ECON 468 Cheat Sheet
Zachary Probst Page:1

1 Basics

Distributions

Normal: $\phi(x) = (2\pi)^{-1/2}\exp(-\frac{1}{2}x^2)$

Normal CDF: $\mathbf{\Phi}(x) = \int_{-\infty}^{x}\phi(y)dy$

Moments

Continuous: $m_k(X) \equiv \int_{-\infty}^{\infty} x^k f(x)dx$

Central:
$\mu_k \equiv E(X-E(X)^k) = \int_{-\infty}^{\infty}(x-\mu)^k f(x)dx$

Discrete Central:
$\mu_k \equiv E(X-E(X)^k) = \sum_{i=1}^m p(x)(x_i-\mu)^k$

Multivariate

Joint Density Func: $f(x_1,x_2) = \frac{\partial^2 F(x_1,x_2)}{\partial x_1 \partial x_2}$

Independence:
$F(x_1,x_2) = F(x_1,\infty)F(\infty,x_2)$
or $f(x_1,x_2)=f(x_1)f(x_2)$

Marginal Density:
$f(x_1) \equiv F_1(x_1,\infty) = \int_{-\infty}^{\infty} f(x_1,x_2)dx_2$

Conditional Density: $f(x_1|x_2) = \frac{f(x_1,x_2)}{f(x_2)}$

Law Iterate Expec: $E(E(X_1|X_2)) = E(X_1)$

Any Deterministic Func h:
$E(X_1 h(X_2) \mid X_2) = h(X_2)E(X_1\mid X_2)$

Matrix Algebra

Dot Product: $\mathbf{a}^\top\mathbf{b} = \sum_{i=1} a_i b_i$

Matrix Multiplication: $C_{ij} = \sum_{k=1}^m A_{ik}B_{ki}$

Transpose: $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$

$\|\mathbf{x}\| = (\mathbf{x}^\top\mathbf{y})^{1/2} = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$

$\langle \mathbf{x},\mathbf{y}\rangle = \|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta)$

Cauchy-Schwartz: $\langle\mathbf{x},\mathbf{y}\rangle \le \|\mathbf{x}\|\|\mathbf{y}\|$

Linear dep: $\mathbf{Xb}=\mathbf{0}$, $\mathbf{b}\ne\mathbf{0}$

Singular: $\exists \mathbf{x}\neq 0 : \mathbf{Ax}=\mathbf{0}$

2 Linear Regression

OLS

Estimator: $\hat\beta = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}$

$E(x_{ti}u_t) = 0 \implies \mathbf{X}^\top(\mathbf{y}-\mathbf{X}\beta) = \mathbf{0}$

SSR(β): $= \sum_{t=1}^n (y_i - \mathbf{X}_t\beta)^2$

$\mathbf{y}^\top\mathbf{y} = \hat\beta^\top\mathbf{X}^\top\mathbf{X}\hat\beta + (\mathbf{y}-\mathbf{X}\hat\beta)^\top(\mathbf{y}-\mathbf{X}\hat\beta)$

Projection: $\mathbf{P_X} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top$

$\mathbf{M_X} = \mathbf{I}-\mathbf{P_X} = \mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top$

$\mathbf{P_X y} = \mathbf{X}((\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y})$; $\mathbf{P_X P_X} = \mathbf{P_X}$

$\mathbf{M_X y} = \hat{\mathbf{u}}$; $\mathbf{M_X X} = \mathbf{0}$; $\mathbf{M_X M_X} = \mathbf{M_X}$

$\mathbf{P_X}+\mathbf{M_X} = \mathbf{I}$; $\mathbf{P_X M_X} = \mathbf{0}$; $\mathbf{P_X}^\top = \mathbf{P_X}$

$\|\mathbf{y}\|^2 = \|\mathbf{P_X y}\|^2 + \|\mathbf{M_X y}\|^2$; $\|\mathbf{P_X y}\| \le \|\mathbf{y}\|$

