MGroups

Finite Group Theory in Mathematica

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MGroups *is a Mathematica Package that implements a part of Finite Group Theory. It facilitates studying group operations, order and inverses of the group elements, Cayley Tables, (ordinary and normal) subgroup structures of a group, and group morphisms easily and quickly.* ***MGroups*** *is licensed under the MIT Open-Source License.*

1. Preface

In **MGroups,** groups are defined in the form of “Associations” that are the Mathematica version of mappings. Quite literally, a group in this package is nothing but its Cayley Table. For example, to define the Z_2 group, all that is needed is to define

1. How 0 operates with 0,
2. How 0 operates with 1,
3. How 1 operates with 0, and
4. How 1 operates with 1.

In Mathematica, Associations are of the form

```
In[1]:= <|a → b, c → d, e → f|>;
```

which means that *a* maps to *b, c* maps to *d,* and *e* maps to *f*. Now, it is only appropriate to see how Z_2 will be defined in such a way:—

```
In[2]:= Z2 = <|0 → <|0 → 0, 1 → 1|>, 1 → <|0 → 1, 1 → 0|>|>;
```

... which is nothing but nested Associations. Now, if we need to see what is 0*0 in Z_2, we can type

```
In[3]:= Z2[0][0]
Out[3]= 0
```

to get the required output. Similarly, other complex groups are also defined.

2. Installation

To install the package, you can head over to my GitHub page (`github.com/zplus11/MGroups`) and download the `MGroups.m` file from there. In any Mathematica Notebook, type

```
In[4]:= $UserBaseDirectory <> "\\Applications" (* run this to get your directory *)
Out[4]= C:\Users\Naman Taggar\AppData\Roaming\Mathematica\Applications
```

which should give you a folder path. Place **MGroups.m** file into this path on your device. Then, whenever you need to use **MGroups** in a Mathematica notebook, type

In[5]:=
```
<< MGroups`
```

which will call the package. Now type

In[6]:=
```
AdditiveGroup[2]
```

Out[6]=
```
<|0 → <|0 → 0, 1 → 1|>, 1 → <|0 → 1, 1 → 0|>|>
```

If you get the same output as above, then congratulations — you have successfully installed **MGroups!**

3. Introduction

This documentation covers each aspect of the package, from basic to advanced. We define a group first:—

Definition *(Group)***.** A group is a non-empty set in Mathematics, elements of which follow 4 properties namely Closure, Associativity, Existence of Identity, and Existence of Inverses under a certain binary operation.

Available Groups

This package provides the groups as tabulated below.

Group	Description
AdditiveGroup	Z_n : The group $\{0, 1, \ldots, n-1\}$ formed under addition modulo *n*.
MultiplicativeGroup	U_n : The group $\{0 \le x < n : \gcd(x, n) == 1\}$ formed under multiplication modulo *n*.
DihedralGroupp	D*n* : The group of symmetries of a regular polygon formed under their composition.
K_4 & Q_8	Klein ' s 4 Group and the Quaternion Group.
ExternalDirectProduct	The External Direct Product of given groups.

They can be called by their respective function names.

Defining a Group

Groups can be freely formed with the following command:—

In[7]:=
```
FormGroup[{0}, #[[1]] + #[[1]] &]
```

Out[7]=
```
<|0 → <|0 → 0|>|>
```

without having to type out the Associations yourself.

Basic Operations

Import the package by running

In[8]:=
```
<< MGroups`
```

Define a group using one of the functions as shown below.

In[9]:=
```
D4 = DihedralGroupp[4]; (* Caution: DihedralGroup is an inbuilt function,
so this package uses DihedralGroupp *)
```

and check the domain of this group:—

In[10]:=
```
FindDomain[D4]
```

Out[10]=
```
{r0, r1, r2, r3, s0, s1, s2, s3}
```

Group operations can be applied just by indexing those elements through the associations.

