Izpeljava tipov

Problem, ki ga rešujemo: (izpeljava tipa)

Če imam izraz e, ali obstaja A, da velja $\emptyset \vdash e : A$

Enostavnejši problem: (preverjanje tipa)

Če imam izraz e in tip A, ali velja $\emptyset \vdash e : A$.

Mi bomo reševali prvega s Hindley-Milnerjevim algoritmom.

Ideja: ko moramo ugoniti tip v izpeljavi, ustvarimo spremenljivko $\alpha, \beta, \ldots$

Če se morajo tipi ujemati, dodamo enačbo

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{f:\alpha \vdash f:\alpha} \qquad \dfrac{}{f:\alpha \vdash 10 : \mathrm{int}}}{f:\alpha \vdash f\ 10 : \beta}
}{\emptyset \vdash \lambda f.\, f\ 10 : \alpha \to \beta}
\qquad
\dfrac{
\dfrac{\dfrac{}{x:\gamma \vdash x:\gamma} \qquad \dfrac{}{x:\gamma \vdash 3 : \mathrm{int}}}{x:\gamma \vdash x > 3 : \delta}
}{\emptyset \vdash \lambda x.\, x > 3 : \alpha}
}{\emptyset \vdash (\lambda f.\, f\ 10)(\lambda x.\, x > 3) : \beta}
$$

$$
\begin{array}{l}
\alpha = \mathrm{int} \to \beta \\
\alpha = \gamma \to \delta \\
\gamma = \mathrm{int} \\
\delta = \mathrm{bool}
\end{array}
\rightsquigarrow
\begin{array}{l}
\alpha = \mathrm{int} \to \beta \\
\alpha = \mathrm{int} \to \mathrm{bool} \\
\gamma = \mathrm{int} \\
\delta = \mathrm{bool}
\end{array}
\rightsquigarrow
\begin{array}{l}
\alpha = \mathrm{int} \to \mathrm{bool} \\
\mathrm{int} \to \mathrm{bool} = \mathrm{int} \to \beta \\
\gamma = \mathrm{int} \\
\delta = \mathrm{bool}
\end{array}
$$

$$
\rightsquigarrow
\begin{array}{l}
\alpha = \mathrm{int} \to \mathrm{bool} \\
\mathrm{int} = \mathrm{int} \\
\mathrm{bool} = \beta \\
\gamma = \mathrm{int} \\
\delta = \mathrm{bool}
\end{array}
\rightsquigarrow
\begin{array}{l}
\alpha = \mathrm{int} \to \mathrm{bool} \\
\beta = \mathrm{bool} \\
\gamma = \mathrm{int} \\
\delta = \mathrm{bool}
\end{array}
$$

$$A ::= \alpha \mid \cdots$$

$\Gamma \vdash e : A \mid \mathcal{E}$ ... v kontekstu Γ za e izpeljemo tip A ob omejitvah $\mathcal{E}$.

$$\frac{}{\Gamma \vdash n : \mathrm{int} \mid \emptyset} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash e_1 + e_2 : \mathrm{int} \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = \mathrm{int}, A_2 = \mathrm{int}}$$

pod. za $-$, $*$

$$\frac{}{\Gamma \vdash \mathrm{true} : \mathrm{bool} \mid \emptyset} \qquad \frac{}{\Gamma \vdash \mathrm{false} : \mathrm{bool} \mid \emptyset}$$

$=, <, >$: D.N.

$$\frac{\Gamma \vdash e : A \mid \mathcal{E} \qquad \Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash \mathrm{if}\ e\ \mathrm{then}\ e_1\ \mathrm{else}\ e_2 : A_1 \mid \mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, A = \mathrm{bool}, A_1 = A_2} \qquad \frac{(x : A \in \Gamma)}{\Gamma \vdash x : A \mid \emptyset}$$

(se ne pojavlja v $\Gamma, B, \mathcal{E}$)

$$\frac{\Gamma, x : \alpha \vdash e : B \mid \mathcal{E} \qquad \alpha \text{ svež}}{\Gamma \vdash \lambda x. e : \alpha \to B \mid \mathcal{E}} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2 \qquad \alpha \text{ svež}}{\Gamma \vdash e_1\, e_2 : \alpha \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha}$$

Primer: $\lambda f. \lambda x.\ \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f\,(f\,x)$

$$\cfrac{\cfrac{\cfrac{}{\Gamma \vdash x : \beta} \qquad \cfrac{\cfrac{}{\Gamma \vdash f : \alpha} \quad \cfrac{}{\Gamma \vdash x : \beta}}{\Gamma \vdash f\,x : \gamma} \qquad \cfrac{\cfrac{}{\Gamma \vdash f : \alpha} \quad \cfrac{\cfrac{}{\Gamma \vdash f : \alpha} \quad \cfrac{}{\Gamma \vdash x : \beta}}{\Gamma \vdash f\,x : \delta}}{\Gamma \vdash f\,(f\,x) : \varphi}}{\underbrace{f : \alpha, x : \beta}_{\Gamma} \vdash \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f\,(f\,x) : \gamma}}{\cfrac{f : \alpha \vdash \lambda x. \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f\,(f\,x) : \beta \to \gamma}{\emptyset \vdash \lambda f. \lambda x.\ \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f\,(f\,x) : \alpha \to (\beta \to \gamma)}}$$