Centering: $\mathbf{M}_\iota \mathbf{x} = \mathbf{z} = \mathbf{x} - \bar{x}\iota$; $\iota^\top\mathbf{z} = 0$

$\mathbf{P}_1 \equiv \mathbf{P}_{\mathbf{X}_1}$; $\mathbf{P}_1\mathbf{P_X} = \mathbf{P_X P}_1 = \mathbf{P}_1$

FWL: β_2 from $\mathbf{y}=\mathbf{X}_1\beta_1+\mathbf{X}_2\beta_2+\mathbf{u}$ and $\mathbf{M}_1\mathbf{y} = \mathbf{M}_1\mathbf{X}_2\beta_2 + \text{res}$ are the same. (+ res)

Seasonal w const: $\mathbf{s}_i' = \mathbf{s}_i - \mathbf{s}_4$, $i=1,2,3$. Avg is const coeff. $\mathbf{M_S y}$ is deseasonalized.

$\beta^{(t)} - \hat\beta = -1\backslash 1 - h_t(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}_t^\top\hat{u}_t$ where h_t denotes the t^{th} diagonal element of $\mathbf{P_X}$.

Bias

vector of (true) model params: θ

Bias: $E(\hat\theta)-\theta_0$, $E((\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{u}) = 0$

estimating eq: $g(\mathbf{y},\theta)=0$ unbiased iff $\forall\mu\in\mathbb{M}, E_\mu g(\mathbf{y},\theta_\mu) = \mathbf{0}$ or $E(\mathbf{X}^\top\mathbf{u}) = \mathbf{0}$

X exogenous $\implies E(\mathbf{u}\mid\mathbf{X}) = \mathbf{0}$ and both $\hat\beta$ and estimating equations unbiased.
Make exog assumpt in cross-sec not time.
regressors predetermined: $E(\mathbf{X}^\top\mathbf{u}) = \mathbf{0}$

Stochastic Limits

Converg in prob: $\lim Pr(|Y_n - Y_\infty| > \epsilon) = 0 \Rightarrow \text{plim}\, Y_n = Y_\infty \implies$ converg dist

Converg dist: $\lim F_n(x) = F(x) \equiv Y_n \to F$

LLN: $\text{plim}\,\overline{Y_n} = \text{plim}\frac{1}{n}\sum_t^n Y_t = \mu_Y$, Y_t IID, $\overline{Y_n}$ sample mean of Y_t, μ_T pop mean.

LLN2: $\text{plim}\frac{1}{n}\sum_t^n Y_t = \lim\frac{1}{n}\sum_t^n E(Y_t)$

$\text{plim}\, Y_n Z_n = \text{plim}\, Y_n \text{plim}\, Z_n$ if converg

$\mathbf{X}^\top\mathbf{X}$ may not have plim so mult by $1\backslash n$.

consistent: $\text{plim}_\mu \hat\beta = \beta_\mu$, may be bias

$E(\mu_t \mid \mathbf{X}_t) = 0 \implies \hat\beta$ consistent.

Covariance and Precision Matrices

$\text{Cov}(b_i,b_j) \equiv E((b_i - E(b_i))(b_j - E(b_j)))$

if $i=j$, $\text{Cov}(b_i,b_j) = \text{Var}(b_i)$

$\text{Var}(\mathbf{b}) \equiv E((\mathbf{b}-E(\mathbf{b}))(\mathbf{b}-E(\mathbf{b}))^\top)$

when $E(\mathbf{b}) = \mathbf{0}$, $\text{Var}(\mathbf{b}) = E(\mathbf{bb}^\top)$ b_i, b_j indep: $\text{Cov}(b_i,b_j)=0$, converse false

correlation: $\rho(b_i,b_j) \equiv \frac{\text{Cov}(b_i,b_j)}{(\text{Var}(b_i)\text{Var}(b_j))^{1\backslash 2}}$

$\text{Var}(\mathbf{b})$ positive semidefinite. cov and corr matrix positive definite most of the time.

positive definite: $\mathbf{x}^\top\mathbf{Ax} > 0$ for $\mathbf{x}\in k\times 1$.

