

# Honours Analysis 3

Zachary Probst \*

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## 1 Borel Sets

We will work for some time on  $\mathbb{R}$  exclusively. Before beginning Measure Theory: a quick recap of Topology.

**Definition 1.1** (Open Set). *A subset  $U \subset \mathbb{R}$  is called open if either  $U = \emptyset$  or else*

$$\forall x \in U, \exists r > 0 \text{ such that } (x - r, x + r) \subset U$$

Some examples of open sets:  $\emptyset, \mathbb{R}, (a, b), (a, \infty), (-\infty, a)$ . There are many more because any union of an open set is still open and any finite intersection of open sets is open.

**Definition 1.2** (Closed Set).  *$F \subset \mathbb{R}$  is called closed if  $\mathbb{R} \setminus F := F^c$  is open.*

*$F$  is closed  $\iff F$  contains all points  $x \in \mathbb{R}$  which have the property that  $\forall r > 0, (x - r, x + r) \cap F \neq \emptyset$ .*

If  $F \subset \mathbb{R}$  is any set, the closure of  $F$ , denoted by  $\overline{F}$ , is the smallest closed set that contains  $F$ .

**Definition 1.3** (Compact). *A subset  $G \subset \mathbb{R}$  is compact if given any collection  $\{U_i\}_{i \in I}$  of open sets  $U_i \subset \mathbb{R}$  with  $G \subset \cup_{i \in I} U_i$ , there exists  $J \subset I$ ,  $J$  finite, such that  $G \subset \cup_{j \in J} U_j$*

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\*Notes from the lectures of Valentino Tosatti

**Theorem 1.1** (Heine-Borel).  $G \subset \mathbb{R}$  is compact  $\iff G$  is closed and bounded. To be bounded means  $G \subset (a, b)$  for some  $a, b \in \mathbb{R}$ .

**Corollary 1.1.1** (Nested Set Theorem). Let  $\{F_n\}_{n=1}^\infty$  be a countable collection of non-empty, bounded, closed sets  $F_n \subset \mathbb{R}$  with  $F_{n+1} \subset F_n \forall n$ , then

$$\bigcap_{n=1}^\infty F_n \neq \emptyset$$

*Proof.* Suppose  $\bigcap_{n=1}^\infty F_n = \emptyset$  so let  $U_n = F_n^c$  be open sets, such that  $\bigcup_{n=1}^\infty U_n = \mathbb{R}$ . We also have that  $U_n \subset U_{n+1}$ , since the  $F_n$  were nested. Now  $F_1$  is compact by Heine-Borel and  $F_1 \subset \bigcup_{n=1}^\infty U_n \Rightarrow$  by compactness I can find a finite subcover of  $F_1$ , say  $F \subset \bigcup_{n=1}^N U_n = U_N = F_N^c$

On the other hand  $F_N \subset F_1$  by the nested property which implies  $F_N = \emptyset$  which is a contradiction.  $\square$

## 2 Measure Theory

We want to measure the size of a set. We will deal with a subset of  $\mathbb{R}$ .

It turns out that one needs to select a class of subsets of  $\mathbb{R}$  that one wants to measure. This class of subsets will have certain properties which are as follows.

**Definition 2.1** ( $\sigma$ -algebra). A collection  $\mathcal{A}$  of subsets of  $\mathbb{R}$  is called a  $\sigma$ -algebra if it satisfies

1.  $\emptyset \in \mathcal{A}$
2. If  $A \in \mathcal{A}$  then  $A^c \in \mathcal{A}$
3. If  $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$  then  $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$

Observe the following:

- $\mathbb{R} \in \mathcal{A}$  always

- If  $\{A_n\}_{n=1}^N \subset \mathcal{A}$  then  $\cup_{n=1}^N A_n \in \mathcal{A}$  (just define  $A_n = \emptyset$  for  $n > N$ )
- If  $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$  then  $\cap_{n=1}^\infty A_n \in \mathcal{A}$  (since  $(\cap_{n=1}^\infty A_n)^c = \cup_{n=1}^\infty A_n^c$ )
- If  $A, B \in \mathcal{A}$  then  $A \setminus B \in \mathcal{A}$  too since  $A \setminus B = A \cap B^c$

