STAT 401 Chapter 2.1–2.5, 2.10

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1 Some probability

Appendices A.3, A.4.

X and Y are random variables.
a and b are constants.

1.1 Expected value

Discrete variable X:

$$E[X] = \sum_{i=1}^{n} x_i \, p(x_i)$$

were $x_1, \ldots, x_n$ are all possible values of X, and $p(x_i)$ is the probability of x_i.

Continuous variable X:

$$E[X] = \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x$$

where $f(x)$ is the pdf.

1.2 Variance

$$\operatorname{var}(X) = E\big[\big(X - E[X]\big)^2\big]$$

Discrete variable X:

$$\operatorname{var}(X) = \sum_{i=1}^{n} \big(x_i - E[X]\big)^2 \, p(x_i)$$

Continuous variable X:

$$\operatorname{var}(X) = \int_{-\infty}^{\infty} \big(x - E(X)\big)^2 f(x) \, \mathrm{d}x$$

1.3 Covariance

$$\operatorname{cov}(X,Y) = E\big[\big(X - E[X]\big)\big(Y - E[Y]\big)\big]$$

One can see that $\operatorname{var}(X) \equiv \operatorname{cov}(X,X)$.

Discrete variables X, Y:

$$\operatorname{cov}(X,Y) = \sum_{i=1}^{n} \big(x_i - E[X]\big)\big(y_i - E[Y]\big)\, p(x_i, y_i)$$

where $(x_1, y_1), \ldots, (x_n, y_n)$ are all possible values of the vector (X, Y), and $p(x_i, y_i)$ is the probability of (x_i, y_i).

Continuous variables X, Y:

$$\operatorname{cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(x - E(X)\big)\big(y - E(Y)\big)\, f(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

where $f(x,y)$ is the pdf.

1.4 Properties

1.

$$\operatorname{var}\{X\} = E\{X^2\} - \big(E\{X\}\big)^2$$

Exercise Prove this.

2.

$$X, Y \text{ independent } \Rightarrow \operatorname{cov}\{X,Y\} = 0$$

If X and Y are jointly normal, then

$$X, Y \text{ independent } \Leftrightarrow \operatorname{cov}\{X,Y\} = 0$$

3.

$$E[aX + b] = aE[x] + b$$

4.

$$E\Big[\sum_{i=1}^{n} a_i X_i\Big] = \sum_{i=1}^{n} a_i\, E[X_i]$$

Interpretation: $E[\cdot]$ is a linear operator.

5.

$$\operatorname{var}(a) = 0$$

$$\operatorname{var}(X + b) = \operatorname{var}(X)$$

$$\operatorname{var}(aX + b) = a^2 \operatorname{var}(X)$$

6.

$$\begin{aligned}\operatorname{var}\left(\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\, a_j \operatorname{cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i) + \sum_{\substack{i,j \\ i\neq j}} a_i\, a_j \operatorname{cov}(X_i, X_j)\end{aligned}$$

7. X_i's are independent ⇒

$$\operatorname{var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i)$$

8.

$$\operatorname{cov}\left(\sum_{i=1}^{n} a_i X_i,\, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i\, b_j \operatorname{cov}(X_i, Y_j)$$

1.5 Some univariate distributions

Normal

Standard normal, often denoted by Z.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

On $(-\infty, \infty)$. Symmetric about 0. Single mode, at 0.

The normal distribution is fully described by it mean and variance.

A linear combination of normal variables is a normal variable.

From standard normal to non-standard normal:

$$Z \sim N(0,1) \Rightarrow X = aZ + b \sim N(b, a^2)$$

So see this, first note that X is a linear function of the normal variable Z, hence X is normal. Then we only need to work out the mean and variance of X. Second, we have $E[X] = aE[Z] + b = b$ and $\operatorname{var}(X) = a^2 \operatorname{var}(Z) = a^2$.

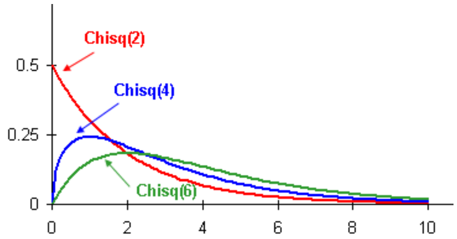
Similarly, from non-standard normal to standard normal (standardization):

$$X \sim N(a, b^2) \Rightarrow Z = \frac{X - a}{b} \sim N(0,1)$$

$\boxed{\chi^2}$

The sum of squared independent standard normal variables is a χ^2 variable:

$$z_1^2 + z_2^2 + \cdots + z_\nu^2 \sim \chi_\nu^2$$

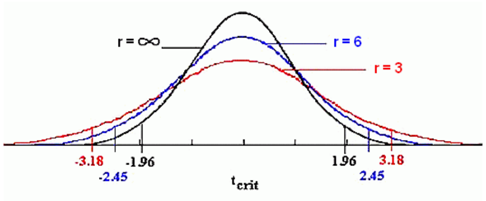


ν: degrees of freedom. Non-negative.

