Homework 2

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I was not able to complete the code exercises this week.

Exercise 1. Use structural induction to prove the result in the slide.

For natural numbers, I cannot see the difference between structural induction and regular induction. To try to modify this, we can frame it as a recurrence relation.

Let s_1 be 1 and $s_n = s_{n-1} + \frac{1}{n^2}$. Then $s_n \leq 2 - \frac{1}{n}$.

For the base case, $(s_1 = 1) \leq 2 - 1$, so the proposition holds.

Assume that the proposition is true for s_{k-1} and consider s_k. By definition,

$$s_k = s_{k-1} + \frac{1}{k^2} \leq 2 - \frac{1}{(k-1)^2} + \frac{1}{k^2}$$

which we manipulate as:

$$\begin{aligned} s_k &= 2 - \frac{k^2}{k(k-1)^2} + \frac{(k-1)^2}{k(k-1)^2} \\ &= 2 - \frac{k^2 + (k-1)^2}{k(k-1)^2} \end{aligned}$$

Now we need to show that $s_k \leq 2 - \frac{1}{k^2}$, which we can rewrite with the denominator as

$$s_k = 2 - \frac{k^2 + (k-1)^2}{k(k-1)^2} \leq 2 - \frac{(k-1)^2}{k(k-1)^2}$$

which, by manipulation, is true if $k^2 + (k-1)^2 > (k-1)^2$, which it is.

Thus $s_k \leq 2 - \frac{1}{k^2}$.

Therefore, by structural induction on the recursion, $s_n \leq 2 - \frac{1}{n^2}$ for all $n \geq 1$.

Therefore, $s_n < 2$ for all $n \geq 1$.

[LCS] Exercise 1.4.15. Use induction on n to prove the theorem

$$(\varphi_1 \wedge (\varphi_2 \wedge (\cdots \wedge \varphi_n)\cdots) \to \psi) \to (\varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\ldots)))$$

For the base case, consider $n = 1$ and thus $(\varphi_1 \to \psi) \to (\varphi_1 \to \psi)$, which is trivially true. Thus, the theorem holds for the base case.

Now, let the inductive hypothesis IH be that

$$(\varphi_1 \wedge (\varphi_2 \wedge (\cdots \wedge \varphi_k)\cdots) \to \chi) \to (\varphi_1 \to (\varphi_2 \to (\cdots(\varphi_k \to \chi)\cdots))).$$

Thus:

1	$(\varphi_1 \wedge (\cdots \wedge (\varphi_k \wedge \varphi_{k+1})\cdots)) \to \psi$	premise
2	$(\varphi_1 \wedge (\cdots \wedge \varphi_k)\cdots)$	assume
3	φ_{k+1}	assume
4	$(\varphi_1 \wedge (\cdots \wedge (\phi_k \wedge \phi_{k+1})))$	$\wedge$i 2 3
5	ψ	$\to$e 4 1
6	$\varphi_{k+1} \to \psi$	$\to$i 3 : 5
7	$(\varphi_1 \to (\cdots(\varphi_k \to (\varphi_{k+1} \to \psi))))$	$\to$e 6 IH $(\chi = 6)$

Therefore, by induction, the theorem holds.

Problem 1. Show that any of the three rules LEM, PBC, and $\neg\neg$E are interderivable.

Using the proof in the book on page 25, we know that PBC can be derived from $\neg\neg$E.

Next, we demonstrate that LEM can be derived from PBC. No premises are needed.

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee$i$_1$ 2
4	$\bot$	$\neg$e 1 3
5	$\neg p$	$\neg$i 2 : 4
6	$p \vee \neg p$	$\vee$i$_2$ 5
7	$\bot$	$\neg$e 1 6
8	$\neg\neg(p \vee \neg p)$	PBC 1 : 7
9	$p \vee \neg p$	$\neg\neg$e 8

Lastly, we demonstrate that $\neg\neg$E is derivable from LEM.

1	$\neg\neg p$	premise
2	$p \vee \neg p$	LEM 1
3	$\neg p$	assumption
4	$\bot$	$\neg$e 1 3
5	p	$\bot$i 4
6	p	assumption
7	p	$\vee$e 3 : 6

Therefore, LEM, PBC, and $\neg\neg$E are interderivable.