STAT 611 Part 2

Stationary Processes

(chapters 4, 6.1–6.3, 6.5–6.6, 7.1–7.5, 8, 9.1–9.6)

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May 4, 2011

1 General linear processes

General formula:

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots , \tag{1}$$

where $e_t \overset{\text{iid}}{\sim} N(0, \sigma^2)$. (However, it would have been more consistent with similar models, in particular MA models, to use '$-$' in front of the ψ's.)

Reasons for talking about this form:

1. It's simple because the e's are iid. For example, it's trivial to find $E(Y_t)$ and $\text{var}(Y_t)$. Covariances are also easy to derive.
2. All the stationary model we're concerned about can be written in this form.

Comments:

1. Y_t is a linear combination of current and past (but not future) e's.
2. There may be infinite or finite (the ψ's at a certain lag in the past and older are 0) terms.
3. The first coefficient, ψ_0, is set to 1 without loss of generality.
4. The e's are NOT observed; they are a model device. The Y's are the observed values. We assume the Y's have occurred this way. (Recall a similar understanding for linear models.)
5. This form is general; it may or may not be the most convenient for a particular purpose. For certain purposes we may want to use alternative (but equivalent) forms.

6. This form implies $E(Y_t) = 0$. This is OK for much discussion of models because adding a constant to it wouldn't affect the correlations. When we fit actual data, we may allow a constant mean to be estimated.

7. We study stationary processes. This needs the ψ's to satisfy certain conditions. In particular, the condition is that the coefficients ψ_j are absolutely summable:

$$\sum_{j=1}^{\infty} |\psi_j| < \infty.$$

   Note that if (1) has a finite number of terms, this condition is satisfied with whatever values of the coefficients.

In general, given a particular form of the series, we'll study

- its ACF (autocorrelation function) ρ_k, $k = 1, 2 \ldots$;
- its variance γ_0;
- its PACF (partial autocorrelation function) ϕ_{kk}, $k = 1, 2 \ldots$;
- discuss how the ρ_k and ϕ_{kk} behaves, in particular, their signs (positive? negative? oscillating?), whether they gradually die off or abruptly cut off, the pace of their decay, and how these behaviours are affected by the coefficients of the model. We always want to look at a plot of ρ_k, ϕ_{kk} versus the lag k.

2 MA models

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

"order q": MA(q). This is just a special form of (1) with finite q.

Because MA is a linear combination of independent e's, it's quite easy to work out its variance and auto-covariances.

Variance

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_e^2$$

ACF

$$\begin{aligned}\gamma_k &= \operatorname{cov}\big(e_t - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q},\, e_{t-k} - \theta_1 e_{t-1-k} - \cdots - \theta_q e_{t-q-k}\big)\\ &= \big(-\theta_k + \theta_{k+1}\theta_1 + \theta_{k+2}\theta_2 + \cdots + \theta_q\theta_{q-k}\big)\sigma_e^2\end{aligned}$$

if $k \le q$ (otherwise there's no common e, hence γ is zero).

Exercise How to identify the overlapping terms?

Subsequently, the ACF is

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \cdots .$$

Note ρ_k is a function of θ's only; it does not involve σ_e^2.

In a MA(q) process, ACF cuts off after lag q. Usually, the coefficients θ_j decreases in absolute value as j increases, hence the ACF decreases in absolute value until it disappears.

Note Verify that MA processes are always stationarity.

Example MA(1) (dropping subscript 1 from θ):

$$\gamma_0 = (1+\theta^2)\sigma_e^2, \quad \gamma_1 = -\theta\sigma_e^2, \quad \rho_1 = -\frac{\theta}{1+\theta^2}, \quad \rho_2 = \rho_3 = \cdots = 0$$

Note 1. The sign of ρ_1 is opposite to that of θ.

2. $|\rho_1| \le \frac{1}{2}$ (To prove this, use $\left|\theta + \frac{1}{\theta}\right| \ge 2$.)

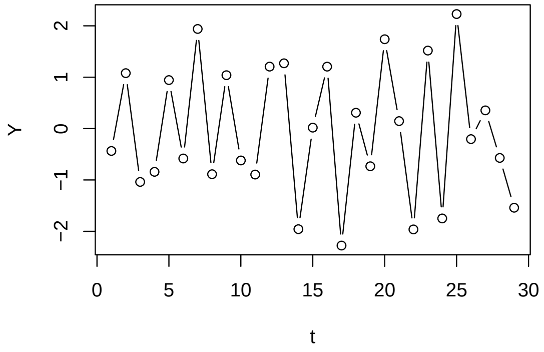
Example

$$Y_t = e_t - 0.8 e_{t-1}$$

How can we we simulate this process?

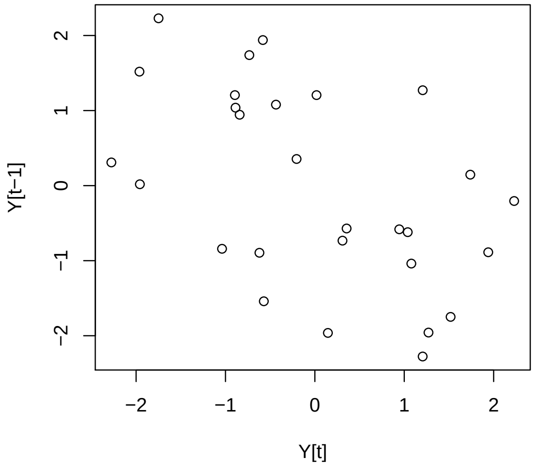
```
> theta <- .8
> e <- rnorm(30)
> e1 <- e[-length(e)]
> e2 <- e[-1]
> y <- e1 - theta * e2
> pdf(file = 'lect02-1.pdf', width = 5, height = 4)
> plot(y, type = 'b', xlab = 't', ylab = 'Y')
> dev.off()
X11cairo
       2
```

Note $\rho_1 = \frac{-0.8}{1+0.64} = -0.49$.



Check lag-1 pairs, they should show some negative correlation.

```
> pdf(file = 'lect02-2.pdf', width = 5, height = 5)
> plot(y[-length(y)], y[-1], xlab = 'Y[t]', ylab = 'Y[t-1]')
> dev.off()
X11cairo
       2
```



We can also check by graph that lag-2 pairs show no correlation.

Example MA(2), p. 62.

3 AR models

$$Y_t = e_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p}$$

Order p: AR(p). This is the "current" random term (e_t) plus a linear regression with its own past—the p most recent past values (hence the name "regression").

Stationarity is assumed (the condition for its stationarity is discussed shortly).

Under the stationarity assumption, taking expectation on both sides, we get

$$E(Y_t) = E(e_t) + \phi_1 E(Y_{t-1}) + \cdots + \phi_p E(Y_{t-p})$$

or

$$\mu = (\phi_1 + \cdots + \phi_p)\mu$$

hence $\mu \equiv E(Y_t) = 0$ unless $1 - \phi_1 - \cdots - \phi_p = 0$. (We'll see that stationarity requires $1 - \phi_1 - \cdots - \phi_p \neq 0$.)

Note The notational conventions (at least in this book) for MA and AR models.

Note We will use this assumption repeatedly: e_t is independent of Y_{t-1}, Y_{t-2}, ..., that is, previous Y's. This claim is backed by the assumption (and requirement) that the Y_t in the AR(p) can be equivalently expressed as

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots ,$$

(note: noise of past, not future times) and the e_j's are independent of each other. This property is called "causality" and will be discussed later.

Note However, e_t is not independent of the current Y_t, and we have

$$E\big[e_t Y_t\big] = E\big[e_t(e_t + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p}\big] = E(e_t^2) = \sigma_e^2$$

This is connected to the property of "causality" to be introduced later.

Note This section of the note stresses consistent strategies whether the order is 1, 2, or more. It's cleaner than counterparts of section 4.3 of the book.

3.1 AR(1)

$$Y_t = e_t + \phi Y_{t-1} \tag{2}$$

ACF

Multiply both sides of (2) by Y_{t-k}:

$$Y_t Y_{t-k} = e_t Y_{t-k} + \phi Y_{t-1} Y_{t-k}$$

Take expectations:

$$E\big[Y_tY_{t-k}\big] = E[e_t]E[Y_{t-k}] + \phi E\big[Y_{t-1}Y_{t-k}\big]$$

(b/c of the independence of e_t and Y_{t-k}). Using the zero-mean assumption, this becomes

$$\gamma_k = \phi\gamma_{k-1}$$

Divide by γ_0:

$$\rho_k = \phi\rho_{k-1}$$

This is a recursive relation that needs the previous ρ to calculate the next ρ. Let's take $k = 1$ to get an equation for ρ_1:

$$\rho_1 = \phi\rho_0 = \phi$$

This is called the Yule-Walker equation. For AR(1), this gives ρ_1 directly. All other ρ's follow by recursion:

$$\begin{aligned}\rho_1 &= \phi\\ \rho_2 &= \phi\rho_1 = \phi^2\\ \rho_3 &= \phi\rho_2 = \phi^3\\ \cdots &= \cdots\\ \rho_k &= \phi^k\end{aligned}$$

(In fact, we don't have to start the recursion with ρ_1. The known $\rho_0 = 1$ already makes the recursion perfectly ready for use.)

Note The ACF of AR(1) decays exponentially fast, but does not reduce to exactly 0.

Variance

Multiply both sides of (2) by Y_t:

$$Y_tY_t = e_tY_t + \phi Y_{t-1}Y_t$$

Take expectations and use the zero-mean assumption:

$$\gamma_0 = E[e_tY_t] + \phi\gamma_1 = \sigma_e^2 + \phi\rho_1\gamma_0$$

hence

$$\gamma_0 = \frac{\sigma_e^2}{1-\phi\rho_1}$$

The known ρ_1 can be substituted to get $\gamma_0 = \frac{\sigma_e^2}{1-\phi^2}$.

