Stochastic Processes in Life Insurance
Assignment 2

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October 9, 2019

1 Introduction

This is the second of to two assignments in the course Stochastic Processes in Life Insurance at Copenhagen University. The course professor is Jesper Lund Pedersen. The course is held in blok 1, 2019.

2 Problem 1

2.1 Question 1.A

$$m^2(t) = E[Y^2(0,n)|F(t)] \tag{1}$$

By 14.4, we have that martingale is on the following form.

$$m(t) = \int_0^t e^{-\int_0^s r(v)dv} dB(s) + e^{-\int_0^t r(v)dv} V(t) \tag{2}$$

Use that $(a+b)^2 = 2ab + a^2 + b^2$ and $a^m * a^n = a^{m+n}$

$$
\begin{aligned}
m^2(t) &= (\int_0^t e^{-\int_0^s r(v)dv} dB(s) + e^{-\int_0^t r(v)dv} V(t))^2 \\
&= \\
&2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\
&+ (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\
&+ (e^{-\int_0^t r(v)dv} V(t))^2 \\
&= \\
&2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\
&+ (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\
&+ e^{-\int_0^t r(v)dv} e^{-\int_0^t r(v)dv} (V(t)V(t)) \\
&= \\
&2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\
&+ (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\
&+ e^{-2\int_0^t r(v)dv} V^2(t)
\end{aligned} \tag{3}
$$

2.2 Question 1.B

$$\begin{aligned} m^{(2)}(t) =& 2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t) \\ & + (\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2 \\ & + e^{-2\int_0^t r(v)dv} V^2(t) \end{aligned} \tag{4}$$

$$\begin{aligned} & d(2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} V(t)) \\ = 2\int_0^t e^{-\int_0^s r(v)dv} dB(s)(-r(t)e^{-\int_0^t r(v)dv} V(t) + e^{-\int_0^t r(v)dv} dV(t) \\ & +2e^{-\int_0^t r(v)dv} dB(t) e^{-\int_0^t r(v)dv} V(t) \end{aligned} \tag{5}$$

$$\begin{aligned} & d((\int_0^t e^{-\int_0^s r(v)dv} dB(s))^2) \\ = & (2\int_0^t e^{-\int_0^s r(v)dv} dB(s)) e^{-\int_0^t r(v)dv} dB(t) \end{aligned} \tag{6}$$

$$\begin{aligned} & d(e^{-2\int_0^t r(v)dv} V^{(2)}(t)) \\ = & -2r(t)e^{-2\int_0^t r(v)dv} V^{(2)}(t) + e^{-2\int_0^t r(v)dv} dV^{(2)}(t) \end{aligned} \tag{7}$$

$$\begin{aligned} dm^{(2)}(t) = & 2\int_0^t e^{-\int_0^s r(v)dv} dB(s)(-r(t)e^{-\int_0^t r(v)dv} V(t)) \\ & +2\int_0^t e^{-\int_0^s r(v)dv} dB(s) e^{-\int_0^t r(v)dv} dV(t) \\ & +2e^{-\int_0^t r(v)dv} dB(t) e^{-\int_0^t r(v)dv} V(t) \\ & +(2\int_0^t e^{-\int_0^s r(v)dv} dB(s)) e^{-\int_0^t r(v)dv} dB(t) \\ & -2r(t)e^{-2\int_0^t r(v)dv} V^{(2)}(t) \\ & +e^{-2\int_0^t r(v)dv} dV^{(2)}(t) \end{aligned} \tag{8}$$

