Honours Analysis 3

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1 Borel Sets

We will work for some time on $\mathbb{R}$ exclusively. Before beginning Measure Theory: a quick recap of Topology.

Definition 1.1 (Open Set). *A subset $U \subset \mathbb{R}$ is called* open *if either $U = \emptyset$ or else*

$$\forall x \in U, \exists r > 0 \text{ such that } (x - r, x + r) \subset U$$

Some examples of open sets: $\emptyset, \mathbb{R}, (a, b), (a, \infty), (-\infty, a)$. There are many more because any union of an open set is still open and any finite intersection of open sets is open.

Definition 1.2 (Closed Set). *$F \subset \mathbb{R}$ is called* closed *if $\mathbb{R} \setminus F := F^c$ is open.*

F is closed $\iff$ F contains all points $x \in \mathbb{R}$ which have the property that $\forall r > 0, (x - r, x + r) \cap F \neq \emptyset$.

If $F \subset \mathbb{R}$ is any set, the closure of F, denoted by $\overline{F}$, is the smallest closed set that contains F.

Definition 1.3 (Compact). *A subset $G \subset \mathbb{R}$ is* compact *if given any collection $\{U_i\}_{i \in I}$ of open sets $U_i \subset \mathbb{R}$ with $G \subset \cup_{i \in I} U_i$, there exists $J \subset I$, J finite, such that $G \subset \cup_{j \in J} U_j$*

*Notes from the lectures of Valentino Tosatti

Theorem 1.1 (Heine-Borel). *$G \subset \mathbb{R}$ is compact $\iff$ G is closed and bounded. To be bounded means $G \subset (a, b)$ for some $a, b \in \mathbb{R}$.*

Corollary 1.1.1 (Nested Set Theorem). *Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of non-empty, bounded, closed sets $F_n \subset \mathbb{R}$ with $F_{n+1} \subset F_n \forall n$, then*

$$\cap_{n=1}^{\infty} F_n \neq \emptyset$$

Proof. Suppose $\cap_{n=1}^{\infty} F_n = \emptyset$ so let $U_n = F_n^c$ be open sets, such that $\cup_{n=1}^{\infty} U_n = \mathbb{R}$. We also have that $U_n \subset U_{n+1}$, since the F_n were nested. Now F_1 is compact by Heine-Borel and $F_1 \subset \cup_{n=1}^{\infty} U_n \Rightarrow$ by compactness I can find a finite subcover of F_1, say $F \subset \cup_{n=1}^{N} U_n = U_N = F_N^c$

On the other hand $F_N \subset F_1$ by the nested property which implies $F_N = \emptyset$ which is a contradiction. $\square$

2 Measure Theory

We want to measure the size of a set. We will deal with a subset of $\mathbb{R}$.

It turns out that one needs to select a class of subsets of $\mathbb{R}$ that one wants to measure. This class of subsets will have certain properties which are as follows.

Definition 2.1 (σ-algebra). *A collection $\mathcal{A}$ of subsets of $\mathbb{R}$ is called a σ-algebra if it satisfies*

1. $\emptyset \in \mathcal{A}$
2. *If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$*
3. *If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$*

Observe the following:

- $\mathbb{R} \in \mathcal{A}$ always

- If $\{A_n\}_{n=1}^N \subset \mathcal{A}$ then $\cup_{n=1}^N A_n \in \mathcal{A}$ (just define $A_n = \emptyset$ for $n > N$)
- If $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$ then $\cap_{n=1}^\infty A_n \in \mathcal{A}$ (since $(\cap_{n=1}^\infty A_n)^c = \cup_{n=1}^\infty A_n^c$)
- If $A, B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$ too since $A \setminus B = A \cap B^c$

Examples:

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$ "Minimal σ-algebra"
2. $\mathcal{A} = \mathcal{P}(\mathbb{R})$ = Collection of all subsets of $\mathbb{R}$. "Maximum σ-algebra"

In fact, if $\mathcal{A}$ is any σ-algebra, then $\{\emptyset, \mathbb{R}\} \subseteq \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$

For better examples, let F be any collection of subsets of $\mathbb{R}$. I want to make F into a σ-algebra. Define $m = \{\mathcal{A} \mid \mathcal{A}$ is a σ-algebra that satisfies $F \subset \mathcal{A}\}$. $m \neq \emptyset$ since it contains $\mathcal{P}(\mathbb{R})$

If $\mathcal{A}, \mathcal{B} \in m$, I can define $\mathcal{A} \cap \mathcal{B} = \{A \subset \mathbb{R} \mid A \in \mathcal{A}$ and $A \in \mathcal{B}\}$ and I can do the same for $\cap_{i \in I} \mathcal{A}$ arbitrary intersection of σ-algebra is still a σ-algebra

Define $\hat{F}_i = \cap_{\mathcal{A} \in m} \mathcal{A}$ as a σ-algebra and $F \subset \hat{F}$ and it is the minimal σ-algebra with these properties. If G is a σ-algebra with $F \subset G$, then $\hat{F} \subset G$. $\hat{F}$ is the σ-algebra generated by F. Concretely, $\hat{F}$ consists of all subsets of $\mathbb{R}$ that can be constructed by applying countable unions, intersections, and complements to elements of F.

