STAT 611 Part 4
Intervention Analysis

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"Intervention analysis" is a special (and simple) case of the so-called "transfer function models". In transfer function models, the process Y_t is affected by an input process X_t, described by

$$Y_t = \sum_{j=0}^{\infty} \tau_j X_{t-j} + N_t,$$

where N_t is an ARMA, or ARIMA, or even SARIMA process. The $\sum_{j=0}^{\infty} \tau_j X_{t-j}$ part is a deterministic function (but can use the AR, MA, or ARMA forms of expression).

In intervention analysis, we focus on two types of X_t input: a <u>pulse</u>,

$$P_t^{(T)} = \begin{cases} 1, & \text{if } t = T, \\ 0, & \text{if } t \neq T \end{cases}$$

(an instantaneous input at time T) and a <u>step function</u>

$$S_t^{(T)} = \begin{cases} 1, & \text{if } t \geq T, \\ 0, & \text{if } t < T \end{cases}$$

(a lasting input starting at time T).

Note

$$P_t^{(T)} = (1 - B)S_t^{(T)}$$

and

$$S_t^{(T)} = (1 - B)^{-1}P_t^{(T)} = (1 + B + B^2 + B^3 + \cdots)P_t^{(T)}$$

Note 1. We only consider the case where the time of intervention, T, is known.

2. There could be multiple interventions with common or different T; just add them up.

3. The intervention model changes the <u>mean</u> of Y_t but not its variance.

4. N_t is called the "background" process (or "noise"). It is assumed that the behavior of N_t is unchanged, hence it is described by one model regardless of the intervention. The observed process is the background process plus the varying level of $E(Y_t)$ that is caused by the intervention.

1   Formulation

We examine several typical patterns of intervention impact and come up with expressions for them. The expressions will contain a small number of parameters, which will be estimated subsequently. In other words, the form of the intervention model is specified based on inspection of graphics of the process (in contrast to the general ARMA modeling, in which we "estimate" the orders before the form of the model can be specified).

Let's write $m_t = \sum_{j=0}^{\infty} \tau_j X_{t-j}$.

1. $m_t = \omega P_t^{(T)}$.

   Variation: say there is a d time unit delay between the intervention event and the appearance of its impact, then $m_t = \omega P_{t-d}^{(T)} = \omega B^d P_t^{(T)}$.

2. $m_t = \omega S_t^{(T)}$.

   It's common that the impact of an intervention that happened at time T was shown only in Y_{T+1}, so it's common to see $BS_t^{(T)}$ or $BP_t^{(T)}$ in the model.

   Figure 11.3(a), p. 253.

3. Figure 11.3(b), p. 253: m_t gradually increases to a stable level.

   A geometric series with rate below 1 is the most common way to achieve this pattern. Think of this general solution:

   $$z_n = a(1+b+b^2+\cdots+b^n) = \frac{a(1-b^{n+1})}{1-b} \xrightarrow{n\to\infty} \frac{a}{1-b}, \qquad 0 < b < 1 \tag{1}$$

   A recursive definition is

   $$z_0 = a$$
   $$z_n = a + bz_{n-1}, \qquad n = 1, 2, \ldots$$

   If we want the accumulation to start at time t instead of time 0, we can adjust this to

   $$z_n = 0, \qquad\qquad n < t$$
   $$z_n = a + bz_{n-1}, \qquad\qquad n \geq t$$

   or using the step function,

   $$z_0 = aS_0^{(T)}$$
   $$z_n = aS_t^{(T)} + bz_{n-1}, \qquad n = 1, 2, \ldots$$

Or if we want the accumulation to start at time $t + 1$, and use the textbook notation, we have

$$m_0 = \omega B S_0^{(T)}$$

$$m_t = \omega B S_t^{(T)} + \delta m_{t-1}, \qquad t = 1, 2, \ldots$$

that is, $m_0 = 0$ and

$$(1 - \delta B) m_t = \omega B S_t^{(T)}$$

or, since we want to express $\underline{m_t}$, we use

$$m_t = \frac{\omega B}{1 - \delta B} S_t^{(T)} \qquad (2)$$

It might be more intuitive to write this as

$$m_t = (1 + \delta B + \delta^2 B^2 + \cdots) \omega B S_t^{(T)}$$

In this process, we see $m_t = 0$, $m_{t+1} = \omega$, $m_{t+2} = \omega(1+\delta)$, $m_{t+3} = \omega(1+\delta+\delta^2)$, and so on. By adjusting ω and δ, we can control the magnitude of the first "jump" and the pace of increase afterwards.