**Examples:**

1.  $\mathcal{A} = \{\emptyset, \mathbb{R}\}$  “Minimal  $\sigma$ -algebra”
2.  $\mathcal{A} = \mathcal{P}(\mathbb{R})$  = Collection of all subsets of  $\mathbb{R}$ . “Maximum  $\sigma$ -algebra”

In fact, if  $\mathcal{A}$  is any  $\sigma$ -algebra, then  $\{\emptyset, \mathbb{R}\} \subseteq \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$

For better examples, let  $F$  be any collection of subsets of  $\mathbb{R}$ . I want to make  $F$  into a  $\sigma$ -algebra. Define  $m = \{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra that satisfies } F \subset \mathcal{A}\}$ .  $m \neq \emptyset$  since it contains  $\mathcal{P}(\mathbb{R})$

If  $\mathcal{A}, \mathcal{B} \in m$ , I can define  $\mathcal{A} \cap \mathcal{B} = \{A \subset \mathbb{R} \mid A \in \mathcal{A} \text{ and } A \in \mathcal{B}\}$  and I can do the same for  $\cap_{i \in I} \mathcal{A}$  arbitrary intersection of  $\sigma$ -algebra is still a  $\sigma$ -algebra

Define  $\hat{F} = \cap_{\mathcal{A} \in m} \mathcal{A}$  as a  $\sigma$ -algebra and  $F \subset \hat{F}$  and it is the minimal  $\sigma$ -algebra with these properties. If  $G$  is a  $\sigma$ -algebra with  $F \subset G$ , then  $\hat{F} \subset G$ .  $\hat{F}$  is the  $\sigma$ -algebra generated by  $F$ . Concretely,  $\hat{F}$  consists of all subsets of  $\mathbb{R}$  that can be constructed by applying countable unions, intersections, and complements to elements of  $F$ .

**Definition 2.2** (Borel Sets). *The  $\sigma$ -algebra  $\mathcal{B}$  of Borel Sets is the  $\sigma$ -algebra  $\hat{F}$  generated by*

$$F = \{U \subset \mathbb{R} \mid U \text{ open} \}$$

**Remark.**  $\mathcal{B}$  is also the  $\sigma$ -algebra generated by the family of all closed subsets of  $\mathbb{R}$

Singletons  $\{x\} \subset \mathbb{R}$  are closed so if  $A \subset \mathbb{R}$  is at most countable then  $A$  is Borel. (e.g  $\mathbb{Q} \subset \mathbb{R}$ ) (e.g  $\mathbb{R} \setminus \mathbb{Q}$ )

Not all Subsets of  $\mathbb{R}$  are Borel. One can actually show that the cardinality of  $\mathcal{B}$  is the same as the cardinality of  $\mathbb{R}$ . On the other hand  $\mathcal{P}(\mathbb{R})$  has strictly larger cardinality.

### 3 Lebesgue Outer Measure

We are hoping to measure the size of subsets of  $\mathbb{R}$ . Ideally we would like to find or construct a function

$$m : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\} = [0, \infty]$$

Which satisfies the following measure requirements:

1. If  $I = [a, b]$  or  $(a, b)$  or  $[a, b)$ , or  $(a, b]$ ,  $a, b \in \mathbb{R}, a \leq b$  then  $m(I) = b - a = \text{measure of interval}$
2.  $m$  is translation invariant. i.e if  $E \subset \mathbb{R}$  and  $x \in \mathbb{R}$ , let  $E + x = \{y + x \mid y \in E\}$  then  $m(E + x) = m(E)$
3. If  $\{E_j\}_{j=1}^n$  is a finite collection of pairwise disjoint  $E_j \subset \mathbb{R}$  then

$$m\left(\bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$

4. The same as (3) except for  $n = \infty$

**Theorem 3.1.** *There is no such  $m$  satisfying all 4 requirements*

The proof for this will come later. The solution for this is that we do not try to measure all subsets of  $\mathbb{R}$ . So we have  $m : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$  but now we will just be happy with  $m : \mathcal{A} \rightarrow [0, \infty]$  where  $\mathcal{A}$  is a  $\sigma$ -algebra which has enough elements. For example  $\mathcal{A} \supset \mathcal{B}$ .