Definition 2.2 (Borel Sets)**.** *The σ-algebra $\mathcal{B}$ of Borel Sets is the σ-algebra $\hat{F}$ generated by*

$$F = \{U \subset \mathbb{R} \mid U \text{ open } \}$$

Remark. *$\mathcal{B}$ is also the σ-algebra generated by the family of all closed subsets of $\mathbb{R}$*

Singletons $\{x\} \subset \mathbb{R}$ are closed so if $A \subset \mathbb{R}$ is at most countable then A is Borel. (e.g $\mathbb{Q} \subset \mathbb{R}$) (e.g $\mathbb{R} \setminus \mathbb{Q}$)

Not all Subsets of $\mathbb{R}$ are Borel. One can actually show that the cardinality of $\mathcal{B}$ is the same as the cardinality of $\mathbb{R}$. On the other hand $\mathcal{P}(\mathbb{R})$ has strictly larger cardinality.

3 Lebesgue Outer Measure

We are hoping to measure the size of subsets of $\mathbb{R}$. Ideally we would like to find or construct a function

$$m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\} = [0, \infty]$$

Which satisfies the following measure requirements:

1. If $I = [a, b]$ or (a, b) or $[a, b)$, or $(a, b]$, $a, b \in \mathbb{R}, a \leq b$ then $m(I) = b - a =$ measure of interval
2. m is translation invariant. i.e if $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, let $E + x = \{y + x \mid y \in E\}$ then $m(E + x) =$ m(E)
3. If $\{E_j\}_{j=1}^n$ is a finite collection of pairwise disjoint $E_j \subset \mathbb{R}$ then
$$m\left(\cup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$
4. The same as (3) except for $n = \infty$

Theorem 3.1. *There is no such m satisfying all 4 requirements*

The proof for this will come later. The solution for this is that we do not try to measure all subsets of $\mathbb{R}$. So we have $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ but now we will just be happy with $m : \mathcal{A} \to [0, \infty]$ where $\mathcal{A}$ is a σ-algebra which has enough elements. For example $\mathcal{A} > \mathcal{B}$.

We will follow H. Lebesgue as we proceed in two steps.

Step 1: construct Lebesgue outer measure $m^\star : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ satisfying requirements 1,2, and 3.

Step 2: Use $m^\star$ to define $\mathcal{A}$ and let $m \subset m^\star \mid \mathcal{A}$

To create this Lebesgue outer measure on $\mathbb{R}$ we satisfy a weakened version of requirement (3) that can be called (3w). For any countably infinite collection $\{E_j\}_{j=1}^\infty$ of arbitrary subsets $E_j \subset \mathbb{R}$

$$m^\star(\cup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty m(E_j)$$

Theorem 3.2 (Lebesgue Outer Measure)**.** *There is a map* $m^\star : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ *that satisfies the measure requirements 1, 2, and 3w.*

This $m^\star$ is called the Lebesgue outer measure on $\mathbb{R}$.

How do we define outer measure $m^\star(A)$?

Observe that any $A \subseteq \mathbb{R}$ can be covered by some countable infinite collection $\{I_j\}_{j=1}^\infty$ of bounded open intervals, which are allowed to be empty, but we do not assume that I_j be pairwise disjoint.

For example: $I_j = (-j, j),\ j = 1, 2, 3 \ldots$

Let

$$\mathcal{C}_A = \{\{I_j\}_{j=1}^\infty \mid I_j \text{ bounded open intervals such that } A \subset \cup_{j=1}^\infty I_j\}$$

$\mathcal{C}_A \neq \emptyset$ by our example so for each $\{I_j\} \in \mathcal{C}_A$, I can consider

$$\sum_{j=1}^\infty \ell(I_j) \in \mathbb{R}_{\geq 0} \cup \{+\infty\} \qquad (\ell \text{ denotes length})$$

Definition 3.1 (Outer Measure)**.**

$$\boxed{m^\star(A) := \inf_{\{I_j\} \in \mathcal{C}_A} \sum_{j=1}^\infty \ell(I_j)} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This defines a map $m^\star : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$

Simple Properties:

- *Monotonicity*: If $A \subseteq B$ then $m^\star(A) \leq m^\star(B)$. Indeed by definition $\mathcal{C}_B \subseteq \mathcal{C}_A$ hence the infimum over $\mathcal{C}_B$ is $\geq$ than the infimum over $\mathcal{C}_A$.
- *Empty Set*: $m^\star(\emptyset) = 0$. Given any $1 > \epsilon > 0$, let $I_j = (-\epsilon^j, \epsilon^j),\ j = 1, 2, \ldots$ $\{I_j\} \in \mathcal{C}_\emptyset$ and $\sum_{j=1}^\infty \ell(I_j) = 2\sum_{j=1}^\infty \epsilon^j = \frac{2\epsilon}{1-\epsilon}$ from the geometric series going to zero so $m^\star(\emptyset) \leq \frac{2\epsilon}{1-\epsilon}$ $\forall 0 < \epsilon < 1$

- If $A \in \mathbb{R}$ is finite or countable infinite then $m^\star(A) = 0$. Indeed enumerate all elements of A by $\{a_j\}_{j=1}^{\infty}$. (If A is finite say $|A| = n$ let $a_j = a_n$ for all $j > n$). For any $0 < \epsilon < 1$, let $I_j = \left(-\epsilon^j + a_j, a_j + \epsilon^j\right)$ so $A \subseteq \cup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} \ell(I_j) = \frac{2\epsilon}{1-\epsilon}$ hence as before, $m^\star(A) = 0$. For example $m^\star(\mathbb{Q}) = 0$

We will now prove that the Lebesgue outer measure satisfies 1, 2, and 3w of the measure requirements.

Proof of Property 1: i.e $m^\star(I) = \ell(I)$ for any interval $I \subseteq \mathbb{R}$

Assume that $I = [a, b]$, $a < b$ are finite numbers. Assume that I is a bounded closed interval. Our goal is to show that $m^\star(I) = b - a$. One direction of inequality is easy to prove, the other is quite tedious and will be left out.

For any $\epsilon > 0$ let $I_1 = (a - \epsilon, b + \epsilon) > I$, let $I_j = \emptyset, j \geq 2$ so $\{I_j\} \in \mathcal{C}_I \Rightarrow m^\star(I) \leq \sum_{j=1}^{\infty} \ell(I_j) = b - a + 2\epsilon$. Let $\epsilon \to 0$ and we obtain $m^\star(I) \leq b - a$.

Proof of Property 2: i.e $\forall A \subset \mathbb{R}, \forall x \in \mathbb{R}$, $m^\star(A + x) = m^\star(A)$

$\mathcal{C}_A$ and $\mathcal{C}_{A+x}$ are naturally in bijection via $\{I_j\} \leftrightarrow \{I_j + x\}$. Furthermore $\ell(I_j + x) = \ell(I_j)$

$$\begin{aligned} m^\star(A + x) &= \inf_{\{I_j + x\} \in \mathcal{C}_{A+x}} \sum_{j=1}^{\infty} \ell(I_j + x) \\ &= \inf_{\{I_j\} \in \mathcal{C}_A} \sum_{j=1}^{\infty} \ell(I_j) = m^\star(A) \end{aligned}$$

Proof of Property 3w: i.e If $\{E_j\}_{j=1}^{n}$ is a finite collection of pairwise disjoint $E_j \subset \mathbb{R}$ then $m^\star\left(\cup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m^\star(E_j)$

If $m^\star(E_j) = +\infty$ for some j, then the property holds. We may assume that $m^\star(E_j) < +\infty \; \forall j$. Let $\epsilon > 0$. By the definition of infimum, for each $j \geq 0$, there is

$$\{I_{j,k}\}_{k=1}^{\infty} \in \mathcal{C}_{E_j} \text{ such that } \sum_{k=1}^{\infty} \ell(I_{j,k}) < m^\star(E_j) + \epsilon 2^{-j}$$

Thus $\{I_{j,k}\}_{k=1}^{\infty}$ is still countable and it covers $\cup_{j=1}^{\infty} E_j$ meaning it belongs to $\mathcal{C}_{\cup_{j=1}^{\infty}} E_j$, so by definition

$$m^{\star}\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \ell(I_{j,k}) < \sum_{j=1}^{\infty}(m^{\star}(E_j) + \epsilon 2^{-j}) = \sum_{j=1}^{\infty} m^{\star}(E_j) + \epsilon$$

Then let $\epsilon \to 0$. Clearly, by taking all $E_j = \emptyset$ except finitely many, we have the same subadditivity 3w for finite collections.