In[11]:=
```
D4["r2"]["s3"]
```

Out[11]=
```
s1
```

In[12]:=
```
D4["s1"]["s1"]
```

Out[12]=
```
r0
```

Trying some other group:—

In[13]:=
```
MultiplicativeGroup[13][3][4]
```

Out[13]=
```
12
```

which is 3 × 4 (mod 13). Doing something more exciting...

In[14]:=
```
DihedralGroupp[240]["s60"]["r190"]
```

Out[14]=
```
s10
```

which is the composition of the 191st rotation and reflection about the 61st axis, in the Dihedral group of order 480.

External Direct Products can be formed as follows:—

In[15]:=
```
edp1 =
  ExternalDirectProduct[AdditiveGroup[12], MultiplicativeGroup[20], QuaternionGroup];
```

The order of this group will be 12 × 8 × 8. Let’s confirm that.

In[16]:=
```
OrderGroup[edp1]
```

Out[16]=
```
768
```

That is perfect! In EDPs, operations are done component-wise. For example,

In[17]:=
```
el1 = {5, 3, "-j"};
el2 = {8, 19, "k"};
edp1[el1][el2]
```

Out[19]=
```
{1, 17, i}
```

Group Properties

You can check whether a group is abelian or cyclic using **AbelianQ** or **CyclicQ** commands respectively.

In[20]:=
```
AbelianQ[AdditiveGroup[10]]
```
Out[20]=
```
True
```

In[21]:=
```
AbelianQ[Klein4Group]
```
Out[21]=
```
True
```

In[22]:=
```
CyclicQ[DihedralGroupp[3]]
```
Out[22]=
```
False
```

In[23]:=
```
CyclicQ[ExternalDirectProduct[
   AdditiveGroup[6],
   AdditiveGroup[7]
  ]]
```
Out[23]=
```
True
```

Cayley Tables

Complete Cayley Tables can be printed for any group using **CayleyTable** command.

In[24]:=
```
CayleyTable[QuaternionGroup]
```
Out[24]//TableForm=

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

or for an EDP:—

In[25]:=
```
CayleyTable[ExternalDirectProduct[AdditiveGroup[2], MultiplicativeGroup[3]]]
```
Out[25]//TableForm=

	{0, 1}	{0, 2}	{1, 1}	{1, 2}
{0, 1}	0 1	0 2	1 1	1 2
{0, 2}	0 2	0 1	1 2	1 1
{1, 1}	1 1	1 2	0 1	0 2
{1, 2}	1 2	1 1	0 2	0 1

Orders and Inverses

Table of Orders and Inverses for a group can also be printed as follows:—

In[26]:=
```
InversesTable[DihedralGroupp[6]]
```

Out[26]//TableForm=

x	x^{-1}	|x|
r0	r0	1
r1	r5	6
r2	r4	3
r3	r3	2
r4	r2	3
r5	r1	6
s0	s0	2
s1	s1	2
s2	s2	2
s3	s3	2
s4	s4	2
s5	s5	2

4. Subgroups

We define a subgroup.

Definition *(Subgroup of a group)*. A subset $H \subseteq G$ is said to be a subgroup of the group G if it forms a group itself under the operation of G.

We can check this in **MGroups** as follows:—

In[27]:=
```
Z20 = AdditiveGroup[20]; (* {0, 1, ..., 19} *)
sub = Table[2 i, {i, 0, 9}]; (* {0, 2, ..., 18} *)
SubgroupQ[Z20, sub]
```

Out[29]= True

In[30]:=
```
SubgroupQ[DihedralGroupp[10], {"s3", "r0", "r2", "r3"}]
```

Out[30]= False

The packages uses the Finite Subgroup Test to check subgroups. All subgroups can be obtained using

In[31]:=
```
Subgroups[DihedralGroupp[5]] // TableForm
```

Out[31]//TableForm=

r0									
r0	s0								
r0	s1								
r0	s2								
r0	s3								
r0	s4								
r0	r1	r2	r3	r4					
r0	r1	r2	r3	r4	s0	s1	s2	s3	s4

