Zermelo-Fraenkel Set Theory

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Contents

1 Language

The symbols of the Zermelo-Fraenkel set theory $\mathcal{T}$ (or simply the theory $\mathcal{T}$) are the following:

- Variable symbols: $x, y, z, \cdots$
- Logical symbols: $\square$, τ, $\neg$, $\implies$
- Predicate symbols: $=$, $\in$
- Auxiliary symbols: $($, $)$

Formulas (relations or terms) in the theory $\mathcal{T}$ are those sequences of symbols which can be constructed by the following rules:

- If x is a variable symbol, then x is a term.
- If T and U are terms, then $(T \in U)$ and $(T = U)$ are relations.
- If R is a relation, then $(\neg R)$ is a relation.
- If R and S are relations, then $(R \implies S)$ is a relation.
- If R is a relation, x is a variable symbol, then $\tau_x R$, which is τR with each occurrence of x replaced by the symbol $\square$ and connected to the front most τ, is a term.

Example 1.

$$\tau \quad (\quad \neg \quad (\quad \square \quad \in \quad y \quad) \quad)$$

is a term abbreviated by $\tau_x(\neg(x \in y))$.

Some abbreviations are made for convenience, which are shown as follows:

- If R is a relation, T is a term, x is a variable symbol, then we use $(T|x)R$ for the relation obtained by replacing each occurrence of x in R with the term T.
- If R, S are relations, then we use $(R \vee S)$ for $((\neg R) \implies S)$, use $(R \wedge S)$ for $(\neg((\neg R) \vee (\neg S)))$ and use $(R \iff S)$ for $((R \implies S) \wedge (S \implies R))$.
- If R is a relation, x is a variable symbol, then we use $\exists_x R$ for $(\tau_x R|x)R$ and use $\forall_x R$ for $(\tau_x(\neg R)|x)R$.
- If R is a relation, T is a term, x is a variable symbol, then we use $\forall_{x \in T} R$ for $\forall_x((x \in T) \implies R)$ and use $\exists_{x \in T} R$ for $\exists_x((x \in T) \wedge R)$.
- If R is a relation, x and y are different variable symbols, then we use $\exists!_x R$ for $\exists_x \forall_y (R \iff (x = y))$.

2 Axioms

Certain relations in theory T are called axioms.
Theory $\mathcal{T}$ contains the following rules to construct logical axioms:

- If R and S are relations, then

  $$(R \implies (S \implies R))$$

  is an axiom.
- If R and S are relations, then

  $$(((\neg R) \implies (\neg S)) \implies (S \implies R))$$

  is an axiom.
- If R, S and G are relations, then

  $$((R \implies (S \implies G)) \implies ((R \implies S) \implies (R \implies G)))$$

  is an axiom.

- If R is a relation, T is a term, x is a variable symbol, then

$$((T|x)R \implies \exists_x R)$$

is an axiom.

- If R is a relation, T and U are terms, x is a variable symbol, then

$$((T = U) \implies ((T|x)R \iff (U|x)R))$$

is an axiom.

- If R and S are relations, x is a variable symbol, then

$$(\forall_x (R \iff S) \implies (\tau_x R = \tau_x S))$$

is an axiom.

In the rest of this section, we assume x, y, z, s are different variable symbols. Theory $\mathcal{T}$ contains the following set-theoretic axioms:

- **Axiom of Empty Set**

$$\exists_s \forall_x (\neg (x \in s))$$

We use $\emptyset$ as an abbreviation of

$$\tau_s \forall_x (\neg (x \in s)).$$

- **Axiom of Extensionality**

$$\forall_x \forall_y \forall_z (((z \in x) \iff (z \in y)) \implies (x = y))$$

- **Axiom of Pairing**

$$\forall_x \forall_y \exists_s \forall_z ((z \in s) \iff ((z = x) \vee (z = y)))$$

If T, U are terms, then we use $\{T, U\}$ as an abbreviation of

$$\tau_s \forall_z ((z \in s) \iff ((z = T) \vee (z = U))).$$

Especially, we use $\{T\}$ as the same as $\{T, T\}$.

- **Axiom of Power Set**

$$\forall_y \exists_s \forall_z ((z \in s) \iff \forall_{x \in z} (x \in y))$$

If T is a term, then we use $\mathcal{P}(T)$ as an abbreviation of

$$\tau_s \forall_z ((z \in s) \iff \forall_{x \in z} (x \in T)).$$

- **Axiom of Union**

$$\forall_y \exists_s \forall_z ((z \in s) \iff \exists_{x \in y}(z \in x))$$

If T is a term, then we use $\bigcup T$ as an abbreviation of

$$\tau_s \forall_z ((z \in s) \iff \exists_{x \in T}(z \in x)).$$

- **Axiom of Infinity**

$$\exists_s ((\emptyset \in s) \wedge \forall_{x \in s}(\bigcup \{x, \{x\}\} \in s))$$

We use Ω_0 as an abbreviation of

$$\tau_s ((\emptyset \in s) \wedge \forall_{x \in s}(\bigcup \{x, \{x\}\} \in s)).$$

- **Axiom of Regularity**

$$\forall_y ((\neg(y = \emptyset)) \implies \exists_{x \in y} \forall_{z \in y}(\neg(z \in x)))$$

Theory $\mathcal{T}$ contains the following rules to construct axioms:

- **Rule of Separation**

If R is a relation which does not contain the variable symbol s, then

$$\forall_y \exists_s \forall_x ((x \in s) \iff ((x \in y) \wedge R))$$

is an axiom.

If R is a relation which does not contain the variable symbol s, T is a term, then we use $\{x \in T | R\}$ as an abbreviation of

$$\tau_s \forall_x ((x \in s) \iff ((x \in T) \wedge R)).$$

- **Rule of Replacement**

If R is a relation which does not contain the variable symbol s, then

$$\forall_z (\forall_{x \in z} \exists !_y R \implies \exists_s \forall_y ((y \in s) \iff \exists_{x \in z} R))$$

is an axiom.

If R is a relation which does not contain the variable symbol s, T is a term, then we use $R[T]$ as an abbreviation of

$$\tau_s \forall_y ((y \in s) \iff \exists_{x \in z} R).$$

3 Rules of Inference

The relation R is called a thoerem of the theory $\mathcal{T}$, if at least one of the following conditions is satisfied:

- R is an axiom of the theory $\mathcal{T}$.

- There are theorems S, G of the theory $\mathcal{T}$ such that G is $(S \implies R)$.