We will follow H. Lebesgue as we proceed in two steps.

Step 1: construct Lebesgue outer measure  $m^* : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$  satisfying requirements 1, 2, and 3.

Step 2: Use  $m^*$  to define  $\mathcal{A}$  and let  $m \subset m^* \mid \mathcal{A}$

To create this Lebesgue outer measure on  $\mathbb{R}$  we satisfy a weakened version of requirement (3) that can be called (3w). For any countably infinite collection  $\{E_j\}_{j=1}^\infty$  of arbitrary subsets  $E_j \subset \mathbb{R}$

$$m^*\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty m(E_j)$$

**Theorem 3.2** (Lebesgue Outer Measure). *There is a map  $m^* : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  that satisfies the measure requirements 1, 2, and 3w.*

This  $m^*$  is called the Lebesgue outer measure on  $\mathbb{R}$ .

How do we define outer measure  $m^*(A)$ ?

Observe that any  $A \subseteq \mathbb{R}$  can be covered by some countable infinite collection  $\{I_j\}_{j=1}^\infty$  of bounded open intervals, which are allowed to be empty, but we do not assume that  $I_j$  be pairwise disjoint.

For example:  $I_j = (-j, j)$ ,  $j = 1, 2, 3 \dots$

Let

$$\mathcal{C}_A = \{\{I_j\}_{j=1}^\infty \mid I_j \text{ bounded open intervals such that } A \subset \cup_{j=1}^\infty I_j\}$$

$\mathcal{C}_A \neq \emptyset$  by our example so for each  $\{I_j\} \in \mathcal{C}_A$ , I can consider

$$\sum_{j=1}^\infty \ell(I_j) \in \mathbb{R}_{\geq 0} \cup \{+\infty\} \quad (\ell \text{ denotes length})$$

**Definition 3.1** (Outer Measure).

$$m^*(A) := \inf_{\{I_j\} \in \mathcal{C}_A} \sum_{j=1}^\infty \ell(I_j) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This defines a map  $m^* : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$

Simple Properties:

- *Monotonicity:* If  $A \subseteq B$  then  $m^*(A) \leq m^*(B)$ . Indeed by definition  $\mathcal{C}_B \subseteq \mathcal{C}_A$  hence the infimum over  $\mathcal{C}_B$  is  $\geq$  than the infimum over  $\mathcal{C}_A$ .
- *Empty Set:*  $m^*(\emptyset) = 0$ . Given any  $1 > \epsilon > 0$ , let  $I_j = (-\epsilon^j, \epsilon^j)$ ,  $j = 1, 2, \dots$   $\{I_j\} \in \mathcal{C}_\emptyset$  and  $\sum_{j=1}^\infty \ell(I_j) = 2 \sum_{j=1}^\infty \epsilon^j = \frac{2\epsilon}{1-\epsilon}$  from the geometric series going to zero so  $m^*(\emptyset) \leq \frac{2\epsilon}{1-\epsilon} \forall 0 < \epsilon < 1$

- If  $A \in \mathbb{R}$  is finite or countable infinite then  $m^*(A) = 0$ . Indeed enumerate all elements of  $A$  by  $\{a_j\}_{j=1}^\infty$ . (If  $A$  is finite say  $|A| = n$  let  $a_j = a_n$  for all  $j > n$ ). For any  $0 < \epsilon < 1$ , let  $I_j = (-\epsilon^j + a_j, a_j + \epsilon^j)$  so  $A \subseteq \cup_{j=1}^\infty I_j$  and  $\sum_{j=1}^\infty \ell(I_j) = \frac{2\epsilon}{1-\epsilon}$  hence as before,  $m^*(A) = 0$ . For example  $m^*(\mathbb{Q}) = 0$

We will now prove that the Lebesgue outer measure satisfies 1, 2, and 3w of the measure requirements.

Proof of Property 1: i.e  $m^*(I) = \ell(I)$  for any interval  $I \subseteq \mathbb{R}$

Assume that  $I = [a, b]$ ,  $a < b$  are finite numbers. Assume that  $I$  is a bounded closed interval. Our goal is to show that  $m^*(I) = b - a$ . One direction of inequality is easy to prove, the other is quite tedious and will be left out.