2.3 Question 1.C

$$
\begin{aligned}
e^{2\int_0^t r(v)dv} dm^2(t) = \\
&+ e^{2\int_0^t r(v)dv} 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)) e^{-\int_0^t r(v)dv} dB(t) \\
&+ \triangle B(t) dB(t) \\
&- 2r(t)V^2(t)dt \\
&+ dV^2(t) \\
&+ 2V(t)dB(t) \\
&- e^{2\int_0^t r(v)dv} 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)) e^{-\int_0^t r(v)dv} r(t)V(t)dt \\
&+ e^{2\int_0^t r(v)dv} 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)) e^{-\int_0^t r(v)dv} dV(t) \\
&= \\
&+ \triangle B(t) dB(t) \\
&- 2r(t)V^2(t) \\
&+ dV^2(t) \\
&+ 2V(t)dB(t) \\
&+ 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)) e^{\int_0^t r(v)dv} (dB(t) + dV(t) - r(t)V(t)dt) \\
&= \\
&+ (\triangle B(t) + 2V(t))dB(t) \\
&- 2r(t)V^2(t) \\
&+ dV^2(t) \\
&+ 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)) e^{\int_0^t r(v)dv} (dB(t) + dV(t) - r(t)V(t)dt)
\end{aligned}
\tag{9}
$$

Note that $\triangle B(t) = \sum_k b^{Z(t-)k}(t) \triangle N^k(t)$ and that $\triangle N^k(t) * b^{Z(t)}(t)$ is zero, because N only counts 1, if in state $k \neq j$. Use that $\sum_k b^{Z(t-)k}(t) =$

$\sum_j \sum_{k \neq j} b^{jk}(t)$. Use that $V^{Z(t)} = V^k$ when $dN^k(t)$

$$
\begin{aligned}
(\triangle B(t) + 2V(t))dB(t) = \\
& (\triangle B(t) + 2V^{Z(t)}(t))(b^{Z(t)}(t)dt + \sum_k b^{Z(t-)k}(t)dN^k(t)) \\
& = \\
& \triangle B(t)(b^{Z(t)}(t)dt + \sum_k b^{Z(t-)k}(t)dN^k(t)) \\
& + (2V^{Z(t)}(t)b^{Z(t)}(t)dt + \sum_k 2V^{Z(t)}(t)b^{Z(t-)k}(t)dN^k(t)) \\
& = \\
& \sum_j 1_{\{Z(t-)=j\}}(2V^j(t)b^j(t)dt + \sum_{k \neq j}(b^{jk}(t) + 2V^k(t))b^{jk}(t)dN^k(t)) \\
& = \\
& 2\sum_j 1_{\{Z(t-)=j\}}V^j(t)b^j(t)dt + \sum_k (b^{Z(t-)}(t) + 2V^k(t))b^{Z(t-)}(t)dN^k(t))
\end{aligned} \tag{10}
$$

Insert into the equation before.

$$
\begin{aligned}
e^{2\int_0^t r(v)dv} dm^2(t) = \\
& + dV^2(t) \\
& - 2r(t)V^2(t) \\
& + 2\sum_j 1_{\{Z(t-)=j\}}V^j(t)b^j(t)dt + \sum_k (b^{Z(t-)}(t) + 2V^k(t))b^{Z(t-)}(t)dN^k(t)) \\
& + 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s))e^{\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt)
\end{aligned} \tag{11}
$$

2.4 Question 1.E

By proposition 16.4, page. 55.

$$
\begin{aligned}
V_j^{(2)}(t) &= (2r(t)+\mu(t))V_j^{(2)}(t) - 2b^j(t)V_j(t) - \sum_{k\neq j}\mu^{jk}(t)\sum_{p=0}^{2}\binom{2}{p}(b^{jk(2)}(t)V^{k(2-p)}(t)) \\
&= 2r(t)V_j^{(2)}(t) - 2b^j(t)V_j(t) - \sum_{k\neq j}\mu^{jk}(t)(b^{jk(2)}(t) + b^{jk}(t)V^k(t) + V^{k(2)}(t) - V_j^{(2)}(t))
\end{aligned}
\tag{12}
$$

