Stochastic Processes in Life Insurance
Assignment 2

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1 Introduction

This is the second of to two assignments in the course Stochastic Processes in Life Insurance at Copenhagen University. The course professor is Jesper Lund Pedersen. The course is held in blok 1, 2019.

2 Problem 1

2.1 Question 1.A

$$m^2(t) = E[Y^2(0,n)|F(t)] \tag{1}$$

By 14.4, we have that martingale is on the following form. We know, that it's continuous, and therefore integrable thus a martingale.

$$m(t) = \int_0^t e^{-\int_0^s r(v)dv} dB(s) + e^{-\int_0^t r(v)dv} V(t) \tag{2}$$

Use that $(a+b)^2 = 2ab + a^2 + b^2$ and $a^m * a^n = a^{m+n}$

$$\begin{aligned}
m^2(t) = &(\int_0^t e^{-\int_0^s r(v)dv} dB(s) + e^{-\int_0^t r(v)dv} V(t))^2 \\
= & \\
& 2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\
& + (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\
& + (e^{-\int_0^t r(v)dv} V(t))^2 \\
= & \\
& 2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\
& + (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\
& + e^{-\int_0^t r(v)dv} e^{-\int_0^t r(v)dv} (V(t)V(t)) \\
= & \\
& 2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\
& + (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\
& + e^{-2\int_0^t r(v)dv} V^2(t)
\end{aligned} \tag{3}$$

2.2 Question 1.B

$$\begin{aligned}
m^{(2)}(t) = &2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\
& + (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\
& + e^{-2\int_0^t r(v)dv} V^2(t)
\end{aligned} \tag{4}$$

Solve the equations with integration by parts. Use that $\int_0^t e^{-\int_0^s} a(s)$ can be written as $e^{-\int_0^t} \int_0^t a(s)$, $e^{-\int_0^t} (\int_0^{t-} a(s) + \int_{t-}^t a(s))$ and $\int_{t-}^t a(s) = a(t) - a(t-) =$

$\triangle a(t)$

$$
\begin{aligned}
&d(2\int_0^t e^{-\int_0^s r(v)dv}dB(s)e^{-\int_0^t r(v)dv}V(t)) \\
&= 2e^{-\int_0^t r(v)dv}dB(t)e^{-\int_0^t r(v)dv}V(t) \\
&+2\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s)(-r(t)e^{-\int_0^t r(v)dv}V(t)dt + e^{-\int_0^t r(v)dv}dV(t))
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&d((\int_0^t e^{-\int_0^s r(v)dv}dB(s))^2) \\
&= (\int_0^t e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}dB(t) \\
&+(\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}dB(t) \\
&= (\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}dB(t) \\
&+(e^{-\int_0^t r(v)dv}\int_{t-}^t dB(s))e^{-\int_0^t r(v)dv}dB(t) \\
&+(\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}dB(t) \\
&= \triangle B(t)e^{-2\int_0^t r(v)dv}dB(t) \\
&+2(\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}dB(t)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&d(e^{-2\int_0^t r(v)dv}V^{(2)}(t)) \\
&= -2r(t)e^{-2\int_0^t r(v)dv}V^{(2)}(t) + e^{-2\int_0^t r(v)dv}dV^{(2)}(t)
\end{aligned} \tag{7}
$$

Merging all the equations gets us and change $t-$ to t for the dt part.

$$\begin{aligned}
dm^{(2)}(t) &= 2e^{-\int_0^t r(v)dv} dB(t) e^{-\int_0^t r(v)dv} V(t) \\
+2\int_0^t e^{-\int_0^s r(v)dv} dB(s)(-r(t)e^{-\int_0^t r(v)dv} V(t)dt) \\
+\triangle B(t) e^{-2\int_0^t r(v)dv} dB(t) \\
+2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)) e^{-\int_0^t r(v)dv} dB(t) \\
-2r(t) e^{-2\int_0^t r(v)dv} V^{(2)}(t) \\
+e^{-2\int_0^t r(v)dv} dV^{(2)}(t)
\end{aligned} \tag{8}$$

