Homework 6

Zachary Probst

November 4, 2021

Chapter 3 Problem 31: Let $\{f_n\}$ be a sequence of measurable functions on E that converge to the real-valued f pointwise on E. Show that $E = \cup_{k=1}^{\infty} E_k$, where for each index k, E_k is measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 1$, and $m(E_1) = 0$.

First we prove the case where $m(E) < \infty$. For $k > 1$ let $E_k \subset E$ be such that E_k is closed and $\{f_n\}$ converges to f uniformly on E_k and $m(E \setminus E_k) < \frac{1}{k}$. Such an E_k exists by Egorov's theorem.

Each E_k is closed and hence measurable. We also have that $\forall \epsilon > 0$

$$\begin{aligned}
\lim_{k\to\infty} |m(E \setminus E_k)| &< \epsilon \\
|m(E) - \lim_{k\to\infty} m(E_k)| &< \epsilon && \text{(Excision)} \\
\implies m(E) &= \lim_{k\to\infty} m(E_k)
\end{aligned}$$

Define $E_1 \coloneqq E \setminus \cap_{k=2}^{\infty} E_k$. Then

$$\begin{aligned}
m(E_1) &= m\left(E \setminus \bigcap_{k=2}^{\infty} E_k\right) \\
&= m(E) - m\left(\bigcap_{k=2}^{\infty} E_k\right) && \text{(Excision)} \\
&= 0
\end{aligned}$$

So E_1 is also measurable and has measure 0. Finally we see that

$$\begin{aligned}
\bigcup_{k=1}^{\infty} E_k &= \left(E \setminus \bigcap_{k=2}^{\infty} E_k\right) \cup \left(\bigcup_{k=2}^{\infty} E_k\right) \\
&= \left(E \cap \left(\bigcap_{k=2}^{\infty} E_k\right)^c\right) \cup \left(E \cap \bigcup_{k=2}^{\infty} E_k\right) && (E_k \subset E) \\
&= E \cap \left(\left(\bigcap_{k=2}^{\infty} E_k\right)^c \cup \bigcup_{k=2}^{\infty} E_k\right) \\
&= E \cap \left(\bigcup_{k=2}^{\infty} E_k \cup E_k^c\right) \\
&= E \cap \mathbb{R} = E
\end{aligned}$$

And our collection $\{E_k\}$ satisfies all the requirements.

For the case where $m(E) = \infty$, we can construct a countable number of subsets of E denoted as $E^i \coloneqq [-i, i] \cap E$. $m(E^i) < \infty$ so we can use our previous result to find a satisfactory collection $\{E_k^i\}$ for each E^i.

We then define $F_0 = \cup_{i=1}^{\infty} E_1^i$ and $F_i = \cap_{k=2}^{\infty} E_k^i$ for $i \geq 1$.

$$\begin{aligned}
\bigcup_{j=0}^{\infty} F_j &= F_0 \cup \bigcap_{i=1}^{\infty} F_i \\
&= F_0 \cup \bigcup_{i=1}^{\infty} \left(\bigcap_{k=2}^{\infty} E_k^i\right) \\
&= \bigcup_{i=1}^{\infty} (E_1^i) \cup \bigcup_{i=1}^{\infty} \left(\bigcap_{k=2}^{\infty} E_k^i\right) \\
&= \bigcup_{i=1} \left(E_1^i \cup \bigcap_{k=2}^{\infty} E_k^i\right) \\
&= \bigcup_{i=1} \left(E^i \setminus \bigcap_{k=2}^{\infty} E_k^i \cup \bigcap_{k=2}^{\infty} E_k^i\right) && \text{(previous result)} \\
&= \bigcup_{i=1}^{\infty} E^i \\
&= E
\end{aligned}$$

Each E_1^i has measure 0 and is measurable hence we also have $m(F_0) = 0$ because of the countable additivity of the measure. Each F_j is a countable intersection of measurable sets, hence it is measurable. Finally $\{f_n\}$ will uniformly converge to f on each F_j because it does so on each E_k^j.

In conclusion, $\{F_j\}_{j=0}^{\infty}$ satisfies the requirements for when $m(E) = \infty$.