Corollary 3.2.1. $m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1 = \ell([0,1])$

Proof.

$$\begin{aligned} m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) &\leq m^{\star}([0,1]) = 1 \\ &\leq m^{\star}([0,1] \cap (\mathbb{Q})) + m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) \\ &\leq 0 + 1 \end{aligned}$$

□

Corollary 3.2.2. $\mathbb{R} \setminus \mathbb{Q}$ *is uncountable*

Proof. If not, then

$$m^{\star}(\mathbb{R} \setminus \mathbb{Q}) = 0 \geq m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1$$ □

4 The σ-Algebra Of Lebesgue Measurable Sets

$m^{\star}$ does not satisfy the third measurability requirement without the weak 3w condition. We can construct some examples to prove this. $A, B \subset \mathbb{R}, A \cap B = \emptyset$, such that $m^{\star}(A \cup B) < m^{\star}(A) + m^{\star}(B)$ later in the class.

The idea to avoid this problem is to look at "reasonable" subsets of $\mathbb{R}$ for which this paradox disappears.

Definition 4.1 (Carathéodory)**.** *$E \subseteq R$ is called (Lebesgue) measurable if* $\forall A \subset \mathbb{R}$

$$m^{\star}(A) = m^{\star}(A \cap E) + m^{\star}(A \cap E^c)$$

Remark. *This is equivalent to Lebesgue's definition: E is measurable if and only if*

$$\exists U \subset \mathbb{R} \text{ such that } E \subset U \text{ and } m^\star(U \setminus E) < \epsilon$$

But we will discuss this later.

Suppose that A is measurable and $B \subset \mathbb{R}$ is any set such that $A \cap B = \emptyset$ then

$$m^\star(A \cup B) = m^\star\left(\underbrace{(A \cup B) \cap A}_{=A}\right) + m^\star\left(\underbrace{(A \cup B) \cap A^c}_{=B}\right)$$

Going back to our counter example for $m^\star$ and measurability requirement 3, A or B would have to be unmeasurable.

Here's another observation: For $E, A \subset \mathbb{R}$ arbitrary sets we have

$$A = (A \cap E) \cup (A \cap E^c)$$

So by 3w $m^\star(A) \leq m^\star(A \cap E) + m^\star(A \cap E^c)$, so E is measurable $\iff$ $\forall A \subset \mathbb{R}$

$$\boxed{m^\star(A) \geq m^\star(A \cap E) + m^\star(A \cap E^c)}$$

This holds trivially for $m^\star(A) = \infty$

Example 1: $\emptyset$ is measurable. $\forall A \subset \mathbb{R}$

$$m^\star(A) = \cancel{m^\star(A \cap \emptyset)} + m^\star(A \cap \mathbb{R})$$

Example 2: $\mathbb{R}$ is measurable. $\forall A \subset \mathbb{R}$

$$m^\star(A) = m^\star(A \cap \mathbb{R}) + m^\star(A \cap E^c)$$

Proposition. *$E \subset \mathbb{R}$ with $m^\star(E) = 0$, then E is measurable.*

Corollary. *Every countable set is measurable. $\mathbb{Q}$ measurable $\to \mathbb{R} \setminus \mathbb{Q}$ are measurable*

Proof. Let $A \subset \mathbb{R}$ be any set

$$A \cap E \subset E \Rightarrow m^\star(A \cap E) \leq m^\star(E) = 0$$
$$A \cap E^c \subset A \Rightarrow m^\star(A \cap E^c) \leq m^\star(A)$$
$$\text{So } m^\star(A) \geq m^\star(A \cap E^c) + \cancel{m^\star(A \cap E)}$$

□

Our goal is to show that Lebesgue measurable sets $\mathcal{L} = \{E \subset \mathbb{R} \mid E \text{ is measurable}\}$ is a σ-algebra on $\mathbb{R}$. We just need to show that if $\{E_j\}_{j=1}^{\infty}$ with $E_j \in \mathcal{L}$, $\forall j$, then $\cup_{j=1}^{\infty} E_j \in \mathcal{L}$

Proposition. *If $\{E_j\}_{j=1}^{n} \subset \mathcal{L}$ then $\cup_{j=1}^{n} E_i \in \mathcal{L}$*

Proof. We use mathematical induction. $n = 1$ is trivial so we set the base case as $n = 2$. E_1, E_2 are measurable, Let $A \subset \mathbb{R}$ be any set

$$\begin{aligned} m^{\star}(A) &= m^{\star}(E_1 \cap A) + m^{\star}(A \cap E_1^c) \\ &= m^{\star}(A \cap E_1) + m^{\star}((A \cap E_1^c) \cap E_2) + m^{\star}((A \cap E_1^c) \cap E_2^c) \\ &= m^{\star}(A \cap E_1) + m^{\star}((A \cap E_1^c) \cap E_2) + m^{\star}(A \cap (E_1^c \cap E_2^c)) \\ &= m^{\star}(A \cap E_1) + m^{\star}((A \cap E_1^c) \cap E_2) + m^{\star}(A \cap (E_1 \cup E_2)^c) \\ &\geq m^{\star}(A \cap (E_1 \cup E_2)) + m^{\star}(A \cap (E_1 \cup E_2)^c) \qquad \text{(3w)} \end{aligned}$$

So $E_1 \cup E_2 \in \mathcal{L}$.