3.2 AR(2)

$$Y_t = e_t + \phi_1Y_{t-1} + \phi_2Y_{t-2} \tag{3}$$

ACF

Multiply both sides of (3) by Y_{t-k}:

$$Y_t Y_{t-k} = e_t Y_{t-k} + \phi_1 Y_{t-1} Y_{t-k} + \phi_2 Y_{t-2} Y_{t-k}$$

Take expectations and divide by γ_0:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$$

This is a recursion that uses two previous ρ's to get the next. Let's take $k = 1, 2$ to get an equation system about ρ_1 and ρ_2:

$$\begin{aligned} \rho_1 &= \phi_1 \rho_0 + \phi_2 \rho_{-1} = \phi_1 + \phi_2 \rho_1 \\ \rho_2 &= \phi_1 \rho_1 + \phi_2 \rho_0 = \phi_1 \rho_1 + \phi_2 \end{aligned}$$

These are called the Yule-Walker equations. Solve to get

$$\begin{aligned} \rho_1 &= \frac{\phi_1}{1 - \phi_2} \\ \rho_2 &= \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1 - \phi_2} \end{aligned}$$

Then the recursion can proceed to get all ρ's one by one. This way we get the values numerically, but not an analytical formula for ρ_k as an explicit function of the coefficients ϕ's.

(In fact, we only need to take $k = 1$ to get the solvable equation for ρ_1; then ρ_0 and ρ_1 will get the recursion rolling. Check that the ρ_2 above follows the recursion.)

Behavior of the ACF (how it decays) will be discussed shortly.

Variance

Multiply both sides of (3) by Y_t:

$$Y_t Y_t = e_t Y_t + \phi_1 Y_{t-1} Y_t + \phi_2 Y_{t-2} Y_t$$

Take expectations and use the zero-mean assumption:

$$\gamma_0 = E[e_t Y_t] + \phi_1 \rho_1 \gamma_0 + \phi_2 \rho_2 \gamma_0$$

hence

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2}$$

The known ρ_1 and ρ_2 can be plugged in to get a formula of γ_0 in terms of model parameters ϕ_1, ϕ_2, and σ_e^2.

Note the analogy with the γ_0 of AR(1).

3.3 AR(p)

$$Y_t = e_t + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} \tag{4}$$

ACF

Multiply both sides of (4) by Y_{t-k}:

$$Y_t Y_{t-k} = e_t Y_{t-k} + \phi_1 Y_{t-1} Y_{t-k} + \cdots + \phi_p Y_{t-p} Y_{t-k}$$

Take expectations and divide by γ_0:

$$\rho_k = \phi_1 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}$$

This is a recursion that uses p previous ρ's to get the next. Take $k = 1, \ldots, p$, we get

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2 \rho_1 + \phi_3 \rho_2 + \cdots + \phi_p \rho_{p-1} \\
\rho_2 &= \phi_1 \rho_1 + \phi_2 + \phi_3 \rho_1 + \cdots + \phi_p \rho_{p-2} \\
&\vdots \\
\rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \cdots + \phi_p
\end{aligned}$$

These are called the Yule-Walker equations. They are p equations with p unknowns. Solve to get $\rho_1, \ldots, \rho_p$. Then the recursion can proceed to get all ρ's one by one, numerically.

(In fact, it suffices to take $k = 1, \ldots, p-1$ to get $p-1$ equations for $\rho_1, \ldots, \rho_{p-1}$. Solve, then $\rho_0, \rho_1, \ldots, \rho_{p-1}$ get the recursion rolling.)

Behavior of the ACF (how it decays) will be discussed shortly.

Variance

Multiply both sides of (4) by Y_t:

$$Y_t Y_t = e_t Y_t + \phi_1 Y_{t-1} Y_t + \cdots + \phi_p Y_{t-p} Y_t$$

Take expectations and use the zero-mean assumption:

$$\gamma_0 = E[e_t Y_t] + \phi_1 \rho_1 \gamma_0 + \cdots + \phi_p \rho_p \gamma_0$$

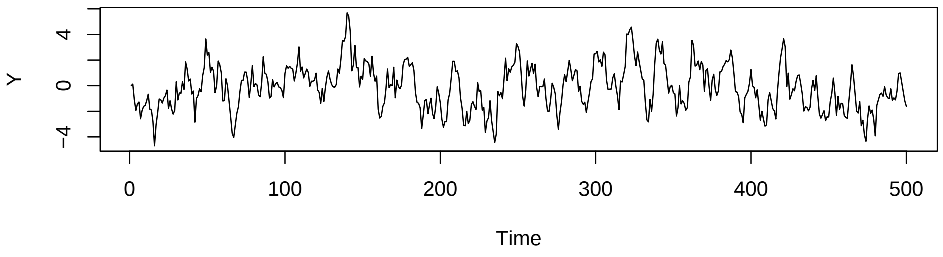
hence

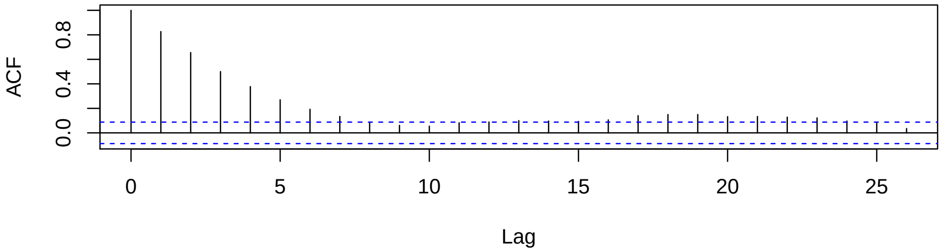
$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi_1 \rho_1 - \cdots - \phi_p \rho_p}$$

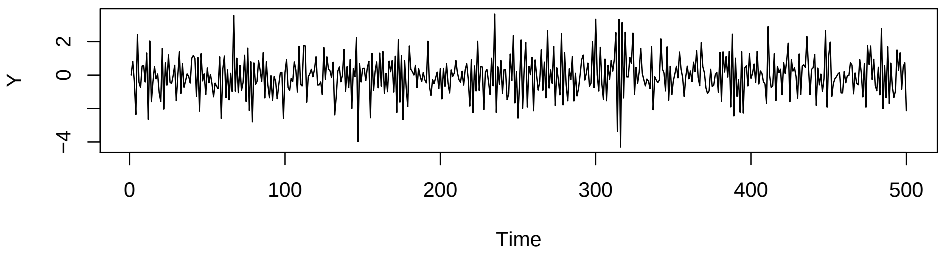
Noticing ρ_1,..., ρ_p are now known (numerically), plug in to get the value of γ_0.

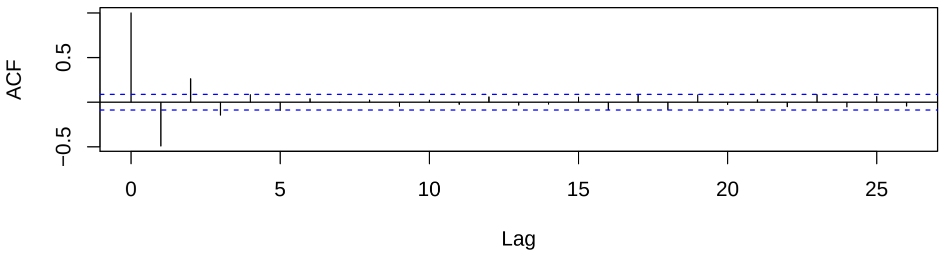
3.4 Simulation

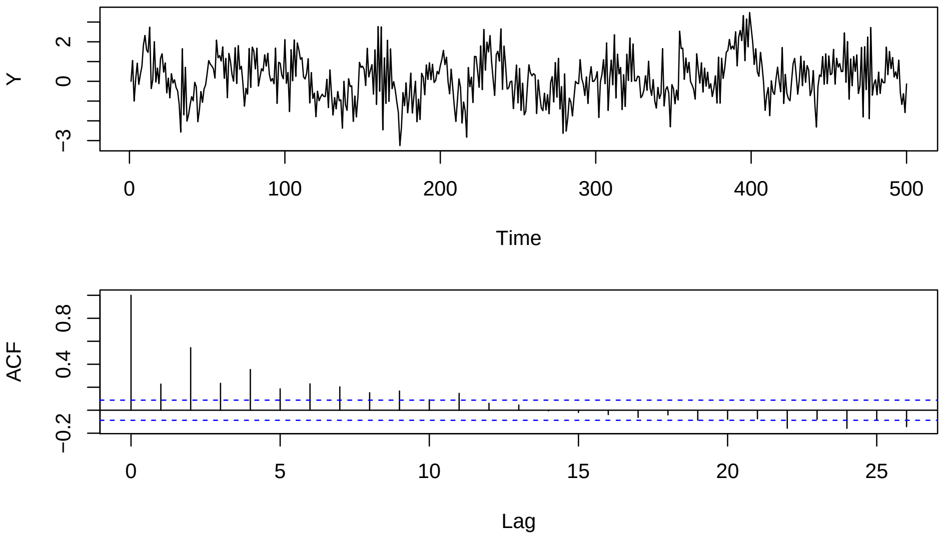
```
> simul.ar <- function(n, phi, sigma = 1)
+ {
+     p <- length(phi)
+
+     et <- rnorm(n) * sigma
+     y <- numeric(n)
+     for (i in 2 : n)
+     {
+         k <- min(p, i-1)
+         y[i] <- et[i] + sum(y[(i-1) : (i-k)] * phi[1:k])
+     }
+     y
+ }

> y <- simul.ar(500, .8)
> pdf(file = ’lect02-3a.pdf’, width = 8, height = 3)
> plot(y, type = ’l’, xlab = ’Time’, ylab = ’Y’, cex = .5)
> dev.off()
X11cairo
       2
> pdf(file = ’lect02-3b.pdf’, width = 8, height = 3)
> acf(y, main = NA)
> dev.off()
X11cairo
       2
>
> y <- simul.ar(500, -.5)
> pdf(file = ’lect02-4a.pdf’, width = 8, height = 3)
> plot(y, type = ’l’, xlab = ’Time’, ylab = ’Y’, cex = .5)
> dev.off()
X11cairo
       2
> pdf(file = ’lect02-4b.pdf’, width = 8, height = 3)
> acf(y, main = NA)
> dev.off()
X11cairo
       2
>
> y <- simul.ar(500, c(.1, .5))
> pdf(file = ’lect02-5a.pdf’, width = 8, height = 3)
> plot(y, type = ’l’, xlab = ’Time’, ylab = ’Y’, cex = .5)
> dev.off()
X11cairo
       2
> pdf(file = ’lect02-5b.pdf’, width = 8, height = 3)
> acf(y, main = NA)
> dev.off()
X11cairo
       2
>
> y <- simul.ar(500, c(.8, -.5))
> pdf(file = 'lect02-6a.pdf', width = 8, height = 3)
> plot(y, type = 'l', xlab = 'Time', ylab = 'Y', cex = .5)
> dev.off()
X11cairo
       2
> pdf(file = 'lect02-6b.pdf', width = 8, height = 3)
> acf(y, main = NA)
> dev.off()
X11cairo
       2
>
```

