

Honours Analysis 3

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1 Borel Sets

We will work for some time on \mathbb{R} exclusively. Before beginning Measure Theory: a quick recap of Topology.

Definition 1.1 (Open Set). *A subset $U \subset \mathbb{R}$ is called open if either $U = \emptyset$ or else*

$$\forall x \in U, \exists r > 0 \text{ such that } (x - r, x + r) \subset U$$

Some examples of open sets: $\emptyset, \mathbb{R}, (a, b), (a, \infty), (-\infty, a)$. There are many more because any union of an open set is still open and any finite intersection of open sets is open.

Definition 1.2 (Closed Set). *$F \subset \mathbb{R}$ is called closed if $\mathbb{R} \setminus F := F^c$ is open.*

F is closed $\iff F$ contains all points $x \in \mathbb{R}$ which have the property that $\forall r > 0, (x - r, x + r) \cap F \neq \emptyset$.

If $F \subset \mathbb{R}$ is any set, the closure of F , denoted by \overline{F} , is the smallest closed set that contains F .

Definition 1.3 (Compact). *A subset $G \subset \mathbb{R}$ is compact if given any collection $\{U_i\}_{i \in I}$ of open sets $U_i \subset \mathbb{R}$ with $G \subset \cup_{i \in I} U_i$, there exists $J \subset I$, J finite, such that $G \subset \cup_{j \in J} U_j$*

*Notes from the lectures of Valentino Tosatti

Theorem 1.1 (Heine-Borel). $G \subset \mathbb{R}$ is compact $\iff G$ is closed and bounded. To be bounded means $G \subset (a, b)$ for some $a, b \in \mathbb{R}$.

Corollary 1.1.1 (Nested Set Theorem). Let $\{F_n\}_{n=1}^\infty$ be a countable collection of non-empty, bounded, closed sets $F_n \subset \mathbb{R}$ with $F_{n+1} \subset F_n \forall n$, then

$$\bigcap_{n=1}^\infty F_n \neq \emptyset$$

Proof. Suppose $\bigcap_{n=1}^\infty F_n = \emptyset$ so let $U_n = F_n^c$ be open sets, such that $\bigcup_{n=1}^\infty U_n = \mathbb{R}$. We also have that $U_n \subset U_{n+1}$, since the F_n were nested. Now F_1 is compact by Heine-Borel and $F_1 \subset \bigcup_{n=1}^\infty U_n \Rightarrow$ by compactness I can find a finite subcover of F_1 , say $F \subset \bigcup_{n=1}^N U_n = U_N = F_N^c$

On the other hand $F_N \subset F_1$ by the nested property which implies $F_N = \emptyset$ which is a contradiction. \square

2 Measure Theory

We want to measure the size of a set. We will deal with a subset of \mathbb{R} .

It turns out that one needs to select a class of subsets of \mathbb{R} that one wants to measure. This class of subsets will have certain properties which are as follows.

Definition 2.1 (σ -algebra). A collection \mathcal{A} of subsets of \mathbb{R} is called a σ -algebra if it satisfies

1. $\emptyset \in \mathcal{A}$
2. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$
3. If $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$ then $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$

Observe the following:

- $\mathbb{R} \in \mathcal{A}$ always

- If $\{A_n\}_{n=1}^N \subset \mathcal{A}$ then $\cup_{n=1}^N A_n \in \mathcal{A}$ (just define $A_n = \emptyset$ for $n > N$)
- If $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$ then $\cap_{n=1}^\infty A_n \in \mathcal{A}$ (since $(\cap_{n=1}^\infty A_n)^c = \cup_{n=1}^\infty A_n^c$)
- If $A, B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$ too since $A \setminus B = A \cap B^c$

Examples:

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$ “Minimal σ -algebra”
2. $\mathcal{A} = \mathcal{P}(\mathbb{R})$ = Collection of all subsets of \mathbb{R} . “Maximum σ -algebra”

In fact, if \mathcal{A} is any σ -algebra, then $\{\emptyset, \mathbb{R}\} \subseteq \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$

For better examples, let F be any collection of subsets of \mathbb{R} . I want to make F into a σ -algebra. Define $m = \{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra that satisfies } F \subset \mathcal{A}\}$. $m \neq \emptyset$ since it contains $\mathcal{P}(\mathbb{R})$

If $\mathcal{A}, \mathcal{B} \in m$, I can define $\mathcal{A} \cap \mathcal{B} = \{A \subset \mathbb{R} \mid A \in \mathcal{A} \text{ and } A \in \mathcal{B}\}$ and I can do the same for $\cap_{i \in I} \mathcal{A}$ arbitrary intersection of σ -algebra is still a σ -algebra

Define $\hat{F} = \cap_{\mathcal{A} \in m} \mathcal{A}$ as a σ -algebra and $F \subset \hat{F}$ and it is the minimal σ -algebra with these properties. If G is a σ -algebra with $F \subset G$, then $\hat{F} \subset G$. \hat{F} is the σ -algebra generated by F . Concretely, \hat{F} consists of all subsets of \mathbb{R} that can be constructed by applying countable unions, intersections, and complements to elements of F .

