

# Homework 5

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**Chapter 3 Problem 1:** Suppose  $f$  and  $g$  are continuous functions on  $[a, b]$ . Show that if  $f = g$  a.e on  $[a, b]$ , then, in fact  $f = g$  on  $[a, b]$ . Is a similar assertion true if  $[a, b]$  is replaced by a general measurable set  $E$ ?

Suppose  $f(x) \neq g(x)$  for some  $x \in [a, b]$ . Let  $\epsilon = |g(x) - f(x)|$ . Since  $f$  is continuous, define  $\delta_f > 0$  such that for  $y \in [a, b]$

$$|x - y| < \delta_f \implies |f(x) - f(y)| < \frac{\epsilon}{2}$$

Define  $\delta_g$  in the same way for  $g$ .

$$|x - y| < \delta_g \implies |g(x) - g(y)| < \frac{\epsilon}{2}$$

Now let  $\delta = \min\{\delta_f, \delta_g\}$ . On the interval  $[x, x + \delta]$  (or  $[x, x - \delta]$  if  $b - x < \delta$ )  $f$  can only converge less than  $\epsilon/2$  closer to  $g$  and likewise  $g$  can only converge less than  $\epsilon/2$  closer to  $f$  for all points in that neighborhood.

$$\forall y \in [x, x + \delta], f(y) \neq g(y)$$

But then the interval  $[x, x + \delta]$  has length  $\delta > 0$  and hence has a non-zero measure so it is not true that  $f = g$  a.e on  $[a, b]$ . This means that if  $f = g$  a.e on  $[a, b] \implies f = g$  on  $[a, b]$ .

This assertion is not true if we replace  $[a, b]$  with a measurable set  $E$  with  $m(E) = 0$ . This is because in fact any functions  $f$  and  $g$  satisfy the property  $f = g$  a.e on  $E$ , in particular functions with  $f(x) \neq g(x)$  for some  $x \in E$ .

**Chapter 3 Problem 3:** Suppose a function  $f$  has a measurable domain and is continuous except at a finite number of points. Is  $f$  necessarily measurable?

Let  $E$  denote the domain of  $f$  and let  $A \subset E$  be the set of all  $x \in E$  where  $f(x)$  is discontinuous.  $A$  is a finite set so we can enumerate every  $x \in A$  as  $x_1 \dots x_n$ .

Define  $I_1 = [-\infty, x_1) \cap E$  and  $I_{n+1} = (x_n, \infty] \cap E$ . For each  $1 \leq j \leq n-1$ , define  $I_j = (x_j, x_{j+1}) \cap E$ . Now  $f$  is continuous on each  $I_j$  by construction.

Define  $g_j : I_j \rightarrow \mathbb{R} = f : I_j \rightarrow \mathbb{R}$  so that each  $g_j$  is continuous. Each  $I_j$  is measurable since it is an intersection of measurable sets, hence each  $g_j$  is measurable.

Finally, define  $h_j : \{x_j\} \rightarrow \mathbb{R} = f(x_j)$  which is trivially measurable.

Since measurable functions are closed under addition, we have

$$f = \sum_{j=1}^{n+1} g_j + \sum_{j=1}^n h_j \in \mathcal{L}$$

So  $f$  is indeed measurable.

**Chapter 3 Problem 5:** Suppose the function  $f$  is defined on a measurable set  $E$  and has the property that  $\{x \in E \mid f(x) > c\}$  is measurable for each rational number  $c$ . Is  $f$  necessarily measurable?

The rationals are dense in  $\mathbb{R}$ , hence for any  $c \in \mathbb{R}$  and any  $n \in \mathbb{N}$ , we can find a  $q \in [c, c + \frac{1}{n}]$  such that  $q \in \mathbb{Q}$ .

$$\{x \in E \mid f(x) > c\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > q > c\}$$

Where  $q \in [c, c + \frac{1}{n}]$

Each element of this union is measurable by assumption since  $q \in \mathbb{Q}$  and it is a countable union, so  $\{x \in E \mid f(x) > c\}$  is measurable and therefore  $f$  is measurable.

**Chapter 3 Problem 14:** Let  $f$  be a measurable function on  $E$  that is finite a.e. on  $E$  and  $m(E) < \infty$ . For each  $\epsilon > 0$ , show that there is a measurable set  $F$  contained in  $E$  such that  $f$  is bounded on  $F$  and  $m(E \setminus F) < \epsilon$ .

Let  $F = \{x \in E : |f(x)| < \infty\}$ . Then

$$m(E \setminus F) = m(\{x \in E \mid f(x) = \pm\infty\}) = 0$$

By definition of  $f$ .

Clearly  $F \subset E$ , now we show that  $F \in \mathcal{L}$

$$F = \bigcup_{n=1}^{\infty} \{x \in E : |f(x)| < n\}$$

Which are all measurable due to the measurability of  $f$ , hence  $F$  is a countable union of measurable sets and is itself measurable.