For any  $\epsilon > 0$  let  $I_1 = (a - \epsilon, b + \epsilon) \supset I$ , let  $I_j = \emptyset, j \geq 2$  so  $\{I_j\} \in \mathcal{C}_I \Rightarrow m^*(I) \leq \sum_{j=1}^\infty \ell(I_j) = b - a + 2\epsilon$ . Let  $\epsilon \rightarrow 0$  and we obtain  $m^*(I) \leq b - a$ .

Proof of Property 2: i.e  $\forall A \subset \mathbb{R}, \forall x \in \mathbb{R}, m^*(A + x) = m^*(A)$

$\mathcal{C}_A$  and  $\mathcal{C}_{A+x}$  are naturally in bijection via  $\{I_j\} \leftrightarrow \{I_j + x\}$ . Furthermore  $\ell(I_j + x) = \ell(I_j)$

$$\begin{aligned} m^*(A + x) &= \inf_{\{I_j+x\} \in \mathcal{C}_{A+x}} \sum_{j=1}^\infty \ell(I_j + x) \\ &= \inf_{\{I_j\} \in \mathcal{C}_A} \sum_{j=1}^\infty \ell(I_j) = m^*(A) \end{aligned}$$

Proof of Property 3w: i.e If  $\{E_j\}_{j=1}^n$  is a finite collection of pairwise disjoint  $E_j \subset \mathbb{R}$  then  $m^*\left(\cup_{j=1}^n E_j\right) = \sum_{j=1}^n m^*(E_j)$

If  $m^*(E_j) = +\infty$  for some  $j$ , then the property holds. We may assume that  $m^*(E_j) < +\infty \forall j$ . Let  $\epsilon > 0$ . By the definition of infimum, for each  $j \geq 0$ , there is

$$\{I_{j,k}\}_{k=1}^\infty \in \mathcal{C}_{E_j} \text{ such that } \sum_{k=1}^\infty \ell(I_{j,k}) < m^*(E_j) + \epsilon 2^{-j}$$

Thus  $\{I_{j,k}\}_{k=1}^{\infty}$  is still countable and it covers  $\cup_{j=1}^{\infty} E_j$  meaning it belongs to  $\mathcal{C}_{\cup_{j=1}^{\infty} E_j}$ , so by definition

$$m^* \left( \bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \ell(I_{j,k}) < \sum_{j=1}^{\infty} (m^*(E_j) + \epsilon 2^{-j}) = \sum_{j=1}^{\infty} m^*(E_j) + \epsilon$$

Then let  $\epsilon \rightarrow 0$ . Clearly, by taking all  $E_j = \emptyset$  except finitely many, we have the same subadditivity 3w for finite collections.

**Corollary 3.2.1.**  $m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1 = \ell([0, 1])$

*Proof.*

$$\begin{aligned} m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) &\leq m^*([0, 1]) = 1 \\ &\leq m^*([0, 1] \cap (\mathbb{Q})) + m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) \\ &\leq 0 + 1 \end{aligned}$$

□

**Corollary 3.2.2.**  $\mathbb{R} \setminus \mathbb{Q}$  is uncountable

*Proof.* If not, then

$$m^*(\mathbb{R} \setminus \mathbb{Q}) = 0 \geq m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1$$

□

## 4 The $\sigma$ -Algebra Of Lebesgue Measurable Sets

$m^*$  does not satisfy the third measurability requirement without the weak 3w condition. We can construct some examples to prove this.  $A, B \subset \mathbb{R}$ ,  $A \cap B = \emptyset$ , such that  $m^*(A \cup B) < m^*(A) + m^*(B)$  later in the class.

The idea to avoid this problem is to look at “reasonable” subsets of  $\mathbb{R}$  for which this paradox disappears.

**Definition 4.1** (Carathéodory).  $E \subseteq \mathbb{R}$  is called (Lebesgue) measurable if  $\forall A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

**Remark.** This is equivalent to Lebesgue's definition:  $E$  is measurable if and only if

$$\exists U \subset \mathbb{R} \text{ such that } E \subset U \text{ and } m^*(U \setminus E) < \epsilon$$

But we will discuss this later.