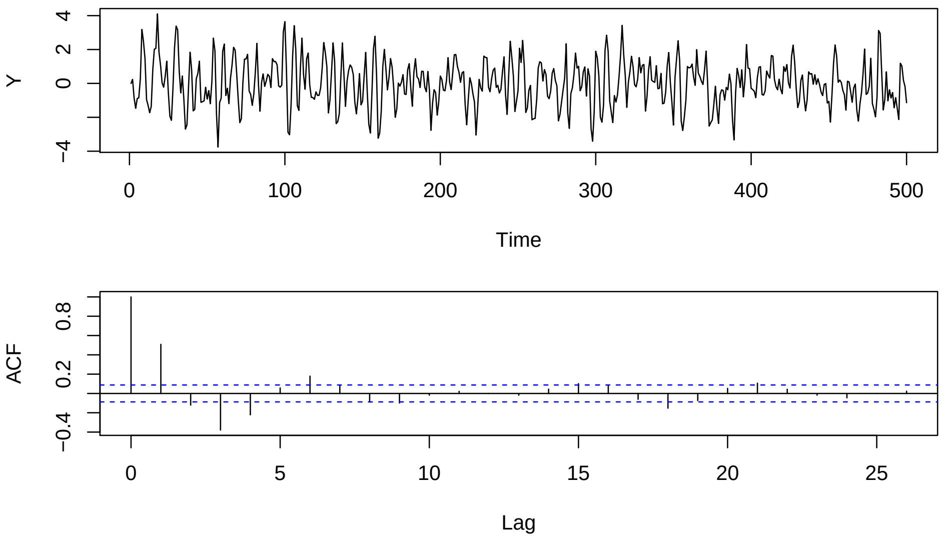












Exercise Play with different values of ϕ's (including some that make the process non-stationary).

4 Backshift operator

(Page 106.)

We use the symbol B. (In some books it's called "lag operator" and denoted by L.)

By definition, let BY_t mean Y_{t-1}.

Then, $B^2 Y_t = BBY_t = BY_{t-1} = Y_{t-2}$.

In general, $B^s Y_t = Y_{t-s}$.

In particular, $B^0 Y_t = Y_t$.

Differencing:

$$\nabla Y_t = Y_t - Y_{t-1} = (1 - B) Y_t$$

$$\nabla^d Y_t = (1 - B)^d Y_t$$

AR(p), $Y_t = e_t + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p}$, can be written as

$$Y_t = e_t + (\phi_1 B + \phi_2 B^2 + \cdots + \phi_p B^p) Y_t$$

or better,

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) Y_t = e_t$$

Definition **Autoregressive operator**

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

Definition **Moving average operator**

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$$

Write the AR(p) model as

$$\phi(B) Y_t = e_t$$

and the MA(q) model as

$$Y_t = \theta(B) e_t$$

To make the notation self-sufficient with order information, we could write $\phi_p(B) Y_t = e_t$ and $Y_t = \theta_q(B) e_t$, such as $\phi_3(B) Y_t = e_t$ and $Y_t = \theta_2(B) e_t$.

Note Now we see whey we use "+" in front of ϕ_i but "-" in front of θ_j. However, some books use "+" in both cases.

Definition AR(p) characteristic polynomial:

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

AR(p) characteristic equation:

$$\phi(x) = 0$$

(These could also be written as $\phi(z)$ or even $\phi(B)$. x or z or B is just a place holder; it doesn't matter what to use here. The real thing is the coefficients.)

Properties of the roots of the characteristic equation are very important in discussion of properties of the AR(p) process. (Note that it's also common to speak of the "roots" of a polynomial, which refer to roots of the equation.)

Similarly define characteristic polynomial and characteristic equation of a MA(q) model.

Definition MA(q) characteristic polynomial:

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^p$$

MA(q) characteristic equation:

$$\theta(x) = 0$$

5 Explosive AR models and causality

Consider $Y_t = e_t + \phi Y_{t-1}$.

We know Y_t is nonstationary if $\phi = 1$ (random walk).

If $|\phi| < 1$, we have

$$Y_t = e_t + \phi(e_{t-1} + \phi Y_{t-2}) = \cdots = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots = \sum_{j=0}^{\infty} \phi^j e_{t-j}$$

Because $|\phi| < 1$, the magnitude of ϕ^j quickly decreases, and so the above series converges, hence it's a meaningful representation.

If $|\phi| > 1$, the series ϕ^j does not converge hence $Y_t = \sum_{j=0}^{\infty} \phi^j e_{t-j}$ is not usable. It's called an "explosive" AR model.

However, if we define $\epsilon_t = -e_t/\phi$, and write $Y_{t-1} = \epsilon_t + \frac{1}{\phi} Y_t$, or better,

$$Y_t = \epsilon_{t+1} + \frac{1}{\phi} Y_{t+1},$$

then since $\left|\frac{1}{\phi}\right| < 1$, we have

$$Y_t = \sum_{j=0}^{\infty} (1/\phi)^j Y_{t+j}.$$

Now the coefficients converges, hence it's a meaningful series. This shows that if it's mathematically equivalent for time to go backward and forward, then $|\phi| > 1$ and $|\phi| < 1$ are equivalent after some transformation.

In practice, however, it's not very useful to express Y_t in terms of the future. Moreover, the $|\phi| > 1$ case has to be somehow transformed to be nice. So we do not like this case.

Definition **Causality**

The process $\{Y_t\}$ represented by the AR(p) model is causal, or is a causal function of $\{e_t\}$, if there exist constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} \quad \text{for all } t.$$

In other words, causality means Y_t can be expressed by current and past e. In still other words, a "causal" AR process is one that can be expressed as a MA process. (This MA process is invariably an infinite series.)

(It seems "causal" and "stationary" AR are interchangeable terms, but I have not seen sources to confirm this.)

Theorem **Condition for causality**

Causality is equivalent to the condition

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots \phi_p x^p \neq 0 \quad \text{for all } |x| \leq 1.$$

In other words, all roots of $\phi(x) = 0$ are outside the unit circle.

Special cases:

For AR(1), a necessary and sufficient condition is $|\phi| < 1$.

For AR(2), a necessary and sufficient condition is (4.3.11), p. 72. Also see Exhibit 4.17, p. 72.

For AR(p) in general, a necessary but not (necessarily) sufficient condition is (4.3.28), p. 76.

When we simulate AR processes, the ϕ's should be chosen with attention to these conditions.

6 Non-uniqueness of MA models and invertibility

Consider $Y_t = e_t - \theta e_{t-1}$.

We have seen that $\gamma_0 = (1+\theta^2)\sigma_e^2$, $\rho_1 = -\frac{\theta}{1+\theta^2}$, $\rho_2 = \rho_3 = \cdots = 0$.

Define model $Y_t = \theta e_t - \frac{1}{\theta} e_{t-1}$. We can see this model has the same γ_0 (note the change of the error variance) and ρ's as the first one. And both models have $E(Y_t) = 0$.

Therefore as far as modeling Y_t is concerned, these two are the same model. (Their noise levels are different but those are not observed; they are an auxiliary device for modeling Y_t.)

Then, which one to choose?

For an AR model, we want a causal one, that is, Y_t can be expressed by current and past e's. Analogously, for a MA model, we want one in which e_t can be expressed by current and past Y's. A reason for this desire is that we want to work with e's; and only when e_t can be expressed by current and past Y's can we calculate the e's up to the current time. Such a MA model is called invertible.

Definition **Invertibility**
The process $\{Y_t\}$ represented by the MA(q) model is invertible if there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$e_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j} \quad \text{for all } t.$$

In other words, invertibility means e_t can be expressed by current and past Y. In still other words, an "invertible" MA process is one that can be expressed as an AR process. (This AR process is invariably an infinite series.)

Re-arrange the MA(1) model as

$$e_t = Y_t + \theta e_{t-1}.$$

Compare with the AR(1) model

$$Y_t = e_t + \phi Y_{t-1}$$

By symmetry, we see the condition for ϕ to make the AR(1) causal must be the same condition for θ to make the MA(1) invertible.

Theorem **Condition for invertibility**

Invertibility is equivalent to the condition

$$\phi(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots \theta_p x^p \neq 0 \quad \text{for all } |x| \leq 1.$$

In other words, all solutions of $\theta(x) = 0$ have modulus greater than 1.