List the cyclic subgroups of U_{30}:—

In[32]:=
```
u30s = Subgroups[MultiplicativeGroup[30]];
Select[u30s, CyclicQ[<|Table[x → <|Table[y → Mod[x y, 30], {y, #}]|>, {x, #}]|>] &]
```

Out[33]= {{1}, {1, 11}, {1, 19}, {1, 29}, {1, 7, 13, 19}, {1, 17, 19, 23}}

In the case of Cyclic subgroups, finding subgroups is easier (by virtue of Fundamental Theorem of Cyclic Groups). For example, even finding subgroups of U_{997} (order 996) is a doable task.

In[34]:=
```
Subgroups[MultiplicativeGroup[997]];
Length[%]
```

Out[35]= 12

5. Subgroup Lattices

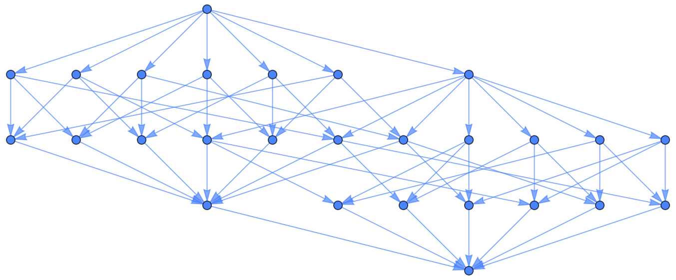
Most interesting, you can also see subgroup lattices of groups using this package. For example, let us see the subgroup lattice of Z_4.

In[36]:=
```
SubgroupLattice[AdditiveGroup[4]]
```

Out[36]=


Here, the top most node is the group Z_4 itself, at the bottom we have {0}, and in the middle we must have {0, 2}. Let us proceed further and see something more exciting:—

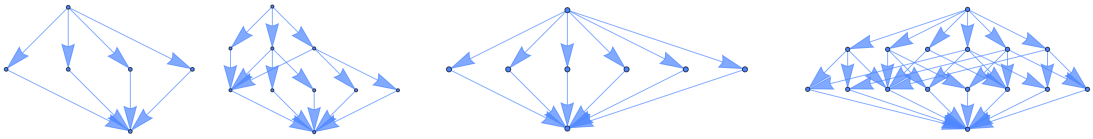
In[37]:=
```
SubgroupLattice[MultiplicativeGroup[40]]
```

Out[37]=


Beautiful! Subgroup Lattices of some Dihedral Groups:—

In[38]:=
```
GraphicsRow[Table[SubgroupLattice[DihedralGroupp[i]], {i, 3, 6}]]
```