Definition 2.2 (Borel Sets). *The σ -algebra \mathcal{B} of Borel Sets is the σ -algebra \hat{F} generated by*

$$F = \{U \subset \mathbb{R} \mid U \text{ open} \}$$

Remark. \mathcal{B} is also the σ -algebra generated by the family of all closed subsets of \mathbb{R}

Singletons $\{x\} \subset \mathbb{R}$ are closed so if $A \subset \mathbb{R}$ is at most countable then A is Borel. (e.g $\mathbb{Q} \subset \mathbb{R}$) (e.g $\mathbb{R} \setminus \mathbb{Q}$)

Not all Subsets of \mathbb{R} are Borel. One can actually show that the cardinality of \mathcal{B} is the same as the cardinality of \mathbb{R} . On the other hand $\mathcal{P}(\mathbb{R})$ has strictly larger cardinality.

3 Lebesgue Outer Measure

We are hoping to measure the size of subsets of \mathbb{R} . Ideally we would like to find or construct a function

$$m : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\} = [0, \infty]$$

Which satisfies the following measure requirements:

1. If $I = [a, b]$ or (a, b) or $[a, b)$, or $(a, b]$, $a, b \in \mathbb{R}, a \leq b$ then $m(I) = b - a = \text{measure of interval}$
2. m is translation invariant. i.e if $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, let $E + x = \{y + x \mid y \in E\}$ then $m(E + x) = m(E)$
3. If $\{E_j\}_{j=1}^n$ is a finite collection of pairwise disjoint $E_j \subset \mathbb{R}$ then

$$m\left(\bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$

4. The same as (3) except for $n = \infty$

Theorem 3.1. *There is no such m satisfying all 4 requirements*

The proof for this will come later. The solution for this is that we do not try to measure all subsets of \mathbb{R} . So we have $m : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ but now we will just be happy with $m : \mathcal{A} \rightarrow [0, \infty]$ where \mathcal{A} is a σ -algebra which has enough elements. For example $\mathcal{A} \supset \mathcal{B}$.

We will follow H. Lebesgue as we proceed in two steps.

Step 1: construct Lebesgue outer measure $m^* : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ satisfying requirements 1, 2, and 3.

Step 2: Use m^* to define \mathcal{A} and let $m \subset m^* \mid \mathcal{A}$

To create this Lebesgue outer measure on \mathbb{R} we satisfy a weakened version of requirement (3) that can be called (3w). For any countably infinite collection $\{E_j\}_{j=1}^\infty$ of arbitrary subsets $E_j \subset \mathbb{R}$

$$m^*\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty m(E_j)$$

Theorem 3.2 (Lebesgue Outer Measure). *There is a map $m^* : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ that satisfies the measure requirements 1, 2, and 3w.*

This m^* is called the Lebesgue outer measure on \mathbb{R} .

How do we define outer measure $m^*(A)$?

Observe that any $A \subseteq \mathbb{R}$ can be covered by some countable infinite collection $\{I_j\}_{j=1}^\infty$ of bounded open intervals, which are allowed to be empty, but we do not assume that I_j be pairwise disjoint.

For example: $I_j = (-j, j)$, $j = 1, 2, 3 \dots$

Let

$$\mathcal{C}_A = \{\{I_j\}_{j=1}^\infty \mid I_j \text{ bounded open intervals such that } A \subset \cup_{j=1}^\infty I_j\}$$

$\mathcal{C}_A \neq \emptyset$ by our example so for each $\{I_j\} \in \mathcal{C}_A$, I can consider

$$\sum_{j=1}^\infty \ell(I_j) \in \mathbb{R}_{\geq 0} \cup \{+\infty\} \quad (\ell \text{ denotes length})$$

Definition 3.1 (Outer Measure).

$$m^*(A) := \inf_{\{I_j\} \in \mathcal{C}_A} \sum_{j=1}^\infty \ell(I_j) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This defines a map $m^* : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$

Simple Properties:

- *Monotonicity:* If $A \subseteq B$ then $m^*(A) \leq m^*(B)$. Indeed by definition $\mathcal{C}_B \subseteq \mathcal{C}_A$ hence the infimum over \mathcal{C}_B is \geq than the infimum over \mathcal{C}_A .
- *Empty Set:* $m^*(\emptyset) = 0$. Given any $1 > \epsilon > 0$, let $I_j = (-\epsilon^j, \epsilon^j)$, $j = 1, 2, \dots$ $\{I_j\} \in \mathcal{C}_\emptyset$ and $\sum_{j=1}^\infty \ell(I_j) = 2 \sum_{j=1}^\infty \epsilon^j = \frac{2\epsilon}{1-\epsilon}$ from the geometric series going to zero so $m^*(\emptyset) \leq \frac{2\epsilon}{1-\epsilon} \forall 0 < \epsilon < 1$

- If $A \in \mathbb{R}$ is finite or countable infinite then $m^*(A) = 0$. Indeed enumerate all elements of A by $\{a_j\}_{j=1}^\infty$. (If A is finite say $|A| = n$ let $a_j = a_n$ for all $j > n$). For any $0 < \epsilon < 1$, let $I_j = (-\epsilon^j + a_j, a_j + \epsilon^j)$ so $A \subseteq \cup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty \ell(I_j) = \frac{2\epsilon}{1-\epsilon}$ hence as before, $m^*(A) = 0$. For example $m^*(\mathbb{Q}) = 0$

We will now prove that the Lebesgue outer measure satisfies 1, 2, and 3w of the measure requirements.

