

# Homework 8

Zachary Probst

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**Problem 1:** Let  $E$  be measurable with  $m(E) < \infty$  and  $f : E \rightarrow \mathbb{R}$  a bounded measurable function. Show that we have

$$\left| \int_E f \right| = \int_E |f|$$

if and only if  $f \geq 0$  a.e on  $E$  or  $f \leq 0$  a.e on  $E$ .

Assume that  $f \geq 0$  a.e on  $E$ . Let  $E' := \{x \in E : f(x) < 0\}$ . Let  $c := \inf\{f(x) : x \in E'\}$ . Define a function  $z : E \rightarrow \{c, 0\}$  as

$$z(x) := \begin{cases} c & x \in E' \\ 0 & x \notin E' \end{cases}$$

Since  $f \geq 0$  a.e on  $E$ , we know  $m(E') = 0$  so

$$\int_E z = \int_{E \setminus E'} 0 + \int_{E'} c = 0$$

So  $z \leq f$  is a simple function and a contender for the supremum in  $\mathcal{L}(f)$

$$0 = \int_E z \leq \mathcal{L}(f) = \int_E f$$

Now we know that  $\int_E f \geq 0$  so

$$\left| \int_E f \right| = \int_E f = \int_{E \setminus E'} f + \int_{E'} f = \int_{E \setminus E'} f = \int_{E \setminus E'} |f| = \int_E |f|$$

Now assume that  $f \leq 0$  a.e on  $E$ . Then  $-f \geq 0$  a.e on  $E$  and we have already proved that

$$\left| \int_E -f \right| = \int_E |-f| = \int_E |f|$$

But the lebesgue integral satisfies linearity so

$$\left| \int_E -f \right| = \left| - \int_E f \right| = \left| \int_E f \right|$$

And putting that together we get

$$\left| \int_E f \right| = \int_E |f|$$

Now assume that  $\left| \int_E f \right| = \int_E |f|$ . Let  $E' := \{x \in E : f(x) > 0\}$  and  $E'' := \{x \in E : f(x) < 0\}$ . By way of contradiction, let us assume that  $m(E') > 0$  and  $m(E'') > 0$ .  $f = 0$  on  $E \setminus E' \cup E''$  so we can decompose the integral of  $f$  like so:

$$\left| \int_E f \right| = \int_E |f| = \int_{E'} |f| + \int_{E''} |f| = \left| \int_{E'} f \right| + \left| \int_{E''} f \right|$$

But

$$\left| \int_E f \right| = \left| \int_{E'} f + \int_{E''} f \right| = \left| \int_{E'} f \right| + \left| \int_{E''} f \right|$$

Which is only possible if both  $\int_{E'} f \geq 0$  and  $\int_{E''} f \geq 0$  or if one term were equal to 0. But  $\int_{E''} f < 0$  and  $\int_{E'} f > 0$  so we have a contradiction.

So it must be the case that either  $m(E') = 0$  or  $m(E'') = 0$ . This means that either  $f \leq 0$  a.e on  $E$  or  $f \geq 0$  a.e on  $E$ .

**Chapter 4 Problem 26:** Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

Let  $E := [0, \infty)$ . Define  $f_n : E \rightarrow \mathbb{R}$  as  $f_n(x) = \frac{1}{n}$ . Clearly  $f_n$  is a decreasing sequence of functions that converges pointwise to the zero function, since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . However, we can write each  $f_n$  as a simple function whose integral is

$$\int_E f_n = \int_{[0,1)} f_n + \int_{[1,2)} f_n + \dots = \sum_{i=1}^{\infty} \frac{1}{n} \geq \sum_{i=1}^n \frac{1}{n} = 1$$

Letting  $n \rightarrow \infty$  we get

$$\lim_{n \rightarrow \infty} \int_E f_n \geq 1$$

But  $\int_E 0 = 0$ , therefore the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

**Chapter 4 Problem 27** Prove the following estimation of Fatou's Lemma: If  $\{f_n\}$  is a sequence of non-negative measurable functions on  $E$ , then

$$\int_E \liminf f_n \leq \liminf \int_E f_n$$

Let  $N \in \mathbb{N}$ , define  $c := \inf\{f_n : n \geq N\}$  then  $c \leq f_n$  on  $E \forall n \geq N$ . Let  $\phi_N : E \rightarrow \{c\}$  be a simple function on  $E$  with  $\phi_N \leq f_n \forall n \geq N$  so

$$\int_E \phi_N \leq \int_E f_n$$

$\int_E \phi_N$  is a lower bound for  $\{\int_E f_n : n \geq N\}$  hence

$$\int_E \phi_N \leq \inf \left\{ \int_E f_n : n \geq N \right\}$$

$\phi_N$  converges pointwise to  $\liminf f_n$  and  $0 \leq \phi_N \leq \liminf f_n$  so  $|\phi_N| \leq \liminf f_n$  and we can apply the Lebesgue Dominated Convergence Theorem.

$$\int_E \liminf f_n = \lim_{N \rightarrow \infty} \int_E \phi_N \leq \lim_{N \rightarrow \infty} \inf \left\{ \int_E f_n : n \geq N \right\} = \liminf \int_E f_n$$