$\mathbf{x}^\top\mathbf{Ax} = \sum_i^k\sum_j^k x_i x_j A_{ij}$. If $\ge 0 \Rightarrow$ semidef.

Any $\mathbf{B}^\top\mathbf{B}$ is pos semidef. If full col rank then pos def. pos def $\Rightarrow$ diag >0 & non-singular. (pos def)$^{-1}$ $\exists$ & is pos def.

Precision mtrx: invers of cov mtrx of estmatr. $\exists$ & pos def iff cov mtrx pos def.

If u IID w Var σ^2 and cov of any pair $=0$: $\text{Var}(\mathbf{u}) = E(\mathbf{uu}^\top) = \sigma^2\mathbf{I}$. If false, $\mathbf{\Omega}$ = err cov mtrx. If diag of $\mathbf{\Omega}$ differ, heteroskedastic. Homoskedastic: all u same Var. Autocorrelated: off-diag $\mathbf{\Omega}\neq 0$.

$\hat\beta$ unbiased & $\mathbf{\Omega} = \sigma^2\mathbf{I}$ so no hetero or autocorr, then $\text{Var}(\hat\beta) = \sigma^2(\mathbf{X}^\top\mathbf{X})^{-1}$.

Precision affected by n, σ^2, X.
Collinearity: precision for β_1 dep on $\mathbf{X}_2$.

Efficiency

$\tilde\beta$ more efficient than $\hat\beta$ iff $\text{Var}(\tilde\beta)^{-1} - \text{Var}(\hat\beta)^{-1}$ is nonzero pos semidef mtrx.

Gauss-Markov: If $E(\mathbf{u}\mid\mathbf{X}) = \mathbf{0}$ and $E(\mathbf{uu}^\top\mid\mathbf{X}) = \sigma^2\mathbf{I}$ then OLS $\hat\beta$ is BLUE (best linear unbiased estimator). Not necessary that u normally distributed.

Residuals & Disturbances

$\hat{\mathbf{u}} = \mathbf{M_X u}$ (hat resid, u dist).

If $E(\mathbf{u}\mid\mathbf{X}) = \mathbf{0} \Rightarrow E(\|\hat{\mathbf{u}}\|^2) \le E(\|\mathbf{u}\|^2)$

$\text{Var}(\hat{u}_t) < \sigma^2$; $\hat\sigma^2 \equiv \frac{1}{n}\sum_t^n \hat{u}_t^2$

$E(\hat\sigma^2) = \frac{n-k}{n}\sigma^2$

$E(\mathbf{u}^\top\mathbf{M_X u}) = E(\text{SSR}(\hat\beta)) = (n-k)\sigma^2$

unbiased: $s^2 \equiv \frac{1}{n-k}\sum_t^n \hat{u}_t^2$; s = std err.

unbias est of $\text{Var}(\hat\beta)$: $\widehat{\text{Var}}(\hat\beta) = s^2(\mathbf{X}^\top\mathbf{X})^{-1}$

s^2 unbiased and consistent.

$\text{MSE}(\hat\beta) \equiv E((\hat\beta-\beta_0)(\hat\beta-\beta_0)^\top)$

I $\hat\beta$ unbiased $\text{MSE}(\hat\beta) = \text{Var}(\hat\beta)$.

Measures of Goodness of Fit

$R_u^2 = \frac{\text{ESS}}{\text{TSS}} = \frac{\|\mathbf{P_X y}\|^2}{\|\mathbf{y}\|^2} = \cos^2\theta$, where θ angle between $\mathbf{y}$ and $\mathbf{P_X y}$. $0\le R_u^2\le 1$.

R_c^2: center all vars first. Invalid if $\iota\notin\mathcal{S}(\mathbf{X})$.

$R_c^2 = 1 - \sum_t^n \hat{u}_t^2 \backslash \sum_t^n (y_t-\bar{y})^2$.