Proof of Property 1: i.e $m^*(I) = \ell(I)$ for any interval $I \subseteq \mathbb{R}$

Assume that $I = [a, b]$, $a < b$ are finite numbers. Assume that I is a bounded closed interval. Our goal is to show that $m^*(I) = b - a$. One direction of inequality is easy to prove, the other is quite tedious and will be left out.

For any $\epsilon > 0$ let $I_1 = (a - \epsilon, b + \epsilon) \supset I$, let $I_j = \emptyset, j \geq 2$ so $\{I_j\} \in \mathcal{C}_I \Rightarrow m^*(I) \leq \sum_{j=1}^\infty \ell(I_j) = b - a + 2\epsilon$. Let $\epsilon \rightarrow 0$ and we obtain $m^*(I) \leq b - a$.

Proof of Property 2: i.e $\forall A \subset \mathbb{R}, \forall x \in \mathbb{R}, m^*(A + x) = m^*(A)$

\mathcal{C}_A and \mathcal{C}_{A+x} are naturally in bijection via $\{I_j\} \leftrightarrow \{I_j + x\}$. Furthermore $\ell(I_j + x) = \ell(I_j)$

$$\begin{aligned} m^*(A + x) &= \inf_{\{I_j+x\} \in \mathcal{C}_{A+x}} \sum_{j=1}^\infty \ell(I_j + x) \\ &= \inf_{\{I_j\} \in \mathcal{C}_A} \sum_{j=1}^\infty \ell(I_j) = m^*(A) \end{aligned}$$

Proof of Property 3w: i.e If $\{E_j\}_{j=1}^n$ is a finite collection of pairwise disjoint $E_j \subset \mathbb{R}$ then $m^*\left(\cup_{j=1}^n E_j\right) = \sum_{j=1}^n m^*(E_j)$

If $m^*(E_j) = +\infty$ for some j , then the property holds. We may assume that $m^*(E_j) < +\infty \forall j$. Let $\epsilon > 0$. By the definition of infimum, for each $j \geq 0$, there is

$$\{I_{j,k}\}_{k=1}^\infty \in \mathcal{C}_{E_j} \text{ such that } \sum_{k=1}^\infty \ell(I_{j,k}) < m^*(E_j) + \epsilon 2^{-j}$$

Thus $\{I_{j,k}\}_{k=1}^{\infty}$ is still countable and it covers $\cup_{j=1}^{\infty} E_j$ meaning it belongs to $\mathcal{C}_{\cup_{j=1}^{\infty} E_j}$, so by definition

$$m^* \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \ell(I_{j,k}) < \sum_{j=1}^{\infty} (m^*(E_j) + \epsilon 2^{-j}) = \sum_{j=1}^{\infty} m^*(E_j) + \epsilon$$

Then let $\epsilon \rightarrow 0$. Clearly, by taking all $E_j = \emptyset$ except finitely many, we have the same subadditivity 3w for finite collections.

Corollary 3.2.1. $m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1 = \ell([0, 1])$

Proof.

$$\begin{aligned} m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) &\leq m^*([0, 1]) = 1 \\ &\leq m^*([0, 1] \cap (\mathbb{Q})) + m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) \\ &\leq 0 + 1 \end{aligned}$$

□

Corollary 3.2.2. $\mathbb{R} \setminus \mathbb{Q}$ is uncountable

Proof. If not, then

$$m^*(\mathbb{R} \setminus \mathbb{Q}) = 0 \geq m^*([0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1$$

□

4 The σ -Algebra Of Lebesgue Measurable Sets

m^* does not satisfy the third measurability requirement without the weak 3w condition. We can construct some examples to prove this. $A, B \subset \mathbb{R}$, $A \cap B = \emptyset$, such that $m^*(A \cup B) < m^*(A) + m^*(B)$ later in the class.

The idea to avoid this problem is to look at “reasonable” subsets of \mathbb{R} for which this paradox disappears.

Definition 4.1 (Carathéodory). $E \subseteq \mathbb{R}$ is called (Lebesgue) measurable if $\forall A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

Remark. This is equivalent to Lebesgue's definition: E is measurable if and only if

$$\exists U \subset \mathbb{R} \text{ such that } E \subset U \text{ and } m^*(U \setminus E) < \epsilon$$

But we will discuss this later.

Suppose that A is measurable and $B \subset \mathbb{R}$ is any set such that $A \cap B = \emptyset$ then

$$m^*(A \cup B) = m^*\left(\underbrace{(A \cup B) \cap A}_{=A}\right) + m^*\left(\underbrace{(A \cup B) \cap A^c}_{=B}\right)$$

Going back to our counter example for m^* and measurability requirement 3, A or B would have to be unmeasurable.

