Homework 2

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Problem 1: Show that a set $E \subset \mathbb{R}$ is measurable if and only if for every interval $I \subset \mathbb{R}$ we have

$$m^\star(I) = m^\star(I \cap E) + m^\star(I \cap E^c) \tag{1}$$

To prove the nontrivial direction, we assume (1) and need to show that given any $A \subset \mathbb{R}$ with $m^\star(A) < \infty$ we have

$$m^\star(A) \geq m^\star(A \cap E) + m^\star(A \cap E^c) \tag{2}$$

First we will show that we have

$$m^\star(U) = m^\star(U \cap E) + m^\star(U + E^c)$$

for any open set $U \subset \mathbb{R}$. Since U is open, it can be written as a countable union of disjoint open intervals $U = \cup_{k=1}^{\infty} I_k$. Let $n \in \mathbb{N}$ be any natural number. Define $F_n \coloneqq \cup_{k=1}^{n} I_k$. Since F_n is a countable union of intervals and intervals are measurable, F_n is measurable by *Proposition 7*.

By *Proposition 6* we have:

$$m^\star(E \cap F_n) = \sum_{k=1}^{n} m^\star(E \cap I_k)$$

$$
\begin{aligned}
m^{\star}(F_n) &= m^{\star}\left(\bigcup_{k=1}^{n} I_k\right) \\
&= \sum_{k=1}^{n} m^{\star}(I_k) && \text{(Proposition 6)} \\
&= \sum_{k=1}^{n} m^{\star}(I_k \cap E) + m^{\star}(I_k \cap E^c) && \text{(Equation (1))} \\
&= m^{\star}(F_n \cap E) + m^{\star}(F_n \cap E^c) && \text{(Proposition 6)}
\end{aligned}
$$

Now take the limit as $n \to \infty$ on both sides of this equation and we arrive at

$$
m^{\star}(U) = m^{\star}(U \cap E) + m^{\star}(U \cap E^c) \tag{3}
$$

Next let $\epsilon > 0$. By the definition of infimum, There exists some $U = \cup_{k=1}^{\infty} I_k \in \mathcal{C}_A$ such that $\sum_{k=1}^{\infty} \ell(I_k) < m^{\star}(A) + \epsilon$. U is a countable union of nonempty open bounded intervals, so it must be its own minimal cover. This means $m^{\star}(U) = \sum_{k=1}^{\infty} \ell(I_k)$. Using monotonicity and the fact that $A \subset U$

$$
m^{\star}(A) \leq m^{\star}(U) < m^{\star}(A) + \epsilon
$$

Since we have $m^{\star}(A) < \infty$ we can subtract $m^{\star}(A)$ to arrive at

$$
0 \leq m^{\star}(U) - m^{\star}(A) < \epsilon
$$

For all ϵ so $m^{\star}(U) = m^{\star}(A)$.

Now to conclude:

$$
\begin{aligned}
m^{\star}(A) &= m^{\star}(U) \\
&= m^{\star}(U \cap E) + m^{\star}(U \cap E^c) && \text{(Equation (3))} \\
&\geq m^{\star}(A \cap E) + m^{\star}(A \cap E^c) && (A \subset U)
\end{aligned}
$$

So putting this result together with the trivial inequality in the other direction

$$
m^{\star}(A) = m^{\star}(A \cap E) + m^{\star}(A \cap E^c)
$$

Problem 2: Let $E \subset \mathbb{R}$ be any set. Show that there is a G_δ set $F \supset E$ (in particular F is measurable) with

$$m^\star(F) = m^\star(E)$$

By monotonicity we have $m^\star(E) \leq m^\star(F)$ for any F.

To show the reverse inequality, $\forall n \in \mathbb{N}$, from the definition of infimum, there must exist a collection $\{I_{n_k}\}_{k=1}^{\infty}$ of open intervals such that $\{I_{n_k}\}_{k=1}^{\infty} \in \mathcal{C}_E$ and

$$m^\star(E) \leq \sum_{k=1}^{\infty} \ell(I_{n_k}) < m^\star(E) + \frac{1}{n}$$

Define F to be

$$F \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{n_k}$$

F is a countable intersection since the natural numbers are countable and it is an open set since it is composed of intervals. Therefore F is a G_δ.

For any $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$.

$$\begin{aligned}
m^\star(F) &= \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{n_k} \\
&\leq m^\star\left(\bigcup_{k=1}^{\infty} I_{N_k}\right) && \text{(Monotonicity)} \\
&\leq \sum_{k=1}^{\infty} m^\star(I_{N_k}) && \text{(Subadditivity)} \\
&< m^\star(E) + \frac{1}{N} \\
&< m^\star(E) + \epsilon
\end{aligned}$$

This must mean that $m^\star(F) = m^\star(E)$

Chapter 2 Problem 10 Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a-b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^{\star}(A \cup B) = m^{\star}(A) + m^{\star}(B)$

$A \cap B = \emptyset$ because otherwise there would be some $a \in A, b \in B$ such that $a = b \Rightarrow |a-b| = 0$ but $|a-b| > 0$ by definition.

Since A and B are disjoint, any minimal collection of bounded open intervals I_k that cover $A \cup B$ can be split up into two disjoint collections $I_k^A \supseteq A$ and $I_k^B \supseteq B$ where $I_k^A \cap I_k^B = \emptyset \; \forall k$

$$\begin{aligned}
m^{\star}(A \cup B) &= \inf_{\{I_k\} \in \mathcal{C}_{A \cup B}} \sum_{k=1}^{\infty} \ell(I_k) \\
&= \inf_{\{I_k^A \cup I_k^B\} \in \mathcal{C}_{A \cup B}} \sum_{k=1}^{\infty} \ell(I_k^A) + \sum_{k=1}^{\infty} \ell(I_k^B) \\
&= \inf_{\{I_k^A\} \in \mathcal{C}_A} \sum_{k=1}^{\infty} \ell(I_k^A) + \inf_{\{I_k^B\} \in \mathcal{C}_B} \sum_{j=1}^{\infty} \ell(I_k^B) \\
&= m^{\star}(A) + m^{\star}(B)
\end{aligned}$$

Chapter 2 Problem 14: Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

Define an open interval $I_j \coloneqq (-j, j) \cap E$. We claim that there is some $N \in \mathbb{N}$ such that $m^\star(I_N) > 0$. If there were no such N, then $m^\star(I_j) = 0 \ \forall j$ and we could let $j \to \infty$

$$\begin{aligned} m^\star(E) &= m^\star\left(\bigcup_{j=1}^{\infty} I_j\right) \\ &\leq \sum_{j=1}^{\infty} m^\star(I_j) && \text{(Countable Subadditivity)} \\ &= 0 \end{aligned}$$

But this is a contradiction since $m^\star(E) > 0$. Therefore there exists an I_N bounded subset of E that also has positive outer measure.