

Honours Analysis 3

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1 Borel Sets

We will work for some time on \mathbb{R} exclusively. Before beginning Measure Theory: a quick recap of Topology.

Definition 1.1 (Open Set). *A subset $U \subset \mathbb{R}$ is called open if either $U = \emptyset$ or else*

$$\forall x \in U, \exists r > 0 \text{ such that } (x - r, x + r) \subset U$$

Some examples of open sets: $\emptyset, \mathbb{R}, (a, b), (a, \infty), (-\infty, a)$. There are many more because any union of an open set is still open and any finite intersection of open sets is open.

Definition 1.2 (Closed Set). *$F \subset \mathbb{R}$ is called closed if $\mathbb{R} \setminus F := F^c$ is open. F is closed $\iff F$ contains all points $x \in \mathbb{R}$ which have the property that $\forall r > 0, (x - r, x + r) \cap F \neq \emptyset$.*

If $F \subset \mathbb{R}$ is any set, the closure of F , denoted by \overline{F} , is the smallest closed set that contains F .

Definition 1.3 (Compact). *A subset $G \subset \mathbb{R}$ is compact if given any collection $\{U_i\}_{i \in I}$ of open sets $U_i \subset \mathbb{R}$ with $G \subset \cup_{i \in I} U_i$, there exists $J \subset I$, J finite, such that $G \subset \cup_{j \in J} U_j$*

Theorem 1.1 (Heine-Borel). *$G \subset \mathbb{R}$ is compact $\iff G$ is closed and bounded. To be bounded means $G \subset (a, b)$ for some $a, b \in \mathbb{R}$.*

Corollary 1.1.1 (Nested Set Theorem). *Let $\{F_n\}_{n=1}^\infty$ be a countable collection of non-empty, bounded, closed sets $F_n \subset \mathbb{R}$ with $F_{n+1} \subset F_n \forall n$, then*

$$\cap_{n=1}^\infty F_n \neq \emptyset$$

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Proof. Suppose $\cap_{n=1}^{\infty} F_n = \emptyset$ so let $U_n = F_n^c$ be open sets, such that $\cup_{n=1}^{\infty} U_n = \mathbb{R}$. We also have that $U_n \subset U_{n+1}$, since the F_n were nested. Now F_1 is compact by Heine-Borel and $F_1 \subset \cup_{n=1}^{\infty} U_n \Rightarrow$ by compactness I can find a finite subcover of F_1 , say $F \subset \cup_{n=1}^N U_n = U_N = F_N^c$. On the other hand $F_N \subset F_1$ by the nested property which implies $F_N = \emptyset$ which is a contradiction. \square

2 Measure Theory

We want to measure the size of a set. We will deal with a subset of \mathbb{R} . It turns out that one needs to select a class of subsets of \mathbb{R} that one wants to measure. This class of subsets will have certain properties which are as follows.

Definition 2.1 (σ -algebra). *A collection \mathcal{A} of subsets of \mathbb{R} is called a σ -algebra if it satisfies*

1. $\emptyset \in \mathcal{A}$
2. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$
3. If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$

Observe the following:

- $\mathbb{R} \in \mathcal{A}$ always
- If $\{A_n\}_{n=1}^N \subset \mathcal{A}$ then $\cup_{n=1}^N A_n \in \mathcal{A}$ (just define $A_n = \emptyset$ for $n > N$)
- If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ then $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$ (since $(\cap_{n=1}^{\infty} A_n)^c = \cup_{n=1}^{\infty} A_n^c$)
- If $A, B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$ too since $A \setminus B = A \cap B^c$

Examples:

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$ “Minimal σ -algebra”
2. $\mathcal{A} = \mathcal{P}(\mathbb{R})$ = Collection of all subsets of \mathbb{R} . “Maximum σ -algebra”

In fact, if \mathcal{A} is any σ -algebra, then $\{\emptyset, \mathbb{R}\} \subseteq \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$

For better examples, let F be any collection of subsets of \mathbb{R} . I want to make F into a σ -algebra. Define $m = \{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra that satisfies } F \subset \mathcal{A}\}$. $m \neq \emptyset$ since it contains $\mathcal{P}(\mathbb{R})$

If $\mathcal{A}, \mathcal{B} \in m$, I can define $\mathcal{A} \cap \mathcal{B} = \{A \subset \mathbb{R} \mid A \in \mathcal{A} \text{ and } A \in \mathcal{B}\}$ and I can do the same for $\cap_{i \in I} \mathcal{A}$ arbitrary intersection of σ -algebra is still a σ -algebra