Here's another observation: For $E, A \subset \mathbb{R}$ arbitrary sets we have

$$A = (A \cap E) \cup (A \cap E^c)$$

So by 3w $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$, so E is measurable $\iff \forall A \subset \mathbb{R}$

$$\boxed{m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)}$$

This holds trivially for $m^*(A) = \infty$

Example 1: \emptyset is measurable. $\forall A \subset \mathbb{R}$

$$m^*(A) = \cancel{m^*(A \cap \emptyset)} + m^*(A \cap \mathbb{R})$$

Example 2: \mathbb{R} is measurable. $\forall A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap E^c)$$

Proposition. $E \subset \mathbb{R}$ with $m^*(E) = 0$, then E is measurable.

Corollary. Every countable set is measurable. \mathbb{Q} measurable $\rightarrow \mathbb{R} \setminus \mathbb{Q}$ are measurable

Proof. Let $A \subset \mathbb{R}$ be any set

$$A \cap E \subset E \Rightarrow m^*(A \cap E) \leq m^*(E) = 0$$

$$A \cap E^c \subset A \Rightarrow m^*(A \cap E^c) \leq m^*(A)$$

$$\text{So } m^*(A) \geq m^*(A \cap E^c) + \cancel{m^*(A \cap E)}$$

□

Our goal is to show that Lebesgue measurable sets $\mathcal{L} = \{E \subset \mathbb{R} \mid E \text{ is measurable}\}$ is a σ -algebra on \mathbb{R} . We just need to show that if $\{E_j\}_{j=1}^\infty$ with $E_j \in \mathcal{L}$, $\forall j$, then $\bigcup_{j=1}^\infty E_j \in \mathcal{L}$

Proposition. *If $\{E_j\}_{j=1}^n \subset \mathcal{L}$ then $\bigcup_{j=1}^n E_j \in \mathcal{L}$*

Proof. We use mathematical induction. $n = 1$ is trivial so we set the base case as $n = 2$. E_1, E_2 are measurable, Let $A \subset \mathbb{R}$ be any set

$$\begin{aligned}
m^*(A) &= m^*(E_1 \cap A) + m^*(A \cap E_1^c) \\
&= m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c) \\
&= m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*(A \cap (E_1^c \cap E_2^c)) \\
&= m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c) \\
&\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \quad (3w)
\end{aligned}$$

So $E_1 \cup E_2 \in \mathcal{L}$.

Induction step $n \geq 2$

$$\bigcup_{j=1}^\infty E_j = \left(\bigcup_{j=1}^{n-1} E_j \right) \cup E_n \in \mathcal{L} \text{ by the } n = 2 \text{ case} \quad \square$$

To prove that this also applies to countable sets, we use

Proposition (Analog of measurability requirement 3 for $m^* \mid \mathcal{L}$). *Suppose $A \subset \mathbb{R}$ is any set and $\{E_j\}_{j=1}^n$ is a finite disjoint collection of sets $E_j \in \mathcal{L}$, then*

$$m^*\left(A \cap \bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m^*(A \cap E_j)$$

In particular take $A = \mathbb{R}$ to get $m^\left(\bigcup_{j=1}^n E_j\right) = \sum m^*(E_j)$*

Proposition. *If $\{E_j\}_{j=1}^\infty$ is a countable family with $E_i \in \mathcal{L} \forall j$, then $\bigcup_{j=1}^\infty E_j \in \mathcal{L}$. In particular, \mathcal{L} is a σ -algebra.*

We would like to have the Borel sets be measurable, i.e $\mathcal{B} \subset \mathcal{L}$. Recall that $\mathcal{B} = \hat{\mathcal{F}}$, where $\mathcal{F} = \{U \subset \mathbb{R} \mid U \text{ is open}\}$ and $\hat{\cdot}$ denotes the σ -algebra.

This results follows from the measurability of intervals combined with the measurability of the union of measurable sets.

Proposition. *If $I \subseteq \mathbb{R}$ is any interval, then I is measurable.*

Theorem 4.1. *$\mathcal{L} = \text{Lebesgue Measurable subsets of } \mathbb{R} \text{ form a } \sigma\text{-algebra that contains the Borel } \sigma\text{-algebra } \mathcal{B}$*

Proof. We already know that \mathcal{L} is a σ -algebra. If we can show that \mathcal{L} contains all open sets $U \subset \mathbb{R}$, then \mathcal{L} (being a σ -algebra) must contain \mathcal{B} which is the σ -algebra generated by open sets. Now if $U \subset \mathbb{R}$ is any (non empty) open set then by definition $\forall x \in U, \exists I_x \ni x$ where I_x is an open interval and $I_x \subset U$.

We want to choose I_x to be the “maximal” such. So by assigning

$$a_x := \inf\{z \in \mathbb{R} \mid (z, x) \subset U\} \text{ satisfies } a_x < x$$

and

$$b_x := \sup\{y \in \mathbb{R} \mid (x, y) \subset U\} \text{ satisfies } x < b_x$$

so $I_x := (a_x, b_x)$ is an open interval that contains x and by construction $I_x \subset U$. It is the largest such, in the sense that if $a_x > -\infty$ then $a_x \notin U$ and symmetrically if $b_x < \infty$ then $b_x \notin U$.

For any $y \in I_x$, we have $y < b_x$, so there is $z > y$ such that $(x, z) \subset U$ so $y \in U$. Indeed, if $a_x \in U$ then since U open, $\exists r > 0$ such that $(a_x - r, a_x + r) \subset U$ contradicting the definition of a_x .

