STAT 300 Chapter 7

Confidence Intervals

Zepu Zhang
April 1, 2012

1 Concept of confidence intervals (CI)

Section 7.1 of the textbook through first half of p. 258 (7th ed) or p. 271 (8th ed) is very readable and makes things quite clear.

Confidence interval is in contrast to point estimation, which uses a single value to estimate a population parameter.

Confidence interval (CI): an interval constructed such that it contains the true value of the population parameter with a specified degree of confidence (a percentage).

This "degree of confidence" is the probability that the CI so constructed contains the true parameter value in repeated samplings.

Imagine a synthetic problem in which we know the true population parameter θ. We obtain a random sample from this distribution, pretending we don't know the true value of θ, and follow a chosen method to construct a CI for θ. The CI is a (bivariate) statistic. Because CI is random, we can talk about probabilities about it. In particular, we are interested in the probability that the CI covers the (unknown, fixed) θ. Suppose a particular method of constructing the CI (that is, how to define this interval, which is a function of the data!) makes this probability equal to 0.95. Then, if we repeatedly draw random samples (of the same size, say n) and obtain the CI based on each sample by the same procedure, some of the CI's will contain θ whereas some others will miss. If we're able to do this repeated sampling many, many, many times, the "success rate" will converge to 0.95 (this is simply the "limiting relative frequency" interpretation of probability). This 95% is called the confidence level.

Confidence level (say 95%), confidence coefficient (0.95). (But I don't think such distinction is anything important.)

A CI is usually around a point estimator, $\hat{\theta}$, symmetrically or not.

2 CI for mean when variance is known

2.1 Normal population

Example 95% CI for μ of a normal population with known σ^2.

Use $\overline{X}$ as an estimator of μ. Construct the interval $(\overline{X} - \Delta, \overline{X} + \Delta)$ such that as we do the sampling repeatedly (hence we'll get many values of the random variable $\overline{X}$, and subsequently many different CI), there is a 0.95 probability that the CI encloses μ.

$$
\begin{aligned}
P(\text{CI encloses } \mu) = 0.95 \quad &\iff \quad P(|\mu - \overline{X}| < \Delta) = 0.95 \\
&\iff \quad P\left(\left|\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right| < \frac{\Delta}{\sigma/\sqrt{n}}\right) = 0.95 \\
&\iff \quad P\Big(|Z| < \frac{\Delta}{\sigma/\sqrt{n}}\Big) = 0.95 \\
&\iff \quad \frac{\Delta}{\sigma/\sqrt{n}} = \Phi^{-1}\big(0.95 + (1 - 0.95)/2\big) \\
&\iff \quad \Delta = \Phi^{-1}(0.975)\frac{\sigma}{\sqrt{n}} = 1.96\frac{\sigma}{\sqrt{n}}
\end{aligned}
$$

Hence the 95% CI is $\left(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}}, \overline{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$. Two ways to word this: top box on p. 257 (7th ed) or p. 270 (8th ed).

Example Ex. 7.2 in Chap. 7.1.

For another confidence level, say $1 - \alpha$, replace 1.96 by $\Phi^{-1}(1 - \alpha/2)$. This is called critical value and is denoted $z_{\alpha/2}$.

Note the definition of critical value $z_{\alpha/2}$: it is defined as the value z such that $P(Z < z) = 1 - \alpha/2$. In other words, $z_{\alpha/2} = \Phi^{-1}(1 - \alpha/2)$.

General pattern:

$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $\frac{\sigma}{\sqrt{n}}$ is actually $\sigma_{\overline{X}}$. (Note we're using lower-case letters because we're supposed to plug in actual values now.)

Interpreting a Confidence Interval: in Chap. 7.1.

Note a somewhat common misinterpretation: 2nd paragraph of that section.

Figure 7.3 in Chap. 7.1.

Example Ex. 7.3 in Chap. 7.1.

2.2 Non-normal population but large sample

Suppose we don't know the distribution of the population, but the sample size n is large (say ≥ 30). In this situation the Central Limit Theorem (CLT) provides the approximate distribution of $\overline{X}$:

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Then, the same derivation leads to the same CI

$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

3 CI for mean when variance is unknown

In reality, we almost never know σ^2, and what we do naturally is to use s^2 in place of σ^2. After this substitution, however, we need to make some other adjustments to the formula.

3.1 Normal population

The distribution of $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ is not $N(0, 1)$. Since $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is $N(0, 1)$, we understand T have more variation than Z because of the randomness in S (although S varies around σ).

Fortunately, the distribution of T is known when X is normal. The distribution is called t with $n-1$ degrees of freedom:

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Properties of t distributions: in Chap. 7.3.

Note: it's a family of distributions, family members being distinguished by the sole parameter ν—degrees of freedom.

By a similar derivation we get the CI for μ:

$$\overline{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2,n-1}$ indicates the critical value on a t curve with df $n-1$.

Box on p. 271, 7th ed, or p. 287, 8th ed.

Learn to use table A.5.

Example Ex. 7.11 in Chap. 7.3.

3.2 Non-normal population but large sample

In this case, $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is approximately $N(0,1)$, and $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ is approximately t_{n-1}. So, we can use the approximate interval

$$\overline{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}$$

In fact, if n is sufficiently large (e.g. ≥ 40), S is a pretty accurate and stable estimator for σ, then the distribution of $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ is approximately $N(0,1)$, hence we could simply use the interval

$$\overline{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$$

at the $1-\alpha$ confidence level.

Example Ex. 7.6 in Chap. 7.2.

4 Useful R functions

`pnorm`, `pt`: CDF.

`qnorm`, `qt`: quantile (i.e. inverse CDF).

To calculate the critical values:

`qnorm(1 - alpha/2)`, `qt(1 - alpha/2, n - 1)`