Homework 1

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Chapter 1 Problem 33:
Show that the Nested Set Theorem is false if F_1 is unbounded.

Let $\{F_n\}_{n=1}^{\infty} = \{[n, \infty), n \in \mathbb{N}\}$. We have $F_1 = [1, \infty)$ so F_1 is unbounded.

Assume that the Nested Set Theorem holds for descending countable collections of nonempty closed sets. Then there is some $x \in \mathbb{R}$ such that

$$x \in \cap_{n=1}^{\infty} F_n$$

Let $n' = \lceil x \rceil$ be the smallest integer larger than x. Then clearly $x \notin [n', \infty)$ and so $x \notin F_{n'}$. But this is a contradiction so it must be that

$$\cap_{n=1}^{\infty} F_n = \emptyset$$

And the Nested Set Theorem does not hold for F_1 unbounded.

Chapter 2 Problem 1:
Prove that if there is a set A in the collection $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

If we have $A \subseteq B$ then we can write $B = A \cup (B \setminus A)$

$$\begin{aligned} \Rightarrow m(B) &= m(A \cup (B \setminus A)) \\ &= m(A) + m(B \setminus A) && \text{(by countable additivity)} \\ &\geq m(A) \end{aligned}$$

Chapter 2 Problem 2:
Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

Assume that $m(\emptyset) > 0$. Define a countable collection of sets $\{E_k\}_{k=1}^{\infty}$ where

$$E_k = \begin{cases} A & k = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Then since A and $\emptyset$ are clearly disjoint, we apply countable additivity

$$m(A) = m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

This sum diverges since we assumed that $m(\emptyset) > 0$, but this is a contradiction since $m(A) < \infty$. Hence, it must be that $m(\emptyset) = 0$.

Chapter 2 Problem 3:
Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$

If $\{E_k\}_{k=1}^{\infty}$ is disjoint then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

And we are done. In the case where $\{E_k\}_{k=1}^{\infty}$ is not pair-wise disjoint then we define a new set:

$$F_k = \{e \in E_k \mid e \notin \cup_{i=1}^{\infty} E_i \setminus E_k\}$$

$\{F_k\}_{k=1}^{\infty}$ is a pair-wise disjoint collection because each F_k has all the elements of E_k except for the ones which are shared in other E_i. This means that

$$m(F_k) \leq m(E_k)\ \forall k$$

By monotonicity since $F_k \subseteq E_k$. Finally, using the fact that we have only removed non-unique values from $\{E_k\}_{k=1}^{\infty}$ to build $\{F_k\}_{k=1}^{\infty}$

$$\begin{aligned} m(\bigcup_{k=1}^{\infty} E_k) = m(\bigcup_{k=1}^{\infty} F_k) &= \sum_{k=1}^{\infty} m(F_k) && (F_k \text{ disjoint}) \\ &\leq \sum_{k=1}^{\infty} m(E_k) \end{aligned}$$

Chapter 2 Problem 4:
A set function c, defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**

To prove *countable additivity*, let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint sets of $\mathbb{R}$. Since $\{E_k\}_{k=1}^{\infty}$ is disjoint, by definition $\cup_{k=1}^{\infty} E_k$ will have the same cardinality as the sum of the cardinality of each E_k. Well this is the same thing as

$$c\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} c(E_k)$$

To prove *translation invariance*, for any set $A \subset \mathbb{R}$ we can create a bijection $f : A \to A + y$ where $A + y = \{x + y \mid x \in A\}$ for a given $y \in \mathbb{R}$.

$$f(x) = x + y$$

Let $x \in A + y$ be arbitrary. Consider the fact that $x - y \in A$ and $f(x - y) = x - y + y = x$ so f is surjective.

Assume that $f(a) = f(b)$. Well then $a + y = b + y \Rightarrow a = b$ and so f is injective.

Since f is injective and surjective, it must be bijective and $|A| = |A + y|$ so

$$c(A) = c(A + y)$$