So $U = \cup_{x \in U} I_x$. It is a huge union, however if $x, x' \in U, x \neq x'$, then either $I_x \cap I_{x'} = \emptyset$, or if not then necessarily $I_x = I_{x'}$, since $I_x \cup I_{x'}$ is then another open interval that contains x & x' and is a subset of U , so by maximality it must equal I_x & $I_{x'}$. So, throwing away all repeated I_x , we can write $U = \cup_{i \in I} I_{x_i}$ for some I where the intervals I_{x_i} are pairwise disjoint. By density of $\mathbb{Q} \subset \mathbb{R}$, each such interval contains a different rational number $r_i \in I_{x_i}$. Since \mathbb{Q} is countable, I is at worst countable.

So every U open is an at most countable disjoint union of open intervals. Since such intervals belong to \mathcal{L} , and \mathcal{L} is a σ -algebra, it follows that every U open is in \mathcal{L} as desired. \square

Proposition (The σ -algebra \mathcal{L} is also translation invariant). *If $E \in \mathcal{L}$ and $x \in \mathbb{R}$ then $E + x \in \mathcal{L}$*

Proof. Given any $A \in \mathcal{L}$,

$$\begin{aligned} m^*(A) &= m^*(A - x) \\ &= m^*((A - x) \cap E) + m^*((A - x) \cap E^c) \\ &= m^*(A \cap E + x) + m^*(A \cap (E + x)^c) \quad (m^* \text{ translation invariant}) \end{aligned}$$

□

Remark. *If $A \in \mathcal{L}$ with $m^*(A) < \infty$, and $B \in \mathcal{L}$ is any set with $A \subset B$, then*

$$m^*(B \setminus A) = m^*(B) - m^*(A)$$

5 Outer and Inner Approximation of Lebesgue Measurable Sets

Definition 5.1 (Gebiet-Durchschnitt). *A subset $A \subset \mathbb{R}$ is called a G_δ if $A = \bigcap_{i=1}^\infty A_i$ where A_i are all open (possibly empty).*

Definition 5.2 (Fermé-Somme). *A subset $A \subset \mathbb{R}$ is called a F_σ if $A = \bigcup_{i=1}^\infty A_i$ where A_i are all closed (possibly empty).*

Clearly, A is $G_\delta \iff A^c$ is F_σ . Also clearly, all G_δ and F_σ sets are Borel. Of course not all G_δ are open, e.g. $[0, 1] = \bigcap_{i=1}^\infty (-\frac{1}{i}, 1 + \frac{1}{i})$ and not all F_σ are closed. e.g. $(0, 1) = \bigcup_{i=1}^\infty [\frac{1}{i}, 1 - \frac{1}{i}]$

\mathbb{Q} is clearly F_σ , so $\mathbb{R} \setminus \mathbb{Q}$ is G_δ . With this, we can give several equivalent formulations of measurability.

Theorem 5.1. *Let $E \subset \mathbb{R}$ be any set, then the following are equivalent:*

1. $E \in \mathcal{L}$
2. $\forall \epsilon > 0, \exists U \supset E, U \text{ open, } m^*(U \setminus E) < \epsilon$

3. $\exists G \subset \mathbb{R}$ a G_δ set, $G \supset E$, with $m^*(G \setminus E) = 0$

4. $\forall \epsilon > 0, \exists F \subset E$, F closed, $m^*(E \setminus F) < \epsilon$

5. $\exists F \subset \mathbb{R}$ a F_σ set, $F \subset E$ with $m^*(E \setminus F) = 0$

Proposition. For an $E \in \mathcal{L}$ with $m^*(E) < \infty$. Then $\forall \epsilon > 0, \exists \{I_j\}_{j=1}^n$ a finite disjoint family of open intervals so that if we let $U = \cup_{j=1}^n I_j$ (open) then $m^*(E \Delta U) < \epsilon$.

6 Lebesgue Measure

We can now take m^* and restrict it to \mathcal{L} . $m^*|_{\mathcal{L}}$.

Definition 6.1 (Lebesgue Measure). This Lebesgue Measure is a function

$$m := m^*|_{\mathcal{L}}: \mathcal{L} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This means that for $E \in \mathcal{L}$ we define $m(E) = m^*(E)$. Clearly, m satisfies the measurability requirements 1, 2, & 3 as we have proved earlier. It also satisfies requirement 4 which was requirement 3 for countably infinite sets.

Proposition. If $\{E_j\}_{j=1}^\infty$ is a countably infinite collection of pairwise disjoint sets $E_j \in \mathcal{L}$ (possibly empty), then $\cup_{j=1}^\infty E_j \in \mathcal{L}$ and

$$m\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty m(E_j)$$

Proof. We proved earlier that $\cup_{j=1}^\infty E_j \in \mathcal{L}$ and that

$$m\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty m(E_j)$$

For the opposite inequality, for each n we proved earlier that

$$m\left(\bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$

But $\cup_{j=1}^n E_j \subset \cup_{j=1}^\infty E_j$, hence

$$m\left(\bigcup_{j=1}^\infty E_j\right) \geq m\left(\bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j) \quad \forall n$$

Take the limit as $n \rightarrow \infty$ to get

$$m\left(\bigcup_{j=1}^\infty E_j\right) \geq \sum_{j=1}^\infty m(E_j)$$

As desired. This argument shows that measurability requirement 3 and 3w together imply 4. \square

7 Non-Measurable Sets

We saw earlier that if $E \subset \mathbb{R}$ satisfies $m^*(E) = 0$ then $E \in \mathcal{L}$. In particular, $\forall F \subset E$, $m^*(F) \leq m^*(E) = 0$, so $F \in \mathcal{L}$ too. This however totally fails when $m^*(E) > 0$.

