Stochastic Processes in Life Insurance

Assignment 1

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1 Introduction

This is the first of to two assignments in the course Stochastic Processes in Life Insurance at Copenhagen University. The course professor is Jesper Lund Pedersen. The course is held in blok 1, 2019.

2 Problem

In this assignment we look at a multi-state contract, with two states Alive=0 and Dead=1. Let $Z(t)$ denote the state of the policy at time t. The contract is issued at time 0 and expires at time n. We assume that the policy commencing in state 0, that is $Z(0) = 0$. Assume $Z(t)$ is a continuous time Markov chain with transition intensity $\mu_{01}(t) = \mu(t)$ where $\mu(t)$ is a Borel function bounded on bounded intervals. Assume interest rate $r(t)$ is deterministic. The life length of the policy holder is the jump time T_x and the survival probability is $p_{00}(s,t) = e^{-\int_s^t \mu(v)dv}$.

2.1 a - term insurance

Interpretation of the payment process. The payment process below is a term insurance, which is a payment upon the death of the policy holder if and only if the time of death T_x is before the expiration of the policy at time n.

$$B(t) = N(t) = 1_{[T_x,\infty)}(t) \tag{1}$$

Intensity process of the payment process. By Proposition 12.9[1], we know that N(t) have a the intensity process $\lambda^B(t)$. Where the hazzard transition intensity α is μ known from the problem definition.

$$\lambda^B(t) = \mu(t) * 1_{\{Z(t-)=0\}} \tag{2}$$

Compute the reserve. From 14.3[1] we know that the reserve $V(t)$ is:

$$V(t) = \int_t^n e^{-\int_t^u} E[\lambda^B(u)|F(t)]du \tag{3}$$

$= \int_t^n e^{-\int_t^u r(v)dv} E[\mu(u) * 1_{\{Z(u-)=0\}}\|F(t)]du$	insert the intensity process.
$= \int_t^n e^{-\int_t^u r(v)dv} \mu(u) E[1_{\{Z(u-)=0\}}\|F(t)]du$	If $\mu(t)$ is deterministic.
$= \int_t^n e^{-\int_t^u r(v)dv} \mu(u) E[1_{\{Z(u)=0\}}\|F(t)]du$	When we integrate with du $u-$ can be replaced with u.
$= \int_t^n e^{-\int_t^u r(v)dv} \mu(u) p_{00}(t,u)du$	$1_{\{Z(u)=0\}}$ can be seen as the survival up to time u. Therefore replaceable with $p_{00}(t,u)$ from the problem definition.
$= \int_t^n e^{-\int_t^u r(v)dv} \mu(u) e^{-\int_t^u \mu(v)dv} du$	We can replace $p_{00}(t,u)$.
$= \int_t^n e^{-\int_t^u (r(v)+\mu(v))dv} \mu(u)du$	Merge exponential expressions.

$$V(t) = \int_t^n e^{-\int_t^u (r(v)+\mu(v))dv} \mu(u)du \tag{4}$$

2.2 b - life annuity

Interpretation of the payment process. The payment process below is a life annuity, which is a continuous payment from time 0 to the policy expires or the policy holder dies $min(T_x, n)$ what ever comes first.

$$B(t) = \int_0^t 1_{\{Z(t)=0\}} ds \tag{5}$$

Intensity process of the payment process. We look at a probability for payment while staying in state 0. therefore we argue, at that for every time s, $0 < s <= t$, there is a risk of leaving state 0 and jumping to state 1, stopping the payment. We could look at this at $p_{00}(s, t)$ and the again doing this for each couple (s, t) before expiration time n.

$$\lambda^B(t) = \int_t^n p_{00}(s, t) \tag{6}$$

The reserve is given. 16.3[1, p.51]

$$V(t) = E[Y(t, n)|F(t)]du \tag{7}$$

$= \int_t^n e^{-\int_t^u r(v)dv} E[1_{\{Z(u)=0\}}\|F(t)]du$	insert the payment process
$= \int_t^n e^{-\int_t^u r(v)dv} P(Z(u) = 0\|Z(t))du$	insert probabilities. Markov property.
$= \int_t^n e^{-\int_t^u r(v)dv}$ $(P(T_x > u\|T_x > t)1_{\{Z(t)=0\}}$ $+P(T_x > u\|T_x <= t)1_{\{Z(t)=1\}})du$	
$= \int_t^n e^{-\int_t^u r(v)dv} p_{00}(t, u)1_{\{Z(t)=0\}}du$	
$= \int_t^n e^{-\int_t^u r(v)dv} e^{-\int_t^u \mu(v)dv} 1_{\{Z(t)=0\}}du$	
$= \int_t^n e^{-\int_t^u (r(v)+\mu(v))dv} 1_{\{Z(t)=0\}}du$	