Adj R^2: unbiased estimators. maybe <0.

$\bar{R}^2 \equiv 1 - \frac{\frac{1}{n-k}\sum_t^n \hat{u}_t{}^2}{\frac{1}{n-1}\sum_t^n(y_t-\bar{y})^2} = 1 - \frac{(n-1)\mathbf{y}^\top\mathbf{M_X y}}{(n-k)\mathbf{y}^\top\mathbf{M}_\iota\mathbf{y}}$

$\bar{R}^2$ does not always $\uparrow$ in regressors.

Hypothesis Testing

If u_t normal, and σ known, test $\beta = \beta_0$ w

$z = \frac{\hat\beta - \beta_0}{(\text{Var}(\hat\beta))^{1\backslash 2}} = \frac{n^{1\backslash 2}}{\sigma}(\hat\beta - \beta_0)$, $z\sim N(0,1)$

NCP: $\lambda = \frac{n^{1\backslash 2}}{\sigma}(\beta_1 - \beta_0)$, $\beta_1 \neq \beta_0$

Reject null if z large enough. 2-tail: $|z|$.
Type 1: reject true null, 2: accept false

left-tail $\Phi(-c_\alpha) = \alpha\backslash 2$, $c_\alpha = \Phi^{-1}(\alpha\backslash 2)$.

$\Phi^{-1}(.975) = 1.96$. Power: prob test rejects the null. Prob of Type 2 $= 1 - P$(power).

Power $\uparrow$ with $(\beta_1 - \beta_0)\uparrow$ or $\sigma\downarrow$ or $n\uparrow$.

$p(z) = 2(1-\Phi(|z|))$

$x\sim N(\mu,\sigma^2) \Rightarrow z = (x-\mu)\backslash\sigma$, $z\sim N(0,1)$.

Lin comb of rand vars that are jointly multivariate normal must be $\sim N$. If $\mathbf{x}$ multivar norm with 0 cov, componenets of $\mathbf{x}$ are mutually indep.

χ^2: $y \equiv \|\mathbf{z}\|^2 = \mathbf{z}^\top\mathbf{z} = \sum_i^m z_i^2$., $y\sim\chi^2(m)$ with $\mathbf{z}\sim N(\mathbf{0},\mathbf{I})$; $E(y) = m$. $\text{Var}(y) = 2m$.

$y_1 \sim \chi^2(m_1)$ & $y_2 \sim \chi^2(m_2)$ indep $\Rightarrow y_1 + y_2 \sim \chi^2(m_1+m_2)$

$m\times 1$ $\mathbf{x}\sim N(\mathbf{0},\mathbf{\Omega})$, then $\mathbf{x}^\top\mathbf{\Omega}^{-1}\mathbf{x}\sim\chi^2(m)$

If $\mathbf{P}$ $n\times n$ w rank $r<n$ and $\mathbf{z}\sim N(\mathbf{0},\mathbf{I})$ then $\mathbf{z}^\top\mathbf{Pz}\sim\chi^2(r)$.

$z\sim N(0,1)$ & $y\sim\chi^2(m)$, z,y indep, then $t \equiv z\backslash(y\backslash m)^{1\backslash 2}$. Or $t\sim t(m)$. Only first $m-1$ moments exist. Cauchy: $t(1)$. $\text{Var}(t) = m\backslash(m-2)$. $t(m)$ tends to std norm.

y_1, y_2 indep rand var $\sim\chi^2(m_1)$ & $\chi^2(m_2)$, then $F \equiv \frac{y_1\backslash m_1}{y_2\backslash m_2}$. $F\sim F(m_1,m_2)$.

As $m_2\to\infty$, $F\sim 1\backslash m_1$ times $\chi^2(m_1)$. $t\sim t(m_2)\Rightarrow t^2\sim F(1,m_2)$.

