Homework 3

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Code for this assignment is at: `https://github.com/zpm-bu/cs511-formal-methods/blob/assignments/lean/Homework/hw3.lean`

Exercise 1, Part 1. Write a natural-deduction proof of the following WFF: $\varphi_1 \triangleq \neg(p \wedge q \wedge r) \to (\neg p \vee \neg q \vee \neg r)$, de Morgan's law (4).

1	$\neg(p \wedge q \wedge r)$	assume
2	$\neg(\neg p \vee \neg q \vee \neg r)$	assume
3	$\neg r$	assume
4	$\neg p \vee \neg q \vee \neg r$	$\vee\mathrm{i}_2$ 3
5	$\bot$	$\bot\mathrm{i}$ $4, 2$
6	$\neg\neg r$	$\neg\mathrm{i}$ $3 : 5$
7	r	$\neg\neg\mathrm{e}$ 6
8	$\neg p \vee \neg q$	assume
9	$\neg p \vee \neg q \vee \neg r$	$\vee\mathrm{i}_1$ 8
10	$\bot$	$\bot\mathrm{i}$ $2, 9$
11	$\neg(\neg p \vee \neg q)$	$\neg\mathrm{i}$ $8 : 10$
12	$\neg q$	assume
13	$\neg p \vee \neg q$	$\vee\mathrm{i}_2$ 12
14	$\bot$	$\bot\mathrm{i}$ $2, 12$
15	$\neg\neg q$	$\neg\mathrm{i}$ $12 : 14$
16	q	$\neg\neg\mathrm{e}$ 15
17	$\neg p$	assume
18	$\neg p \vee \neg q$	$\vee\mathrm{i}_1$ 17
19	$\bot$	$\bot\mathrm{i}$ $2, 18$
20	$\neg\neg p$	$\neg\mathrm{i}$ $17 : 19$
21	p	$\neg\neg\mathrm{e}$ 19
22	$p \wedge q$	$\wedge\mathrm{i}$ $16, 21$
23	$p \wedge q \wedge r$	$\wedge\mathrm{i}$ $22, 7$
24	$\bot$	$\bot\mathrm{i}$ $1, 23$
25	$\neg\neg(\neg p \vee \neg q \vee \neg r)$	$\neg\mathrm{i}$ $2 : 24$
26	$\neg p \vee \neg q \vee \neg r$	$\neg\neg\mathrm{e}$ 25
27	$\neg(p \wedge q \wedge r) \to (\neg p \vee \neg q \vee \neg r)$	$\to\mathrm{i}$ $1 : 26$

Exercise 1, Part 2. Write a natural-deduction proof of the most generalized form of de Morgan's law (4),

$$\varphi_2 \triangleq \neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n).$$

1	$\neg\,(p_1 \wedge \cdots \wedge p_n)$	premise
2	$\neg(\neg p_1 \vee \cdots \vee \neg p_n)$	assume
3	$\neg p_n$	assume
4	$(\neg p_1 \vee \dots) \vee \neg p_n$	$\vee\mathrm{i}_2$ 3
5	$\bot$	$\bot\mathrm{i}$ 2, 4
6	$\neg\neg p_n$	$\neg\mathrm{i}$ 3 : 5
7	p_n	$\neg\neg\mathrm{e}$ 6
8	$\neg p_1 \vee \cdots \vee \neg p_{n-1}$	assume
9	$\neg p_1 \vee \cdots \vee \neg p_{n-1} \vee \neg p_n$	$\vee\mathrm{i}_1$ 8
10	$\bot$	$\bot\mathrm{i}$ 2, 9
11	$\neg(\neg p_1 \vee \cdots \vee \neg p_{n-1})$	$\neg\mathrm{i}$ 8 : 10
12	$\neg p_{n-1}$	assume
13	$(\neg p_1 \vee \cdots \vee \neg p_{n-2}) \vee p_{n-1}$	$\vee\mathrm{i}_2$ 12
14	$\bot$	$\bot\mathrm{i}$ 11, 13
15	$\neg\neg p_{n-1}$	$\neg\mathrm{i}$ 12 : 14
16	p_{n-1}	$\neg\neg\mathrm{e}$ 15
17	$\vdots$	$\vdots$
18	$p_1 \wedge \cdots \wedge p_n$	$\wedge\mathrm{i}$
19	$\bot$	$\bot\mathrm{i}$ 1, 18
20	$\neg\neg(\neg p_1 \vee \cdots \vee \neg p_n)$	$\neg\mathrm{i}$ box
21	$\neg p_1 \vee \cdots \vee \neg p_n$	$\neg\neg\mathrm{e}$ 20
22	$\neg\,(p_1 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n)$	$\rightarrow\mathrm{i}$ 1 : 21

Exercise 1, Part 3. Show that there is a natural-deduction proof of the generalized de Morgan's law (4) whose length (defined as the number of lines in the proof) is $O(n)$.

We use the formal proof in Exercise 1, Part 2 as the template for constructing a proof in $O(n)$ steps.