2.3 c - Compute the dynamics of the statewise reserves given in Theorem 16.1

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\begin{aligned}
V^j(t) &= \int_t^n e^{-\int_t^u (\mu^j(v)+r(v))dv}(b^j(u) + \sum_{k\neq j} \mu^{jk}(u)(b^{jk}(u) + V^k(u))du \\
&\qquad + \sum_{t<s<n} e^{-\int_t^u (\mu^j(v)+r(v))dv} \triangle B^j(u) \\
&= \int_t^n e^{-\int_t^u (\mu^j(v)+r(v))dv}(b^j(u) + \sum_{k\neq j} \mu^{jk}(u)(b^{jk}(u) + V^k(u))du \\
&= \int_0^n e^{-\int_t^u (\mu^j(v)+r(v))dv}(b^j(u) + \sum_{k\neq j} \mu^{jk}(u)(b^{jk}(u) + V^k(u))du \\
&\quad - \int_0^t e^{-\int_t^u (\mu^j(v)+r(v))dv}(b^j(u) + \sum_{k\neq j} \mu^{jk}(u)(b^{jk}(u) + V^k(u))du \\
&= e^{-\int_t^n (\mu^j(v)+r(v))dv}(b^j(n) + \sum_{k\neq j} \mu^{jk}(n)(b^{jk}(n) + V^k(n))dn \\
&\quad -d(\int_0^t e^{-\int_t^u (\mu^j(v)+r(v))dv}(b^j(u) + \sum_{k\neq j} \mu^{jk}(u)(b^{jk}(u) + V^k(u))du) \\
&= -d(\int_0^t e^{-\int_t^u (\mu^j(v)+r(v))dv}(b^j(u) + \sum_{k\neq j} \mu^{jk}(u)(b^{jk}(u) + V^k(u))du) \\
&= -(\mu^j(t) + r(t)) * (-1) * \int_0^t e^{-\int_t^u (\mu^j(v)+r(v))dv}(b^j(u) + \sum_{k\neq j} \mu^{jk}(u)(b^{jk}(u) + V^k(u))du \\
&\quad -(b^j(t) + \sum_{k\neq j} \mu^{jk}(t)(b^{jk}(t) + V^k(t)) \\
&= (\mu^j(t) + r(t))V^j(t) - (b^j(t) + \sum_{k\neq j} \mu^{jk}(t)(b^{jk}(t) + V^k(t)) \\
&= r(t)V^j(t) - b^j(t) - \sum_{k\neq j} \mu^{jk}(t)(b^{jk}(t) + V^k(t) - V^j(t))
\end{aligned}
\tag{8}
$$

First off, $\triangle B^j(t) = 0$, due to no endowments. known from Theorem 16.2. Second we split upper and lower bound of the integral $\int_t^n = \int_0^n * - \int_0^t$. We see that the dynamics of the integral with upper bound n, goes towards the expiration time, where the value at time n is equal to zero $V^j(n) = 0$.

2.4 d - Show the dynamics of the predictable compensator of the payment process is given by

$$\begin{aligned} d\Lambda^B(t) &= \sum_j 1_{\{Z(t-)=j\}} dB^j(t) + \sum_k b^{Z(t-)k}(t)\mu^{Z(t-)k}(t) 1_{\{(Z(t-)\neq k\}} dt \\ &= \sum_j 1_{\{Z(t-)=j\}} (dB^j(t) + \sum_{k\neq j} b^{jk}(t)\mu^{jk}(t)dt) \end{aligned} \tag{9}$$

$$dB(t) = \sum_j 1_{\{Z(t)=j\}} dB^j(t) + \sum_k b^{Z(t-)k}(t) dN^k(t) \tag{10}$$