Exact Tests ($\mathbf{u}\sim N(\mathbf{0},\sigma^2\mathbf{I})$)

$\frac{\mathbf{x}_2^\top\mathbf{M}_1\mathbf{y}}{s(\mathbf{x}_2^\top\mathbf{M}_1\mathbf{x}_2)^{1/2}} = \left(\frac{\mathbf{y}^\top\mathbf{M_X y}}{n-k}\right)^{-\frac{1}{2}}\frac{\mathbf{x}_2^\top\mathbf{M}_1\mathbf{y}}{(\mathbf{x}_2^\top\mathbf{M}_1\mathbf{x}_2)^{1\backslash 2}}$ is t-stat $t_{\beta_2}\sim t(n-k)$ for testing $\beta_2 = 0$.

$\beta_2\in\mathbb{R}\Rightarrow$ test for $\beta_2 = \beta_{20}$: $(\hat\beta_2 - \beta_{20})/s_2$.

$F_{\beta_2} \equiv \frac{(\text{RSSR - USSR})/r}{\text{USSR}/(n-k)} = \frac{\mathbf{y}^\top\mathbf{P}_{\mathbf{M}_1\mathbf{X}_2}\mathbf{y}/r}{\mathbf{y}^\top\mathbf{M_X y}/(n-k)}$ is F-stat $\sim F(r,n-k)$, used for multiple hyp on β_2. Under null, $\mathbf{M}_1\mathbf{y} = \mathbf{M}_1\mathbf{u} \Rightarrow F_{\beta_2} = \frac{\epsilon^\top\mathbf{P}_{\mathbf{M}_1\mathbf{X}_2}\epsilon/r}{\epsilon^\top\mathbf{M_X}\epsilon/(n-k)}$, where $\epsilon\equiv\mathbf{u}/\sigma$, $\mathbf{P}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{P_X} - \mathbf{P}_1$. P-value for F is $1 - F_{r,n-k}(F_{\beta_2})$. When only 1 restriction, F and 2-tailed t test are the same. If testing all $\beta = 0$, $F = \frac{n-k}{k-1}\times\frac{R_c^2}{1-R_c^2}$. If testing $\beta_1 = \beta_2$, let $\gamma = \beta_2 - \beta_1$ then $F_\gamma = \frac{(\text{RSSR}-\text{SSR}_1-\text{SSR}_2)/k}{(\text{SSR}_1+\text{SSR}_2)/(n-2k)}$

Asymptotic Theory

EDF: $\hat{F}(x) \equiv \frac{1}{n}\sum_t^n\mathbb{I}(x_t \le x)$. FTS: $\text{plim}\hat{F}(x) = F(x)$. CLT: $z_n \equiv \frac{1}{\sqrt{n}}\sum_t^n\frac{x_t-\mu}{\sigma}$ asymptotically $\sim N(0,1)$ if x_t IID.

Uncorrelated x_t with $E(x_t) = 0 \Rightarrow n^{-1/2}\sum_t^n x_t$ goes to $N(0,\lim\frac{1}{n}\sum_t^n\text{Var}(x_t))$.

If $\mathbf{u}\sim IID(\mathbf{0},\sigma^2\mathbf{I})$, $E(u_t\mid\mathbf{X}_t) = 0$, $E(u_t^2\mid\mathbf{X_t} = \sigma^2)$, $\text{plim}\frac{1}{n}\mathbf{X}^\top\mathbf{X} = S_{\mathbf{X}^\top\mathbf{X}}$ where S finite, deterministic, pos def mtrx, then $n^{1/2}(\hat\beta - \beta_0) \xrightarrow{d} N(\mathbf{0},\sigma^2\mathbf{S}_{\mathbf{X}^\top\mathbf{X}}^{-1})$ and $\text{plim}\, s^2(n^{-1}\mathbf{X}^\top\mathbf{X})^{-1} = \sigma^2\mathbf{S}_{\mathbf{X}^\top\mathbf{X}}^{-1}$.

