Honours Analysis 3

Zachary Probst *

September 7, 2021

1 Borel Sets

We will work for some time on $\mathbb{R}$ exclusively. Before beginning Measure Theory: a quick recap of Topology.

Definition 1.1 (Open Set). *A subset $U \subset \mathbb{R}$ is called* open *if either $U = \emptyset$ or else*

$$\forall x \in U, \exists r > 0 \text{ such that } (x - r, x + r) \subset U$$

Some examples of open sets: $\emptyset, \mathbb{R}, (a, b), (a, \infty), (-\infty, a)$. There are many more because any union of an open set is still open and any finite intersection of open sets is open.

Definition 1.2 (Closed Set). *$F \subset \mathbb{R}$ is called* closed *if $\mathbb{R} \setminus F := F^c$ is open. F is closed $\iff$ F contains all points $x \in \mathbb{R}$ which have the property that $\forall r > 0, (x - r, x + r) \cap F \neq \emptyset$.*

If $F \subset \mathbb{R}$ is any set, the closure of F, denoted by $\overline{F}$, is the smallest closed set that contains F.

Definition 1.3 (Compact). *A subset $G \subset \mathbb{R}$ is* compact *if given any collection $\{U_i\}_{i \in I}$ of open sets $U_i \subset \mathbb{R}$ with $G \subset \cup_{i \in I} U_i$, there exists $J \subset I$, J finite, such that $G \subset \cup_{j \in J} U_j$*

Theorem 1.1 (Heine-Borel). *$G \subset \mathbb{R}$ is compact $\iff$ G is closed and bounded. To be bounded means $G \subset (a, b)$ for some $a, b \in \mathbb{R}$.*

Corollary 1.1.1 (Nested Set Theorem). *Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of non-empty, bounded, closed sets $F_n \subset \mathbb{R}$ with $F_{n+1} \subset F_n \forall n$, then*

$$\cap_{n=1}^{\infty} F_n \neq \emptyset$$

*Valentino Tosatti

Proof. Suppose $\cap_{n=1}^{\infty} F_n = \emptyset$ so let $U_n = F_n^c$ be open sets, such that $\cup_{n=1}^{\infty} U_n = \mathbb{R}$. We also have that $U_n \subset U_{n+1}$, since the F_n were nested. Now F_1 is compact by Heine-Borel and $F_1 \subset \cup_{n=1}^{\infty} U_n \Rightarrow$ by compactness I can find a finite subcover of F_1, say $F \subset \cup_{n=1}^{N} U_n = U_N = F_N^c$
On the other hand $F_N \subset F_1$ by the nested property which implies $F_N = \emptyset$ which is a contradiction. $\square$

2 Measure Theory

We want to measure the size of a set. We will deal with a subset of $\mathbb{R}$.
It turns out that one needs to select a class of subsets of $\mathbb{R}$ that one wants to measure. This class of subsets will have certain properties which are as follows.

Definition 2.1 (σ-algebra)**.** *A collection $\mathcal{A}$ of subsets of $\mathbb{R}$ is called a σ-algebra if it satisfies*

1. $\emptyset \in \mathcal{A}$
2. *If* $A \in \mathcal{A}$ *then* $A^c \in \mathcal{A}$
3. *If* $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ *then* $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$

Observe the following:

- $\mathbb{R} \in \mathcal{A}$ always
- If $\{A_n\}_{n=1}^{N} \subset \mathcal{A}$ then $\cup_{n=1}^{N} A_n \in \mathcal{A}$ (just define $A_n = \emptyset$ for $n > N$)
- If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ then $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$ (since $(\cap_{n=1}^{\infty} A_n)^c = \cup_{n=1}^{\infty} A_n^c$)
- If $A, B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$ too since $A \setminus B = A \cap B^c$

Examples:

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$ "Minimal σ-algebra"
2. $\mathcal{A} = \mathcal{P}(\mathbb{R}) =$ Collection of all subsets of $\mathbb{R}$. "Maximum σ-algebra"

In fact, if $\mathcal{A}$ is any σ-algebra, then $\{\emptyset, \mathbb{R}\} \subseteq \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$

For better examples, let F be any collection of subsets of $\mathbb{R}$. I want to make F into a σ-algebra. Define $m = \{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra that satisfies } F \subset \mathcal{A}\}$.
$m \neq \emptyset$ since it contains $\mathcal{P}(\mathbb{R})$

If $\mathcal{A}, \mathcal{B} \in m$, I can define $\mathcal{A} \cap \mathcal{B} = \{A \subset \mathbb{R} \mid A \in \mathcal{A} \text{ and } A \in \mathcal{B}\}$ and I can do the same for $\cap_{i \in I} \mathcal{A}$ arbitrary intersection of σ-algebra is still a σ-algebra

Define $\hat{F}_i = \cap_{\mathcal{A} \in m} \mathcal{A}$ as a σ-algebra and $F \subset \hat{F}$ and it is the minimal σ-algebra with these properties. If G is a σ-algebra with $F \subset G$, then $\hat{F} \subset G$. $\hat{F}$ is the σ-algebra generated by F. Concretely, $\hat{F}$ consists of all subsets of $\mathbb{R}$ that can be constructed by applying countable unions, intersections, and complements to elements of F.

Definition 2.2 (Borel Sets)**.** *The σ-algebra $\mathcal{B}$ of Borel Sets is the σ-algebra $\hat{F}$ generated by*

$$F = \{U \subset \mathbb{R} \mid U \text{ open } \}$$

Remark. *$\mathcal{B}$ is also the σ-algebra generated by the family of all closed subsets of $\mathbb{R}$*

Singletons $\{x\} \subset \mathbb{R}$ are closed so if $A \subset \mathbb{R}$ is at most countable then A is Borel. (e.g $\mathbb{Q} \subset \mathbb{R}$) (e.g $\mathbb{R} \setminus \mathbb{Q}$)
Not all Subsets of $\mathbb{R}$ are Borel. One can actually show that the cardinality of $\mathcal{B}$ is the same as the cardinality of $\mathbb{R}$. On the other hand $\mathcal{P}(\mathbb{R})$ has strictly larger cardinality.

3 Legesgue Outer Measure

We are hoping to measure the size of subsets of $\mathbb{R}$. Ideally we would like to find or construct a function

$$m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\} = [0, \infty]$$

Which satisfies the following:

1. If $I = [a, b]$ or (a, b) or $[a, b)$, or $(a, b]$, $a, b \in \mathbb{R}, a \leq b$ then $m(I) = b - a =$ measure of interval

2. m is translation invariant. i.e if $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, let $E + x = \{y + x \mid y \in E\}$ then $m(E + x) =$ m(E)

3. If $\{E_j\}_{j=1}^n$ is a finite collection of pairwise disjoint $E_j \subset \mathbb{R}$ then

$$m\left(\cup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$

4. The same as (3) except for $n = \infty$

Theorem 3.1. *There is no such m satisfying all 4 requirements*

The proof for this will come later. The solution for this is that we do not try to measure all subsets of $\mathbb{R}$. So we have $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ but now we will just be happy with $m : \mathcal{A} \to [0, \infty]$ where $\mathcal{A}$ is a σ-algebra which has enough elements. For example $\mathcal{A} > \mathcal{B}$.

We will follow H. Lebesgue as we proceed in two steps.

<u>Step 1:</u> construct Lebesgue outer measure $m^{\star} : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ satisfying requirements 1,2, and 3.

<u>Step 2:</u> Use $m^{\star}$ to define $\mathcal{A}$ and let $m \subset m^{\star} \mid \mathcal{A}$