Homework 10

Zachary Moring

November 13, 2024

Code for the Lean portion is here: `https://github.com/zpm-bu/cs511-formal-methods/blob/assignments/lean/Homework/hw10.lean`

Exercise 1. If $\mathcal{M}$ is a relational structure, the first-order theory of $\mathcal{M}$ is:

$$\mathrm{Th}(\mathcal{M}) \stackrel{\text{def}}{=} \{\varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{M} \vDash \varphi\}$$

Is $\mathrm{Th}(\mathcal{M})$ deductively closed?

Because we are dealing with first-order sentences, completeness tells us that $\mathcal{M} \vDash \varphi$ is equivalent to $\mathcal{M} \vdash \varphi$. Thus, every statement $\varphi \in \mathrm{Th}(\mathcal{M})$ is also in $\overline{\mathcal{M}}$.

The same is true *mutatis mutandis* to show that every statement $\psi \in \overline{\mathcal{M}}$ is also an element of $\mathrm{Th}(\mathcal{M})$.

Thus by double containment, $\mathrm{Th}(\mathcal{M})$ is deductively closed.

Exercise 2.1. Write a first-order sentence φ_1 which, in any Σ' structure $\mathcal{M}$ satisfying Γ, asserts "every vertex has at least one of the colors: blue, green, purple, or yellow."

$$\varphi_1 \overset{\text{def}}{=} \forall v.\ B(v) \vee G(v) \vee P(v) \vee Y(v)$$

Exercise 2.2. Write a first-order sentence φ_2 which asserts "every vertex has at most one color."

This is a little brute-force, but it seems like the most straightforward way to handle it:

$$\begin{aligned}\varphi_2 \overset{\text{def}}{=} \forall v.\ &\neg(B(v) \wedge G(v)) \ \wedge\ \neg(B(v) \wedge P(v)) \ \wedge\ \neg(B(v) \wedge Y(v)) \ \wedge \\ &\neg(G(v) \wedge P(v)) \ \wedge\ \neg(G(v) \wedge Y(v)) \ \wedge\ \neg(P(v) \wedge Y(v))\end{aligned}$$

Exercise 2.3. Write a first-order sentence φ_3 which asserts "no two adjacent vertices have the same color."

Again, the most straightforward rule is kind of a brute-force approach, but it definitely *works*:

$$\begin{aligned}\varphi_3 \overset{\text{def}}{=} \forall u \forall v.\ R(u,v) \to &\neg(B(u) \wedge B(v)) \ \wedge\ \neg(G(u) \wedge G(v)) \ \wedge \\ &\neg(P(u) \wedge P(v)) \ \wedge\ \neg(Y(u) \wedge Y(v))\end{aligned}$$

Exercise 2.4. Show that if $\mathcal{M}$ is an infinite planar graph then there is a Σ-structure $\mathcal{M}\prime$ which expands $\mathcal{M}$ with four unary relations $B^{\mathcal{M}\prime}$, $G^{\mathcal{M}\prime}$, $P^{\mathcal{M}\prime}$, $Y^{\mathcal{M}\prime}$ and which satisfies $\varphi_1 \wedge \varphi_2 \wedge varphi_3$; that is, $\mathcal{M}\prime$ is four-colorable and thus $\mathcal{M}$ is also four-colorable.

In Chapter 4, we are given the fact that every finite planar graph is 4-colorable. We will rely on that fact here.

Consider a finite subset of $\mathcal{M}\prime$, the finite graph M. Because $\mathcal{M}\prime$ is planar, M is also planar. So M is a finite planar graph and thus M is four-colorable. Thus, $M \vDash \varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

Therefore since M is an arbitrary finite submodel of $\mathcal{M}\prime$, by compactness, there exists a finite model of a four-coloring for $\mathcal{M}\prime$.