Induction step $n \geq 2$

$$\bigcup_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{n-1} E_j\right) \cup E_n \in \mathcal{L} \text{ by the } n = 2 \text{ case} \qquad \square$$

To prove that this also applies to countable sets, we use

Proposition (Analog of measurability requirement 3 for $m^{\star} \mid \mathcal{L}$)**.** *Suppose $A \subset \mathbb{R}$ is any set and $\{E_j\}_{j=1}^{n}$ is a finite disjoint collection of sets $E_j \in \mathcal{L}$, then*

$$m^{\star}\left(A \cap \bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m^{\star}(A \cap E_j)$$

In particular take $A = \mathbb{R}$ to get $m^{\star}\left(\bigcup_{j=1}^{n} E_j\right) = \sum m^{\star}(E_j)$

Proposition. *If $\{E_j\}_{j=1}^{\infty}$ is a countable family with $E_i \in \mathcal{L}\ \forall j$, then $\cup_{j=1}^{\infty} E_j \in \mathcal{L}$. In particular, $\mathcal{L}$ is a σ-algebra.*

We would like to have the Borel sets be measurable, i.e $\mathcal{B} \subset \mathcal{L}$. Recall that $\mathcal{B} = \hat{\mathcal{F}}$, where $\mathcal{F} = \{U \subset \mathbb{R} \mid U \text{ is open }\}$ and $\hat{}$ denotes the σ-algebra.

This results follows from the measurability of intervals combined with the measurability of the union of measurable sets.

Proposition. *If $I \subseteq \mathbb{R}$ is any interval, then I is measurable.*

Theorem 4.1. *$\mathcal{L}$ = Lebesgue Measurable subsets of $\mathbb{R}$ form a σ-algebra that contains the Borel σ-algebra $\mathcal{B}$*

Proof. We already know that $\mathcal{L}$ is a σ-algebra. If we can show that $\mathcal{L}$ contains all open sets $U \subset \mathbb{R}$, then $\mathcal{L}$ (being a σ-algebra) must contain $\mathcal{B}$ which is the σ-algebra generated by open sets. Now if $U \subset \mathbb{R}$ is any (non empty) open set then by definition $\forall x \in U, \exists I_x \ni x$ where I_x is an open interval and $I_x \subset U$.

We want to choose I_x to be the "maximal" such. So by assigning

$$a_x \coloneqq \inf\{z \in \mathbb{R} \mid (z, x) \subset U\} \text{ satisfies } a_x < x$$

and

$$b_x \coloneqq \sup\{y \in \mathbb{R} \mid (x, y) \subset U\} \text{ satisfies } x < b_x$$

so $I_x \coloneqq (a_x, b_x)$ is an open interval that contains x and by construction $I_x \in U$. It is the largest such, in the sense that if $a_x > -\infty$ then $a_x \notin U$ and symmetrically if $b_x < \infty$ then $b_x \notin U$.

For any $y \in I_x$, we have $y < b_x$, so there is $z > y$ such that $(x, z) \subset U$ so $y \in U$. Indeed, if $a_x \in U$ then since U open, $\exists r > 0$ such that $(a_x - r, a_x + r) \subset U$ contradicting the definition of a_x.

So $U = \cup_{x \in U} I_x$. It is a huge union, however if $x, x' \in U, x \neq x'$, then either $I_x \cap I_{x'} = \emptyset$, or if not then necessarily $I_x = I_{x'}$, since $I_x \cup I_{x'}$ is then another open interval that contains x & x' and is a subset of U, so by maximality it must equal I_x & $I_{x'}$. So, throwing away all repeated I_x, we can write $U = \cup_{i \in I} I_x$ for some I where the intervals I_{x_i} are pairwise disjoint. By density of $\mathbb{Q} \subset \mathbb{R}$, each such interval contains a different rational number $r_i \in I_{x_i}$. Since $\mathbb{Q}$ is countable, I is at worst countable.

So every U open is an at most countable disjoint union of open intervals. Since such intervals belong ot $\mathcal{L}$, and $\mathcal{L}$ is a σ-algebra, it follows that every U open is in $\mathcal{L}$ as desired. $\square$

Proposition (The σ-algebra $\mathcal{L}$ is also translation invariant)**.** *If $E \subset \mathcal{L}$ and $x \in \mathbb{R}$ then $E + x \in \mathcal{L}$*

Proof. Given any $A \subset \mathbb{R}$,

$$\begin{aligned} m^{\star}(A) &= m^{\star}(A - x) \\ &= m^{\star}((A - x) \cap E) + m^{\star}((A - x) \cap E^{c}) \\ &= m^{\star}(A \cap E + x) + m^{\star}(A \cap (E + x)^{c}) \quad (m^{\star} \text{ translation invariant}) \end{aligned}$$