$\boxed{t}$

A t variable is constructed from independent standard normal variable z and χ_ν^2 variable:

$$\frac{z}{\sqrt{\chi_\nu^2/\nu}} \sim t_\nu$$

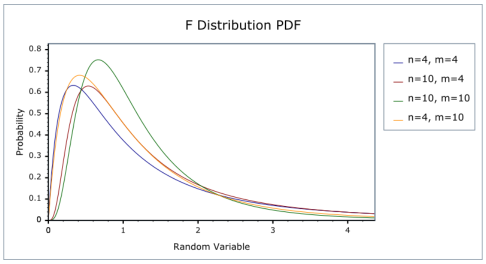


ν: degrees of freedom. On $(-\infty, \infty)$. Symmetric about 0. Single mode, at 0. Similar to, but "fatter" than standard normal.

$\boxed{F}$

The quotient of two independent χ^2 variables is a F variable:

$$\frac{\chi_{\nu_1}^2}{\nu_1} \bigg/ \frac{\chi_{\nu_2}^2}{\nu_2} \sim F_{\nu_1,\nu_2}$$



ν_1, ν_2: degrees of freedom (the order of the two matters). Non-negative.

The square of a t variable is a F variable:

$$t_\nu^2 \sim F_{1,\nu}$$

2 Sampling distributions of $\hat\beta_0$, $\hat\beta_1$, S^2

Continue our analysis of the simple regression model with normal errors $\epsilon_i \overset{iid}{\sim} N(0, \sigma^2)$.

We will make inferences about an unknown parameter, such as β_1, based on data. This is usually carried out starting with an estimator of the parameter, such as

$$\hat\beta_1 = \frac{\sum \big(X_i - \overline{X}\big)\big(Y_i - \overline{Y}\big)}{\sum \big(X_i - \overline{X}\big)^2}.$$

An estimator is a statistic (that is, function) of the data. The value of the estimator (that is, the estimate) changes with the particular data, which are assumed to have arisen according to our model which contains random components. As we obtain another set of data under identical conditions, the random mechanism will work to give rise to a different dataset. Each of such datasets is called a sample. The value of the estimator changes from sample to sample, hence the estimator is a random variable. The distribution of this random variable is called its sampling distribution.

In the context of linear regression, re-sampling entails getting a new set of Y_i values, keeping the set of X_i values unchanged.

Always remember that $\hat\beta_0$ and $\hat\beta_1$ are random variables. (Their values vary as we re-sample the Y's.) We'll use the upper-case S^2 to mean MSE, i.e. SSE$/(n-2)$, emphasizing its being a variable as well, and use s^2 for its value given by a particular data set.

2.1 $\hat\beta_0$ and $\hat\beta_1$ are linear estimators

$$\begin{aligned}
\hat\beta_1 = \frac{S_{xy}}{S_{xx}} &= S_{xx}^{-1} \sum_{i=1}^{n} (x_i - \overline{x})(Y_i - \overline{Y}) \\
&= S_{xx}^{-1} \left(\sum_{i=1}^{n} (x_i - \overline{x})\, Y_i - \overline{Y} \sum_{i=1}^{n} (x_i - \overline{x}) \right) \\
&= S_{xx}^{-1} \sum_{i=1}^{n} (x_i - \overline{x})\, Y_i \\
&= \sum_{i=1}^{n} k_i Y_i,
\end{aligned} \tag{1}$$

where $k_i = (x_i - \overline{x})/S_{xx}$ and we have used the factor $\sum_{i=1}^{n}(x_i - \overline{x}) = 0$.

We have used different cases for X and Y to emphasize that the x's are constants but Y's are variables.

Conclusion: $\hat{\beta}_1$ is a linear function of the Y's.

Similarly,

$$\begin{aligned}\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{x} &= \frac{\sum_i Y_i}{n} - \overline{x}\sum_i k_i Y_i \\ &= \sum_i \Big(\frac{1}{n} - \overline{x}k_i\Big)Y_i,\end{aligned} \tag{2}$$

which is another linear function of the Y's.

Exercise Prove the following properties of the k_i's (page 42):
(1) $\sum k_i = 0$;
(2) $\sum k_i x_i = 1$;
(3) $\sum k_i^2 = 1/S_{xx}$.

2.2 Sampling distributions

Theorem

1. $\hat{\beta}_1 \sim N(\beta_1, \sigma^2/S_{xx})$.
2. $\hat{\beta}_0 \sim N\big(\beta_0, \big(1/n + \overline{x}^2/S_{xx}\big)\sigma^2\big)$.
3. $\mathrm{cov}(\hat{\beta}_0, \hat{\beta}_1) = -\overline{x}\sigma^2/S_{xx}$.
4. S^2 is independent of $\hat{\beta}_0$ and $\hat{\beta}_1$.
5. $\frac{(n-2)S^2}{\sigma^2} \sim \chi^2_{n-2}$. (Note $(n-2)S^2$ is SSE.)
6. $E(S^2) = \sigma^2$.

Partial proof:

Extensively use the model assumptions that the Y_i's are independent normal variables, as well as the properties of the k_i's.

1.

Because $\hat{\beta}_1$ is a linear function of the normal variables Y_i, $\hat{\beta}_1$ is normal. (The independence between the Y_i is not needed for this statement.)

$$E(\hat{\beta}_1) = \sum_i k_i E(Y_i) = \sum_i k_i(\beta_0 + \beta_1 x_i) = \beta_0 \sum_i k_i + \beta_1 \sum_i k_i x_i = \beta_1$$

$$\mathrm{var}(\hat{\beta}_1) = \sum_i k_i^2 \,\mathrm{var}(Y_i) = \sigma^2 \sum_i k_i^2 = \frac{\sigma^2}{S_{xx}}$$

2.