Theorem 7.1 (Vitali). *For any $E \subset \mathbb{R}$ with $m^*(E) > 0$, there is an $F \subset E$ which is NOT measurable. The construction uses the axiom of choice (and it is really needed).*

The proof of this theorem and construction of a Vitali set are currently omitted due to length.

8 Cantor Set

We showed earlier that if $A \subset \mathbb{R}$ is countable then $A \in \mathcal{L}$ and $m(A) = 0$. How about the converse; if $A \in \mathcal{L}$ has $m(A) = 0$, is A countable? No!

Theorem 8.1 (Cantor). *There is a closed, uncountable set \mathcal{C} with $m(\mathcal{C}) = 0$*

Start with an interval $I = [0, 1]$ and remove the middle $\frac{1}{3}$, namely $(\frac{1}{3}, \frac{2}{3})$.

$$\begin{aligned}\mathcal{C}_1 &:= I \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\ \mathcal{C}_2 &:= \mathcal{C}_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right) \\ \mathcal{C}_k &:= \mathcal{C}_{k-1} \setminus \bigcup_{j=0}^{3^{k-1}-1} \left(\frac{3j+1}{3^k}, \frac{3j+2}{3^k}\right) \\ &= [0, 1] \setminus \bigcup_{l=1}^k \bigcup_{j=0}^{3^{l-1}-1} \left(\frac{3j+1}{3^l}, \frac{3j+2}{3^l}\right)\end{aligned}$$

Thus $\{\mathcal{C}_k\}_{k=1}^{\infty}$ is a very large descending (i.e nested $\mathcal{C} \subset \mathcal{C}_{k-1}$) sequence of closed sets, and \mathcal{C}_k is a disjoint union of 2^k closed intervals of length $\frac{1}{3^k}$. Let then $\mathcal{C} = \bigcap_{k=1}^{\infty} \mathcal{C}_k$, so \mathcal{C} is closed, and hence also measurable.

Since $m(\mathcal{C}_k) = \left(\frac{2}{3}\right)^k$, $m(\mathcal{C}) \leq m(\mathcal{C}_k) \leq \left(\frac{2}{3}\right)^k \forall k$. Taking the limit as $k \rightarrow \infty$ we get $m(\mathcal{C}) = 0$.

Suppose that \mathcal{C} was countable, let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of all it's elements. Then writing \mathcal{C}_1 = the disjoint union of 2 intervals, we must have that c_1 belongs to precisely one of them. Say $c_1 \notin F_1$. Now $F_1 \subset \mathcal{C}_2$ is made of 2 disjoint intervals, and one of them does not contain c_2 , say $c_2 \notin F_2$.

Continue this way until we get a sequence of $\{F_k\}_{k=1}^{\infty}$, where F_k is a closed interval, $F_{k+1} \subset F_k$, and $F_k \subset \mathcal{C}_k$, and $c_k \notin F_k$. By the nested set theorem, let $x \in \bigcap_{k=1}^{\infty} F_k$. Then

$$x \in \bigcap_{k=1}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} \mathcal{C}_k = \mathcal{C}$$

So $x \in \mathcal{C}$ but $\{c_k\}_{k=1}^{\infty}$ enumerates ALL points of \mathcal{C} so $\exists n$ such that $x = c_n$. Hence $x \notin F_n$ but this is a contradiction so we conclude that \mathcal{C} is uncountable.

Finally observe that \mathcal{C} is closed and $\mathcal{C} \subset [0, 1]$, so \mathcal{C} is compact by Heine-Borel.

There are two variations of this theorem.

1. If instead of removing the middle third, we removed the middle $p\%$ where $0 < p < 100$, then we also get a Cantor set which has the same properties as \mathcal{C} .
2. We could also remove a *smaller* proportion at each step, instead of a fixed one. At each step we remove 2^{n-1} intervals of length a^n for some $0 < a \leq \frac{1}{3}$. Then the total length removed is $\sum_{n=1}^{\infty} 2^{n-1} a^n = \frac{a}{1-2a}$. So, for this “fat” Cantor set $m(\mathcal{C}_{\text{fat}}) = 1 - \frac{a}{1-2a} = \frac{1-3a}{1-2a}$. Which is indeed 0 when $a = \frac{1}{3}$ (standard Cantor), and $m(\mathcal{C}_{\text{fat}}) > 0$ for $0 < a < \frac{1}{3}$

Remark. $|\mathcal{L}| = |\mathcal{P}(\mathbb{R})| \leq$ is trivial so $\forall A \subset \mathcal{C}, A \in \mathcal{L}$ but $|\mathcal{C}| = \mathbb{R} \Rightarrow |\mathcal{L}| = |\mathcal{P}(\mathbb{R})|$

