Homework 12

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Code for the Lean portion is here: `https://github.com/zpm-bu/cs511-formal-methods/blob/assignments/lean/Homework/hw12.lean`

Exercise 1. Define $X \sim Y$ in second-order logic using a unary function $F : XY$ which is both injective and surjective.

$$\begin{aligned} X \sim Y \longleftrightarrow & \exists F(\forall x \in X(F(x) \in Y) \\ & \wedge \forall x_1, x_2 \in X(F(x_1) \approx F(x_2) \to x_1 \approx x_2) \\ & \wedge \forall y \in Y(\exists x \in X(F(x) \approx y))) \end{aligned}$$

Exercise 2. A set Y is "countably infinite" if Y is infinite and for every infinite subset X of Y, there is a bijection from X to Y.

(a) Define a second-order sentence $\Psi_{\text{countably inft}}$ such that $\mathcal{A} \models \Psi_{\text{countably inft}}$ if and only if $\mathcal{A}$ is countably infinite.

(b) Define a second-order sentence $\Psi_{\text{uncountably inft}}$ such that $\mathcal{A} \models \Psi_{\text{uncountably inft}}$ if and only if $\mathcal{A}$ is uncountably infinite.

Consider

$$\begin{aligned}\Psi_{\text{countably inft}}(Y) \stackrel{\text{def}}{=} & \Psi_{\text{infinite}}(Y) \\ & \wedge \forall S \subseteq Y(\Psi_{\text{infinite}}(S) \rightarrow \\ & \exists f : S \rightarrow Y(\forall s_1, s_2(f(s_1) \approx f(s_2) \rightarrow s_1 \approx s_2) \\ & \forall y \in Y(\exists s \in S(f(s) \approx y))))\end{aligned}$$

and

$$\Psi_{\text{uncountably inft}}(Y) \stackrel{\text{def}}{=} \Psi_{\text{infinite}}(Y) \wedge \neg\Psi_{\text{countably inft}}(Y)$$

Since countable infinity is the "smallest" infinite cardinality, any infinite set which is not countable is, by definition, uncountably infinite.