□

Remark. *If $A \in \mathcal{L}$ with $m^{\star}(A) < \infty$, and $B \subset \mathbb{R}$ is any set with $A \subset B$, then*

$$m^{\star}(B \setminus A) = m^{\star}(B) - m^{\star}(A)$$

5 Outer and Inner Approximation of Lebesgue Measurable Sets

Definition 5.1 (Gebiet-Durchshnitt)**.** *A subset $A \subset \mathbb{R}$ is called a G_{δ} if $A = \cap_{i=1}^{\infty} A_i$ where A_i are all open (possibly empty).*

Definition 5.2 (Fermé-Somme)**.** *A subset $A \subset \mathbb{R}$ is called a F_{σ} if $A = \cup_{i=1}^{\infty} A_i$ where A_i are all closed (possibly empty).*

Clearly, A is $G_{\delta} \iff A^{c}$ is F_{δ}. Also clearly, all G_{δ} and F_{σ} sets are Borel. Of course not all G_{δ} are open, e.g $[0,1] = \cap_{i=1}^{\infty} \left(-\frac{1}{i}, 1 + \frac{1}{i}\right)$ and not all F_{σ} are closed. e.g. $(0,1) = \cup_{i=1}^{\infty} \left[\frac{1}{i}, 1 - \frac{1}{i}\right]$

$\mathbb{Q}$ is clearly F_{σ}, so $\mathbb{R} \setminus \mathbb{Q}$ is G_{δ}. With this, we can give several equivalent formulations of measurability.

Theorem 5.1. *Let $E \subset \mathbb{R}$ be any set, then the following are equivalent:*

1. *$E \in \mathcal{L}$*
2. *$\forall \epsilon > 0$, $\exists U \supset E$, U open, $m^{\star}(U \setminus E) < \epsilon$*

3. *$\exists G \subset \mathbb{R}$ a G_δ set, $G \supset E$, with $m^\star(G \setminus E) = 0$*

4. *$\forall \epsilon > 0$, $\exists F \subset E$, F closed, $m^\star(E \setminus F) < \epsilon$*

5. *$\exists F \subset \mathbb{R}$ a F_σ set, $F \subset E$ with $m^\star(E \setminus F) = 0$*

Proposition. *For an $E \in \mathcal{L}$ with $m^\star(E) < \infty$. Then $\forall \epsilon > 0$, $\exists \{I_j\}_{j=1}^n$ a finite disjoint family of open intervals so that if we let $U = \cup_{j=1}^n I_j$ (open) then $m^\star(E \Delta U) < \epsilon$.*

6 Lebesgue Measure

We can now take $m^\star$ and restrict it to $\mathcal{L}$. $m^\star|_{\mathcal{L}}$.

Definition 6.1 (Lebesgue Measure)**.** *This Lebesgue Measure is a function*

$$m := m^\star|_{\mathcal{L}} \colon \mathcal{L} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This means that for $E \in \mathcal{L}$ we define $m(E) = m^\star(E)$. Clearly, m satisfies the measurability requirements 1, 2, & 3 as we have proved earlier. It also satisfies requirement 4 which was requirement 3 for countably infinite sets.

Proposition. *If $\{E_j\}_{j=1}^\infty$ is a countably infinite collection of pairwise disjoint sets $E_j \in \mathcal{L}$ (possibly empty), then $\cup_{j=1}^\infty E_j \in \mathcal{L}$ and*

$$m\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty m(E_j)$$

Proof. We proved earlier that $\cup_{j=1}^\infty E_j \in \mathcal{L}$ and that

$$m\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty m(E_j)$$

For the opposite inequality, for each n we proved earlier that

$$m\left(\bigcup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$

But $\cup_{j=1}^{n} E_j \subset \cup_{j=1}^{\infty} E_j$, hence

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \geq m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j) \ \ \forall n$$

Take the limit as $n \to \infty$ to get

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \geq \sum_{j=1}^{\infty} m(E_j)$$

As desired. This argument shows that measurability requirement 3 and 3w together imply 4. $\square$

7 Non-Measurable Sets

We saw earlier that if $E \subset \mathbb{R}$ satisfies $m^\star(E) = 0$ then $E \in \mathcal{L}$. In particular, $\forall F \subset E$, $m^\star(F) \leq m^\star(E) = 0$, so $F \in \mathcal{L}$ too. This however totally fails when $m^\star(E) > 0$.

Theorem 7.1 (Vitali)**.** *For any $E \subset \mathbb{R}$ with $m^\star(E) > 0$, there is an $F \subset E$ which is NOT measurable. The construction uses the axiom of choice (and it is really needed).*

The proof of this theorem and construction of a Vitali set are currently omitted due to length.