4. Figure 11.3(c), p. 253: m_t increases linearly without bound (less realistically).

   Just let $b = 1$ in (1) or $\delta = 1$ in the previous definition:

   $$m_t = \frac{\omega B}{1 - B} S_t^{(T)}$$

5. Figure 11.4(a), p. 254: m_t has a jump after the intervention, then gradually falls back to the original level.

   This obviously is characterized by

   $$m_t = 0$$

   $$m_{t+1} = \omega$$

   $$m_{t+k} = \delta m_{t+k-1}, \qquad t > 1, 0 < \delta < 1.$$

   Using the pulse function, we can write this as $m_0 = 0$ and

   $$m_t = \omega B P_t^{(T)} + \delta m_{t-1}$$

   or

   $$m_t = \frac{\omega B}{1 - \delta B} P_t^{(T)} \qquad (3)$$

   This is equivalent to

   $$m_t = (1 + \delta B + \delta^2 B^2 + \cdots) \omega B P_t^{(T)}$$

   $$= \omega \sum_{j=0}^{\infty} \delta^j P_{t-1-j}^{(T)}$$

   $$= \omega \delta^{t-T-1} \qquad \text{(b/c we get something only when } t - 1 - j = T)$$

This process clearly decreases from ω towards 0 as time elapse after T.

On a second thought, we can define

$$m_t = \omega \delta^{t-T} B S_t^{(T)}$$

which seems to be intuitive.

6. Figure 11.4(b), p. 254: jumps and then gradually falls to a level above the original.

Simply add a step function onto the previous one:

$$m_t = \frac{\omega_1 B}{1 - \delta B} P_t^{(T)} + \omega_2 B S_t^{(T)} \qquad (4)$$

to uniformly lift the post-intervention level.

This is the same as the one in the book, noticing $P_t^{(T)} = (1 - B) S_t^{(T)}$.

7. Gradually decreases after intervention towards a certain stable level.

Simply lower (3) to some level below 0 at time $t + 1$:

$$
\begin{aligned}
m_t &= \frac{\omega_1 B}{1 - \delta B} P_t^{(T)} - (\omega_1 + \omega_2) B S_t^{(T)} \\
&= \frac{\delta \omega_1 - \omega_2 + (\delta \omega_2 - \omega_1) B}{1 - \delta B} S_t^{(T)} \qquad \text{(using } P = (1 - B)S)
\end{aligned}
$$

Better yet, simply negate (2):

$$m_t = -\frac{\omega B}{1 - \delta B} S_t^{(T)}, \qquad \omega > 0, 0 < \delta < 1.$$

8. Figure 11.4(c), p. 254: a one-time jump, then dips and gradually rises back.

Use a pulse to represent the one-time jump, followed by a negated (4):

$$m_t = \omega_0 P_t^{(T)} - \omega_1 B S_t^{(T)} - \frac{\omega_2 B}{1 - \delta B} P_t^{(T)}, \qquad \omega_0, \omega_1, \omega_2 > 0; 0 < \delta < 1.$$

2   Estimation

After we have specified the model for m_t (with unknown parameters), the density of $Y_t - m_t$ (that is, N_t) can be written. Hence a MLE can be attempted.

We use explorations of the series prior to the intervention for tentative specification of model for N_t.

If the process looks stationary after removing the intervention effect, we could try a simple least squares estimation of the intervention model parameters. After that, get the residual series and check what model is appropriate for it. (The book does not mention this strategy, and I didn't try it.)

As usual, after estimation of the full model, various diagnostics should be checked, and the model revised (and refitted, and re-checked) if necessary.