Note that, as shown in steps 3-11, we can perform a sequence of conclusions in 9 steps which strips the last p_i from the $\vee$ sequence and reduces the length of the $\vee$ sequence by 1 proposition. We conclude p_i as well as $p_1 \vee \cdots \vee p_{i-1}$, and repeat. Each of these operations is in a constant number of lines, repeated n times, once for each p. Our total steps is $O(8 \times n) = O(n)$.

Then, we need n additional steps to stitch the various p_i together into the conclusion $p_1 \wedge \cdots \wedge p_n$, contradicting the premise.

This proof is overall $O(n)$ steps. Thus, by construction, we build a proof of $O(n)$ steps.

Exercise 1, Part 4. Compare the complexity of a natural-deduction proof of φ_2 and the complexity of a truth-table verification of φ_2.

The natural-deduction proof of φ_2 can be done in a linear number of steps $O(n)$, as demonstrated by construction. The truth-table approach, on the other hand, requires a truth table with 2^n rows for each combination of the n binary-value inputs. As a result, this is a problem for which the natural-deduction approach is significantly easier.

Exercise 2, Part 1. Use the tableaux method to show the validity of the following more general version of de Morgan's law (4):

$$\varphi_1 \triangleq \neg(p \wedge q \wedge r) \rightarrow (\neg p \vee \neg q \vee \neg r)$$

$$\begin{array}{c}
\neg(\neg(p \wedge q \wedge r) \rightarrow (\neg p \vee \neg q \vee \neg r)) \\
| \\
\neg(p \wedge q \wedge r) \\
\neg(\neg p \vee \neg q \vee \neg r) \\
| \\
\neg(p \wedge q \wedge r) \\
\neg(\neg p \vee \neg q) \\
\neg\neg r \\
| \\
\neg(p \wedge q \wedge r) \\
\neg\neg p \\
\neg\neg q \\
\neg\neg r \\
| \\
\neg(p \wedge q \wedge r) \\
p \\
\neg\neg q \\
\neg\neg r \\
| \\
\neg(p \wedge q \wedge r) \\
p \\
q \\
\neg\neg r \\
| \\
\neg(p \wedge q \wedge r) \\
p \\
q \\
r \\
| \\
\neg(p \wedge q \wedge r) \\
p \\
q \\
r
\end{array}$$

$$\begin{array}{ccc}
\neg(p \wedge q) & & \neg r \\
\begin{array}{cc} \neg p & \neg q \\ \times & \times \end{array} & & \times
\end{array}$$

Exercise 2, Part 2. Use the tableaux method to show the validity of de Morgan's law (4) in general:

$$\varphi_2 \triangleq \neg(p_1 \wedge \cdots \wedge p_2) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n)$$

$$\neg(\neg(p_1 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n))$$

$$\neg(p_1 \wedge \cdots \wedge p_n)$$
$$\neg(\neg p_1 \vee \cdots \vee \neg p_n)$$

Repeat n times:

$$\neg(p_1 \wedge \cdots \wedge p_n)$$
$$\neg(\neg p_1 \vee \cdots \vee \neg p_{n-1})$$
$$\neg\neg p_n$$

$$\neg(p_1 \wedge \cdots \wedge p_n)$$
$$\neg\neg p_n$$
$$\vdots$$
$$\neg\neg p_1$$

Simplify n times:

$$\neg(p_1 \wedge \cdots \wedge p_n)$$
$$p_n$$
$$\vdots$$
$$p_1$$

$$\neg(p_1 \wedge \cdots \wedge p_{n-1}) \qquad \neg p_n \quad \times$$

$$\neg(p_1 \wedge \cdots \wedge p_{n-2}) \qquad \neg p_{n-1} \quad \times$$

$$\vdots$$
$$\times$$

Exercise 2, Part 3. Compute the precise size of the tableau in Part 2 above, as a function of n.

We can enumerate the nodes as follows:

The initial node is the negation of the implication, 1.

The next node is the one below it, splitting the implication into two parts, for 2 nodes.

Then, we need to add n additional nodes to get the $\neg\neg p_i$ statements, so $n + 2$ nodes so far.

We then need to cancel each of the $\neg\neg$ nodes, which agains requires n more nodes. That brings the total to $2n + 2$.

That leaves us in a state with $\neg(p_1 \wedge \cdots \wedge p_n)$ and the additional n statements p_i.

Each simplification from here produces 2 nodes, a left and right branch, which immediately close by contradicting one of the p_i statements. That is an additional $2n$ nodes, for a total of $4n + 2$.

Lastly, note that the final cancellation is actually $\neg(p_1 \wedge p_2)$, which splits into two branches which immediately close. Thus, there actually is no node with $\neg p_1$ on the right, so we counted 2 too many nodes.

The final node count is $4n$.

Exercise 2, Problem 4. Compare the comlexity of the tableau proof for φ_2 in part 2 above with the complexity of the natural deduction proof of φ_2 and the truth-table verification of φ_2.

As established, both the natural-deduction proof and the tableau proof have a linear size complexity (number of steps and nubmer of nodes, respectively). This seems to line up with the intuitive expectation that these two methods are roughly equivalent.

In both cases, they are more efficient than the truth table, which requires 2^n rows.