$(\mathrm{bool} \to \mathrm{bool}) \to (\mathrm{bool} \to \mathrm{bool})$

$\alpha = \beta \to \gamma$
$\alpha = \beta \to \delta$
$\alpha = \delta \to \varphi$
$\varphi = \gamma$
$\beta = \mathrm{bool}$

rešimo $\rightsquigarrow$

$\beta = \delta = \gamma = \varphi = \mathrm{bool}$
$\alpha = \mathrm{bool} \to \mathrm{bool}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : \alpha \mid \emptyset} \qquad \dfrac{}{\Gamma \vdash x : \beta \mid \emptyset}}{\Gamma \vdash f\,x : \gamma \mid \alpha = \beta \to \gamma} \qquad \dfrac{}{\Gamma \vdash 3 : int \mid \emptyset}}{\overbrace{f : \alpha, x : \beta}^{\Gamma} \vdash f\,x + 3 : int \mid \alpha = \beta \to \gamma, \gamma = int, int = int}}{\dfrac{f : \alpha \vdash \lambda x.\, f\,x + 3 : \beta \to int \mid \cdots}{\emptyset \vdash \lambda f. \lambda x.\, f\,x + 3 : \alpha \to (\beta \to int) \mid \cdots}}$$

$$\begin{matrix} \alpha = \beta \to \gamma \\ \gamma = int \\ \cancel{int = int} \end{matrix} \leadsto \begin{matrix} \gamma \mapsto int \\ \alpha = \beta \to int \end{matrix} \leadsto \begin{matrix} \alpha \mapsto \beta \to int \\ \gamma \mapsto int \end{matrix}$$

$$\emptyset \vdash \lambda f. \lambda x.\, f\,x + 3 : (\beta \to int) \to (\beta \to int)$$

S σ označujmo substitucije, t.j. končne preslikave $\alpha_1 \mapsto A_1, \ldots, \alpha_n \mapsto A_n$

Def.
$$\sigma(int) = int$$
$$\sigma(bool) = bool$$
$$\sigma(A \to B) = \sigma(A) \to \sigma(B)$$
$$\sigma(\alpha) = \begin{cases} A & \alpha \mapsto A \in \sigma \\ \alpha & \text{sicer} \end{cases}$$

Def $\sigma \models \mathcal{E}$ ... σ reši enačbe $\mathcal{E}$

$$\sigma \models A_1 = A_1', \ldots, A_n = A_n' \iff \sigma(A_1) = \sigma(A_1') \wedge \cdots \wedge \sigma(A_n) = \sigma(A_n').$$

Trditev 5 Če velja $\Gamma \vdash e : A \mid \mathcal{E}$, potem za poljubno $\sigma \models \mathcal{E}$ velja $\sigma(\Gamma) \vdash e : \sigma(A)$.

Trditev 6 Če velja $\Gamma \vdash e : A$ in $\Gamma \vdash e : A' \mid \mathcal{E}$, potem obstaja $\sigma \models \mathcal{E}$, da je $\sigma(A') = A$.

Def $\mathcal{E} \leadsto \sigma$ ... $\mathcal{E}$ ima najbolj splošno rešitev σ $\quad$ unify$(\mathcal{E}) = \sigma$

$$\dfrac{}{\emptyset \leadsto \emptyset} \qquad \dfrac{\mathcal{E} \leadsto \sigma}{int = int, \mathcal{E} \leadsto \sigma}$$ podobno za $bool = bool$, $\alpha = \alpha$

$\alpha \notin FV(A)$ ← vse proste spremenljivke v A

$$\dfrac{A_1 = A_2, B_1 = B_2, \mathcal{E} \leadsto \sigma}{A_1 \to B_1 = A_2 \to B_2, \mathcal{E} \leadsto \sigma} \qquad \dfrac{(\alpha \mapsto A)(\mathcal{E}) \leadsto \sigma}{\alpha = A, \mathcal{E} \leadsto \sigma \circ (\alpha \mapsto A)}$$ podobno za $A = \alpha$

$$(\underbrace{\alpha_1 \mapsto A_1, \dots, \alpha_n \mapsto A_n}_{\sigma}) \circ (\alpha_1' \mapsto A_1', \dots, \alpha_m' \mapsto A_m')$$
$$= \alpha_1' \mapsto \sigma(A_1'), \dots, \alpha_m' \mapsto \sigma(A_m'), \underbrace{\alpha_1 \mapsto A_1, \dots, \alpha_n \mapsto A_n}_{\text{BREZ } \alpha_1', \dots, \alpha_m'}$$

__Trditev 7__ Če $\mathcal{E} \leadsto \sigma$, potem $\sigma \models \mathcal{E}$.

__Trditev 8__ Če $\sigma \models \mathcal{E}$, potem za poljubno σ' velja $\sigma' \circ \sigma \models \mathcal{E}$.

__Trditev 9__ Če $\mathcal{E} \leadsto \sigma$ in velja $\sigma' \models \mathcal{E}$, potem obstaja σ'', da je $\sigma' = \sigma'' \circ \sigma$.