7 Properties of ACF of an AR process

The most important observation about ACF of a MA(q) process is that it cuts off after lag q. ACF usually (or probably always) drops quickly before it becomes zero; but the specific formulae for its nonzero values are less important.

In this section we focus on the behavior of the ACF of an AR(p) process. This ACF does not cut off—it remains nonzero for whatever lag. But we'll show its magnitude drops rather quickly as the lag increases.

Discussion of the behavior of ρ_k is under the causality assumption (which stipulates permissible values of the ϕ's).

Some discussion can be made based on the formulas for AR(1) and AR(2) we have derived.

For AR(1), we know $-1 < \phi < 1$ and $\phi_k = \phi^k$. So,
if $\phi > 0$,...
if $\phi < 0$,...

For AR(2), the magnitude of ρ_k dies out exponentially fast as k increases. In the case of complex roots of the characteristic equation, ρ_k displays a damped sine wave behavior. (p. 73)

These statements about the behavior of ρ in AR is general: they die out exponentially fast, either with the same sign or in a sinusoidal fashion.

In general, suppose

$$\phi(x) = \prod_{i=1}^{p} (1 - g_i x)$$

that is, the roots of the characteristic equation are $g_1^{-1}, \ldots, g_p^{-1}$. The general solution of the autocorrelations are

$$\rho_k = a_1 g_1^k + a_2 g_2^k + \ldots + a_p g_p^k.$$

For causality we require $|g_i| < 1$. Thus two situations can arise if we assume the roots are distinct.

1. A root g_i^{-1} is real, in which case the term $a_i g_i^k$ decays to zero geometrically as k increases. This is called a "damped exponential".

2. A pair of roots, g_i^{-1} and g_j^{-1}, are complex conjugates, in which case they contribute the following term,

   $$a_i g_i^k + a_j g_j^k = d^k \sin(2\pi f k + F),$$

   to ρ_k. This is a damped sine wave, with damping factor d.

In general, the autocorrelation function of a causal AR process will consist of a mixture of damped exponentials and damped sine waves.

8 ARMA models

General form of ARMA(p, q): (4.4.1), p. 77.

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q}$$

This is written concisely as

$$\phi(B) Y_t = \theta(B) e_t$$

Appendix C, p. 85, describes a procedure for finding the γ_0, $\rho_1, ...$ for ARMA(p, q):

1. Find the coefs $\psi_0, \psi_1, \ldots$ for the "general linear process" representation of Y_t; see (4.C.1), p. 85, and (4.4.7), p. 79. (The book uses '$+$' in front of the ψ's; but it's better to use '$-$' to be consistent with the treatment of θ's.)

2. Note Y_t is correlated with past e's but uncorrelated with future e's. Using the ψ form, it's easy to find $E(Y_t e_{t-k})$, $k \geq 0$, because Y_t is expressed as a linear function of iid e's.

3. Multiply Y_{t-k} on both sides of the ARMA form of Y_t, then take expectation. The AR terms give γ's. The MA terms are obtained using the ψ's as described above.

4. Take k to be $0, 1, \ldots, p$, to get a system of p equations for $\gamma_0, \gamma_1, \ldots, \gamma_p$. Solve the system to get $\gamma_0, \ldots, \gamma_p$, hence $\rho_1, \ldots, \rho_p$.

5. If $q > p$, use the same procedure to get $\gamma_{p+1}, \ldots, \gamma_q$ one at a time. This will use the previously obtained $\gamma_0, \ldots, \gamma_p$. One equation (not a system equations) at a time, but not a simple recursion.

6. For $k > \max(p, q)$, multiply Y_{t-k} on both sides of the ARMA form of Y_t, take expectation. The e terms all vanish (because they are all in the future of Y_{t-k}. We get a recursive formula for γ or ρ.

Causality: if $p > 0$, we talk about whether the ARMA process is "causal". The definition is the same as that for an AR process—only the AR part of the ARMA has an impact on its causality.

Invertibility: if $q > 0$, we talk about whether the ARMA process is "invertible". The definition is the same as that for a MA process—only the MA part of the ARMA has an impact on its invertibility.

Similarly, the conditions for causality and invertibility are the same for causal AR and invertible MA, respectively.

In summary,

1. Note the MA and AR formulations are the same except the roles of Y and e are switched. This explain the analogy between the concepts of causality and invertibility, and their conditions on the respective coefficients ϕ's and θ's.

2. When we check causality of ARMA, we check the AR coefficients only (because the MA part is already "causal"). Similarly, when we check invertibility we check the MA coefficients only.

3. Similar analogies/contrasts of the AR and MA models exist for ACF, PACF, and other properties.

Requirements on ARMA

"Stationarity" is explicitly required in the definition of ARMA. In addition we require it to be

1. causal;

2. invertible;

3. in the simplest (lowest-order) form.

The third requirement means there should be no common factors in $\phi(B)$ and $\theta(B)$. We'll have an example (Example 3.5, page 93, and Example 3.6, page 96, Shumway and Stoffer). But this requires knowledge about polynomials which is beyond this course.

9 Conversion between various forms

9.1 MA form, or ψ form

Page 55, (4.1.1):

$$Y_t = e_t - \psi_1 e_{t-1} - \psi_2 e_{t-2} - \cdots \tag{5}$$

or $Y_t = \psi(B)e_t$.

Page 57, (4.2.1):

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \tag{6}$$

or $Y_t = \theta(B)e_t$.

9.2 AR form, or π form

Page 80, (4.5.5):

$$Y_t = e_t + \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \cdots \tag{7}$$

or $\pi(B)Y_t = e_t$.

Page 66, (4.3.1):

$$Y_t = e_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} \tag{8}$$

or $\phi(B)Y_t = e_t$.

9.3 ARMA form

Page 77, (4.4.1):

$$\begin{aligned} Y_t = e_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} \\ - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \end{aligned} \tag{9}$$

or $\phi(B)Y_t = \theta(B)e_t$.

9.4 From AR form to MA form

(Incomplete discussion on page 75.)

Substitute the ψ from (15) into the AR form (8):

$$\begin{aligned}
&e_t - \psi_1 e_{t-1} - \psi_2 e_{t-2} - \psi_3 e_{t-3} - \cdots \\
&= e_t \\
&\quad + \phi_1\big(e_{t-1} - \psi_1 e_{t-2} - \psi_2 e_{t-3} - \cdots\big) \\
&\quad + \phi_2\big(e_{t-2} - \psi_1 e_{t-3} - \psi_2 e_{t-4} - \cdots\big) \\
&\quad + \cdots \\
&\quad + \phi_p\big(e_{t-p} - \psi_1 e_{t-p-1} - \psi_2 e_{t-p-2} - \cdots\big)
\end{aligned}$$

Equate coefficients of the same e on both sides to get the recursion

$$\begin{aligned}
\psi_1 &= -\phi_1 \\
\psi_2 &= \phi_1\psi_1 - \phi_2 \\
\psi_3 &= \phi_1\psi_2 + \phi_2\psi_1 - \phi_3 \\
\cdots &= \cdots \\
\psi_p &= \phi_1\psi_{p-1} + \phi_2\psi_{p-2} + \cdots + \phi_{p-1}\psi_1 - \phi_p \\
\psi_{p+1} &= \phi_1\psi_p + \phi_2\psi_{p-1} + \cdots + \phi_{p-1}\psi_2 + \phi_p\psi_1 \\
\psi_{p+2} &= \phi_1\psi_{p+1} + \phi_2\psi_p + \cdots + \phi_{p-1}\psi_3 + \phi_p\psi_2 \\
\cdots &= \cdots
\end{aligned}$$

Alternatively, from $\phi(B)Y_t = e_t$ we get

$$Y_t = \frac{1}{\phi(B)} e_t$$

But exactly what is $\frac{1}{\phi(B)}$? In general,

$$\frac{1}{\phi(B)} = 1 - \psi_1 B - \psi_2 B^2 - \cdots$$

that is,

$$(1 - \psi_1 B - \psi_2 B^2 - \cdots)(1 - \phi_1 B - \phi_2 B^2 - \cdots) = 1$$

Expand the multiplication, work on the coefficients, and a recursive formulation will emerge.

9.5 From MA form to AR form

(Incomplete discussion on page 208.)

Substitute the π form (7) into the MA form (6):

$$\begin{aligned}
&e_t + \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \cdots \\
&= e_t \\
&\quad - \theta_1\big(Y_{t-1} - \pi_1 Y_{t-2} - \pi_2 Y_{t-3} - \cdots\big) \\
&\quad - \theta_2\big(Y_{t-2} - \pi_1 Y_{t-3} - \pi_2 Y_{t-4} - \cdots\big) \\
&\quad - \cdots \\
&\quad - \theta_q\big(Y_{t-q} - \pi_1 Y_{t-q-1} - \pi_2 Y_{t-q-2} - \cdots\big)
\end{aligned}$$

Equate coefficients of the same Y on both sides to get the recursion

$$\begin{aligned}
\pi_1 &= -\theta_1 \\
\pi_2 &= -\theta_2 + \theta_1\pi_1 \\
\pi_3 &= -\theta_3 + \theta_2\pi_1 + \theta_1\pi_2 \\
\cdots &= \cdots \\
\pi_q &= -\theta_q + \theta_{q-1}\pi_1 + \cdots + \theta_1\pi_{q-1} \\
\pi_{q+1} &= \theta_q\pi_1 + \theta_{q-1}\pi_2 + \cdots + \theta_1\pi_q \\
\pi_{q+2} &= \theta_q\pi_2 + \theta_{q-1}\pi_3 + \cdots + \theta_1\pi_{q+1} \\
\cdots &= \cdots
\end{aligned}$$

Alternatively, from $Y_t = \theta(B)e_t$ we get

$$\frac{1}{\theta(B)} Y_t = e_t$$

Indeed, as we have seen much duality between AR and MA models, techniques for AR and for MA are the same.