By proposition 10.6, $dB(t)$ have the intensity process as the following informal states:

$$\begin{aligned} \lambda^B(t)dt &= E[dB(t)|F(t-)] \\ &= b^{Z(t-)}(t)dt + \sum_k b^{Z(t-)k}(t) E[dN^k(t)|F(t-)] \\ &= (b^{Z(t-)}(t) + \sum_k b^{Z(t-)k}(t)\mu^{Z(t-)k} 1_{\{Z(t-)\neq j\}})dt \end{aligned} \tag{11}$$

By definition 9.11 and 9.14, we have that,$\lambda^B(t)dt = E[dX(t)|F(t-)] = d\Lambda^B(t)$

$$\begin{aligned} d\Lambda^B(t) &= (b^{Z(t-)}(t) + \sum_k b^{Z(t-)k}(t)\mu^{Z(t-)k} 1_{\{Z(t-)\neq j\}})dt \\ d\Lambda^B(t) &= (\sum_j 1_{\{Z(t-)=j\}} b^j(t) + \sum_{k\neq j} b^{jk}(t)\mu^{jk} 1_{\{Z(t-)=j\}})dt \end{aligned} \tag{12}$$

We know that $dB^j(t) = b^j(t)dt + B^j(t) - B^j(t-)$ where $\triangle B^j(t) = B^j(t) - B^j(t-)$ is an endowment, and is equal to zero, in the differential equation, thus $dB^j(t) = b^j(t)dt$

$$\begin{aligned} d\Lambda^B(t) &= \sum_j 1_{\{Z(t-)=j\}} dB^j(t) + \sum_{k\neq j} b^{jk}(t)\mu^{jk} 1_{\{Z(t-)=j\}} dt \\ &= \sum_j 1_{\{Z(t-)=j\}} (dB^j(t) + \sum_{k\neq j} b^{jk}(t)\mu^{jk} dt) \end{aligned} \tag{13}$$

$$dV(t) = V^{Z(t)}(t)dt + \sum_k (V^k(t) - V^{Z(t-)}(t))dN^k(t) \tag{14}$$

By proposition 10.6, $dV(t)$ have the intensity process as the following informal states:

$$\begin{aligned} \lambda^V(t)dt &= E[dV(t)|F(t-)] \\ = V^{Z(t-)}(t)dt + \sum_k (V^k(t) &- V^{Z(t-)}(t))E[dN^k(t)|F(t-)] \\ = (V^{Z(t-)}(t) + \sum_k (V^k(t) - V^{Z(t-)}&(t))\mu^{Z(t-)k}(t)1_{\{Z(t-)\neq k\}})dt \end{aligned} \tag{15}$$

By definition 9.11 and 9.14, we have that, $\lambda^V(t)dt = E[dX(t)|F(t-)] = d\Lambda^V(t)$

$$\begin{aligned} \Lambda^V(t) = (V^{Z(t-)}(t) + \sum_k (V^k(t) - V^{Z(t-)}(t))\mu^{Z(t-)k}(t)1_{\{Z(t-)\neq k\}})dt \\ = \sum_j 1_{\{Z(t-)=j\}} V^j(t)dt \\ + \sum_k V^k(t)\mu^{Z(t-)k}(t)1_{\{Z(t-)\neq k}dt \\ - \sum_j 1_{\{Z(t-)=j\}} V^j(t)dt \sum_{k\neq j} \mu^{jk}(t)1_{\{Z(t-)=j\}} \\ = \sum_j 1_{\{Z(t-)=j\}}(V^j(t)dt + \sum_{k\neq j} \mu^{jk}(V^k(t) - V^j(t))dt \end{aligned} \tag{16}$$

We know that $dV^j(t) = V^j(t)dt + V^j(t) - V^j(t-)$ where $\triangle V^j(t) = V^j(t) - V^j(t-)$ is an endowment, and is equal to zero, in the differential equation, thus $dV^j(t) = V^j(t)dt$

$$\Lambda^V(t) = \sum_j 1_{\{Z(t-)=j\}}(dV^j(t) + \sum_{k\neq j} \mu^{jk}(V^k(t) - V^j(t))dt \tag{17}$$

References

[1] Jesper Lund Pedersen. *Stochastic Processes in Life Insurance: The Dynamic Approach*. Department of Mathematical Sciences.