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Given any partition $\mathcal{P}$ of $[a,b]$ we let $\mathbf{L}(f,\mathcal{P})$ and $\mathbf{U}(f,\mathcal{P})$ denote the lower and upper Darboux sums of f with respect to $\mathcal{P}$. Show that

$$\sup_{\mathcal{P}} \mathbf{L}(f,\mathcal{P}) = \sup\left\{\int_a^b \phi(x)dx \mid \phi \text{ step function with } \phi \le f \text{ on } [a,b]\right\}$$
$$\inf_{\mathcal{P}} \mathbf{U}(f,\mathcal{P}) = \inf\left\{\int_a^b \psi(x)dx \mid \psi \text{ step function with } \psi \ge f \text{ on } [a,b]\right\}$$

Given any partition $\mathcal{P} = \{x_j\}_{j=0}^n$ of $[a,b]$, for $j = 1, \dots n$ let $E_j = (x_{j-1}, x_j)$, $m_j = \inf_{(x_{j-1},x_j)} f$ and $M_j = \sup_{(x_{j-1},x_j)} f$. For $j = n+1, \dots, 2n+1$ let $p = j - n - 1$ so for $0 \le p \le n$ set $E_j = [x_p, x_p]$ and $m_j = M_j = f(x_p)$.

Then $\phi \coloneqq \sum_{j=1}^{2n+1} m_j \chi_{E_j}$ and $\psi \coloneqq \sum_{j=1}^{2n+1} M_j \chi_{E_j}$ are 2 step functions with $\phi \le f \le \psi$ on $[a,b]$, and

$$\mathbf{L}(f,\mathcal{P}) = \mathbf{L}(\phi,\mathcal{P}) = \int_a^b \phi(x)dx \quad \text{(Since } \ell(E_j) = 0 \text{ for } n+1 \le j \le 2n+1)$$
$$\mathbf{U}(f,\mathcal{P}) = \mathbf{U}(\psi,\mathcal{P}) = \int_a^b \psi(x)dx$$

Hence

$$\sup_{\mathcal{P}} \mathbf{L}(f,\mathcal{P}) \le \sup\left\{\int_a^b \phi(x)dx \mid \phi \text{ step function } \phi \le f \text{ on } [a,b]\right\}$$

Because any $\mathcal{P}$ over [a,b] has a corresponding ϕ such that $\mathbf{L}(f,\mathcal{P}) = \int_a^b \phi(x)dx$. We also have

$$\inf\left\{\int_a^b \psi(x)dx \mid \psi \text{ step function } \psi \ge f \text{ on } [a,b]\right\} \le \inf_{\mathcal{P}} \mathbf{U}(f,\mathcal{P})$$

Because any $\mathcal{P}$ over [a,b] has a corresponding ψ such that $\mathbf{U}(f,\mathcal{P}) = \int_a^b \psi(x)dx$.

Now we prove the inequalities in the other direction. Given any ϕ with $\phi \le f$ on $[a,b]$, ϕ step function such that $\phi = \sum_{j=1}^n \alpha_j \chi_{E_j}$ for some disjoint intervals $\{E_j\}$ that cover $[a,b]$ and some $\alpha_j \in \mathbb{R}$.

If $E_j = [x_{j-1}, x_j]$ (can be open or closed) take $E_j^1 \coloneqq [x_{j-1}, \frac{x_{j-1}+x_j}{2}]$ and $E_j^2 \coloneqq [\frac{x_{j-1}+x_j}{2}, x_j]$ (each endpoint can be open or closed depending on E_j) so we can set $\mathcal{Q} \coloneqq \{E_j^1, E_j^2\}$ as a valid partition of $[a, b]$. We denote the minimal value of f on E_j as m_j, and on E_j^i as m_j^i. Clearly $m_j \leq m_j^i$ for both $i = 1, 2$. Then

$$\begin{aligned}
\mathbf{L}(f, \mathcal{Q}) &= \sum_{i=1}^{n} m_i^1 \ell(E_i^1) + m_i^2 \ell(E_i^2) \\
&\geq \sum_{i=1}^{n} m_i(\ell(E_i^1) + \ell(E_i^2)) \\
&= \sum_{i=1}^{n} m_i \ell(E_i) \\
&\geq \sum_{i=1}^{n} \alpha_i \ell(E_i) && (\phi \leq f) \\
&= \int_a^b \phi(x) dx
\end{aligned}$$

Hence

$$\sup_{\mathcal{Q}} \mathbf{L}(f, \mathcal{Q}) \geq \sup \left\{ \int_a^b \phi(x) dx \mid \phi \text{ step function } \phi \leq f \text{ on } [a, b] \right\}$$

Because any step function ϕ over [a,b] with $\phi \leq f$ has a corresponding $\mathcal{Q}$ such that $\mathbf{L}(f, \mathcal{Q}) \geq \int_a^b \phi(x) dx$.

We can now apply a very similar argument to some step function $\psi \geq f$ over intervals $\{E_j\}$ and find a partition $\mathcal{Q}$ that is more refined than $\{E_j\}$ such that

$$\mathbf{U}(f, \mathcal{Q}) \leq \int_a^b \psi(x) dx$$

And we are done.

