Homework 1

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For code see: `https://github.com/zpm-bu/cs511-formal-methods/blob/assignments/lean/Homework/hw1.lean`

Exercise 1.2.1(h). Prove the validity of

$$p \vdash (p \to q) \to q.$$

1	p	premise
2	$p \to q$	assume
3	q	$\to$e 1 2
4	$(p \to q) \to q$	$\to$i 2 : 3

Exercise 1.2.1(i). Prove the validity of

$$(p \to r) \wedge (q \to r) \vdash p \wedge q \to r.$$

1	$(p \to r) \wedge (q \to r)$	premise
2	$p \wedge q$	assume
3	p	$\wedge$e1 2
4	$p \to r$	$\wedge$e1 1
5	r	$\to$e 3 4
6	$p \wedge q \to r$	$\to$i 2 : 5

Exercise 1.2.1(j). Prove the validity of

$$q \to r \vdash (p \to q) \to (p \to r)\,.$$

1	$q \to r$	premise
2	$p \to q$	assume
3	p	assume
4	q	$\to$e 2 3
5	r	$\to$e 1 4
6	$p \to r$	$\to$i 3 : 5
7	$(p \to q)$	$\to$i 2 : 6

Exercise 1.4.2(g). Provide the truth table for

$$((p \to q) \to p) \to p.$$

This statement is a tautology:

p	q	$p \to q$	$(p \to q) \to p$	$((p \to q) \to p) \to p$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Exercise 1.4.2(h). Provide the truth table for

$$((p \vee q) \to r) \to ((p \to r) \vee (q \to r)) \,.$$

This statement is a tautology:

p	q	r	$p \vee q$	$(p \vee q) \to r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

p	q	r	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

p	q	r	$((p \vee q) \to r) \to ((p \to r) \vee (q \to r))$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

Exercise 1.4.2(i). Prove the truth table for

$$(p \to q) \to (\neg p \to \neg q)\,.$$

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p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$(p \to q) \to (\neg p \to \neg q)$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem 1.5.3(b). Show that if $C \subseteq \{\neg, \wedge, \vee, \to, \perp\}$ is adequate for propositional logic, then $\neg \in C$ or $\perp \in C$.

Let $C \subseteq \{\neg, \wedge, \vee, \to, \perp\}$ be a subset and assume that $\neg \notin C$ and $\perp \notin C$; then, $C \subseteq \{\wedge, \vee, \to\}$.

To demonstrate that C is not adequate for propositional logic, we only need to demonstrate that there exists at least one statement of propositional logic which cannot be expressed only through the connectives in C.

Consider the propositional logic statement $\psi = \neg p \wedge \neg q$. Assume that φ is a formula formed only with the connectives in C which is equivalent to ψ. Then φ must produce `F` when p and q are both `T`.

However, φ can only be made of some combination of the connectives $\wedge$, $\vee$, and $\to$, all of which assign `T` when the input atoms are both `T`. Thus, no combination of strictly these symbols can produce `F` when p and q are both true; to do so, we need some mechanism to invert the truth value, and we have none since $\neg \notin C$. Further, since $\perp \notin C$, we cannot even use a contradiction to reach the output we desire.

Thus, there is no formula φ over the connectives in C which is equivalent to ψ, and thus C is not adequate for propositional logic.

Therefore, by the contrapositive, if C is adequate for propositional logic, then $\neg \in C$ or $\perp \in C$.

Problem 1.5.3(c). Is $\{\leftrightarrow, \neg\}$ adequate? Prove your answer.

Consider the combinations of $\leftrightarrow$ and $\neg$. First, observe that $p \leftrightarrow q$ has exactly two `T` outputs and two `F` outputs. Likewise, $p \leftarrow \neg q$, etc., all also have exactly two `T` outputs and two `F` outputs. Applying $\neg$ to any statement composed of $\leftrightarrow$ connections will flip all `T` to `F` and vice versa, producing in turn a statement with two `T` outcomes and two `F` outcomes. Thus, a combination of $\neg$ and $\leftrightarrow$ on two input atoms can only produce statements with two `T` and two `F` outcomes.

Now consider $p \wedge q$, a statement with one `T` output and three `F` outputs. By our observation on the properties of $\neg$ and $\leftrightarrow$, no combination of $\neg$ and $\leftrightarrow$ on two input atoms can produce a valuation with three `F` outputs and one `T` output.

Thus, no combination of $\neg$ and $\leftrightarrow$ on two input atoms can produce the same truth table as $p \wedge q$.

Therefore, there exists at least one statement of formal logic which cannot be modeled using only $\leftrightarrow$ and $\neg$.

Therefore, $\{\leftrightarrow, \neg\}$ is not adequate for formal logic.