Because $\hat\beta_0$ is a linear function of the normal variables Y_i, $\hat\beta_0$ is normal. (The independence between the Y_i is not needed for this statement.)

$$E(\hat\beta_0) = \sum_i \Big(\frac{1}{n} - \overline{x}k_i\Big) E(Y_i) = \sum_i \Big(\frac{1}{n} - \overline{x}k_i\Big)(\beta_0+\beta_1 x_i) = \Big(1-\overline{x}\sum_i k_i\Big)\beta_0 + \Big(\overline{x}-\overline{x}\sum_i k_i x_i\Big)\beta_1$$

$$\operatorname{var}(\hat\beta_0) = \sum_i \Big(\frac{1}{n} - \overline{x}k_i\Big)^2 \operatorname{var}(Y_i) = \sigma^2 \sum_i \Big(\frac{1}{n^2} - \frac{2\overline{x}k_i}{n} + \overline{x}^2 k_i^2\Big) = \sigma^2 \Big(\frac{1}{n} - \frac{2\overline{x}}{n}\sum_i k_i + \overline{x}^2 \sum_i k_i^2\Big)$$

3.

$$\begin{aligned}\operatorname{cov}(\hat\beta_0, \hat\beta_1) &= \sum_i \sum_j \Big(\frac{1}{n} - \overline{x}k_i\Big) k_j \ \operatorname{cov}(Y_i, Y_j) \\ &= \sum_i \Big(\frac{1}{n} - \overline{x}k_i\Big) k_i \operatorname{var}(Y_i) \\ &= \sigma^2 \Big(\frac{1}{n}\sum_i k_i - \overline{x}\sum_i k_i^2\Big)\end{aligned}$$

where we have used $\operatorname{cov}(Y_i, Y_j) = 0$ if $i \neq j$.

4. Do not worry about the proof.

5. Do not worry about the proof.

6.

From $\frac{(n-2)S^2}{\sigma^2} \sim \chi^2_{n-2}$, we have (per a property of the χ^2 distribution)

$$E\Big(\frac{(n-2)S^2}{\sigma^2}\Big) = n-2$$

hence

$$E(S^2) = \sigma^2$$

Note Not only $\hat\beta_0$ and $\hat\beta_1$ are unbiased, but their sampling variances are the smallest among linear, unbiased estimators of β_0 and β_1.

Proposition 1. $\frac{\hat\beta_1 - \beta_1}{S_{\hat\beta_1}} \sim t_{n-2}$, where $S_{\hat\beta_1} = \sqrt{S^2/S_{xx}}$.

2. $\frac{\hat\beta_0 - \beta_0}{S_{\hat\beta_0}} \sim t_{n-2}$, where $S_{\hat\beta_0} = \sqrt{\big(1/n + \overline{x}^2/S_{xx}\big)S^2}$.

Comparing $\sigma^2_{\hat\beta_1}$ (the variance of the sampling distribution of $\hat\beta_1$) and $S_{\hat\beta_1}$, we see that $S^2_{\hat\beta_1}$ is $\sigma^2_{\hat\beta_1}$ with the unknown σ^2 replaced by its estimator S^2.

Proof

Making use of $\sigma^2_{\hat{\beta}_1} = \sigma^2/S_{xx}$ and $S^2_{\hat{\beta}_1} = S^2/S_{xx}$, write

$$\frac{\hat{\beta}_1 - \beta_1}{S_{\hat{\beta}_1}} = \frac{(\hat{\beta}_1 - \beta_1)/\sigma_{\hat{\beta}_1}}{S_{\hat{\beta}_1}/\sigma_{\hat{\beta}_1}} = \frac{(\hat{\beta}_1 - \beta_1)/\sigma_{\hat{\beta}_1}}{\sqrt{S^2/\sigma^2}} = \frac{(\hat{\beta}_1 - \beta_1)/\sigma_{\hat{\beta}_1}}{\sqrt{\frac{(n-2)S^2/\sigma^2}{n-2}}}$$

Denote the above by $U/\sqrt{V/(n-2)}$. Notice $U \sim N(0,1)$ and $V \sim \chi^2_{n-2}$. If U and V are independent, then $U/\sqrt{V/(n-2)}$ is a t_{n-2} variable. (See "Some probability".) The independence is true, following item 4 of the theorem above.

Proof for item 2 is similar.

Note We will see this pattern of going from $N(0,1)$ to t again. If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$. If we replace σ by an estimate, say s, then often $\frac{X-\mu}{s} \sim t$ (of a suitable degrees of freedom). This makes good sense: (1) An estimate of σ introduces more uncertainties, making $\frac{X-\mu}{s}$ more variable than $\frac{X-\mu}{\sigma}$. Incidentally, a t distribution is "fatter" than $N(0,1)$. (2) On the one hand, as n increases, the estimate s should become steadily close to σ, hence $\frac{X-\mu}{s}$ approaches becomes more and more like a standard variable. On the other hand, the t distribution approaches $N(0,1)$ as its "degree of freedom" increases.

3 Review of confidence interval and hypothesis test

(See handout `ch02b.pdf` for additional reading.)