Suppose that  $A$  is measurable and  $B \subset \mathbb{R}$  is any set such that  $A \cap B = \emptyset$  then

$$m^*(A \cup B) = m^*\left(\underbrace{(A \cup B) \cap A}_{=A}\right) + m^*\left(\underbrace{(A \cup B) \cap A^c}_{=B}\right)$$

Going back to our counter example for  $m^*$  and measurability requirement 3,  $A$  or  $B$  would have to be unmeasurable.

Here's another observation: For  $E, A \subset \mathbb{R}$  arbitrary sets we have

$$A = (A \cap E) \cup (A \cap E^c)$$

So by 3w  $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ , so  $E$  is measurable  $\iff \forall A \subset \mathbb{R}$

$$\boxed{m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)}$$

This holds trivially for  $m^*(A) = \infty$

Example 1:  $\emptyset$  is measurable.  $\forall A \subset \mathbb{R}$

$$m^*(A) = \cancel{m^*(A \cap \emptyset)} + m^*(A \cap \mathbb{R})$$

Example 2:  $\mathbb{R}$  is measurable.  $\forall A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap E^c)$$

**Proposition.**  $E \subset \mathbb{R}$  with  $m^*(E) = 0$ , then  $E$  is measurable.

**Corollary.** Every countable set is measurable.  $\mathbb{Q}$  measurable  $\rightarrow \mathbb{R} \setminus \mathbb{Q}$  are measurable

*Proof.* Let  $A \subset \mathbb{R}$  be any set

$$A \cap E \subset E \Rightarrow m^*(A \cap E) \leq m^*(E) = 0$$

$$A \cap E^c \subset A \Rightarrow m^*(A \cap E^c) \leq m^*(A)$$

$$\text{So } m^*(A) \geq m^*(A \cap E^c) + \cancel{m^*(A \cap E)}$$

□



Our goal is to show that Lebesgue measurable sets  $\mathcal{L} = \{E \subset \mathbb{R} \mid E \text{ is measurable}\}$  is a  $\sigma$ -algebra on  $\mathbb{R}$ . We just need to show that if  $\{E_j\}_{j=1}^\infty$  with  $E_j \in \mathcal{L}$ ,  $\forall j$ , then  $\bigcup_{j=1}^\infty E_j \in \mathcal{L}$

**Proposition.** *If  $\{E_j\}_{j=1}^n \subset \mathcal{L}$  then  $\bigcup_{j=1}^n E_j \in \mathcal{L}$*

*Proof.* We use mathematical induction.  $n = 1$  is trivial so we set the base case as  $n = 2$ .  $E_1, E_2$  are measurable, Let  $A \subset \mathbb{R}$  be any set

$$\begin{aligned}
m^*(A) &= m^*(E_1 \cap A) + m^*(A \cap E_1^c) \\
&= m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c) \\
&= m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*(A \cap (E_1^c \cap E_2^c)) \\
&= m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c) \\
&\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \quad (3w)
\end{aligned}$$

So  $E_1 \cup E_2 \in \mathcal{L}$ .

Induction step  $n \geq 2$

$$\bigcup_{j=1}^\infty E_j = \left( \bigcup_{j=1}^{n-1} E_j \right) \cup E_n \in \mathcal{L} \text{ by the } n = 2 \text{ case} \quad \square$$

To prove that this also applies to countable sets, we use

**Proposition** (Analog of measurability requirement 3 for  $m^* \mid \mathcal{L}$ ). *Suppose  $A \subset \mathbb{R}$  is any set and  $\{E_j\}_{j=1}^n$  is a finite disjoint collection of sets  $E_j \in \mathcal{L}$ , then*

$$m^*\left(A \cap \bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m^*(A \cap E_j)$$

*In particular take  $A = \mathbb{R}$  to get  $m^*\left(\bigcup_{j=1}^n E_j\right) = \sum m^*(E_j)$*

**Proposition.** *If  $\{E_j\}_{j=1}^\infty$  is a countable family with  $E_i \in \mathcal{L} \forall j$ , then  $\bigcup_{j=1}^\infty E_j \in \mathcal{L}$ . In particular,  $\mathcal{L}$  is a  $\sigma$ -algebra.*

We would like to have the Borel sets be measurable, i.e  $\mathcal{B} \subset \mathcal{L}$ . Recall that  $\mathcal{B} = \hat{\mathcal{F}}$ , where  $\mathcal{F} = \{U \subset \mathbb{R} \mid U \text{ is open}\}$  and  $\hat{\cdot}$  denotes the  $\sigma$ -algebra.