Chapter 4 Problem 1: Define f on $[0,1]$ by setting $f(x) = 1$ if x is rational and 0 if x is irrational. Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in $[0,1]$. For a natural number n, define f_n on $[0,1]$ by setting $f_n(x) = 1$, if $x = q_k$ with $1 \leq k \leq n$, and $f(x) = 0$ otherwise. Show that $\{f_n\}$ fails to converge to f uniformly on $[0,1]$.

Assume that f_n convergers to f uniformly on $[0,1]$. Them $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in [0,1]$

$$|f_n(x) - f(x)| < 0.5$$

Since the rational numbers are countably infinite, there is some rational number $q_{N+1} \in [0,1]$ that is the $(N+1)^{\text{th}}$ enumeration of $\{q_k\}_{k=1}^{\infty}$. Since $N + 1 > N$, we have $f_N(q_{N+1}) = 0$ while $f(q_{N+1}) = 1$ because q_{N+1} is rational. This implies that

$$|f_N(q_{N+1}) - f(q_{N+1})| = 1 < 0.5$$

Which is a contradiction, hence f_n does not converge to f uniformly on $[0,1]$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Given two partitions $\mathcal{P}, \mathcal{Q}$ of $[a, b]$, let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ be their union, which is another partition of $[a, b]$. Show that we have

$$\mathbf{L}(f, \mathcal{P}) \leq \mathbf{L}(f, \mathcal{R})$$
$$\mathbf{U}(f, \mathcal{Q}) \geq \mathbf{L}(f, \mathcal{R})$$

Assume that $\mathcal{Q}$ has only one point y that is not in $\mathcal{P}$ so that

$$\mathcal{P} = \{x_0, x_1, \dots, x_n\},\ \mathcal{R} = \{x_0, \dots, x_{k-1}, y, x_k, \dots, x_n\}$$

Where $x_{k-1} < y < x_k$. Then

$$\begin{aligned} \mathbf{L}(f, \mathcal{R}) = &\sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) \\ &+ \left[\inf_{x_{k-1} \leq x \leq y} f(x)\right](y - x_{k-1}) + \left[\inf_{y \leq x \leq x_k} f(x)\right](x_k - x_y) \qquad (\star) \\ &+ \sum_{i=k+1}^{n} m_i(x_i - x_{i-1}) \end{aligned}$$

We know that

$$\inf_{x_{k-1} \leq x \leq y} f(x) \geq \inf_{x_{k-1} \leq x \leq x_k} f(x) = m_k$$
$$\inf_{y \leq x \leq x_k} f(x) \geq \inf_{x_{k-1} \leq x \leq x_k} f(x) = m_k$$

So

$$(\star) \geq m_k(y - x_{k-1} + x_k - y) = m_k(x_k - x_{k-1})$$

And it follows that

$$\mathbf{L}(f, \mathcal{P}) \leq \mathbf{L}(f, \mathcal{R})$$

Suppose now that $\mathcal{Q}$ has finitely many points $y_1, \dots, y_m$ which are not in $\mathcal{P}$, and set

$$\mathcal{R}_1 = \mathcal{P} \cup \{y_1\}, \mathcal{R}_2 = \mathcal{R}_1 \cup \{y_2\}, \dots, \mathcal{R}_m = \mathcal{R}_{m-1} \cup \{y_m\}$$

Notice that $\mathcal{R}_m = \mathcal{R}$ and by the result we have already established,

$$\mathbf{L}(f, \mathcal{R}) \geq \mathbf{L}(f, \mathcal{R}_{m-1}) \geq \mathbf{L}(f, \mathcal{R}_{m-2}) \ldots \geq \mathbf{L}(f, \mathcal{R}_1) \geq \mathbf{L}(f, \mathcal{P})$$

To show the second inequality we use our previous result to prove that

$$\mathbf{U}(f, \mathcal{R}) = \mathbf{L}(-f, \mathcal{R}) \leq \mathbf{L}(-f, \mathcal{Q}) = \mathbf{U}(f, \mathcal{Q})$$

Where we have used the fact that

$$\inf_{z \leq x \leq y} -f(x) = \sup_{z \leq x \leq y} f(x)$$

And

$$\mathbf{L}(-f, \mathcal{R}) \leq \mathbf{L}(-f, \mathcal{Q}) \iff \mathbf{L}(f, \mathcal{R}) \geq \mathbf{L}(f, \mathcal{Q})$$

Finally we use the fact shown in class that for any partition $\mathcal{R}$ of $[a, b]$ $\mathbf{U}(f, \mathcal{R}) \geq \mathbf{L}(f, \mathcal{R})$

$$\mathbf{L}(f, \mathcal{R}) \leq \mathbf{U}(f, \mathcal{R}) \leq \mathbf{U}(f, \mathcal{Q})$$