Homework 5

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Chapter 3 Problem 1: Suppose f and g are continuous functions on $[a, b]$. Show that if $f = g$ a.e on $[a, b]$, then, in fact $f = g$ on [a,b]. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E?

Suppose $f(x) \neq g(x)$ for some $x \in [a, b]$. Let $\epsilon = |g(x) - f(x)|$. Since f is continuous, define $\delta_f > 0$ such that for $y \in [a, b]$

$$|x - y| < \delta_f \implies |f(x) - f(y)| < \frac{\epsilon}{2}$$

Define δ_g in the same way for g.

$$|x - y| < \delta_g \implies |g(x) - g(y)| < \frac{\epsilon}{2}$$

Now let $\delta = \min\{\delta_f, \delta_g\}$. On the interval $[x, x+\delta]$ (or $[x, x-\delta]$ if $b-x < \delta$) f can only converge less than $\epsilon\,/\,2$ closer to g and likewise g can only converge less than $\epsilon\ /\ 2$ closer to f for all points in that neighborhood.

$$\forall y \in [x, x + \delta],\ f(y) \neq g(y)$$

But then the interval $[x, x + \delta]$ has length $\delta > 0$ and hence has a non-zero measure so it is not true that $f = g$ a.e on $[a, b]$.

This assertion is not true if we replace $[a, b]$ with a measurable set E with $m(E) = 0$. This is because in fact any functions f and g satisfy the property $f = g$ a.e on E, in particular functions with $f(x) \neq g(x)$ for some $x \in E$.

Chapter 3 Problem 3: Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable?

Let E denote the domain of f and let $A \subset E$ be the set of all $x \in E$ where $f(x)$ is discontinuous. A is a finite set so we can enumerate every $x \in A$ as $x_1 \ldots x_n$.

Define $I_1 = [-\infty, x_1) \cap E$ and $I_{n+1} = (x_n, \infty] \cap E$. For each $1 \leq j \leq n-1$, define $I_j = (x_j, x_{j+1}) \cap E$. Now f is continuous on each I_j by construction.

Define $g_j : I_j \to \mathbb{R} = f : I_j \to \mathbb{R}$ so that each g_j is continuous. Each I_j is measurable since it is an intersection of measurable sets, hence each g_j is measurable.

Finally, define $h_j : \{x_j\} \to \mathbb{R} = f(x_j)$ which is trivially measurable.

Since measurable functions are closed under addition, we have

$$f = \sum_{j=1}^{n+1} g_j + \sum_{j=1}^{n} h_j \in \mathcal{L}$$

So f is indeed measurable.

Chapter 3 Problem 5: Suppose the function f is defined on a measurable set E and has the property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational number c. Is f necessarily measurable?

The rationals are dense in $\mathbb{R}$, hence for any $c \in \mathbb{R}$ and any $n \in \mathbb{N}$, we can find a $q \in \left[c, c + \frac{1}{n}\right]$ such that $q \in \mathbb{Q}$.

$$\{x \in E \mid f(x) > c\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > q > c\}$$

Where $q \in \left[c, c + \frac{1}{n}\right]$

Each element of this union is measurable by assumption since $q \in \mathbb{Q}$ and it is a countable union, so $\{x \in E \mid f(x) > c\}$ is measurable and therefore f is measurable.

Chapter 3 Problem 14: Let f be a measurable funciton on E that is finite a.e. on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E such that f is bounded on F and $m(E \setminus F) < \epsilon$.

Let $F = \{x \in E : |f(x)| < \infty\}$. Then

$$m(E \setminus F) = m(\{x \in E \mid f(x) = \pm\infty\}) = 0$$

By definition of f.

Clearly $F \subset E$, now we show that $F \in \mathcal{L}$

$$F = \bigcup_{n=1}^{\infty} \{x \in E : |f(x)| < n\}$$

Which are all measurable due to the measurability of f, hence F is a countable union of measurable sets and is itself measurable.