Homework 7

Zachary Probst

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Problem 1: Let $E \subset \mathbb{R}$ be a nonmeasurable set. Prove that there is $\epsilon > 0$ such that if $\{E_j\}_{j=1}^{n}$ are measureable sets with $E \subset \bigcup_{j=1}^{n} E_j$ then $\sum_{j=1}^{n} m(E_j) \geq \epsilon$.

Assume by way of contradiction that we had

$$\sum_{j=1}^{n} m(E_j) = 0$$

Then since each E_j is measurable, we can apply countable additivity of the measure to see that

$$m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j) = 0$$

Now we know that $E \subset \bigcup_{j=1}^{n} E_j$ so by monotonicity, we must have that $m(E) = 0$. But this proves that E is measurable since any set of zero measure is indeed measurable. This is a contradiction hence there must be some $\epsilon > 0$ such that $\sum_{j=1}^{n} m(E_j) \geq \epsilon$.

Problem 2: For the next problems let $V \subset [0,1]$ be a Vitali set for $[0,1]$, i.e a choice set for the equivalence relation $\sim$ on $[0,1]$ where $x \sim y \iff x - y \in \mathbb{Q}$. Consider the characteristic function $\chi_V : [0,1] \to \mathbb{R}$ of this Vitali set. Show that the lower Lebesgue integral of χ_V satisfies $\mathcal{L}(\chi_V) = 0$.

Consider a simple function $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ on $[0,1]$ with $\phi \leq \chi_V$. Such a ϕ exists because we can take the zero function on $[0,1]$.

Split the index set $\{1, \ldots, n\}$ into three disjoint parts

1. $\underline{E_j \subset V}$

   In this case $m(E_j) = 0$ since E_j is measurable and $E_j \subset V$. This means that $\int_{E_j} \phi = 0$.

2. $\underline{E_j \subset [0,1] \setminus V}$.

   In this case $\chi_V = 0$ on E_j since $E_j \subset V^c$ and hence we must take $a_j \leq 0$ to still maintain $\phi \leq \chi_V$. This means $\phi\,|_{E_j} \leq 0 \implies \int_{E_j} \phi \leq 0$.

3. $\underline{E_j \text{ has elements in both } V \text{ and } V^c}$

   Then the minimum value that χ_V takes on E_j is 0, and hence we must have $a_j \leq 0 \implies \phi\,|_{E_j} \leq 0 \implies \int_{E_j} \phi \leq 0$.

Now for each of the three cases, $\int_{E_j} \phi \leq 0$ and so since $[0,1] = \cup_{j=1}^{n} E_j$

$$\int_0^1 \phi \leq 0$$

But we know that the zero function is a possible choice for ϕ so

$$0 \leq \sup \left\{ \int_0^1 \phi \mid \phi : [0,1] \to \mathbb{R} \text{ simple with } \phi \leq f \right\} = \mathcal{L}(f) \leq 0$$

Problem 3: Show that the upper Lebesgue integral of χ_V satisfies $\mathcal{U}(\chi_V) \geq \epsilon$, where $\epsilon > 0$ is the number given by problem 1 for the nonmeasurable set V. Combining this with problem 2, we see that χ_V is not Lebesgue integrable on $[0, 1]$.

Consider $\psi = \sum_{j=1}^{n} a_j \chi_{E_j}$ on $[0, 1]$ with $\psi \geq \chi_V$, and split it into 3 disjoint cases like we did in the previous question.

1. $\underline{E_j \subset V}$

   In this case $m(E_j) = 0$ since E_j is measurable and $E_j \subset V$. This means that $\int_{E_j} \psi = 0$.

2. $\underline{E_j \subset [0, 1] \setminus V}$.

   In this case $\chi_V = 0$ on E_j since $E_j \subset V^c$ and hence we must take $a_j \geq 0$ to still maintain $\psi \geq \chi_V$. This means $\psi \mid_{E_j} \geq 0 \implies \int_{E_j} \psi \geq 0$.

3. $\underline{E_j \text{ has elements in both } V \text{ and } V^c}$

   Then the maximum value that χ_V takes on E_j is 1, and hence we must have $a_j \geq 1$.

If we combine all the cases of E_j from (1) and (3), and relabel them from $1, \ldots, m$ then $\cup_{j=1}^{m} E_j$ is a finite cover of V. We can now apply the result from the first problem in this homework to see that there is $\epsilon > 0$ such that

$$\sum_{j=1}^{m} m(E_j) \geq \epsilon$$

We also know that all the $E_j^{(1)}$ from case (1) have $m(E_j^{(1)}) = 0$. Now since each E_j is measurable, we have that for each of the $E_j^{(1)}$ from case (1)

$$0 = m\left(\bigcup_{j=1}^{m_1} E_j^{(1)}\right) = \sum_{j=1}^{m_1} m(E_j^{(1)})$$

Where $m_1 \leq m$ is the number of $E_j^{(1)}$ from case (1) and we have once again relabeled for convenience. But the sum of all E_j from cases (1) and (2) had positive measure, therefore the sum of all m_2 of the $E_j^{(2)}$ from case (2) must satisfy

$$\sum_{j=1}^{m_2} m(E_j^{(2)}) \geq \epsilon$$

We saw that for each $E_j^{(2)}$, we had $a_j \geq 1$. Now if we set $E'' \coloneqq \cup_{j=1}^{m_2} E_j^{(2)}$ and $E' \coloneqq [0,1] \setminus E''$ we can conclude by taking

$$\int_0^1 \psi = \int_{E'} \psi + \int_{E''} \psi \geq \epsilon$$

And since ϵ is a lower bound over the set of all simple functions $\psi \geq f$,

$$\mathcal{U}(\chi_V) \geq \epsilon$$

From this exercise we see that χ_V is not Lebesgue integrable on $[0,1]$