8 Cantor Set

We showed earlier that if $A \subset \mathbb{R}$ is countable then $A \in \mathcal{L}$ and $m(A) = 0$. How about the converse; if $A \in \mathcal{L}$ has $m(A) = 0$, is A countable? No!

Theorem 8.1 (Cantor)**.** *There is a closed, uncountable set $\mathcal{C}$ with $m(\mathcal{C}) = 0$*

Start with an interval $I = [0,1]$ and remove the middle $\frac{1}{3}$, namely $(\frac{1}{3}, \frac{2}{3})$.

$$
\begin{aligned}
\mathcal{C}_1 &:= I \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right] \\
\mathcal{C}_2 &:= \mathcal{C}_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right)\right) \\
\mathcal{C}_k &:= \mathcal{C}_{k-1} \setminus \bigcup_{j=0}^{3^{k-1}-1} \left(\frac{3j+1}{3^k}, \frac{3j+2}{3^k}\right) \\
&= [0,1] \setminus \bigcup_{l=1}^{k} \bigcup_{j=0}^{3^{l-1}-1} \left(\frac{3j+1}{3^l}, \frac{3j+2}{3^l}\right)
\end{aligned}
$$

Thus $\{\mathcal{C}_k\}_{k=1}^{\infty}$ is a very large descending (i.e nested $\mathcal{C} \subset \mathcal{C}_{k-1}$) sequence of closed sets, and $\mathcal{C}_k$ is a disjoint union of 2^k closed intervals of length $\frac{1}{3^k}$. Let then $\mathcal{C} = \bigcap_{k=1}^{\infty} \mathcal{C}_k$, so $\mathcal{C}$ is closed, and hence also measurable.

Since $m(\mathcal{C}_k) = \left(\frac{2}{3}\right)^k$, $m(\mathcal{C}) \leq m(\mathcal{C}_k) \leq \left(\frac{2}{3}\right)^k$ $\forall k$. Taking the limit as $k \to \infty$ we get $m(\mathcal{C}) = 0$.

Suppose that $\mathcal{C}$ was countable, let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of all it's elements. Then writing $\mathcal{C}_1 =$ the disjoint union of 2 interavals, we must have that c_1 belongs to precisely one of them. Say $c_1 \notin F_1$. Now $F_1 \subset \mathcal{C}_2$ is made of 2 disjoint intervals, and one of them does not contain c_2, say $c_2 \notin F_2$.

Continue this way until we get a sequence of $\{F_k\}_{k=1}^{\infty}$, where F_k is a closed interval, $F_{k+1} \subset F_k$, and $F_k \subset \mathcal{C}_k$, and $c_k \notin F_k$. By the nested set theorem, let $x \in \cap_{k=1}^{\infty} F_k$. Then

$$x \in \cap_{k=1}^{\infty} F_k \subset \cap_{k=1}^{\infty} \mathcal{C}_k = \mathcal{C}$$

So $x \in \mathcal{C}$ but $\{c_k\}_{k=1}^{\infty}$ enumerates ALL points of $\mathcal{C}$ so $\exists n$ such that $x = c_n$. Hence $x \notin F_n$ but this is a contradiction so we conclude that $\mathcal{C}$ is uncountable.

Finally observe that $\mathcal{C}$ is closed and $\mathcal{C} \subset [0,1]$, so $\mathcal{C}$ is compact by Heine-Borel.

There are two variations of this theorem.

1. If instead of removing the middle third, we removed the middle $p\%$ where $0 < p < 100$, then we also get a Cantor set which has the same properties as $\mathcal{C}$.

2. We could also remove a *smaller* proportion at each step, instead of a fixed one. At each step we remove 2^{n-1} intervals of length a^n for some $0 < a \leq \frac{1}{3}$. Then the total length removed is $\sum_{n=1}^{\infty} 2^{n-1} a^n = \frac{a}{1-2a}$. So, for this "fat" Cantor set $m(\mathcal{C}_{\text{fat}}) = 1 - \frac{a}{1-2a} = \frac{1-3a}{1-2a}$. Which is indeed 0 when $a = \frac{1}{3}$ (standard Cantor), and $m(\mathcal{C}_{\text{fat}}) > 0$ for $0 < a < \frac{1}{3}$

Remark. $|\mathcal{L}| = |\mathcal{P}(\mathbb{R})|$*:* $\leq$ *is trivial so* $\forall A \subset \mathcal{C}$*,* $A \in \mathcal{L}$ *but* $|\mathcal{C}| = \mathbb{R} \Rightarrow |\mathcal{L}| = |\mathcal{P}(\mathbb{R})|$