This results follows from the measurability of intervals combined with the measurability of the union of measurable sets.

**Proposition.** *If  $I \subseteq \mathbb{R}$  is any interval, then  $I$  is measurable.*

**Theorem 4.1.**  *$\mathcal{L} = \text{Lebesgue Measurable subsets of } \mathbb{R} \text{ form a } \sigma\text{-algebra that contains the Borel } \sigma\text{-algebra } \mathcal{B}$*

*Proof.* We already know that  $\mathcal{L}$  is a  $\sigma$ -algebra. If we can show that  $\mathcal{L}$  contains all open sets  $U \subset \mathbb{R}$ , then  $\mathcal{L}$  (being a  $\sigma$ -algebra) must contain  $\mathcal{B}$  which is the  $\sigma$ -algebra generated by open sets. Now if  $U \subset \mathbb{R}$  is any (non empty) open set then by definition  $\forall x \in U, \exists I_x \ni x$  where  $I_x$  is an open interval and  $I_x \subset U$ .

We want to choose  $I_x$  to be the “maximal” such. So by assigning

$$a_x := \inf\{z \in \mathbb{R} \mid (z, x) \subset U\} \text{ satisfies } a_x < x$$

and

$$b_x := \sup\{y \in \mathbb{R} \mid (x, y) \subset U\} \text{ satisfies } x < b_x$$

so  $I_x := (a_x, b_x)$  is an open interval that contains  $x$  and by construction  $I_x \subset U$ . It is the largest such, in the sense that if  $a_x > -\infty$  then  $a_x \notin U$  and symmetrically if  $b_x < \infty$  then  $b_x \notin U$ .

For any  $y \in I_x$ , we have  $y < b_x$ , so there is  $z > y$  such that  $(x, z) \subset U$  so  $y \in U$ . Indeed, if  $a_x \in U$  then since  $U$  open,  $\exists r > 0$  such that  $(a_x - r, a_x + r) \subset U$  contradicting the definition of  $a_x$ .

So  $U = \cup_{x \in U} I_x$ . It is a huge union, however if  $x, x' \in U, x \neq x'$ , then either  $I_x \cap I_{x'} = \emptyset$ , or if not then necessarily  $I_x = I_{x'}$ , since  $I_x \cup I_{x'}$  is then another open interval that contains  $x$  &  $x'$  and is a subset of  $U$ , so by maximality it must equal  $I_x$  &  $I_{x'}$ . So, throwing away all repeated  $I_x$ , we can write  $U = \cup_{i \in I} I_{x_i}$  for some  $I$  where the intervals  $I_{x_i}$  are pairwise disjoint. By density of  $\mathbb{Q} \subset \mathbb{R}$ , each such interval contains a different rational number  $r_i \in I_{x_i}$ . Since  $\mathbb{Q}$  is countable,  $I$  is at worst countable.

So every  $U$  open is an at most countable disjoint union of open intervals. Since such intervals belong to  $\mathcal{L}$ , and  $\mathcal{L}$  is a  $\sigma$ -algebra, it follows that every  $U$  open is in  $\mathcal{L}$  as desired.  $\square$

**Proposition** (The  $\sigma$ -algebra  $\mathcal{L}$  is also translation invariant). *If  $E \in \mathcal{L}$  and  $x \in \mathbb{R}$  then  $E + x \in \mathcal{L}$*

*Proof.* Given any  $A \in \mathcal{L}$ ,

$$\begin{aligned} m^*(A) &= m^*(A - x) \\ &= m^*((A - x) \cap E) + m^*((A - x) \cap E^c) \\ &= m^*(A \cap E + x) + m^*(A \cap (E + x)^c) \quad (m^* \text{ translation invariant}) \end{aligned}$$