2.5 Question 1.F

$$Var_j(t) = V_j^{(2)}(t) - (V_j(t))^2 \tag{13}$$

$$dV_j(t) = r(t)V_j(t) - b^j(t) - \sum_{k\neq j} \mu^{jk}(t)(b^{jk}(t) + V^k(t) - V^j(t)) \tag{14}$$

$$\begin{aligned} dV_j^{(2)}(t) &= 2r(t)V_j^{(2)}(t) - 2b^j(t)V^j(t) - \sum_{k\neq j} \mu^{jk}(t)(b^{jk(2)}(t) + 2b^{jk}V^k(t) + V^{k(2)}(t) - V^{j(2)}(t)) \\ &= (2r(t) + \mu(t))V_j^{(2)}(t) - 2b^j(t)V^j(t) - \sum_{k\neq j} \mu^{jk}(t)(b^{jk(2)}(t) + 2b^{jk}V^k(t) + V^{k(2)}(t)(t)) \end{aligned} \tag{15}$$

$$R^{jk(2)}(t) = b^{jk(2)}(t) + V^{k(2)}(t) - V^{j(2)}(t) + 2b^{jk}V^kV^j - 2V^kV^j \tag{16}$$

$$\begin{aligned} &(-2)V_j(t)dV_j(t) = \\ &\qquad (-2)V_j(t)(r(t)V_j(t) - b^j(t) - \sum_{k\neq j} \mu^{jk}(t)(b^{jk}(t) + V^k(t) - V^j(t))) \\ &\qquad = (-2r(t) - \mu(t))V_{j(2)}(t) + 2b^j(t)V_j(t) \\ &\qquad - \sum_{k\neq j} \mu^{jk}(t)(-2b^{jk}(t)V^j(t) - 2V^k(t)V^j(t)) - V_{j(2)}(t)) \end{aligned} \tag{17}$$

Search and replace variance and $R^{jk(2)}$. See that $V^{k(2)}(t) - (V^j(t))^2$ can be seen as $V^{k(2)}(t) - (V^k(t))^2 = var_k(t)$

$$
\begin{aligned}
dVar_j(t) &= d(V_j^{(2)}(t) - (V_j(t))^2) \\
&= dV_j^{(2)}(t) - 2V_j(t)dV_j(t) \\
\\
&= \\
&2r(t)(V^{j(2)(t)} - (V_j(t))^2) \\
&- \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) \\
&+ V^{k(2)}(t) - V^{j(2)}(t) - 2V^j(t)V^k(t) - 2(V^j(t))^2 + 2b^{jk}(V^k(t) - V^j(t))) \\
&= \\
&(2r(t) + \mu(t))var_j(t) \\
&- \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) \\
&+ V^{k(2)}(t) - 2V^j(t)V^k(t) - (V^j(t))^2 + 2b^{jk}(V^k(t) - V^j(t))) \\
&= \\
&(2r(t) + \mu(t))var_j(t) \\
&- \sum_{k \neq j} \mu^{jk}(t)(b^{jk(2)}(t) \\
&+ var_k(t) - 2V^j(t)V^k(t) + 2b^{jk}(V^k(t) - V^j(t))) \\
&= \\
&(2r(t) + \mu(t))var_j(t) - \sum_{k \neq j} \mu^{jk}(t)(R^{jk(2)}(t) + var_k(t)) \\
&= \\
&2r(t)var_j(t) - \sum_{k \neq j} \mu^{jk}(t)(R^{jk(2)}(t) + var_k(t) - var_j(t))
\end{aligned} \tag{18}
$$

We see, that the equation looks like Thiele and conclude, we just replace var and R^2 with the original v and R

$$
var_j(t) = \int_t^n e^{-\int_t^s 2r(v)dv} \sum_k p^{jk}(t,s) \sum_{k \neq l} \mu^{kl}(s)(R^{kl})^2(s)ds \tag{19}
$$

3 Problem 2

3.1 Question 2.A

By page 63, we have differential equations for $f^j(t)$ and $g^j(t)$

$$\begin{aligned} df^j(t) &= r(t)f^j(t) - b^j(t) - g^j(t)c_j(t) \\ &- \sum_{k\neq j} \mu^{jk}(t)(bj^{jk}(t) + c^{jk}(t)g^k(t) + f^k(t) - f^j(t)) \end{aligned} \tag{20}$$

Compute f(t) and g(t) in the survival model with two states, alive(0) and dead(1).