An estimator for cov mtrx is consistent if $\text{plim}(n\widehat{\text{Var}}(\hat\theta)) = \mathbf{V}(\theta)$, where $\mathbf{V}(\theta)$ is limiting cov mtrx of $n^{1/2}(\hat\theta - \theta_0)$

If u IID and testing $\beta_2 = \beta_2^0$, $t_{\beta_2} = \frac{\hat\beta_2 - \beta_2^0}{\sqrt{s^2(\mathbf{X}^\top\mathbf{X})_{22}^{-1}}}$ and $t_{\beta_2} \overset{a}{\sim} N(0,1) \Rightarrow t_{\beta_2} = O_p(1)$. Under null $\beta_2 = \mathbf{0}$, w predetermined regressors $rF_{\beta_2}\overset{a}{\sim}\chi^2(r)$ where $r = k_2$ is dim of β_2.

$W(\hat\beta) = (\mathbf{R}\hat\beta - r)^\top(\mathbf{R}\widehat{\text{Var}}(\hat\beta)\mathbf{R})^{-1}(\mathbf{R}\hat\beta - r)$ is Wald where cov mtrx consistent. $W(\hat\beta)\overset{a}{\sim}\chi^2(r)$ under null where r is r-vector.

Multiple Testing

FWER: $\alpha_m = 1-(1-\alpha)^m$.

Bonferroni: $Pr(\cup_i^m P_i \le \alpha/m) \le \alpha$.

Simes: $P_{(j)} \le j\alpha/m$ for increasing P. Bonf more conservative than Simes.

Power

If $\mathbf{z}\sim N(\mu,\mathbf{I})$ then $\mathbf{z}^\top\mathbf{z}\sim$ non-central $\chi^2(m,\Lambda = \mu^\top\mu)$ If $z\sim N(\mu,\mathbf{I})$ then $\mathbf{z}^\top\mathbf{z}\sim$ non-central $\chi^2(m,\Lambda=\mu^\top\mu)$. Under null $\beta_2 = \mathbf{0}\Rightarrow\Lambda = 0$.

$t(n-k,\lambda)\sim\frac{N(\lambda,1)}{(\chi^2(n-k)/(n-k))^{1/2}}$, $\lambda^2 = \Lambda$.

Pretest estimator: $\acute\beta = \mathbb{I}(F_{\gamma=0} > c_\alpha)\hat\beta + \mathbb{I}(F_{\gamma=0}\le c_\alpha)\tilde\beta$ where c_α is critical value for F test with r and $n-k-r$ df at α. $\acute\beta = \hat\beta$ when pretest rejects and $\acute\beta = \tilde\beta$ when reject.

Confidence & Sandwich Cov Matrices

$\theta_0\in$ confidence set iff $\tau(\mathbf{y},\theta_0)\le c_\alpha$, if θ_0 true then prob is $1-\alpha$. Asymp t-stat: $(\hat\theta-\theta)/s_\theta$. Pivot: same distribution $\forall$DGP. CI exact only if τ pivot. If no stat with known finite sample dist, use Wald with k_2 vector $\hat\theta_2$ asym normal: $(\hat\theta_2 - \theta_{20})^\top(\widehat{\text{Var}}(\hat\theta_2))^{-1}(\hat\theta_2-\theta_{20}) \le c_\alpha$.

$\text{Var}(\hat\beta) = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{\Omega}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}$

$\hat{\mathbf{\Omega}}$: $\frac{1}{n}\sum_t^n\hat{u}_t^2 x_{ti}x_{tj}$

$\widehat{\text{Var}}(\hat\beta) = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\hat{\mathbf{\Omega}}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}$

HC_0: Use $\hat{u}_t^2$ in diag of $\hat{\mathbf{\Omega}}$ and 0 else.

HC_1: Use $\hat{u}_t^2$ in $\hat{\mathbf{\Omega}}$ then mult by $n/(n-k)$

HC_2: $\hat{u}_t^2/(1-h_t)$ with $h_t \equiv \mathbf{X_t}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X_t}^\top$.

HC_3: $\hat{u}_t^2/(1-h_t)$ jackknife, for big varianc.

