Homework 9

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Code for the Lean portion is here: `https://github.com/zpm-bu/cs511-formal-methods/blob/assignments/lean/Homework/hw9.lean`

Exercise 1. The notion of *two-colorable simple graphs* coincides with the notion of *bipartite simple graphs*. Write an infinite set Γ_{bip} of first-order sentences such that, for every simple graph G, it holds that $G \vDash \Gamma_{\text{bip}}$ if and only if G is bipartite.

Thanks to the hint, we know that a graph G is bipartite if and only if every cycle in G is even. An equivalent statement is that *no* cycle in G is odd, which will be easier for us to model in sentences.

This allows us to build a simple set Γ_{bip} which encodes this statement. Let φ_n be the sentence "G has a cycle of length n", which then gives us

$$\Gamma_{\text{bip}} \stackrel{def}{=} \{\neg\varphi_3, \neg\varphi_5, \neg\varphi_7, \dots\}$$

Exercise 2. Consider the three sentences

$$\varphi_1 \stackrel{\text{def}}{=} \forall x P(x,x)$$
$$\varphi_2 \stackrel{\text{def}}{=} \forall x \forall y (P(x,y) \to P(y,x))$$
$$\varphi_3 \stackrel{\text{def}}{=} \forall x \forall y \forall z ((P(x,y) \wedge P(y,z)) \to P(x,z))$$

which express that the binary predicate P is reflexive, symmetric, and transitive, respectively. Show that none of these sentences is semantically entailed by the other ones by choosing for each pair of sentences a model which satisfies that pair, but not the third. (That is, find three binary relations, each satisfying only two of the three properties).

(Reflexive and symmetric but not transitive.) Consider the real numbers $\mathbb{R}$ and the simple distance metric $d(x,y) = |x-y|$. Say that two points are related if $d(x,y) < 1$. Let $x, y, z \in \mathbb{R}$.

- The relationship is *reflexive* since each point is distance 0 to itself.
- The relationship is *symmetric* since the distance is absolute.
- The relationship is *not transitive*. Let $x < y$ and $d(x,y) > 0.5$, and $y < z$ and $d(y,z) > 0.5$. Then x and z are on opposite "sides" of y so their distance is $d(x,y) + d(y,z) > 1$ and thus x and z are not related.

(Reflexive and transitive but not symmetric.) Consider the simple relationship $\leq$ on the integers.

- The relationship is *reflexive*, since $n \leq n$ for all $n \in \mathbb{Z}$.
- The relationship is *transitive*: $x \leq y$ and $y \leq z$, then $x \leq z$.
- The relationship is *not symmetric*. Note $2 \leq 3$ but $3 > 2$.

(Symmetric and transitive but not reflexive.) Consider the non-negative integers $\mathbb{N}_{+0}$. Define the relationship $\sim$ as the "nonzero product" relationship: $n \sim m$ if $nm > 0$.

- The relationship is *symmetric*, since integer multiplication is symmetric.
- The relationship is *transitive*. If $n \sim m$ then $n, m > 0$ and likewise if $m \sim \ell$ then $m, \ell > 0$. Thus $n\ell > 0$ and so $n \sim \ell$.
- The relationship is *not reflexive*. $0 \not\sim 0$. (In fact, $0 \not\sim n$ for all n, but this is the specific sub-behavior that is useful here.)