2.3 Question 1.C

$$
\begin{aligned}
e^{2\int_0^t r(v)dv} dm^2(t) = & \\
& + e^{2\int_0^t r(v)dv} 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} dB(t) \\
& + \triangle B(t) dB(t) \\
& - 2r(t)V^2(t)dt \\
& + dV^2(t) \\
& + 2V(t)dB(t) \\
& - e^{2\int_0^t r(v)dv} 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} r(t)V(t)dt \\
& + e^{2\int_0^t r(v)dv} 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} dV(t) \\
& = \\
& + \triangle B(t) dB(t) \\
& - 2r(t)V^2(t) \\
& + dV^2(t) \\
& + 2V(t)dB(t) \\
& + 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{\int_0^t r(v)dv} (dB(t) + dV(t) - r(t)V(t)dt) \\
& = \\
& + (\triangle B(t) + 2V(t))dB(t) \\
& - 2r(t)V^2(t) \\
& + dV^2(t) \\
& + 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{\int_0^t r(v)dv} (dB(t) + dV(t) - r(t)V(t)dt)
\end{aligned}
\tag{9}
$$

Note that $\triangle B(t) = \sum_k b^{Z(t-)k}(t) \triangle N^k(t)$ and that $\triangle N^k(t) * b^{Z(t)}(t)$ is zero, because N only counts 1, if in state $k \neq j$. Use that $\sum_k b^{Z(t-)k}(t) =$

$\sum_j \sum_{k\neq j} b^{jk}(t)$. Use that $V^{Z(t)} = V^k$ when $dN^k(t)$

$$
\begin{aligned}
(\triangle B(t) + 2V(t))dB(t) = \\
&(\triangle B(t) + 2V^{Z(t)}(t))(b^{Z(t)}(t)dt + \sum_k b^{Z(t-)k}(t)dN^k(t)) \\
&= \\
&\triangle B(t)(b^{Z(t)}(t)dt + \sum_k b^{Z(t-)k}(t)dN^k(t)) \\
&+ (2V^{Z(t)}(t)b^{Z(t)}(t)dt + \sum_k 2V^{Z(t)}(t)b^{Z(t-)k}(t)dN^k(t)) \\
&= \\
&\sum_j 1_{\{Z(t-)=j\}}(2V^j(t)b^j(t)dt + \sum_{k\neq j}(b^{jk}(t) + 2V^k(t))b^{jk}(t)dN^k(t)) \\
&= \\
&2\sum_j 1_{\{Z(t-)=j\}}V^j(t)b^j(t)dt + \sum_k(b^{Z(t-)}(t) + 2V^k(t))b^{Z(t-)}(t)dN^k(t))
\end{aligned} \tag{10}
$$

Insert into the equation before.

$$
\begin{aligned}
e^{2\int_0^t r(v)dv}dm^2(t) = \\
&+ dV^2(t) \\
&- 2r(t)V^2(t) \\
&+ 2\sum_j 1_{\{Z(t-)=j\}}V^j(t)b^j(t)dt + \sum_k(b^{Z(t-)}(t) + 2V^k(t))b^{Z(t-)}(t)dN^k(t)) \\
&+ 2(\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt)
\end{aligned} \tag{11}
$$

2.4 Question 1.E

By proposition 16.4, page. 55.

$$\begin{aligned} V_j^{(2)}(t) &= (2r(t) + \mu(t))V_j^{(2)}(t) - 2b^j(t)V_j(t) - \sum_{k \neq j} \mu^{jk}(t) \sum_{p=0}^{2} \binom{2}{p} (b^{jk(2)}(t) V^{k(2-p)}(t)) \\ &= 2r(t)V_j^{(2)}(t) - 2b^j(t)V_j(t) - \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) + b^{jk}(t)V^k(t) + V^{k(2)}(t) - V_j^{(2)}(t)) \end{aligned} \quad (12)$$

