

Homework 11

Zachary Moring

November 21, 2024

Code for the Lean portion is here: <https://github.com/zpm-bu/cs511-formal-methods/blob/assignments/lean/Homework/hw12.lean>

Exercise 2.4.12(a) Find a model which does not satisfy the following formula, or prove that it is valid:

$$(\forall x \forall y (S(x, y) \rightarrow S(y, x))) \rightarrow (\forall x \neg S(x, x))$$

Any equivalence relation will be a model which does not satisfy the formula. In plain language, the formula reads "If S is symmetric then S is not reflexive," yet *every* equivalence relation has both of these properties.

Exercise 2.4.12(b) Find a model which does not satisfy the following formula, or prove that it is valid:

$$\exists y(\forall x P(x)) \rightarrow P(y)$$

1	$\exists y.\forall x.P(x)$	premise
2	$y_0.\forall x.P(x)$	$\exists yi$ 1
3	$P(y_0)$	$\forall xe$ 2
4	$\forall x.P(x) \rightarrow P(y_0)$	$\rightarrow i$ 2 : 3
5	$P(y)$	$\exists ye$ 2 : 4

Exercise 2.4.12(c) Find a model which does not satisfy the following formula, or prove that it is valid:

$$(\forall x(P(x) \rightarrow \exists y Q(y))) \rightarrow (\forall x \exists y(P(x) \rightarrow Q(y)))$$

1	$\forall x.P(x) \rightarrow \exists y Q(y)$	premise
2	x_0	arbitrary
3	$P(x_0) \rightarrow \exists y Q(y)$	$\forall xe$ 1
4	$P(x_0)$	assume
5	$\exists y.Q(y)$	$\rightarrow e$ 3
6	$y_0.Q(y_0)$	$\exists yi$ 4
7	$P(x_0) \rightarrow Q(y_0)$	$\rightarrow i$ 2 : 6
8	$\exists y.P(x_0) \rightarrow Q(y)$	$\exists ye$ 7
9	$\forall x.\exists y.P(x) \rightarrow Q(y)$	$\forall xi$ 2 : 8

Exercise 2.4.12(d) Find a model which does not satisfy the following formula or prove that it is valid:

$$(\forall x \exists y (P(x) \rightarrow Q(y))) \rightarrow (\forall x (P(x) \rightarrow \exists y Q(y)))$$

1	$\forall x. \exists y. P(x) \rightarrow Q(y)$	premise
2	x_0	arbitrary
3	$\exists y. P(x_0) \rightarrow Q(y)$	$\forall x e 1$
4	$y_0. P(x_0) \rightarrow Q(y_0)$	$\exists y i 3$
5	$P(x_0)$	assume
6	$Q(y_0)$	$\rightarrow e 4, 5$
7	$\exists y. P(x_0) \rightarrow Q(y)$	$\exists y e 6$
8	$P(x_0) \rightarrow \exists y. Q(y)$	$\rightarrow i 5 : 7$
9	$\forall x. P(x) \rightarrow \exists y. Q(y)$	$\forall x i 2 : 8$