Remark. $|\mathcal{P}(\mathbb{R}) \setminus \mathcal{L}| = |\mathcal{P}(\mathbb{R})|$: Let V be a Vitali set, $V[0, 1]$, then $\forall A \subset [2, 3], V \cup A \notin \mathcal{L}$ and so $|\mathcal{P}(\mathbb{R})| \geq |\mathcal{P}(\mathbb{R}) \setminus \mathcal{L}| \geq |\mathcal{P}([2, 3])| = |\mathcal{P}(\mathbb{R})|$

Cantor-Lebesgue Function

Let $U_k := [0, 1] \setminus \mathcal{C}_k$, which is $2^k - 1$ disjoint open intervals, of various lengths, and

$$U = [0, 1] \setminus \mathcal{C} = [0, 1] \setminus \bigcap_{k=1}^{\infty} \mathcal{C}_k = \bigcup_{k=1}^{\infty} U_k$$

Thus U is open on $[0, 1]$ and $m(U) = m([0, 1]) = 1$ since $m(\mathcal{C}) = 0$.

Theorem 8.2. *There is a continuous (weakly) increasing function $\phi : [0, 1] \rightarrow [0, 1]$ that is surjective with $\phi(0) = 0$ and $\phi(1) = 1$ such that ϕ is differentiable in U and $\phi'(x) = 0 \forall x \in U$*

First define ϕ on U_k by setting it to be equal to the constants $\{\frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k-1}{2^k}\}$ on it's $2^k - 1$ open intervals. Observe that if we increase $k \rightarrow k + 1$, U_{k+1} has more intervals but some of them are the same that we already had in U_k , and on those, the value of ϕ in the 2 steps agrees!

Taking the union over k defines ϕ on U . To extend ϕ to all of $[0, 1]$, we let $\phi(0) = 0$ and for all $x \in \mathcal{C} \setminus \{0\}$ let $\phi|x| := \sup\{\phi(y) \mid y \in U \cap [0, x]\}$ (this is finite since ≤ 1)

We have defined a function $\phi : [0, 1] \rightarrow [0, 1]$ and it satisfies the specified properties.

Consider now $\psi(x) := \phi(x) + x$ for $x \in [0, 1]$. Some obvious properties:

- ψ is continuous
- ψ is strictly increasing
- $\psi(0) = 0, \psi(1) = 2$
- $\psi([0, 1]) = [0, 2]$ and ψ is a bijection between these
- $\psi^{-1} : [0, 2] \rightarrow [0, 1]$ is continuous

Proposition. $m(\psi(\mathcal{C})) = 1$ and $\exists E \subset \mathcal{C}, E \in \mathcal{L}$ such that $\psi(E) \notin \mathcal{L}$

Corollary. *This set E is measurable but not Borel.*

Proposition (Continuity of Measure). 1. If $\{A_j\}_{j=1}^{\infty}$ are measurable sets with $A_j \subset A_{j+1} \forall j$, then

$$m\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j \rightarrow \infty} m(A_j)$$

2. If $\{B_j\}_{j=1}^{\infty}$ are measurable sets with $B_{j+1} \subset B_j \forall j$, and $m(B_j) < \infty \iff m(B_1) < \infty$ then

$$m\left(\bigcap_{j=1}^{\infty} B_j\right) = \lim_{j \rightarrow \infty} m(B_j)$$

Definition 8.1 (Almost Everywhere). We say some property “ P ” holds almost everywhere on E , or for a.e $x \in E$, if $\exists E_0 \subset E$ with $m^*(E_0) = 0$ such that P holds for all $x \in E \setminus E_0$. We also say “ P holds for almost all x in E ”.

Ex: Almost every real number is irrational.

Proposition (Borel-Cantelli’s Lemma). Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{L}$ be such that $\sum_{j=1}^{\infty} m(E_j) < \infty$. Then almost every $x \in \mathbb{R}$ belongs to at most finitely many E_j ’s.

Proof. For each n ,

$$m\left(\bigcup_{j=n}^{\infty} E_j\right) \leq \sum_{j=n}^{\infty} m(E_j) < \infty$$

and

$$\bigcup_{j=n+1}^{\infty} E_j \subset \bigcup_{j=n}^{\infty} E_j$$

So by the continuity of measure

$$m\left(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j\right) = \lim_{n \rightarrow \infty} m\left(\bigcup_{j=n}^{\infty} E_j\right) \leq \lim_{n \rightarrow \infty} \sum_{j=n}^{\infty} m(E_j) \underbrace{=}_{\text{tails of a convergent series}} 0$$

Hence “almost every” $x \in E$ satisfies $x \notin \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j$. i.e for each such x , $\exists n$ such that $x \notin \bigcup_{j=n}^{\infty} E_j$ so x belongs only to (at most) $E_1 \dots E_{n-1}$ \square

9 Measurable Functions

We shall now study functions $f : E \rightarrow [-\infty, \infty] := \mathbb{R} \cup \{\pm\infty\}$ where $E \subset \mathbb{R}$ is a *measurable* set.

Sublevel sets of f are the sets of the form $f^{-1}([-\infty, c)) = \{x \in E \mid f(x) < c\}$, for some $c \in \mathbb{R}$

Definition 9.1. If we have $f : E \rightarrow [-\infty, \infty]$ with E measurable, then we say that f is measurable if all sublevel sets $f^{-1}([-\infty, c))$ are in \mathcal{L} for all $x \in \mathbb{R}$.

