Homework 8

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Code for the Lean portion is here: `https://github.com/zpm-bu/cs511-formal-methods/blob/assignments/lean/Homework/hw8.lean`

Exercise 2.2.3a. Which of the following strings are formulae in predicate logic? Specify a reason for failure for strings which aren't; draw parse trees for all strings which are.

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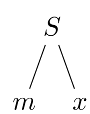
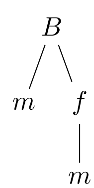
i. Yes.



ii. Yes.



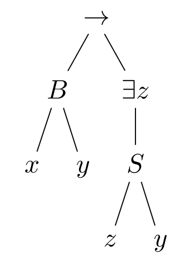
iii. $f(m)$ is not a predicate; it is a function applied to a constant, also producing a constant.

iv. $B(B(m,x),y)$ is not a formula. Our definition in the book for a formula requires that a predicate apply to *terms*, and $B(m,x)$ is not a *term*, it is a predicate.

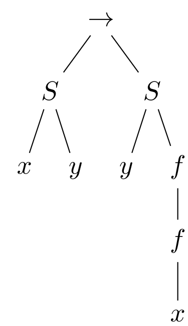
I don't actually UNDERSTAND the difference very well, but the definition 2.3. is pretty clear about this restriction. No nesting predicates.

v. $S(B(m),z)$ is not a formula. S and B are both predicates and we cannot nest predicates inside each other.

vi. Yes.



vii. Yes.



viii. No. You cannot nest predicates.

Exercise 2.2.3b. Let c and d be constants, f a function symbol with one argument, g a function with two arguments, and h a function symbol with three arguments. Also let P and Q be predicate symbols with three arguments. Which of the following are formulae? Draw a parse tree if so. If not, describe why.

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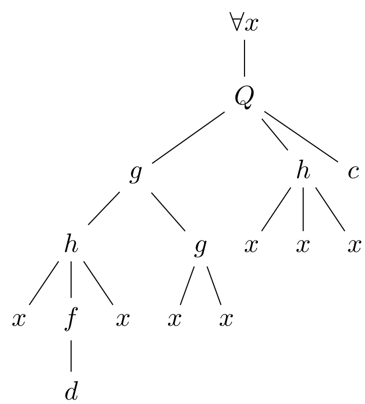
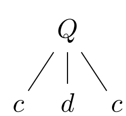
i. No. There is an unbound and undefined y as the third argument for h.
ii. No. As above, there is an unbound and undefined y.
iii. Yes.



iv. No. P requires three arguments, so the statement is not well-formed.
v. No, I don't think so. The left hand of the implication is $g(x, y)$, which is not a predicate – it is a formula evaluation. Thus, there is not a way to assign a 'truth value' to the left hand side and the implication is not well-formed.
vi. Yes.



Exercise 2.3.2 Recall that we use $\approx$ to express the equality of elements in our models. Consider the formula

$$\exists x \exists y\, (\neg(x \approx y) \wedge (\forall z((z \approx x) \vee (z \approx y))))$$

In plain English, what does this formula specify?

In this universe, there are two elements x and y which are not related to each other. Every other element is related to x or to y. Thus, the space is partitioned into two parts: The x part and the y part. Under $\approx$, we can say that the set only has two elements.

Exercise 2.3.3. Try to write down a sentence of predicate logic which intuitively holds in a model if and only if the model has (respectively):

(a) Exactly three distinct elements.

(b) At most three distinct elements.

(c) At least three distinct elements.

(a) Consider

$$\begin{aligned}\exists x \exists y \exists z(&\neg(x \approx y) \wedge \neg(x \approx z) \wedge \neg(y \approx z)\\ &\wedge \forall w((w \approx x) \vee (w \approx y) \vee (w \approx z)))\end{aligned}$$

(b) Here's what I thought of: There are *not* four elements.

$$\begin{aligned}\neg\exists w \exists x \exists y \exists z(\neg(w \approx x)&\\ \neg(w \approx y)&\\ \neg(w \approx z)&\\ \neg(x \approx y)&\\ \neg(x \approx z)&\\ \neg(y \approx z))&\end{aligned}$$

(c) At *least* three elements should be the same as part (a), but without the universal binding.

$$\exists x \exists y \exists z(\neg(x \approx y) \wedge \neg(x \approx z) \wedge \neg(y \approx z))$$