]= s1

In[12]:= D4["s1"]["s1"]

Out[12]= r0
```

Trying some other group:—

```
In[13]:= MultiplicativeGroup[13][3][4]

Out[13]= 12
```

which is 3 × 4 (mod 13). Doing something more exciting...

```
In[14]:= DihedralGroupp[240]["s60"]["r190"]

Out[14]= s10
```

which is the composition of the 191st rotation and reflection about the 61st axis, in the Dihedral group of order 480.

External Direct Products can be formed as follows:—

```
In[15]:= edp1 =
           ExternalDirectProduct[AdditiveGroup[12], MultiplicativeGroup[20], QuaternionGroup];
```

The order of this group will be 12 × 8 × 8. Let's confirm that.

```
In[16]:= OrderGroup[edp1]

Out[16]= 768
```

That is perfect! In EDPs, operations are done component-wise. For example,

```
In[17]:= el1 = {5, 3, "-j"};
         el2 = {8, 19, "k"};
         edp1[el1][el2]

Out[19]= {1, 17, i}
```

Group Properties

You can check whether a group is abelian or cyclic using **AbelianQ** or **CyclicQ** commands respectively.

```
In[20]:= AbelianQ[AdditiveGroup[10]]

Out[20]= True
```

```
In[21]:= AbelianQ[Klein4Group]

Out[21]= True
```

```
In[22]:= CyclicQ[DihedralGroupp[3]]

Out[22]= False
```

```
In[23]:= CyclicQ[ExternalDirectProduct[
           AdditiveGroup[6],
           AdditiveGroup[7]
          ]]

Out[23]= True
```

Cayley Tables

Complete Cayley Tables can be printed for any group using **CayleyTable** command.

```
In[24]:= CayleyTable[QuaternionGroup]
```

```
Out[24]//TableForm=
        │  1     -1    i     -i    j     -j    k     -k
────────┼─────────────────────────────────────────────────
 1      │  1     -1    i     -i    j     -j    k     -k
 -1     │  -1    1     -i    i     -j    j     -k    k
 i      │  i     -i    -1    1     k     -k    -j    j
 -i     │  -i    i     1     -1    -k    k     j     -j
 j      │  j     -j    -k    k     -1    1     i     -i
 -j     │  -j    j     k     -k    1     -1    -i    i
 k      │  k     -k    j     -j    -i    i     -1    1
 -k     │  -k    k     -j    j     i     -i    1     -1
```

or for an EDP:—

```
In[25]:= CayleyTable[ExternalDirectProduct[AdditiveGroup[2], MultiplicativeGroup[3]]]
```

```
Out[25]//TableForm=
         │  {0, 1}    {0, 2}    {1, 1}    {1, 2}
─────────┼─────────────────────────────────────────
         │  0         0         1         1
 {0, 1}  │  1         2         1         2
         │  0         0         1         1
 {0, 2}  │  2         1         2         1
         │  1         1         0         0
 {1, 1}  │  1         2         1         2
         │  1         1         0         0
 {1, 2}  │  2         1         2         1
```

Orders and Inverses

Table of Orders and Inverses for a group can also be printed as follows:—

```
In[26]:= InversesTable[DihedralGroupp[6]]
Out[26]//TableForm=
         x      x^-1    |x|
         r0     r0      1
         r1     r5      6
         r2     r4      3
         r3     r3      2
         r4     r2      3
         r5     r1      6
         s0     s0      2
         s1     s1      2
         s2     s2      2
         s3     s3      2
         s4     s4      2
         s5     s5      2
```

4. Subgroups

We define a subgroup.

Definition *(Subgroup of a group)***.** A subset $H \subseteq G$ is said to be a subgroup of the group G if it forms a group itself under the operation of G.

We can check this in **MGroups** as follows:—

```
In[27]:= Z20 = AdditiveGroup[20]; (* {0, 1, ..., 19} *)
         sub = Table[2 i, {i, 0, 9}]; (* {0, 2, ..., 18} *)
         SubgroupQ[Z20, sub]
Out[29]= True
```

```
In[30]:= SubgroupQ[DihedralGroupp[10], {"s3", "r0", "r2", "r3"}]
Out[30]= False
```

The packages uses the Finite Subgroup Test to check subgroups. All subgroups can be obtained using

```
In[31]:= Subgroups[DihedralGroupp[5]] // TableForm
Out[31]//TableForm=
         r0
         r0     s0
         r0     s1
         r0     s2
         r0     s3
         r0     s4
         r0     r1     r2     r3     r4
         r0     r1     r2     r3     r4     s0     s1     s2     s3     s4
```

List the cyclic subgroups of U_{30}:—

```
In[32]:= u30s = Subgroups[MultiplicativeGroup[30]];
         Select[u30s, CyclicQ[<|Table[x → <|Table[y → Mod[x y, 30], {y, #}]|>, {x, #}]|>] &]
Out[33]= {{1}, {1, 11}, {1, 19}, {1, 29}, {1, 7, 13, 19}, {1, 17, 19, 23}}
```

In the case of Cyclic subgroups, finding subgroups is easier (by virtue of Fundamental Theorem of Cyclic Groups). For example, even finding subgroups of U_{997} (order 996) is a doable task.

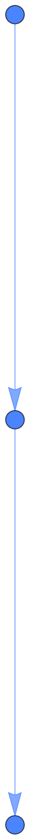
```
In[34]:= Subgroups[MultiplicativeGroup[997]];
         Length[%]
Out[35]= 12
```

5. Subgroup Lattices

Most interesting, you can also see subgroup lattices of groups using this package. For example, let us see the subgroup lattice of Z_4.