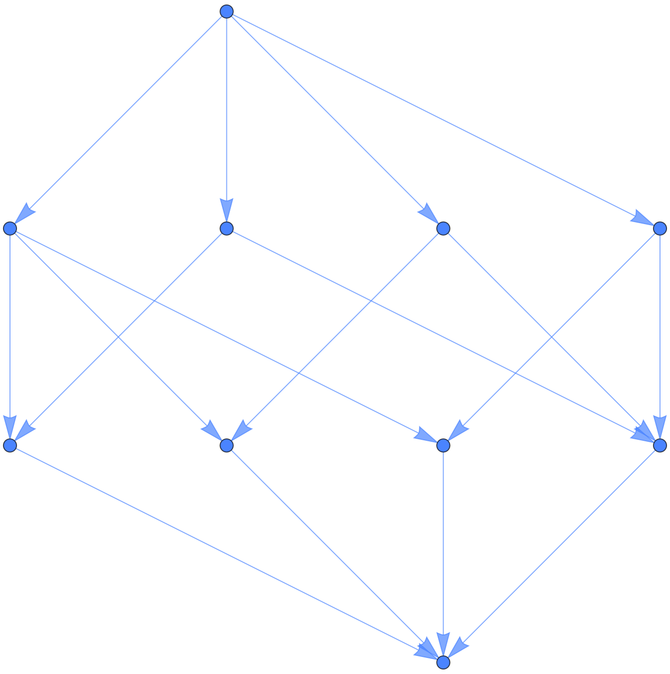
Out[38]=



Some EDPs

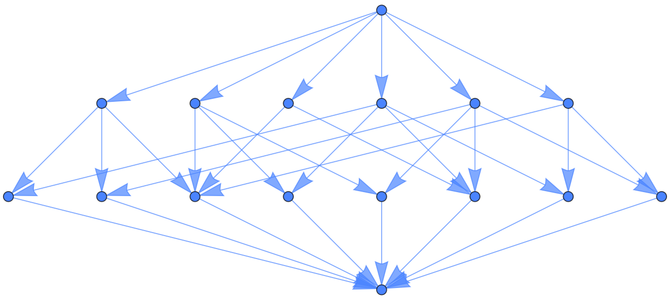
In[39]:=
```
SubgroupLattice[ExternalDirectProduct[AdditiveGroup[6], AdditiveGroup[2]]]
```

Out[39]=



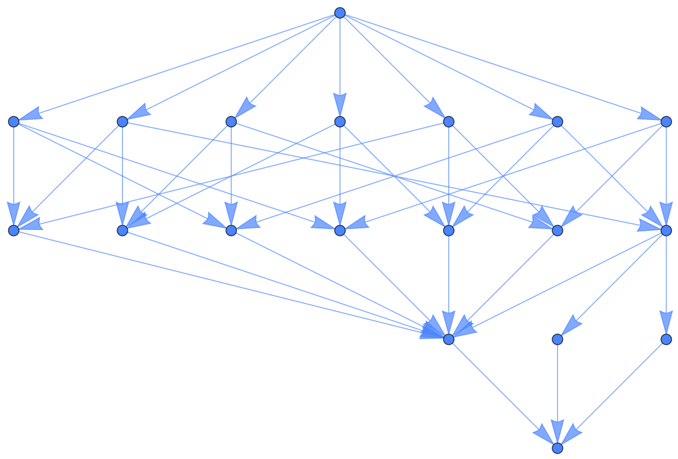
In[40]:=
```
SubgroupLattice[ExternalDirectProduct[DihedralGroupp[3], AdditiveGroup[2]]]
```

Out[40]=



In[41]:=
```
ExternalDirectProduct[QuaternionGroup, AdditiveGroup[2]] // SubgroupLattice
```

Out[41]=



6. Cosets and Normal Subgroups

Definition *(Coset)***.** If H is a subgroup of G, then for some a in G, the set $\{a\,h:\ h \in H\}$ is called the left coset of H in G containing a. Subsequently, corresponding right coset is the set $\{h\,a:\ h \in H\}$.

You can define cosets in this package as follows:—

In[42]:=
```
edp2 = ExternalDirectProduct[MultiplicativeGroup[8], Klein4Group];
s = {{1, "e"}, {1, "c"}};(* any one subgroup *)
```

In[44]:=
```
Coset[edp2, s, {3, "b"}, "l"]
```

Out[44]= {{3, b}, {3, a}}

In[45]:=
```
Coset[edp2, s, {3, "b"}, "r"]
```

Out[45]= {{3, b}, {3, a}}

Both are equal.

Definition *(Normal Subgroup of a group)***.** A subgroup H of G is said to be normal in G if the left coset by every element is equal to the right counterpart.

To find normal subgroups of a group:—

In[46]:=
```
NormalSubgroups[QuaternionGroup] // TableForm
```

Out[46]//TableForm=

1							
1	-1						
1	-1	i	-i				
1	-1	j	-j				
1	-1	k	-k				
1	-1	i	-i	j	-j	k	-k

7. Morphisms

Definition *(Group Homomorphism)***.** A map ϕ from G_1 to G_2 is said to be a homomorphism, if and only if it is operation preserving, *i.e.*, if $\phi\,(xy) = \phi\,(x)\,\phi(y)\,\forall\, x,\ y \in G_1$.