We use β_1 and $\hat{\beta}_1$ as examples to review the concepts of confidence interval and hypothesis test. To begin with, we know $\hat{\beta}_1$ is an estimator of the unknown parameter β_1, and the sampling distribution of $\hat{\beta}_1$ is $N(\beta_1, \sigma^2/S_{xx})$. This is an unbiased estimator because $E(\hat{\beta}_1) = \beta_1$. Given the unbiasedness, the sampling variance indicates how good an estimator it is. (The smaller the sampling variance, the better.)

3.1 Confidence interval

$\hat{\beta}_1$ provides a point estimate, i.e. a single value. However, this single value is essentially guaranteed to be different from the true value β_1 (although we hope it gets close). In contrast, a confidence interval (CI) is an interval constructed such that it contains the true value of the parameter with a specified percentage (say 95%) if we were to re-sample the data and re-construct the CI (which would vary with the data) many, many times.

The "capture" percentage (95%, say) is called the "confidence level". Let $\alpha = 0.05$, then $0.95 = 1-\alpha$ is the "degree of confidence" or "confidence coefficient", and 95% is the "confidence level". We will speak of "constructing a CI at the 95% confidence level" or "constructing a 95% CI".

Suppose σ^2 is known, hence the sampling variance $\sigma^2_{\hat{\beta}_1} = \sigma^2/S_{xx}$ is known.

Let

$$\Delta = z_{\alpha/2}\, \sigma_{\hat{\beta}_1}$$

where $z_{\alpha/2}$ is such that $P(Z > z_{\alpha/2}) = \alpha/2$ for the standard normal variable Z. In `R` we find $z_{0.025} = \texttt{qnorm}(0.975) = 1.96$.

Then,

$$\begin{aligned}
&P(\beta_1 - \Delta < \hat{\beta}_1 < \beta_1 + \Delta) \\
&= 2\,P(\beta_1 < \hat{\beta}_1 < \beta_1 + \Delta) \\
&= 2\,P\Big(\frac{\beta_1 - \beta_1}{\sigma_{\hat{\beta}_1}} < \frac{\hat{\beta}_1 - \beta_1}{\sigma_{\hat{\beta}_1}} < \frac{\beta_1 + \Delta - \beta_1}{\sigma_{\hat{\beta}_1}}\Big) \\
&= 2\,P\big(0 < Z < z_{\alpha/2}\big) \\
&= 2\,(0.5 - \alpha/2) \\
&= 1 - \alpha \quad (= 0.95)
\end{aligned} \tag{3}$$

Note that $\frac{\hat\beta_1-\beta_1}{\sigma_{\hat\beta_1}}$ is a standard normal variable (Z) because $\hat\beta_1 \sim N(\beta_1, \sigma^2_{\hat\beta_1})$.

The relation (3) says that if we sample $\hat\beta_1$ (that is, re-sample data Y and calculate $\hat\beta_1$) repeatedly, then in roughly 95% of the repetitions the estimate $\hat\beta_1$ falls between $\beta_1 - \Delta$ and $\beta_1 + \Delta$.

However, we don't know β_1, and our goal is not to say how $\hat\beta_1$ behaves but to say something about β_1.

The probability statement (3), that is, $P(\beta_1 - \Delta < \hat\beta_1 < \beta_1 + \Delta) = 0.95$, can be re-phrased as

$$P(\hat\beta_1 - \Delta < \beta_1 < \hat\beta_1 + \Delta) = 0.95. \tag{4}$$

This statement holds, because the relations

$$\hat\beta_1 - \Delta < \beta_1 < \hat\beta_1 + \Delta$$

and

$$\beta_1 - \Delta < \hat\beta_1 < \beta_1 + \Delta$$

are equivalent.

How can we interpret (4)?

We can not say something like this: In repeated sampling of $\hat\beta_1$, 95% of the time β_1 falls between $\hat\beta_1 - \Delta$ and $\hat\beta_1 + \Delta$. That sounded like β_1 is changeable, and it could "fall" somewhere. We know β_1 is fixed (although unknown). It sits somewhere still; it won't "fall" here in this sampling and there in another sampling.

Instead, we say this: In repeated sampling of $\hat\beta_1$ and with the interval $(\hat\beta_1 - \Delta,\ \hat\beta_1 + \Delta)$ constructed in each sampling, 95% of the time the interval contains the true value of β_1.

In reality we have just one dataset and one value of $\hat\beta_1$ (let's write this value as b_1). With this single estimate, we build an interval $(b_1 - \Delta,\ b_1 + \Delta)$, and call it the 95% confidence interval for β_1. We don't have another dataset (with the same set of X's), hence we can't "re-sample $\hat\beta_1$". The one interval $(b_1 - \Delta,\ b_1 + \Delta)$ either contains β_1 or it does not; it makes no sense to say "it contains β_1 with probability 0.95". (Only when we have many, we can say "95% of them..."). Therefore, we interpret the CI with the rhetoric "If we were able to re-sample and repeat..., then with 0.95 probability the interval would contain the true value." The logic is, since we happened to have got this dataset, it is just like one of the samples that we could randomly draw; and now we follow the same recipe to construct the interval, there is a good chance that this interval does contain the true value of β_1.

Note that Δ is a known value because we assumed $\hat{\beta}_1$ is unknown. (In reality we don't know σ^2, hence some adjustment is necessary.)