9.6 From ARMA form to AR and MA forms

(Incomplete discussion on pages 79 and 208.)

In this general situation the elegance of the B notation shows most vividly.

Suppose the model is $\phi(B)Y_t = \theta(B)e_t$.

Suppose an alternative form of it is $\pi(B)Y_t = e_t$ (the existence of which is guaranteed by invertibility), then substitute this for e_t in ARMA, we get

$$\phi(B)Y_t = \theta(B)\pi(B)Y_t$$

that is,

$$\phi(B) = \theta(B)\pi(B)$$

Multiply out $\theta(B)\pi(B)$, equate coefficients of corresponding B polynomials, we get recursive relations for π's.

Suppose an alternative form of it is $Y_t = \psi(B)e_t$ (the existence of which is guaranteed by causality), then substitute this for Y_t in ARMA, we get

$$\phi(B)\psi(B)e_t = \theta(B)e_t$$

that is,

$$\phi(B)\psi(B) = \theta(B)$$

Multiply out $\phi(B)\psi(B)$, equate coefficients of corresponding B polynomials, we get recursive relations for ψ's.

We can also write, directly from $\phi(B)Y_t = \theta(B)e_t$,

$$Y_t = \frac{\theta(B)}{\phi(B)}e_t$$

hence $\psi(B) = \theta(B)/\phi(B)$, and

$$\frac{\phi(B)}{\theta(B)}Y_t = e_t$$

hence $\pi(B) = \phi(B)/\theta(B)$. This is handy for discussion. But if we want to calculate the actual values of ψ's and π's, writing out the recursive equations might still be the ultimate way to go.

10 PACF

10.1 Definition and properties

Definition (6.2.1), 6.2.2)

When assuming normal noise, these two definitions are equivalent.

Notation ϕ_{kk}

convention $\phi_{00} = 1$ (The book states $\phi_{11} = 1$ by convention. This appears to be an error. Instead, $\phi_{11} = \rho_1$.)

An operational definition is given as follows (see (6.2.8), p 114).

Definition PACF ϕ_{kk}, $k = 0, 1, \ldots$.
Define $\phi_{00} = 0$ and let ϕ_{kk} be the last element of $\Gamma_k^{-1}\vec{p}_k$, where $\Gamma_k = \left[\rho(i-j)\right]_{i,j=1}^{k}$ and $\vec{p}_k = [\rho(1), \rho(2), \ldots, \rho(k)]'$.

Example ϕ_{22} for a general stationary process: (6.2.3)

Example Applying (6.2.3) for the special case of AR(1), we get $\phi_{22} = 0$.

Example AR(p) at $k > p$. (6.2.4).

Note This result shows that $\phi_{kk} = 0$ for AR(p) when $k > p$. In addition we can show (last paragraph, page 114) that $\phi_{pp} = \phi_p \neq 0$ for AR(p). Both results combined, we see PACF cuts off right after lag p for AR(p), hence the cutting off of (sample) PACF is an indicator for the order of an AR process.

Note PACF for AR behaves much like ACF for MA. ACF for AR behaves much like PACF for MA.

Example Applying (6.2.3) to MA(1). See (6.2.6).

Note PACF for AR(p) cuts off after lag p, because the auto-regression representation extends only to the past p times. PACF for MA(q) does not cut off, because the AR representation for a MA process goes back to infinite history. Similarly, because an invertible ARMA(p,q) model (with $q > 0$) has an infinite AR representation, its PCAF will not cut off.

Note For ARMA(p,q) with $p > 0$ and $q > 0$, neither ACF nor PACF cuts off. Hence there is no very nice, clear-cut indicators for the orders p and q.

Table 6.3, page 116.

10.2 Computation

For a general stationary process, we can find ϕ_{kk} by solving the Yule-Walker equations (6.2.8), which uses ρ's.

Note To get ϕ_{kk}, we'll need to solve k equations and discard $k - 1$ solutions.

A recursive algorithm: Levinson-Durbin, page 115.

11 Specification of ARMA models

Prior to estimating the model parameters, we need to "specify" the model, meaning here coming up with a reasonable choice (determination) for the orders p and q, based on explorations of the data. There are a few semi-formal tools to guide this decision. After the specification, the model parameters (coefficients and noise variance) will be estimated. The estimated model will be examined (by "diagnostics") to see whether it seems to be adequate; if not, how to change and specification (and then re-estimate).

Specification is largely based on the observed (i.e. “sample” or “empirical”) correlation structure, that is, ACF and PACF. Because we know the theoretical correlation structure of specific models, the observed structure tells us what the model should be.

Because ACF of MA(q) cuts off after lag q and PACF of AR(p) cuts off after lag p, a cut-off ACF indicates a MA model (along with its order) whereas a cut-off PACF indicates an AR model (along with its order).

In order to judge whether the ACF or PACF has “cut off”, that is, reduced to zero, we need to know their sampling distributions, which give us thresholds to be used in tests of of significance.

If neither ACF nor PACF cuts off, it’s an ARMA model. To deal with ARMA we learn a new tool called EACF.

11.1 Computation of sample ACF

Recall the definition of ACF, (3.6.2), page 46, or (6.1.1), page 109.

11.2 Sampling distribution of ACF

ACF is a basic graphical tool for us. To use it, for example judging whether it has dropped to 0 after certain lag, we need its sampling distribution and properties.

Theorem As $n \to \infty$,

$$r_k \to N(\rho_k, c_{kk}/n)$$

where c_{kk} is given in (6.1.2).

We won’t deal with (6.1.2) or c_{kk} in the general case. We’ll consider the following special cases.

White noise

(6.1.3), p. 110

$$\begin{aligned} \rho_k &= 0, \qquad k = 1, 2, \ldots \\ \operatorname{var}(r_k) &\approx \frac{1}{n}, \qquad k = 1, 2, \ldots \\ \operatorname{corr}(r_i, r_j) &\approx 0, \qquad i \neq j \end{aligned}$$

Usage

Properties of white noise are used mainly to check whether the residuals, after fitting some ARMA model, are white noise.

1. Check whether r_k, $k = 1, 2, \ldots$, is within $\pm\frac{2}{\sqrt{n}}$.
2. If the first check passes, check whether there are correlations among the residuals by looking at the sample ACF or PACF of the residuals.

If the residuals successfully pass these checks, meaning it's reasonable to claim the model has reduced the residuals to white noise, we move on to consider whether "volatility clustering" exists in the residual series, that is, whether residuals demonstrate varying conditional variance (to be discussed later), given that they are already uncorrelated.

AR

AR(1): (6.1.4), p. 110

Sampling properties of r_k for an AR process are complicated. The important point is that, since ρ_k for AR does not reduce to 0, the expected value of r_k does not reduce to 0.

Usage

If r_k stays significant (by eye-balling the plot) for quite a few lags, it suggests it's a AR or ARMA process.

With sample ACF, we mainly check whether it drops to 0 after a small number of lags, which is an indication of MA (see below). If sample ACF does not conform to the MA behavior, then we move on to non-ACF tools.

MA

MA(q): (6.1.11) for $k > q$, p. 112.

$$\rho_k = 0, \qquad k > q$$

$$\operatorname{var}(r_k) = \frac{1}{n}\Big(1 + 2\sum_{j=1}^{q} \rho_j^2\Big), \qquad k > q$$

These properties suggest the sample r_k is expected to lie within twice $\sqrt{\operatorname{var}(r_k)}$ (about its mean 0) after lag q.

Usage

In practice, we substitute the sample r_k for the theoretical ρ_k in the formula of $\text{var}(r_k)$ to get an estimate of the standard error of r_k:

$$s(r_k) = \frac{1}{\sqrt{n}}\sqrt{1 + 2\sum_{j=1}^{q} r_j^2}, \qquad k > q$$

Note this standard error stays constant after lag q. However, we don't know q! What we do is: assuming $q = 1$ and calculating $s(r_k)$; assuming $q = 2$ and calculating $s(r_k)$; assuming $q = 3$ and calculating $s(r_k)$; and so on. Then we plot these $2s(r_k)$ at the assumed corresponding q, and check whether all subsequent r's are within these bounds. If they are, then it suggests it's a MA process and we've found its q.

Note The ACF of a AR does not cut off. Hence we're not interested in formally testing its significance. To deal with AR, we'll use PACF.

If the r's do not appear to conform to MA behavior, the process may be AR or ARMA.

11.3 Computation of sample PACF

For a general stationary process, we can find ϕ_{kk} by solving the Yule-Walker equations (6.2.8) (p. 114). For theoretical ϕ_{kk}, we use ρ's in (6.2.8); for sample ϕ_{kk}, we use r's in place of ρ's.

Note To get ϕ_{kk} by (6.2.8), we'll need to solve k equations and discard $k-1$ solutions.

A recursive algorithm, Levinson-Durbin, is given on page 115.

11.4 Sampling distributions of PACF

Since PACF for AR(p) cuts off after lag p, but does not cut off for MA, PACF is more useful for AR than for MA. After examination of ACF has indicated the process is not MA, we examine its PACF to see whether it's an AR process.

To tell whether PACF has become insignificant after certain lag, we need its sampling properties.

Theorem (page 115) For AR(p), approximately,

$$\hat{\phi}_{kk} \to N(0, 1/n), \quad k > p$$

Therefore, $\hat{\phi}_{kk}$ outside of $\pm 2/\sqrt{n}$ suggests an insignificant value. If after certain lag, all empirical $\hat{\phi}_{kk}$ are within $\pm 2/\sqrt{n}$, then we can claim the process is AR and we've found its order p.

Note This cutting-off test is simpler than ACF for MA, as the bounds do not depend on p.