Ignore hetero for std err of sample mean.
HAC for when u_t hetero and/or autocorr.

$\hat{\mathbf{\Sigma}} = (1/n)\mathbf{X}^\top\hat{\mathbf{\Omega}}\mathbf{X}$ (Newey-West/H. White)

$f(a+h) = f(a) + hf'(a+\lambda h)$, $h = b-a$, $0<\lambda<1$.

Bootstrap

$\hat{p}^*(\hat\tau) = \frac{1}{B}\sum_b^B\mathbb{I}(\tau_b^* > \hat\tau)$

$\hat{p}_{et}^*(\hat\tau) = 2\min\left(\frac{1}{B}\sum_b^B\mathbb{I}(\tau_b^*\le\hat\tau), \frac{1}{B}\sum_b^B\mathbb{I}(\tau_b^* >\right.$ when biased param est and 2-tailed. $\hat{p}^*(\hat\tau)\to p(\hat\tau)$ as $B\to\infty$. τ pivotal.

Instrumental Variables

Can define $E(u_t\mid\Omega_t) = 0$.

Err in variables: indep vars in regr model measured with err. $E(u_t\mid x_t)\neq 0$, $\text{Cov}(x_t.u_t)\ne 0$. OLS est biased and inconsist. Simultaneity: two or more endog vars jointly determined by sys of simultaneous eq.

Assume, $E(\mathbf{uu}^\top) = \sigma^2\mathbf{I}$ and at least one in $\mathbf{X}$ not predetermined wrt disturb. $n\times k$ mtrx $\mathbf{W}$ with $\mathbf{W_t}\in\mathbf{\Omega_t}$. Col of $\mathbf{W}$ are IV. $E(u_t\mid\mathbf{W}_t) = 0$, $\mathbf{W}^\top(\mathbf{y}-\mathbf{X}\beta) = \mathbf{0}$ are unbiased est eq. $\hat\beta_{IV}\equiv(\mathbf{W}^\top\mathbf{X})^{-1}\mathbf{W}^\top\mathbf{y}$. $\mathbf{W}^\top\mathbf{X}$ must be non-sing. $\hat\beta_{IV}$ generally biased but consistent. Assume $\mathbf{S}_{\mathbf{W}^\top\mathbf{X}} \equiv \text{plim}\frac{1}{n}\mathbf{W}^\top\mathbf{X}$ is deterministic and non-sing. Same for $\mathbf{S}_{\mathbf{W}^\top\mathbf{W}}$. $\hat\beta_{IV}$ consistent iff $\text{plim}\frac{1}{n}\mathbf{W}^\top\mathbf{u} = 0$. Asym cov mtrx of IV est: $\sigma_0^2\text{plim}(n^{-1}\mathbf{X}^\top\mathbf{P_W X})^{-1}$. $\mathbf{J}$: full col rank, asym deterministic, min asym cov mtrx of IV est.

GIVE: $\hat\beta_{IV} = (\mathbf{X}^\top\mathbf{P_W X})^{-1}\mathbf{X}^\top\mathbf{P_W y}$ IV est: Col of $P_W X$ should be lin indep.

IV asym normal like all est.
$\widehat{\mathrm{Var}}(\hat{\beta}_{IV}) = \hat{\sigma}^2(\mathbf{X}^\top \mathbf{P_W X})^{-1}$.
$\hat{\sigma}^2 = ||\mathbf{y} - \mathbf{X}\hat{\beta_{\mathbf{IV}}}||^2/n$.