2.5 Question 1.F

$$Var_j(t) = V_j^{(2)}(t) - (V_j(t))^2 \tag{13}$$

$$dV_j(t) = r(t)V_j(t) - b^j(t) - \sum_{k \neq j} \mu^{jk}(t)(b^{jk}(t) + V^k(t) - V^j(t)) \tag{14}$$

$$\begin{aligned} dV_j^{(2)}(t) &= 2r(t)V_j^{(2)}(t) - 2b^j(t)V^j(t) - \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) + 2b^{jk}V^k(t) + V^{k(2)}(t) - V^{j(2)}(t)) \\ &= (2r(t) + \mu(t))V_j^{(2)}(t) - 2b^j(t)V^j(t) - \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) + 2b^{jk}V^k(t) + V^{k(2)}(t)(t)) \end{aligned} \tag{15}$$

$$R^{jk(2)}(t) = b^{jk(2)}(t) + V^{k(2)}(t) - V^{j(2)}(t) + 2b^{jk}V^kV^j - 2V^kV^j \tag{16}$$

$$\begin{aligned} &(-2)V_j(t)dV_j(t) = \\ &\qquad (-2)V_j(t)(r(t)V_j(t) - b^j(t) - \sum_{k \neq j} \mu^{jk}(t)(b^{jk}(t) + V^k(t) - V^j(t))) \\ &\qquad = (-2r(t) - \mu(t))V_{j(2)}(t) + 2b^j(t)V_j(t) \\ &\qquad - \sum_{k \neq j} \mu^{jk}(t)(-2b^{jk}(t)V^j(t) - 2V^k(t)V^j(t)) - V_{j(2)}(t)) \end{aligned} \tag{17}$$

Search and replace variance and $R^{jk(2)}$. See that $V^{k(2)}(t) - (V^j(t))^2$ can be seen as $V^{k(2)}(t) - (V^k(t))^2 = var_k(t)$

$$
\begin{aligned}
dVar_j(t) &= d(V_j^{(2)}(t) - (V_j(t))^2) \\
&= dV_j^{(2)}(t) - 2V_j(t)dV_j(t) \\
&= \\
& 2r(t)(V^{j(2)(t)} - (V_j(t))^2) \\
& - \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) \\
& + V^{k(2)}(t) - V^{j(2)}(t) - 2V^j(t)V^k(t) - 2(V^j(t))^2 + 2b^{jk}(V^k(t) - V^j(t))) \\
&= \\
& (2r(t) + \mu(t))var_j(t) \\
& - \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) \\
& + V^{k(2)}(t) - 2V^j(t)V^k(t) - (V^j(t))^2 + 2b^{jk}(V^k(t) - V^j(t))) \\
&= \\
& (2r(t) + \mu(t))var_j(t) \\
& - \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) \\
& + var_k(t) - 2V^j(t)V^k(t) + 2b^{jk}(V^k(t) - V^j(t))) \\
&= \\
& (2r(t) + \mu(t))var_j(t) - \sum_{k \neq j} \mu^{jk}(t)(R^{jk(2)}(t) + var_k(t)) \\
&= \\
& 2r(t)var_j(t) - \sum_{k \neq j} \mu^{jk}(t)(R^{jk(2)}(t) + var_k(t) - var_j(t))
\end{aligned} \tag{18}
$$

We see, that the equation looks like Thiele and conclude, we just replace var and R^2 with the original v and R

$$var_j(t) = \int_t^n e^{-\int_t^s 2r(v)dv} \sum_k p^{jk}(t,s) \sum_{k \neq l} \mu^{kl}(s)(R^{kl})^2(s)ds \tag{19}$$

3 Problem 2

3.1 Question 2.A

By page 63, we have differential equations for $f^j(t)$ and $g^j(t)$

$$\begin{aligned} df^j(t) &= r(t)f^j(t) - b^j(t) - g^j(t)c_j(t) \\ &- \sum_{k \neq j} \mu^{jk}(t)(bj^{jk}(t) + c^{jk}(t)g^k(t) + f^k(t) - f^j(t)) \end{aligned} \tag{20}$$

Compute f(t) and g(t) in the survival model with two states, alive(0) and dead(1).