__Trditev 10__ Če $\sigma' \models \mathcal{E}$, potem obstaja σ, da velja $\mathcal{E} \leadsto \sigma$.

$$\frac{}{\Gamma \vdash n : \mathrm{int} \mid \emptyset} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash e_1 + e_2 : \mathrm{int} \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = \mathrm{int}, A_2 = \mathrm{int}} \qquad \text{pod. za } -, *$$

$$\frac{}{\Gamma \vdash \mathrm{true} : \mathrm{bool} \mid \emptyset} \qquad \frac{}{\Gamma \vdash \mathrm{false} : \mathrm{bool} \mid \emptyset} \qquad =, <, > : \text{D.N.}$$

$$\frac{\Gamma \vdash e : A \mid \mathcal{E} \qquad \Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash \mathrm{if}\ e\ \mathrm{then}\ e_1\ \mathrm{else}\ e_2 : A_1 \mid \mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, A = \mathrm{bool}, A_1 = A_2} \qquad \frac{(x : A \in \Gamma)}{\Gamma \vdash x : A \mid \emptyset}$$

$$\frac{\Gamma, x : \alpha \vdash e : B \mid \mathcal{E} \qquad \alpha \text{ svež (se ne pojavlja v } \Gamma, B, \mathcal{E})}{\Gamma \vdash \lambda x. e : \alpha \to B \mid \mathcal{E}} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2 \qquad \alpha \text{ svež}}{\Gamma \vdash e_1 e_2 : \alpha \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha}$$

__Dokaz 5__

Indukcija na $\Gamma \vdash e : A \mid \mathcal{E}$.

- aplikacija:
  imamo $\Gamma \vdash e_1 e_2 : \alpha \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha$.
  in $\sigma \models \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha$.
  Po inverziji velja $\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1$ ter $\Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2$.
  Ker $\sigma \models \mathcal{E}_1$ in $\sigma \models \mathcal{E}_2$, lahko uporabimo I.P. in dobimo
  $\sigma(\Gamma) \vdash e_1 : \sigma(A_1)$ in $\sigma(\Gamma) \vdash e_2 : \sigma(A_2)$. Ker $\sigma(A_1) = \sigma(A_2) \to \sigma(\alpha)$,
  velja $$\frac{\sigma(\Gamma) \vdash e_1 : \sigma(A_2) \to \sigma(\alpha) \qquad \sigma(\Gamma) \vdash e_2 : \sigma(A_2)}{\sigma(\Gamma) \vdash e_1 e_2 : \sigma(\alpha)}$$ $\square$
- ostalo podobno.

__Dokaz 6__ TAPL

__Dokaz 7__

- vse trivialno razen parametrov
- $$\frac{(\alpha \mapsto A)(\mathcal{E}) \leadsto \sigma \qquad \alpha \notin FV(A)}{\alpha = A, \mathcal{E} \leadsto \sigma \circ (\alpha \mapsto A)}$$

  $\sigma \circ (\alpha \mapsto A) \models \alpha = A, \mathcal{E}$

1) L.S. $(\sigma \circ (\alpha \mapsto A))(\alpha) = \sigma(A)$
   D.S. $(\sigma \circ (\alpha \mapsto A))(A) = \sigma(A)$ (ker $\alpha \notin FV(A)$).

2) $(\sigma \circ (\alpha \mapsto A)) \models \mathcal{E} \iff \sigma \models (\alpha \mapsto A)(\mathcal{E}) \leftarrow$ I.P.

<u>Trditev 8</u> Očitno

<u>Trditev 9</u> Indukcija na $\mathcal{E} \rightsquigarrow \sigma$.

- $\emptyset \rightsquigarrow \emptyset$ in $\sigma \models \emptyset$, potem je $\sigma = \sigma \circ \emptyset$
- $\dfrac{A_1 = A_2, B_1 = B_2, \mathcal{E} \rightsquigarrow \sigma}{A_1 \to B_1 = A_2 \to B_2, \mathcal{E} \rightsquigarrow \sigma}$. Če $\sigma' \models A_1 \to B_1 = A_2 \to B_2, \mathcal{E}$, potem $\sigma' \models A_1 = A_2, B_1 = B_2, \mathcal{E}$. Po I.P. $\exists \sigma''. \sigma' = \sigma'' \circ \sigma$.
- ostalo podobno.