To define a Group Homomorphism, you need to define the domain, the codomain, and the map's definition. It can be done as follows:—

In[47]:=
```
phi = Homomorphism[
  AdditiveGroup[20], (* domain *)
  AdditiveGroup[2], (* co-domain *)
  Mod[#, 2] & (* definition in the form of a function *)
 ]
```

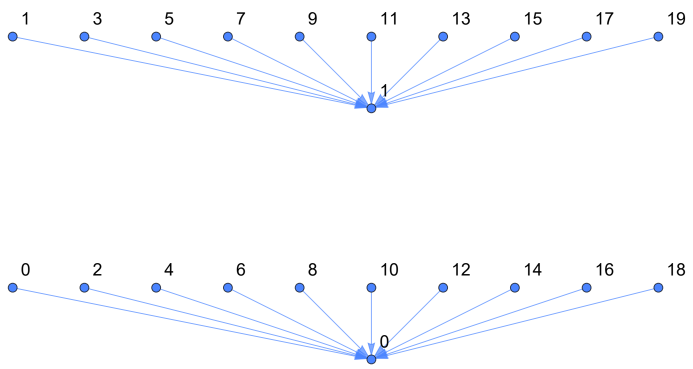
Out[47]=
```
<|0 → 0, 1 → 1, 2 → 0, 3 → 1, 4 → 0, 5 → 1, 6 → 0, 7 → 1, 8 → 0, 9 → 1,
 10 → 0, 11 → 1, 12 → 0, 13 → 1, 14 → 0, 15 → 1, 16 → 0, 17 → 1, 18 → 0, 19 → 1|>
```

Morphisms can be visualised as follows.

In[48]:=
```
<< MGroups`
```

In[49]:=
```
VisualiseMorphism[phi]
```

Out[49]=



We can see that the kernel of phi is {0, 2, ..., 18}.

Definition *(Group Isomorphism)*. A homomorphism ϕ is said to be an isomorphism, if and only if it is one-one.

In[50]:=
```
Isomorphism[
  <|1 → <|1 → 1|>|>,
  <|1 → <|1 → 1|>|>,
  # &
]
```

Out[50]=
```
<|1 → 1|>
```

Definition *(Group Automorphism)*. An isomorphism from a group to itself is said to be an automorphism.

In[51]:=
```
Automorphism[
  <|1 → <|1 → 1|>|>,
  # &
]
```

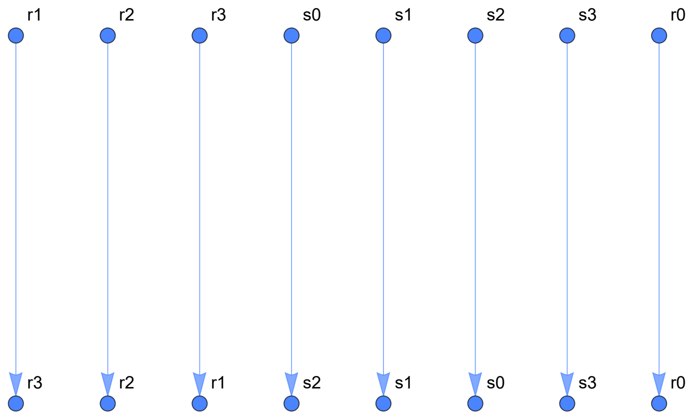
Out[51]=
```
<|1 → 1|>
```

Definition *(Inner Automorphism)*. An inner automorphism is an automorphism that is induced by a group element a, with the definition $x \longrightarrow a\,x\,a^{-1}$.

In[52]:=
```
InnerAutomorphism[DihedralGroupp[4], "s3"]
VisualiseMorphism[%]
```

Out[52]= <| r0 → r0, r1 → r3, r2 → r2, r3 → r1, s0 → s2, s1 → s1, s2 → s0, s3 → s3 |>

Out[53]=



Thank you

Github Repository: https://github.com/zplus11/MGroups.git.