Proposition. $f : E \rightarrow [-\infty, \infty]$, then the following are equivalent:

1. f measurable
2. $\forall c \in \mathbb{R}, f^{-1}([-\infty, c]) = \{x \in E \mid f(x) \leq c\} \in \mathcal{L}$
3. $\forall c \in \mathbb{R}, f^{-1}((c, \infty]) = \{x \in E \mid f(x) > c\} \in \mathcal{L}$
4. $\forall c \in \mathbb{R}, f^{-1}([c, \infty]) \in \mathcal{L}$
5. $\forall U \subset \mathbb{R}$ open, $f^{-1}(U) \in \mathcal{L}$

6. $\forall A \subset \mathbb{R}$ Borel set, $f^{-1}(A) \in \mathcal{L}$

Ex: If E measurable, $f : E \rightarrow \mathbb{R}$ continuous, then f is measurable. Indeed, $\forall U \subset \mathbb{R}$ open, $f^{-1}(U)$ is open in E , i.e $f^{-1}(U) = V \cap E$ where $V \subset \mathbb{R}$ open. Clearly $V \cap E \in \mathcal{L}$, so f is measurable.

Caution: $f : E \rightarrow \mathbb{R}$ continuous and $A \subset \mathbb{R}$ measurable $\not\Rightarrow f^{-1}(A) \in \mathcal{L}$. For example: $E = [0, 1]$, $f = \psi^{-1}$ then we proved earlier that ψ maps a measurable subset onto a non-measurable subset.

Proposition. $f : [a, b] \rightarrow \mathbb{R}$ monotone $\implies f$ measurable

Proof. without loss of generality, we may assume f is monotone increasing $f(x) \leq f(y)$ whenever $x \leq y$. For any $c \in \mathbb{R}$, look at $\{f < c\}$ and assume it is non-empty. We show that $\{f < c\}$ is an interval $\subset [a, b]$. Now, intervals $I \in \mathbb{R}$ are characterized by the property that if $x \leq y \in I$ then the whole segment $tx + (1 - t)y$ is in I , for $0 \leq t \leq 1$. So let $f(x) < c$, $f(y) < c$, then $tx + (1 - t)y \leq y$ so $f(tx + (1 - t)y) \leq f(y) < c$ too.

So $\{f < c\}$ is an interval which means that f is measurable. \square

Proposition. given $E \subset \mathbb{R}$ measurable, $f : E \rightarrow [-\infty, \infty]$ measurable

1. If $g : E \rightarrow [-\infty, \infty]$ is another function and $f = g$ a.e on E . Then g is measurable
2. Suppose $D \subset E$, D measurable. Then f is measurable (as a function on E) $\iff f|_D$ measurable (as a function on D) and $f|_{E \setminus D}$ is measurable (as a function on $E \setminus D$).

Proof. (1): Let $A = \{x \in E \mid f(x) \neq g(x)\}$, which by assumption has $m(A) = 0$. Then $\forall c \in \mathbb{R}$,

$$\begin{aligned} \{x \in E \mid g(x) > c\} &= \{x \in A \mid g(x) > c\} \cup \{x \in E \setminus A \mid f(x) > c\} \\ &= \{x \in A \mid g(x) > c\} \cup \underbrace{\{x \in E \mid f(x) > c\}}_{\in \mathcal{L}} \cap \underbrace{\{E \setminus A\}}_{\in \mathcal{L}} \end{aligned}$$

$\{x \in A \mid g(x) > c\}$ is a subset of A hence it has measure 0 and is also measurable so $\{g > c\} \in \mathcal{L}$.

(2):

$$\begin{aligned}\{x \in E \mid f(x) > c\} &= \{x \in D \mid f(x) > c\} \cup \{x \in E \setminus D \mid f(x) > c\} \\ &= (\{x \in E \mid f(x) > c\} \cap D) \cup (\{x \in E \mid f(x) > c\} \cap (E \setminus D))\end{aligned}$$

□

Sums and Products: If $f, g : E \rightarrow [-\infty, \infty]$ can we consider their sum $f + g$? Well, if $f(x) = \infty$ and $g(x) = -\infty$ then $f(x) + g(x)$ is definitely undefined. Let us then assume that f and g are finite for a.e point in E . Thus, $\exists E_0 \subset E$ with $m(E_0) = 0$, such that f and g are finite on $E \setminus E_0$. We will now show that $f + g : E \setminus E_0 \rightarrow \mathbb{R}$ is measurable (on $E \setminus E_0$). Then if $h : E \rightarrow [-\infty, \infty]$ is any function such that $h|_{E \setminus E_0} = (f + g)|_{E \setminus E_0}$ then h is also measurable by part (2) above. Observe that such an h always exists (e.g set $h = f + g$ on $E \setminus E_0$ and $h = 0$ on E_0), and it is not unique at all. However, as we just said, all such h are measurable. We thus can say $f + g$ is measurable on E .

Proposition. $f, g : E \rightarrow [-\infty, \infty]$ measurable such that f, g are finite a.e on E . Then $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ are measurable on E .