Generalized Least Squares

Consider $E(\mathbf{uu}^\top) = \mathbf{\Omega}$, $\mathbf{\Omega}^{-1} = \mathbf{\Psi\Psi}^\top$
$\hat{\beta}_{GLS} = (\mathbf{X}^\top \mathbf{\Omega^{-1} X})^{-1}\mathbf{X}^\top \mathbf{\Omega^{-1} y}$
$E(\mathbf{\Psi}^\top \mathbf{uu}^\top \mathbf{\Psi}) = \mathbf{I}$.
$\mathrm{Var}(\hat{\beta}^\top_{GLS}) = (\mathbf{X}^\top \mathbf{\Omega^{-1} X})^{-1}$ autocovariance of AR(1): $\mathbf{\Omega}(p) = \frac{\sigma_\epsilon^2}{1-\rho^2}\times$ mtrx with 1 diag and ρ^i increasing away from diag.
$\mathrm{Cov}(u_t, u_{t-1}) = \rho\sigma_u^2$.
$\sigma_u^2 == \sigma_\epsilon^2/(1-\rho^2)$
$u_t = \rho u_{t-1} + \epsilon_t$, $\epsilon_t \sim IID(0, \sigma_\epsilon^2)$

3 Questions

Q:$X \sim N(0,1)$, $Z \equiv \mu + \sigma X$ mean and var?
A:Mean: μ, Var: σ^2.
Q:If X_1, X_2 indep, show $E(X_1 \mid X_2) = 0$. **A**:$f(x_1|x_2) = f(x_1,x_2)f(x_2) = f(x_1)f(x_2)f(x_2) = f(x_1)$.
Q:Prove $\mathbf{P_X M_X = 0}$.
A:$\mathbf{P_X + M_X = I \Rightarrow P_X + P_X M_X = P_X \Rightarrow P_X M_X = 0}$
Q:$\mathcal{S}(W) \neq \mathcal{S}(X)$, show $P \equiv \mathbf{X(W^\top X)^{-1}W^\top}$ idempotent but not sym. **A**:idem easy. Find $P^\top$. P projects onto $S(X)$. $Py = Xb$. Image of P is all $S(X)$ but image of $P^\top$ is $S(W)$. $I-P$ projects to $S^\perp(W)$. $P_W(I-P) = 0$.
Q:$y = \beta_1\iota + X_2 beta_2 + u$ show using FWL that β_i can be written as $\ldots with$M$_\iota$. **A**:OLS resid orthog to regressors $\Rightarrow X_2^\top(M_\iota y - M_\iota X_2\beta_2) = 0$. OLS of β_1 must be resid have mean 0, $\iota^\top(y - \beta_1\iota - X_2\beta_2) = 0$.
Q:Show that $P_X - P_1 = P_{M_1 X_2}$. **A**:vector $\gamma \in s(M_1, X_2)$ is $M_1 X_2\gamma$. Pre-multiply by $P_x - P_1$ and get $M_1 X_2 \gamma$. So invariant under projection. Now consider z orthog to $S(M_1, X_2)$, $X_2^\top M_1 z = 0$. Show $P_x z = P_1 z$.
Q:What about $M_\iota X$ and $P_\iota X$ when $n \times 3$?
A:First col of $M_\iota X$ is 0. Other two are centered X. Each col of $P_\iota X$ has n copies of mean from col in X. First col is ι, 2nd: $\overline{x_2}\iota$. Know $P_\iota P_X = P_\iota$ show that $M_\iota M_X = M_X$. Finally show $P_\iota M_X = 0$.
Q:Show that leverage h_t is square of cos of angle bt e_t and its proj. **A**:proj: $P_X e_t$. $\cos^2\theta$: $e_t^\top P_X e_t$.
Q:What is $\mathrm{Tr}(P_X)$? **A**:$\mathrm{Tr}(P_X) = k$ if full rank or r if less. $\mathrm{Tr}(M_X) = n - r$.
Q:Show $(\mathrm{Cov}(b_1, b_2))^2 \leq \mathrm{Var}(b_1)\mathrm{Var}(b_2)$.
A:determinant of pos semi def matrix is nonnegative. Find det of $\mathrm{Var}(b) = Var(b_1)Var(b_2) - (Cov(b_1, b_2))^2$.
Q:If A pos def matrix, show that A^{-1} is pos def. **A**:non-zero x: $x^\top A^{-1} x = (A^{-1}x)^\top A(A^{-1}x)$. quad form must be pos, and so is this.