$$\begin{aligned} df^0(t) &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) \\ &- \sum_{k \neq j} \mu^{01}(b^{01}(t) + c^{01}(t)g^1(t) + f^1(t) - f^0(t)) \\ c_{01}(t) &= -(b^{01}(t) + V^{1*}(t) - V^{0*}(t)) \end{aligned} \tag{21}$$

$$\begin{aligned} df^0(t) &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) \\ &- \sum_{k \neq j} \mu^{01}(b^{01}(t) + c^{01}(t)g^1(t) + f^1(t) - f^0(t)) \\ &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) \\ &- \mu^{01}(b^{01}(t) + c^{01}(t)g^1(t) + f^1(t) - f^0(t)) \end{aligned} \tag{22}$$

Assume, that the insurance company which issues the policy has no future liabilities within state of death(1). Why $g^1(t) = 0$ and $f^1(t) = 0$.

$$\begin{aligned} df^0(t) &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) - \mu^{01}b^{01}(t) + \mu^{01}f^0(t) \\ &= (r(t) + \mu(t))f^0(t) - b^0(t) - g^0(t)c_0(t) - \mu^{01}(t)b^{01}(t) \end{aligned} \tag{23}$$

$$f^0(t) = \int_t^n e^{-\int_t^s (r(v)+\mu(v))dv}(b^0(s) + g^0(s)c_0(s) + \mu^{01}(s)b^{01}(s))ds \tag{24}$$

Where $f^0(n) = 0$.

$$\begin{aligned}dg^j(t) = (q^j(t) + \sum_{k\neq j} \mu^{jk}(t)q^{jk}(t))g^j(t) - q^j(t) \\ - \sum_{k\neq j}(1 - q^{jk}(t))\mu^{jk}(t)(\frac{q^{jk}(t)}{1 - q^{jk}(t)} + g^k(t) - g^j(t))\end{aligned} \tag{25}$$

We know, that transitions $q^{01}(t) = 0$ for all t and $g^1(t) = 0$ for all t

$$\begin{aligned}dg^0(t) &= (q^0(t) + \mu^{01}(t)q^{01}(t))g^0(t) - q^0(t) \\ &- (1 - q^{01}(t))\mu^{01}(t)(\frac{q^{01}(t)}{1 - q^{01}(t)} + g^1(t) - g^0(t)) \\ &= (q^0(t) + \mu^{01}(t)q^{01}(t) + (1 - q^{01}(t))\mu^{01}(t))g^0(t) - q^0(t) - \mu^{01}(t)q^{01}(t) \\ &= (q^0(t) + \mu^{01}(t))\mu^{01}(t))g^0(t) - q^0(t) - \mu^{01}(t)q^{01}(t)\end{aligned} \tag{26}$$

$$g^0(t) = \int_t^n e^{-\int_t^s (q^0(v)+\mu^{01}(v))dv} q^0(s)ds \tag{27}$$

Where $q^j(t)$ $0 <= q^{jk}(t) < 1$

4 Appendix

4.1 Appendix - Question 2.A

4.2 Question 2.A

By page 60. Contribution function is on the following form. And by page 63, we have differential equations for $f^j(t)$ and $g^j(t)$

$$\begin{aligned} c_j(t) &= (r(t) - r^*(t))V_j^*(t) + \sum_{k \neq j}(b^{jk}(t) + V^{*k}(t) - V^{j*}(t)) \\ c_{jk}(t) &= -(b^{jk}(t) + V^{k*}(t) - V^{j*}(t)) \\ df^j(t) &= r(t)f^j(t) - b^j(t) - g^j(t)c_j(t) \\ &\quad - \sum_{k \neq j} \mu^{jk}(t)(bj^{jk}(t) + c^{jk}(t)g^k(t) + f^k(t) - f^j(t)) \end{aligned} \tag{28}$$

4.3 Appendix - Question 1.B

Use integration by parts, and use, that $\triangle m(t) = m(t) - m(t-)$, and $m(t) = \triangle m(t) + m(t-)$

$$\begin{aligned} dm(t) &= e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt) \\ dm^2(t) &= d(m(t)m(t)) \\ &= dm(t)m(t) + m(t-)dm(t) \\ &= dm(t)(m(t) + m(t-)) \\ &= dm(t)(\triangle m(t) + 2m(t-)) \end{aligned} \tag{29}$$