If neither ACF nor PACF cuts off by the preceding tests, the process is probably a mixture of AR and MA, i.e. an ARMA. Then we need the next tool to guess its p and q.

11.5 EACF

Determination of the orders of an ARMA process involves much exploration, and it is quite an unsettled issue.

The concept: page 116.

"The EACF method uses the fact that if the AR part of a mixed ARMA model is known, 'filtering out' the autoregression from the observed time series results in a pure MA process that enjoys the cutoff property in its ACF".

Because the order p is unknown, and correlation structure is unclear, the "filtering" can't be done easily in one go. An iterative algorithm can do it. Test conclusions on the resultant sample EACF are summarized in a table (p. 117). The estimates of p (row) and q (col) are read off the table.

Note: table of sample EACF will not be clear-cut. Judgement is needed.

Details of computation are not required. Call an `R` function such as `eacf`.

12 Estimating AR models by the method of moments

We'll discuss three methods for estimating the parameters ϕ's, θ's and σ_e^2 after p and q have been determined: method of moments, LS, and MLE.

Of the three methods, the method of moments is most informal and ad hoc. There are extensive results on LS in rather formal studies. MLE has the most formal theoretical foundation.

For the method of moments, we'll consider AR only b/c the method is no good for MA. It can also be used to estimate the AR parameters in an ARMA model (but it's not very clear at this moment how it is done). Its performance is comparable to that of LS and MLE.

LS appears to be the easiest to set up and use for general forms of models. Its performance is comparable to that of MLE.

For MLE, we'll consider rather simple model forms only.

The idea for the method of moments is intuitive: suppose there is some quantity, say T, that we can derive as a function of the unknown model parameters, and this quantity can be estimated from the data. Then equating the theoretical T (as a formula involving unknown model parameters) and its estimated value gives us an equation about the model parameters. If we have as many such quantities as there are unknown parameters, the system of equations can be solved to get estimates of the model parameters.

In the method of moments, we take "moments", e.g. mean, variance, correlation, etc, as such T.

12.1 Estimating ϕ's

The Yule-Walker equations, (4.3.30) on page 76, were used to compute ρ's using ϕ's. Now ϕ's are unknown. Substituting empirical r's for ρ's, they become equations for ϕ's.

12.2 Estimating σ_e^2

First, estimate γ_0 by the sample variance:

$$s^2 = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \overline{Y})^2$$

Then, remember how we calculated γ_0 given the model?

(4.3.31), page 77. Now with ρ's, ϕ's, and γ_0 all estimated, this gives an estimate of σ_e^2.

13 Least squares estimation

13.1 AR models

AR(p):

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \cdots + \phi_p(Y_{t-p} - \mu) + e_t$$

Here we allow a non-zero mean μ, which is a reasonable thing to do in real-world applications.

The goal is to find ϕ's and μ that minimize the following "objective function":

$$Q = \sum_{t=p+1}^{n} e_t^2 = \sum_{t=p+1}^{n} \big[(Y_t - \mu) - \phi_1(Y_{t-1} - \mu) - \cdots - \phi_p(Y_{t-p} - \mu)\big]^2$$

Difficulty: We can not take $Y_t - \mu$ as response and $Y_{t-1} - \mu, \dots, Y_{t-p} - \mu$ as predictors, and use the formulas learned in "Linear Regression Models". The reason lies in $\phi\mu$, which makes it a <u>nonlinear</u> model in the parameters.

Tackle μ first:

$$\frac{\partial Q}{\partial \mu} = \sum_{t=p+1}^{n} 2\big[(Y_t - \mu) - \phi_1(Y_{t-1} - \mu) - \cdots - \phi_p(Y_{t-p} - \mu)\big] \cdot (-1 + \phi_1 + \cdots + \phi_p) \overset{\text{set}}{=} 0$$

hence

$$\sum_{t=p+1}^{n} \big[(Y_t - \mu) - \phi_1(Y_{t-1} - \mu) - \cdots - \phi_p(Y_{t-p} - \mu)\big] = 0$$

because we know $\phi_1 + \cdots + \phi_p < 1$ by (4.3.28) on page 76.

$$\begin{aligned}(n-p)(1 - \phi_1 - \cdots - \phi_p)\,\mu &= \sum_{t=p+1}^{n} Y_t - \phi_1 \sum_{t=p+1}^{n} Y_{t-1} - \cdots - \phi_p \sum_{t=p+1}^{n} Y_{t-p} \\ &\approx (n-p)\overline{Y} - \phi_1(n-p)\overline{Y} - \cdots - \phi_p(n-p)\overline{Y}\end{aligned}$$

hence an estimator for μ is

$$\hat{\mu} = \overline{Y}$$

The approximation makes things much simpler without causing big errors as long as the sample size n is reasonably large.

Now tackle ϕ's (substituting $\overline{Y}$ for μ):

$$\frac{\partial Q}{\partial \phi_i} = \sum_{t=p+1}^{n} -2\big[(Y_t - \overline{Y}) - \phi_1(Y_{t-1} - \overline{Y}) - \cdots - \phi_p(Y_{t-p} - \overline{Y})\big] \cdot (Y_{t-i} - \overline{Y}) \overset{\text{set}}{=} 0$$

that is,

$$\sum_{t=p+1}^{n}(Y_t-\overline{Y})(Y_{t-i}-\overline{Y}) = \phi_1\sum_{t=p+1}^{n}(Y_{t-1}-\overline{Y})(Y_{t-i}-\overline{Y})+\cdots+\phi_p\sum_{t=p+1}^{n}(Y_{t-p}-\overline{Y})(Y_{t-i}-\overline{Y})$$

which gives a system of p equations (one for each $i = 1, \ldots, p$). Solving them gives estimates of the ϕ's.

If we divide both sides by $\sum(Y_t - \overline{Y})^2$, we see the above is approximately

$$r_i = \phi_1 r_{i-1} + \cdots + \phi_p r_{i-p}$$

This again is the Yule-Walker equations. Therefore, if we take this approximation, this will give the "method of moments" estimates. This shows why the estimates by LS and by MM are close for AR.

Using the estimates $\hat{\mu}$ and $\hat{\phi}$'s, we can calculate the residuals:

$$e_t = Y_t - \hat{\phi}_1 Y_{t-1} - \cdots - \hat{\phi}_p Y_{t-p} - (1 - \hat{\phi}_1 - \cdots - \hat{\phi}_p)\hat{\mu}$$

then the error variance is estimated by

$$\hat{\sigma_e^2} = \frac{1}{n-p-(p+1)}\sum_{t=p+1}^{n} e_t^2$$

(I need to find support for this formula. It may need small changes.)

13.2 MA models

$$Y_t - \mu = e_t - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q}$$

First, estimate μ by $\overline{Y}$.

Then, for given values of θ's, our predictor for Y_t is

$$\hat{\mu} - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q}$$

and the prediction error is e_t, hence we can calculate the total squared prediction errors:

$$Q(\theta) = \sum_{t=1}^{n} e_t^2$$

Then the idea is clear: take $Q(\theta)$ as the "objective function", call an optimization routine to find θ's that minimizes $Q(\theta)$.

All the e_t's can be calculated by the MA model provided we have q early e's to get the recursion started. We can do this by setting $e_0 = e_{-1} = \cdots = e_{-q+1} = 0$. This is kind of arbitrary, but setting the noise to their expected value is reasonable. This causes no trouble if n is large.

Note The idea we used for the AR model is the same. But we did not start with e_1. Instead, we started with e_{p+1}, which is the first e that can be calculated with the available Y's. If we started with e_1, we would have to assume the values for some leading Y's (maybe we can assign them to $\overline{Y}$), which is not better than discarding several early e's.

Note We can not make much analytical progress as we did in the AR case. The reason is that the e's in the MA model are calculated using the θ's. If we take derivatives of $Q(\theta)$ with respect to θ, the e's are also functions of the θ's, so it's not that simple.

13.3 ARMA models

By the same idea, we can numerically find (ϕ, θ) that minimizes the target

$$Q(\phi, \theta) = \sum_{t=p+1}^{n} e_t^2$$

Some leading e's may need to be set to 0, depending on how big q is.

Example 7.4, page 170.

Example 7.6, page 171.

Note Implementation of the estimation algorithm may involve nontrivial technical details, e.g. choice of initial values for the unknown parameters. Although crude, it is not unreasonable to use 0 as initial value for all ϕ's and θ's to start the optimization procedure.

14 Maximum likelihood estimation

We only need to get a sense of this method. The details are not required.

The general idea of MLE: we have assumed a joint distribution for the variable, which has been observed; the assumed density involves unknown parameters; write out this density evaluated at the observations; maximize this to give estimates of the unknown parameters.

MLE is usually used on an iid sample, because its joint density is simple: it's the product of the assumed density of each component.

Our observations are the Y's. However, it is not easy to write the joint density of $Y_1, \ldots, Y_n$, because they are dependent.

Instead, the e's are iid, and we have assumed $e_t \sim N(0, \sigma_e^2)$.

(7.3.2) shows $Y_2, \ldots, Y_n$ (considering Y_1 known) are a linear transform of $e_2, \ldots, e_n$. By some probability theory, the joint density of $Y_2, \ldots, Y_n$ is related to the joint density of $e_2, \ldots, e_n$, as shown by (7.3.3).

Now this gives $f(y_2, \ldots, y_n \,|\, y_1)$. We know

$$f(y_1, \ldots, y_n) = f(y_1) \cdot f(y_2, \ldots, y_n \,|\, y_1)$$

So we need $f(y_1)$ to finish the objective function, $f(y_1, \ldots, y_n)$ (or its logarithm).

$f(y_1)$ has been assumed to be normal with mean μ and variance γ_0, which is related to the unknown parameters.

Finally, similar to the situation in linear regression, the MLE is very close to the LSE.

15 Properties of the estimates

Details of the formulas are not required. We should know where to find them when they are needed.

