Homework 8

Zachary Probst

November 18, 2021

Problem 1: Let E be measurable with $m(E) < \infty$ and $f : E \to \mathbb{R}$ a bounded measurable function. Show that we have

$$\Big|\int_E f\Big| = \int_E |f|$$

if and only if $f \geq 0$ a.e on E or $f \leq 0$ a.e on E.

Assume that $f \geq 0$ a.e on E. Let $E' \coloneqq \{x \in E : f(x) < 0\}$. Let $c \coloneqq \inf\{f(x) : x \in E'\}$ Define a function $z : E \to \{c, 0\}$ as

$$z(x) \coloneqq \begin{cases} c & x \in E' \\ 0 & x \notin E' \end{cases}$$

Since $f \geq 0$ a.e on E, we know $m(E') = 0$ so

$$\int_E z = \int_{E \setminus E'} 0 + \int_{E'} c = 0$$

So $z \leq f$ is a simple function and a contender for the supremum in $\mathcal{L}(f)$

$$0 = \int_E z \leq \mathcal{L}(f) = \int_E f$$

Now we know that $\int_E f \geq 0$ so

$$\Big|\int_E f\Big| = \int_E f = \int_{E \setminus E'} f + \int_{E'} f = \int_{E \setminus E'} f = \int_{E \setminus E'} |f| = \int_E |f|$$

Now assume that $f \leq 0$ a.e on E. Then $-f \geq 0$ a.e on E and we have already proved that

$$\left|\int_E -f\right| = \int_E |-f| = \int_E |f|$$

But the lebesgue integral satisfies linearity so

$$\left|\int_E -f\right| = \left|-\int_E f\right| = \left|\int_E f\right|$$

And putting that together we get

$$\left|\int_E f\right| = \int_E |f|$$

Now assume that $\left|\int_E f\right| = \int_E |f|$. Let $E' := \{x \in E : f(x) > 0\}$ and $E'' := \{x \in E : f(x) < 0\}$. By way of contradiction, let us assume that $m(E') > 0$ and $m(E'') > 0$. $f = 0$ on $E \setminus E' \cup E''$ so we can decompose the integral of f like so:

$$\left|\int_E f\right| = \int_E |f| = \int_{E'} |f| + \int_{E''} |f| = \left|\int_{E'} f\right| + \left|\int_{E''} f\right|$$

But

$$\left|\int_E f\right| = \left|\int_{E'} f + \int_{E''} f\right| = \left|\int_{E'} f\right| + \left|\int_{E''} f\right|$$

Which is only possible if both $\int_{E'} f \geq 0$ and $\int_{E''} f \geq 0$ or if one term were equal to 0. But $\int_{E''} f < 0$ and $\int_{E'} f > 0$ so we have a contradiction.

So it must be the case that either $m(E') = 0$ or $m(E'') = 0$. This means that either $f \leq 0$ a.e on E or $f \geq 0$ a.e on E.

Chapter 4 Problem 26: Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

Let $E \coloneqq [0, \infty)$. Define $f_n : E \to \mathbb{R}$ as $f_n(x) = \frac{1}{n}$. Clearly f_n is a decreasing sequence of functions that converges pointwise to the zero function, since $\lim_{n\to\infty} \frac{1}{n} = 0$. However, we can write each f_n as a simple function whose integral is

$$\int_E f_n = \int_{[0,1)} f_n + \int_{[1,2)} f_n + \ldots = \sum_{i=1}^{\infty} \frac{1}{n} \geq \sum_{i=1}^{n} \frac{1}{n} = 1$$

Letting $n \to \infty$ we get

$$\lim_{n\to\infty} \int_E f_n \geq 1$$

But $\int_E 0 = 0$, therefore the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

Chapter 4 Problem 27 Prove the following estimation of Fatou's Lemma: If $\{f_n\}$ is a sequence of non-negative measurable functions on E, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n$$

Let $N \in \mathbb{N}$, define $c \coloneqq \inf\{f_n : n \geq N\}$ then $c \leq f_n$ on E $\forall n \geq N$. Let $\phi_N : E \to \{c\}$ be a simple function on E with $\phi_N \leq f_n$ $\forall n \geq N$ so

$$\int_E \phi_N \leq \int_E f_n$$

$\int_E \phi_N$ is a lower bound for $\{\int_E f_n : n \geq N\}$ hence

$$\int_E \phi_N \leq \inf \left\{ \int_E f_n : n \geq N \right\}$$

ϕ_N converges pointwise to $\liminf f_n$ by the definition of $\liminf$ and ϕ_N is non-negative because f_n is non-negative. We also know that ϕ_N is increasing because a sequence of inf over smaller and smaller sets is weakly increasing hence we can apply the Monotone Convergence Theorem.

$$\int_E \liminf f_n = \lim_{N\to\infty} \int_E \phi_N \leq \lim_{N\to\infty} \inf \left\{ \int_E f_n : n \geq N \right\} = \liminf \int_E f_n$$