```
In[36]:= SubgroupLattice[AdditiveGroup[4]]
```

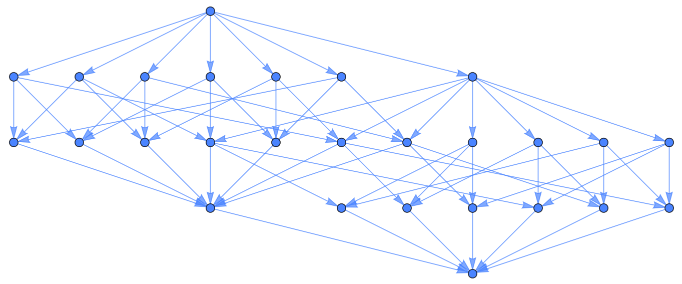
Out[36]=



Here, the top most node is the group Z_4 itself, at the bottom we have {0}, and in the middle we must have {0, 2}. Let us proceed further and see something more exciting:—

```
In[37]:= SubgroupLattice[MultiplicativeGroup[40]]
```

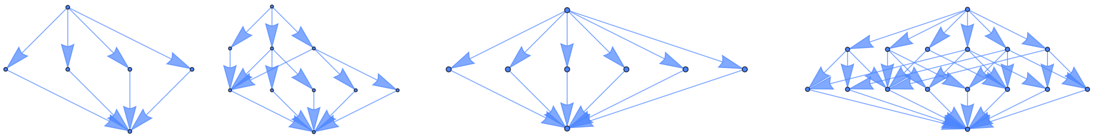
Out[37]=



Beautiful! Subgroup Lattices of some Dihedral Groups:—

In[38]:=
```
GraphicsRow[Table[SubgroupLattice[DihedralGroupp[i]], {i, 3, 6}]]
```

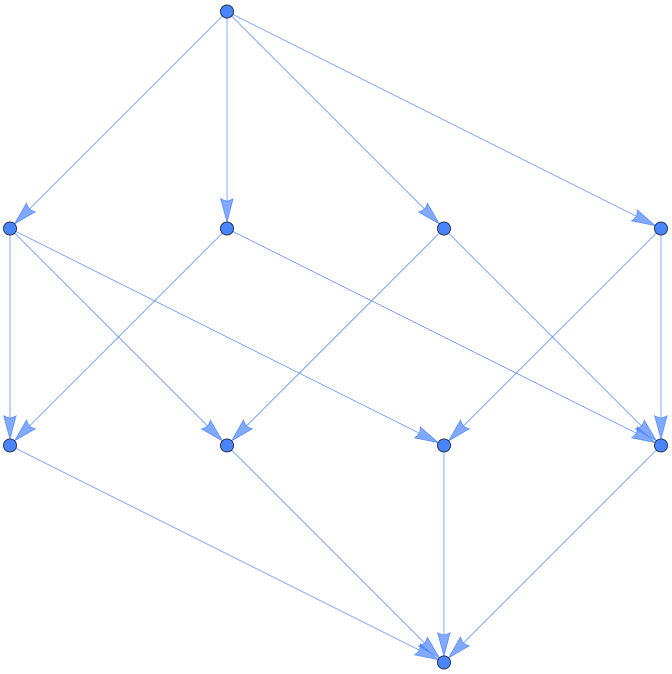
Out[38]=



Some EDPs

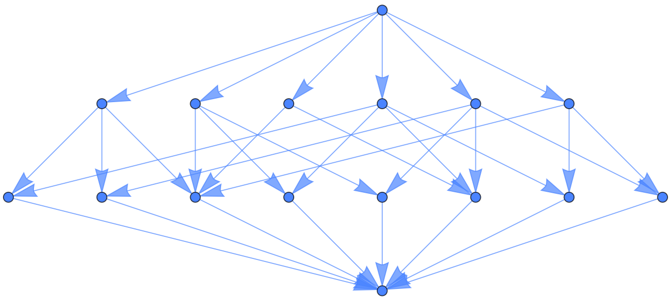
In[39]:=
```
SubgroupLattice[ExternalDirectProduct[AdditiveGroup[6], AdditiveGroup[2]]]
```

Out[39]=



In[40]:=
```
SubgroupLattice[ExternalDirectProduct[DihedralGroupp[3], AdditiveGroup[2]]]
```

Out[40]=



6. Cosets and Normal Subgroups

Definition *(Coset)***.** If H is a subgroup of G, then for some a in G, the set $\{a\,h:\ h \in H\}$ is called the left coset of H in G containing a. Subsequently, corresponding right coset is the set $\{h\,a:\ h \in H\}$.

You can define cosets in this package as follows:—