$$\begin{aligned} df^0(t) &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) \\ &- \sum_{k\neq j} \mu^{01}(b^{01}(t) + c^{01}(t)g^1(t) + f^1(t) - f^0(t)) \\ c_{01}(t) &= -(b^{01}(t) + V^{1*}(t) - V^{0*}(t)) \end{aligned} \tag{21}$$

$$\begin{aligned} df^0(t) &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) \\ &- \sum_{k\neq j} \mu^{01}(b^{01}(t) + c^{01}(t)g^1(t) + f^1(t) - f^0(t)) \\ &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) \\ &- \mu^{01}(b^{01}(t) + c^{01}(t)g^1(t) + f^1(t) - f^0(t)) \end{aligned} \tag{22}$$

Assume, that the insurance company which issues the policy has no future liabilities within state of death(1). Why $g^1(t) = 0$ and $f^1(t) = 0$.

$$\begin{aligned} df^0(t) &= r(t)f^0(t) - b^0(t) - g^0(t)c_0(t) - \mu^{01}b^{01}(t) + \mu^{01}f^0(t) \\ &= (r(t) + \mu(t))f^0(t) - b^0(t) - g^0(t)c_0(t) - \mu^{01}(t)b^{01}(t) \end{aligned} \tag{23}$$

$$f^0(t) = \int_t^n e^{-\int_t^s (r(v)+\mu(v))dv}(b^0(s) + g^0(s)c_0(s) + \mu^{01}(s)b^{01}(s))ds \tag{24}$$

Where $f^0(n) = 0$.

$$dg^j(t) = (q^j(t) + \sum_{k \neq j} \mu^{jk}(t) q^{jk}(t)) g^j(t) - q^j(t) \\ - \sum_{k \neq j} (1 - q^{jk}(t)) \mu^{jk}(t) (\frac{q^{jk}(t)}{1 - q^{jk}(t)} + g^k(t) - g^j(t)) \tag{25}$$

$$\begin{aligned} dg^0(t) &= (q^0(t) + \mu^{01}(t) q^{01}(t)) g^0(t) - q^0(t) \\ &- (1 - q^{01}(t)) \mu^{01}(t) (\frac{q^{01}(t)}{1 - q^{01}(t)} + g^1(t) - g^0(t)) \\ &= (q^0(t) + \mu^{01}(t) q^{01}(t) + (1 - q^{01}(t)) \mu^{01}(t)) g^0(t) - q^0(t) - \mu^{01}(t) q^{01}(t) \\ &= (q^0(t) + \mu^{01}(t)) \mu^{01}(t)) g^0(t) - q^0(t) - \mu^{01}(t) q^{01}(t) \end{aligned} \tag{26}$$

$$g^0(t) = \int_t^n e^{-\int_t^s (q^0(v) + \mu^{01}(v)) dv} (q^0(s) + \mu^{01}(s) q^{01}(s)) ds \tag{27}$$

Where $q^j(t)$ $0 <= q^{jk}(t) < 1$

4 Appendix

4.1 Appendix - Question 2.A

4.2 Question 2.A

By page 60. Contribution function is on the following form. And by page 63, we have differential equations for $f^j(t)$ and $g^j(t)$

$$\begin{aligned} c_j(t) &= (r(t) - r^*(t))V_j^*(t) + \sum_{k \neq j}(b^{jk}(t) + V^{*k}(t) - V^{j*}(t)) \\ c_{jk}(t) &= -(b^{jk}(t) + V^{k*}(t) - V^{j*}(t)) \\ \\ df^j(t) &= r(t)f^j(t) - b^j(t) - g^j(t)c_j(t) \\ &\quad - \sum_{k \neq j} \mu^{jk}(t)(bj^{jk}(t) + c^{jk}(t)g^k(t) + f^k(t) - f^j(t)) \end{aligned} \tag{28}$$

4.3 Appendix - Question 1.B

Use integration by parts, and use, that $\triangle m(t) = m(t) - m(t-)$, and $m(t) = \triangle m(t) + m(t-)$

$$\begin{aligned} dm(t) &= e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt) \\ dm^2(t) &= d(m(t)m(t)) \\ &= dm(t)m(t) + m(t-)dm(t) \\ &= dm(t)(m(t) + m(t-)) \\ &= dm(t)(\triangle m(t) + 2m(t-)) \end{aligned} \tag{29}$$