3.2 Hypothesis tests

Hypothesis

Hypothesis is a statement about an unknown parameter, say β_1, for example, $\beta_1 = 1$; $5.2 < \beta_1 < 5.8$; $\beta_1 > 0$.

A "null" hypothesis, H_0, is a statement that is taken by default unless sample data provide strong evidence against it. An "alternative" hypothesis, H_a, is the opposite of H_0.

A hypothesis test procedure looks into the sample data and determines whether there is strong evidence that H_0 should be rejected.

There are two possible conclusions of a test:
(1) reject H_0;
(2) fail to reject H_0.

Note, we do not say accept H_0; it's a lack of disproof, which is not the same as proof. We do not say "we believe H_0 is true". We're not sure about that; it's just that evidence against it is not strong enough (by a prescribed criterion).

In this course, we always take H_0 to be an equality to a particular number, say β_{1*}:
H_0: $\beta_1 = \beta_{1*}$

H_a has three possible forms:
H_a: $\beta_1 > \beta_{1*}$
H_a: $\beta_1 < \beta_{1*}$
H_a: $\beta_1 \neq \beta_{1*}$

Test procedure

Take a function of the sample data, say T, called "test statistic". (Any function of the sample data is called a "statistic".) For example $\max(y_1, \ldots, y_n)$, $\overline{y}$, etc. This T is a deliberate choice such that it is connected with β_1 and H_0.

We identify a rejection region, i.e. a set of values (discrete numbers or interval). Let's call it R.

Procedure of a test:

1. Define a test statistic T and the rejection region R.
2. Calculate the value of T using the sample data.
3. Draw a conclusion:
   reject H_0 if $T \in R$; do not reject H_0 if $T \notin R$.

Take the β_1 example. We know the estimator $\hat{\beta}_1$ has sampling distribution $N(\beta_1, \sigma^2/S_{xx})$. Suppose σ^2, hence $\sigma^2_{\hat{\beta}_1}$, is known.

How do we define a rejection region R?

This depends on how safe (conservative) we want our conclusion to be. There is always a chance to be wrong (otherwise there is no problem of a "hypothesis"). Our definition of the rejection region is dictated by what kind of mistakes we try to avoid and what kind of mistakes are less devastating, and considerations like that.

Definition Type I error: reject H_0 but it's a mistake (in fact H_0 is true).
Type II error: do not reject H_0 but it's a mistake (in fact H_0 is false).

Suppose H_a is $\beta_1 \neq \beta_{1*}$. This is called a "two-sided" test. We control the chance of making a type I error. Specifically, we will define a rejection region such that the probability of making a type I error is $\alpha = 0.05$. (This α is called the "significance level".) Like the concept of confidence interval, this probability α has to be understood in a "repeatable experiment" setting. Suppose we could re-sample data Y. With each re-sampling, we determine R and draw a conclusion. In many such repetitions, we require our recipe of reaching a conclusion (the center piece of which is how to determined R) is such that at most 5% of the time the conclusion is wrong.

Suppose we simply take $\hat{\beta}_1$ as the test statistic. (This is valid because $\hat{\beta}_1$ is a function of the sample data, hence a statistic.) How should we choose the rejection R?

Let's ponder again how we draw a conclusion and when we make a type I error—We make a type I error when H_0 is true but we reject it, and we reject it only when $\hat{\beta}_1 \in R$. Note that, to control type I error, we only need to care about the situation where H_0 is actually true, and in this situation we make a mistake if and only if the calculated $\hat{\beta}_1$ falls in R. To make the probability of this mistake 0.05, we must have $P(\hat{\beta}_1 \in R) = 0.05$.

In the "confidence interval" section we have seen that

$$P(\beta_1 - \Delta < \hat{\beta}_1 < \beta_1 + \Delta) = 1 - \alpha$$

in other words,

$$P(\hat{\beta}_1 < \beta_1 - \Delta \text{ or } \hat{\beta}_1 > \beta_1 + \Delta) = \alpha \quad (= 0.05)$$

Now we are assuming β_{1*} is indeed the true β_1 value. Replacing β_1 in the above by β_{1*}, we define

$$R = (-\infty, \beta_{1*} - \Delta) \cup (\beta_{1*} + \Delta, \infty)$$

Therefore the decision rule is
`Reject` H_0 `if` $\hat{\beta}_1 \in R$.
or equivalently,
`Reject` H_0 `if` $|\hat{\beta}_1 - \beta_{1*}| > \Delta$.

When actually conducting the test, we should replace $\hat{\beta}_1$ by its actual value b_1 that is provided by the data.

Equivalently, we may choose

$$T = \frac{\hat{\beta}_1 - \beta_{1*}}{\sigma_{\hat{\beta}_1}}$$

to be the test statistic. Then the rejection region becomes

$$R = (-\infty, -z_{\alpha/2}) \cup (z_{\alpha/2}, \infty)$$

The decision rule becomes
`Reject` H_0 `if` $|T| > z_{\alpha/2}$.

Another try to understand the logic in the test procedure—Under the assumption that $\beta_1 = \beta_{1*}$, a low-probability event (i.e. $T \in R$) has happened. Now we (1) have sound probability theory and (2) do not think the only dataset we have is a strange thing. There is a contradiction here, because the observed T is a rare event. We argue that the cause of the awkward situation is the assumption—that $\beta_1 = \beta_{1*}$. We reject the assumption and conclude that the true β_1 is some other value. Starting with that true value (if we knew it) the calculated T would not be a rare event (hence our dataset is not a statistically rare observation).