```
In[41]:= edp2 = ExternalDirectProduct[MultiplicativeGroup[8], Klein4Group];
         s = {{1, "e"}, {1, "c"}};(* any one subgroup *)
```

```
In[43]:= Coset[edp2, s, {3, "b"}, "l"]

Out[43]= {{3, b}, {3, a}}
```

```
In[44]:= Coset[edp2, s, {3, "b"}, "r"]

Out[44]= {{3, b}, {3, a}}
```

Both are equal.

Definition *(Normal Subgroup of a group)***.** A subgroup H of G is said to be normal in G if the left coset by every element is equal to the right counterpart.

To find normal subgroups of a group:—

```
In[45]:= NormalSubgroups[QuaternionGroup] // TableForm

Out[45]//TableForm=
         1
         1    -1
         1    -1    i    -i
         1    -1    j    -j
         1    -1    k    -k
         1    -1    i    -i    j    -j    k    -k
```

7. Morphisms

Definition *(Group Homomorphism)***.** A map ϕ from G_1 to G_2 is said to be a homomorphism, if and only if it is operation preserving, *i.e.*, if $\phi(\mathrm{xy}) = \phi(x)\,\phi(y)\,\forall\, x,\ y \in G_1$.

To define a Group Homomorphism, you need to define the domain, the codomain, and the map's definition. It can be done as follows:—

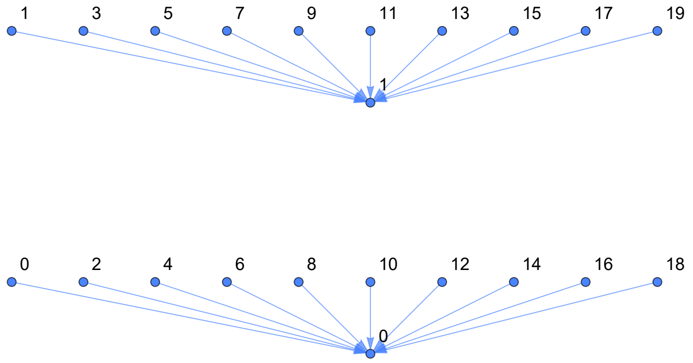
```
In[46]:= phi = Homomorphism[
           AdditiveGroup[20], (* domain *)
           AdditiveGroup[2], (* co-domain *)
           Mod[#, 2] & (* definition in the form of a function *)
          ]

Out[46]= <|0 → 0, 1 → 1, 2 → 0, 3 → 1, 4 → 0, 5 → 1, 6 → 0, 7 → 1, 8 → 0, 9 → 1,
          10 → 0, 11 → 1, 12 → 0, 13 → 1, 14 → 0, 15 → 1, 16 → 0, 17 → 1, 18 → 0, 19 → 1|>
```

Morphisms can be visualised as follows.

```
In[47]:= << MGroups`
```

```
In[48]:= VisualiseMorphism[phi]
```

Out[48]=



We can see that the kernel of phi is {0, 2, ..., 18}.

Definition *(Group Isomorphism)***.** A homomorphism ϕ is said to be an isomorphism, if and only if it is one-one.

```
In[49]:= Isomorphism[
          <|1 → <|1 → 1|>|>,
          <|1 → <|1 → 1|>|>,
          # &
         ]
Out[49]= <|1 → 1|>
```

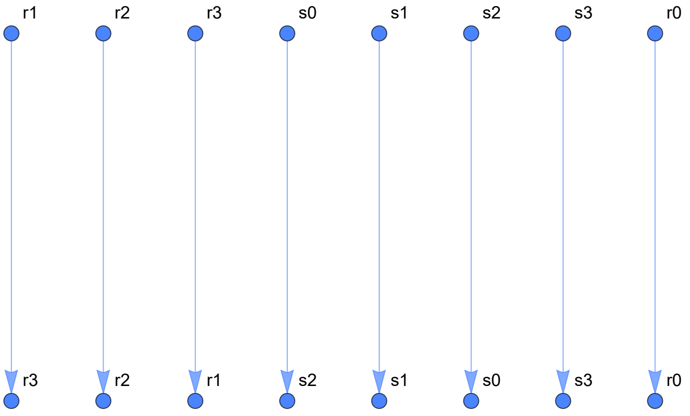
Definition *(Group Automorphism)***.** An isomorphism from a group to itself is said to be an automorphism.

```
In[50]:= Automorphism[
          <|1 → <|1 → 1|>|>,
          # &
         ]
Out[50]= <|1 → 1|>
```

Definition *(Inner Automorphism)***.** An inner automorphism is an automorphism that is induced by a group element a, with the definition $x \longrightarrow a\,x\,a^{-1}$.

```
In[51]:= InnerAutomorphism[DihedralGroupp[4], "s3"]
         VisualiseMorphism[%]
Out[51]= <|r0 → r0, r1 → r3, r2 → r2, r3 → r1, s0 → s2, s1 → s1, s2 → s0, s3 → s3|>
```

Out[52]=



Thank you

Github Repository: `https://github.com/zplus11/MGroups.git`.