Split $m(t)$ in $B^{"}(t) = \int_0^t e^{-\int_0^s r(v)dv} dB(s)$ and $V^{"}(t) = e^{-\int_0^t r(v)dv}V(t)$. Now solve $dm(t)\triangle m(t)$ and $dm(t)2B^{"}(t-)$ and $dm(t)2V^{"}(t-)$, where $dm(t)$

$$dm(t)2B^{"}(t-) = e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt)2\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s) \tag{30}$$

See that $dm^2(t) = 2\int_0^t m(s)dm(s)$ and $dV^2(t) = 2V(t)dV(t)$

$$
\begin{aligned}
dm(t)2V^{"}(t-) &= e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t))2e^{-\int_0^t r(v)dv}V(t) \\
&= 2e^{-2\int_0^t r(v)dv}V(t)(dB(t) + dV(t) - r(t)V(t)dt) \\
&= 2e^{-2\int_0^t r(v)dv}(V(t)dB(t) - r(t)V^2(t)) + e^{-2\int_0^t r(v)dv}dV^2(t)
\end{aligned} \tag{31}
$$

See that $V^{"}(t) - V^{"}(t-) = 0$ And see that $B^{"}(t) - B^{"}(t-) = \int_0^t e^{-\int_0^s r(v)dv}dB(s) - dB(s-) = e^{-\int_0^t r(v)dv}\int_0^t dB(t) - dB(t-) = e^{-\int_0^t r(v)dv}\triangle B(t)$ And no jump payments in $dV(t)$ or $V(t)$

$$
\begin{aligned}
dm(t)\triangle m(t) &= (e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt))(B^{"}(t) + V^{"}(t) - B^{"}(t-) - V^{"}(t-)) \\
&= (e^{-\int_0^t r(v)dv}(dB(t))e^{-\int_0^t r(v)dv}\triangle B(t) \\
&= e^{-2\int_0^t r(v)dv}dB(t)\triangle B(t)
\end{aligned} \tag{32}
$$

And see that $r(t)V(t)dt$ is continuous therefore we can exchange $t-$ with t.

$$
\begin{aligned}
dm^2(t) &= \\
&= dm(t)B^{"}(t) + dm(t)V^{"}(t) + dm(t)\triangle m(t) \\
&e^{-\int_0^t r(v)dv}(dB(t) + dV(t) - r(t)V(t)dt)2\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s) \\
&+ 2e^{-2\int_0^t r(v)dv}(V(t)dB(t) - r(t)V^2(t)) + e^{-2\int_0^t r(v)dv} dV^2(t) \\
&+ e^{-2\int_0^t r(v)dv} dB(t)\triangle B(t) \\
&= \\
&2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s))e^{-\int_0^t r(v)dv} dB(t) \\
&+ e^{-2\int_0^t r(v)dv}\triangle B(t)dB(t) \\
&- 2e^{-2\int_0^t r(v)dv} r(t)V^2(t) \\
&+ e^{-2\int_0^t r(v)dv} dV^2(t) \\
&+ 2e^{-2\int_0^t r(v)dv} V(t)dB(t) \\
&- 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s))e^{-\int_0^t r(v)dv} r(t)V(t)dt \\
&+ 2(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s))e^{-\int_0^t r(v)dv} dV(t)
\end{aligned}
\tag{33}
$$

4.4 Appendix - Question 1.C

$$
\begin{aligned}
dm^2(t) = & \\
& + 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} dB(t) \\
& + e^{-2\int_0^t r(v)dv} \triangle B(t) dB(t) \\
& - 2e^{-2\int_0^t r(v)dv} r(t) V^2(t) dt \\
& + e^{-2\int_0^t r(v)dv} dV^2(t) \\
& + 2e^{-2\int_0^t r(v)dv} V(t) dB(t) \\
& - 2\left(\int_0^{t} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} r(t) V(t) \\
& + 2\left(\int_0^{t-} e^{-\int_0^s r(v)dv} dB(s)\right) e^{-\int_0^t r(v)dv} dV(t)
\end{aligned} \tag{34}
$$

References

[1] Jesper Lund Pedersen. *Stochastic Processes in Life Insurance: The Dynamic Approach*. Department of Mathematical Sciences.