3.3 Fourier Series Computation for G_k

In the words of the great Barry Mazur, "Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist."[1] The holomorphic Eisenstein series, defined as

$$G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(c\tau+d)^k}$$

for k even and $k > 2$, is one such example of their existence. However, proving that $G_k(\tau)$ is a modular form can be a bit tricky, especially when examining its behavior at the cusp. We can make this problem simpler by explicitly writing down $G_k(\tau)$ as a Fourier expansion, which is the motivation for the following theorem.

Theorem 3.1. Let τ in the upper half-plane $\mathbb{H}$

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

for $k > 2$, k even, and $q = e^{2\pi i\tau}$.

In the subsequent proof, we will be differentiating heavily and invoking the cotangent identity. Don't fret, this is a common practice in complex analysis. A proof of this identity can be found in Section 1.5 of Rick Schwartz's notes titled "Modular Forms and Fourier Series".

Proof: Let $\tau \in \mathbb{H}$. We will use the following identity:

$$\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau-d} + \frac{1}{\tau+d}\right) = \pi\cot(\pi\tau) = \pi i - 2\pi i\sum_{m=0}^{\infty} q^m \tag{1}$$

where $q = e^{2\pi i\tau}$. Differentiating (1) $k-1$ times with respect to τ gives:

$$(-1)^{k-1}(k-1)!\sum_{d\in\mathbb{Z}}\frac{1}{(\tau+d)^k} = -(2\pi i)^k\sum_{m=1}^{\infty} m^{k-1}q^m \tag{2}$$

for $k \geq 2$. Readers worry not, we will work through this differentiation together. The explanation, like the Saturn V rocket, comes to you in three parts.[2]

[1] A quote from NOVA's documentary chronicling the solution to Fermat's Last Theorem titled "The Proof".
[2] Credit: "Other John Greens" by John Green.

Part I: Starting with the RHS of (1), we take the first derivative and find:

$$\frac{d}{d\tau}(\pi i - 2\pi i \sum_{m=0}^{\infty} q^m) = -2\pi i \ \frac{d}{d\tau} \sum_{m=0}^{\infty} q^m \tag{3}$$

This is perhaps not as illuminating as we had hoped. Let us break it down further and take the first derivative of one element of the summation:

$$\begin{aligned} \frac{d}{d\tau}(q^m) &= \frac{d}{d\tau}(e^{2\pi i \tau m}) \\ &= 2\pi i \, m \, e^{2\pi i \tau m} \\ &= 2\pi i \, m \, q^m \end{aligned}$$

From this, we can see that after differentiating $k-1$ times, we will have $k-1$ factors of m. Note, we also will have $k-1$ factors of $2\pi i$ from the summation, which, along with the factor of $2\pi i$ outside the summation, gives us a total of k factors of $2\pi i$. Thus, we find that differentiating the RHS of (1) $k-1$ times with respect to τ does in fact yield:

$$(-1)(2\pi i)^k \sum_{m=1}^{\infty} m^{k-1} q^m$$

Part II: Moving to the LHS of (1), we will examine the third, fourth and fifth derivative to establish a pattern. Note, we start with the third derivative because G_k is defined for even $k \geq 4$, so we must take at minimum $4-1=3$ derivatives. Examining these derivatives, we have:

Third Derivative:

$$\frac{d^3}{d\tau^3}\left(\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau-d} + \frac{1}{\tau+d}\right)\right) = \frac{-6}{\tau^4} + \sum_{d=1}^{\infty}\left(\frac{-6}{(\tau-d)^4} + \frac{-6}{(\tau+d)^4}\right)$$

Fourth Derivative:

$$\frac{d^4}{d\tau^4}\left(\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau-d} + \frac{1}{\tau+d}\right)\right) = \frac{24}{\tau^5} + \sum_{d=1}^{\infty}\left(\frac{24}{(\tau-d)^5} + \frac{24}{(\tau+d)^5}\right)$$

Fifth Derivative:

$$\frac{d^5}{d\tau^5}\left(\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau-d} + \frac{1}{\tau+d}\right)\right) = \frac{-120}{\tau^6} + \sum_{d=1}^{\infty}\left(\frac{-120}{(\tau-d)^6} + \frac{-120}{(\tau+d)^6}\right)$$

From these derivatives, we recognize an oscillating -1 term and a recurring $(k-1)!$ term, both of which can be factored out. Also, we note that if we rewrite the $(\tau - d)^k$ denominators as $(\tau + (-d))^k$, we can simply rewrite the sum over all the nonzero integers. In fact, term outside the summation accounts for the $d = 0$ case, so we can sum over all the integers. Thus, we have:

$$\frac{d^{k-1}}{d\tau^{k-1}}\left(\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau - d} + \frac{1}{\tau + d}\right)\right) = (-1)^{k-1}(k-1)! + \sum_{d\in\mathbb{Z}} \frac{1}{(\tau + d)^k}$$

So, (2) is indeed correct.

Part III: Now, notice the following. For even $k > 2$,

$$\sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(c\tau + d)^k} = \underline{\sum_{d\neq 0} \frac{1}{d^k}} + 2\underline{\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau + d)^k}}$$

Orange Segment: This summation takes care of the cases where $c = 0$.
Blue Segment: This term covers all other cases. Again, recall that k is even. Therefore, $(c\tau \pm d)^k = (-c\tau \mp d)^k$. As a result, the cases where $c \neq 0$ can be addressed by simply summing over the positive c-values twice. Hence, two times the sum from $c = 1$ to infinity.

Finally piecing the puzzle together, let ζ denote the Riemann zeta function:

$$\zeta(k) = \sum_{d=1}^{\infty} \frac{1}{d^k}$$

Since k is even, we recognize that:

$$\sum_{d\neq 0} \frac{1}{d^k} = 2\sum_{d=1}^{\infty} \frac{1}{d^k} = 2\zeta(k)$$

From (2) we see that if we divide by $(-1)^{k-1}$ and $(k-1)!$, we get:

$$\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau + d)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1} q^{cm}$$

Now, we can make the appropriate substitutions and find:

$$\sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(c\tau + d)^k} = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{c=1}^{\infty}\sum_{m=1}^{\infty} m^{k-1} q^{cm}$$

Let

$$\sigma_{k-1}(n) = \sum_{\substack{m|n \\ m>0}} m^{k-1}.$$

Recalling that $G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(c\tau+d)^k}$ and making the final substitution, we arrive at the Fourier expansion for $G_k(\tau)$:

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

for $k > 2$, k even, and $q = e^{2\pi i\tau}$. $\square$