□

**Remark.** *If  $A \in \mathcal{L}$  with  $m^*(A) < \infty$ , and  $B \in \mathcal{L}$  is any set with  $A \subset B$ , then*

$$m^*(B \setminus A) = m^*(B) - m^*(A)$$

## 5 Outer and Inner Approximation of Lebesgue Measurable Sets

**Definition 5.1** (Gebiet-Durchschnitt). *A subset  $A \subset \mathbb{R}$  is called a  $G_\delta$  if  $A = \bigcap_{i=1}^\infty A_i$  where  $A_i$  are all open (possibly empty).*

**Definition 5.2** (Fermé-Somme). *A subset  $A \subset \mathbb{R}$  is called a  $F_\sigma$  if  $A = \bigcup_{i=1}^\infty A_i$  where  $A_i$  are all closed (possibly empty).*

Clearly,  $A$  is  $G_\delta \iff A^c$  is  $F_\sigma$ . Also clearly, all  $G_\delta$  and  $F_\sigma$  sets are Borel. Of course not all  $G_\delta$  are open, e.g.  $[0, 1] = \bigcap_{i=1}^\infty (-\frac{1}{i}, 1 + \frac{1}{i})$  and not all  $F_\sigma$  are closed. e.g.  $(0, 1) = \bigcup_{i=1}^\infty [\frac{1}{i}, 1 - \frac{1}{i}]$

$\mathbb{Q}$  is clearly  $F_\sigma$ , so  $\mathbb{R} \setminus \mathbb{Q}$  is  $G_\delta$ . With this, we can give several equivalent formulations of measurability.

**Theorem 5.1.** *Let  $E \subset \mathbb{R}$  be any set, then the following are equivalent:*

1.  $E \in \mathcal{L}$
2.  $\forall \epsilon > 0, \exists U \supset E, U \text{ open, } m^*(U \setminus E) < \epsilon$

3.  $\exists G \subset \mathbb{R}$  a  $G_\delta$  set,  $G \supset E$ , with  $m^*(G \setminus E) = 0$

4.  $\forall \epsilon > 0, \exists F \subset E$ ,  $F$  closed,  $m^*(E \setminus F) < \epsilon$

5.  $\exists F \subset \mathbb{R}$  a  $F_\sigma$  set,  $F \subset E$  with  $m^*(E \setminus F) = 0$

**Proposition.** For an  $E \in \mathcal{L}$  with  $m^*(E) < \infty$ . Then  $\forall \epsilon > 0, \exists \{I_j\}_{j=1}^n$  a finite disjoint family of open intervals so that if we let  $U = \cup_{j=1}^n I_j$  (open) then  $m^*(E \Delta U) < \epsilon$ .

## 6 Lebesgue Measure

We can now take  $m^*$  and restrict it to  $\mathcal{L}$ .  $m^*|_{\mathcal{L}}$ .

**Definition 6.1** (Lebesgue Measure). This Lebesgue Measure is a function

$$m := m^*|_{\mathcal{L}}: \mathcal{L} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This means that for  $E \in \mathcal{L}$  we define  $m(E) = m^*(E)$ . Clearly,  $m$  satisfies the measurability requirements 1, 2, & 3 as we have proved earlier. It also satisfies requirement 4 which was requirement 3 for countably infinite sets.

**Proposition.** If  $\{E_j\}_{j=1}^\infty$  is a countably infinite collection of pairwise disjoint sets  $E_j \in \mathcal{L}$  (possibly empty), then  $\cup_{j=1}^\infty E_j \in \mathcal{L}$  and

$$m\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty m(E_j)$$

*Proof.* We proved earlier that  $\cup_{j=1}^\infty E_j \in \mathcal{L}$  and that

$$m\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty m(E_j)$$

For the opposite inequality, for each  $n$  we proved earlier that

$$m\left(\bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$

But  $\cup_{j=1}^n E_j \subset \cup_{j=1}^\infty E_j$ , hence

$$m\left(\bigcup_{j=1}^\infty E_j\right) \geq m\left(\bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j) \quad \forall n$$

Take the limit as  $n \rightarrow \infty$  to get

$$m\left(\bigcup_{j=1}^\infty E_j\right) \geq \sum_{j=1}^\infty m(E_j)$$

As desired. This argument shows that measurability requirement 3 and 3w together imply 4.  $\square$