We need to know, qualitatively, that the estimators are approximately (especially when sample size is large) unbiased and normally distributed.

The section 7.4 (p. 160–162) has some comments that are worth noting, including the first sentence on page 161, the paragraph after the equations on page 161, and the first two lines of the paragraph above (7.4.14).

Exercise Verify the $\text{var}(\hat{\theta})$ in (7.4.14) is larger than that given by (7.4.11).

16 Model diagnostics

16.1 Residual analysis

The main criterion for the adequacy of the model is that the residuals left by the model fitting are white noise, that is, iid, or at least uncorrelated.

In addition to the usual checks including

1. plot the standardized residuals and look for any systematic patterns;
2. normality: QQ plot, and probably formal tests

we check the sample ACF of the residuals. If the residuals are indeed white noise, the sample ACF has mean 0. The variance of ACF is approximately $\frac{1}{n}$ for large lags but can be substantially smaller for small lags; see p. 180–183. For a rough check, we can use the bounds $\pm\frac{2}{\sqrt{n}}$: if ACF is within the bounds, it presents no evidence against the residuals being white noise.

The Ljung-Box test

This is a further test, which takes the ACF at multiple lags as a whole. The test statistic is

$$Q_* = n(n+2)\sum_{k=1}^{K} \frac{\hat{r}_k^2}{n-k}$$

where n is sample size, $\hat{r}_k$ is the sample autocorrelation of the residuals at lag k, and K is chosen such that the ψ_j weights are negligible for $j > K$. A typical choice is $K = 20$.

When n is large, the distribution of Q_* is approximately chi-square with $K - p - q$ degrees of freedom:

$$Q_* \sim \chi^2_{K-p-q}, \quad n \text{ large}$$

where q and p are the orders of the ARMA model. The null hypothesis that there is no significant autocorrelation would be rejected if

$$Q_* > \chi^2_{\alpha,K-p-q}$$

where $\chi^2_{\alpha,K-p-q}$ is the $1-\alpha$, e.g. 0.95, quantile of the distribution χ^2_{K-p-q}.

16.2 Overfitting and parameter redundancy

17 Forecasting

17.1 Preliminaries

For most discussions in chapter 9, it is assumed the model, including its parameters, are known exactly. In applications we'll plug in estimates. That will change the quantitative statements, but the impact will be small if the sample size (n) is large.

Notation: given Y_t, Y_{t-1},..., Y_1, we forecast (or predict) $Y_{t+\ell}$ for $\ell = 1, 2, \ldots$. The estimate is written as

$$\hat{Y}_t(\ell)$$

The forecast error is

$$e_t(\ell) = Y_{t+\ell} - \hat{Y}_t(\ell)$$

Note: defined by truth − estimate instead of estimate − truth; three should not be any deep reason; just a choice. Usually we'll require an estimator to be unbiased, that is,

$$E\big(e_t(\ell)\big) = 0$$

which is the case with the estimators we choose here. Further, we would like the estimate to be accurate, that is, $\text{var}\big(e_t(\ell)\big)$ is better small.

We will always use this estimator:

$$\hat{Y}_t(\ell) = E\big(Y_{t+\ell} \,|\, Y_t, Y_{t-1}, \ldots, Y_1\big)$$

that is, the conditional expectation (i.e. expectation of the conditional distribution). It depends on the specific model to determine what this conditional expectation is.

It's kind of by definition (or by design) that this estimator is unbiased, because

$$E\big(Y_{t+\ell} - \hat{Y}_t(\ell) \,|\, Y_1, \ldots, Y_t\big) = E\big(Y_{t+\ell} \,|\, Y_1, \ldots, Y_t\big) - \hat{Y}_t(\ell) = 0$$

If we can find a formula to express $e_t(\ell)$, say it's $e_{t+\ell}$, then $\text{var}\big(e_t(\ell)\big)$ is just the variance of it.

As usual, we always assume that e_t and Y_s are independent if $t > s$, that is, noise is independent of the past. Further, we'll use the following results:

$$\begin{aligned} &E\big(e_{t+\ell} \,|\, Y_t, Y_{t-1}, \ldots\big) = E(e_{t+\ell}) = 0 \quad \ell = 1, 2, \ldots \\ &E\big(e_{t-\ell} \,|\, Y_t, Y_{t-1}, \ldots\big) = E\big(e_{t-\ell} \,|\, Y_{t-\ell}, Y_{t-\ell-1}, \ldots\big) = e_{t-\ell} \quad \ell = 0, 1, 2, \ldots \end{aligned} \tag{10}$$

The second statement suggests the e in question is already realized (determined by the observed Y's). It's not random; it's whatever the realized value is. In a AR model, this comes directly from the model ($e_t = Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p}$); in a MA model, this comes from (4.5.5) on page 80, which, however, requires knowledge of the infinite history of Y.

17.2 Deterministic trend plus white noise

Model:

$$Y_t = \mu_t + e_t$$

We have

$$\begin{aligned}\hat{Y}_t(\ell) &= E\big(\mu_{t+\ell} + e_{t+\ell} \,|\, Y_1, \dots, Y_t\big)\\ &= E\big(\mu_{t+\ell} \,|\, Y_1, \dots, Y_t\big) + E\big(e_{t+\ell} \,|\, Y_1, \dots, Y_t\big)\\ &= E\big(\mu_{t+\ell} \,|\, Y_1, \dots, Y_t\big) + E\big(e_{t+\ell}\big)\\ &= \mu_{t+\ell} + 0\\ &= \mu_{t+\ell}\end{aligned}$$

(With the implication that we have a formula for $\mu_{t+\ell}$, as μ is part of the known model.)

The forecast error is

$$e_t(\ell) = e_{t+\ell}$$

Easy to see $\hat{Y}_t(\ell)$ is unbiased. Error variance is

$$\operatorname{var}\big(e_t(\ell)\big) = \operatorname{var}\big(e_{t+\ell}\big) = \sigma_e^2$$

which does not change with ℓ.

It is natural to guess that the forecast error variance increases as ℓ gets larger—the farther ahead you forecast, the less accurate it tends to be—but this is not true in the current situation. This would be true if $\hat{Y}_t(\ell + 1)$ makes use of $\hat{Y}_t(\ell)$, and so on recursively, so that the error builds up. This would be the case if the Y's are dependent on each other. However, in the current model the Y's are independent:

$$\operatorname{var}(Y_{t+1}, Y_t) = \operatorname{var}\big(\mu_{t+1} + e_{t+1}, \mu_t + e_t\big) = \operatorname{var}\big(e_{t+1}, e_t\big) = 0,$$

therefore farther forecasts do not build on near forecasts (then error does not build up).

17.3 AR(p) with constant mean

Model:

$$Y_{t+\ell} - \mu = \phi_1(Y_{t+\ell-1} - \mu) + \phi_2(Y_{t+\ell-2} - \mu) + \cdots + \phi_p(Y_{t+\ell-p} - \mu) + e_{t+\ell} \tag{11}$$

Take expectation on both sides,

$$\begin{aligned}E\big(Y_{t+\ell} - \mu \,|\, Y_1, \dots, Y_t\big) &= E\big(\phi_1(Y_{t+\ell-1} - \mu) + \phi_2(Y_{t+\ell-2} - \mu) + \cdots + \phi_p(Y_{t+\ell-p} - \mu) + e_{t+\ell} \,|\, Y_1, \dots, Y_t\big)\\ \hat{Y}_t(\ell) - \mu &= \phi_1\big(\hat{Y}_t(\ell - 1) - \mu\big) + \phi_2\big(\hat{Y}_t(\ell - 2) - \mu\big) + \cdots + \phi_p\big(\hat{Y}_t(\ell - p) - \mu\big)\end{aligned} \tag{12}$$

where $\hat{Y}_t(\ell) = Y_{t+\ell}$ if $\ell \le 0$.

Note We didn't try to get $\hat{Y}_t(\ell)$ in one step; rather, we took advantage of the recursive relation. It is very straightforward to use (12) to forecast the future recursively.

Note In (11), μ is indeed $E(Y_t)$. This can be seen by taking expectations on both side and letting $E(Y_t) = \theta$: $\theta - \mu = (\phi_1 + \cdots + \phi_p)\theta - (\phi_1 + \cdots + \phi_p)\mu$, leading to $\theta = \mu$.

In (12), move μ to the RHS, wee see $\hat{Y}_t(\ell)$ is μ plus a weighted sum of the forecast or observed deviations from μ in the past. The deviations are called "innovations". We can say the forecast is μ plus forecast innovation. How does the forecast innovation change as ℓ increases? It's not clear in the above. The most tangible innovations are the observed ones. Supposing $\ell > p$, let's try to fold in the recursion:

$$\begin{aligned}
\hat{Y}_t(\ell) - \mu &= \phi_1\Big(\phi_1\big(\hat{Y}_t(\ell-2) - \mu\big) + \phi_2\big(\hat{Y}_t(\ell-3) - \mu\big) + \cdots + \phi_p\big(\hat{Y}_t(\ell-p-1) - \mu\big)\Big) \\
&\quad + \phi_2\big(\hat{Y}_t(\ell-2) - \mu\big) \\
&\quad + \cdots \\
&\quad + \phi_p\big(\hat{Y}_t(\ell-p) - \mu\big) \\
&= (\phi_1^2 + \phi_2)\big(\hat{Y}_t(\ell-2) - \mu\big) \\
&\quad + (\phi_1\phi_2 + \phi_3)\big(\hat{Y}_t(\ell-3) - \mu\big) \\
&\quad + \cdots \\
&\quad + (\phi_1\phi_{p-1} + \phi_p)\big(\hat{Y}_t(\ell-p) - \mu\big) \\
&\quad + \phi_1\phi_p\big(\hat{Y}_t(\ell-p-1) - \mu\big) \\
&= (\phi_1^3 + 2\phi_1\phi_2 + \phi_3)\big(\hat{Y}_t(\ell-3) - \mu\big) \\
&\quad + (\phi_1^2\phi_2 + \phi_2^2 + \phi_1\phi_3 + \phi_4)\big(\hat{Y}_t(\ell-4) - \mu\big) \\
&\quad + \cdots \\
&\quad + (\phi_1^2\phi_{p-2} + \phi_2\phi_{p-2} + \phi_1\phi_{p-1} + \phi_p)\big(\hat{Y}_t(\ell-p) - \mu\big) \\
&\quad + (\phi_1^2\phi_{p-1} + \phi_2\phi_{p-1} + \phi_1\phi_p\big(\hat{Y}_t(\ell-p-1) - \mu\big) \\
&\quad + (\phi_1^2\phi_p + \phi_2\phi_p)\big(\hat{Y}_t(\ell-p-2) - \mu\big) \\
&= \cdots
\end{aligned}$$