To recap, the significance level, α, is the probability of making type-I error. α is specified by us. It affects the value of Δ, hence the rejection region, through the requirement $P(\hat{\beta}_1 \in R) = \alpha$.

In contrast, the probability of type II error is not in our control. In fact, to calculate it requires knowing the true value of β_1.

P value

Assuming $H_0 : \beta_1 = \beta_{1*}$ is true, we can examine the relative standing of the observed estimator, b_1, in the sampling distribution of $\hat{\beta}_1$. Suppose $b_1 > \beta_{1*}$. Let

$$p_1 = P(\hat{\beta}_1 > b_1) = P\left(\frac{\hat{\beta}_1 - \beta_{1*}}{\sigma_{\hat{\beta}_1}} > \frac{b_1 - \beta_{1*}}{\sigma_{\hat{\beta}_1}}\right) = P\left(Z > \frac{b_1 - \beta_{1*}}{\sigma_{\hat{\beta}_1}}\right)$$

where Z is the standard normal variable and b_1, β_{1*}, and $\sigma_{\hat{\beta}_1}$ are all known values. Therefore p_1 can be calculated. This value is the probability in (imagined) repeated samplings that $\hat{\beta}_1$ is more distant from β_{1*} on the larger side (meaning $\hat{\beta}_1 > \beta_{1*}$) than the observed value b_1 is from β_{1*}.

Since the alternative being considered here is two-sided, we want the probability that $\hat{\beta}_1$ is farther away from β_{1*} than the observed b_1 is, on either side (either far below β_{1*} or far above β_{1*}), we take

$$p = P\big(|\hat{\beta}_1 - \beta_{1*}| > |b_1 - \beta_{1*}|\big) = 2p_1$$

called the "p value". p-value is the probability of obtaining a test statistic (here we are using $\hat{\beta}_1$ itself as the test statistic) at least as extreme as the one that was actually observed. A small p value is evidence against H_0, because it indicates the actually observed test statistic (here, b_1) is a rare event in its sampling distribution, if H_0 is true.

In sum, confidence interval and hypothesis test both rely on our knowing the sampling distribution, hence being able to do certain calculations.

We may take either of the following two ways to do hypothesis test:

- Find the rejection region R and examine whether $T \in R$. If yes, reject H_0.

- Calculate the p value for the test statistic and compare p to α. If $p < \alpha$, then reject H_0.

4 Inferences concerning β_1

The "inference" here refers to (1) construction of confidence intervals, and (2) hypothesis tests. You can also think of point estimation (i.e. getting $\hat{\beta}_1$) as another aspect of inference.

If we know σ^2, then since $\hat{\beta}_1 \sim N(\beta_1, \sigma^2_{\hat{\beta}_1})$ where $\sigma^2_{\hat{\beta}_1} = \sigma^2/S_{xx}$, both confidence interval and hypothesis test are based on standard normal. For example, the $(1-\alpha)$ CI of β_1 is

$$(\hat{\beta}_1 - z_{\alpha/2}\sigma_{\hat{\beta}_1},\ \hat{\beta}_1 + z_{\alpha/2}\sigma_{\hat{\beta}_1}),$$

where $z_{\alpha/2}$ is the "critical value" defined by $P(Z > z_{\alpha/2}) = \alpha/2$ for the standard normal variable Z. For a two-sided test $H_0 : \beta_1 = \beta_{1*}$ versus $H_a : \beta_1 \neq \beta_{1*}$, the null hypothesis would be rejected if

$$\frac{|\hat{\beta}_1 - \beta_{1*}|}{\sigma_{\hat{\beta}_1}} > z_{\alpha/2}.$$

In reality σ^2 is unknown, and we have to use an estimate of it, s^2, and use the property $\frac{\hat{\beta}_1 - \beta_1}{S_{\hat{\beta}_1}} \sim t_{n-2}$. The CI is then

$$(\hat{\beta}_1 - t_{\alpha/2,\,n-2}s_{\hat{\beta}_1},\ \hat{\beta}_1 + t_{\alpha/2,\,n-2}s_{\hat{\beta}_1}),$$

where $t_{\alpha/2,\,n-2}$ is the "critical value" defined by $P(T > t_{\alpha/2,\,n-2}) = \alpha/2$ for the t_{n-2} variable T. (The textbook uses different notations for critical values.) For the test above, the null hypothesis would be rejected if

$$\frac{|\hat{\beta}_1 - \beta_{1*}|}{s_{\hat{\beta}_1}} > t_{\alpha/2,\,n-2}.$$

Most often, one wants to test whether there is significant linear association between X and Y. That calls for a two-sided test with $H_0 :\ \beta_{1*} = 0$.

Example Page 46.

One-sided tests

For example,
H_0: $\beta_1 = \beta_{1*}$
H_a: $\beta_1 > \beta_{1*}$

Rejection rule:
$\frac{\hat{\beta}_1 - \beta_{1*}}{s_{\hat{\beta}_1}} > t_{\alpha,\,n-2}$ where $t_{\alpha,\,n-2}$ is such that $P(T > t_{\alpha,\,n-2}) = \alpha$ for the t_{n-2} variable T.