Remark. $|\mathcal{P}(\mathbb{R}) \setminus \mathcal{L}| = |\mathcal{P}(\mathbb{R})|$*: Let* V *be a Vitali set,* $V[0,1]$*, then* $\forall A \subset [2,3], V \cup A \notin \mathcal{L}$ *and so* $|\mathcal{P}(\mathbb{R})| \geq |\mathcal{P}(\mathbb{R}) \setminus \mathcal{L}| \geq |\mathcal{P}([2,3])| = |\mathcal{P}(\mathbb{R})|$

Cantor-Lebesgue Function

Let $U_k \coloneqq [0,1] \setminus \mathcal{C}_k$, which is $2^k - 1$ disjoint open intervals, of various lengths, and

$$U = [0,1] \setminus \mathcal{C} = [0,1] \setminus \bigcap_{k=1}^{\infty} \mathcal{C}_k = \bigcup_{k=1}^{\infty} U_k$$

Thus U is open on $[0,1]$ and $m(U) = m([0,1]) = 1$ since $m(\mathcal{C}) = 0$.

Theorem 8.2. *There is a continuous (weakly) increasing function* $\phi : [0,1] \to [0,1]$ *that is surjective with* $\phi(0) = 0$ *and* $\phi(1) = 1$ *such that* ϕ *is differentiable in* U *and* $\phi'(x) = 0 \ \forall x \in U$

First define ϕ on U_k by setting it to be equal to the constants $\{\frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k-1}{2^k}\}$ on it's $2^k - 1$ open intervals. Observe that if we increase $k \to k+1$, U_{k+1} has more intervals but some of them are the same that we already had in U_k, and on those, the value of ϕ in the 2 steps agrees!

Taking the union over k defines ϕ on U. To extend ϕ to all of $[0,1]$, we let $\phi(0) = 0$ and for all $x \in \mathcal{C} \setminus \{0\}$ let $\phi|x| \coloneqq \sup\{\phi(y) \mid y \in U \cap [0,x)\}$ (this is finite since ≤ 1)

We have defined a function $\phi : [0,1] \to [0,1)$ and it satisfies the specified properties.

Consider now $\psi(x) \coloneqq \phi(x) + x$ for $x \in [0,1]$. Some obvious properties:

- ψ is continuous
- ψ is strictly increasing
- $\psi(0) = 0$, $\psi(1) = 2$
- $\psi([0,1]) = [0,2]$ and ψ is a bijection between these
- $\psi^{-1} : [0,2] \to [0,1]$ is continuous

Proposition. *$m(\psi(\mathcal{C})) = 1$ and $\exists E \subset \mathcal{C}$, $E \in \mathcal{L}$ such that $\psi(E) \notin \mathcal{L}$*

Corollary. *This set E is measurable but not Borel.*

Proposition (Continuity of Measure)**.** *1. If $\{A_j\}_{j=1}^{\infty}$ are measurable sets with $A_j \subset A_{j+1}$ $\forall j$, then*

$$m\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} m(A_j)$$

2. If $\{B_j\}_{j=1}^{\infty}$ are measurable sets with $B_{j+1} \subset B_j$ $\forall j$, and $m(B_j) < \infty \iff m(B_1) < \infty$ then

$$m\left(\bigcap_{j=1}^{\infty} B_j\right) = \lim_{j\to\infty} m(B_j)$$

Definition 8.1 (Almost Everywhere)**.** *We say some property "P" holds almost everywhere on E, or for a.e $x \in E$, if $\exists E_0 \subset E$ with $m^\star(E_0) = 0$ such that P holds for* all $x \in E \setminus E_0$. *We also say "P holds for almost all x in E".*

Ex: Almost every real number is irrational.

Proposition (Borel-Cantelli's Lemma)**.** *Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{L}$ be such that $\sum_{j=1}^{\infty} m(E_j) < \infty$. Then almost every $x \in \mathbb{R}$ belongs to at most finitely many E_j's.*

Proof. For each n,

$$m\left(\bigcup_{j=n}^{\infty}\right) \leq \sum_{j=n}^{\infty} m(E_j) < \infty$$

and

$$\bigcup_{j=n+1}^{\infty} E_j \subset \bigcup_{j=n}^{\infty} E_j$$

So by the continuity of measure

$$m\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} E_j\right) = \lim_{n\to\infty} m\left(\bigcup_{j=n}^{\infty} E_j\right) \leq \lim_{n\to\infty}\sum_{j=n}^{\infty} m(E_j) \underbrace{=}_{\text{tails of a convergent series}} 0$$

Hence "almost every" $x \in E$ satisfies $x \notin \cap_{n=1}^{\infty} \cup_{j=n}^{\infty} E_j$. i.e for each such x, $\exists n$ such that $x \notin \cup_{j=n}^{\infty} E_n$ so x belongs only to (at most) $E_1 \ldots E_{n-1}$ $\square$