Split $m(t)$ in $B^{"}(t) = \int_0^t e^{-\int_0^s r(v)dv}dB(s)$ and $V^{"}(t) = e^{-\int_0^t r(v)dv}V(t)$. Now solve $dm(t)\triangle m(t)$ and $dm(t)2B^{"}(t-)$ and $dm(t)2V^{"}(t-)$, where $dm(t)$

$$dm(t)2B^{"}(t-) = e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt)2\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s) \tag{30}$$

See that $dm^2(t) = 2\int_0^t m(s)dm(s)$ and $dV^2(t) = 2V(t)dV(t)$

$$
\begin{aligned}
dm(t)2V^{"}(t-) &= e^{-\int_0^t r(v)dv}(dB(t)+dV(t)-r(t)V(t))2e^{-\int_0^t r(v)dv}V(t) \\
&= 2e^{-2\int_0^t r(v)dv}V(t)(dB(t)+dV(t)-r(t)V(t)dt) \\
&= 2e^{-2\int_0^t r(v)dv}(V(t)dB(t)-r(t)V^2(t))+e^{-2\int_0^t r(v)dv}dV^2(t)
\end{aligned} \tag{31}
$$

See that $V^{"}(t)-V^{"}(t-)=0$ And see that $B^{"}(t)-B^{"}(t-)=\int_0^t e^{-\int_0^s r(v)dv}dB(s)-dB(s-)=e^{-\int_0^t r(v)dv}\int_0^t dB(t)-dB(t-)=e^{-\int_0^t r(v)dv}\triangle B(t)$ And no jump payments in $dV(t)$ or $V(t)$

$$
\begin{aligned}
dm(t)\triangle m(t) &= (e^{-\int_0^t r(v)dv}(dB(t)+dV(t)-r(t)V(t)dt))(B^{"}(t)+V^{"}(t)-B^{"}(t-)-V^{"}(t-)) \\
&= (e^{-\int_0^t r(v)dv}(dB(t))e^{-\int_0^t r(v)dv}\triangle B(t) \\
&= e^{-2\int_0^t r(v)dv}dB(t)\triangle B(t)
\end{aligned} \tag{32}
$$

And see that $r(t)V(t)dt$ is continuous therefore we can exchange $t-$ with t.

$$
\begin{aligned}
dm^2(t) = \\
&= dm(t)B^{"}(t) + dm(t)V^{"}(t) + dm(t)\triangle m(t) \\
&e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt)2\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s) \\
&+ 2e^{-2\int_0^t r(v)dv}(V(t)dB(t) - r(t)V^2(t)) + e^{-2\int_0^t r(v)dv}dV^2(t) \\
&+ e^{-2\int_0^t r(v)dv}dB(t)\triangle B(t) \\
&= \\
&2(\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}dB(t) \\
&+ e^{-2\int_0^t r(v)dv}\triangle B(t)dB(t) \\
&- 2e^{-2\int_0^t r(v)dv}r(t)V^2(t) \\
&+ e^{-2\int_0^t r(v)dv}dV^2(t) \\
&+ 2e^{-2\int_0^t r(v)dv}V(t)dB(t) \\
&- 2(\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}r(t)V(t)dt \\
&+ 2(\int_0^{t-} e^{-\int_0^s r(v)dv}dB(s))e^{-\int_0^t r(v)dv}dV(t)
\end{aligned}
\tag{33}
$$

4.4 Appendix - Question 1.C

$$
\begin{aligned}
dm^2(t) =& \\
&+ 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} dB(t) \\
&+ e^{-2\int_0^t r(v)dv} \triangle B(t) dB(t) \\
&- 2e^{-2\int_0^t r(v)dv} r(t) V^2(t) dt \\
&+ e^{-2\int_0^t r(v)dv} dV^2(t) \\
&+ 2e^{-2\int_0^t r(v)dv} V(t) dB(t) \\
&- 2\left(\int_0^{t} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} r(t) V(t) \\
&+ 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} dV(t)
\end{aligned}
\tag{34}
$$

References

[1] Jesper Lund Pedersen. *Stochastic Processes in Life Insurance: The Dynamic Approach*. Department of Mathematical Sciences.