For another example,
H_0: $\beta_1 \leq \beta_{1*}$
H_a: $\beta_1 > \beta_{1*}$

Rejection rule is, again,
$\frac{\hat{\beta}_1 - \beta_{1*}}{s_{\hat{\beta}_1}} > t_{\alpha, n-2}$

In the last example, we plug in the hypothesized value (β_{10}) and calculate based on that value. Now α is the probability of type-I error if $\beta_1 = \beta_{1*}$, which is not exactly H_0. (We can calculate the probability of type-I error only if we plug in a specific number for the unknown, that is, assuming the unknown is a certain value; we can not plug in the interval $(-\infty, \beta_{1*}]$.)

Example Tree diameter (X) versus height (Y) example. Suppose $n = 50$, $S_{xx} = 4.0$, $b_1 = 1.2$, $b_0 = 1$, $s^2 = 0.36$, $\overline{x} = 1$.
(1) Construct a 95% CI of β_1.
(2) Test $H_0 : \beta_1 = 0$ vs $H_a : \beta_1 \neq 0$ with $\alpha = 0.05$.
(3) Test $H_0 : \beta_1 = 1.0$ vs $H_a : \beta_1 \neq 1.0$ with $\alpha = 0.01$.
(4) Test $H_0 : \beta_1 \leq 1.0$ vs $H_a : \beta_1 > 1.0$ with $\alpha = 0.01$.

Example 1 on page 47.

Example 2 on page 47.

Two procedures for hypothesis tests:

Procedure 1

1. Choose a test statistic T (whose sample distribution is known under assumption H_0).

2. Determine the "rejection region" R for the test. (This will be related to the sampling distribution, the hypothesized value in H_0, the α level, and the nature of H_0 and H_1, meaning one-sided or two-sided, etc.)

3. Calculate the test statistic, T^*, using the data and fitted model.

4. Check whether $T^* \in R$ and draw a conclusion:

   If $T^* \in R$: reject H_0.
   If $T^* \notin R$: fail to reject H_0.

Procedure 2

1. Choose a test statistic T (whose sample distribution is known under assumption H_0).

2. Calculate the test statistic, T^*, using the data and fitted model.

3. Calculate the p value of T^*. (This will be related to the sampling distribution, the hypothesized value in H_0, and the nature of H_0 and H_1, meaning one-sided or two-sided, etc.)

4. Draw a conclusion for any specified α:

   If $p \leq \alpha$: reject H_0.
   If $p > \alpha$: fail to reject H_0.

Comments

1. Effects of departures from normality

If the normality assumption about ϵ (or equivalently, Y) is violated but n is large, $\hat{\beta}_0$ and $\hat{\beta}_1$ are approx normal, per Central Limit Theorem. The previous inference and test procedures are still usable.

2. Interpretation of the "confidence level" and "significance level"

Again, X is fixed and Y is random. See previous sections of this handout and handout `ch02b.pdf`.

3. Spacing of the X levels

$\sigma_{\hat{\beta}_1}$ and $\sigma_{\hat{\beta}_0}$ are affected by the spread of the X levels as indicated by S_{xx}. For $\hat{\beta}_1$, large S_{xx} reduces its sampling variance.

4. α versus p value in tests

In tests, I prefer to always report the p value. This is more useful than just reporting the conclusion according to a specific "significance level" α. Given the p value, it's up to the user to decide whether it is "significant". Given the conclusion at a specific α, the user does not know how much "extra room" is left.

When reporting the p value, report the H_0 and H_a as well. Otherwise the meaning of the p value may be unclear.

5. Power of tests

Type-I error: reject H_0, wrongly.
Type-II error: accept H_0, wrongly.

Let
$P(\text{type-I error}) = \alpha$
$P(\text{type-II error}) = \beta$

$1-\beta$ is called the "power" of the test, which is the probability that H_0 is rejected when H_a is actually true (hence rejection is the right conclusion). Whereas the "significance level" α is controlled (specified) by us, the power can not be known exactly, because it depends on the true value of the unknown parameter (say β_1).

6. How to obtain the critical value $t_{\alpha/2,\, n-2}$

Method 1: look up table B.2.

Method 2: use `R` function `qt`$(1-\alpha/2, n-2)$.

5 Inferences concerning β_0

Same idea as what we do for β_1. The sampling distribution is normal; only the distribution parameters are different.

6 Inferences on $E(Y)$ at a specific X level

We already have a point estimator for $E(Y \mid X = x_*)$; it is

$$\hat{Y}_* = \hat{\beta}_0 + \hat{\beta}_1 x_*.$$

Since $\hat{\beta}_0$ and $\hat{\beta}_1$ are both random, we want describe the uncertainty in $\hat{Y}_*$. This needs the sampling distribution of $\hat{Y}_*$. Since $\hat{Y}_*$ is a linear function of the <u>normal</u> random variables $\hat{\beta}_0$ and $\hat{\beta}_1$, we know $\hat{Y}_*$ is also normal.

Theorem $E(\hat{Y}_*) = \beta_0 + \beta_1 x_*$
$\operatorname{var}(\hat{Y}_*) = \sigma^2\left(\frac{1}{n} + \frac{(x_* - \overline{x})^2}{S_{xx}}\right)$

Remark on notation: x_* is any value one wishes to specify (within the scope of the observed x); it may or may not be in the observed data set. $\overline{x}$ and S_{xx} are from the data set which has given rise to the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$; they are not recalculated to include the contribution of x_*.

