Homework 3

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Chapter 2 Problem 19: Let E have a finite outer measure. Show that if E is not measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which

$$m^\star(\mathcal{O} \setminus E) > m^\star(\mathcal{O}) - m^\star(E)$$

If $E \notin \mathcal{L}$ then by the negation of the properties of a measurable set there is some $N \in \mathbb{R}$ for which $m^\star(\mathcal{O} \setminus E) \geq N$ for any open set $\mathcal{O}$ that contains E.

By the definition of outer measure, there is a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ which covers E and for which

$$\sum_{k=1}^{\infty} \ell(I_k) < m^\star(E) + N$$

Define $\mathcal{O} \coloneqq \cup_{k=1}^\infty I_k$. Then $\mathcal{O}$ is an open set containing E. By the definition of outer measure of $\mathcal{O}$,

$$m^\star(\mathcal{O}) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^\star(E) + N$$

so that

$$m^\star(\mathcal{O}) - m^\star(E) < N \qquad (\mathrm{m}^\star(E) < \infty)$$

And we conclude with

$$m^\star(\mathcal{O} \setminus E) \geq N > m^\star(\mathcal{O}) - m^\star(E)$$

Chapter 2: Problem 22 For any set A, define $m^{\star\star}(A) \in [0, \infty]$ by

$$m^{\star\star}(A) = \inf\{m^{\star}(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open }\}$$

How is this set function $m^{\star\star}$ related to outer measure $m^{\star}$?

If $m^{\star}(A) = \infty$ then $m^{\star\star}(A) = \infty$ by monotonicity of $A \subset \mathcal{O} \Rightarrow m^{\star}(\mathcal{O}) \geq \infty$. If $m^{\star}(A) < \infty$ and A is measurable then $\forall \epsilon > 0$, $\exists \mathcal{O}$ open such that

$$\begin{aligned} m^{\star}(\mathcal{O} \setminus A) &< \epsilon \\ m^{\star}(\mathcal{O}) - m^{\star}(A) &< \epsilon \\ m^{\star}(\mathcal{O}) &< m^{\star}(A) + \epsilon \end{aligned}$$

So by the definition of infimum

$$m^{\star}(A) = \inf\{m^{\star}(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open }\} = m^{\star\star}(A)$$

Chapter 2 Problem 26: Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A.

$$m^\star\left(A\cap\bigcup_{k=1}^{\infty}E_k\right)=\sum_{k=1}^{\infty}m^\star\left(A\cap E_k\right)$$

By the distributive property of intersections we can write

$$m^\star\left(A\cap\bigcup_{k=1}^{\infty}E_k\right)=m^\star\left(\bigcup_{k=1}^{\infty}A\cap E_k\right)$$

By countable subadditivity we have

$$m^\star\left(\bigcup_{k=1}^{\infty}A\cap E_k\right)\leq\sum_{k=1}^{\infty}m^\star\left(A\cap E_k\right)$$

For the opposite inequality, for each n we proved in class that

$$m^\star\left(\bigcup_{k=1}^{n}A\cap E_k\right)=\sum_{k=1}^{n}m^\star\left(A\cap E_k\right)$$

But $\cup_{k=1}^{n}A\cap E_k\subset\cup_{k=1}^{\infty}A\cap E_k$, hence

$$m^\star\left(\bigcup_{k=1}^{\infty}A\cap E_k\right)\geq m^\star\left(\bigcup_{k=1}^{n}A\cap E_k\right)=\sum_{k=1}^{n}m^\star\left(A\cap E_k\right)\quad\forall n$$

Take the limit as $n\to\infty$ to get

$$m^\star\left(\bigcup_{k=1}^{\infty}A\cap E_k\right)\geq\sum_{k=1}^{\infty}m^\star\left(A\cap E_k\right)$$

As desired.

Chapter 2 Problem 33: Let E be a non-measurable set of finite outer measure. Show that there is a G_δ set G that contains E for which

$$m^\star(E) = m^\star(G)\,, \text{ while } m^\star(G \setminus E) > 0$$

Since E is non-measurable, any G_δ set that covers E will have $m^\star(G \setminus E) > 0$ by the properties of measurability.

Indeed if $\exists G'$ such that G' is G_δ and $E \subseteq G'$ and $m^\star(G' \setminus E) = 0$ then E would be measurable.

In Homework 2 Problem 2 we proved that there is always a G_δ set $G \supset E$ (in particular G is measurable) with

$$m^\star(G) = m^\star(E)$$

For any set $E \subset \mathbb{R}$. We now apply that theory to say there does exist a G_δ set G such that

$$m^\star(E) = m^\star(G)\,, \text{ while } m^\star(G \setminus E) > 0$$