The pattern is not easy to see. But since the ϕ's decrease exponentially, and as ℓ increases we need to do more substitution steps to get the RHS all observed innovations, we can conclude the weights to the observed innovations get smaller and smaller as ℓ increases. Hence the general observation

$$\hat{Y}_t(\ell) \approx \mu \quad \text{for large } \ell \tag{13}$$

When you forecast a long time ahead, you won't put much faith on the current or recent history; the best bet is to return to the (stationary) mean.

Let's examine the one-step-ahead forecast error:

$$\begin{aligned}
e_t(1) &= Y_{t+1} - \hat{Y}_t(1) \\
&= \Big(\mu + \phi_1\big(Y_t - \mu\big) + \phi_2\big(Y_{t-1} - \mu\big) + \cdots + \phi_p\big(Y_{t+1-p} - \mu\big) + e_{t+1}\Big) \\
&\quad - \Big(\mu + \phi_1\big(Y_t - \mu\big) + \phi_2\big(Y_{t-1} - \mu\big) + \cdots + \phi_p\big(Y_{t+1-p} - \mu\big)\Big) \\
&= e_{t+1}
\end{aligned} \tag{14}$$

This is another general result: when you forecast <u>one step ahead</u>, the error is just one term of the white noise. This is understandable, because our model is always some function of the past plus a white noise, and now the past is known.

What is the error in multiple-step-ahead forecast? This is not obvious. Let's turn to the general linear form

$$Y_{t+\ell} = \mu + e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots \tag{15}$$

Now we see our estimator is actually

$$\begin{aligned}
\hat{Y}_t(\ell) &= \mu + E\big(e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots + \psi_\ell e_t + \psi_{\ell+1} e_{t-1} + \cdots \,|\, Y_t, Y_{t-1}, \cdots, Y_1\big) \\
&= \mu + \psi_\ell e_t + \psi_{\ell+1} e_{t-1} + \cdots
\end{aligned} \tag{16}$$

Note Since this estimator requires the infinite history of e's, we do not actually use this form. We use the recursive form (12). The recursive form is in effect equivalent to the one above.

Note We can also understand (13) from the estimator above because the ψ's decrease very rapidly.

Therefore,

$$e_t(\ell) = e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots + \psi_{\ell-1} e_{t+1} \tag{17}$$

This is another general result. It says the error in the forecast is the future e's weighted by the leading ψ's. We see that $e_t(1)$ is a special case of this formula.

Note Again, we use the recursive forecast to do the actual forecasting; the error is in effect the above.

Now obviously $E\big(e_t(\ell)\big) = 0$ and

$$\operatorname{var}\big(e_t(\ell)\big) = \sigma_e^2(1 + \psi_1^2 + \cdots + \psi_{\ell-1}^2) \tag{18}$$

We see the forecast error variance increases with ℓ.

Based on the general linear model (15), we know $\gamma_0 = \sigma_e^2(1 + \psi_1^2 + \cdots)$. Therefore

$$\operatorname{var}\big(e_t(\ell)\big) \approx \gamma_0, \quad \text{for large } \ell \tag{19}$$

because the ψ's decrease very rapidly. This is again a general result and it forms a nice, intuitive pair with (13).

17.4 MA(q) with constant mean

Model:

$$Y_{t+\ell} - \mu = e_{t+\ell} - \theta_1 e_{t+\ell-1} - \cdots - \theta_q e_{t+\ell-q} \tag{20}$$

Note In (20), μ is $E(Y_t$. This can be seen by taking expectation on both sides.

Given the relations (10) as well as (15)–(17), the MA case is simple. It is in the form of a "general linear model" with a finite number of non-zero ψ's.

$\hat{Y}_e(\ell)$ is μ plus the "observed" e's weighted by θ's:

$$\hat{Y}_t(\ell) = \begin{cases} \mu - \theta_\ell e_t - \theta_{\ell+1} e_{t-1} - \cdots - \theta_q e_{t+\ell-q}, & \ell \le q \\ \mu, & \ell > q \end{cases}$$

using relations (10). The $\ell > q$ case is especially interesting.

Note The use of (10) here involves slight approximation. (10) uses (4.5.5) on page 80, which says if we know the infinite history of Y up to time t, then each e up to time t is determined. However, we don't know the infinite history. An approximation is that we set all unavailable, older Y's to 0, which is equivalent to assuming their corresponding π coef in (4.5.5) are negligible. Then all the e's up to time t can be computed. Alternatively, we can set all older, unavailable e's to 0, then use the MA model to compute later e's recursively. Either way, the assignment of old Y's or e's to 0 entails approximation.

The forecast error is

$$e_t(\ell) = \begin{cases} -\theta_1 e_{t+\ell-1} - \cdots - \theta_{\ell-1} e_{t+1} & \ell \le q \\ e_{t+\ell} - \theta_1 e_{t+\ell-1} - \cdots - \theta_q e_{t+\ell-q} & \ell > q \end{cases}$$

The error variance can be calculated. We see that the forecast capability of MA(q) "cuts off" after $\ell > q$. The error variance doesn't even increase after that.

17.5 ARMA(p, q) with constant mean

Model:

$$Y_{t+\ell} = \phi_1 Y_{t+\ell-1} + \cdots + \phi_p Y_{t+\ell-p} + \theta_0 + e_{t+\ell} - \theta_1 e_{t+\ell-1} - \cdots - \theta_q e_{t+\ell-q} \tag{21}$$

Note In (21), θ_0 is there to account for non-zero mean, but θ_0 is not the mean. Let $E(Y_t)$ be μ. Taking expectation on both sides we get $\mu = (\phi_1 + \cdots + \phi_p)\mu + \theta_0$, hence $\mu = \theta_0/(1 - \phi_1 - \cdots - \phi_p)$.

ARMA is a combination of AR and MA. Using similar ideas as before, we see

$$\begin{aligned}\hat{Y}_t(\ell) = {} & \phi_1 \hat{Y}_t(\ell - 1) + \cdots + \phi_p \hat{Y}_t(\ell - p) + \theta_0 \\ & - \theta_1 E(e_{t+\ell-1} \,|\, Y_t, Y_{t-1}, \ldots, Y_1) - \cdots - \theta_q E(e_{t+\ell-q} \,|\, Y_t, Y_{t-1}, \ldots, Y_1)\end{aligned}$$

where $\hat{Y}_t(s) = Y_{t+s}$ if $s \le 0$, and

$$E(e_{t+s} \,|\, Y_t, \ldots, Y_1) = \begin{cases} 0, & s > 0 \\ e_{t+s}, & s \le 0 \end{cases}$$

When $\ell > q$, $\hat{Y}_t(\ell)$ is contributed by the AR part (as well as θ_0) only; see (9.3.33) on page 200. Therefore the general nature of long-term forecast is determined by the AR coefs of the model.

The ARMA model can also be written in the general linear form (15). With that from, we again get (16), (17), (18), and (13), (19).

17.6 Updating forecasts

Suppose at time t we forecast $Y_t(1), \ldots, Y_t(\ell)$. At time 1, we have observed Y_{t+1}, so we have better ideas about $Y_{t+2}, \ldots, Y_{t+\ell}$. Certainly we can re-forecast based on the now current time $t + 1$. But there is a short cut: we can update the previous forecasts by a simply correction.

Take the general form (15) and the forecast (16). We see

$$\begin{aligned}\hat{Y}_t(\ell + 1) &= \mu + \psi_{\ell+1} e_t + \psi_{\ell+2} e_{t-1} + \cdots \\ \hat{Y}_{t+1}(\ell) &= \mu + \psi_\ell e_{t+1} + \psi_{\ell+1} e_t + \cdots \\ \hat{Y}_t(1) &= \mu + \psi_1 e_t + \psi_2 e_{t-1} + \cdots\end{aligned}$$

Also note

$$Y_{t+1} = \mu + e_{t+1} + \psi_1 e_t + \psi_2 e_{t-1} + \cdots$$

Therefore

$$\begin{aligned}\hat{Y}_{t+1}(\ell) &= \hat{Y}_t(\ell + 1) + \psi_\ell e_{t+1} \\ &= \hat{Y}_t(\ell + 1) + \psi_\ell \big(Y_{t+1} - \hat{Y}_t(1)\big)\end{aligned}$$

Hence the previous forecast, $\hat{Y}_t(\ell)$, is updated by the forecast error of the new observation, $Y_{t+1} - \hat{Y}_t(1)$, suitably weighted.

We could get specialized forms if the model is AR or MA.

The same result can also be obtain by looking at the forecast error (17).

17.7 Prediction limits

We often make a plot of forecasts. In addition to the forecast value, it's informative to also give a "confidence bounds" (or "prediction limits"). Computation of these bounds are most generally based on the error variance given by (18).

We could get specialized forms if the model is AR or MA.