Proof

$$E(\hat{Y}_*) = E(\hat{\beta}_0 + \hat{\beta}_1 x_*) = E(\hat{\beta}_0) + x_* E(\hat{\beta}_1) = \beta_0 + \beta_1 x_*$$

$$\hat{Y}_* = \hat{\beta}_0 + \hat{\beta}_1 x_* = \sum \Big(\frac{1}{n} - \overline{x} k_i\Big) Y_i + \sum k_i Y_i x_* = \sum \Big(\frac{1}{n} - \overline{x} k_i + x_* k_i\Big) Y_i$$

(The k_i's were introduced in section 2.1 of this note.) Using the independence of Y_i, and properties of k_i:

$$\operatorname{var}(\hat{Y}_*) = \sum \Big(\frac{1}{n} - \overline{x} k_i + x_* k_i\Big)^2 \operatorname{var}(Y_i) = \cdots = \Big(\frac{1}{n} + \frac{(x_* - \overline{x})^2}{S_{xx}}\Big) \sigma^2$$

Since we don't know σ^2, we substitute the estimate s^2 (that is, MSE) for σ^2. There is a t_{n-2} variable like before, and CI construction and tests follow.

Theorem

$$\frac{\hat{Y}_* - (\beta_0 + \beta_1 x_*)}{s\sqrt{\frac{1}{n} + \frac{(x_* - \overline{x})^2}{S_{xx}}}} \sim t_{n-2}$$

Note 1. We see $\hat{Y}_*$ is an unbiased estimator for $E(Y \mid X = x_*)$.

2. Examine the formula of $\operatorname{var}(\hat{Y}_*)$: the sampling variance is small when x_* is close to $\overline{x}$. Implications for application: if we have a target x_* location for which an estimate of the $E(Y)$ is needed, obtain data with X values surrounding this x_*. (Sounds natural.)

Example 1 on page 54.

Example 2 on page 55.

Example Tree example on page 16 of this note. Derive a 95% CI for $E[Y(x = 1.5)]$.

7 Prediction interval for a new observation

Suppose we are about to observe a new Y at $X = x_*$, and we want to "predict" the Y value prior to the observation. Since the new observation is a particular value a random variable $(Y \mid X = x_*)$ will take by chance, it makes little sense to "predict" what the value will happen to be. What we can do is find the distribution of the random variable; after that we'll be able to make statistical statements like "the new observation will fall in a certain interval with certain probability".

At this point, our best guess of the unobserved Y is

$$\hat{Y}_* = \hat{\beta}_0 + \hat{\beta}_1 x_*,$$

whereas the "real thing" is

$$Y_* = \beta_0 + \beta_1 x_* + \epsilon.$$

Two factors contribute to the deviation of Y_* from the estimate $\hat{Y}_*$: (1) $\hat{Y}_*$ is an estimate of $\beta_0 + \beta_1 x_*$, the mean of the distribution of Y_*; this estimate may be off. (2) Y_* contains random fluctuation ϵ around its mean $\beta_0 + \beta_1 x_*$.

Notice that $Y_* - \hat{Y}_* = \beta_0 + \beta_1 x_* + \epsilon - \hat{Y}_*$ is a normal variable, because $\hat{Y}_*$ and ϵ are both normal variables and $\beta_0 + \beta_1 x_*$ is a constant (although unknown). Furthermore,

$$\begin{aligned} E\big(Y_* - \hat{Y}_*\big) &= \beta_0 - \beta_1 x_* + E(\epsilon) - E\big(\hat{Y}_*\big) = 0, \\ \operatorname{var}\big(Y_* - \hat{Y}_*\big) &= \operatorname{var}\big(\hat{Y}_*\big) + \operatorname{var}(\epsilon) = \Big(\frac{1}{n} + \frac{(x_* - \overline{x})^2}{S_{xx}}\Big)\sigma^2 + \sigma^2, \end{aligned}$$

in which we have used the independence between $\hat{Y}_*$ and ϵ. Substituting the estimate s^2 for σ^2, we have

$$\frac{Y_* - \hat{Y}_*}{s\sqrt{\frac{1}{n} + \frac{(x_* - \overline{x})^2}{S_{xx}} + 1}} \sim t_{n-2}$$

Based on this we construct a $(1 - \alpha)$ "prediction interval" for the future observation, bounded by

$$\hat{Y}_* \mp t_{\alpha/2,\, n-2}\big(1 + 1/n + (x_* - \overline{x})^2/S_{xx}\big)^{1/2} s$$

Note 1. Note the terminology—we call it "prediction interval" instead of "confidence interval".

2. We don't do hypothesis tests for for a future observation, because it does not make sense.

3. Unlike the cases for $\hat{\beta}_0$, $\hat{\beta}_1$, and $E(Y)$, the construction of prediction interval here is sensitive to departures from normality of the random term ϵ. The reason is that a significantly non-normal ϵ would make $\hat{Y}_* - Y_*$ non-normal.

Example on page 59.

Example Tree example on page 16 of this note. Derive a 90% prediction interval for a future observation at $x = 1.5$.

8 Considerations in application

Read section 2.10, page 77.